Munkres 1.3

1. Define two points (x_0, y_0) and (x_1, y_1) of the plane to be equivalent if $y_0 - x_0^2 = y_1 - x_1^2$. Check that this is an equivalence relation and describe the equivalence classes.
   Reflexivity: Let (x_0, y_0). $y_0 - x_0^2 = y_0 - x_0^2$ so $(x_0, y_0) \sim (x_0, y_0)$ for every (x_0, y_0).
   Symmetry: Let (x_0, y_0) and (x_1, y_1) such that $(x_0, y_0) \sim (x_1, y_1)$. $y_0 - x_0^2 = y_1 - x_1^2$. Equality is symmetric, so $y_1 - x_1^2 = y_0 - x_0^2$, so $(x_1, y_1) \sim (x_0, y_0)$. Thus, $(x_0, y_0) \sim (x_1, y_1) \Rightarrow (x_1, y_1) \sim (x_0, y_0)$.
   Transitivity: Let (x_0, y_0), (x_1, y_1), and (x_2, y_2) such that $(x_0, y_0) \sim (x_1, y_1)$ and $(x_1, y_1) \sim (x_2, y_2)$. $y_0 - x_0^2 = y_1 - x_1^2$ and $y_1 - x_1^2 = y_2 - x_2^2$. By the transitive property of equality, $y_0 - x_0^2 = y_2 - x_2^2$, so $(x_0, y_0) \sim (x_2, y_2)$. Thus $((x_0, y_0) \sim (x_1, y_1) \wedge (x_1, y_1) \sim (x_2, y_2)) \Rightarrow (x_0, y_0) \sim (x_2, y_2)$.
   The relation is satisfies all three properties of an equivalence relation.

   An equivalence class of this relation determined by an element (x_0, y_0) is the set of all points (x, y) such that $y - x^2 = y_0 - x_0^2$. In other words, let $a = y_0 - x_0^2$. For any value of x, $(x, x^2 + a)$ is in the equivalence class. Note that for any point (x, y) in the equivalence class, $(-x, y)$ is also in the equivalence class.

2. Let C be a relation on a set A. If $A_0 \subset A$, define the ***restriction*** of C to A_0 to be the relation $C \cap (A_0 \times A_0)$. Show that the restriction of an equivalence relation is an equivalence relation.
   Define C_{A_0} to be the restriction of C to A_0.
   Lemma: $xC_{A_0}y \Rightarrow xCy$.
   $xC_{A_0}y \iff (x, y) \in C \cap (A_0 \times A_0) \iff (x, y) \in C \wedge (x, y) \in (A_0 \times A_0)$. Thus, by simplifying, $(x, y) \in C \iff xCy$.

   Reflexivity: Let $x \in A_0$ (and thus also $x \in A$). $x \in A$, so xCx, so $(x, x) \in C$. $x \in A_0$, so $(x, x) \in (A_0 \times A_0)$. $(x, x) \in C \wedge (x, x) \in (A_0 \times A_0)$, so $(x, x) \in C \cap (A_0 \times A_0)$, so $xC_{A_0}x$ for every $x \in A_0$
   Symmetry: Let $x, y \in A_0$ such that $xC_{A_0}y$. $xC_{A_0}y \Rightarrow xCy$, so yCx as well. Thus $(y, x) \in C$. $x \in A_0 \wedge y \in A_0$, so $(y, x) \in (A_0 \times A_0)$ too. $(y, x) \in C \cap (A_0 \times A_0)$. Therefore, $xC_{A_0}y \Rightarrow yC_{A_0}x$.
   Transitivity: Let $x, y, z \in A_0$ such that $xC_{A_0}y$ and $yC_{A_0}z$. $xCy \wedge yCz$, so xCz, so $(x, z) \in C$. $x \in A_0 \wedge z \in A_0$, so $(x, z) \in (A_0 \times A_0)$. Thus, $(x, z) \in C_{A_0}$. Therefore, $xC_{A_0}y \wedge yC_{A_0}z \Rightarrow xC_{A_0}z$.

3. Here is a "proof" that every relation C that is both symmetric and transitive is also reflexive: "Since C is symmetric, aCb implies bCa. Since C is transitive, aCb and bCa together imply aCa, as desired." Find the flaw in this argument.
   The argument can be rewritten as:
   $aCb \Rightarrow bCa$
   $aCb \wedge bCa \Rightarrow aCa$

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   aCa
   Which simplifies to:
   $aCb \Rightarrow aCa$

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   aCa
   A counterexample can be found in the case where aCb and aCa are both false.
   More specifically: Let $A = \{1, 2\}$. Thus, $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Let $C \subset A$ be $\{(1, 1)\}$. The premise $2C1 \Rightarrow 2C2$ is true, but $2C2$ is false and C is not reflexive.

4. Let $f : A \to B$ be a surjective function. Let us define a relation on A by setting $a_0 \sim a_1$ if $f(a_0) = f(a_1)$.

   (a) Show that this is an equivalence relation.
   Reflexivity: Let $a \in A$. $f(a) = f(a)$ so $a \sim a$ for every $a \in A$.
   Symmetry: Let $a_0, a_1 \in A$ such that $a_0 \sim a_1 . f(a_0) = f(a_1)$, so $f(a_1) = f(a_0)$, so $a_1 \sim a_0$. Thus, $a_0 \sim a_1 \Rightarrow a_1 \sim a_0$.
   Transitivity: Let $a_0, a_1, a_2 \in A$ such that $a_0 \sim a_1 \wedge a_1 \sim a_2$. $f(a_0) = f(a_1) \wedge f(a_1) = f(a_2)$, so $f(a_0) = f(a_2)$, so $a_0 \sim a_2$. Thus, $(a_0 \sim a_1 \wedge a_1 \sim a_2) \Rightarrow a_0 \sim a_2$.

   (b) Let A^* be the set of equivalence classes. Show there is a bijective correspondence of A^* with B.
   A conceptual understanding: Let $x \in A$. Let $E = \{y \mid y \sim x\}$ be the equivalence class determined by x. $E = \{y \mid f(x) = f(y)\}$. Let $b = f(x)$; $E = \{y \mid f(y) = b\}$. In plain English, every equivalence class is the set of all $a \in A$ that map to a specific b under f.
   Showing injectivity: Let $E_1, E_2 \in A^*$ be two equivalence classes that both correspond to b. This means there is some $a_1 \in E_1$ and some $a_2 \in E_2$ such that $f(a_1) = b = f(a_2)$. Thus, $f(a_1) = f(a_2)$, meaning

$a_2 \in E_1 \wedge a_2 \in E_2$. Distinct equivalence classes are disjoint, but E_1 and E_2 overlap, so we can conclude $E_1 = E_2$. Thus, E_1 corresponds to b and E_2 corresponds to b implies that $E_1 = E_2$, showing that the correspondence from A^* to B is injective.
Shwoing surjectivity: Let $b \in B$. f is surjective, so $f(a) = b$ for at least one $a \in A$. Thus, there is a corresponding equivalence class $E = a \mid f(a) = b \in A^*$ for every $b \in B$. The correspondence from A^* to B is also surjective.
Because the correspondence from A^* to B is both injective and surjective, it must be bijective.