Unit 3 Test Cheatsheet

1.1 Set Theory:
Union: $A \cup B = \{x \mid x \in A \vee x \in B\}$; Intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$
A and B are disjoint: $A \cap B = \varnothing$
Difference: $A - B = \{x \mid x \in A \wedge x \notin B\}$
Distribution: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$; $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
DeMorgan: $A - (B \cup C) = (A - B) \cap (A - C)$; $A - (B \cap C) = (A - B) \cup (A - C)$
Power set of A: $\mathcal{P}(A)$ is the set of all subsets of A (including A and $\varnothing$)
Arbitrary union: $\bigcup_{A \in \mathcal{A}} A = \{x \mid x \in A \text{ for at least one } A \in \mathcal{A}\}$ (where $\mathcal{A}$ is a collection of sets)
Arbitrary intersection: $\bigcap_{A \in \mathcal{A}} A = \{x \mid x \in A \text{ for every } A \in \mathcal{A}\}$ (undefined when $\mathcal{A} = \varnothing$)
Cartesian product: $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$ (set of all ordered pairs (a, b) where $a \in A$ and $b \in B$)

1.2 Functions:
Rule of assignment: subset r of the Cartesian product $C \times D$ of two sets, having the property that each element of C appears as the first coordinate of at most one ordered pair in r (i.e. every thing in the domain is mapped to no more than one thing in the range)

- r is a rule of assignment if $(c, d) \in r \wedge (c, d') \in r \Rightarrow d = d'$
- Domain: subset of C consisting of all first coordinates of r; $\{c \mid \text{there exists } d \in D \text{ such that } (c, d) \in r\}$
- Image: subset of C consisting of all second coordinates of r (everything that actually gets mapped to); $\{d \mid \text{there exists } c \in C \text{ such that } (c, d) \in r\}$

Function f: rule of assignment r with a set B that contains the image set of r. The domain A of r is also the domain of f; image set of r is the image set of f; B is the range of f.

- $f(a)$ is the unique element of B such that $(a, f(a)) \in r$
- Let $A_0 \subset A$. Restriction of f to A_0: $\{(a, f(a)) \mid a \in A_0\}$
- Image of A_0 under f: $f(A_0) = \{b \mid b = f(a) \text{ for at least one } a \in A_0\}$
- Changing the domain or range changes the function

Composition of functions: given $f : A \to B$ and $g : B \to C$, $g \circ f : A \to C$ is defined by $g \circ f(a) = g(f(a))$

- Only defined when the range of f equals the domain of g
- Alternately: $g \circ f : A \to C = \{(a, c) \mid \text{For some } b \in B, f(a) = b \wedge g(b) = c\}$

Injective: no two distinct points in the domain map to the same point in the range; every point in the range is mapped to by at most one point in the domain; $f(a) = f(a') \Rightarrow a = a'$
Surjective: every point in the range is mapped to; $b \in B \Rightarrow (b = f(a) \text{ for at least one } a \in A)$
Inverse: exists if f is bijective; $f^{-1}(B_0) = \{a \mid f(a) \in B_0\}$

1.3 Relations:
Relation: a relation on A is a subset C of $A \times A$
Equivalence relation has 3 properties:

- Reflexivity: xCx for every $x \in A$
- Symmetry: If xCy, then yCx
- Transitivity: If xCy and yCz, then xCz

Equivalence class determined by x: $\{y \mid y \sim x\}$

- Always contains x

- Two equivalence classes E and E' are either disjoint or equal

Proof. Let E be the equivalence class determined by x, and let E' be the equivalence class determined by x'. Suppose that $E \cap E'$ is not empty; let y be a point of $E \cap E'$. See Figure 3.1. We show that $E = E'$.

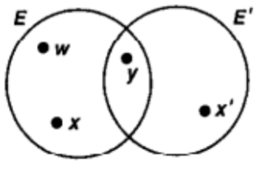

Figure 3.1

By definition, we have $y \sim x$ and $y \sim x'$. Symmetry allows us to conclude that $x \sim y$ and $y \sim x'$; from transitivity it follows that $x \sim x'$. If now w is any point of E, we have $w \sim x$ by definition; it follows from another application of transitivity that $w \sim x'$. We conclude that $E \subset E'$.

The symmetry of the situation allows us to conclude that $E' \subset E$ as well, so that $E = E'$. ■

Partition of set A: collection of disjoint, nonempty sets of A, where the union of these sets is all of A; any partition of A comes from exactly one equivalence relation on A

Studying equivalence relations on a set A and studying partitions of A are really the same thing. Given any partition $\mathcal{D}$ of A, there is exactly one equivalence relation on A from which it is derived.

The proof is not difficult. To show that the partition $\mathcal{D}$ comes from some equivalence relation, let us define a relation C on A by setting xCy if x and y belong to the same element of $\mathcal{D}$. Symmetry of C is obvious; reflexivity follows from the fact that the union of the elements of $\mathcal{D}$ equals all of A; transitivity follows from the fact that distinct elements of $\mathcal{D}$ are disjoint. It is simple to check that the collection of equivalence classes determined by C is precisely the collection $\mathcal{D}$.

To show there is only one such equivalence relation, suppose that C_1 and C_2 are two equivalence relations on A that give rise to the same collection of equivalence classes $\mathcal{D}$. Given $x \in A$, we show that yC_1x if and only if yC_2x, from which we conclude that $C_1 = C_2$. Let E_1 be the equivalence class determined by x relative to the relation C_1; let E_2 be the equivalence class determined by x relative to the relation C_2. Then E_1 is an element of $\mathcal{D}$, so that it must equal the unique element D of $\mathcal{D}$ that contains x. Similarly, E_2 must equal D. Now by definition, E_1 consists of all y such that yC_1x; and E_2 consists of all y such that yC_2x. Since $E_1 = D = E_2$, our result is proved.