Submission 3.2

1. Let $f : A \to B$. Let $A_0 \subset A$ and $B_0 \subset B$.

   (a) Show that $A_0 \subset f^{-1}(f(A_0))$ and that equality holds if f is injective.

   Proof that $A_0 \subset f^{-1}(f(A_0))$:
   By the definition of an inverse, $f^{-1}(C) = \{x \mid f(x) \in C\}$. Thus, $f^{-1}(f(A_0)) = \{x \mid f(x) \in f(A_0)\}$.
   For every $a \in A_0$, $f(a) \in f(A_0)$, fulfilling the conditions of the set.
   Therefore, $A_0 \subset f^{-1}(f(A_0))$.

   Proof that $A_0 = f^{-1}(f(A_0))$ if f is injective:
   By the definition of an inverse, $f^{-1}(f(A_0)) = \{x \mid f(x) \in f(A_0)\}$.
   Let $a \in f^{-1}(f(A_0))$. Therefore, $f(a) \in f(A_0)$. Therefore, there is some $x \in A_0$ such that $f(a) = f(x)$.
   By the definition of injectivity, $f(a) = f(x)$ implies that $a = x$.
   Therefore, for every $a \in f^{-1}(f(A_0))$, $a = x \in A_0$. Thus, $f^{-1}(f(A_0)) \subset A_0$. As proven above, $A_0 \subset f^{-1}(f(A_0))$ as well, so $A_0 = f^{-1}(f(A_0))$.

   (b) Show that $f(f^{-1}(B_0)) \subset B_0$ and that equality holds if f is surjective.

   Proof that $f(f^{-1}(B_0)) \subset B_0$:
   By the definition of a function, $f(f^{-1}(B_0)) = \{x \mid f^{-1}(x) \in f^{-1}(B_0)\}$ and by the definition of an inverse, $f^{-1}(B_0) = \{x \mid f(x) \in B_0\}$.
   For every element $b \in f(f^{-1}(B_0))$, $f^{-1}(b) \in f^{-1}(B_0)$. Therefore (by assigning $x = f^{-1}(b)$), $f(f^{-1}(b)) \in B_0$.
   This means $f(f^{-1}(B_0)) \subset B_0$.

   An alternate (and probably better) proof:
   By the definition of an inverse, $f^{-1}(B_0) = \{x \mid f(x) \in B_0\}$.
   Let $x \in f^{-1}(B_0)$. By definition, $f(x) \in B_0$, so $f^{-1}(B_0) \subset B_0$.

   Proof that $f(f^{-1}(B_0)) = B_0$ if f is surjective:
   Let $b \in B_0$. By the definition of surjectivity, there exists an x such that $b = f(x)$. By the definition of an inverse, $f^{-1}(b) = x$. Therefore, $b = f(x) = f(f^{-1}(b))$. Thus, every element $b \in B_0$ is equal to an element $f(f^{-1}(b)) \in f(f^{-1}(B_0))$, so $B_0 \subset f(f^{-1}(B_0))$. As proven above, $f(f^{-1}(B_0)) \subset B_0$, so $f(f^{-1}(B_0)) = B_0$.

2. Let $f : A \to B$ and let $A_i \subset A$ and $B_i \subset B$ for $i = 0$ and $i = 1$. Show that f^{-1} preserves inclusions, unions, intersections, and differences of sets.

   (a) $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$
   Let $x \in f^{-1}(B_0)$. By the definition of an inverse, $f(x) \in B_0$. Because $B_0 \subset B_1$, $f(x) \in B_1$ as well.
   $f^{-1}(B_1) = \{x \mid f(x) \in B_1\}$. Thus, every element in $f^{-1}(B_0)$ is also in $f^{-1}(B_1)$, so $f^{-1}(B_0) \subset f^{-1}(B_1)$.

   (b) $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$
   Let $x \in f^{-1}(B_0 \cup B_1)$. By the definition of an inverse, $f(x) \in B_0 \cup B_1$, so $f(x) \in B_0 \vee f(x) \in B_1$.
   Let $y \in f^{-1}(B_0) \cup f^{-1}(B_1)$. By the deinition of inverses, this means $f(y) \in B_0 \vee f(y) \in B_1$.
   Since the definitions for being contained in either set are the same, the two sets are equivalent.

   (c) $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$
   Let $x \in f^{-1}(B_0 \cap B_1)$. $f(x) \in B_0 \cap B_1$, so $f(x) \in B_0 \wedge f(x) \in B_1$. This can be rewritten as $f^{-1}(B_0) \cap f^{-1}(B_1)$ and all steps are reversible, showing equality.

   (d) $f^{-1}(B_0 - B_1) = f^{-1}(B_0) - f^{-1}(B_1)$
   Let $x \in f^{-1}(B_0 - B_1)$. $f(x) \in B_0 - B_1$, so $x \in B_0 \wedge x \notin B_1$. This can be rewritten as $f^{-1}(B_0) - f^{-1}(B_1)$ and all steps are reversible, again showing equality.

   Show that f preserves inclusions and unions only:

   (e) $A_0 \subset A_1 \Rightarrow f(A_0) \subset f(A_1)$
   Let $b \in f(A_0)$. There is some $x \in A_0$ such that $f(x) = a$. Because $A_0 \subset A_1$, $x \in A_1$, so $f(x) \in f(A_1)$. Thus, every $a \in f(A_0)$ is also in $f(A_1)$, so $f(A_0) \subset f(A_1)$.

(f) $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$
Let $b \in f(A_0 \cup A_1)$. There exists some $x \in A_0 \cup A_1$ such that $f(x) = b$. $x \in A_0 \cup A_1$ is logically equivalent to $x \in A_0 \vee x \in A_1$. Thus, $f(x) \in f(A_0) \vee f(x) \in A_1$, so $f(x) \in f(A_0) \cup f(A_1)$. Substituting b back in for $f(x)$, we get $b \in f(A_0) \cup f(A_1)$. Because every step is reversible, $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$.

(g) $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$
Let $b \in f(A_0 \cap A_1)$. There exists some $x \in A_0 \cap A_1$ such that $f(x) = b$. This means that $x \in A_0 \wedge x \in A_1$, so $f(x) \in f(A_0) \wedge f(x) \in f(A_1)$, so $f(x) \in f(A_0) \cap f(A_1)$. Thus, every $b \in f(A_0 \cap A_1)$ is also in $f(A_0) \cap f(A_1)$, so $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$.

Show that equality holds if f is injective.
Let $b \in f(A_0) \cap f(A_1)$. $b \in f(A_0) \wedge b \in f(A_1)$. There exists an $x_0 \in A_0$ such that $f(x_0) = b$ and an $x_1 \in A_1$ such that $f(x_1) = b$. Thus, $f(x_0) = f(x_1)$. By the definition of injectivity, $x_0 = x_1$. Thus, $x_0 \in A_0 \wedge x_0 \in A_1$, so $x_0 \in A_0 \cap A_1$, so $f(x_0) \in f(A_0 \cap A_1)$. Thus, every $b \in f(A_0) \cap f(A_1)$ is also in $f(A_0 \cap A_1)$, so $f(A_0) \cap f(A_1) \subset f(A_0 \cap A_1)$. As shown above, $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$, so $f(A_0 \cap A_1) = f(A_0) \cap f(A_1)$.

(h) $f(A_0 - A_1) \supset f(A_0) - f(A_1)$
Let $b \in f(A_0) - f(A_1)$. $b \in f(A_0) \wedge b \notin f(A_1)$. This means there exists an $x_0 \in A_0$ such that $f(x_0) = b$, and there does not exist an $x_1 \in A_1$ such that $f(x_1) = b$. Thus, for any x such that $f(x) = b$, $x \in A_0 \wedge x \notin A_1$, so $x \in A_0 - A_1$. Therefore, every $b \in f(A_0) - f(A_1)$ is also in $f(A_0 - A_1)$, so $f(A_0 - A_1) \supset f(A_0) - f(A_1)$.

Show that equality holds if f is injective.
Let $b \in f(A_0 - A_1)$. There exists an $x \in A_0 - A_1$ such that $f(x) = b$. Thus, $x \in A_0 \wedge x \notin A_1$. Let there be some $a \in A_1$ such that $f(a) = b = f(x)$. However, because f is injective, this implies that $a = x$. $x \notin A_1$, so this presents a contradiction. We can then conclude there is no $a \in A_1$ such that $f(a) = f(x)$, so $f(x) \notin f(A_1)$. Thus, $f(x) \in f(A_0) \wedge f(x) \notin f(A_1)$, so $f(x) \in f(A_0) - f(A_1)$. Therefore, every $b \in f(A_0 - A_1)$ is also in $f(A_0) - f(A_1)$, so $f(A_0) - f(A_1) \supset f(A_0 - A_1)$. As shown above, $f(A_0 - A_1) \supset f(A_0) - f(A_1)$, so $f(A_0 - A_1) = f(A_0) - f(A_1)$.

3. Show that 2b, 2c, 2f, and 2g of Exercise 2 hold for arbitrary unions and intersections.

(b) $f^{-1}(\bigcup_{B \in \mathcal{B}} B) = \bigcup_{B \in \mathcal{B}} f^{-1}(B)$
Let $x \in f^{-1}(\bigcup_{B \in \mathcal{B}} B)$. $f(x) \in \bigcup_{B \in \mathcal{B}} B$, so $f(x) \in B$ for at least one $B \in \mathcal{B}$. Thus, $x \in \bigcup_{B \in \mathcal{B}} f^{-1}(B)$. Every step is reversible, so $f^{-1}(\bigcup_{B \in \mathcal{B}} B) = \bigcup_{B \in \mathcal{B}} f^{-1}(B)$.

(c) $f^{-1}(\bigcap_{B \in \mathcal{B}} B) = \bigcap_{B \in \mathcal{B}} f^{-1}(B)$
Let $x \in f^{-1}(\bigcap_{B \in \mathcal{B}} B)$. $f(x) \in \bigcap_{B \in \mathcal{B}} B$, so $f(x) \in B$ for every $B \in \mathcal{B}$. Thus, $x \in \bigcap_{B \in \mathcal{B}} f^{-1}(B)$. Every step is reversible, so $f^{-1}(\bigcap_{B \in \mathcal{B}} B) = \bigcap_{B \in \mathcal{B}} f^{-1}(B)$.

(f) $f(\bigcup_{A \in \mathcal{A}} A) = \bigcup_{A \in \mathcal{A}} f(A)$
Let $b \in f(\bigcup_{A \in \mathcal{A}} A)$. There exists some $x \in \bigcup_{A \in \mathcal{A}} A$ such that $f(x) = b$. $x \in A$ for at least one $A \in \mathcal{A}$, so $f(x) = b \in f(A)$ for at least one $A \in \mathcal{A}$. Thus, every $b \in f(\bigcup_{A \in \mathcal{A}} A)$ is also in $\bigcup_{A \in \mathcal{A}} f(A)$, and because every step is reversible, $f(\bigcup_{A \in \mathcal{A}} A) = \bigcup_{A \in \mathcal{A}} f(A)$.

(g) $f(\bigcap_{A \in \mathcal{A}} A) \subset \bigcap_{A \in \mathcal{A}} f(A)$
Let $b \in f(\bigcap_{A \in \mathcal{A}} A)$. There exists some $x \in \bigcap_{A \in \mathcal{A}} A$ such that $f(x) = b$. This means $x \in A$ for every $A \in \mathcal{A}$, so $f(x) = b \in f(A)$ for every $A \in \mathcal{A}$. Thus, every $b \in f(\bigcap_{A \in \mathcal{A}} A)$ is also in $\bigcap_{A \in \mathcal{A}} f(A)$, so $f(\bigcap_{A \in \mathcal{A}} A) \subset \bigcap_{A \in \mathcal{A}} f(A)$.

Show that equality holds if f is injective.
Let $b \in \bigcap_{A \in \mathcal{A}} f(A)$. $b \in f(A)$ for every $A \in \mathcal{A}$. For every $A \in \mathcal{A}$, there exists an $x \in A$ such that $f(x) = b$. Because f is injective, all these x's are equal to each other, so there is a singular $x \in \bigcap_{A \in \mathcal{A}} A$. Thus, $f(x) = b \in f(\bigcap_{A \in \mathcal{A}} A)$. $\bigcap_{A \in \mathcal{A}} f(A) \subset f(\bigcap_{A \in \mathcal{A}} A)$, and because the reverse was proven above, $f(\bigcap_{A \in \mathcal{A}} A) = \bigcap_{A \in \mathcal{A}} f(A)$.

4. Let $f : A \to B$ and $g : B \to C$.

(a) If $C_0 \subset C$, show that $(g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0))$.
$(g \circ f)^{-1}(C_0) = \{a \mid (g \circ f)(a) \in C_0\} = \{a \mid g(f(a)) \in C_0\}$.
$f^{-1}(g^{-1}(C_0)) = \{a \mid f(a) \in g^{-1}(C_0)\}$ and $g^{-1}(C_0) = \{b \mid g(b) \in C_0\}$. Therefore, $f^{-1}(g^{-1}(C_0)) = \{a \mid g(f(a)) \in C_0\}$.
Both $(g \circ f)^{-1}(C_0)$ and $f^{-1}(g^{-1}(C_0))$ share a definition, so they must be equivalent.

(b) If f and g are injective, show that $g \circ f$ is injective.
Let a_0 and a_1 such that $(g \circ f)(a_0) = (g \circ f)(a_1)$. Thus, $g(f(a_0)) = g(f(a_1))$. Because g is injective, $f(a_0) = f(a_1)$, and because f is injective, $a_0 = a_1$. Thus, $(g \circ f)(a_0) = (g \circ f)(a_1) \Rightarrow a_0 = a_1$, so $g \circ f$ is injective.

(c) If $g \circ f$ is injective, what can you say about the injectivity of f and g?
Let a_0 and a_1 such that $f(a_0) = f(a_1)$. $g(f(a_0)) = g(f(a_1))$, and because $g \circ f$ is injective, $a_0 = a_1$. Thus, if $g \circ f$ is injective, f is injective as well.
Nothing can be said about the injectivity of g.

(d) If f and g are surjective, show that $g \circ f$ is surjective.
$g \circ f : A \to C$. Let $c \in C$. Because g is surjective, there exists a $b \in B$ such that $g(b) = c$. Because f is surjective, there exists an $a \in A$ such that $f(a) = b$. Thus, for every $c \in C$, there exists an a such that $g(f(a)) = c$, so $g \circ f$ is surjective.

(e) If $g \circ f$ is surjective, what can you say about the surjectivity of f and g?
Let $c \in C$. Because $g \circ f$ is surjective, there is exists $a \in A$ such that $(g \circ f)(a) = g(f(a)) = c$. Let $f(a) = b \in B$; $g(b) = c$. Thus, $c \in C \Rightarrow c = g(b)$ for at least one $b \in B$, so g is surjective.
Nothing can be said about the surjectivity of f.

(f) f and g are injective $\Rightarrow g \circ f$ is injective.
$g \circ f$ is injective $\Rightarrow f$ is injective.
f and g are surjective $\Rightarrow g \circ f$ is surjective.
$g \circ f$ is surjective $\Rightarrow g$ is surjective.

5. Let us denote the ***identity function*** for a set C by i_C, i.e. define $i_C : C \to C$ to be the function $i_C(x) = x$ for all $x \in C$. Given $f : A \to B$, $g : B \to A$ is a ***left inverse*** of f if $g \circ f = i_A$; $h : B \to A$ is a ***right inverse*** for f if $f \circ h = i_B$.

(a) Show that if f has a left inverse, f is injective.
Let g be the left inverse of f. Then, $g(f(x)) = x$. Let $x_0 \in A$ and $x_1 \in A$ such that $f(x_0) = f(x_1)$. $g(f(x_0)) = g(f(x_1))$, but also $g(f(x_0)) = x_0$ and $g(f(x_1)) = x_1$. By substituting, we get $x_1 = x0$, so $f(x_0) = f(x_1) \Rightarrow x_0 = x_1$, so f is injective.

Show that if f has a right inverse, f is surjective.
Let h be the right inverse of f. Then, $f(h(x)) = x \in B$. Let $a \in A = h(x)$. Then, $x \in B \Rightarrow x = f(a)$ for at least one $a \in a$, so f is surjective.

(b) Give an example of a function that has a left inverse but no right inverse.
The function $f(x) = \sqrt{x}$ has a left inverse $g(x) = x^2$, but no right inverse because it is not surjective (there is no value of x such that $f(x) = -1$).

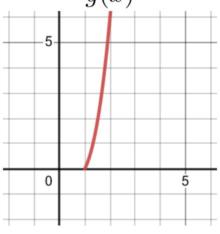
(c) Give an example of a function that has a right inverse but no left inverse.
The function $f(x) = x^2$ has a right inverse $h(x) = \sqrt{x}$, but no left inverse because it is not injective ($x^2 = (-x)^2$).

(d) Can a function have more than one left inverse? More than one right inverse?
A function cannot have more than one left inverse. Let $f : A \to B$ with two left inversesm g_1 and g_2. Thus, $g_1 : B \to A$ and $g_2 : B \to A$. Let $x \in A$ and let $f(x) = b \in B$. $g_1(f(x)) = x = g_2(f(x))$, so for every $b \in B$, $g_1(b) = g_2(b)$. Since g_1 and g_2 both have the same domain (B), range (A), and rule of assignment, $g_1 = g2$. Therefore, no function has 2 or more unique left inverses.
A function can have more than one right inverse. $f(x) = x^2$ has both $h_1(x) = \sqrt{x}$ and $h_2(x) = -\sqrt{x}$ as right inverses.

(e) Show that if f has both a left inverse g and a right inverse h, then f is bijective and $g = h = f^{-1}$.
Because f has a left inverse, f must be injective, and because f has a right inverse f must be surjective (proven in part (a)), meaning f is bijective.
Because f is bijective, it must have an inverse $f^{-1} : B \to A$ such that $f^{-1}(b) = a$, where $f(a) = b$.
By substituting, we get that $f^{-1}(f(a)) = a$ for all $a \in A$. This means f^{-1} is a left inverse of f, and because a function can have only one left inverse, $f^{-1} = g$.
We can also substitute to reach $f(f^{-1}(b)) = b$, meaning f^{-1} is a right inverse of f. $h : B \to A$ is also a right inverse of f, so $f(h(b)) = b$. f is injective, so $h(b) = f^{-1}(b)$ for all $b \in B$, so $h = f^{-1}$. $g = f^{-1}$ and $h = f^{-1}$, so $g = h = f^{-1}$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^3 - x$. By restricting the domain and range of f appropriately, obtain from f a bijective function g. Draw the graphs of g and g^{-1}.
Let $g : [1, \infty) \to [0, \infty)$ be the function $g(x) = x^3 - x$.

$g(x)$



$g^{-1}(x)$