Greatest Common Divisor

The *greatest common divisor* (GCD) of two integers, a and b, which are not both 0, is the largest integer that divides both a and b. We write this $\gcd(a, b)$ and define $\gcd(0, 0)$ to be 0 because otherwise it would be annoying.

We also have a name for two integers that share no factors. If $\gcd(a, b) = 1$, we say that a and b are relatively prime.

1. Find the greatest common divisor for each of the following pairs of integers.
   (a) 15, 35
   5
   (b) 0, 111
   111 (0 is divisible by every integer)
   (c) -12, 18
   6
   (d) 99, 100
   1 (the prime factors of 99 are 3, 3, and 11; the prime factors of 100 are 2, 2, 5, and 5)
   (e) 11, 121
   11
   (f) 100, 102
   2
2. Let a be a positive integer. What is $\gcd(a, 2a)$?
   a
3. Let a be a positive integer. What is $\gcd(a, a^2)$?
   a
4. Let a be a positive integer. What is $\gcd(a, a + 1)$?
   1. Every integer can be divided by 1, so $1 \leq \gcd(a, a + 1)$. Let a have a divisor x such that $x > 1$. $a \bmod x = 0$, so $(a + 1) \bmod x = 1$. Thus, there is no integer greater than 1 that divides both a and $a + 1$.
5. Let a be a positive integer. What is $\gcd(a, a + 2)$?
   If 2 divides a, $\gcd(a, a + 2) = 2$.
   Otherwise, $\gcd(a, a + 2) = 1$.
6. Find the greatest common divisor for each of the following sets of integers.
   (a) 8, 10, 12
   2
   (b) 6, 15, 21
   3
   (c) -7, 28, -35
   7
7. Find a set of three integers that are mutually relatively prime, but any two of which are not relatively prime.
   6, 15, 10
   To achieve this task, I figured I needed 3 numbers, each with 2 distinct prime factors. Each pair of numbers shares a prime factor, so I then reverse-engineered this. The 3 smallest prime numbers are 2, 3, and 5, so I multiplied pairs of these together.
8. Find four integers that are mutually relatively prime such that any three of these integers are not mutually relatively prime.
   105, 70, 42, 30
   This was found with a similar process. From 4 items, I could construct 4 groups of 3 (each group consists of 1 from the initial group removed), so I wanted 4 unique prime factors; I chose 2, 3, 5, and 7. I constructed 4 groups of 3: {3, 5, 7}, {2, 5, 7}, {2, 3, 7}, and {2, 3, 5}. I then took the product of each group.