Lambda Calculus and its Implications in Computer Science

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Preface

The reason I chose to write about this subject, is that it combines two things I enjoy: abstract mathematics and computer science. I have been programming for about five to six years now. I mainly enjoy low-level programming, so—naturally—C is my most used language and I am most familiar with a simple procedural paradigm. Such a paradigm is, however, not always very easy to use when working with very large and complex systems. I, as many, started out with object-oriented programming (OOP), but I did not like that very much. Therefore, I have been exploring alternative paradigms, including data-oriented programming and functional programming. I am quite familiar with data-oriented programming and the Rust programming language by now, but functional programming isn't something I've ever really got into yet. I did find out about lambda calculus and combinatory logic, which intrigued me, but I haven't got into it beyond a basic level of understanding. That is why I decided to research it for this work.

I have some things to note on the structure and style of this work. Firstly, the structure. The work starts with some introductory information about lambda calculus and its history. It goes on to explain the simple syntax of lambda calculus and how it can be used. The next section goes into lambda calculus in-depth. It really shows how lambda calculus is used as a Turing complete mathematical system. The section after that explains functional programming. It shows how lambda calculus can be turned into a programming language, and explains why it is useful by comparing it to other paradigms, namely procedural imperative programming (what that means isn't really of importance right now). The final few sections are smaller; they really just meant to illustrate what is said in the preceding sections.

Secondly, every reference will be denoted with a number in braces which corresponds with a work found in the back on the bibliography page. Many of my sources, however, aren't directly referenced, so it will also contain all the sources I used that weren't referenced.

As for the style in which I explain lambda calculus, I may differ in it from how other literature usually approaches the topic. Lambda calculus is often used in a mathematical context, so people often write in a very mathematical style. People don't usually write pure lambda calculus, they borrow a lot of symbols from other mathematical disciplines, such as logic. Nor do people define it in words, but also with mathematical symbols and definitions. I try to keep the use of these symbols to a minimum so as to lower the barrier to entry. Rather than using mathematical symbols and definitions, I use my words to explain the lambda calculus, thus this work is written in a more prose-like style, rather than a strictly mathematical one. I think that lambda calculus is simple enough that, when explained well, most people should be able to follow it. On a similar note, I think that the way this work is written, most people who are not familiar with mathematical literature should be able to follow it (except for maybe part of the functional programming section).

Lastly, I want to thank Daan Hoogcarspel for reading this work and giving feedback and making suggestions along the way. We discuss similar topics in our PWS's (profielwerkstuk). He talks about formal logic and covers many similar topics. I recommend you read it [15].

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1 Introduction

With the decline of OOP (object-oriented programming), many other paradigms are gaining in popularity. One increasingly popular paradigm is functional programming. Functional programming is fundamentally based on lambda calculus and it has been seeping into other paradigms and into mainstream languages. Most of the popular languages now implement lambda functions and have ways to write in a more declarative style of programming. In this work I will look at all the ways lambda calculus has influenced computer science.

2 Introduction to lambda calculus

Lambda calculus is, as its name suggests, a calculus. A calculus is a system of manipulating symbols, which by themselves don't have any meaning, in a way that is somehow meaningful. We all know algebra. Algebra itself doesn't have an innate meaning, but we can use it to represent and solve real world problems. Algebra, however, is limited. Not every problem can be represented in algebra. There are many branches of mathematics that use different systems. One example would be formal logic, which is used for logical operations on booleans. Another such example is lambda calculus.

2.1 A short history

People always trusted mathematics to be true and relied on it heavily. If something was proven true with mathematical logic, then that must be true. However, starting from the late 19th century, people ran into paradoxes. People made a distinction between reasoning that is rigorous and reasoning that isn't—reasoning that is logical, and reasoning that is psychological. The fact that mathematical logic, which people looked at for rigorousness, is infested with paradoxes and self-reference was very troubling for people at the time.

The concept of mathematics and mathematical logic wasn't well defined, so people started to think about the formalisation of mathematical logic to try to solve these issues. People wanted a system that would encapsulate all of mathematical logic. Preferably this system would be simple, clean and intuitive.

Throughout the late 19th and early 20th centuries, people started formally defining and redefining different aspects of mathematics. Frege [13] wrote about propositional calculus and functions as graphs, and in doing so re-evaluated the concept of functions and was already using concepts like Currying functions (more on this in section 4.2.1) without really giving thought to it. Peano [21] invented the Peano axioms and Peano arithmetic as a way of defining natural numbers. He was not the first to attempt defining natural numbers, but he was the most successful. Schönfinkel [27] invented combinatory logic, which was later rediscovered and improved on by Curry [12], as a way to remove the need for quantified variables in logic.

One major attempt to define all of mathematics was done by Russell and Whitehead [24–26]. They wrote a book that would become well know in all of mathematics and logic. This book is called *Principia Mathematica*. They did, however, run into a few problems, which arose from self-reference. To solve these problems that this paradoxical self-reference brought with it, they invented an elaborate system, the theory of types, to circumvent/eliminate it. It was a very carefully crafted bastion against self-reference ever coming up in their system, which was not very simple, clean or intuitive.

People praised *PM* as they thought they had finally done it; they had formalised all of mathematical logic, they had realised the dream of grounding all of mathematics in logic. But in Vienna, Gödel was sceptical of this book. He started seeing some cracks, he felt that there was something wrong about this attempt. Gödel felt that self-reference was a fundamental part of mathematical logic. Then he went out and actually proved that there is no consistent system of axioms whose theorems can be listed by an effective procedure that is capable of proving all truths about arithmetic of natural numbers[1] [14], meaning that such a system is either inconsistent or incomplete[2], greatly disturbing many mathematicians and upending mathematics as they knew it.

During this time, in this environment of the formalisation of mathematical logic, Church [7] invented the lambda calculus. Lambda calculus is a very simple and minimalistic system of substitution. A little while later, Turing [32, 33] invented Turing machines. Turing machines are conceptual mathematical machines that function based on state—they were state machines. These could perform all kinds of mathematical and logical computations. He was not the first to invent computers, but he was the first to work them out as well as he did (and, as you probably know, he built one which he cracked the German enigma code with).

There was this problem that has a few different names. It is often known as the *halting problem* or the *Entscheidungsproblem*, which is German for *decision problem*. The halting problem and decision problem aren't exactly synonyms, but they come down to the same thing. Basically, it asks whether it is possible to know via an algorithm whether a computation will complete execution or result in an infinite loop. In 1936, Turing [32, 33] spent a long time proving, using his Turing machines, that this isn't possible, but it didn't get published until early 1937. Also in 1936, Church [8] proved the same thing using lambda calculus and happened to publish it before Turing did. When Turing finally got around to publishing his proof, he found out that he was beaten to it by Church. He wasn't too pleased. What is interesting, though, is that lambda calculus and Turing machines take two completely different approaches. Turing machines function entirely on state, while lambda calculus is completely stateless (we'll look at this later). Turing thought this was interesting too, so he researched lambda calculus and how it relates to his Turing machines, and proved that they are formally equivalent [31].

Why do I tell you all this? Well, your main takeaway should be that even though lambda calculus is a very simple system, which at first glance might not seem to be very semantic or seem to have any real world implications, it actually is Turing complete. Lambda calculus and Turing machines take wildly different approaches: one state based, the other stateless. Another difference is that Turing machines can be physically built. We can, however, use lambda calculus on these Turing machines *and* simulate Turing machines with lambda calculus, which is a fundamental part to the thesis of this work. We will look at lambda calculus and how the work of all the previously mentioned mathematicians, and many more, can be applied in lambda calculus to get a useful system.

2.2 The syntax

Lambda calculus is all about first-class higher-order pure (anonymous[3]) unary functions. Such a function takes a single input, and returns a single expression that is only dependent on the input, so it doesn't have any outside state. Such a function can take and return any expression, which in lambda calculus is always a function. A simple function definition in lambda calculus looks as follows:

$$\lambda a.a$$

The lambda signifies a function. Everything following it will be part of that function's definition. The a before the . is the name of the argument. There is only one, because, as I said before, all functions in lambda calculus are unary. Everything following the . is part of the function body, which is the return expression. The function above is the identity function in lambda calculus; it just returns the input. This is the equivalent of multiplying by one, or

[1] This is a definition of the first incompleteness theorem I got from Wikipedia [35]

[2] Incompleteness means that there are things that are true, but are not provable.

[3] The core lambda calculus has no way of naming functions.

defining a function like $f(x) = x$, or multiplying a vector with the identity matrix; it does nothing.

But how do we use this function? Well, just like defining a function, it is quite simple. If you want to apply this function to a symbol, you just put it in front of the symbol. Something like this:

$$(\lambda a.a)x$$

Which evaluates to x, because you remove the x and then replace all the a's in the function body with x and then remove the function signifier and argument list. It all comes down to a simple process of substitution.

In this case you need parentheses around the function, otherwise x would be considered part of the function's body, which it isn't. It's also important to note that lambda calculus is left-associative, that is, it evaluates an expression from left to right. This means that the function on the far left of an expression gets invoked first.

I have now basically explained the entire lambda calculus, it is really that simple. I have explained abstraction (functions), application (applying functions), and grouping (parentheses), which is basically all we need. You can also give names to expressions. We could name our identity function I as follows:

$$I := \lambda a.a$$

But this isn't really part of the core lambda calculus any more, just some syntactic sugar. This way, instead of constantly having to write $\lambda a.a$, we can just write I. So instead of writing:

$$(\lambda a.a)x = x$$

We could use our previous definition of I and write:

$$Ix = x$$

We have now covered identifiers too.

But if this is all there is, how can this possibly be Turing complete? How do we do boolean logic, or algebra? How can we do things with only unary functions? What are a and x supposed to represent? If there is no concept of value, how do we even use this meaningfully? Well, the key is this: a function can return any expression (remember?), which is always a function[4], not just a single symbol. We can start combining these simple functions into more complex functions. Let's say that we wanted to have a function that takes two arguments, and then applies the first argument to the second one. You are probably asking yourself a few questions. For example, what does it mean for one argument to be applied to another? Well, as I said, everything is a function. But the biggest question you are probably asking yourself is: how can you have a function that takes two arguments?

We actually can't, but what we can do is to have a function that takes one argument and returns another function that takes one argument. We can define that function as follows:

$$\lambda a.\lambda b.ab$$

We currently have a function definition inside the body of another function. If we now apply this function to a symbol like x, we get this:

$$(\lambda a.\lambda b.ab)x = \lambda b.xb$$

We get a new function that takes an argument and applies x to it. If we now apply this function to a symbol like y, we get this:

$$(\lambda b.xb)y = xy$$

[4] Everything is.

Alternatively, we could write it all on one line:

$$(\lambda a.\lambda b.ab)xy = (\lambda b.xb)y = xy$$

xy in this case is what we would call the *β-normal form* of the preceding expressions. That just means that it is in the simplest form and isn't able to be evaluated any further. Reducing a lambda expression to the β-normal form is called *β-reduction.*

You can start to see how we can combine unary functions to create more complex functions[5]. In this example we used two nested unary functions to get the same result you would with a binary function. Such a nested function is often called a *Curried function.* You might think to yourself that having this many nested functions can be quite convoluted and not very readable, and you're quite right. That's why people often use a shorthand notation. They would basically write it as if it is a single binary function (as with any n-ary function). They would write the example function above as:

$$\lambda ab.ab$$

Do keep in mind, that even though this looks and, for the most part, acts as if it is a single binary function, it really isn't. It still is a curried function that feeds in the arguments one by one, but this way the expression becomes more readable and easier to think about conceptually. We will use this notation from now on.

Congratulations, you now know the very basics of lambda calculus. You may still not see how this is Turing complete or how this can be useful and meaningful. You might also already see some of the intrigues of lambda calculus; how simple it is, how it doesn't have a concept of value or data, how everything is an expression, how it is stateless, etc. But we'll get to all of that eventually. If you get this, everything else will follow naturally (mostly).

2.3 Combinatory logic

Combinatory logic is a notation to eliminate the need for quantified variables in mathematical logic [34]. That basically means a form of logic without values, just like with lambda calculus, but just pure logical expressions, using so called *combinators.* The idea of combinators first came from Schönfinkel [27], and was later rediscovered by Curry [12]. Combinators are just symbols, in this case letters, that perform operations on symbols that succeed it. We've actually looked at one of these combinators already.

We will be using Curry's names for combinators, since his names are most widely used.

2.3.1 Identity

The first combinator we'll cover is I. It does exactly the same thing as our I function we defined in lambda calculus in the previous subsection ($I := \lambda a.a$). In fact, all combinators can be defined in lambda calculus. Lambda calculus is really just 90% combinatory logic, but without identifiers. This combinator may seem quite useless, but it is actually quite useful when composing combinators, which we'll come to soon.

2.3.2 The omega combinator

The next combinator we'll cover is M. All it does is repeat its one argument twice. It can be defined in lambda calculus as follows:

$$M := \lambda f.ff$$

We could, for example, look at what happens when you apply M to I. We get:

$$MI = II = I$$

[5] That's what makes them higher-order (and first-class).

Or, written out in lambda calculus:

$$(\lambda f.ff)\lambda a.a = (\lambda a.a)\lambda a.a = \lambda a.a$$

What happens if you apply M to M? You get:

$$MM = MM = MM = ...$$

ad infinitum. Or in lambda calculus:

$$(\lambda f.ff)\lambda f.ff = (\lambda f.ff)\lambda f.ff = (\lambda f.ff)\lambda f.ff = ...$$

This expression cannot be evaluated. We say that it doesn't have a β-normal form. In lambda calculus and combinatory logic not every expression is reducible. As we've seen in the second to last paragraph of section 2.1: there is no single algorithm to decide whether a lambda expression has a β-normal form[6].

MM is sometimes called the Ω combinator. Omega, because it is the end of the Greek alphabet. The M combinator is sometimes called the ω combinator because of this. Combinators often have many different names. Sometimes because scientists discovered them separately, unaware of each other, sometimes because they preferred a different name, sometimes because scientists like to give them pet names [7].

The omega combinator will be discussed further in section 3.2.2.

2.3.3 The constant combinator

The next combinator we'll cover is K. It is a combinator that takes two arguments and returns the first. We can easily define it in lambda calculus as follows:

$$K := \lambda ab.a$$

Remember that we defined this as a curried function. This means we can give it just one argument and get a new function out of it. Let's say we apply K to 5:

$$K5 = (\lambda ab.a)5 = \lambda b.5$$

Our new function, $K5$, is a function that takes an argument and returns 5. This means that whatever we apply this function to, we always get 5. K gets its name from the German word *Konstant*, meaning constant. You can probably see why.

Just like with the previous combinators, it'll prove very useful, much more so than you'd expect.

2.3.4 The kite

Here is where things get a little spicier. Our next combinator is KI. It takes two arguments and returns the latter. We can define it in lambda calculus as follows:

$$KI := \lambda ab.b$$

You may already be thinking about its name. Why does it have two letters? And why are they two letters we've talked about already? Well, the answer is very simple. If you apply K to I, you get KI. Don't believe me? Let's try!

If we use our definition of KI and apply it to xy we get:

$$KIxy = (\lambda ab.b)xy = y$$

[6] This is not exactly what is written, but it means the same thing.

[7] I have a theory they are just trying to throw us off

But if we use the K and I combinators separately, we get the following:

$$KIxy = (\lambda ab.a)Ixy = (\lambda b.I)xy = Iy = y$$

If you think about it, it is very logical. If K takes two arguments and returns the first, then, in this case, it uses up both I and x and returns I, which will just return the next argument, in this case y. KI will always return the second symbol after the I, because—again—the first gets used up by K.

We can also just see what function we get when we apply K to I:

$$KI = (\lambda ab.a)I = \lambda b.I = \lambda b.\lambda a.a = \lambda ba.a$$

We get our definition of KI (except the names of the arguments are switched).

We're starting to define combinators as combinations of other combinators. Every combinator, in fact, can be defined as a combination of other combinators. That's why they are called combinators.

2.3.5 The flip combinator

The next combinator is the C combinator. The C combinator is definable in lambda calculus as:

$$C := \lambda fab.fba$$

What it basically does is switch the arguments to the next combinator around.

If we apply C to K and two random symbols, we get the same result we would get if we had applied KI to those same symbols:

$$CKxy = Kyx = y$$

$$KIxy = y$$

$$CK = KI$$

Let's see what happens when we apply C to K in lambda calculus (I have changed the names of K's arguments as to avoid confusion with those of C[8]):

$$(\lambda fab.fba)\lambda xy.x = \lambda ab.(\lambda xy.x)ba = \lambda ab.b$$

We again see that it reduces to our definition of KI.

It could so happen that you get a different function, but which produces the same outputs for the same inputs. If that happens, then that new function is *extensionally equal* to the original definition. When two functions are extensionally equal, you can't β-reduce one, you can *η-reduce* one to the other. η-reduction is when you reduce a function to another function that is extensionally equal to it when β-reduction isn't possible.

You can do the same thing to find out that $CKI = K$. It really does make sense. K and KI both "select" one of two arguments. One selects the first, the other selects the latter. Flipping their arguments make them select the opposite of what they normally would, so they select the argument that the other combinator usually would.

[8] A common rule people use is that when a variable name is used multiple times in the same scope, the most inner scope is the definition it's bound to

2.3.6 The composition combinator

Our next combinator, B, is defined as follows:

$$B := \lambda fga.f(ga)$$

It applies g to a before applying f to the result. This combinator is used for function composition. When a function applies g to a and f to the result, we say that this function composes f with g. Function composition is used to apply a sequence of operations on a variable (but the order in which you write them is reversed).

In mathematics we usually write function composition as follows:

$$f \circ g$$

So

$$(f \circ g)\ a = f(ga)$$

If we define a function h as

$$h = f_4 \circ f_3 \circ f_2 \circ f_1$$

it means that h applies f_1, f_2, f_3 and f_4 in that order. It is really important to keep in mind this reversed order, or you are in for a confusing ride.

In combinatory logic we write function composition of f and g as follows:

$$Bfg$$

which reduces to

$$(\lambda fga.f(ga))fg = \lambda a.f(ga)$$

in lambda calculus.

2.3.7 The thrush

Our next combinator is T_h. It is defined as follows:

$$T_h := \lambda af.fa$$

It swaps around two functions. It is basically a very simple form of storage; when you apply it to an argument, it stores that argument to have some other function applied to it later.

2.3.8 The vireo

The V-combinator is basically the T_h-combinator with an extra argument. If you consider T_h a variable, V could be considered a *pair* (section 3.1.4).

It is defined as

$$V := \lambda abf.fab$$

2.3.9 The starling

Our last combinator is S. It can be defined as follows

$$S := \lambda fga.fa(ga)$$

It applies f to a and its result to the result of the application of g to a.

2.3.10 *S* and *K*

As I've said, every combinator can be defined as a combination of other combinators. The question arises: how many combinators do we need to define every other combinator? It turns out you need just two. You can define every other combinator using just S and K. I will not go over it now, but it's interesting to look into. What's also interesting is that you can define all formal logical operations with just two operators.

2.3.11 To Mock a Mockingbird

At the start of this section about combinatory logic, I said that Schönfinkel [27] invented combinatory logic as a way of removing the need for quantified variables. He started with propositional logic and stripped it down until there was a very pure and simple form of logic left. But how can we use this form of logic in the real world, if he even removes things like propositions? You already know that it is Turing complete, so it must be able to do any computation, but I haven't explain how yet (see section 3). But we can use combinatory logic in the real world already.

You may have noticed some of the previous subsections have bird names as titles. This is because they are the names given to the combinators, discussed in the respective subsections, by an author named Smullyan. He is a mathematician who likes to write puzzle books. His book *To Mock a Mockingbird* [28] is practically a large metaphor for combinatory logic. There are some unrelated puzzles in the beginning of the book, but the rest is about a big forest with birds. The birds represent the combinators. The begin letters of the bird names are the names of the combinators. The way the birds interact reflects the way the combinators interact. The reason he chose birds for his metaphor is that Curry was an avid bird watcher. If you feel like you still don't understand the notation of combinatory logic completely, I would recommend you give this book a read, because it explains it very simply and clearly.

In Smullyan's world, there is a forest with birds. These birds represent combinators. If you call out a bird name to a bird, it will give you the name of another bird. Calling out bird names to birds represents application.

I think it would be fun if we had a look at one of the puzzles to see if we can solve it using our newfound knowledge of combinatory logic. I think the first puzzle is sufficiently interesting. So far, Smullyan has only introduced the mockingbird, which is the omega operator (section 2.3.2), and the idea of function composition, but not yet the bluebird, which is the composition operator (section 2.3.6). I have taken the puzzle directly from the book:

> It could happen that if you call out B to A, A might call the same bird B back to you. If this happens, it indicates that A is fond of the bird B. In symbols, A is fond of B means that: $AB = B$
>
> We are now given that the forest satisfies the following two conditions.
>
> C_1 (the composition condition): For any two birds A and B (whether the same or different) there is a bird C such that for any bird x, $Cx = A(Bx)$. In other words, for any birds A and B there is a bird C that composes A with B.
>
> C_2 (the mockingbird condition): The forest contains a mockingbird M.
>
> One rumor has it that every bird of the forest is fond of at least one bird. Another rumor has it that there is at least one bird that is not fond of any bird. The interesting thing is that it is possible to settle the matter completely by virtue of the given conditions C_1 and C_2.
>
> Which of the two rumors is correct?

Do note that in this case, C and B do not refer to the C and B combinators we've looked at previously.

The answer will be shown on the next page.

Because of C_1 and C_2, we know that for every bird A, there's a bird—we'll call it C—that composes A with M. We can say the following:

$$Cx = A(Mx) = A(xx)$$

If we now fill in C in place of x, we get:

$$A(CC) = CC$$

We thus know that for any bird A, A is fond of the bird CC, where C composes A with M. Therefore rumour one is true and rumour two is false.

The answer Smullyan [28] gives is a bit more verbose, but it comes down to the same thing.

3 Using lambda calculus for computation

Now that we've gone through lambda calculus and combinatory logic, it is finally time to get to the good parts. I hope the way here wasn't too boring or difficult. We will now look at how to use lambda calculus for computation.

3.1 Church encoding

To do computation with lambda calculus, we need to be able to do a few things such as boolean logic and arithmetic. Lambda calculus itself doesn't have these things built in to it, but we can define things such as boolean logic and arithmetic in lambda calculus. In section 2.1, I talked about how mathematicians were defining everything in mathematics. We can build of of their work and see how we can implement their definitions in lambda calculus. This is exactly what Church did.

3.1.1 Simple boolean operations

Let's look at a simple form of computation before jumping into arithmetic. We'll start with binary logic. In it's simplest form, binary logic is really just control flow: *if A then B else C* ($A\,?\,B\,:\,C$). We want to have some condition that chooses between two expressions. We know how to do that already (see section 2.3). K chooses the first of two expressions and KI the latter. We can define true to be K and false to be KI:

$$T := K = \lambda ab.a$$

$$F := KI = \lambda ab.b$$

But how can we do logic gates? Let's look at negation. We have already looked at the flip combinator—C (section 2.3.5), which does exactly what we want. Thus, we can say:

$$NOT := C = \lambda fab.fba$$

We could also do it another way. We already know how to do control flow, so we could define a function that chooses the opposite of what the input is. In simple programming terms:

$$if\ p\ then\ F\ else\ T$$

Or in a C-like expression:

$$p\ ?\ F\ :\ T$$

Or in lambda calculus:

$$NOT := \lambda p.pFT$$

It depends which one's preferable. The C combinator is a bit more elegant and performant, because it takes less function applications, but you get a function that is only extensionally equal to T or F, while the other definition literally returns T or F.

How do we define AND? We could define it in simple programming terms, and then translate it to lambda calculus. In simple programming terms, AND would look like this:

$$if\ p\ then\ (if\ q\ then\ T\ else\ F)\ else\ f$$

$$p\ ?\ q\ ?\ T : F : F$$

In lambda calculus we would get:

$$AND := \lambda pq.p(qTF)F$$

But this is a very naïve way of defining AND. If you look closely at the qTF part, you notice that it actually just returns whatever q is anyway. Thus, we can say:

$$AND := \lambda pq.pqF$$

You can also replace the remaining F with q if you want, because the F will only be returned if q is F—in other words: q in that case is always false. So we can say:

$$AND := \lambda pq.pqp$$

This is the equivalent of:

$$if\ p\ then\ q\ else\ p$$

$$p\ ?\ q\ :\ p$$

We can define OR in a very similar way. If the first argument is true, we just return it, else we return the second argument:

$$OR := \lambda pq.ppq$$

Defining XOR is very easy too. If the first argument is true, then we want to return what the second argument is not, else we want to return what the second argument is. In lambda calculus:

$$XOR := \lambda pq.p(NOT\ q)q$$

Giving:

$$XOR := \lambda pq.p(qFT)q$$

Alternatively:

$$XOR := \lambda pq.p((\lambda fab.fba)q)q = \lambda pq.p(\lambda ab.qba)q$$

We can define boolean equality (BEQ) similarly:

$$BEQ := \lambda pq.pq(qFT)$$

Or:

$$BEQ := \lambda pq.pq(\lambda ab.qba)$$

Which you can read as: "If p is true, then return what q is, else return what q is not."

Let's look at an example expression and solve it in lambda calculus. We'll solve the following expression:

$$!\,(x\ \&\&\ y)\ == !\,x\ ||\ !\,y$$

If you're not familiar with C-like expressions: ! means NOT, && means AND, == means BEQ and || means OR. ! has a higher and == a lower precedence than the rest.

We can write and reduce this expression using the functions/combinators we defined (using $NOT := \lambda p.pFT$, because else we wouldn't be able to write out NOT without using lambda calculus):

$$\begin{aligned}
& BEQ\ \ (NOT\ (AND\, x\, y))\ \ (OR\ (NOT\, x)\ (NOT\, y)) \\
=\ & BEQ\ \ (NOT\ (xyx))\ \ (OR\ (xFT)\ (yFT)) \\
=\ & BEQ\ \ (xqxFT)\ \ (xFT\ (xFT)\ (yFT)) \\
=\ & xyxFT\ \ (xFT\ (xFT)\ (yFT))\ \ (xFT\ (xFT)\ (yFT)\ FT)
\end{aligned}$$

If we've done this correctly, then according to *De Morgan's Law*, we should always get T for any substitution of x and y for T and F in this final expression.

$x = F$ and $y = F$ gives us:

$$\begin{aligned}
& FFFFT\ \ (FFT\ (FFT)\ (FFT))\ \ (FFT\ (FFT)\ (FFT)\ FT) \\
=\ & FFT\ \ (TTT)\ \ (TTTFT) \\
=\ & TT\ \ (TFT) \\
=\ & TTF \\
=\ & T
\end{aligned}$$

$x = F$ and $y = T$ gives us:

$$\begin{aligned}
& FTFFT\ \ (FFT\ (FFT)\ (TFT))\ \ (FFT\ (FFT)\ (TFT)\ FT) \\
=\ & FFT\ \ (TTF)\ \ (TTFFT) \\
=\ & TT\ \ (TFT) \\
=\ & TTF \\
=\ & T
\end{aligned}$$

$x = T$ and $y = F$ gives us:

$$\begin{aligned}
& TFTFT\ \ (TFT\ (TFT)\ (FFT))\ \ (TFT\ (TFT)\ (FFT)\ FT) \\
=\ & FFT\ \ (FFT)\ \ (FFTFT) \\
=\ & TT\ \ (TFT) \\
=\ & TTF \\
=\ & T
\end{aligned}$$

$x = T$ and $y = T$ gives us:

$$\begin{aligned}
& TTTFT\ \ (TFT\ (TFT)\ (TFT))\ \ (TFT\ (TFT)\ (TFT)\ FT) \\
=\ & TFT\ \ (FFF)\ \ (FFFFT) \\
=\ & FF\ \ (FFT) \\
=\ & FFT \\
=\ & T
\end{aligned}$$

We can also try to solve this expression in lambda calculus (using the shorthand notation and using $NOT := C$, because it is easier to write out):

$$
\begin{aligned}
&\lambda xy.(\lambda pq.pq(\lambda cd.qdc))((\lambda fab.fba)((\lambda pq.pqp)xy))((\lambda pq.ppq)((\lambda fab.fba)x)((\lambda fab.fba)y)) && (1)\\
=\ &\lambda xy.(\lambda pq.pq(\lambda cd.qdc))((\lambda fab.fba)(xyx))((\lambda pq.ppq)(\lambda ab.xba)(\lambda ab.yba)) && (2)\\
=\ &\lambda xy.(\lambda pq.pq(\lambda cd.qdc))(\lambda ab.xyxba)((\lambda ab.xba)(\lambda ab.xba)(\lambda ab.yba)) && (3)\\
=\ &\lambda xy.(\lambda pq.pq(\lambda cd.qdc))(\lambda ab.xyxba)(x(\lambda ab.yba)(\lambda ab.xba)) && (4)\\
=\ &\lambda xy.(\lambda ab.xyxba)(x(\lambda ab.yba)(\lambda ab.xba))(\lambda cd.x(\lambda ab.yba)(\lambda ab.xba)dc) && (5)\\
=\ &\lambda xy.xyx(\lambda cd.x(\lambda ab.yba)(\lambda ab.xba)dc)(x(\lambda ab.yba)(\lambda ab.xba)) && (6)\\
\equiv\ &\lambda xy.xyx(\lambda cd.x(\lambda ab.yba)(\lambda ab.a)dc)(x(\lambda ab.yba)(\lambda ab.a)) && (7)\\
\equiv\ &\lambda xy.xyx(xy(\lambda ab.b))(x(\lambda ab.yba)(\lambda ab.a)) && (8)\\
\equiv\ &\lambda xy.xyx(xy(\lambda ab.b))(\lambda ab.a) && (9)\\
\equiv\ &\lambda xy.xyx(\lambda ab.a)(\lambda ab.a) && (10)\\
\equiv\ &\lambda ab.a && (11)
\end{aligned}
$$

We know that expressions 6 and 7 are extensionally equal;

$$\lambda x.x(\lambda ab.yba)(\lambda ab.xba) \equiv \lambda x.x(\lambda ab.yba)(\lambda ab.a)$$

because x picks between $\lambda ab.yba$ and $\lambda ab.xba$. $\lambda ab.xba$ only gets picked when x is F, so $\lambda ab.xba$ is always $\lambda ab.Fba{=}\lambda ab.a$. Therefore, a $x(\lambda ab.yba)(\lambda ab.xba)$ that occurs at the start of its grouping (we do need to keep left-associativity in mind) can be replaced with $x(\lambda ab.yba)(\lambda ab.a)$.

We also know that expressions 7 and 8 are extensionally equal. We know that

$$\lambda xcd.x(\lambda ab.yba)(\lambda ab.a)dc$$

selects between

$$\lambda cd.(\lambda ab.yba)dc = \lambda cd.ycd \equiv y$$

and

$$\lambda cd.(\lambda ab.a)dc = \lambda cd.d = \lambda ab.b$$

therefore

$$\lambda xcd.x(\lambda ab.yba)(\lambda ab.a)dc \equiv \lambda x.xy(\lambda ab.b)$$

Expressions 8 and 9 are also extensionally equal, because xyx selects between the expressions $xy(\lambda ab.b)$ and $x(\lambda ab.yba)(\lambda ab.a)$. The expression $x(\lambda ab.yba)(\lambda ab.a)$ only gets selected if $xyx = F$. There are two ways for $xyx = F$ to be true; either $x = F$, or both $x = T$ and $y = F$. $x = F$ gives

$$x(\lambda ab.yba)(\lambda ab.a) = F(\lambda ab.yba)(\lambda ab.a) = \lambda ab.a$$

and $x = T$ and $y = F$ gives

$$x(\lambda ab.yba)(\lambda ab.a) = T(\lambda ab.Fab)(\lambda ab.a) = \lambda ab.Fab = \lambda ab.a$$

meaning that for all the possibilities of

$$xyx(xy(\lambda ab.b))(x(\lambda ab.yba)(\lambda ab.a)) = x(\lambda ab.yba)(\lambda ab.a)$$

also

$$x(\lambda ab.yba)(\lambda ab.a) \equiv \lambda ab.a$$

meaning

$$\lambda xy.xyx(xy(\lambda ab.b))(x(\lambda ab.yba)(\lambda ab.a)) \equiv \lambda xy.xyx(xy(\lambda ab.b))(\lambda ab.a)$$

We can apply the same reasoning to prove that equations 9 and 10 are extensionally equal. We know xyx picks between $xy(\lambda ab.a)$ and $\lambda ab.a$. For xyx to pick the first, xyx must equal T, which is only possible if both $x = T$ and $y = T$. Which gives

$$xy(\lambda ab.a) = TT(\lambda ab.a) = T = \lambda ab.a$$

The last step should be quite self-explanatory; xyx can only select between $\lambda ab.a$ and $\lambda ab.a$.

3.1.2 Natural numbers

Now that we've covered boolean logic, it is finally time to move on to arithmetic. Remember that in section 2.1, I said that Peano [21] defined natural numbers? He basically defined zero and then defined every number as a successor of the previous number, starting from zero. He also defined the properties of natural numbers. We can do exactly the same in lambda calculus. The key concept is function composition (section 2.3.6).

We'll define functions that compose a given function f, n times with themselves, where n is the natural number the function is supposed to represent. Thus we'll say

$$\begin{aligned} N0 &:= \lambda fa.a \\ N1 &:= \lambda fa.fa \\ N2 &:= \lambda fa.f(fa) \\ N3 &:= \lambda fa.f(f(fa)) \\ N4 &:= \lambda fa.f(f(f(fa))) \end{aligned}$$

and so on, or in a more standard mathematical notation

$$\begin{aligned} 0f &:= 0 \\ 1f &:= f \\ 2f &:= f \circ f \\ 3f &:= f \circ f \circ f \\ 4f &:= f \circ f \circ f \circ f \end{aligned}$$

You can also say

$$\begin{aligned} N0 &:= F \\ N1 &:= \lambda f.f = I \\ N2 &:= \lambda f.Bff \\ N3 &:= \lambda f.B(Bff)f \\ N4 &:= \lambda f.B(B(Bff)f)f \end{aligned}$$

We can't define an infinite number of functions by hand, so we'll use this shorthand definition:

$$n := \lambda fa.f^{\circ n}a$$

meaning: applying a number n to a function f is the same as composing f with f, n times. This is meaningful, because it allows us to do something a given number of times.

We can use this in an example. Let's say we wanted to do a negation multiple times. We can use our newly defined numbers:

$$\begin{aligned} &N0\ \ NOT\ \ T = T \\ &N1\ \ NOT\ \ T = NOT\ \ T = F \\ &N2\ \ NOT\ \ T = NOT\ \ (NOT\ \ T) = T \\ &N3\ \ NOT\ \ T = NOT\ \ (NOT\ \ (NOT\ \ T)) = F \end{aligned}$$

Numbers are just function composition. Function composition, and thus the B combinator (section 2.3.6), is fundamental to all arithmetic in lambda calculus. What happens when we compose two numbers, say $N3$ with $N3$?

$$\begin{aligned}
& B\ \ N3\ \ N3 \\
= \ & (\lambda fgh.f(gh))\, N3\ \ N3 \\
= \ & \lambda h.N3\, (N3\ \ h) \\
= \ & \lambda h.N3\, ((\lambda fb.f(f(fb))h)) \\
= \ & \lambda h.N3\, (\lambda b.h(h(hb))) \\
= \ & \lambda h.(\lambda fa.f(f(fa)))(\lambda b.h(h(hb))) \\
= \ & \lambda h.\lambda a.(\lambda b.h(h(hb)))((\lambda b.h(h(hb)))((\lambda b.h(h(hb)))a)) \\
= \ & \lambda ha.(\lambda b.h(h(hb)))((\lambda b.h(h(hb)))(h(h(ha)))) \\
= \ & \lambda ha.(\lambda b.h(h(hb)))(h(h(h(h(h(ha)))))) \\
= \ & \lambda ha.h(h(h(h(h(h(h(h(ha)))))))) \\
= \ & N9
\end{aligned}$$

We can tell that composition with natural numbers is the same as multiplication, which makes a lot of sense, because function composition is associative. The ninefold composition of f is the same as the threefold composition of the threefold composition of f:

$$f \circ f \circ f \circ f \circ f \circ f \circ f \circ f \circ f = (f \circ f \circ f) \circ (f \circ f \circ f) \circ (f \circ f \circ f)$$

We know that if we were to define a $MULT$ combinator, the following should be true:

$$MULT\ \ N3\ \ N3\ \ f\ \ a = (f \circ f \circ f \circ f \circ f \circ f \circ f \circ f \circ f)a$$

and in extension

$$\begin{aligned}
MULT\ \ N3\ \ N3\ \ f\ \ a &= ((f \circ f \circ f) \circ (f \circ f \circ f) \circ (f \circ f \circ f))\, a \\
&= ((N3\ \ f) \circ (N3\ \ f) \circ (N3\ \ f))\, a \\
&= N3\, (N3\ \ f)\, a \\
MULT\ \ N3\ \ N3\ \ f\ \ &= N3\, (N3\ \ f) \\
&= B\ \ N3\ \ N3\ \ f \\
MULT &= B
\end{aligned}$$

thus we can say

$$MULT := B = \lambda nkf.n(kf)$$

What happens when we apply $N3$ to $N2$? We get the following:

$$\begin{aligned}
& N3\ \ N2 \\
= \ & \lambda fa.f(f(fa))\, N2 \\
= \ & \lambda a.N2(N2(N2\ \ a)) \\
= \ & \lambda a.N2(N4\ \ a)) \\
= \ & \lambda a.N8\ \ a \\
\equiv \ & N8
\end{aligned}$$

in other ”words”

$$\begin{aligned}
& 3\ \ 2 \\
= \ & 2 \circ 2 \circ 2 \\
= \ & 8
\end{aligned}$$

so application of natural numbers is the same exponentiation, but the numbers are reversed. Thus, we can say:

$$POW := \lambda nk.kn$$

which is just our T_h combinator (section 2.3.7).

$$POW := T_h$$

How do we do addition? Let's start with adding one—in other words—finding the successor. All we need to do is compose the function once more. We could define the $SUCC$ function as follows:

$$SUCC := \lambda nfa.f(nfa)$$

Alternatively:

$$SUCC := \lambda nf.Bf(nf)$$

which some find prettier, because it makes it clear that we're doing function composition, but it also takes a bit longer to compute, because the second definition reduces down to the first:

$$\begin{aligned} & \lambda nf.Bf(nf) \\ = \ & \lambda nf.(\lambda gha.g(ha))f(nf) \\ = \ & \lambda nf.\lambda a.f((nf)a) \\ = \ & \lambda nfa.f(nfa) \end{aligned}$$

so if you use the second, you need to perform that reduction every time you invoke it.

If we now try to find the successor of $N3$, we get

$$\begin{aligned} & SUCC\ \ N3 \\ = \ & (\lambda nfa.f(nfa))\, N3 \\ = \ & \lambda fa.f(N3\ \ fa) \\ = \ & \lambda fa.f((\lambda hb.h(h(hb)))fa) \\ = \ & \lambda fa.f(f(f(fa))) \\ = \ & N4 \end{aligned}$$

or

$$\begin{aligned} & SUCC\ \ N3 \\ = \ & (\lambda nf.Bf(nf))\, N3 \\ = \ & \lambda f.Bf(N3\ \ f) \\ = \ & \lambda f.Bf((\lambda ha.h(h(ha)))f) \\ = \ & \lambda f.Bf(\lambda a.f(f(fa))) \\ = \ & \lambda f.Bf(\lambda a.f(f(fa))) \\ = \ & \lambda f.(\lambda ghb.g(hb))f(\lambda a.f(f(fa))) \\ = \ & \lambda f.(\lambda b.f((\lambda a.f(f(fa)))b)) \\ = \ & \lambda f.\lambda b.f(f(f(fb))) \\ = \ & \lambda fa.f(f(f(fa))) \\ = \ & N4 \end{aligned}$$

To add two numbers, we can now just call this successor function multiple times. Say we wanted to add three to four. We can call the successor function three times on four:

$$\begin{aligned}
& N3\ \ SUCC\ \ N4 \\
=\ & SUCC\,(SUCC\,(SUCC\ \ N4)) \\
=\ & SUCC\,(SUCC\ \ N5) \\
=\ & SUCC\ \ N6 \\
=\ & N7
\end{aligned}$$

Thus we can say

$$ADD := \lambda nk.(n\ \ SUCC\ \ k)$$

$SUCC$ is really just an infix ADD.

We could also say that if we want to add two numbers n and k, we can compose a function n times and k times and compose the results to get a $(n+k)$-fold composition of the function. In lambda calculus:

$$ADD := \lambda nkf.B(nf)(kf)$$

or

$$ADD := \lambda nkfa.(nf)(kfa)$$

which looks a lot like our $SUCC$ function, just with an extra argument to decide the number of times to compose the function, and it's prefix instead of infix. In a more mathematical notation, you can say

$$ADD\ \ n\ \ k\ \ f := f^{\circ n} \circ f^{\circ k}$$

3.1.3 Boolean comparison

The C programming language has only added booleans in the C99 standard. In C, booleans are really just integers. All boolean expressions are really just arithmetic. In C, $false$ is synonymous with 0 and every non-zero number means $true$. In lambda calculus, this is the same. Our definition of $N0$ is exactly the same as our definition of F. This section talks about things that are important to both boolean logic and arithmetic, which are really just the same.

I want to define a function that takes a church numeral and returns T if it's $N0$ or F if it's not. I will call this function $ISZERO$. To do this, we need to remember what a church numeral really is; it's a function that takes a function and an argument and applies that function a given number of times to that argument. The unique thing about $N0$ is that it applies that given function zero times to the given argument, meaning it just returns the argument. That means that if we give T as the second argument, $N0$ will return T. So far, we have this:

$$ISZERO := \lambda n.n...T$$

We still need to fill in the dots. If n isn't $N0$, whatever is on the dots will be applied to T, n times. We want this to always return F. We know a function that can do this: the constant function (section 2.3.3). Thus, we get:

$$ISZERO := \lambda n.n(KF)T$$

Here is an example for if it is not yet clear to you why we use the constant combinator:

$$ISZERO\ \ N3 = (\lambda n.n(KF)T)N3 = N3(KF)T = K\ \ F\ \ (KF(KF\ \ T)) = F$$

Using this in combination with previously defined arithmetic operations, we can quite easily define other numerical comparisons. I consider it futile to go over all of them, because they're easy enough to figure out for yourself, besides I'm not using them in later sections, so I don't feel the need to define them when I don't use them.

3.1.4 Data structures

A fundamental concept in programming is that of data structures. Data structures are ways of storing data together. There are different data structures, which are sort of like different data "layouts". You could argue that one combinator we've looked at, the thrush (section 2.3.7), is basically a data structure already; it takes an argument and holds on to it until you give it a function to apply to it. Although this isn't really a data structure yet; it only stores one thing. We can turn it into a data structure by "upgrading" it; let's give it another argument.

$$V := \lambda abf.fab$$

Which we have already seen in section 2.3.8.

We have basically defined something that is known as a (Church) pair. It holds on to two things. We could put two values into it:

$$Vxy = \lambda f.fxy$$

If we want to look at the first value, we can input K:

$$(\lambda f.fxy)\ K = Kxy = x$$

If we want to look at the second value, we can input KI:

$$(\lambda f.fxy)\ KI = KIxy = y$$

Thus, we could define the following functions:

$$FST := \lambda n.nK$$

$$SND := \lambda n.n(KI)$$

From now on, we'll also use:

$$PAIR := V$$

Pairs are very powerful, we use them for all kinds of things. One thing we can do is create linked lists. We do this by putting a value in the first index of the pair and another pair in the second. You can continue this pattern to make infinitely large lists. We could, for example, make a list from $N1$ through $N4$:

$$PAIR\ N1\ (PAIR\ N2\ (PAIR\ N3\ N4))$$

If we want to find the first index of the array, we would do:

$$FST\ (PAIR\ N1\ (PAIR\ N2\ (PAIR\ N3\ N4))) = N1$$

For the second index, we would do:

$$\begin{aligned} & FST\ (SND\ (PAIR\ N1\ (PAIR\ N2\ (PAIR\ N3\ N4)))) \\ = & FST\ (PAIR\ N2\ (PAIR\ N3\ N4)) \\ = & N2 \end{aligned}$$

We can easily write a function that prepends something to a list. The following function prepends a list with $N0$:

$$\lambda p.PAIR\ N0\ p$$

Applying it gives:

$$\begin{aligned} & (\lambda p.PAIR\ N0\ p)\ (PAIR\ N1\ (PAIR\ N2\ (PAIR\ N3\ N4))) \\ = & PAIR\ N0\ (PAIR\ N1\ (PAIR\ N2\ (PAIR\ N3\ N4))) \end{aligned}$$

Appending to a list is a bit harder; it involves stepping through the entire list and changing the last pair to link to a new pair. We'll talk more about later, when we cover functional programming and meta programming.

3.1.5 Natural numbers continuation

We've covered addition, multiplication and exponentiation, which were quite simple, but what about subtraction, division and finding the square root? Well, in contrast to the operations we've covered, these are actually quite complex. We'll quickly look at subtraction, but it gets very complicated very quickly, and it's really out of the scope of this text to cover all of arithmetic in lambda calculus. I just wanted to show that it's possible and that lambda calculus is Turing complete.

Just like we defined a successor function before defining addition, we'll define a predecessor function before defining subtraction.

The reason subtraction is so much harder than addition, is that applying a function once more is very easy, but removing an application is hard. The trick is this: you don't remove an application, you reapply all the functions from the start, until you get to the number you want the predecessor of. In other words, you reapply all the functions, except for the last.

How do we know when to stop? If we create an algorithm that just reapplies our function until we get to the number we want to know the predecessor of, we will have already overshot the predecessor when we get to that number. What we actually want to do, is not to create an algorithm that returns a number after each iteration, but to create a pair containing the new number and the previous number carried over from the last iteration. That way, we can compare the second number of the pair to the input, and if they are equal, we return the first number of the pair.

Let's start by defining a function that takes a pair, and returns the next pair. Given a pair, we can make a new pair, of which the first item is the same as the second item of the input pair, and the second item is the successor of the second item of the input pair. This function is usually denoted with the letter Φ.

$$\Phi := \lambda p.PAIR\ (SND\ p)\ (SUCC\ (SND\ p))$$

It can be reduced to

$$\begin{aligned}
&\lambda p.PAIR\ (SND\ p)\ (SUCC\ (SND\ P))\\
&= \lambda p.(\lambda abf.fab)\ (SND\ p)\ (SUCC\ (SND\ p))\\
&= \lambda pf.f\ (SND\ p)\ (SUCC\ (SND\ p))\\
&= \lambda pf.f\ (p\ KI)\ (SUCC\ (p\ KI))\\
&= \lambda pf.f\ (p\ KI)\ ((\lambda nga.g(nga))\ (p\ KI))\\
&= \lambda pf.f\ (p\ KI)\ (\lambda ga.g(p\ KI\ g\ a))\\
&= \lambda pf.f(p(\lambda ab.b))(\lambda ga.g(p(\lambda ab.b)ga))
\end{aligned}$$

in lambda calculus.

Using this, we can quite easily define a predecessor function. If we want to find the predecessor of n, we can apply Φ, n times to the pair $(0, 0)$. The second item will be equal to n, so the first item will be n's predecessor.

$$PRED := \lambda n.FST\ (n\ \Phi\ (PAIR\ N0\ N0))$$

which reduces to

$$\begin{aligned}
&\lambda n.FST\ (n\ \Phi\ (PAIR\ N0\ N0))\\
&= \lambda n.n\ \Phi\ (PAIR\ N0\ N0)\ K\\
&= \lambda n.n\ \Phi\ (\lambda f.f\ N0\ N0)\ K\\
&= \lambda n.n\ \Phi\ (\lambda f.f(\lambda ba.a)(\lambda ba.a))\ K\\
&= \lambda n.n(\lambda pf.f(p(\lambda ab.b))(\lambda ga.g(p(\lambda ab.b)ga)))(\lambda f.f(\lambda ba.a)(\lambda ba.a))(\lambda ab.a)
\end{aligned}$$

in lambda calculus.

Just like how addition is just the successor function applied multiple times, subtraction is just the predecessor function applied multiple times.

$$SUB := \lambda nk.n\ PRED\ k$$

which reduces to

$$\lambda nk.n(\lambda m.m(\lambda pf.f(p(\lambda ab.b))(\lambda ga.g((p(\lambda ab.b))ga)))(\lambda f.f(\lambda ba.a)(\lambda ba.a))(\lambda ab.a))k$$

There are many alternative ways of going about defining subtraction, but this is probably the most straightforward.

3.1.6 Other kinds of numbers

We have as of yet only covered natural numbers. They are the easiest to define and use. There are, however, many other types of numbers, such as integers, rationals, reals, complex numbers, etc. I will quickly cover how you'd go about defining them, but I won't go into it deeply. I have shown how natural numbers work, and using them with pairs you can create all sorts of numbers, but it can be slightly difficult to wrap your head around, and it isn't in the scope of this text. I just wanted to show how simple arithmetic works in lambda calculus and show it is possible. I am not, however, trying to define all arithmetic. We are just touching on the topic.

That said, the way you define these other, more complicated types of numbers is very elegant in my opinion, and I really wanted to show it. Another reason I wanted to show it, is that it makes use of pairs extensively. By showing these elegant solutions to defining more complicated types of numbers, I can really show the usefulness and power of pairs.

Let's start with the easiest of the bunch: integers. They are just like natural numbers, except they can be negative. There are different ways of achieving this. The simplest would be to define an integer as a pair of a boolean and a natural number, where the boolean decides whether the integer is positive or negative. You can then define special arithmetic operators for integers, based on those for natural numbers, to take into account the sign of the number (whether it's positive or negative).

$$(sign, n)$$

This makes multiplication very straightforward; you just perform a regular multiplication on the natural numbers in the pairs, and XOR the signs.

$$MULT_s := \lambda ab.PAIR\ (XOR\ (FST\ a)\ (FST\ b))\ (MULT\ (SND\ a)\ (SND\ b))$$

Addition is a bit more complicated, since it requires a lot of conditionals (section 3.1.3).

$$ADD_s := \lambda ab. \begin{cases} (T, ADD\ (SND\ a)\ (SND\ b)) & \text{if } a \geq 0 \land b \geq 0 \\ (F, ADD\ (SND\ a)\ (SND\ b)) & \text{if } a \leq 0 \land b \leq 0 \\ (T, SUB\ (SND\ a)\ (SND\ b)) & \text{if } a \geq 0 \land b \leq 0 \land |a| \geq |b| \\ (F, SUB\ (SND\ b)\ (SND\ a)) & \text{if } a \geq 0 \land b \leq 0 \land |a| < |b| \\ (T, SUB\ (SND\ b)\ (SND\ a)) & \text{if } a \geq 0 \land b \leq 0 \land |a| < |b| \\ (F, SUB\ (SND\ a)\ (SND\ b)) & \text{if } a \geq 0 \land b \leq 0 \land |a| \geq |b| \end{cases}$$

Converting this to pure lambda calculus will be left as an exercise to the reader.

This makes subtraction very simple though, it is just addition, but you negate the sign of one of the numbers.

$$NEG_s := \lambda a.PAIR\ (NOT\ (FST\ a))\ (SND\ a)$$
$$SUB_s := \lambda ab.ADD_s\ a\ (NEG_s\ b)$$

However, this is not the only way of defining integers. Instead of defining an integer as a pair of a boolean and a natural number, you define it as a pair of two natural numbers so that if you subtract the second from the first, you get your integer. In other words, an integer k is represented as a pair (a, b), where a and b are natural numbers and $k = a - b$. This means there are multiple ways (infinite in fact) of representing a given integer, but it also means that our operators are greatly simplified. We can define our operators as follows

$$\begin{aligned}
ADD_i :=& \lambda ab.PAIR\ (ADD\ (FST\ a)\ (FST\ b))\ (ADD\ (SND\ a)\ (SND\ b))\\
NEG_i :=& \lambda a.PAIR\ (SND\ a)\ (FST\ a)\\
SUB_i :=& \lambda ab.ADD\ a\ (NEG_i\ b)\\
MULT_i :=& \lambda ab.PAIR\\
& \qquad (ADD\ (MULT\ (FST\ a)\ (FST\ b))\ (MULT\ (SND\ a)\ (SND\ b)))\\
& \qquad (ADD\ (MULT\ (FST\ a)\ (SND\ b))\ (MULT\ (SND\ a)\ (FST\ b)))
\end{aligned}$$

Multiplication is relatively complex, but it's still simpler and more elegant than our previous definition of addition using a cubic ton of convoluted conditionals.

We can use integers to create rationals. How you ask? Well with pairs of course! Just like how we used a pair of naturals to define integers, we can use a pair of integers to define rationals. You define a rational number q as a pair (k, a), where k is an integer and a a natural number such that $q = k/(1 + a)$.

For encoding real numbers I will refer you to [1]. This and [2] is where I got most information for encodings of other kinds of numbers from.

3.2 Recursion

We already know how to do some control flow using conditionals, but how about looping? We can achieve this using *recursion.*

3.2.1 Understanding recursion

To understand recursion, we must first understand recursion. Recursion is when we make use of self-reference, usually through defining something in terms of itself. A common example is the following factorial function (in C):

```
int fac(unsigned int n)
{
        if (i == 0)
                return 1;
        else
                return n * fac(n - 1);
}
```

or if you prefer ternary operators:

```
int fac(unsigned int i)
{
        return n == 0 ? 1 : n * fac(n - 1);
}
```

Our definition of `fac()` contains calls to `fac()`, i. e. we have defined it in terms of itself. This is a very effective (although sometimes tricky, if you're not used to it) way of looping. Recursion, in fact, completely eliminates the need for things such as *for* or *while* loops commonly found in programming languages.

We've covered boolean logic and arithmetic in section 3. Thus, we can also define our C-function in pseudo lambda calculus[9].

$$FAC := \lambda n.ISZERO\ n\ N1\ (MULT\ n\ (FAC\ (PRED\ n)))$$

or if you want to eliminate some braces using function composition (section 2.3.6)

$$FAC := \lambda n.ISZERO\ n\ N1\ (MULT\ n\ (B\ FAC\ PRED\ n))$$

where n is a natural number as defined in section 3.1.2.

Because of something called *laziness* (section 4.1.3), we don't have to be afraid that computation won't finish. In short, laziness means that the computer won't try to evaluate something until it has to. If it were to try to evaluate this function, it would become an infinitely long expression

$$\begin{aligned}
FAC :=\ & \lambda n.ISZERO\ n\ N1\ (MULT\ n\ (FAC\ (PRED\ n)))\\
=\ & \lambda n.ISZERO\ n\ N1\ (MULT\\
& \qquad n\\
& \qquad ((\lambda m.ISZERO\ m\ N1\ (MULT\ m\ (FAC\ (PRED\ m))))\\
& \qquad\quad (PRED\ n)))\\
=\ & \lambda n.ISZERO\ n\ N1\ (MULT\\
& \qquad n\\
& \qquad ((\lambda m.ISZERO\ m\ N1\ (MULT\\
& \qquad\quad m\\
& \qquad\quad ((\lambda l.ISZERO\ l\ N1\ (MULT\ l\ (FAC\ (PRED\ l))))\\
& \qquad\qquad (PRED\ m))))\\
& \qquad\qquad\quad (PRED\ n)))\\
=\ & ...
\end{aligned}$$

but if we input a value, computation does terminate. Say we want to know the factorial of 0

$$\begin{aligned}
FAC\ N0 &= (\lambda n.ISZERO\ n\ N1\ (MULT\ n\ (FAC\ PRED\ n)))\ N0\\
&= ISZERO\ N0\ N1\ (MULT\ N0\ (FAC\ PRED\ N0))\\
&= N1
\end{aligned}$$

or the factorial of 1

$$\begin{aligned}
FAC\ N1 &= (\lambda n.ISZERO\ n\ N1\ (MULT\ n\ (FAC\ PRED\ n)))\ N1\\
&= ISZERO\ N1\ N1\ (MULT\ N1\ (FAC\ PRED\ N1))\\
&= MULT\ N1\ (FAC\ PRED\ N1)\\
&= MULT\ N1\ (FAC\ N0)\\
&= MULT\ N1\ N1\\
&= N1
\end{aligned}$$

[9] Lambda calculus has no identifiers, so we can't just define an identifier in terms of itself to achieve recursion

or the factorial of 2

$$\begin{aligned} FAC\ N2 &= (\lambda n.ISZERO\ n\ N1\ (MULT\ n\ (FAC\ PRED\ n)))\ N2 \\ &= ISZERO\ N2\ N1\ (MULT\ N2\ (FAC\ PRED\ N2)) \\ &= MULT\ N2\ (FAC\ PRED\ N2) \\ &= MULT\ N2\ (FAC\ N1) \\ &= MULT\ N2\ N1 \\ &= N2 \end{aligned}$$

or the factorial of 3

$$\begin{aligned} FAC\ N3 &= (\lambda n.ISZERO\ n\ N1\ (MULT\ n\ (FAC\ PRED\ n)))\ N3 \\ &= ISZERO\ N3\ N1\ (MULT\ N3\ (FAC\ PRED\ N3)) \\ &= MULT\ N3\ (FAC\ PRED\ N3) \\ &= MULT\ N3\ (FAC\ N2) \\ &= MULT\ N3\ N2 \\ &= N6 \end{aligned}$$

We don't need to evaluate the recursive branch if our input is 0, so we don't. In most languages, like C, every expression gets evaluated immediately[10].

The way we are achieving recursion in this example is by defining an identifier in terms of itself, which doesn't work in pure lambda calculus, since it doesn't have identifiers. What we need to do is to somehow create a combinator that can do recursion without the use of self-referencing identifiers. We will slowly work towards this combinator.

3.2.2 General recursion

Let's start with the simplest recursive combinator you can create in lambda calculus

$$LOOP := (\lambda x.xx)\ \lambda x.xx$$

which will result in an infinite loop if you try to reduce it:

$$\begin{aligned} &(\lambda x.xx)\ \lambda x.xx \\ =\ &(\lambda x.xx)\ \lambda x.xx \\ =\ &(\lambda x.xx)\ \lambda x.xx \\ =\ &... \end{aligned}$$

We have seen this before, this is the *omega combinator* (section 2.3.2).

This is not very useful, it just hangs without doing anything. We could define a slightly more useful function as

$$REC\ f := f\ (REC\ f)$$

which would be

$$REC := \lambda f.f\ (REC\ f)$$

in pseudo lambda calculus. It can apply any function f recursively. This is called *general recursion* in computer science—it is the simplest way to perform an operation recursively.

We can use this new function to redefine our $LOOP$ combinator.

$$LOOP := REC\ \lambda a.a$$

[10] In this case it doesn't matter, because in C functions don't get evaluated, but called.

But we still haven't achieved general recursion in pure lambda calculus, because we are still defining identifiers—which pure lambda calculus doesn't have—in terms of themselves instead of defining a combinator which can do recursion by itself.

We can also use this new REC function to redefine our factorial function

$$\begin{aligned}
FAC :=\; & REC\ (\lambda fn.ISZERO\ n\ N1\ (MULT\ n\ (f\ (PRED\ n))))\\
=\; & (\lambda fn.ISZERO\ n\ N1\ (MULT\ n\ (f\ (PRED\ n))))\\
& \quad (REC\ (\lambda fn.ISZERO\ n\ N1\ (MULT\ n\ (f\ (PRED\ n)))))\\
=\; & (\lambda fn.ISZERO\ n\ N1\ (MULT\ n\ (f\ (PRED\ n))))\ FAC\\
=\; & \lambda n.ISZERO\ n\ N1\ (MULT\ n\ (FAC\ (PRED\ n)))
\end{aligned}$$

which reduces to our previous definition as you can see.

3.2.3 The Y-combinator

We have now defined FAC not in terms of itself using a specialised recursion combinator. This combinator *is* still defined in terms of itself though. If we succeed to define general recursion in lambda calculus without the use of self-referring identifiers, we can finally write our recursive functions in pure lambda calculus.

Luckily for us, this combinator exists, it's called the *Y-combinator.* Instead of self-reference, it uses self-application, much like how our earlier LOOP combinator is really a function applied to itself resulting in that same function being applied to itself. The Y-combinator looks much like our LOOP combinator, except it has and extra function as argument which is applied to the application of the arguments of the inner functions to themselves.

$$Y := \lambda f.(\lambda x.f(xx))\lambda x.f(xx)$$

If we try to evaluate this, we again get an infinitely large expression

$$\begin{aligned}
Y :=\; & \lambda f.(\lambda x.f(xx))\lambda x.f(xx)\\
=\; & \lambda f.f((\lambda x.f(xx))\lambda x.f(xx))\\
=\; & \lambda f.f(f((\lambda x.f(xx))\lambda x.f(xx)))\\
=\; & \lambda f.f(f(f((\lambda x.f(xx))\lambda x.f(xx))))\\
=\; & \lambda f.f(f(f(f(f(f(f(f(f(f(...))))))))))
\end{aligned}$$

We can see it does in fact function as expected.

We have successfully encoded recursion without recursion, and can now implement our FAC function in lambda calculus. The lambda expression is really long, so I've tried to break it up across multiple lines in a way that somewhat makes sense. We have already seen the reduction of the $PRED$ function in section 3.1.4, so I'll omit that here.

$$\begin{aligned}
FAC :=\ & Y(\lambda fn.ISZERO\ n\ N1\ (MULT\ n\ (f\ (PRED\ n))))\\
=\ & Y(\lambda fn.n(KF)T\ N1\ (\lambda g.n\ (f\ (PRED\ n)\ g)))\\
=\ & (\lambda f.(\lambda x.f(xx)(\lambda x.f(xx))))\\
& (\lambda fn.n((\lambda ab.a)(\lambda ab.b))(\lambda ab.a)(\lambda ab.ab)\\
& \qquad (\lambda g.n\ (f\ (PRED\ n)\ g)))\\
=\ & (\lambda f.(\lambda x.f(xx)(\lambda x.f(xx))))\\
& (\lambda fn.n(\lambda abc.c)(\lambda ab.a)(\lambda ab.ab)\\
& \qquad (\lambda g.n\\
& \qquad\qquad (f\\
& \qquad\qquad (n(\lambda ph.h(p.(\lambda ab.b))(\lambda ia.i(p(\lambda bc.c)ia)))(\lambda h.h(\lambda ab.b)(\lambda ab.b))(\lambda ab.a))\\
& \qquad\qquad g)))
\end{aligned}$$

4 Functional programming (applied lambda calculus)

Functional programming is really just lambda calculus as a programming paradigm. The fundamental ideas and properties of lambda calculus are also the fundamental ideas of functional programming. Many functional languages compile to an intermediate language first, which is usually just some form of lambda calculus.

Functional programming does use some other concepts and abstractions borrowed from mathematical disciplines such as set theory, category theory and type theory. The notions of types in functional programming are a bit more like those found in mathematics rather than in mainstream programming languages, where types are often just a way of telling the computer how certain data should be interpreted. Most of the more abstract ideas in functional programming come from category theory, which was created as a sort of general language for abstractions. I will cover the concepts most important to us, in the context of functional programming, as we need them.

I would have liked to talk about these other disciplines separately too, because a lot of interesting stuff (some of which we have mentioned in section 2.1) comes into play. All of the mathematical ideas that were developed in the historical context of the formalisation of mathematical logic is interconnected and very important to logic, mathematics and computer science as a whole. Ultimately, I'd have liked to talk about all of these, but sadly I couldn't.

I spent an entire section (section 3) to show you that lambda calculus is Turing complete. The reason I did this was to show you that lambda calculus and machine language are formally equivalent, while taking two wildly different approaches. Lambda calculus is very simple and has some desirable properties which allow us to make expressive code which is easy to reason about. The idea behind functional programming is programming in a lambda calculus inspired style and translating it to machine code. We can do this, because lambda calculus is Turing complete, thus formally equivalent with machine code. The difficulty is with optimising the translation process.

4.1 Fundamental ideas from lambda calculus

Throughout this work, I have mentioned some of the desirable properties of lambda calculus. This will be a more in-depth look at these properties and why we care for them. Namely statelessness, functions and data being practically the same, and laziness. These properties are very useful. I'll compare functional programming with other paradigms at the end of this section (section 4), which really tries to illustrate the power of this paradigm and also some of its weaknesses.

4.1.1 Statelessness

A big problem in dealing with complexity is keeping track of state. Our minds are very bad at keeping track of multiple moving parts at the same time. We can't always imagine all the relationships between parts and who owns what data at what time. It is hard to keep track of constantly changing context.

The nice thing about lambda calculus is that it doesn't have state. Every function is dependent and *only* dependent on its inputs. The notion of functions in lambda calculus and functional programming is much closer to the mathematical one this way. In procedural/imperative languages, you never know for sure what a function does besides the output it gives. A common example is calling a function to do arithmetic, but accidentally firing the missiles, starting a nuclear war. In procedural/imperative languages, functions aren't the same thing, they describe actions.

Of course there is still some complexity—there should be or you wouldn't be able to create anything at all. The thing is that there is no manipulation of global data, which might have an impact on some completely different part of your program. You don't have any unexpected behaviour happening, because you forgot about some variable.

I talk more about the benefits of statelessness in section 4.3.

4.1.2 Functions as data

In lambda calculus, there is no distinction between functionality and data. Functions are data and data is functions. This may be hard to wrap your head around at first, but it isn't too difficult. We have already seen that lambda calculus has only functions, and that we can encode anything else with it. You can also look at functions as a mapping between sets, thus a function could be written as a giant lookup table, which is just data.

This allows us to do some very powerful things. We can throw functions around just as easily as if it were a value, because it is. We can parse functions as arguments, create functions from functions, curry functions, etc. Functions are first-class citizens.

All functions in functional programming are curried unary functions like in lambda calculus. This means we can create functions by parsing only part of its required arguments for example. It also allows us to create functions with more arguments by parsing another function as its first argument.

The functional programming concept of functions is so powerful in fact, that most modern programming language offer lambda functions. You see them used quite a bit in languages such as C++ and Rust now.

4.1.3 Laziness

Laziness isn't an idea from lambda calculus per se, but it is one of the fundamental ideas of functional programming and it has at least some connection to lambda calculus. The idea of laziness is that expressions aren't evaluated until it is required. A good example would be the Y-combinator we saw in section 3.2. If the Y-combinator were always evaluated immediately, we would never be able to use it in a program, because execution would never terminate. Laziness is a necessity in functional programming, really, but it has its benefits. Laziness was probably even the biggest driving force behind the earlier research into functional programming.

One of the most obvious benefits of laziness is that it means we don't compute what we don't have to, which is called *short-circuit computation*, obviously resulting in a performance boost, but this isn't always the case (as we'll look into later). The real benefit with laziness is the ability to work with infinities. The most common example probably being working with infinite sets. We can define infinite sets, but only use parts of it when we need it. When we create the set, it doesn't store the entire, infinite thing in memory, but only evaluates the part we need when we need it.

I will give some examples of how we can achieve laziness in non-functional languages (in this case Rust) and why we would want to do so.

We can achieve laziness by wrapping everything in lambda functions. Let's look at a simple example. We'll add two numbers lazily first.

```
fn lazy_add<A, B>(a: A, b: B) -> Box<dyn Fn() -> i32>
where
    A: Fn() -> i32 + 'static,
    B: Fn() -> i32 + 'static,
{
    Box::new(move || a() + b())
}

fn main() {
    println!("5 + 6 = {}", lazy_add(|| 5, || 6)());
}
```

There's a lot going on here. Don't worry about all the syntax, conceptually we're just wrapping everything in lambda functions. Instead of parsing and returning integers, we use lambda functions that evaluate to integers, but only when we want them to instead of immediately. There's some stuff going on with the trait system, lifetimes, etc. which you should just ignore. The reason I'm using Rust and not C++ is that C++ doesn't play nicely with converting lambda functions that capture certain values to other types. Other reasons are that Rust lambda syntax is a bit simpler and the type system is more intuitive and clear[11].

As you can see, lambda functions in Rust have a different syntax. In Rust, lambda functions are denoted with the arguments between pipes ('|') followed by the return expression. In this case our lambda functions don't have any arguments, so we parse the lambda functions `||5` and `||6`. These get parsed as the arguments `a` and `b`. We return the lambda function `||a() + b()`. The types of these lambda functions are `Fn() -> i32`. The `i32` is just a 32-bit integer. I'll talk more about types in section 4.2.2.

Conceptually, we're basically doing

```
fn lazy_add(a: Fn() -> i32, b: Fn() -> i32) -> Fn() -> i32 {
    ||a() + b()
}

fn main() {
    println!("5 + 6 = {}", lazy_add(||5, ||6)());
}
```

but that's not how Rust works, sadly. You should read the definition of `lazy_add` as: "`lazy_add` is a function that takes an `a` of type `Fn() -> i32` and a `b` of type `Fn() -> i32`, which returns a lambda function which returns the sum of the evaluations of `a` and `b`. This returned lambda function is also of type `Fn() -> i32`, which makes the return type of `lazy_add` also `Fn() -> i32`." A `Fn() -> i32` is just a function without arguments that returns a 32-bit integer. We define it as such, because this way, `lazy_add` doesn't just return an integer, but a function that evaluates to an integer, which we can decide when to evaluate by calling it. That's why we have the extra parentheses at the end when we call it and print out the result.

I said that concepts such as functions as first-class citizens and lambda functions are a functional concept, yet here we're doing the same thing in Rust. That's because functional concepts are seeping into other paradigms. Most modern programming languages have lambda

[11] When I was experimenting with C++, I very quickly ran into this convoluted mess: `Lazy<T>::Lazy<int>::<lambda()>`

functions, or at least some way of parsing functions as arguments. This already shows how useful these concepts are if every other language is copying it.

In this example, laziness is pretty useless. We only evaluate things later, but still do. A slightly more useful example would be the following.

```
fn hang() -> i32 {
    hang()
}

fn first<A, B>(a: A, b: B) -> Box<dyn Fn() -> i32>
where
    A: Fn() -> i32 + 'static,
    B: Fn() -> i32 + 'static,
{
    Box::new(a)
}

fn main() {
    println!("first 5 hang: {}", first(||5, hang)());
}
```

Again, this is what we really mean with our definition of `first`.

```
fn first(a: Fn() -> i32, b: Fn() -> i32) -> Fn() -> i32
{
    a
}
```

In this case, laziness is preventing us from getting stuck in an infinite recursive loop when trying to evaluate `hang`. This is still a very silly example though. Here is an actually useful example. We'll define our own logical and, but lazy, very similar to how we did it in lambda calculus (section 3.1.1).

```
fn and<A, B>(a: A, b: B) -> Box<dyn Fn() -> bool>
where
    A: Fn() -> bool + 'static,
    B: Fn() -> bool + 'static,
{
    if a() { Box::new(b) } else { Box::new(a) }
}

fn main() {
    println!("T && T: {}", and(||true, ||true)());
    println!("T && F: {}", and(||true, ||false)());
    println!("F && T: {}", and(||false, ||true)());
    println!("F && F: {}", and(||false, ||false)());
}
```

Again the more conceptual pseudo-Rust.

```
fn and(a: Fn() -> bool, b: Fn() -> bool) -> Fn() -> bool
{
    if a() { b } else { a }
}
```

The reason this is a better example, is because we don't evaluate `b` if `a` is already false. If we didn't define this to be lazily, any expression we would parse in for `b` would have to be evaluated before the call to the function, but this is useless when `a` is already known to be false.

But this is still child's play. The real magic is when we start using infinite data structures as I hinted at earlier in this section. Now this gets very complicated with all the extra Rust syntax that we don't care about, so you'll have to bare with me.

```rust
use std::fmt::Display;

struct List<T> {
    head: Option<Box<dyn Fn() -> T>>,
    tail: Option<Box<dyn Fn() -> List<T>>>,
}

impl<T: Copy + Display> List<T> {
    fn new() -> Box<dyn Fn() -> Self> {
        Box::new(move || List { head: None, tail: None })
    }

    fn from_slice(slice: &'static [T]) -> Box<dyn Fn() -> Self> {
        if slice.len() == 0 {
            List::new()
        } else {
            let index: T = slice[0];
            Box::new(
                move || List {
                    head: Some(Box::new(move || index)),
                    tail: Some(List::from_slice(&slice[1..])),
                }
            )
        }
    }

    fn print(&self) {
        if let Some(head) = &self.head {
            print!("{} ", head());
        } else {
            println!();
        }

        if let Some(tail) = &self.tail {
            tail().print();
        }
    }

    fn print_n(&self, n: usize) {
        if n == 0 {
            println!();
            return;
        }

        if let Some(head) = &self.head {
            print!("{}␣", head());
        }

        if let Some(tail) = &self.tail {
            tail().print_n(n - 1);
        }
    }
}

fn range_from(begin: i32) -> Box<dyn Fn() -> List<i32>> {
    Box::new(
        move || List {
            head: Some(Box::new(move || begin)),
            tail: Some(range_from(begin + 1)),
        }
    )
}

fn main() {
    List::from_slice(&[3, 5, 4, 6, 9, 2])().print();
    range_from(0)().print_n(10);
}
```

If you want to play with the code, you can find it in the accompanying code folder.

I don't expect you to understand everyting. You just need to know that we are making a lazy linked list, just like how I explained it in lambda calculus in section 3.1.4. I added functions for converting slices to our lazy list, printing our list and printing the first n values of our list. I also added a `range_from` function which makes an infinite list starting from a given number to infinity.

Laziness allows us to describe a list as a recursive function that only evaluates the elements we need. We are basically defining a list mathematically and evaluating only the part we need.

This is just one example of an infinite list. This one may be a bit pointless, but they can be quite usefull. You could make a list of all possible outputs of a functions and graph it or turn it into a binary string. You can basically do set theory.

You may be asking yourself why `range_from` takes a regular integer rather than a lazy integer. Well, the real reason is that I wasn't able to get it to work using lazy integers, but there's actually a good reason why you wouldn't want it to be lazy. If it were lazy, then `begin` would only be evaluated for an element when you evaluate that element, but it has to be reevaluated for every element. If `begin` is immediately forced, you only have to calculate it once, resulting in a significant performace boost.

Laziness is a double edged sword. It allows you to get more performance in some places and do cool things like infinite data structures, but it may also have significant performance drawbacks. In this case, we were able to decide what to make lazy, but in functional programming everything is lazy and we don't have this luxury. Laziness is just a different approach, it doesn't solve all problems.

4.2 Introduction to Haskell

Haskell is probably one of the earliest, most advanced and most popular functional programming language. It is the mother of functional programming language. When people talk about functional programming, they often talk about Haskell.

4.2.1 Basic syntax

The basic syntax of Haskell is relatively simple and intuitive. There is not much point in explaining everything. I'll explain the most basic and important things. I'll reference you to a book called *Learn You a Haskell for Great Good!* by Lipovaca [18] for a full guide to the Haskell programming language. I think that most syntax is simple and intuitive enough that, if you know the basics, should be self explanatory when I introduce it. My goal is not to teach you Haskell, but the ideas behind functional programming. For a quick cheat sheet of the Haskell syntax, I'll refer you to [22]. I'll be going over the basic syntax from a bird's eye view.

In Haskell, everything is constant. Once you bind an identifier to something, it will be forever bound to it. In Haskell, everything has a type. We denote the type of something with the `::` operator, followed by its type (section 4.2.2). We can define a new identifier with the `=` operator. We follow it with the definition of our identifier. This can be a function or a value (it's all the same to us functional programmers), as long as it has the right type. You don't have to explicitly denote the type of something when it can be inferred from your definition by the compiler, but it's good practice to still do.

You have your normal types as you'd expect them. Haskell types usually use upper camel case or Pascal case, while functions use lower camel case.

There are different kinds of literals like whole numbers, floating point numbers, characters, booleans, lists, strings and the whole lot. They're pretty much as how you'd expect. I won't show them here, because they're pretty obvious when you see them. Boolean values are capitalised.

You have your normal arithmetic operators and boolean operators which work just like they do in pretty much every other language. The only thing to note is that not equal to, `!=` in most languages, is `/=` in Haskell, which is supposed to look like $\neq$. Negation isn't done with a `!` operator like in most languages, but with a `not` function.

Arithmetic operators have function counterparts. These functions are words rather than symbols and are prefix rather than suffix, just like any other function. You can turn functions into infix functions by surrounding them with back ticks (`). As an example:

```
11 ‘div‘ 8
```

as opposed to

```
div 11 8
```

`div` is operates on integers, so these expressions return `1`.

There are some string manipulation operators. The one you'll use most is concatination (`++`), which is an infix operator.

Regular functions are defined by writing their name, followed by their arguments, followed by the `=` sign, followed by the return expression. An example, explicitly denoting the type (section 4.2.2), would be the following.

```
concatenate3 :: String -> String -> String -> String
concatenate3 x y z = x ++ y ++ z
```

Haskell has lambda functions of course. Lambda function in Haskell look very much like regular lambda functions, but instead of a lambda we use a backslash (\)[12]. The argument names are separated by spaces. Instead of a dot we use an arrow (`->`). Thus, a lambda expression in Haskell would look like the following.

```
(\x y -> x + y) 2 3
```

which reduces to

```
(\y -> 2 + y) 3
```

[12] Who has a lambda on his keyboard, right?

and in turn to

```
2 + 3
```

which just results in 5.

We can of course define a function as a lambda function.

```
concatenate3 :: String -> String -> String -> String
concatenate3 = \x y z -> x ++ y ++ z
```

Function composition is done with a dot. There is also a $ operator, which basically puts parentheses around anything right of it. It can be used in place of the dot operator when the argument is known, else you would need to put parentheses around the function composition. So the next two expressions are the same.

```
(f . g) x
f $ g x
```

When we define a function, we may want to define some identifiers and use them in the return expression. We can do this with *where notation.*

```
addFour w x y z = z + b
    where a = w + x
          b = y + a
```

We have different kind of conditionals like this.

```
fib n
  | n < 2 = 1
  | otherwise = fib (n - 1) + fib (n - 2)

fib n =
  case n of
    0 -> 1
    1 -> 1
    _ -> fib (n - 1) + fib (n - 2)

fib n =
  if n < 2
   then 1
   else fib (n - 1) + fib (n - 2)
```

They speak for themselves.

We can create type synonyms

```
type Email = String
type Name = String
type BigFunction = (c -> d) -> (b -> c) -> (a -> b) -> (a -> d)
```

We can create enumeration types.

```
data Bool = True | False
```

We can create monadic functions and use *do notation* to basically write in a procedural style when this is necessary (section 4.2.4).

There is range notation which works as you'd expect.

```
[1..10]
```

Which evaluates to the following.

```
[1,2,3,4,5,6,7,8,9,10]
```

It's inclusive.

Last but not least, there are list comprehensions. They basically allow us to create infinite lists (section 4.1.3) and use notation that is very similar to set notation in set theory. One example would be the following.

```
[ x | x <- [50..100], x `mod` 7 == 3]
```

Which evaluates to the following.

```
[52,59,66,73,80,87,94]
```

This is really powerful, lists in haskell are in general. I recommend you read the chapters about it in [18], I couldn't explain it better.

4.2.2 Haskell's type system

The concept of types originates in mathematics. Types were originally invented as a way of preventing paradoxes in set theory (section 2.1). Many type theories have been invented, some of the most notable ones being Russell's *Theory of Types* [23], Löf's *Type Theory* [19] and Church's *Typed Lambda Calculus* [6] and other typed lambda calculi.

In most programming languages, types are just a way of denoting how data in memory should be interpreted. In functional programming, however, types are much closer to their mathematical meaning. Ideas about types in functional programming come from typed lambda calculi I just mentioned.

Since all functions are first class citizens—i.e. there is no real distinction between data and functionality—and everything has a type, functions must also have types. I'm not talking about return types found in other languages, but the type of the function itself.

In functional programming, you have compound types. There are different types of compound types. I won't go over all of them, because only a few are important to us. You have a pair type, which has the form `(A, B)`, where `A` and `B` are any other type. We also have a list, which just has the form `[A]`, where `A` is again any other type. But probably the most interesting type is the function type, which has the form `A -> B`. A function of this type takes an `A` and returns a `B`.

In functional programming, all functions are curried, so types of multi variable functions look like the following.

```
A -> B -> C -> D
```

You can read this a a function that takes an `A`, then a `B`, then a `C` and finally returns a `D`. But functions are first class citizens, so what would the type of a function that takes an `A -> B` and a `C` and returns a `D` look like? Well, it looks as follows.

```
A -> B -> C -> D
```

What about a function that takes an `A` and a `B -> C` and returns a `D`? It looks as follows.

```
A -> B -> C -> D
```

What about a function that takes an `A` and a `B` and returns a `C -> D`? It looks as follows.

```
A -> B -> C -> D
```

What about a function that takes an `A -> B -> C` and returns a `D`?

```
A -> B -> C -> D
```

What about a function that takes an `A` and returns a `B -> C -> D`?

```
A -> B -> C -> D
```

What about a function that returns a `A -> B -> C -> D`?

A —> B —> C —> D

But it's all the same? Well of course, it's all the same because of function currying!

But what if we want to force the user to parse in a function? We could do the following.

(A —> B) —> C —> D

This way, we can't parse an `A` and then a `B`, but *must* provide a function with the type `A -> B` as a first argument. It explicitly states that the first argument must be a function.

Haskell is *statically typed*, which means types of things don't change. This may not be true for other languages, but in Haskell, this isn't just a design choice, it has to be like this. Everything is a constant, so if things don't change, how would their types change?

Haskell's type system is very powerful and deserves a section of its own to do it justice. We can see what functions require and guess what they do based on it. It makes it possible to reason about programming problems from just the types.

I may be the only one who does this, but if I don't know a formula in some physics problem, I usually look at all the different units of values and figure out how I get a value of the right unit. This makes it easy to get a basic idea of what you should do with each given. You can reason about the semantics of certain calculations later, maybe you need to multiply with some unitless constant to calculate something correctly, but you know which given should be used in what way.

The same is true for writing functional programs. You can look at just the types to get an idea how functions relate. We'll see some good examples of this in section 4.2.4.

4.2.3 Example program

This example program was me experimenting with Haskell. I watched a video by Tscoding [29] where he implemented a simple program that could produce melodies. I played with it a bit more and was able to very quickly introduce new things and make it able to also play chords. I can take this further still, but I don't have the time right now, and I'm not sure if I'm going to expand on this anyway; there are probably more interesting projects I'd want to invest my time in. That said, it might be interesting to add overtones and timbre, and maybe a system in which you pick a key and can easily pick notes within the key, so you don't have to denote every note as number of half-steps from A_1.

I'm not going to explain the code, there really is no point. The video explains it well, so why would I still do it? I just wanted to show the result of my playing with Haskell and the ability to introduce your own ideas very naturally.

I'm already making use of monads (section 4.2.4). Don't worry about them, just think of them as a way of writing necessary procedural code.

I'm sorry the font has to be so small. It had to be like this because the code wouldn't fit otherwise. I'm also going to admit that the melody sounds quite wrong. I was trying to recreate part of a song I made a while back, but I just couldn't figure out a certain part of the melody. I kept trying different things and gave up in the end. Now it's probably worse than it started of as. Just in case anyone ends up running this program (although I consider the chance small), don't say I didn't warn you.

You should be able to find the program in the accompanying code folder. I figured that it was small enough to put it straight in here so you can just quickly glance at it, but if you want to play with it, you can find the code separately.

```
import qualified Data.ByteString.Lazy as B
import qualified Data.ByteString.Builder as B
import Data.Foldable
import Data.List
import System.Process
import Text.Printf
```

```
type Pulse      = Float
type Seconds    = Float
type SampleRate = Float
type Hertz      = Float
type Semitones  = Float
type TET        = Float
type Volume     = Float
type Sound      = [Pulse]
type Chord      = Sound

outputFilePath :: FilePath
outputFilePath = ”output.bin”

volume :: Volume
volume = 0.1

sampleRate :: SampleRate
sampleRate = 48000.0

pitchStandard :: Hertz
pitchStandard = 440.0

temperament :: TET
temperament = 12.0

noteFreq :: Semitones -> Hertz
noteFreq n = pitchStandard * (2 ** (1.0 / temperament)) ** n

freqSound :: Seconds -> Float -> Float -> Hertz -> Sound
freqSound duration attack release freq = map (* volume) $ zipWith3 (\x y z -> x * y * z)
                                                           attackMultipliers releaseMultipliers pureFreqSound
    where
        step = (freq * 2 * pi) / sampleRate
        pureFreqSound :: Sound
        pureFreqSound = map sin $ map (* step) [0.0 .. sampleRate * duration]
        attackMultipliers :: [Volume]
        attackMultipliers = map (min 1.0) [0.0, attack ..]
        releaseMultipliers :: [Volume]
        releaseMultipliers = reverse $ take (length pureFreqSound) $ map (min 1.0) [0.0, release ..]

note :: Semitones -> Seconds -> Float -> Float -> Sound
note duration attack release n = freqSound duration attack release $ noteFreq n

combineSounds :: [Sound] -> Sound
combineSounds = map sum . transpose

chord :: Seconds -> [Semitones] -> Chord
chord duration ns = combineSounds [note duration 0.0001 0.001 n | n <- ns]

chords :: Sound
chords = concat $ take 16 $ repeat $ concat $ [ chord 1.500 [-7,-4,0,3]
                                              , chord 1.125 [-2,2,-7,-4]
                                              , chord 2.625 [-9,-5,-2,2]
                                              , chord 0.750 [-9,-5,-2,2]
                                              ]

melody :: Sound
melody = concat $ take 4 $ repeat $ concat [ note 0.3750 0.001 0.001 17
                                           , note 0.3750 0.001 0.001 20
                                           , note 0.3750 0.001 0.001 24
                                           , note 0.3750 0.001 0.001 27
                                           , note 0.3750 0.001 0.001 10
                                           , note 0.3750 0.001 0.001 22
                                           , note 0.3750 0.001 0.001 17
                                           , note 2.6250 0.001 0.001 14
                                           , note 0.7500 0.001 0.001 12

                                           , note 0.3750 0.001 0.001 17
                                           , note 0.3750 0.001 0.001 20
                                           , note 0.3750 0.001 0.001 24
                                           , note 0.3750 0.001 0.001 27
                                           , note 0.3750 0.001 0.001 10
                                           , note 0.3750 0.001 0.001 22
                                           , note 0.3750 0.001 0.001 17
                                           , note 2.6250 0.001 0.001 12
                                           , note 0.7500 0.001 0.001 10

                                           , note 0.3750 0.001 0.001 17
                                           , note 0.3750 0.001 0.001 20
                                           , note 0.3750 0.001 0.001 24
                                           , note 0.7500 0.001 0.001 27
                                           , note 0.3750 0.001 0.001 24
                                           , note 0.7500 0.001 0.001 20
                                           , note 0.3750 0.001 0.001 15
                                           , note 0.3750 0.001 0.001 19
                                           , note 0.3750 0.001 0.001 20
                                           , note 0.9375 0.001 0.001 24
                                           , note 0.1875 0.001 0.001 22
                                           , note 0.7500 0.001 0.001 26

                                           , note 0.3750 0.001 0.001 17
                                           , note 0.3750 0.001 0.001 20
                                           , note 0.3750 0.001 0.001 24
                                           , note 0.3750 0.001 0.001 27
                                           , note 0.7500 0.001 0.001 22
                                           , note 0.7500 0.001 0.001 31
                                           , note 3.0000 0.001 0.001 27
                                           ]
```

```haskell
song :: Sound
song = combineSounds [chords, melody]

save :: FilePath -> IO ()
save filePath = B.writeFile filePath $ B.toLazyByteString $ fold $ map B.floatLE song

play :: IO ()
play = do
    save outputFilePath
    _ <- runCommand $ printf "ffplay -showmode 1 -f f32le -ar %f %s" sampleRate outputFilePath
    return ()
```

4.2.4 Monads

For the longest time there was a problem in functional programming that programmers just couldn't figure out. How do you do side effects in a language that doesn't allow side effects? The ideas of functional programming are cool and all, but what's the point if it's completely useless? The whole point of a program is for it to do side effects, it needs to have an effect on the world around it. Well, there actually *is* a way of doing side effects in functional programming with monads. The idea of monads was developed in category theory and turned out to be very useful for functional programmers.

Haskell has this magical thing called monad. Haskell also has this magical do notation. When we make a function that returns a monad, we can use this do notation to basically write in a procedural and effectful style. One example of this would be this simple program that does input and output using an `IO` monad.

```haskell
whatIsYourName :: IO ()
whatIsYourName = do
    putStrLn "What is your name?"
    name <- getLine
    putStrLn ("Hello, " ++ name)
```

This looks just like a procedural program, but there's a lot of things going on in the background. I'll explain how monads work and will be defining my own `IO` monad in the process. This section is largely inspired by a video by Tscoding [30], my code examples will be from this, although be it maybe with some modifications. I'll go over it relatively quickly, so if you want to understand this better, I recommend you to watch the video. There is also a Computerphile [11] video about monads, but I don't think that video really explains what monads are and it introduces a bunch of things without much thought or explanation, but it may be useful. Lastly, there is a very useful video by Beckman [4] called *Don't fear the Monad*. It really goes in-depth and explains things very precisely and clearly.

The basic idea behind a monad is very simple. We want to change the state of the outside world, but we aren't able to do that, because all our functions are pure, so we're not allowed to change global state. The thing is, we don't have to. Functions are just mappings between things. We can create a world type that encapsulates the world's state and map it to a new state where our effect was performed. This way we're we can have an effect on the world, while still programming with pure functions. Let's look at this in code.

```haskell
data World = World

printStr :: String -> World -> World
printStr = undefined

readStr :: World -> (String, World)
readStr = undefined
```

I've defined the function `printStr` to take a string and a world and produce a new world where the string has been printed to the screen. I also define a function `readStr` which takes a world and produces a pair of the read string and the new world where the string has been read.

I will not yet go over the implementations of these functions, because there is some other stuff we need to talk about first.

Imagine for now that these functions are in fact implemented. We can use them to write a new version of `whatIsYourName`.

```
whatIsYourPureName :: World -> World
whatIsYourPureName w1 = w4
    where w2          = printStr "What␣is␣your␣name?" w1
          (name, w3) = readStr w2
          w4          = printStr ("Hello" ++ name) w3
```

Our new function manipulates the world, so of course it should take a world and produce a world. We take our input world and run it through our `printStr` function and save the new world. We then run that world through the `readStr` function and save the resulting new world and name, and lastly we print out our message, again running our old new world through the function and returning the final world.

The next step is to make the world inaccessible to the user, we're going to hide it, because the user doesn't need to know about it. What we're going to do is create a world transformation, which is just an type synonym for a function taking a world and returning a pair of some result and the new world.

```
type WorldT a = World -> (a, World)
```

Using this, we can quite easily wrap our `readStr` in a new function returning `WorldT`. The return type of `readStr` is already a `WorldT`, i.e. it's already compatible.

Adapting `printStr` isn't too hard either. Since it doesn't return anything, we can make our wrapper return a pair of nothing and a call to our regular `printStr` resulting in our new world.

```
readStrT :: WorldT String
readStrT = readStr

printStrT :: String -> WorldT ()
printStrT s w = ((), printStr s w)
```

We have abstracted away our world and can now reason in terms of world transformations instead.

We want to be able to do operations sequentially. In other words, we want to be able to compose world transformations. We can't just compose them as we normally would. I'd like to very quickly look at a small snippet in category theory. Don't worry too much about what it means, I just want to represent visually the fact that regular function composition and composition with these world transformations is different.

In category theory, two arrows must have a composition if they follow each other.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g \circ f"] & B \arrow[d, "g"] \\
& C
\end{tikzcd}$$

In this case, the objects A and B represent types and the arrows represent functions.

But our world transformations don't follow each other, but we want to still be able to compose them like this. (I will represent `WorldT` as T instead.)

$$\begin{tikzcd}
A \arrow[r, "f"] & T(B) & & \\
& & A \arrow[r, "g \circ f"] & T(C) \\
B \arrow[r, "g"] & T(C) & &
\end{tikzcd}$$

This is where the idea of monads comes from. The basic idea behind a monad in category theory is a way of composing these types of arrows.

We'll invent a new operator to do composition with world transformations. This operator is known by different names, the official name in Haskell is *bind*, but I've heard it called *shove* too, which I like because it fires off my imagination and sounds a bit comical to me. Haskell already has such an operator, of course, but we'll define our own version.

The first argument will be a world transformation which returns some type `a`. The next argument will be our first transformation which takes our type `a` and returns a new world transformation which returns some type `b`. Then we return the result of this last world transformation which returns `b`. The following is what we have thus far.

```
(>>>=) :: WorldT a
       -> (a -> WorldT b)
       -> WorldT b
wt >>>= f = undefined
```

Remember that our `WorldT` is just a type alias for `World -> (a, World)`, where `a` is some given type. It's just an alias, it's the same thing, they're interchangeable. This means that our new operator's type can also be written as the following.

```
(>>>=) :: World -> (a, World)
       -> (a -> World -> (b, World))
       -> World -> (b, World)
```

We know what function currying is, but we also have uncurried functions in functional programming. An uncurried function doesn't feed in the arguments seperately, but takes everything in at once in a pair. We can see that the function on the second line (`wt`) of our fully expanded type of our new operator is curried, but the function on the first line (`wt`) returns a pair. This means that if we uncurry this second function, we can just apply it to the pair directly. Luckily for us, Haskell provides a function that can uncurry functions.

The type of `uncurry` is as follows.

```
uncurry :: (a -> b -> c) -> (a, b) -> c
```

It's quite obvious what it does from it's type alone.

This means we can uncurry `f` and compose that with `wt`. Thus, we can implement our operator as follows.

```
(>>>=) :: WorldT a
       -> (a -> WorldT b)
       -> WorldT b
wt >>>= f = uncurry f . wt
```

In short, to do function composition with two world transformations that don't normally compose, you uncurry the first function and then, as if it were magic, they suddenly do.

We can use this to reimplement `whatIsYourName` again.

```
whatIsYourPureNameT :: WorldT ()
whatIsYourPureNameT =
printStrT "What␣is␣your␣name?" >>>= \_ ->
readStrT                       >>>= \name ->
printStrT ("Hello␣" ++ name)
```

We can compose a world transformation with a lambda function that takes the result and feeds it as an argument to the next function. You might be confused, because composition happens from right to left, so we don't yet know what name is when we call the last function, but we don't have to because of lazy evaluation. Name gets evaluated before the function gets called.

What we have done so far is create effectful pure functions which don't modify global state directly, but get fed in the world state and produce a new world state. We succeeded in abstracting away this world behind world transformations. We have then successfully invented a way of sequencing these effectful functions by defining a new operator to do a special kind of function composition. We have effectively emulated a procedural programming paradigm in a pure functional language.

If you want to use this as an actual monad, there are a few things I need to note. Right now, we have no way of printing our world to the screen. We need to define our `printStr` and `readStr` too. This will be left as an exercise to the reader. We also need to derive `Show` for a few things for this to work so we can actually look at our states. But what about do notation?

Since we've already achieved monads, there is no real point in me going indepth with explaining how to enable do notation for our monad, since it's just a nicety and doesn't improve our understanding. I will tell you how to do it in broad terms, but if you really want to know, you should look at the Tscoding [30] video.

To enable do notation, we need to prove to the Haskell compiler that `WorldT` is in fact a monad. There is a difficulty with this. `WorldT` is just a type synonym, which makes this very hard. We should wrap it in a new `WorldM` type and prove that *that* is a monad. To prove this, we must prove that our new operator for composition of world transformations is equivalent to Haskell's operator for monadic composition (bind/shove). I'll refer you to the video again if you want to see how.

To reiterate, to do side effects, we do operations on world that encaptulates all outside state. We then abstract away our world behind world transformations. We then invent a way of doing function composition with those world transformations so you can run these effectful functions sequentially, and Bob's your uncle.

4.3 Comparison to other paradigms

Every programming language has its pros and cons (although sometimes disproportionally). Functional programming has many pros, but also some cons. We've looked at laziness extensively already (section 4.1.3), so I won't need to go over that again. We will be looking at a few differences between functional programming and your usual procedural/imperative languages.

4.3.1 Declarative vs imperative

Functional programming is *declarative*. Most languages are *imperative* in nature. Declarative programming provides a more expressive style of programming at the cost of being able to optimise the underlying technicalities.

Imperative means you have to explain to the computer what to do. You don't talk about abstract ideas. Rather, you tell the computer to read some bytes at one memory address and write some bytes at another. You tell it to load certain things into registers, compare values, jump to code at certain addresses. Most languages aren't completely imperative. Machine language is really the only "pure" imperative language, all other languages provide some abstraction. C has functions, loops and variables, which are already some higher level abstractions, but they translate almost directly to machine code, so it's still considered imperative. Even higher level languages such as Java, C#, C++, Rust, etc. are still considered imperative, because there is still a lot of sequentiality and commands rather that descriptive code, although some is present.

Declarative programming is when you describe how things are. You don't think about the internals of your computer, but about abstract mathematical ideas. This provides a level of expressiveness which is very powerful. Haskell code is notoriously short and packs a lot of ideas in just a few lines of code. You can easily add new ideas and change relationships between things. It is very easy to reason about your code, while in imperative programming, you may have made a small typo somewhere, significantly altering the state of your programming and breaking everything. Such a mistake can be very hard to find, because you have to go through your code line by line and check all state to see what is as expected and what not.

Let's take trees as an example. Say we want our computer to be able to recognise trees. In a declarative paradigm, we try to define what trees are. We say that trees have branches and leaves, they're made wood and other organic material, they're tall. We may also define branches and leaves. Eventually our definitions must be somehow connected to terms the computer understands. The difference is, we hide it behind abstractions. We can build abstractions upon abstractions. We do need to be careful to explain things right. We can usually build upon already existing abstractions, so we don't have to think about the computer at all. Once we've explained branches and leaves and colours in terms of things the computer understands, we can reason in terms of these abstractions rather than terms the computer understands. We can mix and match different kinds of branches, leaves and colours to create all sorts of trees.

In a procedural/imperative paradigm, we don't have this luxury of abstraction. We must always speak to the computer in computer terms. We don't think about trees in computer terms, so we need to constantly translate our ideas into computer speech. This can be quite difficult. We'd much rather create layers of abstractions and just write our ideas how we mean them, not in computer terms. When we think of trees, we don't think of reading some memory address into a CPU register, then copying some number of bytes at that register to some other place and compare numbers and jumping to different addresses of our code based on those comparisons. No, we think of branches and leaves and bark and wood.

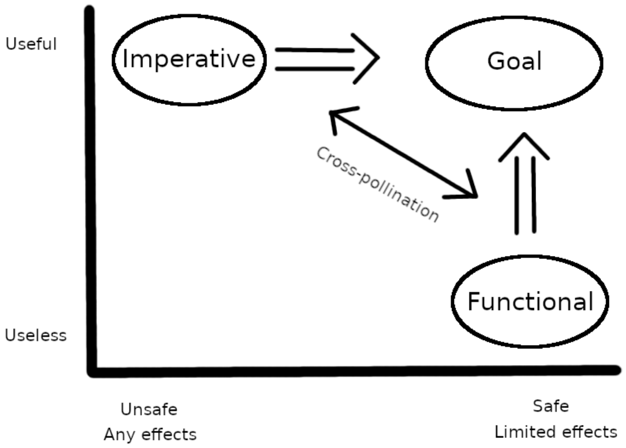
In section 4.1.3 and section 4.2.1, I also talk about describing things with infinities, which is perfectly fine, as long as you don't try to evaluate an infinity.

4.3.2 Usefulness vs conceptual purity and safety

There is, however, a drawback to this. We can't write very useful programs. We've talked about how to solve this in section 4.2.4. Basically, we need to modify state to be able to do things. If we want do draw something on the screen, we need to modify its state, which we normally can't. This was a problem that took programmers an mathematicians long to solve. They came up with monads, which allowed the programmer to write a function with side effects. The basic idea is that you have some "dirty" functions that interface with the outside world, which you can then use in a "pure" program. We separate the dirty things from the pure things, and try to minimise the dirty things as much as possible. We can use these two different types of functions without them polluting each other. The type system keeps them apart.

Ultimately we would have a pure and useful language. Functional programming starts very pure and tries to be more useful by inventing things like monads, while other languages start very useful and try to be more pure by inventing abstractions.

Simon Peyton Jones, the creator of Haskell, did a very well known and funny video called *Haskell is useless*. In there he talks about these two groups trying to be both useful and safe and draws a little graph and talks and makes jokes. The graph is something like the following.



He talks about how both paradigms want to be safe, expressive and useful, but they take two different approaches. Although they take two different approaches, they borrow a lot of ideas from each other.

I personally quite like Rust. It's a low level language, but with a lot of cheap abstractions. In the laziness examples I used Rust (section 4.1.3). It was quite clear in those examples that there was more going on then what I was explaining. All this extra was technical details the Rust compiler requires, but Rust also provides a vocabulary for more abstract ideas. Rust takes a middle way, it tries to have expressiveness and abstraction without losing out on speed and technical details. This does make Rust quite a complicated and large language—it's not as elegant as C or Haskell. Haskell is relatively simple and elegant, and so is C. C is very nice if you have to perform relatively simple tasks where speed and optimisation is important, while Haskell is very nice when you have to write very complicated systems where performance isn't an issue. Rust provides a vocabulary to solve big and complicated problems where performance is an issue. It tries to fill the same gap C++ does, but C++ has been hacked on and added to for what feels like centuries. It is becoming a convoluted monster. Rust starts with a clean slate and new ideas and experience from C++.

One place where the safety—which lambda calculus and functional provide by statelessness—is increasingly important, is in *parallel programming.* Parallel programming is when you run code on multiple CPU threads, so you can speed up your code by running multiple things at the same time, rather than sequentially, when sequentiality doesn't matter (i.e. Some things are not dependent on each other and can run separately). Parallel programming adds extra complexity to keeping track of state, because you don't only have to worry who owns what, but also what thread or instance owns what. It's very hard to avoid so called *data races*, where multiple threads try to modify and/or read the same data, meaning they end up messing up each other's states. When our programs are stateless, we can safely perform any operation, because it can't perform side effects which alter the state of our program, and with that the state of other threads. It may be limiting when you actually *want* to change state in parallel, but it also means you don't have to worry when you just want to perform some big computation in parallel.

4.3.3 Meta programming

There is another paradigm I want to have a quick look at. This is *meta programming.* Most programmers are probably familiar with meta programming. It's when you write code that generates code. Most languages have macro's and something along the lines of C++'s template system. These are all forms of meta programming, but there are languages that have *only* meta programming. Meta programming allows for a lot of abstraction. You can easily abstract repetitive code.

Probably the biggest meta programming language (family) is Lisp. Lisp is very simple, just like the lambda calculus. It only has lists. Your program is entirely made up of linked lists, just like we defined them in section 3.1.4. Lists can be evaluated. The resulting language has quite some similarities to lambda calculus and functional programming. Namely, in lambda calculus everything is a function that can be evaluated, while in Lisp, everything is a list that can be evaluated.

Lisp is also a declarative language. Sometimes people call Lisp a functional programming language, but Lisp itself really isn't. There are Lisp dialects that are functional, such as Scheme, which add concepts like function currying and laziness.

Lisp was originally developed to be able to program AI. The idea was that meta programming allows for easy abstraction, and abstraction would be necessary to write good AI, because it needs to be able to reason abstractly. There are many parallels between Lisp and functional programming languages. Lisp even uses the prefix notation[13], which much resembles lambda calculus and combinatory logic (although be it only in a visual way).

Another funny coincidence is that Lisp, just like lambda calculus, is notorious for having a lot of convoluted parentheses, because your programs are entirely made up of lists, which are denoted with parentheses.

There is an infamous book called *Structure and Interpretation of Computer Programs*[14] [3], or SICP for short, which explains ideas of program structure from both functional programming and Lisp in the context of Scheme. It has been very influential in the way we think about programming. Lispers often joke that everyone else does programming wrong.

A very basic Scheme program (or any Lisp program for that matter) that does some calculations looks as follows.

```
(+ 2 3)
(abs -4)
(+ (* 2 3) 8)
(+ 3 4 5 1)
```

I will show you a slightly larger Scheme program, which I didn't write, because I'm not an epic Lisper (yet?), but got from a random online Scheme course from some random university by an unknown author, to give you an idea how it looks.

```
(if (= 5 (+ 2 3)) 10 20)     => 10
(if (= 0 1) (/ 1 0) (+ 2 3)) => 5
; note that the (/ 1 0) is not evaluated

(define (my-max x y)
    (if (> x y) x y))

(my-max 10 20)                    => 20

(define (my-max3 x y z)
    (if (and (> x y) (> x z))
        x
        (if (> y z)
            y
            z)))
```

[13] Lists have to start with a function to be able to evaluate it.

[14] In hacker culture also known as the *Wizard Book*

You can see that conditionals work very similarly to those in lambda calculus, where the different branches are arguments parsed after the condition itself.

5 A practical example of functional programming

Section 4.3 compares functional programming with other paradigms. I thought it would be interesting to do a practical example. In section 4.3.3, I talked about the Lisp programming language family, in particular Scheme. It isn't all to hard to implement a Lisp-like language, but implementing a programming language is very conceptual. Programming languages have many abstract concepts that are more easily expressed in a declarative/functional language like Haskell, rather than a procedural/imperative language like C. This section is really a practical continuation of section 4.3, so I recommend you read it before you read this. This section tries to paint a picture of what is said in section 4.3.

I sadly didn't have the time to completely finish the interpreters, but I have written some code that already shows what I'm trying to get at. I'll go over what an interpreter should do and how you would go about implementing that in C and Haskell. I got pretty far with the C interpreter, though.

5.1 How an interpreter works

We first have to understand how an interpreter actually works before we start writing one. I will quickly and globally explain the basic steps an interpreter should perform.

Firstly, we need to recognise the atoms our program is made up of. We perform a *lexical analysis* which recognises *tokens*. This is also called the *tokenisation* step. We recognise special characters and words and we recognise other words as identifiers. We make some sort of list of all the tokens in our program. We then need to do *syntactic analysis*.

In the *syntactical analysis* phase, we structure our tokens in what is called an *abstract syntax tree* or *parse tree*. We recognise the order in which things should be performed and what is an argument to what. We put all our expressions in a hierarchical structures, the most inner expressions are at the ends of branches, and the outer expressions are closer to the root. This tells our next phase in what order things should be evaluated and what depends on what.

We then perform a *semantic analysis*. We look at our *parse tree* and try to perform the operations our program is supposed to do. We need to keep track of some state, for example the identifiers that are defined. This state is sometimes called an *environment*. In short, this is the phase where the interpreter tries to perform the commands of the code.

This is the general idea of what an interpreter does. There may be some more steps in other interpreters, such as optimisations, but we're keeping it as simple as possible.

5.2 Lisp interpreter in C

With my C version, I was able to get pretty far and write a functioning program. I was already at the semantic analysis phase, I'd started with implementing an *environment*.

I can't just paste all the code in here, that would be stupid, so you can find it in the accompanying code folder. You're not going to understand most of the code, but that's the point. I'm trying to show that the semantics of code are much more obvious in Haskell than in C. I will take some snippets of code from it and show it when I feel it's useful.

I've refrained from using a grammar library like Yacc. Yacc provides a way of writing a grammar for your programming language in a way that resembles formal grammar. Such a grammar has a declarative style, because you're defining what your grammar is, rather than programming procedurally. The whole point of writing this program was to show off the procedural/imperative paradigm, so it wouldn't make sense to use a declarative grammar library.

When we don't use a grammar, it immediately becomes obvious that procedural/imperative programming is very hard to follow in the lexical analysis phase alone. In our Haskell counterpart, we can just define what are the possibilities of a token or keyword and that words are split by spaces or tokens. This gives the compiler enough information to perform our tasks, but in our C code, we must go through the code character by character. If a character is unimportant, like unnecessary white space, we should ignore it. We then need to test our character against every token. If it's one of our tokens, we save it in our list of tokens and restart the process. Otherwise, need to recognise that it's the start of a new identifier. Things start getting more difficult when there are multi-character keywords or identifiers involved. We need to be aware when we're still reading a multi-character word and check whether we finally ran into some character that signifies the end of the word. We need to store all the characters into a string, one by one, and check that string against all keywords in our program. This is a very painful process. We need to write very large conditionals and think of every possible meaning of each character and its context. Whenever we read a single character, we need to think of it in the context of characters past to know what this character is supposed to be.

Luckily for us, Lisp is very simple and doesn't have many tokens, so the complexity is quite limited, but it's still enough to serve as an example in our comparison. My Lisp interpreter, in it's current state, reads in a single list input by the user. I will show the main function, which contains the main conditional structures of the lexical and syntactical analyses. I then run the function `eval_list`, which contains the semantic analysis, for as far it's finished. This main functions branches out to many other functions, many of which just have to do with memory management, but if you truly want to understand the program, you should look at the entire code. This main function just contains the basic control flow.

You can see from the function names and comments that I'm thinking of what the program does in abstract terms, but I still have to explain everything in computer terms in the actual code.

You can also see I'm already doing things with the environment, but it doesn't really function yet.

I apologise for the tiny font size.

```
int main(int argc, char *argv[])
{
    struct env env;
    struct env_index env_add = { .id = "+", .type = ENV_FUNC, .data = &add };

    init_env(&env, 64);
    push_env(&env, env_add);

    puts("Lispy␣Version␣0.0.1\n");

    /* Main loop */
    for (;;) {
        struct token_stack token_stack;
        char input[2048];

        init_token_stack(&token_stack, 1024); // initialise new token stack

        /* Get input from user */
        fputs("Lispy>␣", stdout);
        try(fgets(input, 2048, stdin), "Error␣reading␣input␣from␣stdin:␣%s.\n", strerror(errno));
        input[strcspn(input, "\n")] = '\0';
        input[strcspn(input, "\t")] = '␣';

        /* Split input up into tokens */
        for (uint i = 0; i < strlen(input);) {
            uint token_len = 1;
            struct token new_token;

            switch (input[i]) {
            case '(':
                new_token.type = OPENING_PAREN_TOKEN;
                new_token.str = (char *)allocate(2, "token_string"); // two including null byte
                strcpy(new_token.str, "(");
                push_token_stack(&token_stack, new_token);
                break;
            case ')':
                new_token.type = CLOSING_PAREN_TOKEN;
                new_token.str = (char *)allocate(2, "token_string"); // two including null byte
                strcpy(new_token.str, ")");
                push_token_stack(&token_stack, new_token);
                break;
            case '␣':
                break;
            default:
                new_token.type = ID_TOKEN;
                for (bool breakl = false; i + token_len < strlen(input) && !breakl; token_len++) {
                    if (input[i + token_len] == '␣')
                        break;
                    for (uint j = 0; j < sizeof(tokens) / sizeof(tokens[0]) && !breakl; j++)
                        if (input[i + token_len] == tokens[j][0])
                            breakl = true;
                    if (breakl)
                        break;
                }
                new_token.str = (char *)allocate(token_len + 1, "token␣string");
                strncpy(new_token.str, input+i, token_len);
                push_token_stack(&token_stack, new_token);
            }

            i += token_len;
        }

        /* Create lists from the token stack and add them to a program */
        struct program program;
        init_program(&program, 1);
        for (uint i = 0; i < token_stack.n; i++) {
            struct list new_list;
            switch (token_stack.ts[i].type) {
            case OPENING_PAREN_TOKEN:
                new_list = list_from_token_stack(token_stack, i + 1, &i);
                push_program(&program, new_list);
                break;
            case CLOSING_PAREN_TOKEN:
                fputs("Error␣parsing␣program:␣unexpected␣end␣of␣list.\n", stderr);
            default:
                fputs("Error␣parsing␣program:␣not␣a␣list.\n", stderr);
            }
        }

        /* Print out program */
        for (uint i = 0; i < program.n; i++) {
            print_list(program.lists[i], 0);
        }

        /* Run program */
        for (uint i = 0; i < program.n; i++) {
            eval_list(program.lists[i], env);
        }

        /* Free memory */
        free(token_stack.ts);
        for (uint i = 0; i < program.n; i++) {
            free_list(program.lists[i]);
        }
        free(program.lists);
    }

    return EXIT_SUCCESS;
}
```

5.3 Lisp interpreter in Haskell

Sadly I didn't have time to create one. I don't have the time to create and show you some code snippets, so you have to imagine some. Our interpreter does the exact same as the one in C, of course, but instead of procedural, it would be functional. We would describe our grammar and different types representing different abstract ideas instead of reading each character one by one.

Afterword

There are a lot of other things I would have liked to talk about more like type theory and category theory. Sadly, this would take a lot of time. I also wrote a few sections very last minute. Most of these weren't too important anyway, but it would have been nice to spend a little more time on them and streamline them a bit more.

The biggest bummer of all was probably that I wasn't able to get any code for the Lisp interpreter in Haskell done. I do have some other code examples in Haskell that make up for it though.

Bibliography

[1] Andrej Bauer (https://cs.stackexchange.com/users/1329/andrej-bauer). *Representing Negative and Complex Numbers Using Lambda Calculus.* URL: `https://cs.stackexchange.com/q/2279` (visited on 25/01/2021).

[2] jmad (https://cs.stackexchange.com/users/82/jmad). *Representing Negative and Complex Numbers Using Lambda Calculus.* URL: `https://cs.stackexchange.com/q/2276` (visited on 25/01/2021).

[3] Harold Abelson and Gerald Jay Sussman. *Structure and interpretation of computer programs.* The MIT Press, 1996.

[4] Brian Beckman. *Don't fear the Monad.* Youtube. 2012. URL: `https://youtu.be/ZhuHCtR3xq8`.

[5] Brilliant.org. *Lambda Calculus.* URL: `https://brilliant.org/wiki/lambda-calculus/` (visited on 25/01/2021).

[6] Alonzo Church. "A formulation of the simple theory of types". In: *The journal of symbolic logic* 5.2 (1940), pp. 56–68.

[7] Alonzo Church. "A set of postulates for the foundation of logic". In: *Annals of mathematics* (1932), pp. 346–366.

[8] Alonzo Church. "An unsolvable problem of elementary number theory". In: *American journal of mathematics* 58.2 (1936), pp. 345–363.

[9] Computerphile. *Essentials: Functional Programming's Y Combinator.* Youtube. 2017. URL: `https://youtu.be/9T8A89jgeTI`.

[10] Computerphile. *Lambda Calculus.* Youtube. 2017. URL: `https://youtu.be/eis11j_iGMs`.

[11] Computerphile. *What is a Monad?* Youtube. 2017. URL: `https://youtu.be/t1e8gqXLbsU`.

[12] Haskell Brooks Curry. "Grundlagen der kombinatorischen Logik". In: *American journal of mathematics* 52.4 (1930), pp. 789–834.

[13] Gottlob Frege. *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens.* Nebert, 1879.

[14] Kurt Gödel. "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I". In: *Monatshefte für mathematik und physik* 38.1 (1931), pp. 173–198.

[15] Daan F.P. Hoogcarspel. "Deze Titel is Onwaar, Een Introductie tot de Formele Logica". In: (2021).

[16] Gabriel Lebec. *A Flock of Functions, Combinators, Lambda Calculus, & Church Encodings in JS - Part I.* Youtube. 2017. URL: `https://youtu.be/3VQ382QG-y4`.

[17] Gabriel Lebec. *A Flock of Functions, Combinators, Lambda Calculus, & Church Encodings in JS - Part II.* Youtube. 2017. URL: `https://youtu.be/pAnLQ9jwN-E`.

[18] Miran Lipovaca. *Learn you a haskell for great good!: a beginner's guide.* no starch press, 2011.

[19] Per Martin-Löf and Giovanni Sambin. *Intuitionistic type theory.* Vol. 9. Bibliopolis Naples, 1984.

[20] Bartosz Milewski. *Category Theory.* Youtube. 2016. URL: `https://youtube.com/playlist?list=PLbgaMIhjbmEnaH_LTkxLI7FMa2HsnawM_`.

[21] Giuseppe Peano. *Arithmetices principia: Nova methodo exposita.* Fratres Bocca, 1889.

[22] Prajit Ramachandran. *A Flock of Functions, Combinators, Lambda Calculus, & Church Encodings in JS - Part II.* URL: http://web.archive.org/web/20201223141125/http://prajitr.github.io/quick-haskell-syntax/ (visited on 23/12/2020).

[23] Bertrand Russell. *The principles of mathematics. 1 (1903).* University Press, 1903.

[24] Bertrand Russell and Alfred North Whitehead. *Principia mathematica.* Vol. 1. Cambridge University Press Cambridge, UK, 1910.

[25] Bertrand Russell and Alfred North Whitehead. *Principia mathematica.* Vol. 2. Cambridge University Press Cambridge, UK, 1912.

[26] Bertrand Russell and Alfred North Whitehead. *Principia mathematica.* Vol. 3. Cambridge University Press Cambridge, UK, 1913.

[27] Moses Schönfinkel. "Über die Bausteine der mathematischen Logik". In: *Mathematische annalen* 92.3-4 (1924), pp. 305–316.

[28] Raymond Merrill Smullyan. *To Mock a Mockingbird: and other logic puzzles including an amazing adventure in combinatory logic.* Oxford University Press, USA, 2000.

[29] Tscoding. *Making Music with Haskell From Scratch.* Youtube. 2020. URL: https://youtu.be/FYTZkE5BZ-0.

[30] Tscoding. *What is IO monad?* Youtube. 2018. URL: https://youtu.be/fCoQb-zqYDI.

[31] Alan Mathison Turing. "Computability and λ-definability". In: *The Journal of Symbolic Logic* 2.4 (1937), pp. 153–163.

[32] Alan Mathison Turing. "On computable numbers, with an application to the Entscheidungsproblem". In: *J. of Math* 58.345-363 (1936), p. 5.

[33] Alan Mathison Turing. *On computable numbers, with an application to the Entscheidungsproblem; a correction.* Royal Society, 1937.

[34] Wikipedia. *Combinatory logic — Wikipedia, The Free Encyclopedia.* 2020. URL: https://web.archive.org/web/20200920164914if_/https://en.wikipedia.org/wiki/Combinatory_logic (visited on 20/09/2020).

[35] Wikipedia. *Gödel's incompleteness theorems — Wikipedia, The Free Encyclopedia.* 2020. URL: https://web.archive.org/web/20200927072027/https://en.wikipedia.org/wiki/G%C3%B6del%27s_incompleteness_theorems (visited on 27/09/2020).

[36] Wikipedia. *Halting problem — Wikipedia, The Free Encyclopedia.* 2020. URL: https://web.archive.org/web/20201005163636/https://en.wikipedia.org/wiki/Halting_problem (visited on 05/10/2020).