Linear Algebra 2

Assignments 6: Diagonalization and Simultaneous Diagonalization of Operators, Power Vectors, and the Jordan Canonical Form (§4.5, 4.7-4.9)

Alexander Heinlein

Q1 2022/2023

Delft University of Technology

Assignment 6.1

Let

$$A = \begin{pmatrix} -3 & -4 \\ 2 & 3 \end{pmatrix}.$$

(a) *Verify that A is diagonalizable.*

(b) *Find a diagonalizable matrix B such that $AB = BA$.*

(c) *Can you diagonalize A and B by the same matrix of eigenvectors?*

Assignment 6.2

Exercises (2) and (4) from page 85 of the book:

Check if the following pars of matrices commute. If so, simultaneously diagonalize them.

(2)

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ -1 & 2 & 1 \end{pmatrix}$$

(4)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Assignment 6.3 (for submission)

Consider the ordinary differential equation

$$2\frac{d^2 f}{dt^2} + 12\frac{df}{dt} - 14f = 0.$$

Find a general solution of the form $f(t) = c_1 e^{a_1 t} + c_2 e^{a_2 t}$.

Assignment 6.4

Let L *be a diagonalizable linear operator on a finite-dimensional vector space, and let* m *be any positive integer. Show that* L *and* L^m *are simultaneously diagonalizable.*

Assignment 6.5

Let two linear operators L_1 *and* L_2 *on the same finite-dimensional vector space* V *be conjugates, that is* $[L_2]_{\mathcal{B}} = P[L_1]_{\mathcal{B}}P^{-1}$ *with some invertible matrix* P *and an arbitrary basis* $\mathcal{B}$ *for* V. *Show that their powers* L_1^m *and* L_2^m *as well as their exponentials* e^{L_1} *and* e^{L_2} *are conjugates.*

Assignment 6.6 (for submission)

Let us define the exponential function of an $n \times n$ matrix B by the power series

$$\exp(B) = \sum_{k=0}^{\infty} \frac{B^k}{k!}.$$

(a) *Let matrix D be a diagonal matrix. Show that*

$$\exp(D) = \operatorname{diag} \exp(d_1), \exp(d_2), \ldots, \exp(d_n),$$

where d_i are the diagonal entries of matrix D.

(b) *Let matrix A be diagonalizable. That is, there exists an invertible matrix P such that $A = PDP^{-1}$. Show that*

$$\exp(A) = P \exp(D) P^{-1}.$$

Assignment 6.6 (contd.)

(c) *Let matrix N be nilpotent with index q. Show that*

$$\exp(N) = I + N + \frac{1}{2!}N^2 + \frac{1}{3!}N^3 + \cdots + \frac{1}{(q-1)!}N^{q-1}.$$

(d) *Let matrix $A = D + N$ be the sum of a diagonal matrix D and a nilpotent matrix N. Moreover, let D and N be commuting, i.e. $DN = ND$. Show that*

$$\exp(A) = \exp(D) \cdot \exp(N).$$

(e) *Let matrix J_{λ_i} be an $n \times n$ Jordan block corresponding to the eigenvalue λ_i. Show that*

$$e^{tJ_{\lambda_i}} = e^{t\lambda_i} \begin{pmatrix} 1 & \frac{t^1}{1!} & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \frac{t^1}{1!} \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

Assignment 6.7

Compute the Jordan canonical form of the matrix

$$A = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

following the procedure described in the sixth lecture.

You can check the correctness by using MATLAB (Symbolic Toolbox required):

```
A = [3 0 -2 ; -2 0 1 ; 2 1 0];
jordan(A);
```

Assignment 6.8 (for submission)

Consider the linear differential equation

$$y(t)' = Ay(t), \qquad A = \begin{pmatrix} 1 & -3 & -1 \\ 1 & 5 & 1 \\ -2 & -6 & 0 \end{pmatrix}$$

with initial condition

$$y(0) = y_0, \qquad y_0 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

The exact solution reads

$$y(t) = \exp(tA)y_0.$$

(a) *Compute the Jordan canonical form of matrix A.*

(b) *Give an explicit expression for the exact solution using assignment 6.5.*