Real analysis exercises week 1

September 6, 2022

Exercise 3.1

Show that $d(x, y) = |\frac{1}{x} - \frac{1}{y}|$ defines a metric on $(0, \infty)$.

Clearly, $\forall x, y \in \mathbb{R} : d(x, y) = 0 \iff x = y$ and $d(x, y)$, and

$\forall x, y \in \mathbb{R} : d(x, y) \geq 0$, and

$\forall x, y \in \mathbb{R} : d(x, y) = d(y, x)$.

For the triangle inequality we use $|a + b| \leq |a| + |b|$. Taking $a = \frac{1}{x} - \frac{1}{z}$ and $b = \frac{1}{z} - \frac{1}{y}$ immediately yields the desired result.

Exercise 3.3

Show that $\forall x, y \in \mathbb{R} : d(x, y) = 0 \iff x = y$ and the triangle inequality are sufficient to denote a metric.

Say there exist $x, y \in M$ such that $d(x, y) \neq d(y, x)$. This implies there are $x, y \in M$ such that $d(x, y) > d(y, x)$. Take $z = x$. Then $d(x, y) > d(x, x) + d(x, y)$, which is a contradiction. Hence $\forall x, y \in M$, $d(x, y) = d(y, x)$.

Say there exist $x, z \in M$ such that $d(x, z) < 0$. Then taking $y = x$ and plugging it into the triangle inequality yields $d(x, x) > d(x, z) + d(z, x)$, which is a contradiction. Hence $\forall x, y \in M$, we have $d(x, y) \geq 0$.

Exercise 3.6

If d is any metric on M, show that $\rho(x, y) = \sqrt{d(x, y)}$, $\sigma(x, y) = \frac{d(x,y)}{d(x,y)+1}$, and $\tau(x, y) = min\{d(x, y), 1\}$ are also metrics on M.

Given as all functions are 0 iff $d(x, y)$ is 0, this requirement is trivial. For the triangle equality it is then sufficient that the second derivative is negative. This is the case for the first two functions: $\frac{d^2\rho}{dd^2} = -\frac{1}{4}d^{-\frac{3}{2}}$, $\frac{d^2\sigma}{dd^2} = -\frac{2}{(1+d)^3}$ which are both negative.

$min\{d(x, y), 1\} \leq d(x, y) \leq d(x, z) + d(z, y)$ which holds if both of $d(x, z)$ and $d(z, y)$ are less than 0. If not, then either of them is one, which means $min\{d(x, y), 1\} \leq min\{d(x, z), 1\} + min\{d(z, y), 1\}$ definitely holds.

Exercise 3.21
Literally can't be bothered.