# Linear Algebra 2

## Assignments 3: Direct Sums, Equivalence Classes, Quotient Spaces, and Linear Transformations (§2.5, 3.1-3.2)

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# Direct Sums (§2.5)

## Assignment 3.1

*Show that* $\mathbb{R}_2[t]$ *is the internal direct sum of the polynomials that vanish at* $t = 0$ *and the span of* $t^2 + 2t + 1$.

## Assignment 3.2

*Exercise (3) from page 34 of the book:*

*Show that* $M_{3,3}$, *the space of* $3 \times 3$ *real matrices, is the direct sum of the subspace of symmetric matrices* $A^\top = A$, *and the subspace of antisymmetric matrices* $A^\top = -A$. *What are the dimensions of these subspaces? What is the dimension of* $M_{3,3}$?

## Assignment 3.3

*Exercise (4) from page 34 of the book:*

*Let* $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. *In the decomposition of the previous exercise, what is* $P_1 A$ *and* $P_2 A$?

**Assignment 3.4 (extra)**

*Let $L(x) = Ax$ define a linear operator on the vector space $V = \mathbb{R}^n$ with $n \times n$ matrix $A$. Show that $A$ is diagonalizable if and only if $V$ is the direct sum of the eigenspaces of $A$. For each eigenvalue $\lambda_i$ of the matrix $A$ the associated eigenspace is defined as follows:*

$$E_{\lambda_i} = \{x \in V : Ax = \lambda_i x\}$$

**Assignment 3.5**

*Exercise (6) from page 35 of the book:*

*Show that, for general $V$ and $W$, the addition operation $[\mathbf{x}] + [\mathbf{y}] = [\mathbf{x} + \mathbf{y}]$ as it is defined in the definition of the quotient space is well defined. That is, show that, if $\mathbf{x}$ and $\mathbf{x}'$ are both in $[\mathbf{x}]$, and if $\mathbf{y}$ and $\mathbf{y}'$ are both in $[\mathbf{y}]$, then $[\mathbf{x} + \mathbf{y}] = [\mathbf{x}' + \mathbf{y}']$.*

**Assignment 3.6**

*Exercise (7) from page 35 of the book:*

*Let $V = \mathbb{R}[t]$, and let $W$ be the subspace consisting of all polynomials divisible by $(t-1)^2$. Show that $W$ is a subspace of $V$ and that $V/W$ is 2-dimensional, and exhibit a basis for $V/W$.*

*[Hint: A point in $V/W$ is the set of all polynomials that have a specific value and derivative at $t = 1$.]*

## Assignment 3.7 (for submission)

*Let $V$ be a vector space and let $W$ be a subspace. Show that the equivalence class of $\mathbf{x} \in V$*

$$[\mathbf{x}] = \{\mathbf{y} \in V : \mathbf{x} \sim \mathbf{y}\}$$

*forms an* ***affine subspace*** *of $V$, which is defined as the subspace $W$ translated by the vector $\mathbf{x} \in V$ (Think about how an affine subspaces differs from a regular one!). We denote affine subspaces as $U = \mathbf{x} + W$. In essence, you have to show that $[\mathbf{x}] = \mathbf{x} + W$, which we will use from now on as an alternative notation.*

# Quotient Spaces (§2.5)

**Assignment 3.8 (for submission)**

*With the notation from Assignment 3.7, we can express the quotient space, which is the set of all equivalence classes of $\sim$, as follows*

$$V/W = \{\mathbf{x} + W : \mathbf{x} \in V\}.$$

*Define the addition and multiplication with a scalar operations in $V/W$ such that the canonical mapping*

$$\sigma : V \to V/W, \qquad \mathbf{x} \mapsto \sigma(\mathbf{x}) = \mathbf{x} + W$$

*is linear. Compare your answer to the result in Assignment 3.5.*

**Assignment 3.9 (for submission)**

*We know that $V/W$ being a vector space has to satisfy the eight axioms of a vector space. Give the zero vector (additive identity) of $V/W$ and the additive inverse of an arbitrary element of $V/W$.*

## Assignment 3.10

*Exercise (3) from page 47 of the book:*

*On* $\mathbb{R}_3[t]$ *with basis* $\mathcal{B} = (1, t, t^2, t^3)$*, compute the matrix form of the linear transformation* $d^2/dt^2$ *and compare it to the square of the matrix*

$$[d/dt]_{\mathcal{B}\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

## Assignment 3.11

*Exercises (4) and (6) from page 47 of the book:*

*Let $V$ be a vector space with basis $\mathcal{B}$ and let $L$ and $M$ be operators on $V$. Show that the composition $L \circ M$ defined by $L \circ M(\mathbf{v}) = L(M(\mathbf{v}))$ is also an operator on $V$. Prove that*

$$[L \circ M]_{\mathcal{B}} = [L]_{\mathcal{B}}[M]_{\mathcal{B}}.$$

*In other words, show that compositions of operators in $V$ correspond to multiplications of square matrices (matrix forms of linear operators).*

*We now extend the above result to multiple vector spaces and linear transformations: Let $U$, $V$ and $W$ be vector spaces, with bases $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, respectively. Let $L_1 : U \to V$ and $L_2 : V \to W$ be linear transformations. We can define the composition $L_2 \circ L_1 : U \to W$ by $L_2 \circ L_1(\mathbf{u}) = L_2(L_1(\mathbf{u}))$. Show that $L_2 \circ L_1$ is a linear transformation from $U$ to $W$. Show that the matrix of $L_2 \circ L_1$ is given by*

$$[L_2 \circ L_1]_{\mathcal{D}\mathcal{B}} = [L_2]_{\mathcal{D}\mathcal{C}}[L_1]_{\mathcal{C}\mathcal{B}}.$$

## Assignment 3.12 (extra)

*Let $T$ be a linear operator on the vector space $V$ and let $W$ be a $T$-**invariant subspace** of $V$. That is, $T(\boldsymbol{x}) \in W$ for all $\boldsymbol{x} \in W$, or $T(W) \subseteq W$.*

*Let us define the mapping*

$$\bar{T} : V/W \to V/W$$

*by*

$$\bar{T}(\boldsymbol{x} + W) = T(\boldsymbol{x}) + W \qquad \text{for any } \boldsymbol{x} + W \in V/W.$$

*Prove that the mapping $\bar{T}$ is well-defined and that it is a linear operator on the vector space $V/W$. Moreover, show that $\sigma \circ T = \bar{T} \circ \sigma$ with $\sigma : V \to V/W$ being the mapping defined in Assignment 3.8. Sketch the commuting diagram.*