1. Rewrite each of the following functions into the form $f(x+iy) = u(x,y) + iv(x,y)$:

(a) $f(z) = 2z^3 - 3z$;

(b) $f(z) = \dfrac{1}{z}$;

(c) $f(z) = \dfrac{i+z}{i-z}$.

2. Rewrite each of the following functions into the form $f(z)$ with z being a complex variable:
[TIP: If you are having difficulties, take $x = \frac{1}{2}(z+\overline{z})$ and $y = \frac{1}{2i}(z-\overline{z})$.]

(a) $f(z) = x^2 - y^2 + 2ixy$;

(b) $f(z) = x(x^2-3y^2) - iy(3x^2-y^2)$;

(c) $f(z) = \dfrac{2(x^2+y^2)-(x+iy)}{4(x^2+y^2)-4x+1}$.

3. Give an accurate description of the geometric effect of the function $f(z) = az+b$, where a and b are complex constants.

4. The function $f : \mathbb{C} \longrightarrow \mathbb{C}$ is given as $f(z) = z^2$. As we have seen before, we say the image of z is w, meaning we can describe the function as $w = z^2$. The real and imaginary part of z we call x and y and those of w we call u and v; thus, $z = x+iy$ and $w = u+iv$. When researching and understanding functions, we will use this set up often.

(a) (i) Determine the image of the lines $x = 0$, $x = -1$ and $x = 1$. In doing so, make a sketch of the lines in the z-plane, while sketching the images in the w-plane.

(ii) What is the image of the region $-1 \leq x \leq 1$?

(b) Answer the same questions for the lines $y = 0$, $y = -1$, $y = 1$ and the region $-1 \leq y \leq 1$.

(c) (i) Determine the (complete) preimages of the lines $u = 0$, $u = -1$ and $u = 1$ In doing so, make a sketch of the lines in the w-plane, while sketching the preimages in the z-plane.

(ii) What is the preimage of the region $-1 \leq u \leq 1$?

(d) Answer the same questions for the lines $v = 0$, $v = -1$, $v = 1$ and the region $-1 \leq v \leq 1$.

5. (a) Do exercise 14.8 from B-C if you haven't done so already.

(b) What happens to the image of the "pie slice" in (a) if we replace the upperbound $\frac{\pi}{4}$ of $\arg z$ with larger numbers? And what happens when we change the modulus?

6. Given is the function $w = e^z$.
Sketch a rectangle $a \leq x \leq b$, $c \leq y \leq d$ in the z-plane, and sketch its image in the w-plane.

7. Given is the function $w = \dfrac{1}{z}$.

(a) Sketch the disc $|z| \leq 2$ in the z-plane, and its image in the w-plane.

(b) Sketch the image of $y = 1$ in the w-plane.

(c) Sketch the image of the rectangle (and its interior) with corners $A(0,0)$, $B(\frac{1}{2},0)$, $C(\frac{1}{2},1)$, $D(0,1)$ in the w-plane.

(d) Sketch the preimage of the line $u = 1$ (so $\operatorname{Re} w = 1$) in the z-plane.

(e) Sketch the preimage of the line $v = -2$ in the z-plane

8. Show that $\lim\limits_{z\to 0} \dfrac{|z|^2}{z^2}$ does not exist (see the slides of lecture 2.1).

9. (a) Later we will calculate so-called residues. These are limits of functions. In one of the exercises, we look at

$$\lim_{z\to 1+i} \frac{z^4-(1+i)z^3}{z^4+4} e^{iz}.$$

Calculate this limit.

Before you can calculate residues you have to know the type of a so called singularity. This type depends for example on the value of a limit:

(b) $z=0$ is a singularity of $f(z)=e^{1/z}$. What is $\lim_{z\to 0} f(z)$?

(c) $z=1+i$ is a singularity of $f(z)=\dfrac{z^3e^{iz}}{z^4+4}$. What is $\lim_{z\to 1+i} f(z)$?

10. Argue that lines in the complex plane correspond to circles on the Riemann sphere. Parallel lines in the complex plane have no intersection, while non-parallel lines have one. What about the circles on the Riemann sphere?

11. What is $\lim\limits_{z\to\infty} e^z$ if z moves along the parabola $y=x^2$?

12. Determine the continuity of the following functions in the point 0:

(a) $f(z)=\begin{cases} \dfrac{\operatorname{Re} z}{|z|} & \text{if } z\neq 0 \\ 0 & \text{if } z=0 \end{cases}$

(b) $f(z)=\begin{cases} \dfrac{(\operatorname{Re} z)^2}{|z|} & \text{if } z\neq 0 \\ 0 & \text{if } z=0 \end{cases}$

13. For each of the following functions f, determine the largest domain D where upon f is analytic, if it exists. Also determine the derivatives on that domain.

(a) $f(z)=z|z|$;

(b) $f(z)=f(x+iy)=x^2-y^2-3x+2+i(2x-3)y$;

(c) $f(z)=\dfrac{x+y}{x^2+y^2}+i\dfrac{x-y}{x^2+y^2}$.

14. Find, if it exists, an entire function $f(z)$ with real part $u(x,y)=y^3-3x^2y$ and $f(i)=1+i$.

15. The functions $u(x,y)=x^2-y^2$ and $v(x,y)=\dfrac{y}{x^2+y^2}$ are both harmonic. Is the function $f(x+iy)=u(x,y)+iv(x,y)$ analytic? Why?

And a little more challenging for the student that wants more:

16. The CR-equations are defined for rectangle-coordinates by substituting $z=x+iy$. We can also work with polar coordinates by substituting $z=r\cos\phi+ir\sin\phi$. This gives

$$w=f(z)=u^*(r,\phi)+iv^*(r,\phi).$$

Prove that if f is differentiable in the point $z_0=r_0\cos\phi_0+ir_0\sin\phi_0$, then in the point (r_0,ϕ_0) we see that

$$r\frac{\partial u^*}{\partial r}=\frac{\partial v^*}{\partial \phi} \quad\text{and}\quad \frac{\partial u^*}{\partial \phi}=-r\frac{\partial v^*}{\partial r}.$$

This is called the polar form of the CR-equations.

Furthermore, prove that in this situation

$$f'(z_0) = e^{-i\phi}\left(\frac{\partial u^*}{\partial r} + i\frac{\partial v^*}{\partial r}\right).$$

17. A complex function $f : D \to \mathbb{C}$, with D a domain, often can be seen as a function of z and $\overline{z}$. In that situation we can rewrite this function as

$$f(z) = g(z, \overline{z}) = u(x, y) + iv(x, y).$$

Suppose that u and v are differentiable real functions of two real variables (on D). Then we can differentiate f with respect to x and y, but also with respect to z and $\overline{z}$ (see (b)). In that situation we mean with for example $\dfrac{\partial f}{\partial \overline{z}}$ the partial derivative of $g(z, \overline{z})$ with respect to $\overline{z}$ and with $\dfrac{\partial f}{\partial \overline{x}}$ the partial derivative of $u(x, y) + iv(x, y)$ with respect to x.

(a) Find $\dfrac{\partial f}{\partial z}$, $\dfrac{\partial f}{\partial \overline{z}}$, $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ if $f(z) = 2\,\mathrm{Re}(z) + i|z|^2$ (with $D = \mathbb{C}$).

(b) Show

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$
$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

(c) Show that f satisfies the Cauchy-Riemann equations in $z_0 = x_0 + iy_0$ if and only if $\dfrac{\partial f}{\partial \overline{z}} = 0$ in z_0.

(d) Show that f is analytic on D if and only if $\dfrac{\partial f}{\partial \overline{z}} = 0$ on D.

(e) Where is the function $f : \mathbb{C} \setminus \{0\}$ defined by $f(z) = z\overline{z} + z/\overline{z}$ complex differentiable?

Exercises of Brown Churchill:

B-C 14.1. For each of the functions below, describe the domain of definition that is understood:

(a) $f(z) = \dfrac{1}{z^2+1}$;

(b) $f(z) = \operatorname{Arg}\left(\dfrac{1}{z}\right)$;

(c) $f(z) = \dfrac{z}{z+\overline{z}}$;

(d) $f(z) = \dfrac{1}{1-|z|^2}$.

B-C 14.8. Sketch the region onto which the sector $r \leq 1$, $0 \leq \theta \leq \pi/4$ is mapped by the transformation

(a) $w = z^2$;

(b) $w = z^3$;

(c) $w = z^4$.

B-C 18.5. [Formulated in another way] Find $\lim_{z\to 0} \left(\dfrac{z}{\overline{z}}\right)^2$.

B-C 18.10. Use the theorem in sec. 17 (i.e. the theorem on slide 2.1.6) to show that

(a) $\lim_{z\to\infty} \dfrac{4z^2}{(z-1)^2} = 4$;

(b) $\lim_{z\to 1} \dfrac{1}{(z-1)^3} = \infty$;

(c) $\lim_{z\to\infty} \dfrac{z^2+1}{z-1} = \infty$;

B-C 18.11. With the aid of the theorem in Sec. 17 (i.e. the theorem on slide 2.1.6), show that when

$$T(z) = \frac{az+b}{cz+d} \qquad (ad - bc \neq 0),$$

(a) $\lim_{z\to\infty} T(z) = \infty$ if $c = 0$;

(b) $\lim_{z\to\infty} T(z) = \dfrac{a}{c}$ and $\lim_{z\to -d/c} T(z) = \infty$ if $c \neq 0$.

B-C 20.2. Use results in Sec. 20 (standard rules for differentiation) to find $f'(z)$ when

(a) $f(z) = 3z^2 - 2z + 4$;

(b) $f(z) = (2z^2 + i)^5$;

(c) $f(z) = \dfrac{z-1}{2z+1}$ $(z \neq -1/2)$;

(d) $f(z) = \dfrac{(1+z^2)^4}{z^2}$ $(z \neq 0)$.

B-C 20.8. Use the method in example 2, Sec. 19 (that is with the definition of derivatives), to show that $f'z)$ does not exist at any point z when

(a) $f(z) = \operatorname{Re} z$;

(b) $f(z) = \operatorname{Im} z$.

B-C 20.9. [Formulated in another way] The function $f : \mathbb{C} \to \mathbb{C}$ is defined by

$$f(z) = \begin{cases} \overline{z}^2/z & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$$

(a) Show that f is not differentiable at $z = 0$.

(b) [extra] Show that f satisfies the Cauchy-Riemann equations at $z = 0$.

This is an example of a function that has a point where the CR-equations are satisfied, but where f is not differentiable!

B-C 24.1. Use the theorem in Sec. 21 (Cauchy-Riemann 1, see slide 2.1.10) to show that $f'(z)$ does not exist at any point if

(a) $f(z) = \overline{z}$;

(b) $f(z) = z - \overline{z}$;

(c) $f(z) = 2x + ixy^2$;

(d) $f(z) = e^x e^{-iy}$.

B-C 24.2. Use the theorem in Sec. 23 (Cauchy-Riemann 2, see slide 2.1.12) to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere, and find $f''(z)$ when

(a) $f(z) = iz + 2$;

(b) $f(z) = e^{-x} e^{-iy}$;

(c) $f(z) = z^3$;

(d) $f(z) = \cos x \cosh y - i \sin x \sinh y$.

B-C 24.3. From results obtained in Secs. 21 and 23 (Cauchy-Riemann 1 and 2), determine where $f'(z)$ exists and find its value when

(a) $f(z) = 1/z$;

(b) $f(z) = x^2 + iy^2$;

(c) $f(z) = z \operatorname{Im} z$.

B-C 26.1. Apply the theorem in Sec. 23 (Cauchy-Riemann 2, slide 2.1.12) to verify that each of these functions is entire:

(a) $f(z) = 3x + y + i(3y - x)$;

(b) $f(z) = \cosh x \cos y + i \sinh x \sin y$;

(c) $f(z) = e^{-y} \sin x - ie^{-y} \cos x$;

(d) $f(z) = (z^2 - 2)e^{-x} e^{-iy}$.

B-C 26.2. With the aid of the theorem in Sec. 21 (Cauchy-Riemann 1, slide 2.1.10), show that each of these functions is nowhere analytic:

(a) $f(z) = xy + iy$;

(b) $f(z) = 2xy + i(x^2 - y^2)$;

(c) $f(z) = e^y e^{ix}$.

B-C 26.4. In each case, determine the singular points of the function and state why the function is analytic anywhere else:

(a) $f(z) = \dfrac{2z + 1}{z(z^2 + 1)}$;

(b) $f(z) = \dfrac{z^3 + i}{z^2 - 3z + 2}$;

(c) $f(z) = \dfrac{z^2 + 1}{(z + 2)(z^2 + 2z + 2)}$.

B-C 26.7. Let a function f be analytic everywhere in a domain D. Prove that if $f(z)$ is real-valued for all z in D, then $f(z)$ must be constant throughout D.

B-C 30.2. State why the function $f(z) = 2z^2 - 3 - ze^z + e^{-z}$ is entire.

B-C 30.3. Use the Cauchy-Riemann equations and the theorem in Sec. 21 (so use Cauchy-Riemann 1 and 2, slides 2.1.10 and 2.1.12) to show that the function $f(z) = \exp(z^2)$ is entire. What is its derivative.

B-C 30.6. Show that $|\exp(z^2)| \leq \exp(|z|^2)$.

B-C 30.8. Find all values of z such that

(a) $e^z = -2$;

(b) $e^z = 1 + i$;

(c) $\exp(2z - 1) = 1$.

B-C 106.4. Find the image of the semi-infinite strip $x \geq 0$, $0 \leq y \leq \pi$ under the transformation $w = \exp(z)$, and label corresponding portions of the bounderies.

B-C 115.1. Show that $u(x, y)$ is harmonic in some domain and follow the steps used in Example 2, Sec. 115 (or the last example of slide 2.2.6), to find a harmonic conjugate $v(x, y)$ when

(a) $u(x, y) = 2x - x^3 + 3xy^2$;

(b) $u(x, y) = \sinh x \sin y$;

(c) $u(x, y) = \dfrac{y}{x^2 + y^2}$.

B-C 115.3. Suppose that v is a harmonic conjugate of u in a domain D and also that u is a harmonic conjugate of v in D. Show how it follows that both $u(x, y)$ and $v(x, y)$ must be constant throughout D.

B-C 115.5. Show that if v and V are harmonic conjugates of $u(x, y)$ in a domain D, then $v(x, y)$ and $V(x, y)$ can differ at most by an additive constant.