Real analysis exercises week 1

September 5, 2022

Exercise 3.1

Show that $d(x,y) = |\frac{1}{x} - \frac{1}{y}|$ defines a metric on $(0, \infty)$.

Clearly, $\forall x, y \in \mathbb{R} : d(x,y) = 0 \iff x = y$ and $d(x,y)$, and

$\forall x, y \in \mathbb{R} : d(x,y) \geq 0$, and

$\forall x, y \in \mathbb{R} : d(x,y) = d(y,x)$.

For the triangle inequality we use $|a+b| \leq |a| + |b|$. Taking $a = \frac{1}{x} - \frac{1}{z}$ and $b = \frac{1}{z} - \frac{1}{y}$ immediately yields the desired result.

Exercise 3.3

Show that $\forall x, y \in \mathbb{R} : d(x,y) = 0 \iff x = y$ and the triangle inequality are sufficient to denote a metric.

Say there exist $x, y \in M$ such that $d(x,y) \neq d(y,x)$. This implies there are $x, y \in M$ such that $d(x,y) > d(y,x)$. Take $z = x$. Then $d(x,y) > d(x,x) + d(x,y)$, which is a contradiction. Hence $\forall x, y \in M$, $d(x,y) = d(y,x)$.

Say there exist $x, z \in M$ such that $d(x,z) < 0$. Then taking $y = x$ and plugging it into the triangle inequality yields $d(x,x) > d(x,z) + d(z,x)$, which is a contradiction. Hence $\forall x, y \in M$, we have $d(x,y) \geq 0$.

Exercise 3.6

If d is any metric on M, show that $\rho(x,y) = \sqrt{d(x,y)}$, $\sigma(x,y) = \frac{d(x,y)}{d(x,y)+1}$, and $\tau(x,y) = min\{d(x,y), 1\}$ are also metrics on M.