ANSWERS REAL ANALYSIS Q1

Exercise 3.3: Assume $d : M \times M \to \mathbb{R}$ satisfies

(1) $d(a, b) = 0$ if and only if $a = b$

(2) for all $a, b, c \in M$, $d(a, b) \le d(a, c) + d(b, c)$.

We show that d satisfies the properties (i), (ii), (iii), (iv) of the definition of a metric. Here (ii) is clear from the assumption (1).

(i): Let $x, y \in M$ be arbitrary. Taking $a = b = x$ and $c = y$ in (2) gives

$$0 \overset{(1)}{=} d(x, x) \overset{(2)}{\le} d(x, y) + d(x, y) = 2d(x, y).$$

Therefore, $d(x, y) \ge 0$.

(iii): Let $x, y \in M$ be arbitrary. Taking $a = x$ and $b = y$ and $c = x$ gives

$$d(x, y) \overset{(2)}{\le} d(x, x) + d(y, x) \overset{(1)}{=} d(y, x).$$

This implies $d(x, y) \le d(y, x)$. Reversing the roles of x and y, we obtain $d(y, x) \le d(x, y)$ as well, and thus (iii) follows.

(iv). This follows from

$$d(x, y) \overset{(2)}{\le} d(x, z) + d(y, z) \overset{\text{(iii)}}{=} d(x, z) + d(z, x).$$

Exercise 3.18: We first check that for $x \in \mathbb{R}^n$, $\|x\|_\infty \le \|x\|_2 \le \|x\|_1$. To prove the first estimate note that for each $k \in \{1, \dots, n\}$, one has

$$|x_k| \le \Big(\sum_{j=1}^{n} |x_j|^2\Big)^{1/2} = \|x\|_2.$$

Therefore, $\|x\|_\infty = \max_{k\in\{1,\dots,n\}} |x_k| \le \|x\|_2$.

To prove $\|x\|_2 \le \|x\|_1$, first note that the case $\|x\|_1 = 0$ is clear. If $\|x\|_1 > 0$, we can use a so-called homogeneity argument. Let $\lambda = \|x\|_1$. Consider the vector $y = \lambda^{-1}x$. Then by the properties of a norm we obtain $\|y\|_1 = \lambda^{-1}\|x\|_1 = 1$. Therefore, $|y_k| \le 1$ for all $k \in \{1, \dots, n\}$. It follows that $|y_k|^2 \le |y_k|$, and thus

$$\|y\|_1^2 = \sum_{k=1}^{n} |y_k|^2 \le \sum_{k=1}^{n} |y_k| = \|y\|_1 = 1.$$

We can conclude that $\|y\|_2 \le 1$. Since $x = \lambda y$, we obtain

$$\|x\|_2 = |\lambda|\, \|y\|_2 \le |\lambda| = \|x\|_1.$$

Next we will show that $\|x\|_1 \le n\|x\|_\infty$. Observe that for all $k \in \{1, \dots, n\}$, one has $|x_k| \le \|x\|_\infty$. Summing over all $k \le n$, we find that

$$\|x\|_1 = \sum_{k=1}^{n} |x_k| \le n\|x\|_\infty.$$

Finally we check that $\|x\|_1 \leq \sqrt{n}\|x\|_2$. From the Cauchy–Schwarz inequality we obtain

$$\|x\|_1 = \sum_{k=1}^n |x_i| \leq \sum_{k=1}^n |x_i| \cdot 1 \leq \Big(\sum_{i=1}^n |x_i|^2\Big)^{1/2}\Big(\sum_{i=1}^n 1^2\Big)^{1/2} = \sqrt{n}\|x\|_2.$$

Exercise 3.22: The estimate $\|x\|_\infty \leq \|x\|_2$ can be proved as in Exercise 3.18. To obtain $\|x\|_2 \leq \|x\|_1$ note that

$$\|x\|_2 = \lim_{n\to\infty}\Big(\sum_{k=1}^n |x_k|^2\Big)^{1/2} \overset{\text{Ex 3.18}}{\leq} \lim_{n\to\infty}\sum_{k=1}^n |x_k| = \|x\|_1.$$

Exercise 3.29: We prove that A is bounded $\Leftrightarrow$ $\operatorname{diam}(A) < \infty$.

"$\Rightarrow$" Assume A is bounded. Then we can find $x \in M$ and $r > 0$ such that $A \subseteq B(x,r)$. Thus for all $a, b \in A$, we have $a, b \in B(x,r)$, and thus

$$d(a,b) \leq d(a,x) + d(x,b) < r + r = 2r.$$

Thus $\operatorname{diam}(A) = \sup_{a,b\in A} d(a,b) \leq 2r < \infty$.

"$\Leftarrow$" Assume $\operatorname{diam}(A) < \infty$. Let $r = \operatorname{diam}(A) + 1$ and fix $x \in A$. For all $a \in A$, one has $d(a,x) \leq \operatorname{diam}(A) < r$. Therefore, $A \subseteq B(x,r)$ and we can conclude that A is bounded.

Exercise 3.33: Assume $x_n \to x$ and $x_n \to y$. If $x \neq y$, then $d(x,y) > 0$ and we will use this to deduce a contraction. Let $\varepsilon = \frac{1}{2}d(x,y)$. Choose $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $d(x,x_n) < \varepsilon$. Choose $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $d(y,x_n) < \varepsilon$. Letting $n = \max\{N_1, N_2\}$, we find

$$d(x,y) \leq d(x,x_n) + d(x_n,y) < 2\varepsilon = d(x,y).$$

This cannot be true, and hence $x = y$.

Exercise 3.41:[1] Below we use the fact that $\langle x,y\rangle - \langle x,z\rangle = \langle x, y-z\rangle$. Assume $x^{(k)} \to x$ and $y^{(k)} \to y$. Then

$$\begin{aligned}|\langle x,y\rangle - \langle x^{(k)},y^{(k)}\rangle| &\leq |\langle x,y\rangle - \langle x,y^{(k)}\rangle| + |\langle x,y^{(k)}\rangle - \langle x^{(k)},y^{(k)}\rangle| \\ &= |\langle x, y-y^{(k)}\rangle| + |\langle x-x^{(k)}, y^{(k)}\rangle| \\ &\leq \|x\|_2\|y-y^{(k)}\|_2 + \|x-x^{(k)}\|_2\|y^{(k)}\|_2\end{aligned}$$

where we used the Cauchy–Schwarz inequality in the last step. Since $(y^{(k)})_{k\geq 1}$ is convergent it is bounded (see Exercise 3.36) by some constant M. It follows that

$$|\langle x,y\rangle - \langle x^{(k)},y^{(k)}\rangle| \leq \|x\|_2\|y-y^{(k)}\|_2 + M\|x-x^{(k)}\|_2.$$

Since the right-hand side tends to zero as $k \to \infty$, the required result follows.

Exercise 4.4: Let $A \subseteq M$. In the lectures we have proved that for any $x \in M$, the set $M \setminus \{x\}$ is open. Thus the result follows if we prove $A = \bigcap_{x\in A^c} M \setminus \{x\}$.

[1] To prove this you could first look again at the prove that the limit of a product equal the products of the limits for real numbers. The exercise is a generalization of this fact

To prove this claim, first let $a \in A$. Then for all $x \in A^c$, we have $x \neq a$ so $a \in M \setminus \{x\}$. Thus $a \in \bigcap_{x\in A^c} M \setminus \{x\}$. For the converse we use contraposition. If $a \notin A$, then $a \in A^c$ and $a \notin M \setminus \{a\}$. Therefore $a \notin \bigcap_{x\in A^c} M \setminus \{x\}$.

Exercise 4.11: Let $F = \{e^{(k)} : k \geq 1\}$. By Theorem 4.9 it suffices to take a sequence (x_n) in A such that $x_n \to x$ for some $x \in \ell^1$ and show that $x \in F$.

Convergent sequences in this set F are a bit special as we will see! Indeed, for $n \neq m$ we have $\|e^{(n)} - e^{(m)}\|_1 = 2$. Since $x_n \in F$ for every $n \geq 1$, for every $n, m \geq 1$ we have either $x_n = x_m$ or $\|x_n - x_m\|_1 = 2$ $(*)$. Since $(x_n)_{n\geq 1}$ is a Cauchy sequence, $(*)$ can only happen for finitely many m, n. Therefore, there is an index $N \geq 1$ such that $x_n = x_N$ for all $n \geq N$. It follows that $x = x_N$, and hence $x \in F$ as required.

Exercise 4.15: We first show that A is closed. Let $y \in A^c$, so $d(x, y) > r$. Let $\varepsilon = d(x, y) - r$. We claim that $B_\varepsilon(y) \subseteq A^c$. Indeed, if $z \in B_\varepsilon(y)$, then

$$d(y, x) - d(x, z) \leq |d(y, x) - d(x, z)| \leq d(y, z) < \varepsilon.$$

This implies $d(x, z) > d(y, x) - \varepsilon = r$. Therefore, $z \in A^c$.

Next we an example such that $\overline{B_r(x)} \neq A$. Let M be any set consisting of at least 2 elements. Let d be the discrete metric on M. Let $x \in M$ and $r = 1$. Then $A = M$. On the other hand $B_1(x) = \{x\}$. As $\{x\}$ is closed, it follows that $\overline{B_1(x)} = \overline{\{x\}} = \{x\}$. Since $A = M$ contains at least two elements, we find that $\overline{B_1(x)} \neq A$.

Exercise 4.33: A limit point is also called accumulation point. Let x be a limit point of A. Let U be a neighborhood of x. We will construct a sequence (x_n) of distinct points in $U \cap A$ in a recursive way. Let $\varepsilon_1 > 0$ be such that $B_{\varepsilon_1}(x) \subseteq U$. Since x is a limit point of A, we can find $x_1 \in (B_{\varepsilon_1}(x) \setminus \{x\}) \cap A$. Let

$$\varepsilon_2 = d(x_1, x)\} > 0.$$

Applying the definition of a limit point to x again but this time with ε_2, it follows that we can find $x_2 \in (B_{\varepsilon_2}(x) \setminus \{x\}) \cap A$. Let $\varepsilon_3 = d(x_2, x)$, etc... In this way we find a sequence of real numbers (ε_n) and (x_n) in A such that $0 < \varepsilon_{n+1} < \varepsilon_n$ and

$$0 < d(x_{n+1}, x) < \varepsilon_n = d(x_n, x).$$

In particular, $x_n \neq x_m$ if $n \neq m$, and for every $n \geq 1$, $x_n \in B_{\varepsilon_n}(x) \subseteq B_{\varepsilon_1}(x) \subseteq U$. This shows that $U \cap A$ contains infinitely many distinct points in A, concluding the proof.

Exercise 4.40: The first assertion follows from the negation of the definition of x being a limit point of A. The second part we explain in detail. Let $A \subseteq \mathbb{R}$. Let $S \subseteq A$ be the set of isolated points of A. For each $x \in S$ let $\varepsilon_x > 0$ be such that $(B_{\varepsilon_x}(x) \setminus \{x\}) \cap A = \varnothing$. For every $x \in S$ choose $q_x \in \mathbb{Q} \cap B_{\varepsilon_x/2}(x)$. We claim that $q_x \neq q_y$ whenever $x, y \in S$ with $x \neq y$. To prove the claim, let $x, y \in S$ with $x \neq y$. Renaming x and y we can assume that $\varepsilon_x \geq \varepsilon_y$. Since $(B_{\varepsilon_x}(x) \setminus \{x\}) \cap A = \varnothing$, we have $y \notin (x - \varepsilon_x, x + \varepsilon_x)$. Therefore, $|x - y| \geq \varepsilon_x$. It follows that

$$\varepsilon_x \leq |x - y| \leq |x - q_x| + |q_x - q_y| + |q_y - y| < \frac{\varepsilon_x}{2} + |q_x - q_y| + \frac{\varepsilon_y}{2} \leq |q_x - q_y| + \varepsilon_x.$$

This clearly implies $q_x \neq q_y$ and the claim follows.

The countability of $\mathbb{Q}$ implies that $T := \{q_x : x \in S\}$ is at most countable. Let $\phi : T \to \mathbb{N}$ be an injection. The claim yields that mapping $\psi : S \to T$ given by $\psi(x) = q_x$ is an injection (it is even a bijection). Therefore, the composition $\phi \circ \psi$ is an injection from S into $\mathbb{N}$. This implies that S is at most countable.

Exercise 4.50: Recall that the family of subsets of $\mathbb{N}$ (this is the power set) is uncountable. For $I \subseteq \mathbb{N}$, let $e_I : \mathbb{N} \to \{0,1\}$ be given by $e_I(i) = 1$ if $i \in I$ and $e_I(i) = 0$ if $i \notin I$. Then each e_I is a sequence of numbers in $\{0,1\}$. Moreover, if $I \neq J$, then e_I and e_J differ by at least one coordinate and it follows that $\|e_I - e_J\|_\infty = 1$. Now assume A is such that A is dense in ℓ^∞. We will show that A has to be uncountable. Since A is dense, it follows that for every $I \subseteq \mathbb{N}$ there is an element in A, say x_I such that $\|x_I - e_I\| < 1/4$. We claim that $x_I \neq x_J$ for $I \neq J$. Indeed,

$$1 = \|e_I - e_J\|_{\ell^\infty} \leq \|e_I - x_I\|_{\ell^\infty} + \|x_I - x_J\|_{\ell^\infty} + \|x_J - e_J\|_{\ell^\infty} < \|x_I - x_J\|_{\ell^\infty} + 1/2.$$

Therefore, $\|x_I - x_J\|_{\ell^\infty} > 1/2$. In particular, this gives $x_I \neq x_J$. It follows that all the elements $x_I \in A$ for $I \subseteq \mathbb{N}$ are distinct, and hence A is uncountable.