1. [This exercise actually belongs to chapter 4] Calculate

(a) $\displaystyle\int_{|z|=2} \frac{\sin z}{3z-\pi}\,dz.$

(b) $\displaystyle\int_{|z+1|=4} \frac{z^5}{(z-2)^3}\,dz.$

2. Determine the radius of convergence of the following power series

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{5^n - 2^n}.$

(b) [difficult] $\displaystyle\sum_{n=1}^{\infty} [3+(-1)^n]^n\, z^n.$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n.$

3. In this exercise you have to expand $\dfrac{1}{z^2+2z-3}$ into a power series around 0 in several ways.

(a) By applying a partial fraction decomposition.

(b) By multiplying the power series for $\dfrac{1}{z+3}$ and $\dfrac{1}{z-1}$ together (Cauchy product).

(c) By defining a general power series $a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$ and requiring that

$$\left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots\right)\left(z^2+2z-3\right) = 1$$

4. [Compare with B-C exercise 56.4]

(a) Show that, given the well-known power series for $\dfrac{1}{1-z}$ expanded around 0, there exists an $r \in \mathbb{R}$ so that for every $\theta \in \mathbb{R}$:

$$\sum_{k=0}^{\infty} r^k \sin k\theta = \frac{r\sin\theta}{1-2r\cos\theta + r^2}$$

(b) For which values of r does this identity hold true?

(c) What would be the analogous identity for $\displaystyle\sum_{k=0}^{\infty} r^k \cos k\theta$?

5. Determine the coefficient of the $(z-5)^{683}$ term of the power series of $\dfrac{1}{(z-2)^2}$ expanded around 5.

6. Expand into a power series and give the radius of convergence:

(a) $\dfrac{1}{z-2+i}$ around 0.

(b) $\dfrac{1}{z+3}$ around i.

(c) $\dfrac{1}{z^2+1}$ around 3.

(d) $\dfrac{1}{(3-z)^2}$ around -1.

7. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n} = -\operatorname{Log}(1-z)$ for $|z|<1$.

8. What is the radius of convergence for the MacLaurin series of $f(z) = \text{Log}(1 + e^z)$? You don't have to actually calculate the series!

9. Determine the MacLaurin series for $f(z) = e^{-z} \sin z$ up to the z^4 term and give the radius of convergence.

10. Determine the MacLaurin series for

(a) $f(z) = \dfrac{1-z}{1+z+z^2+z^3}$ up to the 5th degree term;

(b) $f(z) = \dfrac{\sin z}{1+z^2}$ up to the 5th degree term;

(c) $f(z) = \displaystyle\int_0^z \frac{1-\cos\zeta}{\zeta}\, d\zeta$ up to the 6th degree term

11. Show that $\displaystyle\sum_{n=0}^{\infty} \binom{\alpha}{n} z^n = (1+z)^\alpha$ for $|z| < 1$. α is arbitrary real number, where

$$\binom{\alpha}{n} = \frac{\alpha\,(\alpha-1)\,(\alpha-2)\cdot\cdots\cdot(\alpha-n+1)}{n!}$$

(see Stewart). For $(1+z)^\alpha$ we take the principle value (so that $1^\alpha = 1$)

12. Expand $\dfrac{1}{(1-2z)^5}$ into a power series around 0 up to the z^4 term. What is the radius of convergence?

13. Expand e^z into a power series around 1. What is the radius of convergence?

14. The real function $f(x) = \dfrac{1}{1+x^2}$ has $\mathbb{R}$ as its domain, can be differentiated infinitely many times, and can be expanded into a power series around 0

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

Still, the radius of convergence is 1, not infinity. Why?

15. Determine the Laurent series of

(a) $f(z) = \dfrac{1}{(z-1)(z-2)}$ around 0.

(b) $f(z) = \dfrac{1}{z^2(z-i)}$ around i.

16. Determine the Laurent series of the following functions around z_0 on the given regions.

(a) $f(z) = \dfrac{1}{(z-i)(z-2)}$

(i) $z_0 = 0$ on $1 < |z| < 2$;

(ii) $z_0 = 0$ on $2 < |z| < \infty$;

(iii) $z_0 = i$ on $0 < |z-i| < \sqrt{5}$;

(iv) Now determine $\displaystyle\oint_C \frac{1}{(z-i)(z-2)}\, dz$ where C is a simple closed contour around i but not around 2.

(b) $f(z) = \dfrac{1}{z(z^2-1)}$ with $z_0 = 1$ on $1 < |z-1| < 2$.

17. Given is the function $S(z) = \sum_{n=1}^{\infty} \frac{1}{n! z^n}$.

(a) Show that S is analytic everywhere, except for $z = 0$.

(b) Calculate the contour integral of $S(z)$ over the positively-oriented unit circle.

Exercises of Brown Churchill:

B-C 65.3. Find the Maclaurin series expansion of the function $f(z) = \dfrac{z}{z^4+4}$.

B-C 65.8. Rederive the Maclaurin series (4) in sec. 64 for the function $f(z) = \cos z$ by

(a) using the definition $\cos z = \dfrac{1}{2}(e^{iz} + e^{-iz})$ and appealing to the Maclaurin series for e^z in sec. 64;

(b) showing that $f^{(2n)}(0) = (-1)^n$ and $f^{(2n+1)}(0) = 0 \quad (n = 0, 1, 2, \ldots)$.

B-C 65.9. Use the Maclaurin series expansion of $\sin z$ to write the Maclaurin series for the function $f(z) = \sin(z^2)$ and point out how it follows that $f^{(4n)}(0) = 0$ and $f^{(2n+1)}(0) = 0 \quad (n = 0, 1, 2, \ldots)$.

B-C 65.10. Derive the expansions

(a) $\dfrac{e^z}{z^2} = \dfrac{1}{z^2} + \dfrac{1}{z} + \displaystyle\sum_{n=0}^{\infty} \dfrac{z^n}{(n+2)!} \quad (0 < |z| < \infty)$

(b) $\dfrac{\sin(z^2)}{z^4} = \dfrac{1}{z^2} - \dfrac{z^2}{3!} + \dfrac{z^6}{5!} - \dfrac{z^{10}}{7!} + \cdots \quad (0 < |z| < \infty)$.

B-C 65.11. Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

B-C 68.1. Find the Laurent series that represents the function

$$f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$$

in the domain $0 < |z| < \infty$.

B-C 68.2. Find a representation for the function

$$f(z) = \frac{1}{1+z}$$

in negative powers of z that is valid when $1 < |z| < \infty$.

B-C 68.3. Find the Laurent series that represents the function

$$f(z) = \frac{1}{z(1+z^2)}$$

of example 1, Sec. 68, when $1 < |z| < \infty$.

B-C 68.4. Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)}$$

and specify the regions in which those expansions are valid.

B-C 68.5. The function

$$f(z) = \frac{-1}{(z-1)(z-2)},$$

which has the two singular points $z = 1$ and $z = 2$, is analytic in the domains

$$D_1 : |z| < 1, \quad D_2 : 1 < |z| < 2, \quad D_3 : 2 < |z| < \infty.$$

Find the series representation in powers of z for $f(z)$ in each of those domains.

B-C 68.6. Show that when $0 < |z-1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

B-C 72.1. By differentiating the Maclaurin series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1),$$

obtain the expansions

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1)z^n \qquad (|z| < 1)$$

and

$$\frac{2}{(1-z)^3} = \sum_{n=0}^{\infty} (n+1)(n+2)z^n \qquad (|z| < 1).$$

(Compare with example 2, sec. 71)

B-C 72.2. By substituting $1/(1-z)$ for z in the expansion

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1)z^n \qquad (|z| < 1),$$

found in exercise 72.1, derive the Laurent series representation

$$\frac{1}{z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n(n-1)}{(z-1)^n} \qquad (1 < |z-1| < \infty).$$

B-C 72.3. Find the Taylor series for the function $1/z$ about the point $z_0 = 2$. Then, by differentiating that series term by term, show that

$$\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty} (-1)^n(n+1)\left(\frac{z-2}{2}\right)^n \qquad (|z-2| < 2).$$

B-C 72.4. Show that the function defined by means of the equation

$$f(z) = \begin{cases} (1-\cos z)/z^2 & \text{when } z \neq 0 \\ 1/2 & \text{when } z = 0 \end{cases}$$

is entire.

B-C 72.8. Prove that if f is analytic at z_0 and $f(z_0) = f'(z_0) = \cdots = f^{(m)}(z_0) = 0$, then the function g defined by means of the equations

$$g(z) = \begin{cases} \dfrac{f(z)}{(z-z_0)^{m+1}} & \text{when } z \neq z_0, \\ \dfrac{f^{(m+1)}(z_0)}{(m+1)!} & \text{when } z = z_0 \end{cases}$$

is analytic at z_0.