Answers of the extra exercises for chapter 3

Complex Analysis (EE2M11-2021v1)

1. $z = \ln 5 + i\left((2k-1)\pi + \arctan\dfrac{4}{3}\right)$.

2. $\cos 2 \cosh 3 - i \sin 2 \sinh 3$, $\sin 2 \cosh 3 + i \cos 2 \sinh 3$.

3. $-i \ln\left(2 \pm \sqrt{3}\right) + 2k\pi$.

4. when z is real or when $\operatorname{Re} z = \frac{1}{2}\pi + k\pi$.

5. Write out the left-hand-side of the equations using the definitions of the trigonometric functions.

6. Everywhere except $z = 0$ and $z = \frac{\pi}{2} + k\pi$.

7. (a) $\frac{1}{2}\ln 13 + i(-\arctan\frac{3}{2} + 2k\pi)$, $\frac{1}{2}\ln 13 + i(\pi - \arctan\frac{3}{2} + 2k\pi)$, $\ln 5 + 2k\pi i$, $\ln 2 + i(\pi + 2k\pi)$.

(b) All of $\mathbb{C}$ (Namely, the range of $\log z$ equals the domain of e^z).

(c) $\frac{1}{2}\ln 13 - i\arctan\frac{3}{2}$, $\frac{1}{2}\ln 13 + i\left(\pi - \arctan\frac{3}{2}\right)$, $\ln 5$, undefined.

(d) All complex numbers w with $-\pi < \operatorname{Im} w < \pi$.

(f) $\frac{1}{2}\ln 13 + i(2\pi - \arctan\frac{3}{2})$, $\frac{1}{2}\ln 13 + i\left(3\pi - \arctan\frac{3}{2}\right)$, $\ln 5 + 2\pi i$, undefined.

(g) All complex numbers w with $\pi < \operatorname{Im} w < 3\pi$.

(i) The branch cut consists of 0 and the positive real numbers. The domain thus consists of all numbers except $x + 0i$ where $x \geq 0$.

(j) All complex numbers w with $0 < \operatorname{Im} w < 2\pi$.

(k) $\frac{1}{2}\ln 13 + i(2\pi - \arctan\frac{3}{2})$, $\frac{1}{2}\ln 13 + i\left(\pi - \arctan\frac{3}{2}\right)$, undefined, $\ln 2 + \pi i$.

8. (a) $e^{-\left(\frac{\pi}{2} + 2k\pi\right)}$.

(b) $e^{-\frac{\pi}{2}}$.

(c) $\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}$ and $-\frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}$.

(e) $\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}$.

(f) The negative real numbers and zero are not part of its domain.

9. (a) 2, $-1 \pm i\sqrt{3}$, principle value of 2;

(b) -2, $1 \pm i\sqrt{3}$, principle value of $1 + i\sqrt{3}$ when using $-\pi < \operatorname{Arg} z \leq \pi$.
If we want to keep the principle branch of $z^{\frac{1}{3}}$ analytic, we must take $-\pi < \operatorname{Arg} z < \pi$. In this instance, $(-8)^{\frac{1}{3}}$ is undefined.

(c) i, $\pm\frac{1}{2}\sqrt{3} - \frac{1}{2}i$, principle value of $\frac{1}{2}\sqrt{3} - \frac{1}{2}i$.

10. (a) ± 1, $\pm i\sqrt{2}$, $\pm\sqrt[4]{2}e^{\frac{\pi i}{8}}$, $\pm\left(-\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{6}\right)$, principle value of: 1, $i\sqrt{2}$, $\sqrt[4]{2}e^{\frac{\pi i}{8}}$ and $\frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{6}$, respectively. Note: If we want to keep the principle branch of $z^{\frac{1}{2}}$ analytic, we must take $-\pi < \operatorname{Arg} z < \pi$. In this instance, the principal value of $(-2)^{\frac{1}{2}}$ is undefined.

(b) $i\left(\dfrac{\pi}{2} + 2k\pi\right)$, $\ln 2 + i\left(\dfrac{\pi}{2} + 2k\pi\right)$, $2e^{-2k\pi}e^{i\ln 2}$, principle value of: $\dfrac{\pi i}{2}$ and $\ln 2 + \dfrac{\pi i}{2}$, $2e^{i\ln 2}$, respectively.

(c) 8, -8, $|a|^k e^{i\operatorname{Arg} a}$. Notice there is no multiple-values in these examples.

11. $-\sqrt[3]{2}$.

12. Take an arbitrary branch of $\log z$, with appropriate branch cut $\alpha < \arg(z) < \alpha + 2\pi$. The domain of f is open and if we already know that f is analytic (in this situation thats the same as differentiable in every point of the domain) then applying the chain rule on the identity $e^{\log z} = z$ gives

$$\frac{d \log z}{dz} = \frac{1}{z}.$$

the question that remains is if f is differentiable in every point of the domain. We can answer this by using the polar form of CR (see extra exercise 16 of chapter 2). Using that exercise easily shows that f is differentiable on its domain and it even gives another way to find the derivative.

14. (a) The branch cut is the part of the real axis from $-\infty$ to -5 (including -5).

(b) The branch cut is the part of the real axis from 5 to ∞ (including 5).

(c) The branch cut is the "star" consisting of the three halflines from zero to ∞ that have arguments $\frac{1}{3}\pi$, π, $-\frac{1}{3}\pi$ (including 0).

(d) The branch cut consists of the five halflines starting on the circle $|z| = 1$ to ∞ that have arguments $\frac{1}{5}\pi$, $\frac{3}{5}\pi$, π, $-\frac{1}{5}\pi$, $-\frac{3}{5}\pi$.

17. (a) $\sin z = \sin w$ if and only if $z = w + 2k\pi$ or $z = \pi - w + 2k\pi$. So if z and w are both in the strip, they must be equal.

(b) All of $\mathbb{C}$ without the real numbers $x \geq 1$ and $x \leq -1$.

(c) $\dfrac{1}{i} \operatorname{Log}\left(iz + \sqrt{1 - z^2}\right)$ where we take the principle value of the square root.

(d) $g'(z) = \dfrac{1}{\sqrt{1 - z^2}}$.

B-C 33.8. $z = i$.

B-C 36.2. (a) $\exp(\pi/2)$;

(b) $-\exp(2\pi^2)$;

(c) $e^{\pi}[\cos(2 \ln 2) + i \sin(2 \ln 2)]$.