Linear Algebra 2

Assignments 5: Eigenvalues and Eigenvectors, and Diagonalization of Operators (§4.1-4.5)

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Q1 2022/2023

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Assignment 5.1

For the following operators L defined on the vector space V determine the characteristic polynomial, eigenvalues and corresponding eigenvectors. Check whether L is diagonalizable by looking for a bases $\mathcal{B}$ of eigenvectors that admit the representation of $[L]_{\mathcal{B}}$ as in **Theorem 5.4***:*

(a) $V = \mathbb{R}_2[t] \quad \text{and} \quad L(\boldsymbol{p}(t)) = \boldsymbol{p}(t-1)$

(b) $V = \mathbb{R}_2[t] \quad \text{and} \quad L(\boldsymbol{p}(t)) = \boldsymbol{p}(t) + (t+1)\boldsymbol{p}'(t)$

(c) $V = M_{2,2}(\mathbb{R}) \quad \text{and} \quad L\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -7a-4b+4c-4d & b \\ -8a-4b+5c-4d & d \end{pmatrix}$

(d) $V = M_{2,2}(\mathbb{R}) \quad \text{and} \quad L(A) = A^\top + 2\,\mathrm{tr}\,(A)\, I_2$

Assignment 5.2 (for submission)

Consider $V = \mathbb{C}^2$ as a vector space over the field $\mathbb{R}$ and define the linear operator L on V as follows:

$$L \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \operatorname{im}(b) - i \operatorname{re}(a) \\ \operatorname{im}(a) - i \operatorname{re}(b) \end{pmatrix} \quad \textit{with} \quad \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2.$$

(a) *Give a possible basis for V. What is the dimension of V?*

(b) *Compute the eigenvalues and corresponding eigenvectors of L.*

(c) *Is the operator L diagonalizable?*

Assignment 5.3

Let L be a linear operator on a finite-dimensional vector space V. We define the ***determinant of the operator*** *L by choosing a basis $\mathcal{B}$ for V and setting:*

$$\det(L) = \det([L]_{\mathcal{B}}).$$

(a) *Show that the above definition is independent of the chosen basis $\mathcal{B}$.*

(b) *Show that L is invertible if and only if $\det(L) \neq 0$.*

(c) *Show that if L is invertible, then $\det(L^{-1}) = \det(L)^{-1}$.*

(d) *Show that of L_1 and L_2 are linear operators on V, then*

$$\det(L_1 \circ L_2) = \det(L_1)\det(L_2).$$

Assignment 5.4 (for submission)

*Let L be a linear operator on a finite-dimensional vector space V. We define the **trace of the operator** L by choosing a basis $\mathcal{B}$ for V and setting:*

$$\operatorname{tr}(L) = \operatorname{tr}([L]_{\mathcal{B}}).$$

(a) *Show that the above definition is independent of the chosen basis $\mathcal{B}$.*

(b) *Show that the trace of any diagonalizable operator L is equal to the sum of its eigenvalues independent of the chosen basis $\mathcal{B}$.*

(c) *Let*

$$[L]_{\mathcal{B}} = \begin{pmatrix} 3+\frac{1}{2}i & -\frac{1}{2}-2i & 0 \\ \frac{1}{2}+2i & 3+\frac{1}{2}i & 0 \\ 0 & 0 & 1-i \end{pmatrix}$$

be the transformation matrix of some diagonalizable linear operator L in some basis $\mathcal{B}$. What are the eigenvalues of L?

Assignment 5.5

Prove the following variant of the Cayley-Hamilton theorem:

If $A \in M_{n,n}$ *is a diagonal matrix, then*

$$p_A(A) = 0_{n,n},$$

where $0_{n,n}$ *denotes the zero matrix in* $M_{n,n}$ *and the term* A^r *in the matrix polynomial* $p_A(A)$ *stands for the* r*-times multiplication by matrix* A*.*

Assignment 5.6 (extra)

Generalize the above variant of the Cayley-Hamilton theorem to diagonalizable matrices and give a proof.

Assignment 5.7 (extra)

Generalize the above variant of the Cayley-Hamilton theorem to diagonalizable linear operators defined on a finite-dimensional vector space V.

(a) *What is 0_V in this case?*

(b) *What is the meaning of L^r in this case?*

(c) *Give a proof of the Cayley-Hamilton theorem for diagonalizable operators.*

Assignment 5.8

Compute the eigenvalues and corresponding eigenvectors of the following matrices

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}, \qquad B = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Check if these matrices are diagonalizable and if this is the case give matrices P_A and D_A and P_B and D_B, respectively, such that

$$A = P_A D_A P_A^{-1}, \qquad B = P_B D_B P_B^{-1}.$$

Assignment 5.9 (for submission)

Let A be a 2×2 real matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and a corresponding (complex) eigenvector $\boldsymbol{v}$.

(a) *Show the following relations*

$$\begin{aligned} A(\mathrm{re}\,(\boldsymbol{v})) &= +a(\mathrm{re}\,(\boldsymbol{v})) + b(\mathrm{im}\,(\boldsymbol{v})) \\ A(\mathrm{im}\,(\boldsymbol{v})) &= -b(\mathrm{re}\,(\boldsymbol{v})) + a(\mathrm{im}\,(\boldsymbol{v})) \end{aligned}$$

(b) *Show that* $\mathrm{re}\,(\boldsymbol{v})$ *and* $\mathrm{im}\,(\boldsymbol{v})$ *are linearly independent vectors in* $\mathbb{R}^2$.

(c) *Show that* $A = PCP^{-1}$, *where the columns of* P *are the vectors* $\mathrm{re}\,(\boldsymbol{v})$ *and* $\mathrm{im}\,(\boldsymbol{v})$ *and*

$$C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Assignment 5.10 (for submission)

Check if the operator L defined on the vector space V is diagonalizable using **Theorem 5.8**. *In those cases, where V is a real vector space you first have to verify explicitly that the characteristic polynomial splits over $\mathbb{R}$. If L is diagonalizable, give a basis $\mathcal{B}$ of eigenvectors such that $[L]_{\mathcal{B}}$ is a diagonal matrix.*

(a) $V = \mathbb{R}_2[t] \quad \textit{and} \quad L(\boldsymbol{p}(t)) = \boldsymbol{p}(1) + \boldsymbol{p}'(0)t + (\boldsymbol{p}'(0) + \boldsymbol{p}''(0))t^2$

(b) $V = M_{2,2}(\mathbb{R}) \quad \textit{and} \quad L(A) = A + A^\top$

(c) $V = M_{2,2}(\mathbb{C}) \quad \textit{and} \quad L(A) = A - A^\top$