1. Calculate, if possible, with real integration.

(a) $\displaystyle\int_{-\infty}^{\infty} \sin x \, dx;$

(b) $\displaystyle\lim_{T\to\infty} \int_{-T}^{T} \sin x \, dx;$

(c) $\displaystyle\lim_{T\to\infty} \int_{-T}^{2T} \sin x \, dx;$

(d) P.V. $\displaystyle\int_{-\infty}^{\infty} \sin x \, dx.$

2. Calculate with real integration:

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{x+1}{x^2+1} \, dx.$

(b) $\displaystyle\lim_{T\to\infty} \int_{-T}^{T} \frac{x+1}{x^2+1} \, dx.$

(c) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{x+1}{x^2+1} \, dx.$

3. Calculate

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+4x+13)^2} \, dx;$

(b) $\displaystyle\int_{0}^{\infty} \frac{x^2}{(x^2+a^2)^2} \, dx$ for $a > 0$;

(c) $\displaystyle\int_{0}^{\infty} \frac{1}{(x^2+1)^n} \, dx$ with positive integer n;

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)} \, dx$ with $a > 0$ and $b > 0$;

(e) $\displaystyle\int_{0}^{\infty} \frac{x^2+1}{x^4+1} \, dx.$

(f) $\displaystyle\int_{0}^{\infty} \frac{1}{1+x^n} \, dx$ with positive integer $n \geq 2$.

4. Calculate

(a) $\displaystyle\int_{0}^{\infty} \frac{x^2\sqrt{x}}{(x+1)^4} \, dx.$

(b) $\displaystyle\int_{0}^{\infty} \frac{x^2\sqrt[3]{x}}{(x+1)^4} \, dx.$

(c) $\displaystyle\int_{0}^{\infty} \frac{x^a}{(x^2+1)^2} \, dx$ with $-1 < a < 3$.

5. Calculate $\displaystyle\int_{0}^{\infty} \frac{x^3 \sin x}{x^4+4} \, dx.$

6. Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{e^{i\omega t} \, dt}{(t^2+1)(t^2+4)}$ for every $\omega \in \mathbb{R}$.

7. (a) With the help of techniques from your first year's Analysis course (Stewart's book), the integral

$$\int_0^{2\pi} \sin^2\theta \, d\theta$$

can be calculated. Do you remember how?

(b) Now calculate this integral with the help of a contour integral.

8. Calculate $\displaystyle\int_0^{2\pi} \frac{d\theta}{(5+4\cos\theta)^2}$.

9. In examples 3 and 5 from lecture 8.1, we start to work on

$$\int_0^{\infty} \frac{\ln x}{x^3+1} \, dx.$$

(a) Finish example 3.

(b) Finish example 5.

10. Show that

$$\int_0^{\infty} \frac{\sqrt[3]{x}\ln x}{x^2+1} \, dx = \frac{\pi^2}{6}.$$

In doing so, you solve another integral for free. Which integral would that be?
Note: the last step (step 5) is very computationally intensive!

11. Calculate $\displaystyle\int_0^{\infty} \frac{x}{x^6+1} \, dx$.

12. (a) Compute the Laplace transform of the function

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-2t} & \text{for } t \geq 0 \end{cases}$$

In other words, calculate $F(s) = \int_{-\infty}^{\infty} e^{-st} f(t) \, dt$. What is the region of convergence in $\mathbb{C}$ of this integral?

(b) Compute the inverse Laplace transform of $F(s)$ found in (a). In other words, calculate

$$g(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{L_T} F(s) e^{st} \, ds,$$

where $L_T = \{a + i\tau \mid -T \leq \tau \leq T\}$ for an appropriate a.

(c) What do you notice when comparing $g(t)$ found in (b) with $f(t)$ found in (a)?

13. (a) Compute the Laplace transform of the function $f(t) = \begin{cases} 0 & \text{for } t < 0 \\ \cos t & \text{for } t \geq 0 \end{cases}$

(b) Compute the inverse Laplace transform of the answer found in (a).

For those that want to calculate a few more intensive integrals:

14. Fourier integrals:

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{x\cos x}{x^2-2x+10}\,dx$;

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+4x+20}\,dx$;

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+b^2}\,dx$ with a and b both positive and real;

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin ax}{x^2+b^2}\,dx$ with a and b both positive and real;

(e) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin ax}{x(x^2+1)}\,dx$.

15. Choose from the following problems (difficult, but useful):

(a) Show that

$$\int_0^{\infty} \cos(x^2)\,dx = \int_0^{\infty} \sin(x^2)\,dx = \frac{1}{4}\sqrt{2\pi}$$

by integrating the function $f(z) = e^{iz^2}$ along the positively oriented circle sector $0 \le r \le R$ and $0 \le \theta \le \frac{1}{4}\pi$, while letting R go to ∞. See exercise 12 on page 276 of B-C.

These are called the Fresnel integrals. They play an important role in optics. In this problem, you can use that

$$\int_0^{\infty} e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi},$$

something that you may remember from your first year Analysis class. For help, see Stewart, 6th ed, exercise 36 from §15.4.

(b) Calculate

(i) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2-\pi^2}\,dx$

(ii) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{imx}}{(x-1)(x-2)}\,dx$ for $m > 0$

(c) Show that $\displaystyle\int_0^{\infty} \frac{\cos(\ln x)}{1+x^2}\,dx = \frac{\pi}{2\cosh\frac{\pi}{2}}$. In solving this, you solve another integral for free. Which integral would this be?

[TIP: take $f(z) = z^i/(z^2-1)$ and take the contour consisting of the line segment between Ri and ϵi, the half circle from ϵi via ϵ to $-\epsilon i$, the line segment between $-\epsilon i$ and $-Ri$, and the half circle from $-Ri$ via R to Ri.]

(d) Show that $\displaystyle\int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$ by performing complexification using

$$f(z) = \frac{1-e^{2iz}}{z^2}.$$

Use a similar technique to calculate $\displaystyle\int_0^{\infty} \frac{\sin^3 x}{x^3}\,dx$.