Linear Algebra 2

Assignments 1: Vector Spaces and Bases (§2.1-2.3)

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Assignment 1.1

Show that $V = C(\mathbb{R}, \mathbb{R})$ *together with the operations*

- *addition and*
- *multiplication with a real number*

forms a real vector space.

Assignment 1.2

Explain why each of the following sets is or is not a vector space:

- $\{\boldsymbol{x} \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 = 0$ *and* x_1 *is rational.*$\}$
- $\{\boldsymbol{x} \in \mathbb{R}^n : x_1^2 \geq x_2^2 + \cdots + x_n^2\}$
- *All polynomials* $p(t) \in \mathbb{R}[t]$ *such that* $p(3) = 0$
- *All continuously differentiable functions* $f(x) \in C^1(\mathbb{R}, \mathbb{R})$ *such that* $f'(0) = 1$

Assignment 1.3

Prove that the additive inverse of each element of a vector space is unique.

Assignment 1.4

Let V be the real numbers. Show that V is a real vector space with the operations addition and scalar multiplication defined as $x \oplus y = x + y + 1$ and $c \otimes x = cx + c - 1$, respectively. Give the additive identity of V. What does "$-x$" mean in this context?

Assignment 1.5

Let V be a vector space and let W be a subspace according to definition (§2.1, p.11). Show that this implies that $\mathbf{0} \in W$. Notice that this property is not enforced in the definition of a subspace but that it follows as a consequence.

Assignment 1.6

Let V be a vector space, I be an arbitrary index set, and let a subspace $W_i \subset V$ be given for each $i \in I$. Show that the intersection

$$W := \bigcap_{i \in I} W_i \subset V$$

is a subspace of V. What about $\cup_{i \in I} W_i$? Either show that the union of a finite number of subspaces $W_i \subset V$ is again a subspace of V or give a counterexample.

Assignment 1.7

Give an explicit expression for the subspace

$$\bigcap_{n\in\mathbb{N}} \mathbb{R}_n[t] \subset V = \mathbb{R}[t].$$

Assignment 1.8 (for submission)

Prove Theorem 2.7 (§2.3, p.21) from the book:

Let V be a vector space with basis $\mathcal{B}$.

1. *If $\boldsymbol{v}, \boldsymbol{w} \in V$ and c is a scalar, then $[\boldsymbol{v} + \boldsymbol{w}]_{\mathcal{B}} = [\boldsymbol{v}]_{\mathcal{B}} + [\boldsymbol{w}]_{\mathcal{B}}$ and $[c\boldsymbol{v}]_{\mathcal{B}} = c[\boldsymbol{v}]_{\mathcal{B}}$.*
2. *Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ and $\boldsymbol{w}$ be elements of V. Then, $\boldsymbol{w}$ is a linear combination of the $\boldsymbol{v}_i$'s if, and only if, $[\boldsymbol{w}]_{\mathcal{B}}$ is a linear combination of the $[\boldsymbol{v}_i]_{\mathcal{B}}$'s.*
3. *The $\boldsymbol{v}_i$'s are linearly independent in V if, and only if, the $[\boldsymbol{v}_i]_{\mathcal{B}}$'s are linearly independent in $\mathbb{R}^n$.*

Assignment 1.9

Check whether the three vectors

$$\boldsymbol{b}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \boldsymbol{b}_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \quad \boldsymbol{b}_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

are a basis of $\mathbb{R}^3$. *Furthermore, check whether the three vectors*

$$\boldsymbol{b}_1 = 1 + 2t + 3t^2, \quad \boldsymbol{b}_2 = 2 + 3t + 4t^2, \quad \boldsymbol{b}_3 = 3 + 4t + 5t^2$$

are a basis of $\mathbb{R}_2[t]$.

Assignment 1.10

Let $W = \{A \in M_{2,2} : A = A^\top\}$ *be the subset of* $V = M_{2,2}$ *that consists of symmetric matrices. Show that it is in fact a subspace of* V *and give a basis for* W. *Give the coordinates of* $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \in W$ *in the chosen basis.*