**Answers of the extra exercises for chapter 2**

**Complex Analysis (EE2M11-2021v1)**

**1.** (a) $u(x,y) = 2x^3 - 6xy^2 - 3x$, $v(x,y) = 6x^2y - 2y^3 - 3y$;

(b) $u(x,y) = \dfrac{x}{x^2+y^2}$, $v(x,y) = \dfrac{-y}{x^2+y^2}$;

(c) $u(x,y) = \dfrac{-x^2-y^2+1}{x^2+(y-1)^2}$, $v(x,y) = \dfrac{-2x}{x^2+(y-1)^2}$;

**2.** (a) $f(z) = z^2$,

(b) $f(z) = \overline{z}^3$;

(c) $f(z) = \dfrac{z}{2z-1}$;

**3.** Stretching of $z$ by a factor $|a|$ and a rotation around 0 over the angle $\arg a$, followed by a translation over "vector" $b$.

**4.** (a) The image of the line $x = a$ (for $a \neq 0$) is the parabola $u = a^2 - \dfrac{v^2}{4a^2}$. Notice that $x = a$ and $x = -a$ have the same image! If $a = 0$, the image is the real interval $(-\infty, 0]$.

The image of the line $y = b$ (for $b \neq 0$) is the parabola $u = \dfrac{v^2}{4b^2} - b^2$. If $b = 0$, the image is the real interval $[0, \infty)$.

(b) The preimage of the line $u = c$ (for $c \neq 0$) is the hyperbola $x^2 - y^2 = c$. If $c = 0$, the preimage is $y = \pm x$.

The preimage of the line $v = d$ (for $d \neq 0$)is the hyperbola $xy = \dfrac{d}{2}$. If $d = 0$, the preimage is $xy = 0$.

**5.** (b) Through $z^2$, the pie slice "doubles"; the angle becomes twice as large. From the moment that the upper boundary $\alpha$ of $\arg z$ passes $\pi$, you will have covered the entire disc and you start covering the disc multiple times. The larger $\alpha$ becomes, the more of the disc is multiply covered.
Something similar happens for $w = z^3$ and $w = z^4$, but they start requiring a lower and lower $\alpha$ to cover the whole disc once ($\alpha = \frac{2}{3}\pi$ for $w = z^3$ and $\alpha = \frac{1}{2}\pi$ for $w = z^4$).
Changing the radius of the pie slice changes the radius of the image. If the radius of the preimage was $r$, then the radius of the image is $r^2$, $r^3$, and $r^4$, respectively.

**6.** See figure 21 of §14 from B-C. Also compare with exercise 14.5 from B-C.

**7.** (a) The circle with center 0 and radius $\frac{1}{2}$, and its exterior.

(b) The circle with center $-\frac{1}{2}i$ and radius $\frac{1}{2}$, without 0.

(c) From infinity over the $x$-axis to 2, then "down" via the circle $(u-1)^2 + (v)^2 = 1$ to the intersection with the circle in part (b), via that circle to the $y$-axis, and finally down the $y$-axis to infinity.

(d) The circle with center $\frac{1}{2}$ and radius $\frac{1}{2}$, without 0.

(e) The circle with center $\frac{1}{4}i$ and radius $\frac{1}{4}$, without 0.

**9.** $\dfrac{1}{4e}(\cos 1 + i \sin 1)$

**10.** Say $l$ is a line in the complex plane. For a point $P$ on $l$, a corresponding point $P'$ is found on the sphere as the intersection of the line passing through the north pole $N$ and the point $P$. All lines $NP$ lie in the plane $V$ though $N$ and line $l$. All points $P'$ must then lie on the intersection circle of plane $V$ and the sphere. Parallel lines in the complex plane correspond to circles on the sphere that touch each other in the point $N$. Non-parallel lines intersecting at point $S$ correspond to circles on the sphere that intersect at $N$ and and a point $S'$.

**11.** This limit doesn't exist!

**12.** (a) Not continuous in $z = 0$.
(b) Continuous in $z = 0$.

**13.** (a) nowhere analytic;
(b) the entire plane, $f'(z) = 2z - 3$;
(c) only non-analytic in $z = 0$, $f'(z) = -\dfrac{1+i}{z^2}$.

**14.** $f(z) = iz^3 + i$.

**15.** No, because $v$ is not a harmonic conjugate of $u$.

**B-C 14.1.** (a) $\mathbb{C} \setminus \{-i, i\}$;
(b) $\mathbb{C} \setminus \{0\}$
(c) All $z \in \mathbb{C}$ where $\operatorname{Re} z \neq 0$.
(d) All $z \in \mathbb{C}$ where $|z| \neq 1$.

**B-C 18.5.** Does not exist.

**B-C 24.2.** (b) $f''(z) = f(z)$;
(d) $f''(z) = -f(z)$.

**B-C 24.3.** (a) $f'(z) = -1/z^2$ $(z \neq 0)$;
(b) $f'(x + ix) = 2x$;
(c) $f'(0) = 0$.

**B-C 26.4.** (a) $z = 0, \pm i$;
(b) $z = 1, 2$;
(c) $z = -2, -1 \pm i$.

**B-C 30.3.** $f'(z) = 2z \exp(z^2)$.

**B-C 30.8.** (a) $z = \ln 2 + (2k + 1)\pi i$ $(k = 0, \pm 1, \pm 2, \ldots)$;
(b) $z = \frac{1}{2} \ln 2 + (2k + \frac{1}{4})\pi i$ $(k = 0, \pm 1, \pm 2, \ldots)$;
(c) $z = \frac{1}{2} + k\pi i$ $(k = 0, \pm 1, \pm 2, \ldots)$.

**B-C 115.1.** (a) $v(x, y) = 2y - 3x^2 y + y^3$;
(b) $v(x, y) = -\cosh x \cos y$;
(c) $v(x, y) = \dfrac{x}{x^2 + y^2}$.