Extra exercises for chapter 3

Complex Analysis (EE2M11-2021v1)

1. Solve the equation $e^z = -3 - 4i$ for $z \in \mathbb{C}$.

2. Compute $\cos(2 + 3i)$ and $\sin(2 + 3i)$.

3. (a) Solve the equation $\cos z = 2$ for z.
(b) [New] Let a be a real number. Solve the equation $\cos z = a$ for z.

4. For which z is $\sin z$ real?

5. Prove
(a) $\cos^2 z + \sin^2 z = 1$ for every $z \in \mathbb{C}$;
(b) $\sin\left(\frac{\pi}{2} - z\right) = \cos z$ for every $z \in \mathbb{C}$;
(c) $\cosh^2 z - \sinh^2 z = 1$ for every $z \in \mathbb{C}$.
(d) $\sinh z = -i\sin(iz)$ and $\cosh z = \cos(iz)$ for every $z \in \mathbb{C}$;
(e) $\sin(z + w) = \sin z \cos w + \cos z \sin w$ for every $z \in \mathbb{C}$;
(f) $\sinh(z + w) = \sinh z \cosh w + \cosh z \sinh w$ for every $z \in \mathbb{C}$;

6. Give the largest domain D on which function $f(z) = \dfrac{e^z}{z\cos z}$ is analytic.

7. During the lecture, we saw that the inverse of $w = e^z$ is the multiple-valued function

$$w = \log z = \ln|z| + i\arg z$$

with the domain $\{z \in \mathbb{C} : z \neq 0\}$.
(a) Determine $\log(2 - 3i)$ and $\log(-2 + 3i)$, $\log 5$ and $\log(-2)$..
(b) What is the range of $\log z$?
If for $\arg z$ we choose the value $\operatorname{Arg} z$ (remember that $-\pi < \operatorname{Arg} z \leq \pi$), then we are freed from the multiple-valued nature of $\arg z$. If we restrict ourselves even further to $-\pi < \operatorname{Arg} z < \pi$ (we call this a branch cut), we acquire a singular-valued analytic function $\operatorname{Log} z$ which we call the principle value of $\log z$.
(c) Compute $\operatorname{Log}(2 - 3i)$ and $\operatorname{Log}(-2 + 3i)$, $\operatorname{Log} 5$ and $\operatorname{Log}(-2)$.
(d) What is the range of $\operatorname{Log} z$?
(e) [Difficult] Show that $\dfrac{d}{dz}\operatorname{Log} z = \frac{1}{z}$.
We now take the same branch cut but a different branch of the logarithm function, where $\pi < \arg z < 3\pi$.
(f) Compute $\log(2 - 3i)$ and $\log(-2 + 3i)$, $\log 5$ and $\log(-2)$ for this branch.
(g) What is the range of this branch of the logarithm function?
(h) [Difficult] Show that for this branch, $\dfrac{d}{dz}\log z = \frac{1}{z}$.
We now take a different branch of the logarithm function, where $0 < \arg z < 2\pi$.
(i) Which branch cut are we dealing with? In other words, what is the domain of this branch?
(j) What is the range of this branch of the logarithm function?
(k) Compute $\log(2 - 3i)$ and $\log(-2 + 3i)$, $\log 5$ and $\log(-2)$ for this branch.
(l) [Difficult] Show that for this branch, $\dfrac{d}{dz}\log z = \frac{1}{z}$.

8. For $z \neq 0$, we define $z^w = e^{w \log z}$. In principle, this is multiple-valued. When talking about the principle value of a^b, we speak of $e^{b \operatorname{Log} a}$.

(a) Compute all possible values of i^i.

(b) What is the principle values of i^i?

(c) Compute $i^{\frac{1}{2}}$.

(d) Why is $z^{\frac{1}{2}}$ only double-valued?

(e) What is the principle value of $i^{\frac{1}{2}}$?

(f) Which branch cut corresponds to the principle value of $f(z) = z^{\frac{1}{2}}$?

9. Give all possible values of the following expressions and indicate which is the principle value:

(a) $8^{\frac{1}{3}}$

(b) $(-8)^{\frac{1}{3}}$

(c) $(-i)^{\frac{1}{3}}$

10. Give all possible values of the following expressions and indicate which is the principle value:

(a) $1^{\frac{1}{2}}$, $(-2)^{\frac{1}{2}}$, $(1+i)^{\frac{1}{2}}$, $\left(-1-i\sqrt{3}\right)^{\frac{1}{2}}$;

(b) $\log i$, $\log 2i$, 2^{1+i};

(c) 2^3, $(-2)^3$, a^k where $a \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{Z}$

11. On $\mathbb{C}$, excluding the real numbers ≥ 0, we take the branch for $f(z) = z^{\frac{1}{3}}$ that satisfies $f(i) = e^{\frac{5}{6}\pi i}$. Determine $f(-2)$.

12. Show that $\log z$, z^b and a^z have the derivatives $\dfrac{1}{z}$, bz^{b-1} and $a^z \log a$ respectively, regardless of which branch you take.
[TIP: use the real and imaginary parts of the functions, or use exercise B-C 33.6 for $\log z$.]

13. Show that

(a) $\tan z = -i\dfrac{\exp(2iz)-1}{\exp(2iz)+1}$;

(b) $\tan(-z) = \tan z$;

(c) $\tan(z) = \tan(z+\pi)$.

14. [Nieuwe som!!!] Give the maximal domains such that the following functions are analytic:

(a) $\operatorname{Log}(z+5)$;

(b) $\operatorname{Log}(5-z)$;

(c) $\operatorname{Log}(z^3)$;

(d) $\operatorname{Log}(z^5+1)$.

15. [Nieuwe som!!!] Calculate the principle values of $\big(i(i-1)\big)^i$ and $i^i \cdot (i-1)^i$. If you compare the answers, is it what you expected?

And a little more challenging for the student that wants more:

16. Let $z^{\frac{1}{2}}$ be defined as $\exp\left(\frac{1}{2}\log z\right)$, where $\log z$ is defined with $0 < \arg z < 2\pi$. Determine and sketch the image of the upper half plane $\{z : \operatorname{Im} z > 0\}$ under the map $f(z) = (z^2-1)^{\frac{1}{2}}$.

17. Consider the function $f(z) = \sin z$ on the vertical strip $S = \{z \in \mathbb{C} \mid -\frac{1}{2}\pi < \operatorname{Re} z < \frac{1}{2}\pi\}$.

(a) Verify that f is injective on S.

(b) Determine the image I of S under the map f.

f is invertible on S. Its inverse is the function $g(z) = \arcsin z$ with domain I. With the (during this course non treated) inverse function theorem it is not hard to prove that g is analytic on I (you don't have to prove this).

(c) Find a formula for $g(z) = \arcsin(z)$ in terms of the logarithm and the square root.

(d) With this find the derivative of $\arcsin z$.

18. (a) Show that there does not exist a function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ with both of the following properties:

$$f(zw) = f(z)f(w) \text{ for all } z \neq 0 \text{ and } w \neq 0$$

$$\big(f(z)\big)^2 = z \text{ for all } z \neq 0$$

(b) Show that there is no continuous function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ such that

$$\big(f(z)\big)^2 = z \text{ for all } z \neq 0.$$

(c) Show that there is no continuous function $g : \mathbb{C} \to \mathbb{C}$ such that

$$\big(g(z)\big)^2 = z \text{ for all } z \in \mathbb{C}.$$

(d) Show that there is no continuous function $\phi : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ such that

$$|z| \exp\big(i\phi(z)\big) = z \text{ for all } z \in \mathbb{C} \setminus \{0\}.$$

(e) Show that there is no continuous function $\ell : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that

$$\exp\big(\ell(z)\big) = z \text{ for all } z \in \mathbb{C} \setminus \{0\}.$$

Exercises of Brown Churchill:

B-C 33.1. Show that

(a) $\operatorname{Log}(-ei) = 1 - \frac{\pi}{2}i$;

(b) $\operatorname{Log}(1-i) = \frac{1}{2}\ln 2 - \frac{\pi}{4}i$.

B-C 33.3. Show that $\operatorname{Log}(i^3) \neq 3\operatorname{Log} i$.

B-C 33.4. Show that $\log(i^2) \neq 2\log i$ when the branch

$$\log z = \ln r + i\theta \qquad \left(r > 0, \frac{3\pi}{4} < \theta < \frac{11\pi}{4}\right)$$

is used (compare this with the example in Sec. 33).

B-C 33.8. Find all roots of the equation $\log z = i\pi/2$.

B-C 36.1. Show that

(a) $(1+i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right)\exp\left(i\frac{\ln 2}{2}\right) \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $\frac{1}{i^{2i}} = \exp[(4n+1)\pi] \quad (n = 0, \pm 1, \pm 2, \ldots)$.

B-C 36.2. Find the principal value of

(a) $(-i)^i$;

(b) $\left[\frac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i}$;

(c) $(1-i)^{4i}$.