1. (a) $\dfrac{\pi i}{\sqrt{3}}$

(b) $160\pi i$

2. (a) 5

(b) 1/4

(c) e (For this, you're going to need the limit $\lim_{n\to\infty}(1+\frac{x}{n})^n$, see Stewart's book for its value).

3. All three methods (partial fraction decomposition, Cauchy product, and general power series with requisites) result in:

$$-\frac{1}{3}-\frac{2}{9}z-\frac{7}{27}z^2-\frac{20}{81}z^3-\frac{61}{243}z^4-\cdots$$

or, even better:

$$\sum_{k=0}^{\infty}\left(-\frac{3^{k+1}+(-1)^k}{4\cdot 3^{k+1}}\right)z^k.$$

It should be said that my opinion is that this last equation is determined most easily through partial fraction decomposition.

4. (a) We are given that $\displaystyle\sum_{k=0}^{\infty} z^k=\frac{1}{1-z}$. Replace z with $re^{i\theta}$.
Then equate the imaginary parts of both the left- and right-hand-side.

(b) For $-1<r<1$

(c) $\displaystyle\sum_{k=0}^{\infty} r^k\cos k\theta=\frac{1-r\cos\theta}{1-2r\cos\theta+r^2}$

5. $-\dfrac{76}{3^{683}}$

6. (a) $\displaystyle\sum_{n=0}^{\infty}\frac{-1}{(2-i)^{n+1}}z^n$, radius of convergence is $\sqrt{5}$.

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(3+i)^{n+1}}(z-i)^n$, radius of convergence is $\sqrt{10}$.

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{i(-1)^n}{2}\left(\frac{1}{(3+i)^{n+1}}-\frac{1}{(3-i)^{n+1}}\right)(z-3)^n$, radius of convergence is $\sqrt{10}$.

(d) $\displaystyle\sum_{n=0}^{\infty}\frac{n+1}{4^{n+2}}(z+1)^n$, radius of convergence is 4.

7. We know that $\displaystyle\sum_{k=0}^{\infty} z^k=\frac{1}{1-z}$. Take the antiderivative of the left- and right-hand-side, remembering the constant of integration.
Now recall that $\operatorname{Log} 1=0$ and that the branch cut associated with $-\operatorname{Log}(1-z)$ (principle value!) is the interval $[1,\infty)$. This interval lies outside the region of convergence $|z|<1$ for our power series.

8. π

9. $z - z^2 + \frac{1}{3}z^3 + \cdots$, radius of convergence is ∞.

10. (a) $1 - 2z + z^2 + z^4 - 2z^5 + z^6 + z^8 - 2z^9 + z^{10} + \cdots$
(b) $z - \frac{7}{6}z^3 + \frac{47}{40}z^5 - \cdots + \cdots$
(c) $\frac{1}{4}z^2 - \frac{1}{96}z^4 + \frac{1}{4320}z^6 - \cdots$

11. See Stewart, chapter 11, and replace x with z in the book's proof.

12. $1 + 10z + 60z^2 + 280z^3 + 1120z^4 + \cdots$, radius of convergence is $\dfrac{1}{2}$.

13. [Compare with B-C exercise 59.2] $e + e\,(z-1) + \dfrac{e}{2}\,(z-1)^2 + \dfrac{e}{6}\,(z-1)^3 + \dfrac{e}{24}\,(z-1)^4 + \cdots$
A little more formally: $\displaystyle\sum_{n=0}^{\infty} \frac{e}{n!}\,(z-1)^n$.

14. The complex function $\dfrac{1}{1+z^2}$ has a similar power series and has singularities in i and $-i$. Thus the largest disc around 0 whereupon this function is analytic has a radius of 1.

15. (a) On $|z| < 1$: $(1 + z + z^2 + \cdots) - \frac{1}{2}(1 + \frac{1}{2}z + \frac{1}{4}z^2 + \cdots)$;
On $1 < |z| < 2$: $-\frac{1}{z}(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots) - \frac{1}{2}(1 + \frac{1}{2}z + \frac{1}{4}z^2 + \cdots)$;
On $2 < |z| < \infty$: $\frac{1}{z}(1 + \frac{2}{z} + \frac{4}{z^2} + \cdots) - \frac{1}{z}(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots)$.
(b) On $0 < |z-i| < 1$: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n(z-i)^{n-2}}{i^{n+1}}$;

On $1 < |z-i| < \infty$: $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)i^n}{(z-i)^{n+3}}$.

16. If the Laurent series is rewritten into the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n}$$

then
(a) (i) $a_n = \frac{1}{2^{n+1}(i-2)}$ when $n \geq 0$ and $b_n = \frac{i^{n-1}}{i-2}$ when $n \geq 1$.
(ii) $a_n = 0$ when $n \geq 0$ and $b_n = \frac{i^{n-1} - 2^{n-1}}{i-2}$ when $n \geq 1$.
(iii) $a_n = \frac{-1}{(2-i)^{n+2}}$ when $n \geq 0$ and $b_1 = \frac{1}{i-2}$ and $b_n = 0$ when $n \geq 2$.
(iv) $\frac{2\pi i}{i-2}$.
(b) $a_n = \left(-\frac{1}{2}\right)^{n+2}$ when $n \geq 0$ and $b_1 = -\frac{1}{2}$ and $b_n = (-1)^n$ when $n \geq 2$.

17. We know that $\sum_{n=1}^{\infty} \frac{1}{n!} w^n$ has an infinite radius of convergence and thus converges for all w. If we substitute $w = 1/z$, we get a series that converges for all $z \neq 0$. S is therefore analytic for all $z \neq 0$, since S can be written as a converging Laurent series for all $z \neq 0$. The coefficient for the $\dfrac{1}{z}$ term in the Laurent series is equal to 1, meaning that the contour integral is equal to $2\pi i$.

B-C 65.3. $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+2}} z^{4n+1} \quad (|z| < \sqrt{2})$.

B-C 68.1. $1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\cdot\frac{1}{z^{4n}}$.

B-C 68.2. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^n}$.

B-C 68.3. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}$.

B-C 68.4. $\sum_{n=0}^{\infty}z^n+\frac{1}{z}+\frac{1}{z^2}\quad(0<|z|<1)$; $-\sum_{n=3}^{\infty}\frac{1}{z^n}\quad(1<|z|<\infty)$.

B-C 68.5. $\sum_{n=0}^{\infty}(2^{-n-1}-1)z^n$ in D_1; $\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}+\sum_{n=1}^{\infty}\frac{1}{z^n}$ in D_2; $\sum_{n=1}^{\infty}\frac{1-2^{n-1}}{z^n}$ in D_3.