1. Given is the function $f(z) = \operatorname{Re} z + 2i \operatorname{Im} z$.

(a) K is the curve in the complex plane defined by $y = x^2$ where $0 \leq x \leq 2$. Compute the integral
$$\int_K f(z)\ dz$$
In this, K is traversed in the direction of positively-growing x.

(b) L is the line segment starting at 0 and ending at $2 + 4i$. Notice that L has the same begin and end as the curve K from the previous exercise. Compute
$$\int_L f(z)\ dz.$$

2. C is the circle $|z| = 1$. For every $k \in \mathbb{Z}$, compute the integral $\displaystyle\oint_C z^k dz$, where C is traversed once in the positive direction.

3. Given is $|z| = R$ with $R > 3$. Bound $\left|z^2 - 2z + 3\right|$ with an upper and a lower bound.

4. Given is the curve $|z| = r$ with radius $0 < r < 1$.

(a) For z on the curve, give a good bound M so that $\left|\dfrac{z^{\frac{1}{2}}}{z^2 + z}\right| \leq M$.

(b) Give a good ML-bound of $\left|\displaystyle\int_{|z|=r} \frac{z^{\frac{1}{2}} dz}{z^2 + z}\right|$. Assume that the curve has only be traversed once.

(c) What can we say about $\displaystyle\lim_{r \downarrow 0} \int_{|z|=r} \frac{z^{\frac{1}{2}} dz}{z^2 + z}$?

5. (a) Do exercise 49.3 from B-C. [Compare with exercise 2 of this set]

(b) Why doesn't the theorem work for $n = 0$? $(z - z_0)^{-1}$ has an antiderivative too!

6. Given is a function $f(z) \neq 0$ that is analytic on a open, simple, connected domain D. Prove that for all closed contours C in D,
$$\int_C \frac{f'(z)}{f(z)}\ dz = 0.$$
Give an example why this is not true for an open, connected, not necessarily simple domain.

7. C is the circle $|z - 1| = 2$ that has been traversed once in the positive direction. For every $k \in \mathbb{N}$, calculate the integral
$$\int_C \frac{z^3 dz}{(z - i)^k}.$$

8. (a) Compute $\displaystyle\int_{|z|=2} \frac{e^z dz}{z^2 - 1}$ in two ways:

(i) by modifying the contour

(ii) by applying partial fraction decomposition to $\dfrac{1}{z^2 - 1}$.

9. Take $R > 1$. The contour Γ_R is the part of the circle $|z| = R$ that lies on top of the real axis. Furthermore, C_R is the simple closed contour consisting of Γ_R and the segment of the real axis that lies between $-R$ and R.

(a) Calculate $\oint\limits_{C_R} \frac{dz}{z^2+1}$, where C_R is traversed in the positive direction.

(b) Give a good ML-bound for $\left| \int\limits_{\Gamma_R} \frac{dz}{z^2+1} \right|$.

(c) Let $R \to \infty$ and conclude that $\oint\limits_{C_R} \frac{dz}{z^2+1}$ converges to the real improper integral $\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$. What is the solution to this integral?

10. Take $R > 1$. The contour C_R is the semicircle $|z| = R$ with $\operatorname{Im} z \geq 0$, oriented in the positive direction. Take a suitable branch of $\log z$. Show that

$$\lim_{R\to\infty} \int\limits_{C_R} \frac{\log z}{z^2+1}\, dz = 0.$$

11. Calculate $\int\limits_{|z|=1} \operatorname{Log}(z+2)\, dz$ where the contour is oriented in the positive direction.

12. Come up with a function that is analytic on an open region D (and thus differentiable on D) but isn't twice differentiable everywhere in D.

13. Say $f(z)$ is analytic on the domain $0 < |z| < 1$ and that

$$\int_{|z|=r} f(z)\, dz = 0$$

for all $0 < r < 1$. Is f analytic in 0?

14. Through Liouville's theorem, explain why $\cos\left(z^2+z\right)$ on $\mathbb{C}$ is unbounded.

15. The fundamental theorem of Algebra says that $z^3 + i$ has three zeros. Calculate all three.

16. Compute the average value of $\left(z^5 + 3z + 7\right) e^{2z}$ on the circle $|z-1| = 2$.

17. Show that the function $x^2 - y^2 + x$ is harmonic on $\mathbb{R}^2$ and compute the average value of this function on the circle $x^2 + (y-1)^2 = 4$.

And a little more challenging for the student that wants more:

18. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and assume there exist positive real numbers M and N such that

$$|f(z)| \leq M + N|z|^{\frac{2}{3}}$$

for all z. Prove that f is constant.

19. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and assume there exist positive real numbers L, M and N such that

$$L|z|^{\frac{3}{2}} \leq |f(z)| \leq M + N|z|^{\frac{5}{2}}$$

for all z. Prove that there are complex numbers a, b and c such that $c \neq 0$ and $f(z) = a + bz + cz^2$.

Exercises of Brown Churchill:

B-C 42.2. Evaluate the following integrals:

(a) $\displaystyle\int_0^1 (1+it)^2\,dt$;

(b) $\displaystyle\int_1^2 \left(\frac{1}{t}-i\right)^2 dt$;

(c) $\displaystyle\int_0^{\pi/6} e^{i2t}\,dt$;

(d) $\displaystyle\int_0^{\infty} e^{-zt}\,dt \qquad (\operatorname{Re} z > 0)$.

B-C 42.4. According to definition (2), Sec. 42, of definite integrals of complex-valued functions of a real variable,

$$\int_0^{\pi} e^{(1+i)x}\,dx = \int_0^{\pi} e^x \cos x\,dx + i\int_0^{\pi} e^x \sin x\,dx.$$

Evaluate the integral on the right here by evaluating the single integral on the left and then using the real and imaginary parts of the value found.

B-C 46.3. Evaluate

$$\int_C f(z)\,dz$$

for $f(z) = \pi\exp(\pi\bar{z})$ and C the boundary of the square with vertices at the points 0, 1, $1+i$ and i, the orientation of C being in the counterclockwise direction.

B-C 46.6. Evaluate

$$\int_C f(z)\,dz$$

for $f(z)$ the principal branch of z^i and C the semicircle $z = e^{i\theta}$ $(0 \le \theta \le \pi)$.

B-C 46.7. Evaluate

$$\int_C f(z)\,dz$$

for $f(z)$ the principal branch of z^{-i-2i} and C the contour $z = e^{i\theta}$ $(0 \le \theta \le \frac{1}{2}\pi)$.

B-C 46.9. This exercise demonstrates how the value of an integral of a power function depends in general on the branch that is used.
Let C denote the positively oriented unit circle $|z| = 1$ about the origin.

(a) Show that if $f(z)$ is the principal branch of $z^{-3/4}$, then

$$\int_C f(z)\,dz = 4\sqrt{2}i.$$

(b) Show that if $g(z)$ is the branch of $z^{-3/4}$ defined by $|z| > 0$, $0 < \arg z < 2\pi$ of $z^{-3/4}$, then

$$\int_C f(z)\,dz = -4+4i.$$

B-C 46.10. Evaluate the integral

$$\int_C z^m \bar{z}^n\,dz,$$

where m and n are integers and C is the unit circle $|z| = 1$, taken counterclockwise.

B-C 47.1. Without evaluating the integral, show that

(a) $\left|\int_C \frac{z+4}{z^3-1}\,dz\right| \leq \frac{6\pi}{7}$;

(b) $\left|\int_C \frac{dz}{z^2-1}\right| \leq \frac{\pi}{3}$

when C is the arc that was used in example 1, sec. 47, i.e. the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant.

B-C 47.2. Let C denote the line segment from $z = i$ to $z = 1$, and show that

$$\left|\int_C \frac{dz}{z^4}\right| \leq 4\sqrt{2}$$

without evaluating the integral. *Suggestion:* Observe that of all the points on the line segment, the midpoint is closest tot the origin, that distance being $d = \sqrt{2}/2$.

B-C 49.2. By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration.

(a) $\int_0^{1+i} z^2\,dz$;

(b) $\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right)\,dz$;

(c) $\int_1^3 (z-2)^3\,dz$.

B-C 49.5. Show that

$$\int_{-1}^{1} z^i\,dz = \frac{1+e^{-\pi}}{2}(1-i),$$

where the integrand denotes the principle branch of z^i and where the path of integration is any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis.

B-C 53.1. Apply the Cauchy-Goursat theorem to show that

$$\int_C f(z)\,dz = 0$$

when the contour C is the unit circle $|z| = 1$, in either direction, and when

(a) $f(z) = \frac{z^2}{z+3}$;

(b) $f(z) = ze^{-z}$;

(c) $f(z) = \frac{1}{z^2+2z+2}$;

(d) $f(z) = \text{sech} z$;

(e) $f(z) = \tan z$;

(f) $f(z) = \text{Log}(z+2)$.

B-C 53.2. Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1$ and $y = \pm 1$ and let C_2 be the positively oriented circle $|z| = 4$. Point out why

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

when

(a) $f(z) = \dfrac{1}{3z^2+1}$;

(b) $f(z) = \dfrac{z+2}{\sin(z/2)}$;

(c) $f(z) = \dfrac{z}{1-e^z}$.

B-C 53.6. Let C denote the positively oriented boundary of the half disk $0 \le |z| \le 1$, $0 \le \arg z \le \pi$, and let $f(z)$ be a continuous function defined on that half disk by writing $f(0) = 0$ and using the branch of $z^{1/2}$ defined by $|z| > 0$ and $-\pi/2 < \arg(z) < 3\pi/2$.

(a) Why does the Cauchy-Goursat theorem not apply here?

(b) Show that $\displaystyle\int_C f(z)\,dz = 0$.

B-C 57.1. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

(a) $\displaystyle\int_C \frac{e^{-z}}{z-(\pi i/2)}\,dz$;

(b) $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}\,dz$;

(c) $\displaystyle\int_C \frac{z\,dz}{2z+1}$;

(d) $\displaystyle\int_C \frac{\cosh z}{z^4}\,dz$;

(e) $\displaystyle\int_C \frac{\tan(z/2)}{(z-x_0)^2}\,dz \quad (-2 < x_0 < 2)$.

B-C 57.2. Find the value of the integral of $g(z)$ around the circle $|z-i| = 2$ in the positive sense when

(a) $g(z) = \dfrac{1}{z^2+4}\,dz$;

(b) $g(z) = \dfrac{1}{(z^2+4)^2}\,dz$.

B-C 57.3. Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2-s-2}{s-z}\,ds \quad (|z| \ne 3),$$

then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?