Linear Algebra 2

Assignments 4: Linear Transformations, Kernels, Ranges, and Quotient Maps, the Isomorphosm Theorem (§3.3, 3.5)

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Assignment 4.1

Exercise (5) from page 50 of the book.

On $\mathbb{R}_2[t]$, *let* L *be the operator* $(L\boldsymbol{p})(t) = \boldsymbol{p}(t-1)$. *Find the matrix form of* L *relative to the basis* $\mathcal{B} = (t^2 + t + 1, t^2 + 3t + 2, t^2 + 2t + 1)$.

Assignment 4.2

Let the linear transformation

(a) $L : M_{2,2}(\mathbb{R}) \to M_{2,2}(\mathbb{R})$ *be defined by* $L(A) = A^\top$. *Compute* $[L]_\mathcal{E}$ *in the standard basis of* $M_{2,2}(\mathbb{R})$. *Is the so-defined* L *a bijective mapping?*

(b) $L : M_{2,2}(\mathbb{R}) \to M_{2,2}(\mathbb{R})$ *be defined by* $L(A) = A + A^\top$. *Compute* $[L]_\mathcal{E}$ *in the standard basis of* $M_{2,2}(\mathbb{R})$. *Is the so-defined* L *a bijective mapping?*

(c) $L : M_{2,2}(\mathbb{R}) \to \mathbb{R}$ *be defined by* $L(A) = \operatorname{tr}(A)$, *where* $\operatorname{tr}(A)$ *is the* ***trace*** *of a matrix* A, *which is defined as the sum of the diagonal entries. Compute* $[L]_{\mathcal{E}'\mathcal{E}}$ *in the standard bases of* $M_{2,2}(\mathbb{R})$ *and* $\mathbb{R}$, *respectively. Is the so-defined* L *a bijective mapping?*

Assignment 4.3 (for submission)

Let the linear transformation $L : \mathbb{R}_3[t] \to \mathbb{R}_2[t]$ *be defined by*

$$L(\boldsymbol{p}(t)) = \frac{d}{dt}\boldsymbol{p}(t) + t\frac{d^2}{dt^2}\boldsymbol{p}(t)$$

(a) *Give the linear transformation matrices of* L *relative to*

- *the standard bases for* $\mathbb{R}_2[t]$ *and* $\mathbb{R}_3[t]$, *and*
- *the bases*

$$\mathcal{B}_2 = (t^2 - t + 1, t + 1, t^2 + 1) \quad \textit{for} \quad \mathbb{R}_2[t],$$

and

$$\mathcal{B}_3 = (t^2 - t + 1, t + 1, t^2 + 1, t^3) \quad \textit{for} \quad \mathbb{R}_3[t].$$

(b) *Give the coordinates of a polynomial* $\boldsymbol{p}(t) \in \mathbb{R}_3[t]$ *relative to the standard basis for* $\mathbb{R}_3[t]$ *such that it satisfies* $[L(\boldsymbol{p}(t))]_{\mathcal{B}_2} = (-12, -8, 21)^\top$.

(c) *Is* $\boldsymbol{p}(t)$ *determined uniquely? Give an argument why or why not.*

Assignment 4.4

Prove the following statements:

(a) *A linear mapping* $L : V \to W$ *is 1-1 (injective) if and only if* $\ker L = \{\mathbf{0}\}$.

(b) *A linear mapping* $L : V \to W$ *is onto (surjective) if and only if* $\operatorname{range}(L) = W$.

Assignment 4.5

Exercise (3) from page 56 of the book:

Consider the operator $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L(\boldsymbol{x}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \boldsymbol{x}$.

Find the kernel and the range of L*, and construct an explicit isomorphism between* $\mathbb{R}^2 / \ker L$ *and* $\operatorname{range}(L)$.

Assignment 4.6

Exercise (5) from page 56 of the book:

Let L *be the operator on* $M_{2,2}$ *given by* $L(A) = A + A^\top$. *Find bases for* $\ker L$ *and* $\operatorname{range}(L)$. *What is the rank of* L*?*

Assignment 4.7 (for submission)

Let $L : \mathbb{R}_2[t] \to \mathbb{R}_3[t]$ *be the linear transformation defined by*

$$L(\boldsymbol{p}(t)) = 2\boldsymbol{p}'(t) + \int_0^t 3\boldsymbol{p}(\tau)d\tau.$$

Show that L *is 1-1 (injective) but not onto (surjective).*

Assignment 4.8

Let $L : M_{3,3} \to \mathbb{R}$ *be the linear transformation defined by*

$$L(A) = \operatorname{tr}(A) = \sum_{i=1}^{3} A_{ii}.$$

Show that L *is not 1-1 (injective) but onto (surjective).*

Assignment 4.9 (for submission)

Let the vector space $V = W_1 \oplus W_2$ *be given by the direct sum of the subspaces* W_1 *and* W_2 *and define the operator* $L : V \to V$ *as the projection on* W_1 *along* W_2.

(a) *Show that* L *is linear and* $W_1 = \{\boldsymbol{v} \in V : L(\boldsymbol{v}) = \boldsymbol{v}\}$.

(b) *Show that* $W_1 = \operatorname{range}(L)$ *and* $W_2 = \ker L$.

Assignment 4.10

With the knowledge of the previous assignment define a linear mapping $L : M_{2,2}(\mathbb{R}) \to M_{2,2}(\mathbb{R})$ *such that its kernels is given by the set of all symmetric matrices. What is* $\operatorname{rank}(L)$*?*