1. (a) divergent.
(b) 0.
(c) divergent.
(d) 0.

2. (a) divergent.
(b) π.

3. (a) $-\dfrac{1}{27}\pi$.
(b) $\dfrac{1}{4a}\pi$.
(c) $\dfrac{1\cdot 3\cdot 5\cdots(2n-3)}{2\cdot 4\cdot 6\cdots(2n-2)}\frac{\pi}{2}$ if $n>1$ and $\dfrac{\pi}{2}$ if $n=1$.
For example, the answer for $n=3$ equals $\dfrac{3\pi}{16}$.
(d) $\dfrac{\pi}{ab(a+b)}$. You must handle the situation where $a=b$ separately.
(e) $\dfrac{\pi}{\sqrt{2}}$.
(f) $\dfrac{\pi}{n\sin\frac{\pi}{n}}$.

4. (a) $\dfrac{5\pi}{16}$ (just like in example 2 in lecture 6.1, use a key-hole contour).
(b) $\dfrac{28}{243}\pi\sqrt{3}$
(c) $\dfrac{(1-a)\pi}{4\cos(a\pi/2)}$.

5. $\dfrac{\pi\cos 1}{2e}$ (because the degree of x^4+4 is only 1 greater than the degree of x^3, you have to use Jordan's lemma).

6. $\dfrac{\pi}{6}\left(2e^{-|\omega|}-e^{-2|\omega|}\right)$ (distinguish between $\omega\geq 0$ and $\omega\leq 0$ for the choice of contour; because the degree of $(t^2+1)(t^2+4)$ is 4 greater than the zeroth-degree polynomial 1, the use of Jordan's lemma is not necessary. You can instead bound the integral over the contour using the ML-bound).

7. (a) First year Analysis: for every $\theta\in\mathbb{R}$, we see that $\cos 2\theta = 1-2\sin^2\theta$, or equivalently $\sin^2\theta = \dfrac{1}{2}-\dfrac{1}{2}\cos 2\theta$. Thus

$$\int_0^{2\pi}\sin^2\theta\,d\theta = \int_0^{2\pi}\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)\,d\theta = \left[\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \pi.$$

(b) Complex numbers:

$$\begin{aligned}\int_0^{2\pi} \sin^2\theta\, d\theta &= \int_0^{2\pi} \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^2 d\theta \\ &= -\frac{1}{4}\int_0^{2\pi} \left(e^{2i\theta}-1+e^{-2i\theta}\right) d\theta \\ &= -\frac{1}{4}\left[\frac{e^{2i\theta}}{2i}-\theta-\frac{e^{-2i\theta}}{2i}\right]_0^{2\pi} \\ &= \pi.\end{aligned}$$

(also see exercise 2 from the set of practice exercises from lecture 5).

(c) Contour integration: just like in example 3 in lecture 6.1,

$$\sin^2\theta = \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^2 = \frac{\left(z-\frac{1}{z}\right)^2}{-4} = \cdots$$

etc....

8. $\dfrac{10\pi}{27}$.

9. $-\dfrac{2\pi^2}{27}$.

10. When determining the "pie slice" for your integral, use half of a whole "pie". This is a lot of work! The integral that you solve for free is

$$\int_0^\infty \frac{\sqrt[3]{x}}{x^2+1}\, dx = \frac{\pi}{\sqrt{3}}.$$

11. $\dfrac{\pi}{3\sqrt{3}}$ (can be found very easily with the right "pie slice". This also indicates that when including an extra $\log z$ in the integrand, the resulting new integral can still be easily solved when using a "key-hole" contour).

12. (a) The Laplace transform $\mathcal{F}(s)$ of the function $f(t) = \begin{cases} 0 & \text{for } t<0 \\ e^{-2t} & \text{for } t \geq 0 \end{cases}$ is

$$\int_0^\infty f(t)e^{-st}\, dt$$

where $s = \sigma + i\tau \in \mathbb{C}$. Thus

$$\begin{aligned}\mathcal{F}(s) &= \int_0^\infty e^{-2t}e^{-st}\, dt = \int_0^\infty e^{-(s+2)t} dt = \left[-\frac{1}{s+2}e^{-(s+2)t}\right]_{t=0}^{t=\infty} \overset{\text{provided that } \sigma>-2}{=} \\ &= 0-\left(-\frac{1}{s+2}\right) = \frac{1}{s+2}.\end{aligned}$$

where $\sigma = \operatorname{Re} s$. The region of convergence for $\mathcal{F}(s)$ is $\{s \in \mathbb{C} \mid \operatorname{Re} s > -2\}$.

(b) We need to calculate

$$g(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{L_T} F(s)e^{st}\, ds,$$

where $L_T = \{a + i\tau \mid -T \leq \tau \leq T\}$ for an appropriate a.

We require that $a > -2$, since a must insure that L_T lies in the region of convergence of $\mathcal{F}(s)$. We choose $a = 0$, making computations easier.

We now investigate three possible scenarios: $t > 0$, $t = 0$ and $t < 0$:

- $\boxed{t>0}$ We use contour C_T, consisting of the line segment L_T and the left hand side Γ_T of the circle $|s|=T$, traversed counterclockwise.

  We choose $T>2$. Then -2 is the only singularity of $G(s)=\dfrac{e^{st}}{s+2}$ inside the contour and $\operatorname{Res}\left(\dfrac{e^{st}}{s+2},-2\right)=e^{-2t}$, so that $\displaystyle\int_{C_T} G(s)\,ds=2\pi i e^{-2t}$.
  Now we bound the integral over the arc Γ_T : the ML-bound doesn't work here, but $\displaystyle\int_{\Gamma_T}\frac{e^{st}}{s+2}ds \overset{\text{say } s=iz}{=} \int_{\Gamma_T'}\frac{e^{izt}}{iz+2}d(iz)=i\int_{\Gamma_T'}\frac{e^{izt}}{iz+2}dz$, where Γ_T' is the upper half of the circle $|z|=T$. This last integral, because $t>0$, goes to 0 as $T\longrightarrow\infty$ according to Jordan's lemma.
  $\displaystyle\int_{C_T} G(s)\,ds=2\pi i e^{-2t}$ we rewrite as $\displaystyle\int_{L_T} G(s)\,ds+\int_{\Gamma_T} G(s)\,ds=2\pi i e^{-2t}$; we then let T go to ∞ and we see that:

  $$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{L_T}\frac{e^{st}}{s+2}ds=e^{-2t} \text{ for } t>0.$$

- $\boxed{t<0}$ We use contour C_T, consisting of the line segment L_T and the right hand side Γ_T of the circle $|s|=T$, traversed counterclockwise.
  $G(s)=\dfrac{e^{st}}{s+2}$ has no singularities in this contour, meaning that $\displaystyle\int_{C_T} G(s)\,ds=0$.
  Now we bound the integral over the arc Γ_T: the ML-bound doesn't work here either, but $\displaystyle\int_{\Gamma_T}\frac{e^{st}}{s+2}ds \overset{\text{say } s=iz}{=} \int_{\Gamma_T'}\frac{e^{izt}}{iz+2}d(iz)=i\int_{\Gamma_T'}\frac{e^{izt}}{iz+2}dz$, where Γ_T' is the lower half of the circle $|z|=T$. This last integral, because $t<0$, goes to 0 as $T\longrightarrow\infty$ according to Jordan's lemma.
  $\displaystyle\int_{C_T} G(s)\,ds=0$ we rewrite as $\displaystyle\int_{L_T} G(s)\,ds+\int_{\Gamma_T} G(s)\,ds=0$; we then let T go to ∞ and we see that:
  $$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{L_T}\frac{e^{st}}{s+2}ds=0 \text{ for } t<0.$$

- $\boxed{t=0}$ Now, not even Jordan's lemma can help in bounding the integral (whether we take the left or right half of the circle $|s|=T$), but we actually don't need it. Working with residues isn't necessary: $G(s)$ is now equal to $\dfrac{1}{s+2}$, for which we know an antiderivative: the function $\operatorname{Log}(s+2)$. So

  $$\begin{aligned}\int_{L_T} G(s)\,ds &= \int_{L_T}\frac{1}{s+2}ds \overset{\text{say } s=it}{=} \int_{t=-T}^{t=T}\frac{1}{it+2}d(it)=i\int_{-T}^{T}\frac{1}{it+2}dt=\left[\operatorname{Log}(it+2)\right]_{-T}^{T}= \\ &= \operatorname{Log}(iT+2)-\operatorname{Log}(-iT+2)=\operatorname{Log}\frac{iT+2}{-iT+2}\end{aligned}$$

  and $\lim_{T\to\infty}\operatorname{Log}\dfrac{iT+2}{-iT+2}=\operatorname{Log}(-1)=\pi i$, so that

  $$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{L_T}\frac{e^{st}}{s+2}ds=\frac{1}{2} \text{ for } t=0.$$

- If we combine the results found in each of these 3 situations, we see that:

  $$g(t)=\lim_{T\to\infty}\frac{1}{2\pi i}\int_{L_T}\frac{e^{st}}{s+2}ds=\begin{cases}0 & \text{for } t<0\\ \frac{1}{2} & \text{for } t=0\\ e^{-2t} & \text{for } t>0\end{cases}$$

(c) Notice that the functions $f(t)$ and $g(t)$ aren't completely the same.
To be honest, this is to be expected.
There are an infinite number of functions $f(t)$ that have $\dfrac{1}{s+2}$ as their Laplace transform, but only one of them is called the inverse Laplace transform.

13. (a) $\dfrac{s}{s^2+1}$, for $\operatorname{Re} s > 0$

(b) $g(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{2} & \text{for } t = 0 \\ \cos t & \text{for } t > 0 \end{cases}$

14. (a) $\frac{\pi}{3e^3}(\cos 1 - 3\sin 1)$

(b) $\frac{\pi}{2e^4}(2\cos 2 + \sin 2)$

(c) $\frac{\pi e^{-ab}}{b}$

(d) πe^{-ab}

(e) $(1 - e^{-a})\frac{\pi}{2}$

15. (b) (*i*) $-i\pi$

(*ii*) $\pi i(e^{2mi} - e^{mi})$

(d) $\frac{3\pi}{8}$