Linear Algebra 2

Assignments 7: Jordan Canonical Form, Inner Product Spaces, Normed Vector Spaces, and the Cauchy-Schwarz Inequality (§4.9, 6.1-6.2)

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Assignment 7.1

The matrix

$$A = \begin{pmatrix} -2 & 0 & 0 & -1 \\ -1 & -2 & 0 & 0 \\ -1 & 1 & -3 & 0 \\ 3 & -1 & 0 & -5 \end{pmatrix}$$

has the eigenvalue $\lambda = -3$ *with* $m_a(\lambda) = 4$.

(a) *Give a basis of the eigenspace* E_λ *and show that* A *is not diagonalizable.*

(b) *Compute a basis of power vectors that puts* A *into Jordan canonical form.*

Assignment 7.2

Let $V = \mathbb{R}_2[t]$ *and define the real-valued function*

$$\langle a_0 + a_1 t + a_2 t^2 | b_0 + b_1 t + b_2 t^2 \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$$

for any pair of polynomials of order up to 2.

(a) *Show that* $\langle \cdot | \cdot \rangle$ *defines a real inner product.*

(b) *Compute* $\langle -3 + t + t^2 | 5 - 2t + 3t^2 \rangle$ *using the above inner product.*

(c) *Compute the coordinates of* $\boldsymbol{p}(t) = -3 + t + t^2$ *and* $\boldsymbol{q}(t) = 5 - 2t + 3t^2$ *with respect to the standard basis* $\mathcal{E}$ *of* $\mathbb{R}_2[t]$ *and compute the standard inner product defined on* $\mathbb{R}^3$ *of the coordinates* $[\boldsymbol{p}(t)]_{\mathcal{E}}$ *and* $[\boldsymbol{q}(t)]_{\mathcal{E}}$*.*

(d) *Compute the coordinates of* $\boldsymbol{p}(t)$ *and* $\boldsymbol{q}(t)$ *with respect to the basis* $\mathcal{B} = (t^2 - t + 1, t + 1, t^2 + 1)$.

(e) *Define an inner product* $\langle \cdot | \cdot \rangle_{\mathcal{B}} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ *on* $\mathbb{R}^3$ *such that* $\langle [\boldsymbol{p}(t)]_{\mathcal{B}} | [\boldsymbol{q}(t)]_{\mathcal{B}} \rangle_{\mathcal{B}} = \langle [\boldsymbol{p}(t)]_{\mathcal{E}} | [\boldsymbol{q}(t)]_{\mathcal{E}} \rangle$.

Assignment 7.3 (for submission)

Let V be a real inner product space with norm $\|\cdot\|$ induced by the inner product $\langle\cdot|\cdot\rangle$. Show the following properties:

(a)

$$\langle \boldsymbol{x}+\boldsymbol{y}|\boldsymbol{x}-\boldsymbol{y}\rangle = \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2$$

(b) *Generalized Pythagoras' theorem / law of cosine*

$$\begin{aligned}\|\boldsymbol{x}-\boldsymbol{y}\|^2 &= \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2\langle \boldsymbol{x}|\boldsymbol{y}\rangle \\ &= \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2\|\boldsymbol{x}\| \cdot \|\boldsymbol{y}\| \cos\theta\end{aligned}$$

(c)

$$\|\boldsymbol{x}+\boldsymbol{y}\|^2 - \|\boldsymbol{x}-\boldsymbol{y}\|^2 = 4\langle \boldsymbol{x}|\boldsymbol{y}\rangle$$

(d) *Parallelogram equality*

$$\|\boldsymbol{x}+\boldsymbol{y}\|^2 + \|\boldsymbol{x}-\boldsymbol{y}\|^2 = 2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2$$

Assignment 7.4 (for submission)

Let $V = M_{n,n}(\mathbb{C})$, *and define*

$$\langle A|B\rangle_F = \mathrm{tr}\left(A^\dagger B\right) \quad \textit{for} \quad A, B \in V,$$

where $A^\dagger$ *is the* ***conjugate transpose*** *or* ***adjoint*** *of matrix* A, *that is,*

$$(A^*)_{ij} = \overline{A_{ji}}.$$

(a) *Show that* V *together with the so-define* ***Frobenius inner product*** $\langle\cdot|\cdot\rangle_F$ *defines a complex inner product space.*

(b) *Compute the Frobenius inner products* $\langle A|B\rangle_F$ *and* $\langle B|A\rangle_F$ *of the matrices*

$$A = \begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix}, \quad B = \begin{pmatrix} 1+i & 0 \\ i & -i \end{pmatrix}.$$

Assignment 7.4 (contd.)

(c) *Compute the Frobenius norm* $\|A\|_F$ *and* $\|B\|_F$*, which is induced by the Frobenius inner product.*

Assignment 7.5 (for submission)

Let V *be a complex inner product space. Show that the Cauchy-Schwarz inequality*

$$|\langle \boldsymbol{x}|\boldsymbol{y}\rangle| \leq \|\boldsymbol{x}\| \cdot \|\boldsymbol{y}\|,$$

with equality if and only if $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are linearly dependent also holds in this case.*

Assignment 7.6

Let $\langle\cdot|\cdot\rangle_1$ *and* $\langle\cdot|\cdot\rangle_2$ *two complex inner products on the complex vector space* V*. Show that*

$$\langle\cdot|\cdot\rangle = \langle\cdot|\cdot\rangle_1 + \langle\cdot|\cdot\rangle_2$$

is also a complex inner product.

Assignment 7.7

Exercise (8) from page 151 of the book:

Let V be a real inner product space and let V_c be its complexification. If $\langle\cdot|\cdot\rangle$ is an inner product on V, we define a form $\langle\cdot|\cdot\rangle_c$ on V_c be the formula

$$\langle \boldsymbol{x}|\boldsymbol{y}\rangle_c = \langle \boldsymbol{x}_R|\boldsymbol{y}_R\rangle + \langle \boldsymbol{x}_I|\boldsymbol{y}_I\rangle + i(\langle \boldsymbol{x}_R|\boldsymbol{y}_I\rangle - \langle \boldsymbol{x}_I|\boldsymbol{y}_R\rangle),$$

where $\boldsymbol{x}_R$ and $\boldsymbol{x}_I$ are the real and imaginary parts of $\boldsymbol{x}$, and similarly with $\boldsymbol{y}$. Show that V_c is a complex inner product space.

Assignment 7.8

Exercise (9) from page 152 of the book:

Let V be Euclidean $\mathbb{R}^n$. Show that the procedure of the previous exercise yields $\mathbb{C}^n$ with $\langle\cdot\cdot_c$ being the standard complex inner product

$$\langle \boldsymbol{x}|\boldsymbol{y}\rangle = \bar{\boldsymbol{x}}^\top \boldsymbol{y} = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n.$$