Extra exercises for chapter 1

Complex Analysis (EE2M11-2021v1)

In a certain sense, the course Complex Analysis is the sequel to the course Analysis in the first year. If you notice your knowledge is lacking in some areas, use Stewart's book to remind yourself of the details.
Complex Analysis, like most mathematics courses, can only be learned through the combination of understanding and practicing the material. These concepts strengthen each other: through practice we can understand better, and through understanding we can practice easier. This is why we present the following exercises. They are pulled from the book of Brown and Churchill, other books, or the professors themselves.

1. Compute (in other words, rewrite into the form $x + iy$):
(a) i^{17};
(b) $\dfrac{5}{-3+4i}$;
(c) $(1+2i)^3$;
(d) $\left(-1+i\sqrt{3}\right)^4$;
(e) e^{3+2i}.

2. Compute $\operatorname{Re} z$, $\operatorname{Im} z$, $\bar{z}$, $z\bar{z}$, $|z|$ and $\operatorname{Arg} z$ for $z =$
(a) $3 - 4i$
(b) $\dfrac{1}{1+i}$
(c) $(3-i)(-2+4i)$.
(d) $\dfrac{i}{-2-2i}$
(e) $(\sqrt{3}-i)^6$
(f) $-i$

3. Compute the modulus of $\dfrac{(3+4i)(1+i)^6}{i^5(2+4i)^2}$.

4. Write the following complex numbers into the form $x + iy$:
(a) $(1+i)^n + (1-i)^n$ where n is a nonzero natural number.
(b) $\dfrac{i}{1+i} + \dfrac{1+i}{i}$.

5. In the complex plane, sketch the set of points z that satisfy:
(a) $|z+1+i| > |z+2|$
(b) $|\mathrm{Im} z| + |\mathrm{Re} z| \leq 1$
(c) $|z+1| > |z-1|$
(d) $|z| \geq 1$ and $|\mathrm{Arg} z| \leq \frac{\pi}{4}$.

6. Sketch all points z that satisfy $\operatorname{Im}\left[(1-i)\, z\right] \leq 0$.

7. Give the exact value of $\operatorname{Arg}\left(e^{-1+7i}\right)$ [from the January 22, 2010 exam].

8. Solve for z where $z^2 + 10z + 30 = 0$.

9. Solve for $\theta \in \mathbb{R}$ where
(a) $e^{i\theta} = -\dfrac{1}{2} - \dfrac{1}{2} i\sqrt{3}$.
(b) $e^{i\theta} = -1 - i\sqrt{3}$.

10. Solve for z where $e^z = 1 + i$.

11. Solve for z where

(a) $z^2 = 2 + 2i\sqrt{3}$ (two possible methods).

(b) $z^4 = \dfrac{1-i}{1+i}$

12. Solve for z where $z^2 = 2\bar{z}$.

13. For complex numbers z_1 and z_2, we know that $z_1 + z_2$ and $z_1 z_2$ are both real. This is of course true if z_1 and z_2 are both real. Are there any other possibilities?

14. Which complex numbers have an imaginary part that is purely imaginary?

15. Show that for every $z \in \mathbb{C}$ and $w \in \mathbb{C}$: $|zw| = |z|\,|w|$.

16. Show that for every $z \in \mathbb{C}$ and $w \in \mathbb{C}$: $\overline{zw} = \bar{z}\,\bar{w}$.

And a little more challenging for the student that wants more:

17. (a) Write in standard form: $\displaystyle\sum_{n=1}^{7} \left(\frac{1-i}{\sqrt{2}}\right)^n$.

(b) Find the modulus and an argument for $\dfrac{1+ia}{1-ia}$ where a is a real number.

18. Assume $a, b \in \mathbb{C}$ with $b \neq 0$. Describe the set of all complex numbers z with the property that $\operatorname{Im}\left(\dfrac{z-a}{b}\right) = 0$ geometrically.

19. Let $z = x + iy$ be such that $z^2 = a + bi$ where a and b are given real numbers. Find x and y in terms of a and b, without using arguments, sines and cosines. If you want to get a feeling for the way how to find this formula, you may solve it first for $a = -5$ and $b = 12$.

20. At the end of this course we reach the following question: Find the real numbers I_1 and I_2 (actually integrals) such that

$$(1 - e^{\frac{2}{3}\pi i})I_1 - \frac{2}{3}\pi i e^{\frac{2}{3}\pi i} I_2 = \frac{2}{9}\pi^2 e^{\frac{1}{3}\pi i}.$$

Now you are already able to calculate I_1 and I_2. Do that.

21. Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a polynomial (which means that all coefficients a_i are in $\mathbb{C}$ and that n is a nonnegative integer). A real or complex number z_0 is a root or zero of p if $p(z_0) = 0$.
Prove: If all coefficients are real, then z_0 is a root of p if and only if $\overline{z_0}$ is a root of p. In other words, if the polynomial p has only real coefficients, then the non real roots of p occur as pairs of complex conjugates.

22. Let $H = \{z \mid \operatorname{Im} z > 0\}$ be the upper half plane of $\mathbb{C}$. Show: $z \in H \iff -1/z \in H$.

23. Which of the following subsets of $\mathbb{C}$ are domains?

(a) $\{z \in \mathbb{C} : |z^2 - 3| < 1\}$

(b) $\{z \in \mathbb{C} : |z^2 - 1| < 3\}$

(c) $\{z \in \mathbb{C} : |z^2 - 1| < 1\}$

(d) $\{z \in \mathbb{C} : \big||z|^2 - 2\big| < 1\}$

(e) $\{z \in \mathbb{C} : z + |z| \neq 0\}$

Exercises of Brown Churchill:

B-C 5.2. Verify inequalities (3), Sec. 4, involving $\operatorname{Re} z$, $\operatorname{Im} z$, and $|z|$, i.e. $\operatorname{Re} z \leq |\operatorname{Re} z| \leq |z|$ and $\operatorname{Im} z \leq |\operatorname{Im} z| \leq |z|$

B-C 5.4. Verify that $\sqrt{2}|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$. Suggestion: Reduce this inequality to $(|x| - |y|)^2 \geq 0$.

B-C 5.5. In each case, sketch the set of points determined by the given condition:
(a) $|z - 1 + i| = 1$;
(b) $|z + i| \leq 3$;
(c) $|z - 4i| \geq 4$.

B-C 9.2. Show that
(a) $|e^{i\theta}| = 1$;
(b) $\overline{e^{i\theta}} = e^{-i\theta}$.

B-C 9.10. Use de Moivre's formula (Sec. 8) to derive the following trigonometric identities:
(a) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$;
(b) $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

B-C 11.2. Find the three cube roots c_k $(k = 0, 1, 2)$ of $-8i$, express them in rectangular coordinates, and draw them in the complex plane.

B-C 11.3. Find $(-8 - 8\sqrt{3}i)^{1/4}$, express the roots in rectangular coordinates, exhibit them as the vertices of a certain square, and point out which is the principal root.

B-C 11.4. In each case, find all of the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root.
(a) $(-1)^{1/3}$;
(b) $8^{1/6}$.

B-C 11.6. Find the four zeros of the polynomial $z^4 + 4$, one of them being $z_0 = \sqrt{2}e^{i\pi/4} = 1 + i$. Then use those zeros to factor $z^4 + 4$ into quadratic factors with real coefficients.

B-C 12.1. Scetch the following sets and determine which are domains:
(a) $|z - 2 + i| \leq 1$;
(b) $|2z + 3| > 4$;
(c) $\operatorname{Im} z > 1$;
(d) $\operatorname{Im} z = 1$;
(e) $0 \leq \arg z \leq \pi/4$ $(z \neq 0)$;
(f) $|z - 4| \geq |z|$.

B-C 12.2. Which sets in exercise B-C 12.1 are neither open nor closed?

B-C 12.3. Which sets in exercise B-C 12.1 are bounded?

B-C 12.4. In each case, sketch the closure of the set:
(a) $-\pi < \arg z < \pi$ $(z \neq 0)$;
(b) $|\operatorname{Re} z| < |z|$;
(c) $\operatorname{Re}\left(\frac{1}{z}\right) \leq \frac{1}{2}$;
(d) $\operatorname{Re}(z^2) > 0$.

B-C 12.5. Let S be the open set consisting of all points z such that $|z| < 1$ or $|z - 2| < 1$. Is S connected?