Answers of the extra exercises for chapter 6
Complex Analysis (EE2M11-2021v1)

1. Zeros are 0 (second order) and 2 (first order).

2. This is a second order zero. Namely, if $f(z) = e^z - z - 1$, then $f(0) = f'(0) = 0$ while $f''(0) = 1$.
Alternatively: $f(z) = e^z - z - 1 = \left(1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots\right) - z - 1 = \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots = z^2\left(\frac{1}{2} + \frac{1}{6}z + \cdots\right)$ which shows that 0 is a zero of order 2.

3. The only zero is -1 (first order)

4. All zeros of $f(z) = \dfrac{\sin z}{z}$ are also zeros of $\sin z$, which are the numbers $k\pi$ $(k \in \mathbb{Z})$, because

$$\sin z = 0 \iff \frac{e^{iz} - e^{-iz}}{2i} = 0 \iff e^{iz} = e^{-iz} \iff iz = -iz + 2k\pi i \iff$$
$$\iff 2iz = 2k\pi i \iff z = k\pi$$

The number 0 is not a zero of f since $f(0) \neq 0$.
The other zeros of $\sin z$, the numbers $k\pi$ $(k \in \mathbb{Z}, k \neq 0)$ are first order zeros for $f(z)$ since

$$f'(z) = \frac{z\cos z - \sin z}{z^2} \implies f'(k\pi) = \frac{k\pi\cos k\pi - \sin k\pi}{(k\pi)^2} = \frac{k\pi(-1)^k - 0}{(k\pi)^2} = \frac{(-1)^k}{k\pi} \neq 0.$$

5. (a) This is an entire function, so it does not have any singularities.

(b) First, notice that $e^z = 1 \iff z = 2k\pi i$ $(k \in \mathbb{Z})$ and that all numbers $2k\pi i$ are first order zeros of $e^z - 1$, namely because $\left[\dfrac{d}{dz}(e^z - 1)\right]_{z=2k\pi i} = 1 \neq 0$.
Now rewrite $f(z)$ as

$$\frac{1 + z - e^z}{z(e^z - 1)}.$$

We then see that the numbers $2k\pi i$ $(k \neq 0)$ are first order poles.
Now we determine whether 0 is a singularity or not:

$$\begin{aligned}
\lim_{z\to 0} f(z) &= \lim_{z\to 0} \frac{1 + z - e^z}{z(e^z - 1)} \\
&= \lim_{z\to 0} \frac{1 + z - \left(1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots\right)}{z\left(z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots\right)} \\
&= \lim_{z\to 0} \frac{-\frac{1}{2}z^2 - \frac{1}{6}z^3 + \cdots}{z^2 + \frac{1}{2}z^3 + \cdots} \\
&= \lim_{z\to 0} \frac{-\frac{1}{2} - \frac{1}{6}z + \cdots}{1 + \frac{1}{2}z + \cdots} \\
&= -\frac{1}{2}
\end{aligned}$$

We see that 0 is a removable singularity of f.

(c) $z = 0$, essential.

6. (a) Essential.

(b) Pole of order 6.

7. $f(z) = \dfrac{1}{\sin\dfrac{\pi}{z}}$. First order poles of f are: $\dfrac{1}{k}$ with $k \in \mathbb{Z}, k \neq 0$.
0 is a singularity of f, and in every deleted neighborhood of 0 you can find a singularity (infinitely many, actually) of f, so 0 is a non-isolated singularity.

8. (a) $\dfrac{-1+i\sqrt{3}}{6}$, 1, $-\dfrac{1}{6}$, $\dfrac{1}{24}$.
(b) 0, 2, -2, 0.

9. (a) $\operatorname{Res}(f,i) = -1$, $\operatorname{Res}(f,-i) = -1$, $\operatorname{Res}(f,0) = 2$.
(b) $\operatorname{Res}\left(f, \dfrac{\pm\beta(1\pm i)}{\sqrt{2}}\right) = \mp\dfrac{\sqrt{2}}{8\beta^3}(1\pm i)e^{\alpha\beta(i\mp 1)/\sqrt{2}}$
(c) $\operatorname{Res}(f,k\pi) = (-1)^k\dfrac{1}{k\pi}$ if $k \neq 0$ and whole. $\operatorname{Res}(f,0) = 0$.
(d) $\operatorname{Res}(f,0) = 1$.

10. The Laurent series has been developed over the wrong ring. You must develop the series over the ring $0 < |z-1| < 1$.

12. (a) $10\pi i$.
(b) $-\frac{6}{5}\pi i$.
(c) $\frac{e}{3}\pi i$.
(d) $2\pi i$.
(e) $\dfrac{2\pi i}{(n-1)!}\sin\left(i + (n-1)\dfrac{\pi}{2}\right)$ with positive whole number n.

13. Say $f(z) = \dfrac{1}{z^4+1}$.

(a) $R > 1$, meaning that the poles of f inside the contour C_R are $e^{\frac{\pi i}{4}}$ and $e^{\frac{3\pi i}{4}}$, both first order. We see that

$$\operatorname{Res}\left(f, e^{\frac{\pi i}{4}}\right) = \lim_{z\to e^{\frac{\pi i}{4}}} \frac{z - e^{\frac{\pi i}{4}}}{z^4+1} = \frac{1}{4\left(e^{\frac{\pi i}{4}}\right)^3} = \frac{e^{\frac{5\pi i}{4}}}{4}$$

and

$$\operatorname{Res}\left(f, e^{\frac{3\pi i}{4}}\right) = \lim_{z\to e^{\frac{3\pi i}{4}}} \frac{z - e^{\frac{3\pi i}{4}}}{z^4+1} = \frac{1}{4\left(e^{\frac{3\pi i}{4}}\right)^3} = \frac{e^{-\frac{\pi i}{4}}}{4}.$$

so that according to the Residue Theorem:

$$\begin{aligned}\int\limits_{C_R} \frac{dz}{z^4+1} &= 2\pi i\left(\operatorname{Res}\left(f, e^{\frac{\pi i}{4}}\right) + \operatorname{Res}\left(f, e^{\frac{3\pi i}{4}}\right)\right)\\ &= 2\pi i\left(\frac{e^{\frac{5\pi i}{4}}}{4} + \frac{e^{-\frac{\pi i}{4}}}{4}\right)\\ &= 2\pi i\left(\frac{-2i\sin\frac{\pi}{4}}{4}\right) &= \frac{\pi}{\sqrt{2}}\end{aligned}$$

(b) On Γ_R, we see that $|f(z)| = \dfrac{1}{|z^4+1|} \leq \dfrac{1}{R^4-1}$ so that

$$\left| \int\limits_{\Gamma_R} \frac{dz}{z^4+1} \right| \leq \frac{\pi R}{R^4-1}$$

and this goes to 0 when $R \to \infty$.

(c) We know from (a) that

$$\int\limits_{-R}^{R} \frac{dx}{x^4+1} + \int\limits_{\Gamma_R} \frac{dz}{z^4+1} = \frac{\pi}{\sqrt{2}}.$$

Now we let R go to ∞ and as a result we see that

$$\int\limits_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$$

Because $\dfrac{1}{x^4+1}$ is an even function, we also see that

$$\int\limits_{0}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}.$$

14. Because $a \in D$ is a zero (of order n) of f, $f(z)$ can be rewritten as $(z-a)^n g(z)$ where g is analytic on D and $g(a) \neq 0$. But if g is analytic in a, g must be continuous in a and thus there exists a deleted neighborhood of a where $g(z) \neq 0$. But then in this deleted neighborhood of a, $f(z) \neq 0$.

15. Given that g is analytic in a neighborhood of z_0 and that g has an n-th order zero in z_0, we see that $g(z)$ can be written as $(z-z_0)^n h(z)$ in a neighborhood of z_0, where h is analytic in a neighborhood of z_0 and $h(z_0) \neq 0$.
So $\lim\limits_{z\to z_0} (z-z_0)^n f(z) = \lim\limits_{z\to z_0} \dfrac{(z-z_0)^n}{(z-z_0)^n h(z)} = \lim\limits_{z\to z_0} \dfrac{1}{h(z)} = \dfrac{1}{h(z_0)} \neq 0$ and so z_0 is an n-th order pole of f.

16. (a) $h(z) = (z-z_0)k(z)$ with $k(z_0) \neq 0$ and k an analytic function in a neighborhood of 0.
We then see that $\lim\limits_{z\to z_0} (z-z_0)f(z) = \lim\limits_{z\to z_0} (z-z_0)\dfrac{g(z)}{h(z)} = \lim\limits_{z\to z_0} (z-z_0)\dfrac{g(z)}{(z-z_0)k(z)} = \lim\limits_{z\to z_0} \dfrac{g(z)}{k(z)} = \dfrac{g(z_0)}{k(z_0)} \neq 0$, so z_0 is a first order pole of f.

(b) z_0 is a first order zero of h and thus $h(z_0) = 0$ while $h'(z_0) \neq 0$.
$\mathrm{Res}(f, z_0) = \lim\limits_{z\to z_0} (z-z_0)f(z) = \lim\limits_{z\to z_0} \dfrac{(z-z_0)}{h(z)}g(z) = \lim\limits_{z\to z_0} \dfrac{(z-z_0)}{h(z)-h(z_0)}g(z) = \dfrac{g(z_0)}{h'(z_0)}$.
Alternatively with the help of l 'Hôpital's theorem:
$\mathrm{Res}(f, z_0) = \lim\limits_{z\to z_0} (z-z_0)f(z) = \lim\limits_{z\to z_0} \dfrac{(z-z_0)}{h(z)}g(z) = \lim\limits_{z\to z_0} \dfrac{(z-z_0)}{h(z)} \cdot \lim\limits_{z\to z_0} g(z) = \lim\limits_{z\to z_0} \dfrac{1}{h'(z)} \cdot \lim\limits_{z\to z_0} g(z) = \dfrac{1}{h'(z_0)} \cdot g(z_0) = \dfrac{g(z_0)}{h'(z_0)}$.

B-C 77.1. (a) 1;
(b) $-1/2$;
(c) 0;
(d) $-1/45$;
(e) $7/6$.

B-C 79.1. (a) essential;
(b) pole of order 1;
(c) removable;
(d) pole of order 1;
(e) pole of order 3.