1. Determine the zeros of $z^3 - 2z^2$ and their orders.

2. 0 is a zero of $e^z - z - 1$. Determine the order of this zero.

3. Determine the zeros of $\dfrac{z^2-1}{z^3-1}$ and their orders.

4. Determine the zeros of $\dfrac{\sin z}{z}$ and their orders.

5. For the following functions, determine the isolated singularities and their types.

(a) $\dfrac{z^2+1}{e^z}$;

(b) $\dfrac{1}{e^z-1} - \dfrac{1}{z}$;

(c) $e^{-z}\cos\left(\dfrac{1}{z}\right)$;

6. For the following functions, determine the type of the given singularity:

(a) $\sin\dfrac{1}{z-1}$ in $z = 1$;

(b) $\dfrac{1}{z^3(e^{z^3}-1)}$ in $z = 0$.

7. Given function $\dfrac{1}{\sin\frac{\pi}{z}}$, determine the singularities of this function, along with their types.

8. Calculate

(a) $\operatorname{Res}\left(\dfrac{1}{z^3-1}, e^{\frac{2\pi i}{3}}\right)$, $\operatorname{Res}\left(e^{\frac{1}{z}}, 0\right)$, $\operatorname{Res}\left(\dfrac{\sin z}{z^4}, 0\right)$, $\operatorname{Res}\left(\dfrac{e^z}{z^5}, 0\right)$.

(b) $\operatorname{Res}\left(\dfrac{e^z-1}{z}, 0\right)$, $\operatorname{Res}\left(\dfrac{z+1}{z^3-z^2}, 1\right)$, $\operatorname{Res}\left(\dfrac{z+1}{z^3-z^2}, 0\right)$, $\operatorname{Res}\left(\dfrac{z+1}{z^3-z^2}, -1\right)$.

9. For the following functions, determine all singularities and their residues.

(a) $\dfrac{z^2-1}{z^3(z^2+1)}$;

(b) $\dfrac{e^{i\alpha z}}{z^4+\beta^4}$ (α, β real, $\beta \neq 0$);

(c) $\dfrac{1}{z\sin z}$;

(d) $\dfrac{e^{1/z}}{z}$.

10. What is wrong with the following thought process? The function $f(z) = \dfrac{1}{z(z-1)^2}$ has an isolated singularity in $z = 0$. The Laurent series is equal to

$$f(z) = \frac{1}{(z-1)^3} - \frac{1}{(z-1)^4} + \frac{1}{(z-1)^5} - \cdots + \cdots$$

if $|z-1| > 1$. Apparently, $z = 1$ is an essential singularity with a residue of 0.

11. Show that the following integrals are all equal to 0:

(a) $\displaystyle\int_{|z|=1} ze^{2z}\, dz$;

(b) $\displaystyle\int_{|z|=1} \frac{1}{\cos z}\, dz$;

(c) $\displaystyle\int_{|z|=3} \frac{1}{z^2+1}\, dz$;

12. Calculate the following integrals:

(a) $\displaystyle\int_{|z|=2} \frac{z^4+z}{(z-1)^2}\, dz$;

(b) $\displaystyle\int_{|z|=2} \frac{z^3+3z+1}{z^4-5z^2}\, dz$;

(c) $\displaystyle\int_{|z-i|=2} \frac{e^z+z}{(z-1)^4}\, dz$;

(d) $\displaystyle\int_{|z-i|=2} \frac{e^{-z}\sin z}{z^2}\, dz$;

(e) $\displaystyle\int_{|z-i|=2} \frac{\sin z}{(z-i)^n}\, dz$ with positive integer n.

13. Let $R > 1$. Now look at the contour C_R consisting of the real interval $I_R = [-R, R]$ and the half circle $\Gamma_R = \left\{Re^{i\theta}\, |0 \leq \theta \leq \pi\right\}$.

(a) Compute

$$\int_{C_R} \frac{dz}{z^4+1}.$$

Of course, the contour integral is traversed counter-clockwise.

(b) Show that $\displaystyle\lim_{R\to\infty} \int_{\Gamma_R} \frac{dz}{z^4+1} = 0.$

(c) Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$ and $\displaystyle\int_{0}^{\infty} \frac{dx}{x^4+1}$.

14. Let the function f be analytic on a domain D and suppose that a is an n-th order zero of f. Show that a is an isolated zero of f. In other words, there exists a deleted neighborhood of a where $f(z) \neq 0$.

15. The function g is analytic in a neighborhood of z_0 and has an n-th order zero in z_0. Show that the function $f(z) = \dfrac{1}{g(z)}$ has an n-th order pole in z_0.

16. The functions g and h are analytic in a region of z_0, $g(z_0) \neq 0$ and z_0 is a first order zero of h. Now look at the function $f(z) = \dfrac{g(z)}{h(z)}$.

(a) Show that z_0 is a first order pole of f.

(b) Prove that $\operatorname{Res}(f, z_0) = \dfrac{g(z_0)}{h'(z_0)}$.

And a little more challenging for the student that wants more:

17. Suppose z_0 is an isolated singularity of the function $f(z)$. Prove:

(a) z_0 is removable if and only if $\lim_{z\to z_0} f(z)$ exists (so is finite) if and only if $\lim_{z\to z_0}(z - z_0)f(z) = 0$;

(b) z_0 is a pole if and only if $\lim_{z\to z_0} f(z) = \infty$.

(c) z_0 is essential if and only if $\lim_{z\to z_0} f(z) \neq \infty$ and does not exist.

18. Let $f : D \to \mathbb{C}$ be analytic with D a domain. Let a be a singularity of f.

(a) Assume that there is an $r > 0$ such that $|f(z)| \le |z-a|^{-\frac{3}{4}}$ when $0 < |z-a| < r$. Show that a is a removable singularity of f.

(b) Assume that there are an $r > 0$ and positive numbers M and N such that

$$N|z-a|^{-\frac{5}{2}} \le |f(z)| \le M|z-a|^{-\frac{7}{2}}$$

when $0 < |z-a| < r$. Show that a is a pole of f and determine its order.

B-C 77.1. Find the residue at $z = 0$ of the function

(a) $\dfrac{1}{z+z^2}$;

(b) $z\cos\left(\dfrac{1}{z}\right)$;

(c) $\dfrac{z-\sin z}{z}$;

(d) $\dfrac{\cot z}{z^4}$;

(e) $\dfrac{\sinh z}{z^4(1-z^2)}$.

B-C 79.1. In each case, write the principal part of the funtion at its isolated singular point and determine whether that point is a removable singular point, an essential singular point, or a pole:

(a) $z\exp\left(\dfrac{1}{z}\right)$;

(b) $\dfrac{z^2}{1+z}$;

(c) $\dfrac{\sin z}{z}$;

(d) $\dfrac{\cos z}{z}$;

(e) $\dfrac{1}{(2-z)^3}$.