Array Processing
for MIMO communications and radar

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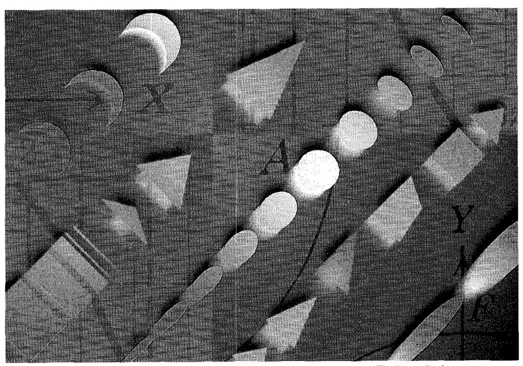
Two Decades of Array Signal Processing Research

The Parametric Approach

HAMID KRIM and MATS VIBERG



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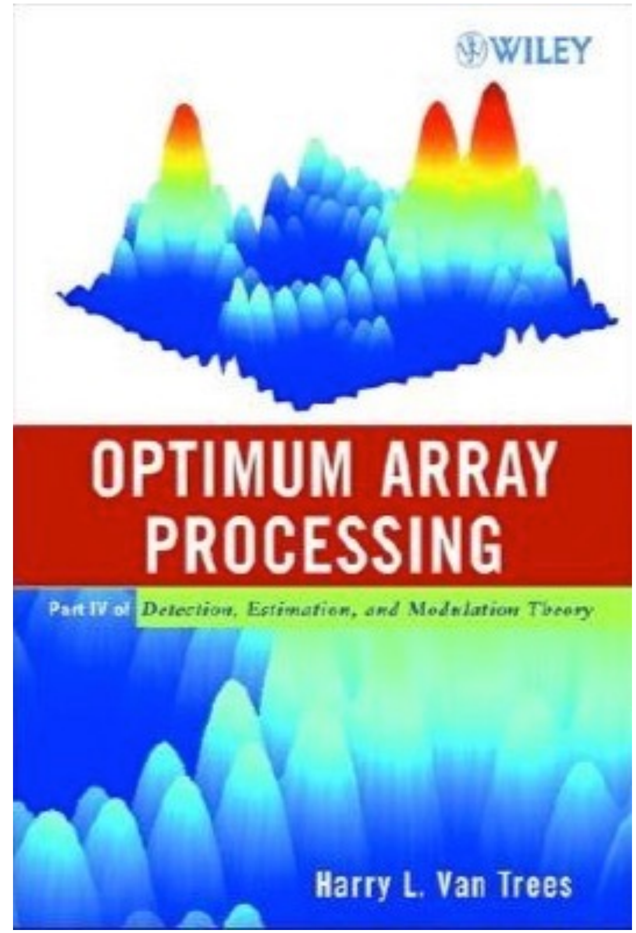
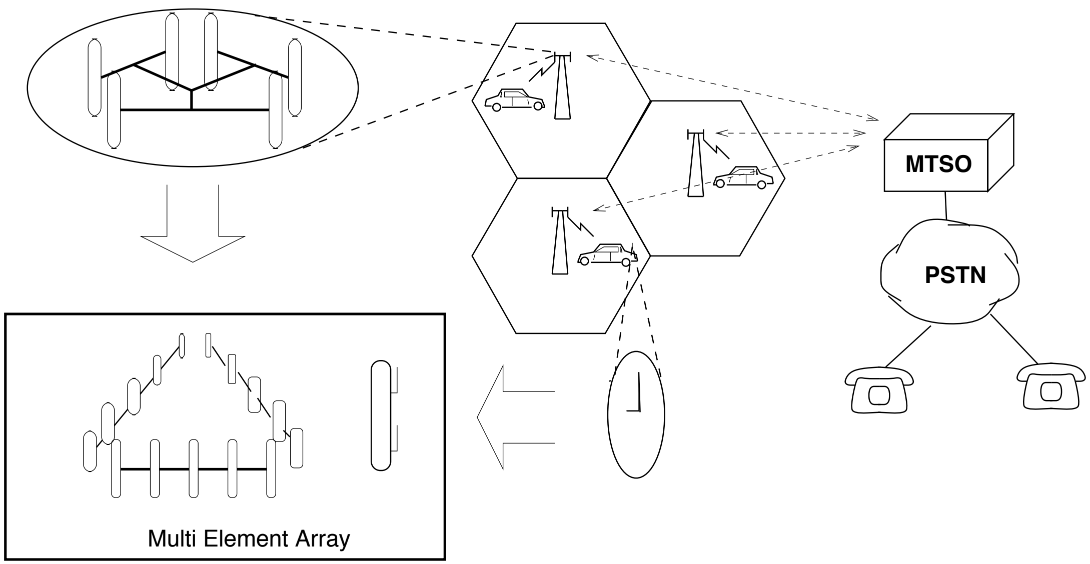


*Slides are based on the reader and from the course ET 4147, TU Delft

Applications – wireless communications

Space-time processing

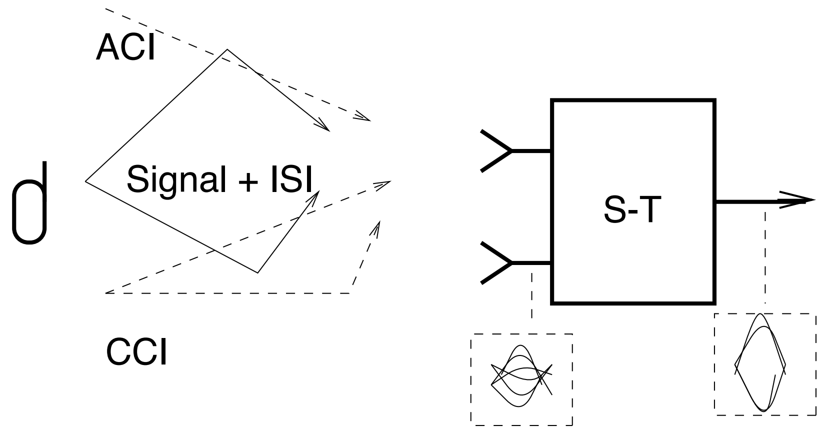
- Use multiple antennas and space-time processing to enhance performance.
- Can provide diversity and interference reduction



Applications – wireless communications

Why Space-Time processing?

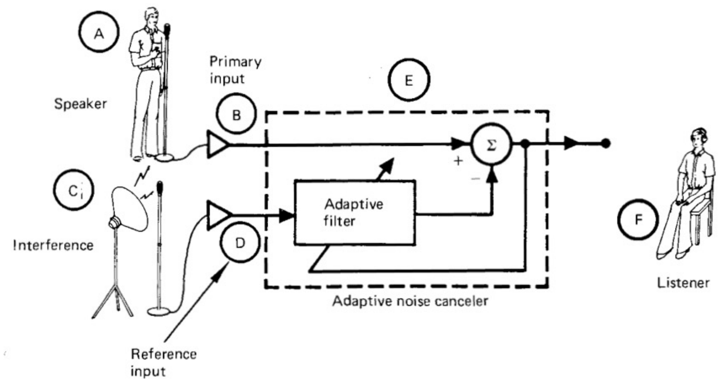
- **Enhance signal** (increase average power and reduce effect of fades)
- **Reduce** adjecent channel, co-channel, and intersymbol **interference**



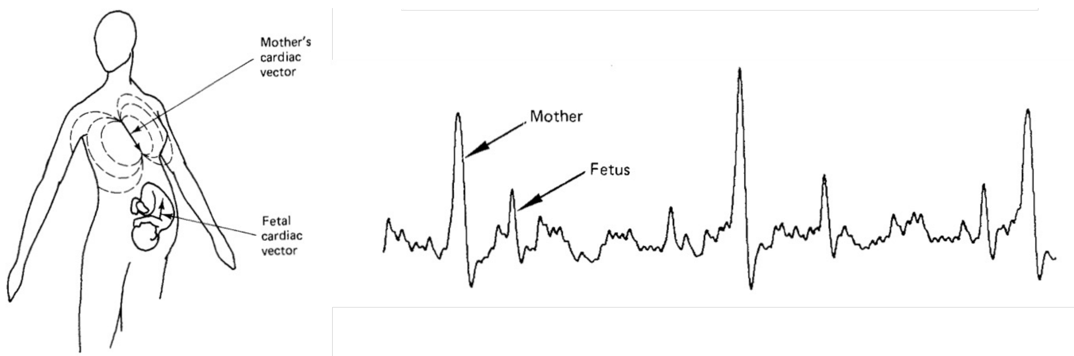
Conclusion: S-T processing promises significant improvements in coverage, capacity, data rate and quality of wireless networks

Applications

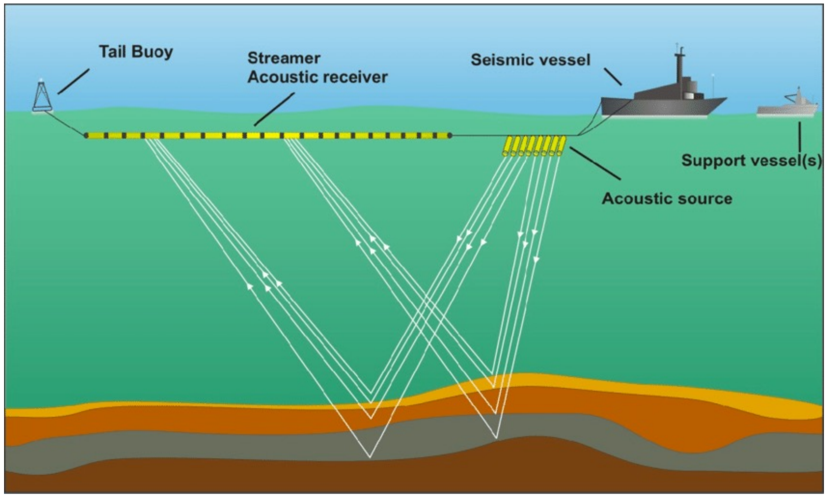
Interference cancellation

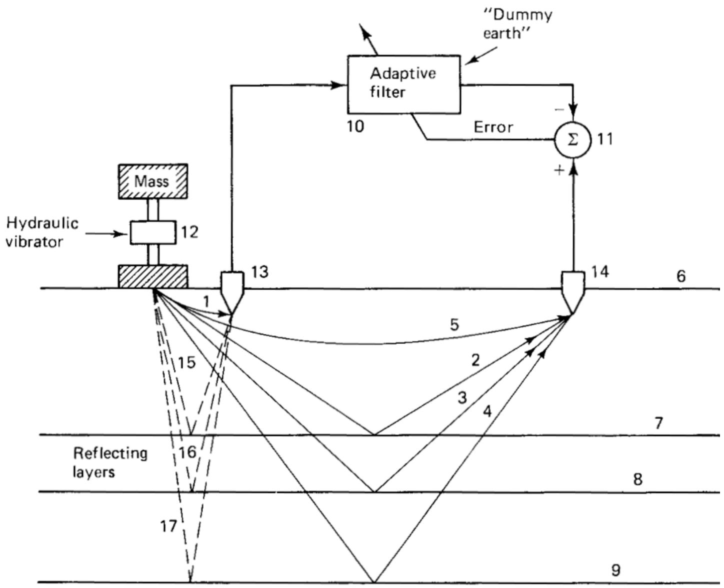


Source separation



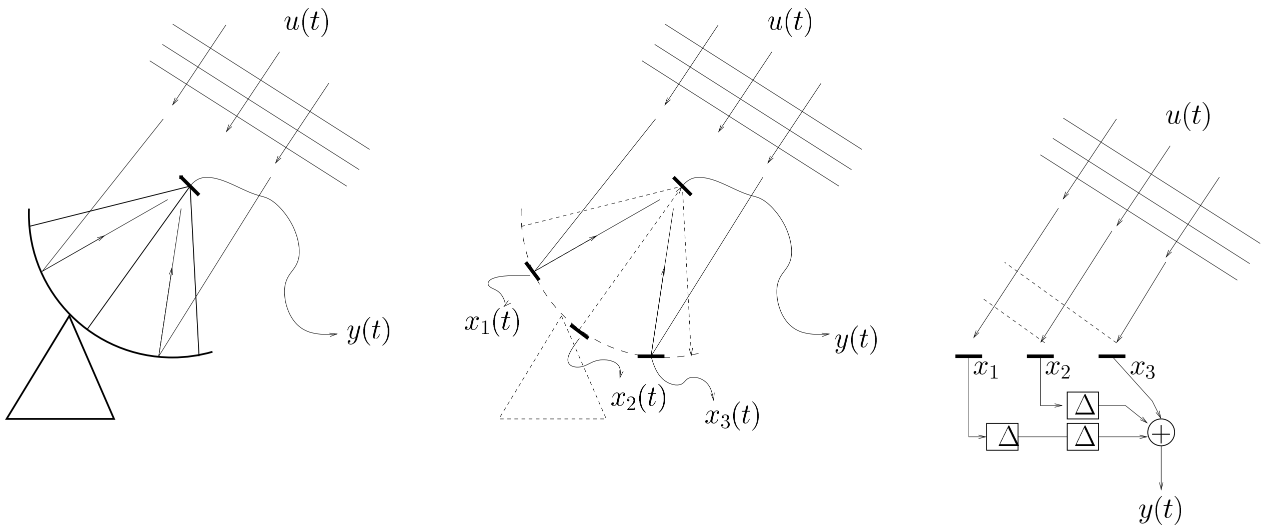
Subsurface imaging





Array processing- introduction

Coherent adding



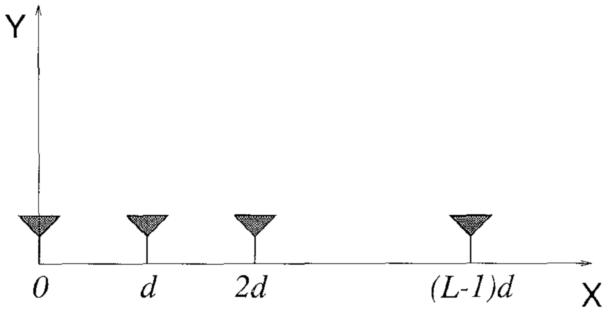
With an array of sensors ($m = 1, \cdots, M$):

$$x_m(t) \;=\; u(t) + n_m(t)\;; \qquad \text{noise variance: } \sigma^2$$

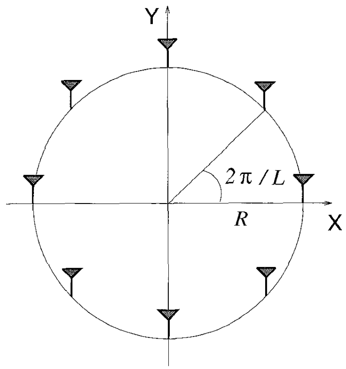
If the noise on the antennas is uncorrelated, then

$$y(t) \;=\; \frac{1}{M}\sum_1^M x_m(t) \;=\; u(t) + \frac{1}{M}\sum_1^M n_m(t) \;=\; u(t) + n(t)\;; \qquad \text{noise variance: } \frac{1}{M}\sigma^2$$

Array configuration



Uniform linear array



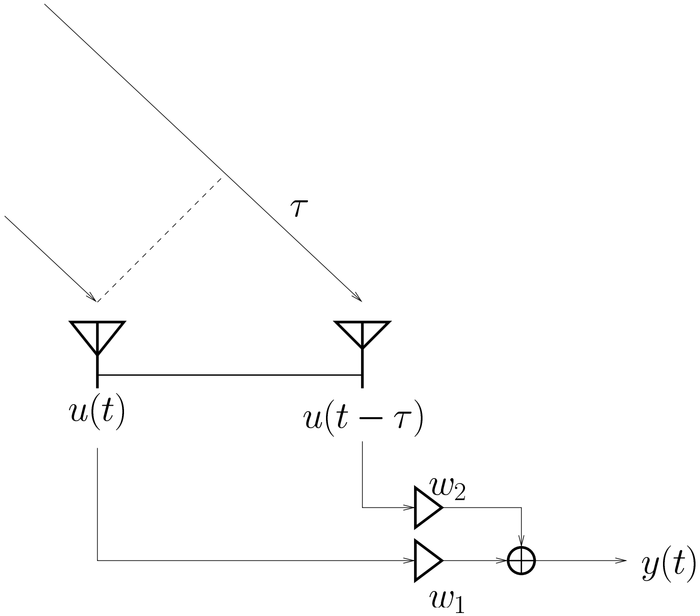
Uniform circular array

Very large array (New Mexico)

LOFAR radio telescope

Null steering

Null-steering



$$
\begin{aligned}
y(t) &= w_1 u(t) + w_2 u(t-\tau) \\
Y(\omega) &= U(\omega)\big(w_1 + w_2 e^{-j\omega\tau}\big)
\end{aligned}
$$

The signal is nulled out, $U(\omega) = 0$, at a certain frequency ω_0 if

$$w_2 = -w_1 e^{j\omega_0\tau}$$

Baseband signal

Baseband signal

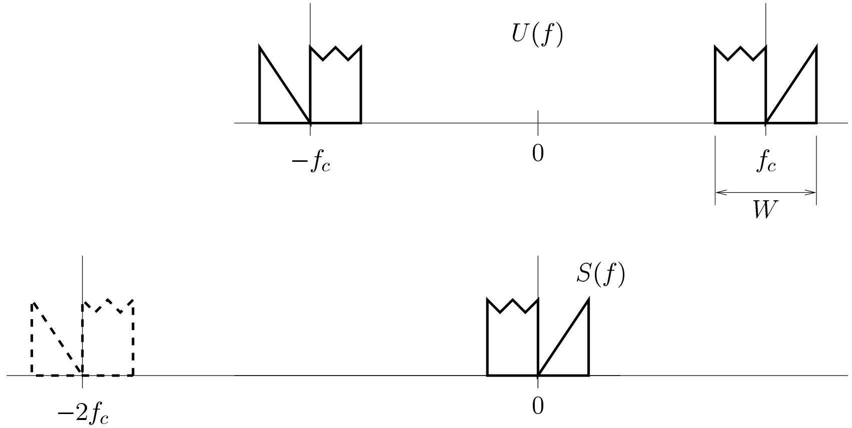
An antenna receives a real-valued bandpass signal with center frequency f_c

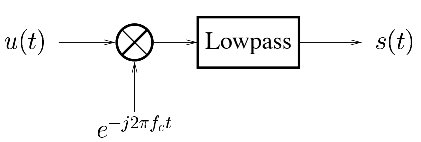
$$u(t) \;=\; \text{real}\{s(t)e^{j2\pi f_c t}\} \;=\; x(t)\cos 2\pi f_c t - y(t)\sin 2\pi f_c t$$

The *baseband signal* or *complex envelope* is

$$s(t) \;=\; x(t) + jy(t)$$

$s(t)$ is recovered from $u(t)$ by *demodulation*:
multiply $u(t)$ with $\cos 2\pi f_c t$ and $\sin 2\pi f_c t$ and low-pass filter the resulting signals





Narrowband assumption

Small delays of narrow band signals

We are interested in the effect of small delays of $u(t)$ on the baseband signal $s(t)$

$$u_\tau(t) \;:=\; u(t-\tau) \;=\; \text{real}\{s(t-\tau)e^{-j2\pi f_c \tau} e^{j2\pi f_c t}\}$$

The complex envelope of the delayed signal is

$$s_\tau(t) = s(t-\tau)e^{-j2\pi f_c \tau}$$

Let W be the bandwidth of $s(t)$. If $\exp(j2\pi f \tau) \approx 1$ for all frequencies $|f| \leq \frac{W}{2}$, then

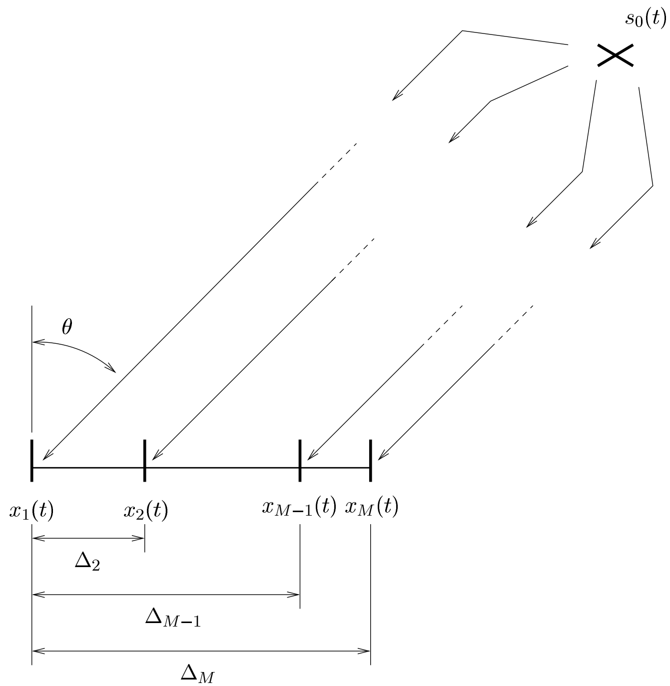
$$s(t-\tau) \;=\; \int_{-W/2}^{W/2} S(f) e^{j2\pi f(t-\tau)} \mathrm{d}f \;\approx\; \int_{-W/2}^{W/2} S(f) e^{j2\pi f t} \mathrm{d}f \quad = s(t)$$

$$\Rightarrow \quad s_\tau(t) \;\approx\; s(t) e^{-j2\pi f_c \tau} \qquad \text{for } W\tau \ll 1$$

Conclusion: for narrowband signals, time delays shorter than the inverse bandwidth amount to phase shifts of the complex envelope.

Array response

Antenna array response



- Far field assumption: planar waves
- θ is the direction of arrival
- A is the attenuation
- T_i is propagation time to i-th element

$$x_i(t) = a(\theta) A s_0(t - T_i) e^{-j2\pi f_c T_i}$$

Define $s(t) = s_0(t - T_1)$, $\tau_i = T_i - T_1$, and $\beta = Ae^{-j2\pi f_c T_1}$, then

$$x_i(t) = a(\theta) \beta s(t - \tau_i) e^{-j2\pi f_c \tau_i}$$

$$2\pi f_c \tau_i = -2\pi f_c \frac{\delta_i \sin(\theta)}{c} = -2\pi \frac{\delta_i}{\lambda} \sin(\theta) = -2\pi \Delta_i \sin(\theta)$$

Antenna array response

Antenna array response

$$\mathbf{x}(t) = \begin{bmatrix} 1 \\ e^{j2\pi\Delta_2 \sin(\theta)} \\ \vdots \\ e^{j2\pi\Delta_M \sin(\theta)} \end{bmatrix} a(\theta)\beta s(t) =: \mathbf{a}(\theta)\beta s(t)$$

$\mathbf{a}(\theta)$ is the **array response vector**

For a uniform linear array, $\Delta_i = (i-1)\Delta$, we obtain

$$\mathbf{a}(\theta) = \begin{bmatrix} 1 \\ e^{j2\pi\Delta \sin(\theta)} \\ \vdots \\ e^{j2\pi(M-1)\Delta \sin(\theta)} \end{bmatrix} a(\theta) = \begin{bmatrix} 1 \\ \phi \\ \vdots \\ \phi^{M-1} \end{bmatrix} a(\theta), \quad \phi = e^{j2\pi\Delta \sin(\theta)}$$

The factor $a(\theta)$ is often ommitted (*omnidirectional* and *normalized* antennas)

Array manifold

The *array manifold* is the curve that the vector $\mathbf{a}(\theta)$ describes when θ is varied:

$$\mathcal{A} = \{\mathbf{a}(\theta) : 0 \le \theta < 2\pi\}$$

- one source

$$\mathbf{x}(t) \;=\; \mathbf{a}(\theta)\beta s(t)$$

For varying $s(t)$, the vector $\mathbf{x}(t)$ is confined to a **line**

➠ θ can be estimated from $\mathbf{x}(t)$ (direction finding)

- two sources

$$\mathbf{x}(t) = \mathbf{a}(\theta_1)\beta_1 s_1(t) \, + \, \mathbf{a}(\theta_2)\beta_2 s_2(t) = [\mathbf{a}(\theta_1) \quad \mathbf{a}(\theta_2)] \begin{bmatrix} \beta_1 & \\ & \beta_2 \end{bmatrix} \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix}$$

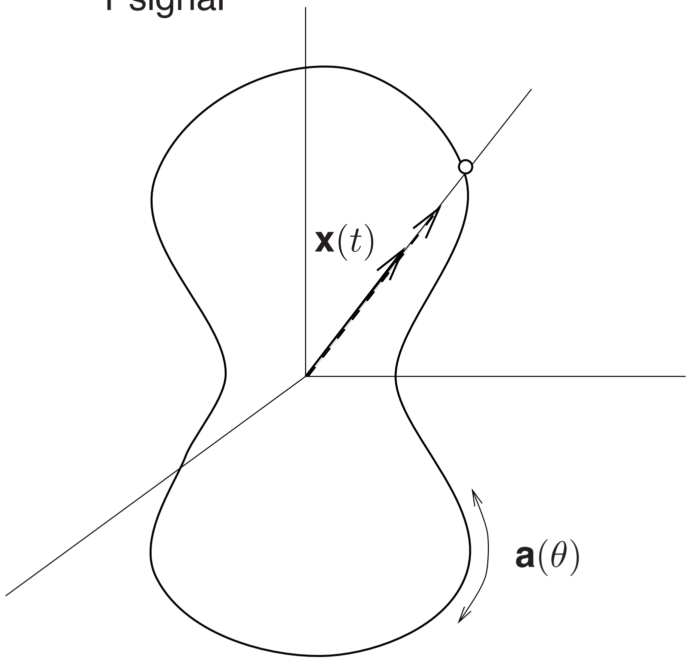
For varying $s(t)$, the vector $\mathbf{x}(t)$ is now confined to a **plane**

➠ θ_1 and θ_2 can be estimated from $\mathbf{x}(t)$ (direction finding)

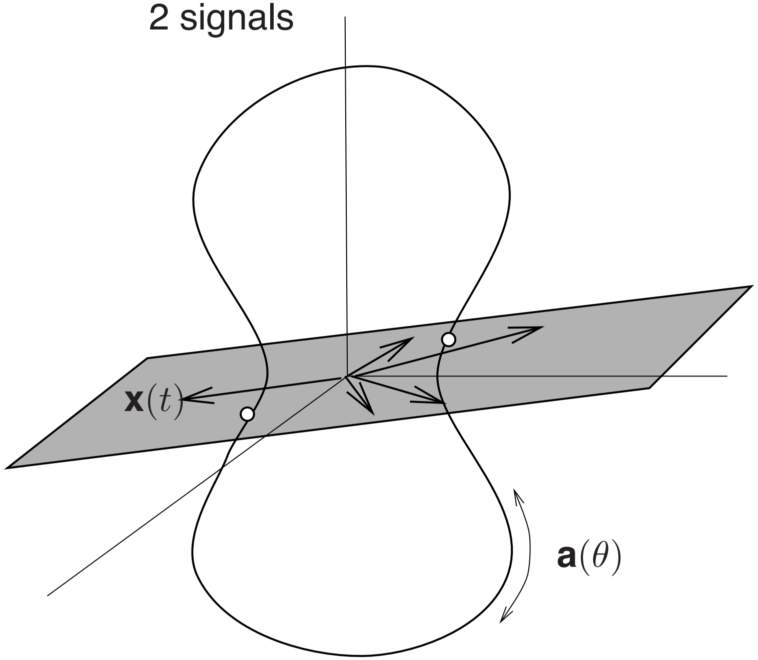
Direction finding

Principle of direction finding

1 signal

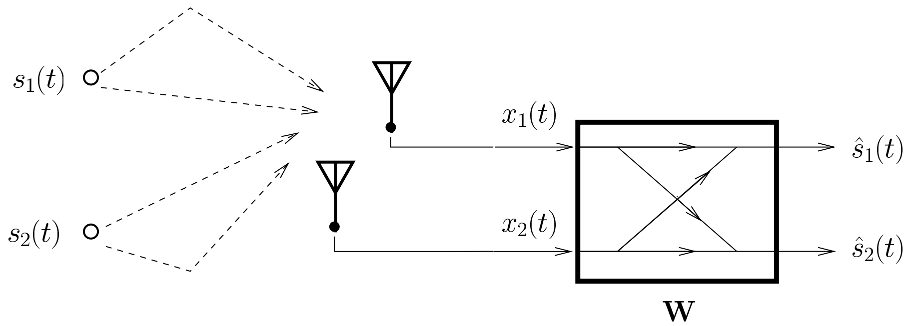


2 signals



Many sources – instantaneous mixtures

Instantaneous mixtures



- For narrowband signals, a delay translates into a phase shift

  ➠ the received data is an **instantaneous linear mixture**:

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t)$$

- Collect N samples: $\mathbf{X} = [\mathbf{x}(0), \cdots, \mathbf{x}(N-1)]$ and $\mathbf{S} = [\mathbf{s}(0), \cdots, \mathbf{s}(N-1)]$

$$\mathbf{X} = \mathbf{A}\mathbf{S}$$

- We look for a beamformer such that

$$\mathbf{W}^{\mathrm{H}}\mathbf{x}(t) = \mathbf{s}(t) \qquad \Leftrightarrow \qquad \mathbf{W}^{\mathrm{H}}\mathbf{A} = \mathbf{I}$$

Array response – beam pattern

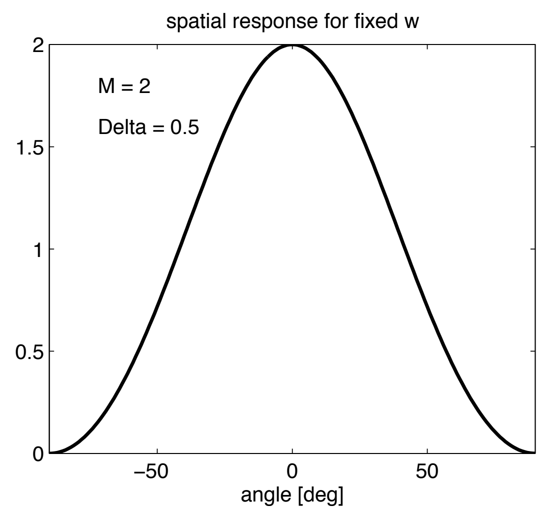
Array response graph

Suppose we choose a beamforming vector **w**, e.g., $\mathbf{w} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$:

$$y(t) \;=\; \mathbf{w}^{\mathrm{H}}\mathbf{x}(t) \;=\; \mathbf{w}^{\mathrm{H}}\mathbf{a}(\theta)\beta s(t)$$

The response of the array to a unit-amplitude signal, $|\beta s(t)| = 1$, from direction θ is

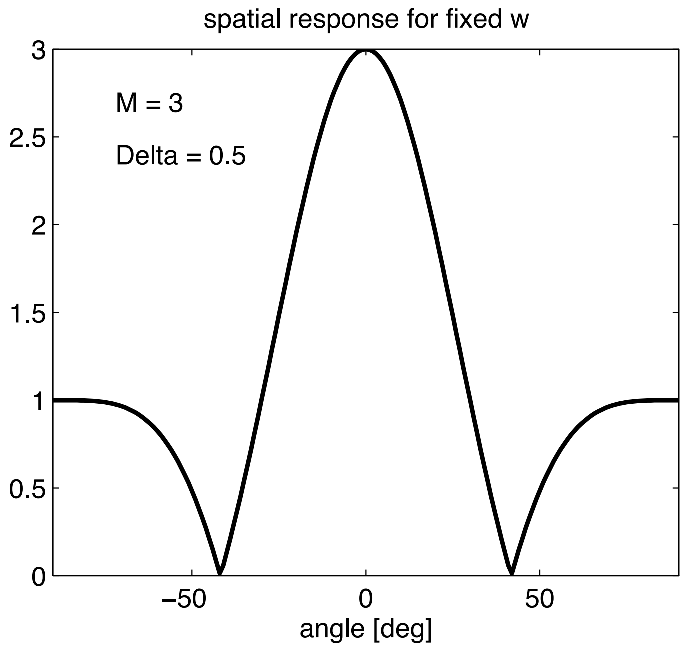
$$|y(t)| = |\mathbf{w}^{\mathrm{H}}\mathbf{a}(\theta)|$$

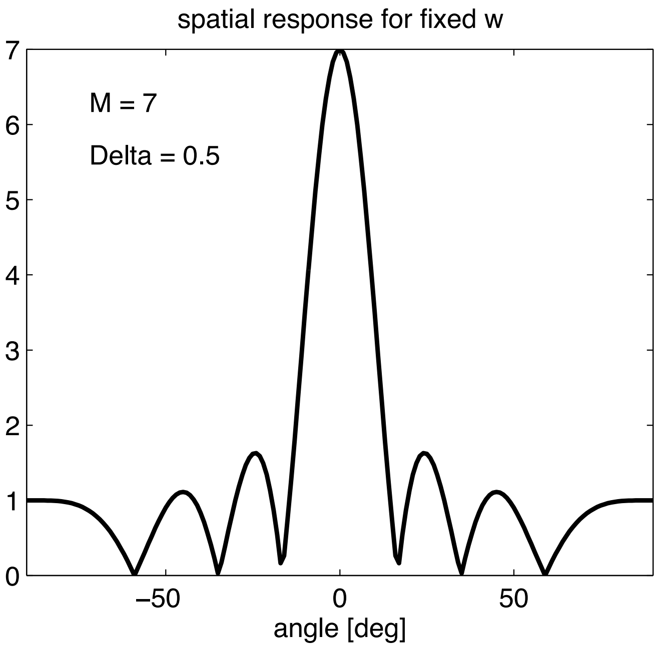


Beampattern

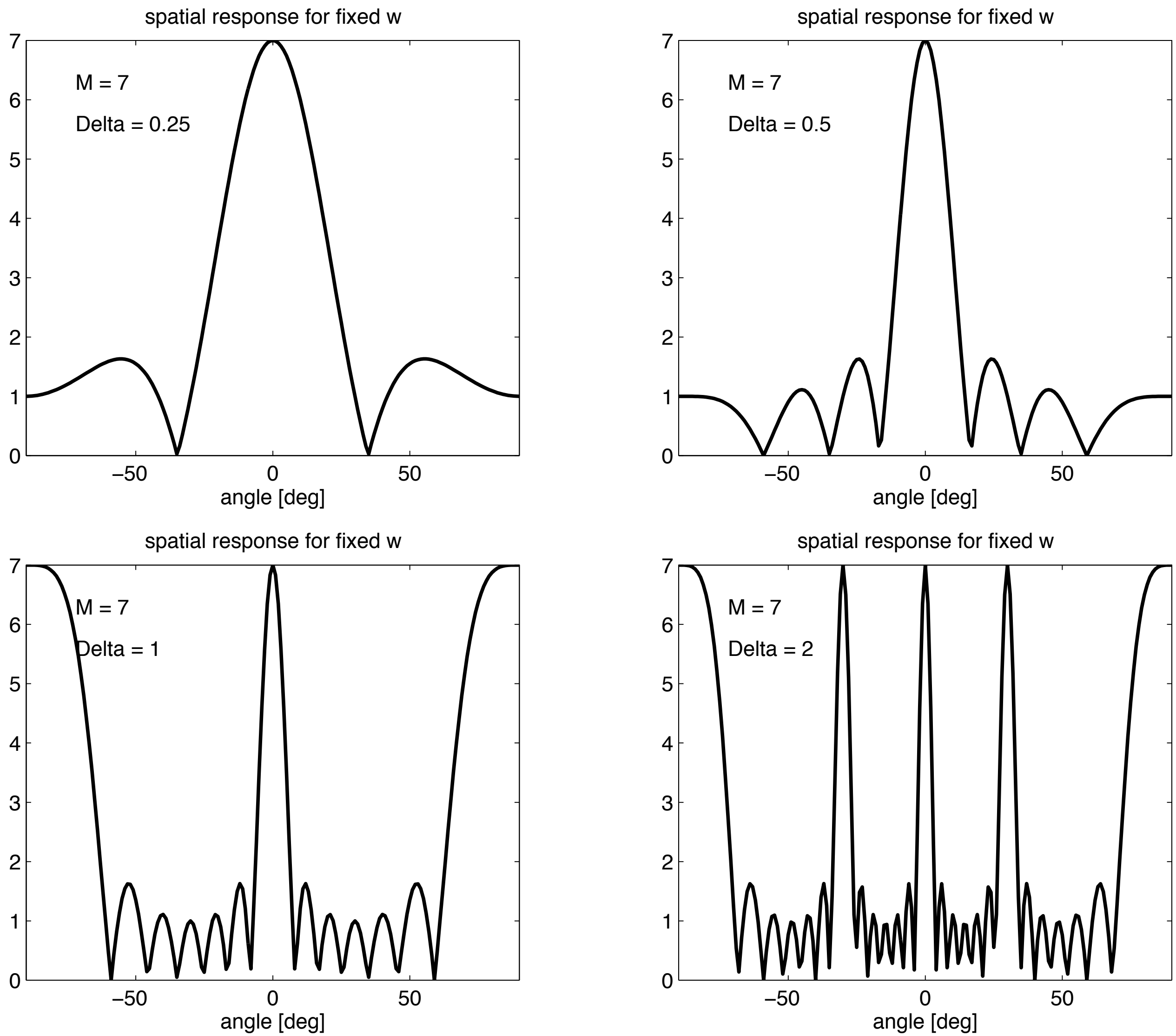
Sidelobes

With a larger number of antennas, resolution improves but sidelobes occur:



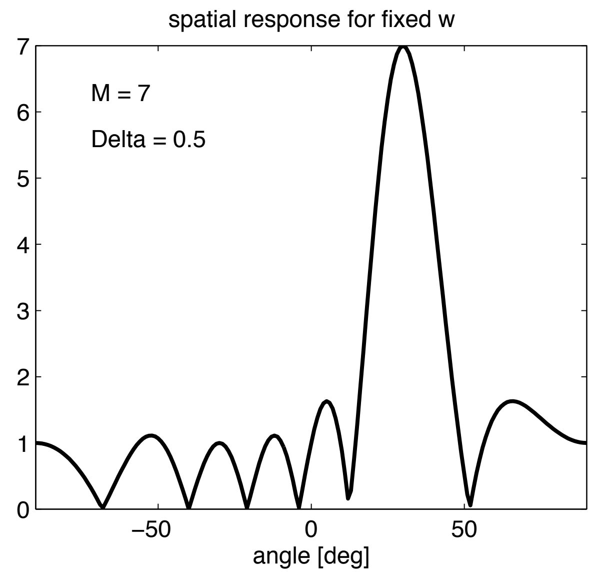


Spatial aliasing



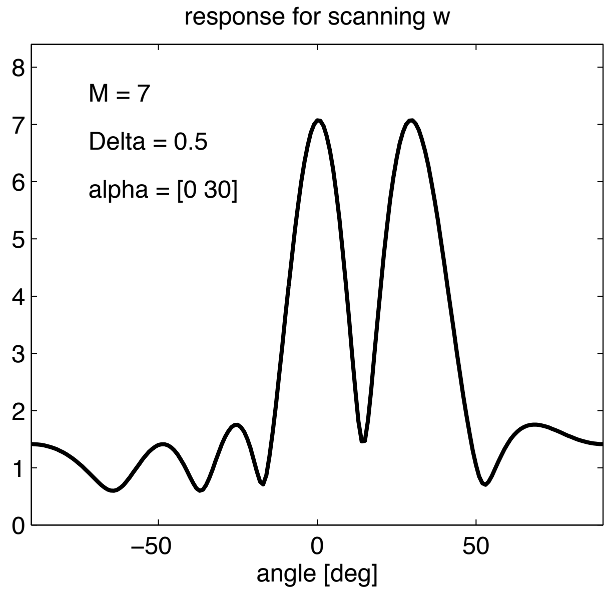
Array response vector

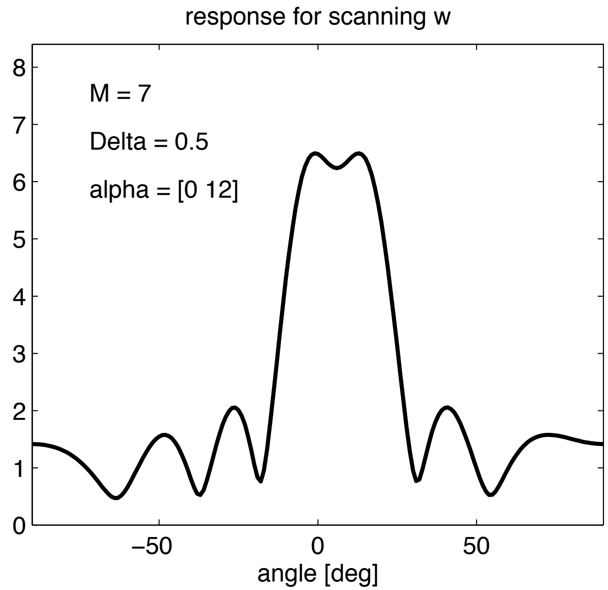
- **w** can be used to steer the beam in other directions
- Choose e.g., $\mathbf{w} = \mathbf{a}(30^\circ)$ and look at $|y(t)| = |\mathbf{w}^{\mathrm{H}}\mathbf{a}(\theta)|$



Spatial spectrum

- To estimate directions of sources, we can scan **w**
- Simple scan would be $\{\mathbf{w} = \mathbf{a}(\theta);\ -\pi/2 \leq \theta \leq \pi/2\}$ and look at $|y(t)| = |\mathbf{w}^{\mathrm{H}}\mathbf{x}(t)|$.
- For a single source, this produces precisely the same plots as before
- If two sources are well separated, they can be resolved





Direction estimation

Model: $\mathbf{x}_k = \mathbf{a}(\theta_0)s_k + \mathbf{n}_k$

Objective: estimate θ_0: *direction finding*

The classical beamformer

- The *classical beamformer* (Bartlett beamformer) is $\mathbf{w} = \mathbf{a}(\theta)$.
- This corresponds to the matched filter assuming spatially white noise.
- Find $\mathbf{w} = \mathbf{a}(\theta)$ that maximizes the output power

$$\hat{\theta}_0 = \max_{\theta} \frac{\mathbf{a}(\theta)^{\mathrm{H}} \mathbf{R}_x \mathbf{a}(\theta)}{\mathbf{a}(\theta)^{\mathrm{H}} \mathbf{a}(\theta)} .$$

  - For finite data, replace $\mathbf{R}_x$ by the sample covariance matrix $\hat{\mathbf{R}}_x$.
  - With known colored noise, replace denominator by $\mathbf{a}(\theta)^{\mathrm{H}} \mathbf{R}_n \mathbf{a}(\theta)$.
  - For multiple signals, choose the d largest local maxima.

  Interference and thus noise color generally not known $\Rightarrow$ *biased estimates*

Direction estimation

MVDR

- In MVDR we try to minimize the output power, while constraining the power towards the direction θ:

$$\hat{\theta}_0 = \max_{\theta}\{\min_{\mathbf{w}} \mathbf{w}^{\mathrm{H}}\hat{\mathbf{R}}_x\mathbf{w} \qquad \text{subject to} \quad \mathbf{w}^{\mathrm{H}}\mathbf{a}(\theta) = 1\}\,.$$

This yields

$$\mathbf{w} = \frac{\hat{\mathbf{R}}_x^{-1}\mathbf{a}(\theta)}{\mathbf{a}(\theta)^{\mathrm{H}}\hat{\mathbf{R}}_x^{-1}\mathbf{a}(\theta)}$$

$$\hat{\theta}_0 = \max_{\theta} \frac{1}{\mathbf{a}(\theta)^{\mathrm{H}}\hat{\mathbf{R}}_x^{-1}\mathbf{a}(\theta)}$$

  - For multiple signals, choose again the d largest local maxima.

Eigenvalue analysis of covariance matrix

EVD of a data matrix

- Suppose we collect a data matrix $\mathbf{X} = \mathbf{A}\mathbf{S}$ and compute its correlation matrix

$$\hat{\mathbf{R}} = \frac{1}{N}\mathbf{X}\mathbf{X}^{\mathrm{H}} = \mathbf{A}(\frac{1}{N}\mathbf{S}\mathbf{S}^{\mathrm{H}})\mathbf{A}^{\mathrm{H}} = \mathbf{A}\hat{\mathbf{R}}_s\mathbf{A}^{\mathrm{H}}$$

- Eigenvalue decomposition: $\hat{\mathbf{R}} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{\mathrm{H}}$

- *Rank property:*

  If the number of sources d is smaller than the number of antennas M

  ➠ $\boldsymbol{\Lambda}$ has d eigenvalues unequal to 0 and $M-d$ equal to zero.

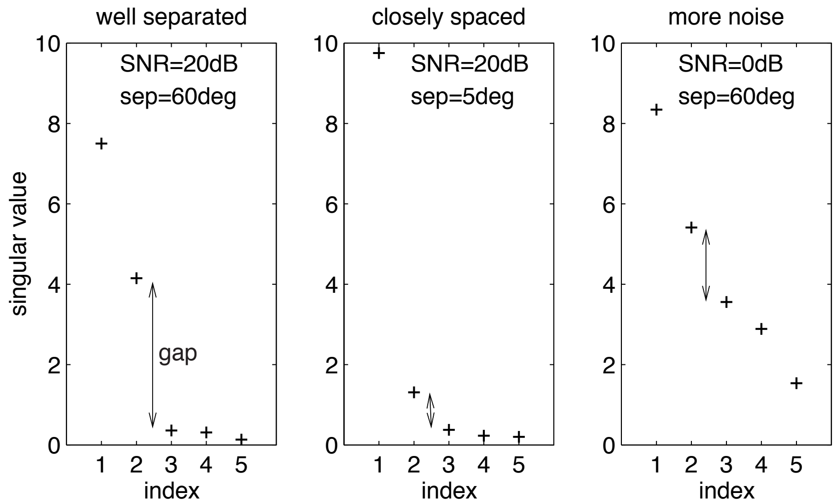
- Add i.i.d. noise: $\mathbf{X} = \mathbf{A}\mathbf{S} + \mathbf{E}$.

$$\begin{aligned} \hat{\mathbf{R}} = \frac{1}{N}\mathbf{X}\mathbf{X}^{\mathrm{H}} &\simeq \mathbf{A}\hat{\mathbf{R}}_s\mathbf{A}^{\mathrm{H}} + \hat{\mathbf{R}}_e \\ &\simeq \mathbf{U}\,\boldsymbol{\Lambda}\,\mathbf{U}^{\mathrm{H}} + \sigma^2\mathbf{I} \\ &= \mathbf{U}(\boldsymbol{\Lambda} + \sigma^2\mathbf{I})\mathbf{U}^{\mathrm{H}} \end{aligned}$$

  All eigenvalues are raised by σ^2, but the eigenvectors stay the same.

Eigenvalue analysis of covariance matrix

SVD of a data matrix



$$\mathbf{X} = \mathbf{AS} + \mathbf{E}, \qquad \mathbf{A} = [\mathbf{a}(\theta_1) \quad \mathbf{a}(\theta_2)]$$

Singular values of **X** for $d = 2$ sources, $M = 5$ antennas, $N = 10$ samples.

(a) Well separated case: large gap between signal and noise singular values,

(b) signals from close directions results in a small signal singular value,

(c) increased noise level increases noise singular values.

Direction estimation

MUSIC (Multiple SIgnal Classification) algorithm

- Eigenvalue-based technique (assume $d < M$):

$$\mathbf{R}_x = \mathbf{A}\mathbf{R}_s\mathbf{A}^{\mathrm{H}} + \sigma^2\mathbf{I}_M = \mathbf{U}_s(\mathbf{\Lambda}_s + \sigma^2\mathbf{I}_d)\mathbf{U}_s^{\mathrm{H}} + \mathbf{U}_n(\sigma^2\mathbf{I}_{M-d})\mathbf{U}_n^{\mathrm{H}}$$

$$\text{span}(\mathbf{U}_s) = \text{span}(\mathbf{A}), \qquad \mathbf{U}_n^{\mathrm{H}}\mathbf{A} = 0, \qquad \text{where } \mathbf{A} = [\mathbf{a}(\theta_1), \cdots, \mathbf{a}(\theta_d)].$$

- Choose $[\theta_1, \cdots, \theta_d]$ to make $\mathbf{A}$ *fit* span($\mathbf{U}_s$):

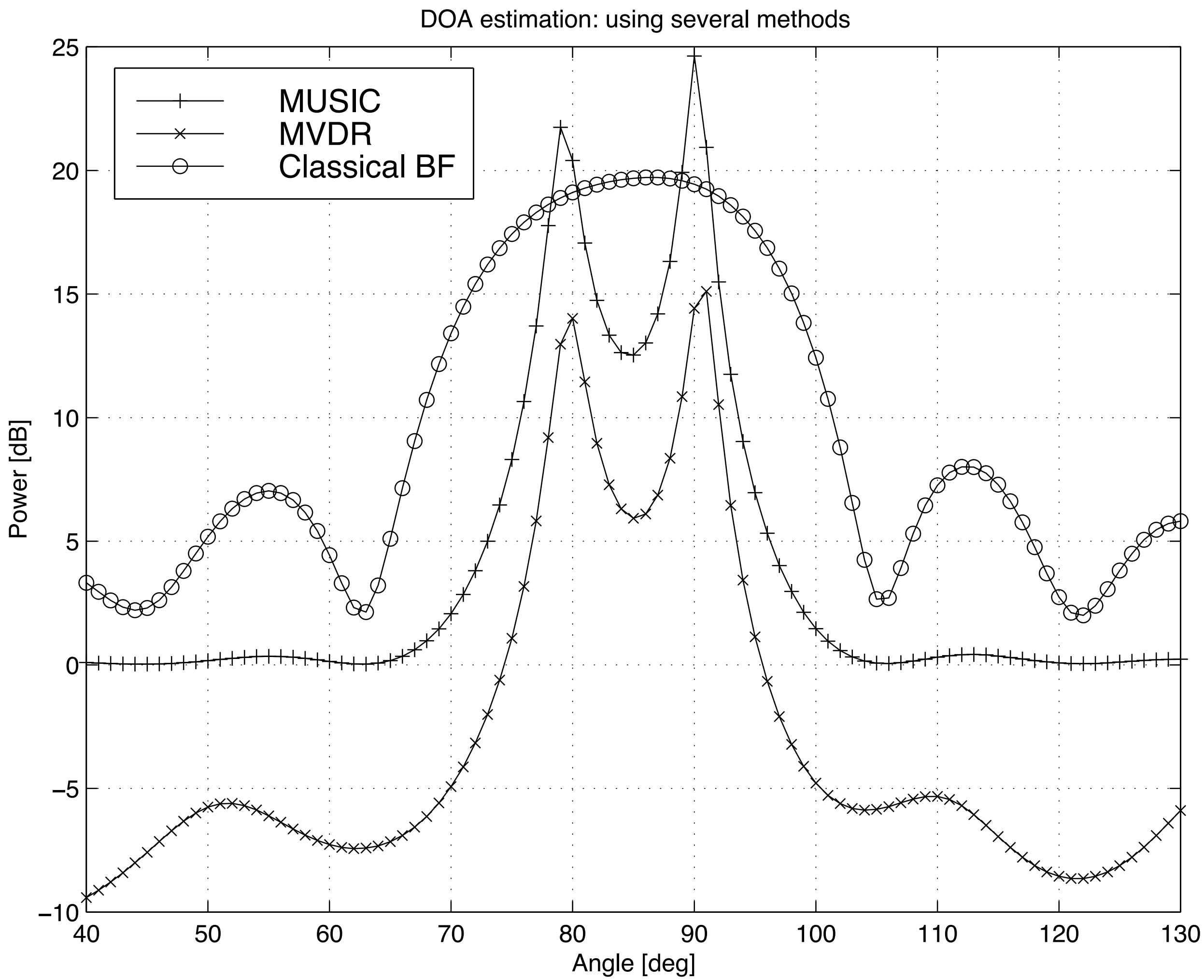
$$\mathbf{U}_n^{\mathrm{H}}\mathbf{a}(\theta_i) = 0, \qquad (1 \leq i \leq d)$$

- Choose the d lowest local minima of the cost function

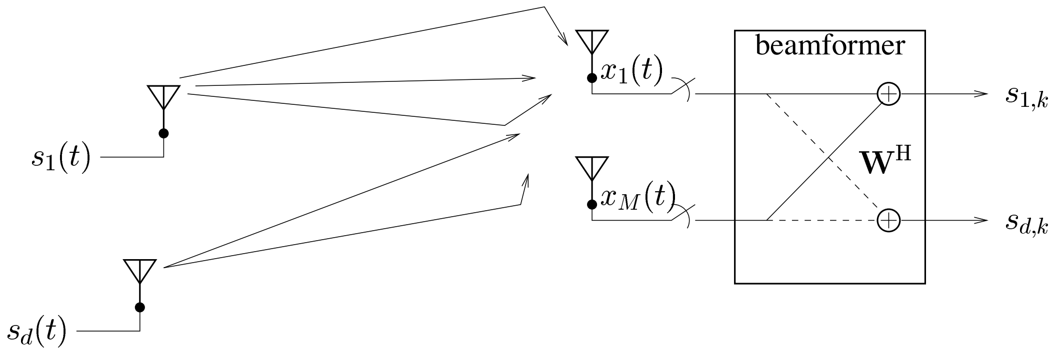
$$J_{MUSIC}(\theta) = \frac{\|\hat{\mathbf{U}}_n^{\mathrm{H}}\mathbf{a}(\theta)\|^2}{\|\mathbf{a}(\theta)\|^2} = \frac{\mathbf{a}(\theta)^{\mathrm{H}}\hat{\mathbf{U}}_n\hat{\mathbf{U}}_n^{\mathrm{H}}\mathbf{a}(\theta)}{\mathbf{a}(\theta)^{\mathrm{H}}\mathbf{a}(\theta)}$$

  - In a graph, we plot the inverse of $J_{MUSIC}(\theta)$.
  - If number of sources smaller than number of sensors ($d < M$), we get the *exact* DOAs for $N \to \infty$ or SNR $\to \infty \Rightarrow$ *statistically consistent* estimates.

Direction estimation



Data model



- Assume we receive d signals on an antenna array, narrow-band case:

$$\mathbf{x}_k := \mathbf{x}(k) = \sum_{i=1}^{d} \mathbf{a}_i s_i(k) + \mathbf{n}(k) := \sum_{i=1}^{d} \mathbf{a}_i s_{i,k} + \mathbf{n}_k \; = \; \mathbf{A}\mathbf{s}_k + \mathbf{n}_k$$

- Objective:
  - Construct a receiver weight vector $\mathbf{w}_i$ such that

$$\mathbf{w}_i^{\mathrm{H}} \mathbf{x}_k = \hat{s}_{i,k}$$

  - Construct a receiver weight matrix $\mathbf{W}$ such that

$$\mathbf{W}^{\mathrm{H}} \mathbf{x}_k = \hat{\mathbf{s}}_k$$

Deterministic approach

Noiseless case $\mathbf{x}_k = \mathbf{A}\mathbf{s}_k \quad \Leftrightarrow \quad \mathbf{X} = \mathbf{A}\mathbf{S}$

- Objective: find $\mathbf{W}$ such that $\mathbf{W}^{\mathrm{H}}\mathbf{X} = \mathbf{S}$
- With $\mathbf{A}$ known (e.g. after channel estimation):

$$\mathbf{X} = \mathbf{A}\mathbf{S} \;\Rightarrow\; \mathbf{S} = \mathbf{A}^{\dagger}\mathbf{X} = (\mathbf{A}^{\mathrm{H}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{H}}\mathbf{X}$$

  Hence we set

$$\mathbf{W}^{\mathrm{H}} = \mathbf{A}^{\dagger}$$

- With $\mathbf{S}$ known (e.g. after synchronization and training):

$$\mathbf{W}^{\mathrm{H}}\mathbf{X} = \mathbf{S} \;\Rightarrow\; \mathbf{W}^{\mathrm{H}} \;=\; \mathbf{S}\mathbf{X}^{\dagger} \;=\; \mathbf{S}\mathbf{X}^{\mathrm{H}}(\mathbf{X}\mathbf{X}^{\mathrm{H}})^{-1}$$

  Further, we have that

$$\mathbf{A} = (\mathbf{W}^{\mathrm{H}})^{\dagger}$$

- In both cases: $\mathbf{W}^{\mathrm{H}}\mathbf{A} = \mathbf{I}$: all interference is cancelled.

Deterministic approach

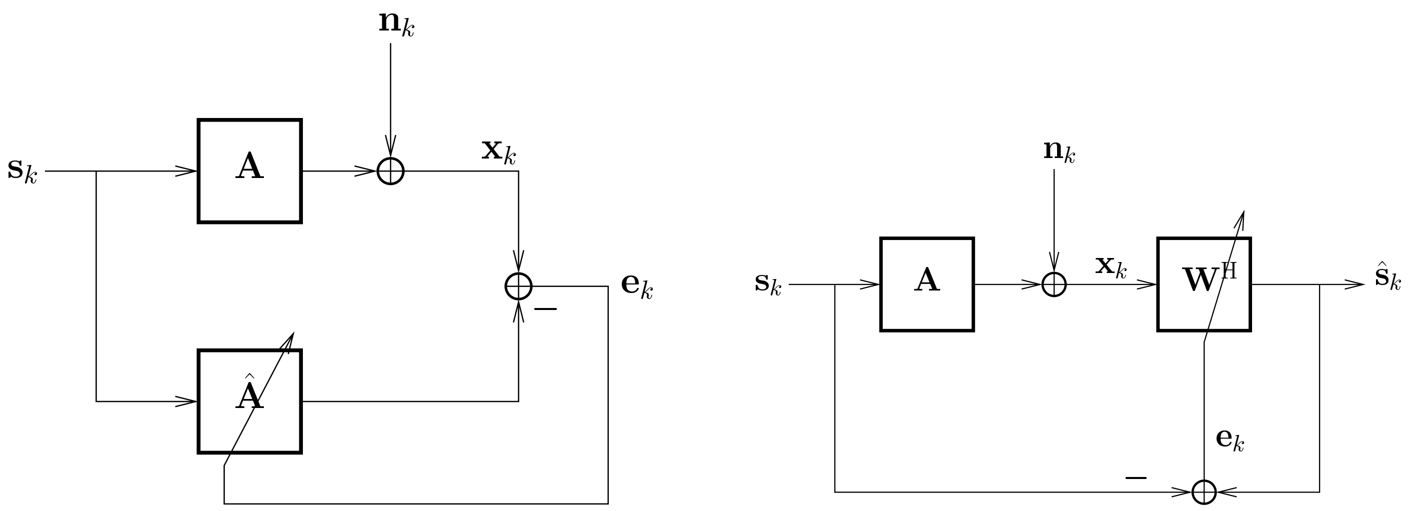
Noisy case: $\mathbf{X} = \mathbf{AS} + \mathbf{N}$

- *Model matching:* minimize residual

$$\min_{\mathbf{S}} \|\mathbf{X} - \mathbf{AS}\|_{\mathrm{F}}^2, \qquad \text{or} \qquad \min_{\mathbf{A}} \|\mathbf{X} - \mathbf{AS}\|_{\mathrm{F}}^2$$

- *Output error minimization:*

$$\min_{\mathbf{W}} \|\mathbf{W}^{\mathrm{H}}\mathbf{X} - \mathbf{S}\|_{\mathrm{F}}^2,$$



Deterministic approach

Model matching

- *With* **A** *known:*

$$\hat{\mathbf{S}} = \arg\min_{\mathbf{S}} \|\mathbf{X} - \mathbf{A}\mathbf{S}\|_{\mathrm{F}}^2 \quad \Rightarrow \quad \hat{\mathbf{S}} = \mathbf{A}^{\dagger}\mathbf{X} \quad \Rightarrow \quad \mathbf{W}^{\mathrm{H}} = \mathbf{A}^{\dagger}$$

This is the *Zero-Forcing (ZF) receiver*.

- It maximizes the output Signal-to-Interference Ratio (SIR).
- It might boost the noise:

  Since $\hat{\mathbf{S}} = \mathbf{W}^{\mathrm{H}}\mathbf{X} = \mathbf{S} + \mathbf{A}^{\dagger}\mathbf{N}$, the output noise depends on $\mathbf{A}^{\dagger}$

$$\mathbf{A} = \mathbf{U}_A \Sigma_A \mathbf{V}_A^{\mathrm{H}} \quad \rightarrow \quad \mathbf{A}^{\dagger} = \mathbf{V}_A \Sigma_A^{-1} \mathbf{U}_A^{\mathrm{H}},$$

  If Σ_A^{-1} is large (i.e., **A** is ill conditioned), the output noise is large.

- *With* **S** *known:*

$$\hat{\mathbf{A}} = \arg\min_{\mathbf{A}} \|\mathbf{X} - \mathbf{A}\mathbf{S}\|_{\mathrm{F}}^2 \quad \Rightarrow \quad \hat{\mathbf{A}} = \mathbf{X}\mathbf{S}^{\dagger} = \mathbf{X}\mathbf{S}^{\mathrm{H}}(\mathbf{S}\mathbf{S}^{\mathrm{H}})^{-1}$$

This does not specify the beamformer, but it is natural to set $\mathbf{W}^{\mathrm{H}} = \hat{\mathbf{A}}^{\dagger}$.

Deterministic approach

Output error minimization

- *With* $\mathbf{S}$ *known*:

$$\mathbf{W}^{\mathrm{H}} = \arg\min_{\mathbf{W}} \| \mathbf{W}^{\mathrm{H}}\mathbf{X} - \mathbf{S} \|_{\mathrm{F}}^2 = \mathbf{S}\mathbf{X}^{\dagger} = \mathbf{S}\mathbf{X}^{\mathrm{H}}(\mathbf{X}\mathbf{X}^{\mathrm{H}})^{-1} = \hat{\mathbf{R}}_{xs}^{\mathrm{H}}\hat{\mathbf{R}}_x^{-1}, \quad \mathbf{W} = \hat{\mathbf{R}}_x^{-1}\hat{\mathbf{R}}_{xs}$$

$\hat{\mathbf{R}}_x = \frac{1}{N}\mathbf{X}\mathbf{X}^{\mathrm{H}}$: sample data covariance matrix

$\hat{\mathbf{R}}_{xs} = \frac{1}{N}(\mathbf{X}\mathbf{S}^{\mathrm{H}})$: sample correlation between the sources and the received data

- *With* $\mathbf{A}$ *known*, and assuming $\frac{1}{N}\mathbf{S}\mathbf{S}^{\mathrm{H}} \to \mathbf{I}$, $\frac{1}{N}\mathbf{N}\mathbf{N}^{\mathrm{H}} \to \sigma^2\mathbf{I}$, and $\frac{1}{N}\mathbf{S}\mathbf{N}^{\mathrm{H}} \to \mathbf{0}$:

$$\hat{\mathbf{R}}_x = \frac{1}{N}\mathbf{X}\mathbf{X}^{\mathrm{H}} = \frac{1}{N}\mathbf{A}\mathbf{S}\mathbf{S}^{\mathrm{H}}\mathbf{A}^{\mathrm{H}} + \frac{1}{N}\mathbf{N}\mathbf{N}^{\mathrm{H}} + \frac{1}{N}\mathbf{A}\mathbf{S}\mathbf{N}^{\mathrm{H}} + \frac{1}{N}\mathbf{N}\mathbf{S}^{\mathrm{H}}\mathbf{A}^{\mathrm{H}} \quad \to \quad \mathbf{A}\mathbf{A}^{\mathrm{H}} + \sigma^2\mathbf{I}$$

$$\hat{\mathbf{R}}_{xs} = \frac{1}{N}\mathbf{X}\mathbf{S}^{\mathrm{H}} = \frac{1}{N}\mathbf{A}\mathbf{S}\mathbf{S}^{\mathrm{H}} + \frac{1}{N}\mathbf{N}\mathbf{S}^{\mathrm{H}} \quad \to \quad \mathbf{A}$$

$$\mathbf{W} = (\mathbf{A}\mathbf{A}^{\mathrm{H}} + \sigma^2\mathbf{I})^{-1}\mathbf{A}$$

This is the *Linear Minimum Mean Square Error (LMMSE)* or *Wiener receiver*.

- It makes a compromise between interference and noise cancellation.
- It maximizes the output Signal-to-Interference-plus-Noise Ratio (SINR).

Maximum ratio combining

- Single signal in white noise: $\mathbf{x}_k = \mathbf{a}s_k + \mathbf{n}_k\,, \quad \mathrm{E}[\mathbf{n}_k\mathbf{n}_k^{\mathrm{H}}] = \sigma^2\mathbf{I}$
- The ZF beamformer is given by

$$\mathbf{w} \;=\; \mathbf{a}(\mathbf{a}^{\mathrm{H}}\mathbf{a})^{-1} \;=\; \gamma_1\mathbf{a}$$

- Single signal in colored noise: $\mathbf{x}_k = \mathbf{a}s_k + \mathbf{n}_k\,, \quad \mathrm{E}[\mathbf{n}_k\mathbf{n}_k^{\mathrm{H}}] = \mathbf{R}_n$
- The ZF beamformer is given by

$$\mathbf{w} \;=\; \mathbf{R}_n^{-1}\mathbf{a}(\mathbf{a}^{\mathrm{H}}\mathbf{R}_n^{-1}\mathbf{a})^{-1} \;=\; \gamma_2\mathbf{R}_n^{-1}\mathbf{a}$$

- Note: a scalar multiplication does not change the output SNR.
- $\mathbf{w} \;=\; \mathbf{a}$ (white noise) and $\mathbf{w} \;=\; \mathbf{R}_n^{-1}\mathbf{a}$ (non-white noise) are known as:

  matched filter, *classical beamformer*, or *Maximum Ratio Combining* (MRC)

Maximum ratio combining

- Also the Wiener filter will lead to MRC
  - Wiener receiver in white noise

$$\mathbf{w} = \mathbf{R}_x^{-1}\mathbf{r}_{xs} \;=\; (\mathbf{a}\mathbf{a}^{\mathrm{H}} + \sigma^2\mathbf{I})^{-1}\mathbf{a} \;=\; \mathbf{a}(\mathbf{a}^{\mathrm{H}}\mathbf{a} + \sigma^2)^{-1} \quad \sim \quad \mathbf{a}$$

  - Wiener receiver in colored noise

$$\mathbf{w} \;=\; \mathbf{R}_x^{-1}\mathbf{r}_{xs} \;=\; (\mathbf{a}\mathbf{a}^{\mathrm{H}} + \mathbf{R}_n)^{-1}\mathbf{a} \;=\; \mathbf{R}_n^{-1}\mathbf{a}(\mathbf{a}^{\mathrm{H}}\mathbf{R}_n^{-1}\mathbf{a} + 1)^{-1} \quad \sim \quad \mathbf{R}_n^{-1}\mathbf{a}$$

- The colored noise case is relevant also for the following reason:

  with more than one signal, we can write the model as

$$\mathbf{x}_k \;=\; \mathbf{A}\mathbf{s}_k + \mathbf{n}_k \;=\; \mathbf{a}_1 s_{1,k} + (\mathbf{A}'\mathbf{s}'_k + \mathbf{n}_k)$$

  This is of the form

$$\mathbf{x}_k \;=\; \mathbf{a}s_k + \mathbf{n}_k\,, \qquad \mathbf{R}_n \;=\; \mathbf{A}'\mathbf{A}'^{H} + \sigma^2\mathbf{I}$$

  where the “noise” is colored due to the contribution of the interfering sources.