Supplementary Information

Introspection dynamics in asymmetric multiplayers games

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The general model for introspection dynamics

We consider an asymmetric normal form game with $N(>2)$ players. Each player, i, has access to their action set, $\mathbf{A}^i$, in which there are m_i possible actions that they can play, $\mathbf{A}^i = \{a^i_i, a^i_2, ..., a^i_{m_i}\}$. The payoff for an individual i depends on what everyone plays and is represented by $\pi^i(\mathbf{a}^1, \mathbf{a}^2, ..., \mathbf{a}^N)$, where $\mathbf{a} := (\mathbf{a}^1, \mathbf{a}^2, ..., \mathbf{a}^N) \in \mathbf{A}^1 \times \mathbf{A}^2 \times ... \times \mathbf{A}^N$ represents a state of the game. In this model we only consider pure strategies for players. So, player i has only m_i possible strategies. Therefore, the total number of states possible for the game is the product of all m_i.

We represent a state of the game, $\mathbf{a}$, from the perspective of player j as $\mathbf{a} =: (\mathbf{a}^j, \mathbf{a}^{-j})$. In the state $(\mathbf{a}^j, \mathbf{a}^{-j})$, player j plays the action $\mathbf{a}^j \in \mathbf{A}^j$ and all the other co-players of j play $\mathbf{a}^{-j} \in \Pi_{k\neq j}\mathbf{A}^k$. The payoff of player j in this state is also represented as $\pi^j(\mathbf{a}^j, \mathbf{a}^{-j})$.

The players update their strategies over time using the introspection dynamics[1]. At every time step, one randomly chosen player can update their strategy. The randomly chosen player, say j, currently playing strategy a^j_k, compares their current payoff to all the payoffs they can achieve if they played an alternate action from their action set $\mathbf{A}^j$. This comparison is done while assuming that the co-players do not change from $\mathbf{a}^{-j}$. Then, they adopt a new strategy $a^j_l \neq a^j_k$ with the probability:

$$p^j_{a^j_k \to a^j_l}(\mathbf{a}^{-j}) = \frac{1}{1 + e^{-\beta(\pi^j(a^j_l, \mathbf{a}^{-j}) - \pi^j(a^j_k, \mathbf{a}^{-j}))}} \tag{1}$$

Where β is the selection parameter. The probability that no update is made by the randomly chosen player is given by the normalization condition:

$$p^j_{a^j_k \to a^j_k}(\mathbf{a}^{-j}) = 1 - \sum_{k\neq l} p^j_{a^j_k \to a^j_l}(\mathbf{a}^{-j}) \tag{2}$$

Now, with the help of Eq. 1 and 2, we can define the transition matrix $\mathbf{T}$ of the process. The transition matrix is a square matrix of size $\Pi_i m_i$. The element $\mathbf{T}_{\mathbf{a}_p,\mathbf{a}_q}$ of the matrix represents the probability that the game will go to state $\mathbf{a}_q$ from state $\mathbf{a}_p$ in a time step. Before defining the transition matrix, we define the notion of neighbouring states.

Definition 1. ***Neighbouring states****:*
The state $\mathbf{a}_q \in Neb(\mathbf{a}_p)$, the neighbourhood of state $\mathbf{a}_p$, if and only if $\exists j$ such that $\mathbf{a}_p^{-j} = \mathbf{a}_q^{-j}$ and $\mathbf{a}_p^j \neq \mathbf{a}_q^j$.

Note that if $\mathbf{a}_q \in Neb(\mathbf{a}_p)$ then by definition $\mathbf{a}_p \in Neb(\mathbf{a}_q)$. We give an example to explain the notion of neighbourhood. Consider three players, player $1, 2$ and 3. Each of these players choose actions from an identical binary action set $\{\mathit{0}, \mathit{1}\}$. In this example, the state $(\mathit{0}, \mathit{0}, \mathit{0})$ is a neighbour of $(\mathit{0}, \mathit{0}, \mathit{1})$ and vice versa. The index of difference between the two states is player 3. The states $(\mathit{0}, \mathit{0}, \mathit{0})$ and $(\mathit{0}, \mathit{1}, \mathit{1})$ are not neighbouring states because more than one player's action are changed.

Definition 2. ***Index of difference for neighbouring states****:*

If the states $\mathbf{a}_p$ and $\mathbf{a}_q$ are neighbours then, the unique j for which $\mathbf{a}_p^j \neq \mathbf{a}_q^j$, is called the index of difference. Or $I(\mathbf{a}_p, \mathbf{a}_q) := j$

For the earlier example, let $\mathbf{a}_p = (\mathit{0}, \mathit{0}, \mathit{0})$ and $\mathbf{a}_q = (\mathit{0}, \mathit{0}, \mathit{1})$. Then, $I(\mathbf{a}_p, \mathbf{a}_q) = 3$.

Using the above definitions of neighbourhood and index of difference, we can define the transition matrix $\mathbf{T}$ for the process as the follows:

$$\mathbf{T}_{\mathbf{a}_p,\mathbf{a}_q} = \begin{cases} \frac{1}{N} \cdot p^j_{\mathbf{a}_p^j \to \mathbf{a}_q^j}(\mathbf{a}_p^{-j}) & \mathbf{a}_p \in Neb(\mathbf{a}_q) \quad \text{and,} \quad j = I(\mathbf{a}_p, \mathbf{a}_q) \\ 0 & \mathbf{a}_p \notin Neb(\mathbf{a}_q) \\ 1 - \sum_{\forall \mathbf{a}_k \neq \mathbf{a}_p} \mathbf{T}_{\mathbf{a}_p,\mathbf{a}_k} & \mathbf{a}_p = \mathbf{a}_q \end{cases} \tag{3}$$

The stochastic transition matrix $\mathbf{T}$ is row-stochastic. This implies that its stationary distribution is a left eigenvector of $\mathbf{T}$ corresponding to eigenvalue 1. We denote the stationary distribution of $\mathbf{T}$ as $\mathbf{u}$. Furthermore, since no state of the process is isolated (i.e., every state has atleast one neighbour), a finite β assures that a state is reachable from any other state with positive probability in a finite number of steps. Therefore, $\mathbf{T}$ is also primitive. The stationary distribution $\mathbf{u}$ is unique and strictly positive (from the Perron-Frobenius Theorem). The following conditions are satisfied by the stationary distribution $\mathbf{u}$ when it is additionally also a probability distribution.

$$\mathbf{u}\mathbf{T} = \mathbf{u} \tag{4}$$

$$\mathbf{u} \cdot \mathbf{1} = 1 \tag{5}$$

Examples of multiplayer games and results

Linear public goods game

We consider the linear public goods game with N asymmetric players. Each player has two possible actions, to contribute ($a = 1$), or to not contribute ($a = 0$). The players differ in their cost of cooperation and their productivities. The cost of cooperation and the productivity of a player i are denoted by c_i and r_i. The payoff for player i when the state of the game is $\mathbf{a}$ is given by:

$$\pi^i(\mathbf{a}) = \sum_{j=1}^{N} \frac{r_j a_j c_j}{N} - a_i c_i \tag{6}$$

The difference in payoffs between playing the two actions in a linear public goods game is independent of what the other co-players play in the game. That is,

$$\pi^i(\mathbf{a}^i = 0, \mathbf{a}^{-i}) - \pi^i(\mathbf{a}^i = 1, \mathbf{a}^{-i}) = c_i \left(1 - \frac{r_i}{N}\right) =: f(c_i, r_i) \tag{7}$$

is independent of $\mathbf{a}^{-i}$. This property of the game results in dominated strategies for player i depending on the relationship between their productivity r_i and number of players N. When $r_i < N$, the action of not contributing ($a = 0$) dominates the action of contributing ($a = 1$) and vice-versa.

Proposition 1. *When β is finite, the unique stationary distribution of an asymmetric linear public goods game under introspection dynamics is given by:*

$$\mathbf{u}_{\mathbf{a}} = \prod_{j=1}^{N} \frac{1}{1 + e^{sign(\mathbf{a}^j)\beta f(c_j, r_j)}} \quad , \forall \mathbf{a} \in \{0, 1\}^N \tag{8}$$

where,

$$sign(a) = 2a - 1 \tag{9}$$

Appendix: Proofs

Proof. **Proof of Proposition** 1

The stationary transition matrix $\mathbf{T}$ for the linear public goods game is primitive when β is finite (i.e., there is a positive power k such that $\mathbf{T}^k$ is a strictly positive matrix). Therefore, the stationary distribution of

$\mathbf{T}$ will always be unique. We define the following short cut notations for the ease of the proof:

$$\bar{\mathbf{a}}^j := \{0,1\} \setminus \mathbf{a}^j \tag{10}$$

$$p^j := \frac{1}{1+ e^{\beta f(c_j,r_j)}} \tag{11}$$

Using these notations and Eq. 1 and 7 we can write the probability that a player j updates from $\mathbf{a}^j$ to $\bar{\mathbf{a}}^j$ while their co-players play $\mathbf{a}^{-j}$ as:

$$p^j_{\bar{\mathbf{a}}^j \to \mathbf{a}^j}(\mathbf{a}^{-j}) = p^j sign(\mathbf{a}^j) + \bar{\mathbf{a}}^j \tag{12}$$

The candidate stationary distribution $\mathbf{u}$ given in Eq 8 can be written down using our shortcut notation as:

$$\mathbf{u}_{\mathbf{a}} = \prod_{k=1}^{N} p^k sign(\mathbf{a}^k) + \bar{\mathbf{a}}^k \quad ,\forall \mathbf{a} \in \{0,1\}^N \tag{13}$$

This stationary distribution must satisfy the following properties, which are also given in Eq 4 and 5:

$$\mathbf{u}_a = \mathbf{T}_{\mathbf{a},\mathbf{a}}\mathbf{u}_a + \sum_{\mathbf{a}_q \neq \mathbf{a}} \mathbf{T}_{\mathbf{a}_q,\mathbf{a}}\mathbf{u}_{\mathbf{a}_q} \tag{14}$$

$$\sum_{\forall \mathbf{a}_q} u_{\mathbf{a}_q} = 1 \tag{15}$$

Where, the terms in the right hand side of Eq. 14 can be simplified using Eq 1 and 3 as follows:

$$\mathbf{T}_{\mathbf{a},\mathbf{a}} = 1 - \sum_{k=1}^{N} \mathbf{T}_{(\mathbf{a}^k,\mathbf{a}^{-k}),(\bar{\mathbf{a}}^k,\mathbf{a}^{-k})} = 1 - \frac{1}{N}\sum_{k=1}^{N} p^k sign(\bar{\mathbf{a}}^k) + \mathbf{a}^k \tag{16}$$

and additionally, also using Eq. 13 the second term can be simplified too:

$$\begin{aligned}
\sum_{\mathbf{a}_q \neq \mathbf{a}} \mathbf{T}_{\mathbf{a}_q,\mathbf{a}}\mathbf{u}_{\mathbf{a}_q} &= \sum_{k=1}^{N} \mathbf{T}_{(\bar{\mathbf{a}}^k,\mathbf{a}^{-k}),(\mathbf{a}^k,\mathbf{a}^{-k})}\mathbf{u}_{(\bar{\mathbf{a}}^k,\mathbf{a}^{-k})} && (17)\\
&= \frac{1}{N}\sum_{k=1}^{N} \left(p^k sign(\mathbf{a}^k) + \bar{\mathbf{a}}^k\right) \mathbf{u}_{(\bar{\mathbf{a}}^k,\mathbf{a}^{-k})} && (18)\\
&= \frac{\mathbf{u}_{\mathbf{a}}}{N}\sum_{k=1}^{N} p^k sign(\bar{\mathbf{a}}^k) + \mathbf{a}^k && (19)
\end{aligned}$$

Now, using Eq. 16, 19 one can show that the right hand side of Eq. 14 is the element of the stationary distribution, corresponding to the state $\mathbf{a}$, $\mathbf{u}_a$. Now, to complete the proof, we must show that Eq. 15 is also true for our candidate stationary distribution. This can be done by decomposing the sum of the elements of the stationary distribution as follows:

$$\sum_{\forall \mathbf{a}_q} u_{\mathbf{a}_q} = \sum_{\forall \mathbf{a}_q} \prod_{k=1}^{N} p^k sign(\mathbf{a}_q^k) + \bar{\mathbf{a}}_q^k \tag{20}$$

$$= \sum_{\forall \mathbf{a}_q} (1-p^N) \prod_{k=1}^{N-1} p^k sign(\mathbf{a}_q^k) + \bar{\mathbf{a}}_q^k + p^N \prod_{k=1}^{N-1} p^k sign(\mathbf{a}_q^k) + \bar{\mathbf{a}}_q^k \tag{21}$$

$$= \sum_{\forall \mathbf{a}_q} \prod_{k=1}^{N-1} p^k sign(\mathbf{a}_q^k) + \bar{\mathbf{a}}_q^k \tag{22}$$

When the above decomposition is perfomed $N-1$ more times, the sum of the right hand side becomes 1. This prooves that the candidate stationary distribution is also normalized. $\square$

Supplementary References

References

[1] Marta Couto, Stefano Giaimo, and Christian Hilbe. Introspection dynamics: A simple model of counterfactual learning in asymmetric games. *New Journal of Physics*, 2022.