XI - Chapter 3.2

Vectors: Concepts and Applications

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This chapter introduces the fundamental concepts of vectors, their arithmetic, and real-world applications in physics, mathematics, and artificial intelligence.

Learning Objectives

After studying this chapter, students will be able to: [a]

- Understand the concept of **vectors** and how they differ from scalars.
- Recognize the relevance of vectors in AI/ML contexts.
- Perform **basic vector arithmetic** such as addition, subtraction, and scalar multiplication.
- Compute the **magnitude and direction** of a vector.
- Understand and apply the concept of the **dot product** and its applications.
- Learn the **cross product** and its applications in AI.
- Represent vectors in **2D and 3D coordinate systems**.
- Applications of vector algebra in different fields.

[a] Unit 3: Mathematics for AI

1 Introduction to Vectors

A **vector** is a quantity that has both *magnitude* and *direction*. Vectors are typically represented by arrows, where the length of the arrow represents the magnitude and the arrowhead shows the direction.

1.1 Scalars vs. Vectors

- **Scalars**: Quantities that have only magnitude (e.g., temperature, mass, speed, time)
- **Vectors**: Quantities that have both magnitude and direction (e.g., velocity, force, displacement, acceleration)

Example:

- Speed is a scalar: "The car is moving at 60 km/h"
- Velocity is a vector: "The car is moving at 60 km/h towards the north"

1.2 Relevance of Vector Algebra in AI Domain

Vector algebra plays a fundamental role in Artificial Intelligence (AI), forming the backbone of various subfields such as machine learning, computer vision, and natural language processing. It enables the efficient computation, representation, and learning required in intelligent systems. Below are key areas where vector algebra is highly relevant.

1.2.1 Data Representation

In AI, different types of data (such as text, images, or audio) are represented using vectors.

- A grayscale image can be represented as a vector of pixel intensities.
- Words can be embedded into vector space using models like Word2Vec or GloVe.

1.2.2 Mathematical Foundation of ML Models

Most machine learning algorithms are built upon vector and matrix operations.

- Algorithms like linear regression and support vector machines use vector dot products and norms.
- Neural networks represent weights, inputs, and outputs as vectors and perform operations between them.

1.2.3 Operations in Neural Networks

- In deep learning, the data flow between layers is vectorized.
- Key operations like dot products and vector addition are used in Forward propagation, Backpropagation for gradient computation, etc.

1.2.4 Distance and Similarity Measures

Vectors allow the measurement of similarity and distance between data points.

- Commonly used metrics include: Euclidean Distance, Cosine Similarity, etc.
- Applications include clustering, recommendation systems, and facial recognition.

1.2.5 Optimization

- Algorithms like Gradient Descent use vector calculus to optimize model parameters.
- Vectors are essential in computing gradients and updating weights in learning processes.

1.3 Vector Notation

Vectors can be represented in several ways:

- Bold lowercase letters: $\mathbf{a}$, $\mathbf{b}$, $\mathbf{v}$
- Arrows over letters: $\vec{a}$, $\vec{b}$, $\vec{v}$
- Component form: (x, y) or $\langle x, y \rangle$ in 2D, (x, y, z) or $\langle x, y, z \rangle$ in 3D
- Column matrix form: $\begin{bmatrix} x \\ y \end{bmatrix}$ in 2D, $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in 3D
- Unit vector notation: $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$

1.4 Magnitude

The magnitude (or length) of a vector $\vec{v} = (x, y, z)$ is calculated using the Euclidean norm:

$$|\vec{v}| = \sqrt{x^2 + y^2 + z^2}$$

Example: If $\vec{v} = (3, 4, 0)$, then $|\vec{v}| = \sqrt{3^2 + 4^2} = 5$

1.5 Direction

Direction is the orientation of a vector in space. It is often described using angles or by the unit vector in its direction.

The direction of a 2D vector $\vec{v} = (v_x, v_y)$ is given by the angle θ it makes with the positive x-axis:

$$\theta = \arctan\left(\frac{v_y}{v_x}\right)$$

Note: Consider the quadrant when determining the angle.

1.6 Unit Vector

A **unit vector** has a magnitude of 1 and indicates direction only. It is computed as:

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

Example: For $\vec{v} = (6, 8)$, magnitude $|\vec{v}| = \sqrt{6^2 + 8^2} = 10$ and the unit vector $\hat{v} = (\frac{6}{10}, \frac{8}{10}) = (0.6, 0.8)$

2 Arithmetic Operations on Vectors

2.1 Vector Addition

Two vectors $\vec{a} = (a_x, a_y)$ and $\vec{b} = (b_x, b_y)$ are added component-wise:

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y)$$

example If $\vec{a} = (3, 4)$ and $\vec{b} = (1, 2)$, then:

$$\vec{a} + \vec{b} = (3 + 1, 4 + 2) = (4, 6)$$

2.2 Vector Subtraction

Vector subtraction is performed component-wise:

$$\vec{a} - \vec{b} = (a_x - b_x, a_y - b_y)$$

example If $\vec{a} = (5, 7)$ and $\vec{b} = (2, 3)$, then:

$$\vec{a} - \vec{b} = (5 - 2, 7 - 3) = (3, 4)$$

2.3 Multiplication

2.3.1 Scalar Multiplication

A vector $\vec{v}$ multiplied by a scalar k results in:

$$k\vec{v} = k(v_x, v_y) = (kv_x, kv_y)$$

Properties:

- If $k > 1$: vector magnitude increases, direction unchanged
- If $0 < k < 1$: vector magnitude decreases, direction unchanged
- If $k < 0$: vector direction reverses, magnitude changes by $|k|$

example If $\vec{v} = (3, 4)$ and $k = 2$, then:

$$2\vec{v} = 2(3, 4) = (6, 8)$$

2.3.2 Dot Product

Definition and Calculation

The dot product (scalar product) of two vectors $\vec{a} = (a_x, a_y)$ and $\vec{b} = (b_x, b_y)$ is:

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y$$

Alternative formula (Optional): $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$. Where, θ is the angle between the vectors.

Properties of Dot Product

- Commutative: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- Distributive: $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- If $\vec{a} \cdot \vec{b} = 0$, then $\vec{a} \perp \vec{b}$ (vectors are perpendicular)
- If $\vec{a} \cdot \vec{b} > 0$, then $\theta < 90$ (acute angle)
- If $\vec{a} \cdot \vec{b} < 0$, then $\theta > 90$ (obtuse angle)

example For $\vec{a} = (3, 4)$ and $\vec{b} = (1, 2)$:

$$\vec{a} \cdot \vec{b} = (3)(1) + (4)(2) = 3 + 8 = 11$$

Applications of Dot Product

- Finding the angle between vectors
- Projection of one vector onto another
- Determining orthogonality
- **Similarity Measure:** The dot product helps measure the similarity between two vectors.
- **Neural Networks:** At the core of every neuron, the input vector and weight vector are combined using a dot product

2.3.3 Cross Product

Definition and Calculation

The cross product is defined only for 3D vectors. For $\vec{a} = (a_x, a_y, a_z)$ and $\vec{b} = (b_x, b_y, b_z)$:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\vec{a} \times \vec{b} = (a_y b_z - a_z b_y)\mathbf{i} - (a_x b_z - a_z b_x)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$$

Properties of Cross Product

- Anti-commutative: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

- The result is perpendicular to both original vectors
- Magnitude: $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$
- If vectors are parallel: $\vec{a} \times \vec{b} = \vec{0}$

example For $\vec{a} = (1, 2, 3)$ and $\vec{b} = (4, 5, 6)$:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = (2 \cdot 6 - 3 \cdot 5, 3 \cdot 4 - 1 \cdot 6, 1 \cdot 5 - 2 \cdot 4) = (-3, 6, -3)$$

Applications of Cross Product

- Finding normal vectors to planes
- Determining area of parallelograms and triangles
- **Reinforcement Learning:** When AI agents are trained in simulated environments (e.g., for robotics or games), the cross product is used in the underlying physics for motion and collision calculations.
- **Computer Vision and Graphics:** Used in 3D modeling, pose estimation, and detecting orientation of objects (since the cross product gives a vector perpendicular to a plane).

3 Linear Combination of Vectors (Optional)

A vector $\vec{v}$ is a **linear combination** of vectors $\vec{a}$ and $\vec{b}$ if: $\vec{v} = c_1\vec{a} + c_2\vec{b}$ where c_1 and c_2 are scalars.

Example:

Let $\vec{a} = (1, 0)$ and $\vec{b} = (0, 1)$. Then, any vector $\vec{v} = (x, y)$ can be expressed as a linear combination of $\vec{a}$ and $\vec{b}$:

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Expanding, we get:

$$x = 1 \cdot c_1 + 0 \cdot c_2$$
$$y = 0 \cdot c_1 + 1 \cdot c_2$$

Hence,

$$c_1 = x, \quad c_2 = y$$

This shows that any vector $\vec{v} = (x, y)$ can be written as:

$$\vec{v} = x\vec{a} + y\vec{b}$$

4 Applications of Vectors

- Physics: Representing force, velocity, and acceleration.
- Engineering: Direction and magnitude of physical quantities.
- Computer Graphics: Position and motion of objects.
- Navigation: Determining direction and displacement.

5 Practice Problems

5.1 Basic Operations

1. Find the magnitude of $\vec{v} = (-3, 4)$.
2. Given $\vec{a} = (3, -2)$ and $\vec{b} = (1, 4)$, find:
   (a) $\vec{a} + \vec{b}$
   (b) $\vec{a} - \vec{b}$
   (c) $3\vec{a} - 2\vec{b}$
   (d) $|\vec{a}|$ and $|\vec{b}|$
3. Find the unit vector in the direction of $\vec{v} = (6, 8)$.
4. Calculate the angle between vectors $\vec{p} = (1, 1)$ and $\vec{q} = (1, -1)$.
5. Find a linear combination of $\vec{a} = (2, 0)$, $\vec{b} = (0, 2)$ to express $\vec{v} = (6, 4)$.
6. Write a short note on vector multiplication (dot and cross product) with examples.

5.2 Applications

1. A particle moves from point A(2, 3) to point B(7, 8). Find the displacement vector and its magnitude.
2. Two forces $\vec{F_1} = (10, 0)$ N and $\vec{F_2} = (0, 15)$ N act on an object. Find the resultant force and its direction.
3. In a computer game, a character at position (100, 200) moves with velocity (5, -3) pixels per frame. Where will the character be after 10 frames?

5.3 Advanced Vector Operations

1. Given vectors $\vec{u} = (2, -1, 3)$, $\vec{v} = (1, 4, -2)$, and $\vec{w} = (-3, 0, 1)$:
   (a) Calculate $\vec{u} \cdot \vec{v}$ and determine if the vectors are perpendicular
   (b) Find $\vec{u} \times \vec{v}$ and determine if the vectors are parallel
   (c) Calculate $(\vec{u} + \vec{v}) \cdot \vec{w}$
   (d) Find the magnitude of $\vec{u} \times \vec{w}$

6 Problem-Solving Examples (Not in the Syllabus)

6.1 Example 1: Force Equilibrium

Problem: Three forces act on an object: $\vec{F_1} = (5, 3)$ N, $\vec{F_2} = (-2, 4)$ N, and $\vec{F_3} = (a, b)$ N. Find $\vec{F_3}$ for equilibrium.

Solution: For equilibrium, $\sum \vec{F} = \vec{0}$

$$\vec{F_1} + \vec{F_2} + \vec{F_3} = \vec{0}$$

$$(5, 3) + (-2, 4) + (a, b) = (0, 0)$$
$$(3 + a, 7 + b) = (0, 0)$$

Therefore: $a = -3$, $b = -7$, so $\vec{F_3} = (-3, -7)$ N

6.2 Example 2: Vector Projection

Problem: Find the projection of $\vec{a} = (4, 3)$ onto $\vec{b} = (1, 2)$.

Solution: The projection formula is:

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\vec{b}$$

Calculate components:

- $\vec{a} \cdot \vec{b} = (4)(1) + (3)(2) = 4 + 6 = 10$
- $|\vec{b}|^2 = 1^2 + 2^2 = 5$

$$\text{proj}_{\vec{b}}\vec{a} = \frac{10}{5}(1, 2) = 2(1, 2) = (2, 4)$$

6.3 Example 3: AI Application

Problem: In a recommendation system, user A has preferences $\vec{A} = (5, 3, 4, 2)$ and user B has preferences $\vec{B} = (4, 4, 3, 3)$. Calculate their similarity using cosine similarity.

Solution:

$$\text{similarity} = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$$

Calculate components:

- $\vec{A} \cdot \vec{B} = 5(4) + 3(4) + 4(3) + 2(3) = 20 + 12 + 12 + 6 = 50$
- $|\vec{A}| = \sqrt{5^2 + 3^2 + 4^2 + 2^2} = \sqrt{25 + 9 + 16 + 4} = \sqrt{54}$
- $|\vec{B}| = \sqrt{4^2 + 4^2 + 3^2 + 3^2} = \sqrt{16 + 16 + 9 + 9} = \sqrt{50}$

$$\text{similarity} = \frac{50}{\sqrt{54}\sqrt{50}} = \frac{50}{\sqrt{2700}} \approx 0.96$$

This high similarity suggests the users have similar preferences.