Class XI – Artificial Intelligence

Unit 3 - Chapter 1: Matrices

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1. Introduction to Matrices

An m-by-n **matrix** is a rectangular array of numbers or functions arranged in m rows and n columns, enclosed in square brackets [ ] or, parentheses () (maintain consistency of the bracket whichever you choose). The individual numbers in a matrix are known as its **elements** or **entries**.

Notation: A matrix with m rows and n columns is called an $m \times n$ matrix (read as "m by n"). It is called the order of the matrix.

Example:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{bmatrix} \quad \text{is a matrix of order } 3 \times 2.$$

The first row has elements 1 and 3, the second row has elements 2 and 4, and the third row has elements 5 and 6. The first column has elements 1, 2 and 5; the second column has elements 3, 4 and 6.

A useful notation for writing a general m-by-n matrix A is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Here, the matrix element of A in the ith row and the jth column is denoted as a_{ij}.

2. Types of Matrices

Column and row matrices are especially important and are referred to as vectors. A column vector is typically an $n \times 1$ matrix, and a row vector is a $1 \times n$ matrix.

- **Row Matrix:** A matrix with only one row.

$$A = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$$

- **Column Matrix:** A matrix with only one column.

$$A = \begin{bmatrix} 3 \\ 7 \\ 9 \end{bmatrix}$$

- **Square Matrix:** A matrix in which the number of rows equals the number of columns.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

- **Diagonal Matrix:** A diagonal matrix is a square matrix and it has its only nonzero elements on the diagonal. For example,

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 8 \end{bmatrix}$$

- **Scalar Matrix:** A diagonal matrix in which all diagonal elements are equal.

$$S = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

- **Identity Matrix (I):** The identity matrix, denoted by I, is a square matrix (number of rows equals number of columns) with ones down the main diagonal and zeros elsewhere.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- **Zero (Null) Matrix:** The zero matrix, denoted by 0, can be any size and is a matrix consisting of all zero elements.

$$Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3. Matrix Operations

A. Addition of Matrices

Two matrices can be added if they have the same dimensions. Addition proceeds element by element. For example,

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1+5 & 3+7 \\ 2+6 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ 8 & 12 \end{bmatrix}$$

B. Subtraction of Matrices

For two matrices to be subtracted, they must have the same dimensions (or order). This means they must have the same number of rows and the same number of columns. Subtract corresponding elements. If their dimensions are different, subtraction is not defined. For example,

$$A - B = \begin{bmatrix} 1-5 & 3-7 \\ 2-6 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

C. Multiplication of Matrices

Two matrices can be multiplied if the number of columns in the first equals the number of rows in the second. In other words, an m-by-n matrix on the left can only be multiplied by an n-by-k matrix on the right. The resulting matrix will be m-by-k. In general, an element in the resulting product matrix, say in row i and column j, is obtained by multiplying and summing the elements in row i of the left matrix with the elements in column j of the right matrix.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} (1 \cdot 2 + 3 \cdot 1) & (1 \cdot 0 + 3 \cdot 2) \\ (2 \cdot 2 + 4 \cdot 1) & (2 \cdot 0 + 4 \cdot 2) \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 8 & 8 \end{bmatrix}$$

Scalar multiplication: Matrices can also be multiplied by a scalar. The rule is to just multiply every element of the matrix. For example,

$$k \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} k & 3k \\ 2k & 4k \end{bmatrix}$$

0.1 Key Properties of Matrix Multiplication.

- Matrix multiplication is not commutative: $AB \neq BA$ in general.
- Matrix multiplication is associative: $(AB)C = A(BC)$

D. Transpose of a Matrix

The transpose of a matrix is obtained by interchanging its rows and columns. The transpose of a matrix A, denoted by A^T and spoken as A-transpose. That is,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Evidently, A is 2-by-3 tand A^T is 3-by-2. In other words, we write $a_{ij}^T = a_{ji}$

Key Properties of Matrix Transposes

The transpose operation on matrices, denoted by a superscript T, possesses several fundamental properties. Two straightforward and easily verifiable facts are:

- The transpose of a transpose returns the original matrix: $(A^T)^T = A$.
- The transpose of a sum of matrices is the sum of their transposes: $(A + B)^T = A^T + B^T$.

A more intricate, yet crucial, property concerns the transpose of a matrix product. When transposing the product of two matrices, the result is the product of their individual transposes, but with the order of multiplication reversed. This is expressed as:

$$(AB)^T = B^T A^T$$

Furthermore, the transpose operation helps define special types of square matrices:

- A square matrix A is deemed **symmetric** if it is equal to its own transpose, i.e., $A^T = A$.
- Conversely, a square matrix A is considered **skew-symmetric** if its transpose is equal to its negative, i.e., $A^T = -A$.

To illustrate, a general 3×3 symmetric matrix takes the form:

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

Notice that elements across the main diagonal are identical ($a_{ij} = a_{ji}$).

In contrast, a general 3×3 skew-symmetric matrix appears as:

$$\begin{pmatrix} 0 & b & c \\ -b & 0 & d \\ -c & -d & 0 \end{pmatrix}$$

For skew-symmetric matrices, the diagonal elements must be zero, and elements across the main diagonal are negations of each other ($a_{ij} = -a_{ji}$).

Summary

Matrices are fundamental in mathematics and have practical applications in computer science, physics, engineering, economics, and artificial intelligence. Understanding matrix types and operations is essential for solving linear equations, modeling systems, and performing various computations efficiently.

Exercise Problems

Identify Matrix Type

a) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$: Zero matrix and square matrix.

b) $B = \begin{bmatrix} 5 \end{bmatrix}$: Scalar matrix and square matrix (singleton matrix).

c) $C = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$: Scalar matrix (since diagonal and all same).

d) $D = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$: Row matrix.

Matrix Operations

1. **Write down a 3×3 matrix with elements defined as follows:** $a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

   This defines the identity matrix: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. **What is the order of the matrix** $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$?

   The order of matrix A is 2×3 (2 rows, 3 columns).

3. **Write an example of a row matrix.**
   $R = \begin{bmatrix} 7 & 8 & 9 \end{bmatrix}$

4. **What is the transpose of** $A = \begin{bmatrix} 2 & 4 \end{bmatrix}$?
   $A^T = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

5. **State whether the following is a scalar matrix:** $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

   Yes, it is a scalar matrix as all diagonal elements are equal and non-diagonal elements are zero.

6. **Given:** $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}$**; Find out (i)** $A + B$**, (ii)** $A - B$**.**

   (i) $A + B$

$$A + B = \begin{bmatrix} 1+2 & 3+1 \\ 2+0 & 4+5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 9 \end{bmatrix}$$

   (ii) $A - B$

$$A - B = \begin{bmatrix} 1-2 & 3-1 \\ 2-0 & 4-5 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

7. **If** $C = \begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix}$**, find** $3C$**.**

$$3C = \begin{bmatrix} 6 & -3 \\ 12 & 0 \end{bmatrix}$$

8. **Multiply the matrices:** $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 \cdot 1 + 0 \cdot 0 & 2 \cdot 4 + 0 \cdot 2 \\ 1 \cdot 1 + 3 \cdot 0 & 1 \cdot 4 + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 1 & 10 \end{bmatrix}$$

9. **Show that the addition of matrices is commutative.**
   Given: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

$$B + A = \begin{bmatrix} 5+1 & 6+2 \\ 7+3 & 8+4 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

   Since $A + B = B + A$, matrix addition is commutative.

10. **Find the transpose of the matrix:** $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

11. **Symmetric Check:**

$$M = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}, \quad M^T = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \Rightarrow M = M^T, \text{ so it is symmetric.}$$

Conceptual Questions

12. Let, $P = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ **Find:**

i) $P \times P^T$

ii) $P^T \times P$

Comment on the dimensions of the resulting matrices.

Solution: $P^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

i)

$$P \cdot P^T = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 & 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \\ 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 & 4 \cdot 4 + 5 \cdot 5 + 6 \cdot 6 \end{bmatrix} = \begin{bmatrix} 14 & 32 \\ 32 & 77 \end{bmatrix}$$

ii)

$$P^T \cdot P = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 17 & 22 & 27 \\ 22 & 29 & 36 \\ 27 & 36 & 45 \end{bmatrix}$$

Dimensions: $P \cdot P^T$ is 2×2, $P^T \cdot P$ is 3×3

13. A company tracks monthly sales in two regions using matrices:

$$R_1 = \begin{bmatrix} 25000 & 30000 & 28000 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 22000 & 31000 & 29000 \end{bmatrix}$$

a Represent this data as a 2×3 matrix.

b Find the total sales in each month.

c Find the total annual sales in each region.

Solution:

(a) Represent as a 2×3 matrix:

$$R = \begin{bmatrix} 25000 & 30000 & 28000 \\ 22000 & 31000 & 29000 \end{bmatrix}$$

(b) Total sales in each month:

Month 1: 25000+22000 = 47000   Month 2: 30000+31000 = 61000   Month 3: 28000+29000 = 57000

(c) Total annual sales in each region:

$$R_1\colon\ 25000 + 30000 + 28000 = 83000 \quad R_2\colon\ 22000 + 31000 + 29000 = 82000$$
