Simulating periodic solutions of the three-body problem

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Abstract

The three-body problem is one of the most famous problems in physics and has been puzzled over since centuries. This term paper provides a brief introduction to the three-body problem before showing the difficulties that occur when trying to solve the problem analytically. It presents the general three-body problem mathematically using both the Newtonian and the Hamiltonian formulations and shows their equivalence. It shows the importance of the periodic solutions of the three-body problem. In the end, it presents the simulated orbits for a few of these discovered periodic solutions along with the code.

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1 Introduction

In 2009, Laskar and Gastineau [2] ran over 2000 numerical simulations to calculate where every planet in our solar system would be up to 5 billion years in the future. Each simulation started with the same exact initial conditions except for one difference: the distance between Mercury and the Sun was modified by less than a millimetre from one simulation to the next. Shockingly, in about 1 percent of their simulations, Mercury's orbit changed so drastically that it could plunge into the Sun or collide with Venus. In one simulation, it even destabilized the entire inner solar system! This experiment showed that even the slightest change in the initial conditions can drastically alter the dynamics of a system of many gravitating objects. This astonishing property of gravitational systems is called the n-body problem.

It is actually impossible to write down all the terms of a general formula that can exactly describe the motion of three or more gravitating objects. The issue lies in how many unknown variables an n-body system contains. Consider a two-body system, initially, it appears to have more unknown variables for position and velocity than equations of motion. However, considering the relative position and velocity of the two bodies with respect to the system's center of mass reduces the number of unknowns and gives us a solvable system.

The general three-body problem, on the other hand, has no closed solutions. This was proved by Poincaré at the end of the nineteenth century. For each of the three bodies, we have six unknowns: three to specify its position and three to specify its velocity. We thus have 18 unknowns for the whole system. Poincaré showed that a three-body system, in a 3D space, has only 12 constant quantities, or integrals of motion, useful to determine the unknowns. As the unknowns are more, the system is unsolvable.

Although the problem has no closed solutions, solutions have been found when the three bodies occupy some special spatial configurations. One such special form of the general three-body problem was proposed by Euler. He considered three bodies of arbitrary (finite) masses and placed them along a straight line. Euler showed that the bodies would always stay on this line for suitable initial conditions, and that the line would rotate about the center of mass of the bodies, leading to periodic motions of all three bodies along elliptical orbits. Around the same time, Lagrange found a second class of periodic orbits in the general three-body problem. He showed that if the bodies were positioned in such a way that they form a triangle of equal sides, they would move along elliptical orbits for certain initial conditions while always preserving their original configuration. The Euler and Lagrange solutions are now known as particular solutions of the general three-body problem. While remarkable, these solutions do not solve the general problem.

2 The general three-body problem

2.1 Basic formulation

We denote the masses by m_i, where $i = 1, 2$ and 3, and their positions with respect to the origin of an inertial Cartesian coordinate system by the vectors $\vec{r}_i$, and define the position of one body with respect to another by $\vec{r}_{ij} = \vec{r}_j - \vec{r}_i$, where $\vec{r}_{ij} = -\vec{r}_{ji}$, $j = 1, 2, 3$. When the interaction between the bodies is purely gravitational, the equations of motion are given by

$$m_i \frac{d^2\vec{r}_i}{dt^2} = G \sum_{j \neq i}^{3} \frac{m_i m_j}{r_{ij}^3} \vec{r}_{ij}, \tag{1}$$

where G is the universal gravitational constant.

The above set of 3 mutually coupled, second-order, ordinary differential equations (ODEs) can be written explicitly in terms of the components of the vector $\vec{r}_i$. This gives us a set of 9 second-order ODEs. We can further convert these into a set of 18 first-order ODEs which fully describe the general three-body problem.

2.2 Energy and stability

As the system is isolated, the total energy of the system must be conserved.
The kinetic energy T is given by

$$T = \frac{1}{2} \sum_{i=1}^{3} m_i \frac{d\vec{r}_i}{dt} \cdot \frac{d\vec{r}_i}{dt} \tag{2}$$

and the potential energy V is given by

$$V = -\frac{G}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{m_i m_j}{r_{ij}} \tag{3}$$

where $i \neq j$. So, the total energy $E = T + V$ is conserved.

Since $T > 0$ and $V < 0$, E can either be positive, negative or zero, and it can be used to classify motions of the general three-body problem. In the case $E > 0$, the three-body system must split, which means that one body is ejected while the remaining two bodies form a binary system. The special case of $E = 0$ is unlikely to take place in Nature; however, if it indeed occurs, it would result in one body escaping the system. Finally, the case $E < 0$ may lead to either escape or periodic orbits with the result depending on the value of the moment of inertia of the system.

The virial theorem $\langle T \rangle = -\frac{1}{2}\langle V \rangle$, where $\langle T \rangle$ and $\langle V \rangle$ are the time average kinetic and potential energies, can be used to determine the stability of the three-body problem. The system is unstable if its time average kinetic energy is more than two times higher than its time average potential energy.

2.3 Hamiltonian formulation

We consider the natural units and introduce $G = 1$ in Eqs (1) and (3). Moreover, we write $\vec{r}_i = (r_{1i}, r_{2i}, r_{3i}) \equiv q_{ki}$, where r_{1i}, r_{2i} and r_{3i} are components of the vector $\vec{r}_i$ in the inertial Cartesian coordinate system, and $k = 1, 2$ and 3. Using this notation, we define the momentum, p_{ki}, as

$$p_{ki} = m_i \frac{dq_{ki}}{dt}, \tag{4}$$

and the total kinetic energy as

$$T = \sum_{k,i=1}^{3} \frac{p_{ki}^2}{2m_i}, \tag{5}$$

and introduce the Hamiltonian $H = T + V$.
We can now write down the Hamilton's equations of motion as

$$\frac{dq_{ki}}{dt} = \frac{\partial H}{\partial p_{ki}} \quad \text{and} \quad \frac{dp_{ki}}{dt} = -\frac{\partial H}{\partial q_{ki}}, \tag{6}$$

which again gives us a set of 18 first-order ODEs equivalent to the set we got from eq.1.

3 Numerical search for periodic solutions

In 1890, Poincaré showed that the trajectories of a general three-body problem display chaos, i.e, they have sensitive dependence to initial conditions. This realization led to Poincaré's famous dictum: 'what makes these periodic solutions so precious to us, is that they are, so to say, the only opening through which we can try to penetrate in a place which, up to now, was supposed to be inaccessible'.

A practical way to find the periodic solutions is with the help of numerical methods. The infinitesimal quantities present in the original set of equations are replaced with finite, although small, quantities. The resulting, simplified, equations are then solved iteratively. These methods are fast and able to accurately approximate the real solution.

In 2012, a search for periodic orbits was performed numerically by Šuvakov. 13 new distinct periodic orbits were found by Šuvakov and Dmitrašinović [3], who considered the planar general three-body problem with equal masses and zero angular momentum. The authors also presented a new classification of periodic orbits. The 13 distinct orbits (15 in all; 13 of these correspond to distinct orbits) discovered by Šuvakov and Dmitrašinović are shown below:

Class, number and name	$\dot{x}_1(0)$	$\dot{y}_1(0)$	T
I.A.1 butterfly I	0.30689	0.12551	6.2356
I.A.2 butterfly II	0.39295	0.09758	7.0039
I.A.3 bumblebee	0.18428	0.58719	63.5345
I.B.1 moth I	0.46444	0.39606	14.8939
I.B.2 moth II	0.43917	0.45297	28.6703
I.B.3 butterfly III	0.40592	0.23016	13.8658
I.B.4 moth III	0.38344	0.37736	25.8406
I.B.5 goggles	0.08330	0.12789	10.4668
I.B.6 butterfly IV	0.350112	0.07934	79.4759
I.B.7 dragonfly	0.08058	0.58884	21.2710
II.B.1 yarn	0.55906	0.34919	55.5018
II.C.2a yin-yang I	0.51394	0.30474	17.3284
II.C.2b yin-yang I	0.28270	0.32721	10.9626
II.C.3a yin-yang II	0.41682	0.33033	55.7898
II.C.3b yin-yang II	0.41734	0.31310	54.2076

TABLE: Initial conditions and periods of the three-body orbits. $\dot{x}_1(0)$, $\dot{y}_1(0)$ are the first particle's initial velocities in the x and y directions, respectively, T is the period. The other two particles' initial conditions are specified by these two parameters, as follows, $x_1(0) = -x_2(0) = -1, x_3(0) = 0, y_1(0) = y_2(0) = y_3(0) = 0, \dot{x}_2(0) = \dot{x}_1(0), \dot{x}_3(0) = -2\dot{x}_1(0), \dot{y}_2(0) = \dot{y}_1(0), \dot{y}_3(0) = -2\dot{y}_1(0)$. The value of G is taken as $G = 1$ and all the masses are equal, $m_1 = m_2 = m_3 = 1$.

4 Simulation

We now simulate the 15 orbits discovered by Šuvakov and Dmitrašinović [3] using the initial values of the velocities provided in the table above.

We want to solve the following set of differential equations:

$$\frac{d^2\vec{r}_i}{dt^2} = \sum_{j\neq i}^{3} \frac{m_j}{r_{ij}^3}\vec{r}_{ij},$$

where we have set G=1.

Also, we note that any second order ODE

$$\frac{d^2y}{dt^2} = F\left(y, \frac{dy}{dt}, t\right),$$

can be written as a set of two first order ODEs

$$\frac{dy}{dt} = v_y$$
$$\frac{dv_y}{dt} = F(y, v_y, t)$$

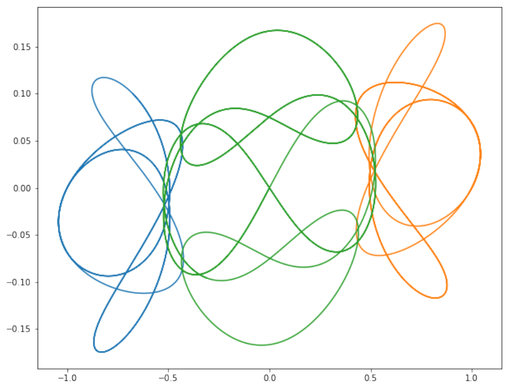
We need to solve for the components of the following matrix given the initial conditions using Python:

$$S = (x_1, y_1, x_2, y_2, x_3, y_3, v_{x1}, v_{y1}, v_{x2}, v_{y2}, v_{x3}, v_{y3})$$

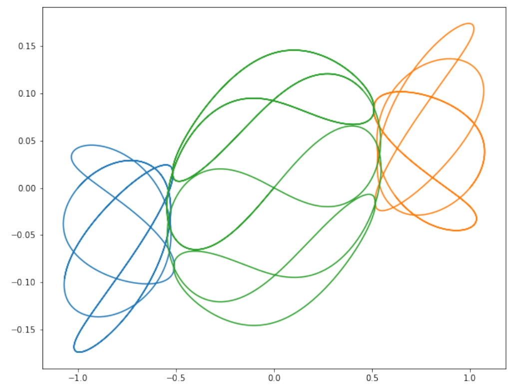
4.1 Python code

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import odeint, solve_ivp

v1 = 0.41734 #setting initial x-velocity
v2 = 0.31310 #setting initial y-velocity

m1 = 1
m2 = 1
m3 = 1
x1_0 = -1
y1_0 = 0
x2_0 = 1
y2_0 = 0
x3_0 = 0
y3_0 = 0
vx1_0 = v1
vy1_0 = v2
vx2_0 = v1
vy2_0 = v2
vx3_0 = -2*v1/m3
vy3_0 = -2*v2/m3

def dSdt(t, S):
    x1, y1, x2, y2, x3, y3, vx1, vy1, vx2, vy2, vx3, vy3 = S
    r12 = np.sqrt((x2-x1)**2 + (y2-y1)**2)
    r13 = np.sqrt((x3-x1)**2 + (y3-y1)**2)
    r23 = np.sqrt((x2-x3)**2 + (y2-y3)**2)
    return [vx1, vy1, vx2, vy2, vx3, vy3,
            m2/r12**3 * (x2-x1) + m3/r13**3 * (x3-x1),
            m2/r12**3 * (y2-y1) + m3/r13**3 * (y3-y1),
            m1/r12**3 * (x1-x2) + m3/r23**3 * (x3-x2),
            m1/r12**3 * (y1-y2) + m3/r23**3 * (y3-y2),
            m1/r13**3 * (x1-x3) + m2/r23**3 * (x2-x3),
            m1/r13**3 * (y1-y3) + m2/r23**3 * (y2-y3)]

t = np.linspace(0, 60, 20000)

sol = solve_ivp(dSdt, (0,60), y0=[x1_0, y1_0, x2_0, y2_0, x3_0, y3_0,
                                  vx1_0, vy1_0, vx2_0, vy2_0, vx3_0, vy3_0],
                method = 'DOP853', t_eval=t, rtol=1e-10, atol=1e-13)

t = sol.t
x1 = sol.y[0]
y1 = sol.y[1]
x2 = sol.y[2]
y2 = sol.y[3]
x3 = sol.y[4]
y3 = sol.y[5]

plt.figure(figsize=(10,8))
plt.plot(x1,y1,x2,y2,x3,y3)
```

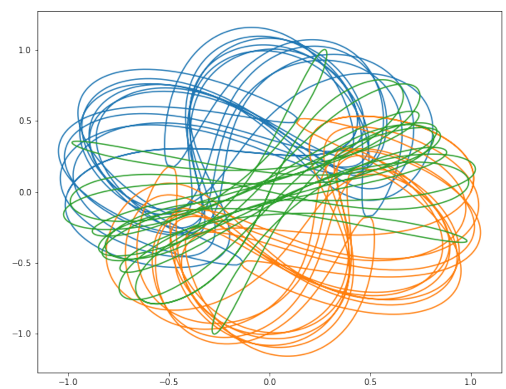
4.2 Simulated orbits



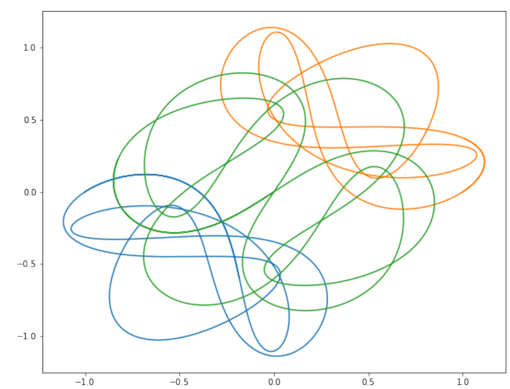
I.A.1 butterfly I



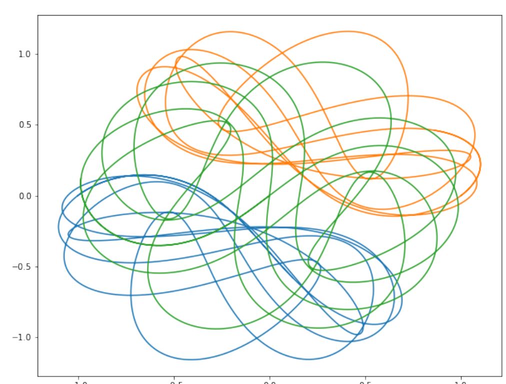
I.A.2 butterfly II



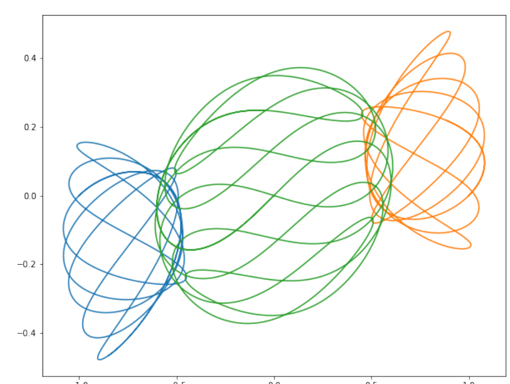
I.A.3 bumblebee



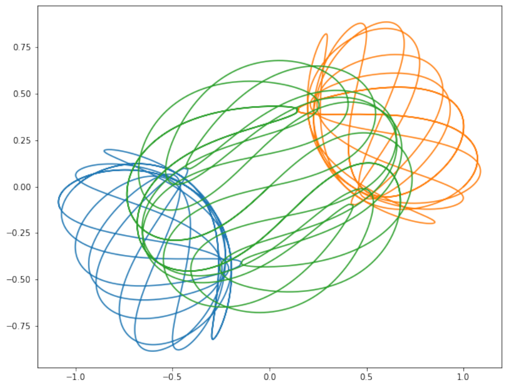
I.B.1 moth I



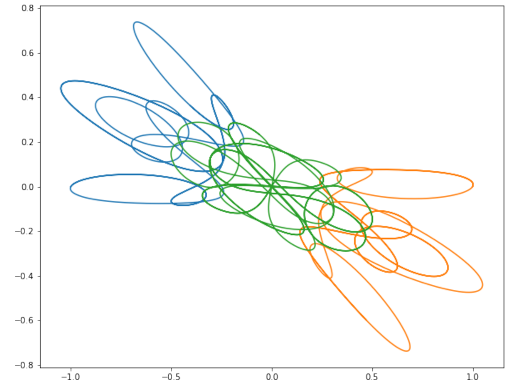
I.B.2 moth II



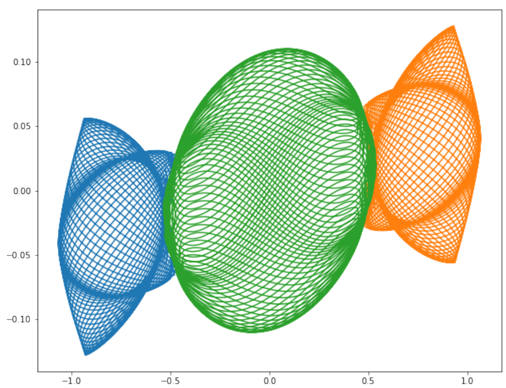
I.B.3 butterfly III



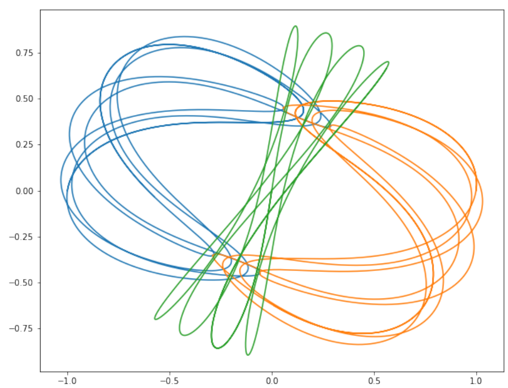
I.B.4 moth III



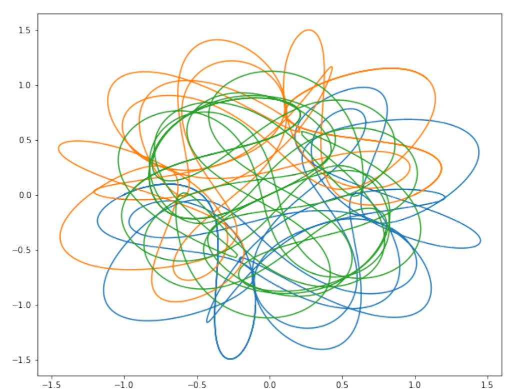
I.B.5 goggles



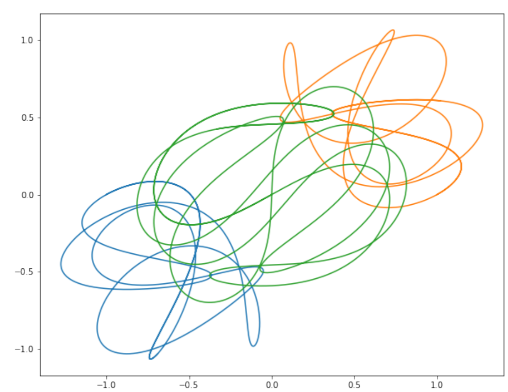
I.B.6 butterfly IV



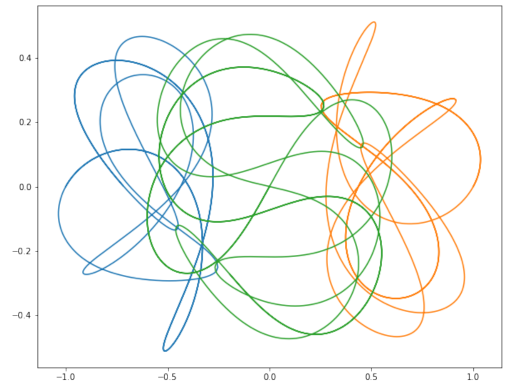
I.B.7 dragonfly



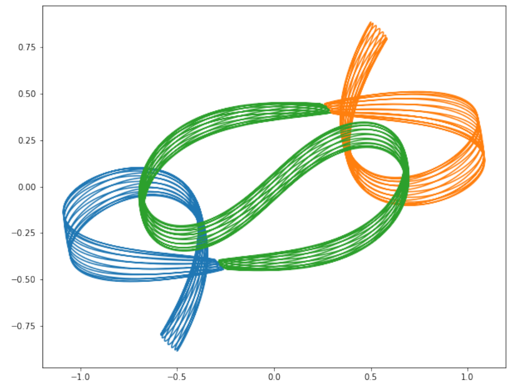
II.B.1 yarn



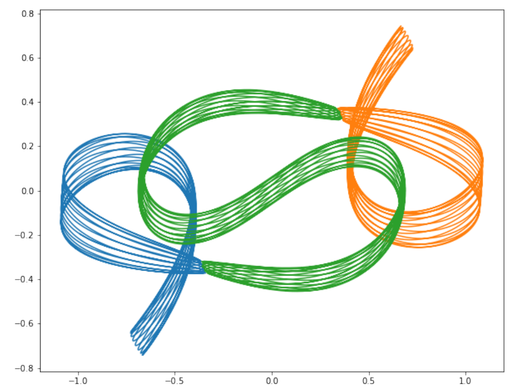
II.C.2a yin-yang I



II.C.2b yin-yang I



II.C.3a yin-yang II



II.C.3b yin-yang II

Animated versions of some of the orbits can be found here: `https://www.youtube.com/playlist?list=PLRwNOtYt5gpr9xx4t5lI8hCHT36tFGo9h`

The code used can be found here: `https://github.com/SaptarshiSrkr/Periodic_Solutions_of_the_Three-Body_Problem`

5 Concluding remarks

The 2013 paper by Šuvakov and Dmitrašinović [3] was one of the first papers to discover the existence of several new periodic solutions of the three-body problem with the help of numerical methods. However, these results have not yet been verified observationally because so far, no astronomical systems with the considered properties of the three bodies (equal mass and zero angular momentum) are known. Since 2013, hundreds of other such periodic orbits have been found through similar numerical methods, including 669 of the equal-mass zero-angular-momentum three-body problem [4] and 1223 solutions for a zero-momentum system of unequal masses [5].

6 References

[1] Musielak, Zdzislaw E., and Billy Quarles. “The three-body problem.” (2014)

[2] Laskar, Jacques, and Mickaël Gastineau. “Existence of collisional trajectories of Mercury, Mars and Venus with the Earth.” (2009)

[3] Šuvakov, Milovan, and Veljko Dmitrašinović. “Three classes of Newtonian three-body planar periodic orbits.” (2013)

[4] Li, XiaoMing, and ShiJun Liao. “More than six hundred new families of Newtonian periodic planar collisionless three-body orbits.” (2017)

[5] Li, Xiaoming, Yipeng Jing, and Shijun Liao. “The 1223 new periodic orbits of planar three-body problem with unequal mass and zero angular momentum.” (2017)