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Fine Properties of Functions with Applications in Brownian Motion

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Abstract:

This report entails the work done by me as a Summer Research Fellow under the guidance of Dr. Anup Biswas at IISER Pune. The topic of our study which was initially fine properties of functions later grew to include some applications of it in the area of Brownian motion. We start with a discussion of the Hausdorff Measure, along with some of its properties. Next, is the remarkable Rademacher's Theorem and some results that lead us to the change of variables formula of integration in a general setting, particularly catering to the cases where the dimensions of the domain and range of the substituted function are not same. This version also does not require the substituted function to be one-one. Then we glance at the concept of weak partial derivatives of functions that leads to function spaces, known as the Sobolev spaces that comes up in connections with solutions to partial differential equations. We see some approximation, extension and inequality results for functions belonging to Sobolev spaces called Sobolev functions. In continuity, we explore Capacity theory, that provides a natural measure of sets from a Potential theoretic view point and its use in characterizing fine properties of Sobolev functions. Next, we look at "functions of bounded variations" that leads to the celebrated Aleksandrov's theorem stating that convex functions are twice differentiable almost everywhere. Finally, we look at dimensions of paths of Brownian motion as an interesting application of the Hausdorff measure. Certain other applications of the topic studied, in areas of partial differential equations and geometry have been briefly mentioned. Since the proofs of most of the results noted in this report are standard, they have been excluded, however proofs to some results of chapter 5 have been included in the Appendix.

Contents

Hausdorff Measure

We are familiar with the Lebesgue measure. It assigns volumes to shapes in $\mathbb{R}^n$. However we can observe that the Lebesgue measure is inherently dependent on the space in which the object is situated. For instance, the Lebesgue measure (volume) of a triangle in a 3-D space is 0, so is that of the surface of a sphere. The definition of Lebesgue measure does not allow us a generalized method to measure these sets, despite the fact that we can, intuitively assign areas to them. Hence arises the need of another measure which takes care of these issues. For the above shapes, we can also, intuitively assign dimensions to them.For instance, a line segment in 3-D space would have dimension 1, where-as a triangle would have dimension 2, hence it would not be natural to assume that the same measure which assigned an area to the triangle also assign length to the line. So we can 'hope' that there is a parameter, based on which the measure is defined, and fine tuning the parameter gives us the measures that we expect of sets. The parameter, which we shall see later, in essence acts like the dimension of the set being measured.

Definition:

The Hausdorff measure of sets in $\mathbb{R}^n$ is defined as:

(i) Let $A \subseteq \mathbb{R}^n, 0 \leq s < \infty, 0 < \delta \leq \infty$. We write

$$H^s_\delta(A) := \inf\left\{ \sum_{j=1}^{\infty} \alpha(s)\left(\frac{diamC_j}{2}\right)^s \middle| A \subseteq \bigcup_{j=1}^{\infty} C_j, diamC_j \leq \delta \right\} \text{ where } \alpha(s) := \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}$$

(ii) For A and s as above, define $H^s(A) := \lim_{\delta \to 0} H^s_\delta(A) = \sup_{\delta > 0} H^s_\delta(A)$.

s, in the above defintion, acts as the parameter discussed earlier and we call H^s the 's-dimensional Hausdorff measure' on $\mathbb{R}^n$

Some properties:

1. The s-dimensional Hausdorff measure for all $s \in [0, \infty)$, H^s is an outer measure in $\mathbb{R}^n$ i.e. $\forall$ collections $\{A_j\}$ of sets in $\mathbb{R}^n$: $H^s(\bigcup_{j=1}^{\infty} A_j) \leq \sum_{j=1}^{\infty} H^s(A_j)$

2. H^s is a borel regular measure in $\mathbb{R}^n$ $\forall$ $s \in [0, \infty)$, i.e. if in a collection of sets $\{A_j\}$, all are borel and disjoint, then $H^s(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} H^s(A_j)$ (Volume of a set which constitutes of disjoint sets should equal the sum of the volumes of the constituent sets.)

3. H^0, i.e. the 0-dimensional Hausdorff measure of a set, gives the cardinality of the set.

4. if $A \subseteq \mathbb{R}^n$ and $s > n$ then $H^s(A) = 0$.

5. Let $A \subseteq \mathbb{R}^n$ and $0 \le s < t < \infty$

   - If $H^s(A) < \infty$ then $H^t(A) = 0$.
   - If $H^t(A) > 0$ then $H^s(A) = \infty$.

We define the Hausdorff dimension of a set $A \subseteq \mathbb{R}^n$ as $H_{dim}(A) := inf\{0 \le s < \infty | H^s(A) = 0\}$. We observe, using 3,4 and 5 that the Hausdorff dimension of a set in $\mathbb{R}^n$ is always defined and that it $\in [0, n]$. This definition of dimension covers the cases that we mentioned earlier, infact it goes beyond, interestingly, we observe that Hausdorff dimension of a set can be non-integers.

Some more properties

1. *Isodiamteric Inequality:* $\forall$ sets A: $L^n(A) \le \alpha(n)\left(\frac{diam A}{2}\right)^n$.(The volume of a set with a given diameter is maximum when the set is a sphere with that diameter). It is interesting also because not everytime a set with a given diameter can be fit inside a sphere with the same diameter but the implication that the above statement would have is true.

2. $H^n(A) = L^n(A)$ $\forall$ sets A. (The Hausdorff measures are an extension of the Lebesgue measure)

Density of sets with respect to the Hausdorff measure

As a natural way, we define the density of a set E at a point x, with respect to an outer-measure μ to be $\lim_{r \to 0} \frac{\mu(E \cap B(x,r))}{\mu(B(x,r))}$,if the limit exists(where $B(x, r)$ denotes the ball of radius r centred at x). We have the following results for density with respect to Lebesgue measure, for Lebesgue measurable sets E:

$$\lim_{r \to 0} \frac{L^n(E \cap B(x,r))}{\alpha(n) r^n} = \begin{cases} 1 \text{ for } L^n\text{-a.e.} x \in E \\ 0 \text{ for } L^n\text{-a.e.} x \notin E \end{cases}$$

Density of sets (with respect to Lebesgue measure) at points inside the set is 1 while at points outside the set is 0.

Along similar lines, we have the following for the s-dimensional Hausdorff measure and H^s measurable set E such that $H^s(E) < \infty$:

$$\lim_{r \to 0} \frac{H^s(E \cap B(x,r))}{\alpha(s) r^s} = 0 \text{ for } H^s\text{-a.e. x} \notin E$$

$$\frac{1}{2^s} \le \limsup_{r \to 0} \frac{H^s(E \cap B(x,r))}{\alpha(s) r^s} \le 1 \text{ for } H^s\text{-a.e. x} \in E$$

Unfortunately, nothing specific about the limit can be said in the second case.

Hausdorff Measure of images and graphs of Functions:

We shall mostly look at a specific class of functions, called Lipschitz functions.

A function f:$\mathbb{R}^n \to \mathbb{R}^m$ is called Lipschitz cotinuous or simply, a Lipschitz function if there exists a constant C such that

$$|f(x) - f(y)| \leq C|x - y| \ \forall x, y \in \mathbb{R}^n$$

for such f, we define

$$\text{lip(f)} := \sup_{x \neq y \in A} \frac{|f(x) - f(y)|}{|x - y|}$$

which, is the smallest constant C for which the inequality above holds.

Hausdorff measure of images of Lipschitz functions:

1. if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function and $A \subseteq \mathbb{R}^n$ and $s \in [0, \infty)$, then:

$$H^s(f(A)) \leq (Lip(f))^s H^s(A)$$

For any $f : \mathbb{R}^n \to \mathbb{R}^m$ and $A \subseteq \mathbb{R}^n$, we define $G(f; A) := \{(x, f(x)) | x \in A\} \subset \mathbb{R}^n \times \mathbb{R}^m$ and we call $G(f; A)$-the graph of f over A

Hausdorff dimension of graphs of functions: if $f : \mathbb{R}^n \to \mathbb{R}^m$ and $L^n(A) > 0$ then

1. $H_{dim}(G(f; A)) \geq n.$
2. If f is Lipschitz, $H_{dim}(G(f; A)) = n.$

Hausdorff Measure of the set where a given function is "large"

if $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ (f is a locally integrable function i.e. f is a real valued function such that for any point x, $\exists$ a compact set containing that point,such that integral of absolute value of f over that compact set is finite), we say f is 'large' at a point y, if $\limsup_{r \to 0} \frac{1}{r^s} \int_{B(y,r)} |f| dx > 0.$

Result:

1. for $0 \leq s < n$, define $\Lambda_s := \left\{ x \in \mathbb{R}^n \middle| f \text{ is large at } x \right\}$. Then $H^s(\Lambda_s) = 0$

Change of Variables Formula:

The change of variables formula is a fundamental result in integration. The change of variables, in a simpler version is stated as:

Let $g : A \rightarrow B$ be a diffeomorphism of open sets in $\mathbb{R}^n$, then for any continuous function $f : B \rightarrow \mathbb{R}$, $f \circ g$ is integrable over A and $\int_A f \circ g |Dg| dx = \int_B f dx$

Here we look at its generalization in the case where the map g is a diffeomorphism between open sets residing in $\mathbb{R}^n$ and $\mathbb{R}^m$ respecitively, where n and m might not be same. Our analysis will be in 2 parts, first where $n \leq m$ and next where $n \geq m$.

We call a function $f : A \rightarrow \mathbb{R}^m$ locally Lipschitz, if f, restricted to any compact subset of A, is Lipschitz.

Rademacher's Theorem: If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is locally Lipschitz, then f is differentiable L^n-a.e.

This is a remarkably general result saying that a function belonging to a specific class of continuous functions, namely the Lipschitz functions although may not be differentiable everywhere (there might be corner like points),but are differentiable at all points except on a set of volume 0.

Some related results:

1. if $f\colon \mathbb{R}^n \rightarrow \mathbb{R}^m$ is locally Lipschitz, and $Z := \{x \in \mathbb{R}^n | f(x) = 0\}$ then $Df(x) = 0$ for L^n-a.e. $x \in Z$.

2. if $f, g\colon \mathbb{R}^n \rightarrow \mathbb{R}^n$ be locally Lipschitz, and $Y := \{x \in \mathbb{R}^n | g(f(x)) = x\}$ Then $Dg(f(x))Df(x) = I$ for L^n-a.e. $x \in Y$.

Defintion of Jacobian

For a linear map $L\colon \mathbb{R}^n \rightarrow \mathbb{R}^m$, we define its Jacobian

$$[[L]] = \sqrt{det(L^* \circ L)} \text{ if } n \leq m \text{ and}$$
$$= \sqrt{det(L \circ L^*)} \text{ if } n \geq m$$

For a locally Lipschitz function $f\colon \mathbb{R}^n \rightarrow \mathbb{R}^m$, we define at x,if f is differentiable there, the **Jacobian** of f to be $Jf(x) := [[Df(x)]]$. Due to Rademacher's Theorem, Jacobian of f is defined L^n-a.e.. We also note that if n=m, then $Jf(x) = |Df(x)|$.

Area Formula:

If $f\colon \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz, $n \leq m$, then for each L^n measurable $A \subset \mathbb{R}^n$:

$$\int_A Jf(x)dx = \int_{\mathbb{R}^m} H^0(A \cap f^{-1}\{y\})dH^n(y)$$

Co-area Formula:

If $f\colon \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz, $n \geq m$. Then for each L^n-measurable set $A \subseteq \mathbb{R}^n$,

$$\int_A Jf(x)dx = \int_{\mathbb{R}^m} H^{n-m}(A \cap f^{-1}\{y\})dy$$

Change of Variables Formula:

1. $n \leq m$: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz, Then for each L^n-summable function $g : \mathbb{R}^n \to \mathbb{R}$,

$$\int_A g(x)Jf(x)dx = \int_{\mathbb{R}^m}\left[\sum_{x\in f^{-1}\{y\}} g(x)\right] dH^n(y)$$

2. $n \geq m$: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz, $n \geq m$, then for each L^n-summable function $g : \mathbb{R}^n \to \mathbb{R}$,

   (i) $g|_{f^{-1}\{y\}}$ is H^{n-m} summable for L^m-a.e. y

   (ii) and

$$\int_{\mathbb{R}^n} g(x)Jf(x)dx = \int_{\mathbb{R}^m}\left[\int_{f^{-1}\{y\}} g(x)dH^{n-m}(x)\right] dy$$

Sobolev Space

We start by defining the concept of weak partial derivative:

Let U be an open subset of $\mathbb{R}^n$. Assume $f \in L^1_{\text{loc}}(U)$ (integral of f over any compact subset of U is finite). For $i \in \{1, 2, \cdots, n\}$. We say $g_i \in L^1_{\text{loc}}(U)$ is the weak partial derivative of f with respect to x_i in U if

$$\int_U f\phi_{x_i} dx = -\int_U g_i\phi dx \tag{†}$$

$\forall\phi \in C^1_c(U)$ ($C^1_c(U)$ denotes all compactly supported continuously differentiable functions in U and ϕ_{x_i} denotes the partial derivative of ϕ with respect t the i^{th} co-ordinate)

Whenever such g_i's exist, they are called the weak partial derivatives of f. It can also be observed that weak partial derivatives-if they exist are uniquely defined L^n-a.e.. If weak partial derivatives ($g'_i s$) exist for a function f then we write:

$$f_{x_i} := g_i$$

$$\text{Df:=}(f_{x_1}, f_{x_2}, \cdots, f_{x_n})$$

It is worthwhile to note that, had f been differentiable then g_i in † would have the i^{th} partial derivative of f and the equation † would have denoted the formula of integration by-parts.

Definition of Sobolev Space and Sobolev function

Let $p \in [1, \infty]$

(i) The real valued function f belongs to the *Sobolev Space*-$W^{1,p}(U)$ if $f \in L^p(U)$ and if $\forall\, i \in \{1, 2, \cdots, n\}$ the weak partial derivative f_{x_i} exist and $\in L^p(U)$.

(ii) The real valued function f belongs to $W^{1,p}_{\text{loc}}(U)$ if $f \in W^{1,p}(V)$ $\forall$ open sets V compactly contained inside U.

(iii) We say f is a *Sobolev function* if $f \in W^{1,p}_{\text{loc}}(U)$ for some $1 \leq p \leq \infty$

We can define a norm on the Sobolev space $W^{1,p}(U)$ such that for a function f in it,

$$\|f\|_{W^{1,p}(U)} := \left(\int_U |f|^p + |Df|^p dx\right)^{\frac{1}{p}} \text{ if } 1 \leq p < \infty, \text{ and}$$

$$\|f\|_{W^{1,\infty}}(U) := \operatorname*{ess\,sup}_U (|f| + |Df|)$$

The Sobolev spaces are actually Banach spaces, with the above norm.

As is expected, we say $f_k \to f$ in $W^{1,p}(U)$ if $\|f_k - f\|_{W^{1,p}(U)} \to 0$ and,

$$f_k \to f \text{ in } W^{1,p}_{\text{loc}}(U) \text{ if } \|f_k - f\|_{W^{1,p}_{\text{loc}}(V)} \to 0 \text{ for each open set } V \subset\subset U$$

Approximation of Sobolev functions by smooth functions

Just like all real numbers can be sppproximated by rational numbers, which is a smaller subset of real numbers, we wish to approximate functions in a Sobolev Space by functions of a smaller subset of the space. In this we use a special technique called mollification.

Mollification: It is a tool which effectively 'smoothens out' a function.

(i) For $\epsilon > 0$, let $U_\epsilon := \{x \in U | \operatorname{dist}(x, \partial U) > \epsilon\}$

(ii) Define the $C^\infty-$ function $\eta : \mathbb{R}^n \to \mathbb{R}$ by

$$\eta(x) := \begin{cases} c \exp\left(\frac{1}{|x|^2 - 1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

c adjusted such that $\int_{\mathbb{R}^n} \eta(x) dx = 1$

(iii)
$$\eta_\epsilon(x) := \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right) \left(\epsilon > 0, x \in \mathbb{R}^n\right)$$

η_ϵ is called the *standard mollifier*.

(iv) If $f \in L^1_{\text{loc}}(U)$,

$$f^\epsilon := \eta_\epsilon * f; \text{ that is}$$
$$f^\epsilon(x) := \int_U \eta_\epsilon(x - y) f(y) dy$$

These f^ϵ's are,as we shall see, the smoothed out functions obtained from f. They also approximate f

Some properties of Mollification:

(i) $\forall \epsilon > 0, f^\epsilon \in C^\infty(U_\epsilon)$

(ii) If $f \in C(U)$, then $f^\epsilon \to f$ uniformly on compact subsets of U.

(iii) If $f \in L^p_{\text{loc}}(U)$ for some $1 \le p < \infty$, then $f^\epsilon \to f$ in $L^p_{\text{loc}}(U)$

(iv) If x is a Lebesgue point of f, then $f^\epsilon \to f$, hence, by Lebesgue differentiation theorem: $f^\epsilon \to f$ L^n-a.e.

(v) If $f \in W^{1,p}_{\text{loc}}(U)$ for some $p \in [1, \infty]$, then $f^\epsilon_{x_i} = \eta_\epsilon * f_{x_i}$ $(i = 1, 2, \cdots, n)$ on U_ϵ

(vi) In particular, if $f \in W^{1,p}_{\text{loc}}(U)$ for some$p \in [1, \infty)$,then $f^\epsilon \to f$ in $W^{1,p}_{\text{loc}}(U)$

Now, although these are approximation results, f^ϵ's might not belong to the Sobolev spaces, since all of their nice properties are valid only inside U_ϵ, hence we have not achieved what we are looking for. Approximation results:

1. Assume $f \in W^{1,p}(U)$ for some $1 \le p < \infty$. Then there exists a sequence $\{f_k\}_{k=1}^\infty \subset W^{1,p}(U) \cap C^\infty(U)$ such that:

$$f_k \to f \text{ in } W^{1,p}(U).$$

Now we wish to approximate any Sobolev function by functions which are smooth all the way up to the boundary, essentiallyby functions in $C^\infty(\bar{U})$

We say the boundary ∂U is Lipschitz if for each $x \in \partial U$, $\exists r > 0$ and a Lipschitz mapping $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, upon our rotating and relabeling the co-ordinate axes (if necessary), we have

$$U \cap Q(x, r) = \{y | \gamma(y_1, \cdots, y_{n-1}) < y_n\} \cap Q(x, r),$$

where $Q(x, r)$ is the cube with center x and side lengths $2r$.

Simply put, near each point $x \in \partial U$, the boundary is the graph of a Lipschitz continuous function.

We can observe, due to Rademacher's Theorem, the outer unit normal $\nu(y)$ to U exists for $H^{n-1}-$a.e. $y \in \partial U$

2. Assume U is bounded and ∂U is Lipschitz.

   (i) If $f \in W^{1,p}(U)$ for some $1 \le p < \infty$, there exists a sequence $\{f_k\}_{k=1}^\infty \subset W^{1,p}(U) \cap C^\infty(\bar{U})$ such that

$$f_k \to f \text{ in } W^{1,p}(U).$$

(ii) If $f \in C(\bar{U})$, then

$$f_k \to f \text{ uniformly}$$

Some additional results:

1. *Product rule of weak derivatives:* If $f, g \in W^{1,p}(U) \cap L^\infty(U)$,then

$$fg \in W^{1,p}(U) \cap L^\infty(U)$$

   and

$$(fg)_{x_i} = f_{x_i} g + f g_{x_i} \ (L^n - a.e.)$$

2. *Chain rule for weak derivatives:* If $f \in W^{1,p}(U)$ and $F \in C^1(\mathbb{R})$, $F' \in L^\infty(\mathbb{R})$, $F(0) = 0$, then

$$F(f) \in W^{1,p}(U)$$

   and

$$F(f)_{x_i} = F'(f) f_{x_i} \text{ for } i = 1, \cdots, n \text{ and } L^n - a.e.x$$

3. *Lipschitz continuity and $W^{1,\infty}$:* Assume $f : U \to \mathbb{R}$ then f is locally Lipschitz in U iff $f \in W^{1,\infty}(U)$

Traces of Sobolev functions:

For Sobolev functions we want to extend them to their boundaries based on their values in the open set, such that the extension for $C^\infty(\bar{U})$ functions match, on the boundaries, with the functional values. Also we can define the trace in such a way that the function along with its trace satisfy the integration byparts (not just with compactly supported functions).

Assume U is bounded and ∂U is Lipschitz, $1 \leq p < \infty$. Then

(i) $\exists$ a bounded linear operator

$$T : W^{1,p}(U) \to L^p(\partial U; H^{n-1})$$

such that

$$Tf = f \text{ on } \partial U$$

$\forall f \in W^{1,p}(U) \cap C(\bar{U})$

(ii) $\forall \phi \in C^1(\mathbb{R}^n; R^n)$ and $f \in W^{1,p}(U)$,

$$\int_U f\, div\phi dx = -\int_U Df.\phi dx + \int_{\partial U} (\phi.\nu) Tf dH^{n-1}.$$

(ν denotes the unit outer normal to ∂U)

Extension of Sobolev functions

Given a Sobolev function in U, we wish to extend it to a Sobolev function in $\mathbb{R}^n$.

Assume U is bounded and ∂U is Lipschitz, $1 \leq p < \infty$. Let V be any open set such that $U \subset\subset V$, then $\exists$ a bounded linear operator

$$E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$$

such that

$$Ef = f \text{ on } U$$

and

$$\text{spt(Ef)} \subset \text{V}$$

$\forall f \in W^{1,p}(U)$

Some inequalities:

For $1 \leq p < n$, define the *Sobolev conjugate-p^** of p as

$$p^* := \frac{np}{n-p}$$

Gagliardo-Nirenberg-Sobolev inequality: For $1 \leq p < n$, $\exists$ a constant C_1, depending only on p and n, such that

$$\left(\int_{\mathbb{R}}^{n} |f|^{p^*} dx\right)^{\frac{1}{p^*}} \leq C_1 \left(\int_{\mathbb{R}}^{n} |Df|^p dx\right)^{\frac{1}{p}}$$

$\forall f \in W^{1,p}(\mathbb{R}^n)$.

Poincaré's inequality: For each $1 \leq p < n \exists$ a constant C_2, depending only on p and n, such that

$$\left(⨍_{B(x,r)} |f - (f)_{x,r}|^{p^*} dy\right)^{\frac{1}{p^*}} \leq C_2 r \left(⨍_{B(x,r)} |Df|^p dy\right)^{\frac{1}{p}}$$

$\forall B(x,r) \subseteq \mathbb{R}^n$, $f \in W^{1,p}(B^0(x,r))$, where $(f)_{x,r} = ⨍_{B(x,r)} f dy$

Morrey's inequality:

(i) For each $p \in (n, \infty)$ $\exists C_3$, depending only on p and n, such that

$$|f(y)-f(z)| \le C_3 r\left(⨍_{B(x,r)}|Df|^p dy\right)^{\frac{1}{p}}$$

$\forall B(x,r) \subseteq \mathbb{R}^n, f \in W^{1,p}(B^0(x,r)), L^n - a.e. y, z \in B(x,r).$

(ii) If $f \in W^{1,p}(\mathbb{R}^n)$, then the limit

$$\lim_{r\to 0}(f)_{x,r} =: f^*(x)$$

exists $\forall x \in \mathbb{R}^n$, and f^* is Holder continuous with exponent $\alpha = 1 - \frac{n}{p}$. (a function $f : U \to \mathbb{R}^m$ is said to be Hölder continuous with exponent α if $\exists C \in \mathbb{R}$ such that $\forall x, y \in U$: $|f(x) - f(y)| \le C|x-y|^\alpha$)

Capacity

Capacity is defined on sets, and it acts like an outer measure. It is used to identify "small" sets, and is suited for the study of fine properties of Sobolev functions.

Definition:

fix $1 \le p < n$, let $K^p := \{f : \mathbb{R}^n \to \mathbb{R} \big| f \ge 0, f \in L^{p^*}(\mathbb{R}^n), Df \in L^p(\mathbb{R}^n, \mathbb{R}^n)\}$

For $A \subset \mathbb{R}^n$, $Cap_p(A) := \inf\left\{ \int_{\mathbb{R}^n} |Df|^p dx \Big| f \in K^p, A \subseteq \{f \ge 1\}^0 \right\}$

$Cap_p(A)$ is called the p-capacity of A

Properties:

1. Capacity is an outer measure. However it is not a borel measure. In fact sets with non zero finite p-capacity are not Cap_p measurable.
2. p-capacity of a set is the infimum of p-capacity of open sets containing it.
3. $Cap_p(\lambda A) = \lambda^{n-p} Cap_p(A)$ $\lambda > 0$
4. p-capacity is preserved under isometry.
5. $Cap_p(A) \le C H^{n-p}(A)$ where the constant C depends only on n, p.
6. $L^n(A) \le C Cap_p(A)^{\frac{n}{n-p}}$ where constant C depends only on n, p
7. $Cap_p(A \cup B) + Cap_p(A \cap B) \le Cap_p(A) + Cap_p(B)$
8. p-capacity satisfies the monotone convergence from below for all sets and the monotone convergence from above for compact sets

Capacity and Hausdorff dimension:

1. $1 < p < n$. If $H^{n-p}(A) < \infty$, then $Cap_p(A) = 0$
2. $1 \leq p < \infty$. If $Cap_p(A) = 0$ then $H^s(A) = 0\ \forall s > n - p$
3. $Cap_1(A) = 0 \iff H^{n-1}(A) = 0$

Quasicontinuity:

Capacity variant of Markov inequality:

For $f \in K^p$, $\epsilon > 0$, Let $A := \{x \in \mathbb{R}^n | (f)_{x,r} > \epsilon$ for some $r > 0\}$, Then

$$Cap_p(A) \leq \frac{C}{\epsilon^p} \int_{\mathbb{R}^n} |Df|^p dx$$

where C only depends on n and p.

Definition of a p-quasiconinuous function:

A function f is *p-quasicontinuous* if $\forall \epsilon > 0, \exists$ an open set V such that

$$Cap_p(v) \leq \epsilon$$

and

$$f\Big|_{\mathbb{R}^n - V} \text{ is continuous.}$$

Precise representatives of Sobolev Functions:

Suppose $f \in W^{1,p}(\mathbb{R}^n)$, $1 \leq p < n$.

(i) There is a borel set $E \subset \mathbb{R}^n$ such that

$$Cap_p(E) = 0$$

and

$$\lim_{r \to 0} (f)_{x,r} =: f^*(x)$$

exists $\forall x \in \mathbb{R}^n - E$ f^* is called the *precise representative* of f

(ii) Also,

$$\lim_{r \to 0} ⨍_{B(x,r)} |f - (f)_{x,r}|^{p^*} dy = 0$$

$\forall x \in \mathbb{R}^n - E$

(iii) The precise represntative f^* is p-quasicontinuous

Differentiability along canonical lines of precise representatives of Sobolev functions:

We recall a result from Analysis: If $h : \mathbb{R} \to \mathbb{R}$ islocally absolutely continuous,then h is differentiable L^1−a.e. and it is denoted by h'.

Theorem:

(i) If $f \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$, then for each $k = 1, \cdots, n$ the functions

$$f^*_k(x', t) := f^*(.., x_{k-1}, t, x_{k+1}, ..)$$

are absolutely continuous in t on compact subsets of $\mathbb{R}$ for L^{n-1}−a.e. points $x' = (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots) \in \mathbb{R}^{n-1}$. Also:

$$(f^*_k)' \in L^p_{\text{loc}}(\mathbb{R}^n)$$

(ii) Conversely, if $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ and $f = g\ L^n - a.e.$, where foreach $k = 1, 2, \cdots, n$, the functions

$$g_k(x', t) := g(x_1, \cdots, x_{k-1}, t, x_{k+1}, \cdots)$$

are locally absolutely continuous in t for$L^{n-1} - a.e.$ points $x' = (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots) \in \mathbb{R}^{n-1}$ and $g'_k \in L^p_{\text{loc}}(\mathbb{R}^n)$. Then $f \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$

The function f^* is defined for points in the plane on the line passing through those points, perpendicular to the plane (parallel to the k^{th} canonical basis).

Functions of Bounded Variations, Aleksandrov's Theorem and some results

We now look at a specific class of functions, whose weak partial derivatives are, in a sense measure. This class of function is bigger than the Sobolev space and functions in this space are differentiable in the weakest (measure theoretic) sense.

Definition:

A function $f \in L^1(U)$ has *bounded variation* in U if

$$\sup\left\{ \int_U f div\phi dx \middle| \phi \in C^1_c(U; \mathbb{R}^n), |\phi| \leq 1 \right\} < \infty$$

$BV(U)$ denotes the space of functions of bounded variation in U.

A function $f \in L^1_{\text{loc}}$ is said to be in $BV_{\text{loc}}(U)$ if $f \in BV(V)$ $\forall$ open $V \subset\subset U$.

Structure Theorem for BV_{loc} functions:

Assume $f \in BV_{\text{loc}}(U)$ then $\exists$ a Radon measure μ on U and a μ−measurable function $\sigma : U \to \mathbb{R}^n$ such that

(i) $|\sigma(x)| = 1$ $\mu-$a.e.

(ii) $\forall \phi \in C_c^1(U;\mathbb{R}^n)$, we have

$$\int_U f div\phi dx = -\int_U \phi.\sigma d\mu$$

If f has weak partial derivatives, then the lebesgue integral of their norm is the radon measure μ

Twice differentiability of convex functions and an extension result:

1. $f: \mathbb{R}^n \to \mathbb{R}$ is called convex if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \forall \lambda \in [0,1], x, y \in \mathbb{R}^n$. Then f is locally Lipschitz.

2. **Aleksandrov's Theorem:** If $f: \mathbb{R}^n \to \mathbb{R}$ is convex, then f is twice differentiable for L^n-a.e. x.(Once differentiability follows from above result and Rademacher's Theorem, but this goes beyond to say that such f will be twice differentiable)

3. Let $C \subset \mathbb{R}^n$ be closed and $f: C \to \mathbb{R}$,$d: C \to \mathbb{R}^n$ are given, we define for compact $K \subseteq C, \delta > 0$

$$\rho_K(\delta) := \sup\left\{ \frac{|f(y) - f(x) - d(x).(y-x)|}{|x-y|} \middle| 0 < |x-y| \leq \delta, x \neq y \in K \right\}$$

Whitney's Extension theorem: If f, d are continuous and for each compact set $K \subseteq C$, $\rho_K(\delta) \to 0$ as $\delta \to 0$, then $\exists \bar{f}: \mathbb{R}^n \to \mathbb{R}$ such that

- $\bar{f}$ is C^1.
- $\bar{f} = f, D\bar{f} = d$ on C.

($\rho_K(\delta) \to 0$ as $\delta \to 0$ gives a kind-of strong differentiability property of f inside C, and then this theorem extends the function as a properly differentiable function to $\mathbb{R}^n$)

Brownian motion

Brownian motion are a model to track random trajectories of pollen grains (first observed in depth by Robert Brown).

Definition

They are formally defined as a stochastic process.

The d-dimensional Brownian motion: $B_t(\omega) : \Omega \times \mathbb{R}^+ \to \mathbb{R}^d$. Thus B_t is a random variable. It should satisfy:

(i) $B_0 = 0$ $\forall \omega$

(ii) B_t is almost surely continuous.

(iii) B_t has independent increments.

(iv) $B_t - B_s \; N(0, t-s)$ for $0 \le s \le t$

Some properties:

1. Brownian paths are almost surely Hölder continuous with any index $0 < \gamma < \dfrac{1}{2}$:
   $\exists\Omega_0$ such that $\mathbb{P}(\Omega - \Omega_0) = 0$ and $\forall\omega \in \Omega_0$ and $\gamma \in (0, \frac{1}{2})\exists L_\gamma(\omega){:}B_t(\omega) - B_s(\omega) \le L_\gamma(\omega)(t-s)^\gamma$
   where $\mathbb{P}(L_\gamma < \infty) = 1$

2. Brownian paths are nowhere differentiable.

Some useful results

1. *Energy Criterion*(A1): Let μ be a probability measure on E.Then

$$\iint_{E\times E} \frac{d(\mu(x))d(\mu(y))}{|x-y|^\alpha} < \infty \Rightarrow \text{dim(E)} \ge \alpha$$

2. *Frostman's Theorem*(A2): Let K be a compact set. Then

$$\alpha < \dim K \Rightarrow H^\alpha(K) > 0$$
$$\Rightarrow \exists \text{ finite measure } \sigma \text{ such that } \forall\beta < \alpha : \iint_{K\times K} \frac{d(\sigma(x))d(\sigma(y))}{|x-y|^\beta} < \infty$$

3. (A3) If f is Hölder continuous with index $\gamma \in (0,1)$ then $\dim f(E) \le \gamma^{-1}\dim E$.

Hausdorff Dimension of Brownian paths:

1. (A4) Let $(B_t)_{t\ge 0}$ be a BM^d and $F \subset [0,1]$, a closed set. Then

$$\dim B(F)\text{=}\min\{d, 2\dim F\} \text{ a.s.}$$

2. (A5) If $d \ge 2$ in above result, then the null set, does not depend on F, i.e., if $(B_t)_{t\ge 0}$ be a BM^d and $E \subset [0,1]$, a Borel set. Then

$$\mathbb{P}\big(\dim B(E) = 2\dim E \; \forall E \in \mathcal{B}[0,1]\big) = 1$$

3. (A6) Let $(B_t)_{t\ge 0}$ be a BM^d and $F \subset [0,1]$, a closed set. Then

$$\dim \mathrm{Gr}B(F) \text{ =}\min\{2\dim F\text{,}\dim F + \frac{1}{2}\} \text{ a.s.}$$

Some applications:

1. Application of Sobolev Spaces in Partial Differential equations and Mathematical physics: for a body moving with velocity function f, one defines the kinetic energy associated with it as $\int_I \frac{(\nabla f)^2}{\|f\|^2} dt$. Our target is to minimise it over certain sets of functions. If we consider the set to be that of all differentiable functions, then an infima of the values attained by the energy function would exist but that value might not be attained by a differentiable function. Same holds, if we consider the set to be C^∞ functions. However, it can be shown that the infima is attained in the Sobolev Space. Also weak derivatives share some of the well known properties of actual derivatives.

2. Application of Hausdorff Measure in Geometry: Fractals are complicated and elegant geometrical figures. Their sizes may not always be captured by Lebesgue measure, for instance the Lebesgue measure of Cantors set or the Sierpinski's Gasket is 0, despite both being uncountable. However due to Hausdorff measure, we can assign dimensions to them.

Appendix

A1: Energy Criteria:Let μ be a probability measure on E.Then

$$\iint_{E\times E} \frac{d(\mu(x))d(\mu(y))}{|x-y|^\alpha} < \infty \Rightarrow \dim(\mathrm{E}) \geq \alpha$$

Proof:

$\iint_{E\times E} \frac{d(\mu(x))d(\mu(y))}{|x-y|^\alpha} < \infty \Rightarrow \exists$ borel set $B \subseteq E$ such that $\mu(B) > 0$ and $\int_E \frac{d(\mu(y))}{|x-y|^\alpha} < M$ $(\forall x \in B)$

Let (B_j) be any covering of B, we have $|B_j| \geq |x-y| \forall x, y \in B_j$

$\Rightarrow \mu(B_j) \leq |B_j|^\alpha \int_{B_j} \frac{d(\mu(y))}{|x-y|^\alpha}$

$\Rightarrow \sum_{j=1}^{\infty} |B_j|^\alpha \geq \sum_{j=1}^{\infty} \frac{\mu(B_j)}{M} \geq \frac{\mu(B)}{M} > 0,$

$\Rightarrow \mathcal{H}^\alpha(E) \geq \mathcal{H}^\alpha(B) \geq \frac{\mu(B)}{M} > 0 \Rightarrow \dim E \geq \alpha.$ ■

A2: Frostman's Theorem: Let K be a compact set. Then

$$\alpha < \dim K \Rightarrow H^\alpha(K) > 0$$
$$\Rightarrow \exists \text{ finite measure } \sigma \text{ such that } \forall \beta < \alpha : \iint_{K\times K} \frac{d(\sigma(x))d(\sigma(y))}{|x-y|^\beta} < \infty$$

Frostman's Lemma: If $A \subset \mathbb{R}^d$ is a closed set such that $\mathcal{H}^s > 0$, then there exists a Borel probability measure μ supported on A and a constant $C > 0$ such that $\mu(D) \leq C|D|^s$ for all borel sets D.

Proof: We prove it using a result in graph theory called the Maxflow-Mincut Theorem.

Given a,possibly infinite, tree with vetex set V and edge set E and root node r, we define a cutset as a set of edges such that any path-infinite or terminating has an edge in this set. We also define a flow function $f(e)$ on the edges subject to $f(e) \leq b(e)$ where $b(e)'s$ are bounds or capacities of the edges, the flow function should also satisfy the Kirchoff criterion-

$f(e) = \sum f(e_v)$ ($\forall$ vertices $v \neq r$) where e is the edge terminating at vertex v and e_v's are the edges starting from v. The maxflow-mincut theorem states that

$$\|f\| := \sup_{f \leq b} f(e_r) = \inf_{\text{all cutsets}-\pi} \sum_{e \in \pi} b(e)$$

Construction of the tree: we construct the tree in steps, at first step let the tree consist of only one vertex-the centre of the unit cube. Call this tree T_1. Next we emanate edges from this vertex to centers of all (2^d) dyadic cubes of side length $\frac{1}{2}$. We call this structure T_2. Then for all vertices newly created, we emanate edges to centres of dyadic cubes of side $\frac{1}{4}$ inside each of the newly formed cubes. We call this T_3. We go on, in this fashion to create T_n $\forall n \in \mathbb{N}$.The tree that we are interested in is the limiting tree T_∞. We define the bounds/capacities as: if e ends in vertex, whose corresponding cube has empty intersection with A, then $b(e) := 0$ else, if e got introduced in T_n, i.e. $e \in E(T_n) - E(T_{n-1})$ then $b(e) := 2^{-ns}$.Consider a cut-set π, then the cubes associated with end points of these edges cover A, as for any $x \in A$, there is an infinite path emanating from root, such that all cubes associated with end points of edges in this path contain x.Let $\{C_j\}_{j=1}^\infty$ denote the cubes associated with end points of cut-set π

$$\sum_{e \in \pi} b(e) = \sum_{j=1}^{\infty} |C_j|^s \geq \frac{\mathcal{H}^s(A)}{\alpha(s)} > 0 \Rightarrow \exists \text{ flow with } \|f\| > 0$$

We observe that the set of all dyadic cubes, cubes arising out of construction of T_∞ form a semi-algebra, and the sigma algebra generated by them would be $\mathcal{B}([0,1]^d)$, we define a measure on the semi-algebra-$\mu^*(C) = f(e)$ where e is the edge ending at C(This is a measure as f satisfies the kirchoff property). By Caratheodory's extension theorem, this can be extended to a measure μ on $\mathcal{B}([0,1]^d)$, as because μ^* is non-zero, thus μ is also non-zero, also it is easy to see that μ is supported inside A. For any $D \in \mathcal{B}(\mathbb{R}^d)$, we define $\mu(D) := \mu(D \cap [0,1]^d)$.

Let $n \in \mathbb{N}$ such that $2^{-n} \leq |D \cap [0,1]^d| \leq 2^{-(n-1)}$, thus 3^d cubes of side 2^{-n}, arising in construction of T_∞ cover $D \cap [0,1]^d \Rightarrow \mu(D) \leq 3^d 2^{-ns} \leq 3^d |D|^s$. Since μ is non-zero, Renormalising gives us a borel probability measure, with the desired property.

Proof of Frostman's Theorem: consider $x \in K$, define $S_n(x) = \{y : 2^{-n} \leq |x-y| \leq 2^{-(n-1)}\}$ and let μ be the probability measure with the property in the lemma for $s = \dim K > \beta > \alpha$.

$$\begin{aligned}\int_K \frac{d(\mu(y))}{|x-y|^\alpha} &= \sum_{j=1}^{\infty} \int_{S_j} \frac{d(\mu(y))}{|x-y|^\alpha} \\ &\leq \sum_{j=1}^{\infty} \int_{S_j} 2^{n\alpha} d(\mu(y))\end{aligned}$$

$$= \sum_{j=1}^{\infty} 2^{n\alpha}\mu(S_j)$$

$$\leq \sum_{j=1}^{\infty} 2^{n\alpha}|S_j|^{\beta}$$

$|S_j| \leq \frac{4}{2^n}$, hence
$$\leq 4^{\beta} \sum_{j=1}^{\infty} 2^{n(\alpha-\beta)}$$

hence, as μ is a probability measure : $\iint_{K\times K} \frac{d(\mu(x)\mu(y)}{|x-y|^{\alpha}} \leq 4^{\beta} \sum_{j=1}^{\infty} 2^{n(\alpha-\beta)} < \infty.$ ■

Results pertaining to Brownian Motion

A3: Dimesnion of image of Hölder continuous functions: If f is Hölder continuous with index $\gamma \in (0,1)$ then $\dim f(E) \leq \gamma^{-1}\dim E$.

Proof: Let f be Hölder continuous with exponent γ, then $|f(E_j)| \leq L|E_j|^{\gamma}$. Thus any δ-cover of E induces a $L\delta^{\gamma}$-cover of $f(E)$. If $\alpha > \frac{\dim E}{\gamma} \Rightarrow \alpha\gamma > \dim E \Rightarrow \mathcal{H}^{\alpha\gamma}(E) = 0$

Fix $\epsilon > 0$, we get a δ covering $\{E_j\}$of E, such that $L^{-\alpha}\sum |f(E_j)|^{\alpha} \leq \sum |E_j|^{\alpha} < \epsilon \Rightarrow \mathcal{H}^{\alpha}(f(E)) = 0$

$$\Rightarrow \dim f(E) \leq \alpha$$

taking $\alpha \rightarrow \frac{\dim E}{\gamma}$ gives us $\dim f(E) \leq \frac{\dim E}{\gamma}$. ■

A4: Dimension of image of Brownian motion: Let $(B_t)_{t\geq 0}$ be a BM^d and $F \subset [0,1]$, a closed set. Then

$$\dim B(F) = \min\{d, 2\dim F\} \text{ a.s.}$$

proof: We complete the proof in 2 steps,we first prove the quantity given is the upper bound and then we prove that any number lesser than the quantity is a lower bound.

Upper bound: $\because B_t$ is Hölder continuous with any exponent $\gamma\frac{1}{2}$, $\dim B(F) \leq \gamma^{-1}\dim F$, a.s. making $\gamma \rightarrow \frac{1}{2}$ along a countable sequence gives $\dim B(F) \leq 2\dim F$ a.s.. $\dim B(F) \leq d$ holds trivially, hence $\dim B(F) \leq \min(d, 2\dim F)$

Lower bound:Pick α such that $\lambda := 2\alpha < \min(d, 2\dim F) \because \mathcal{H}^{\alpha}(F) > 0 \Rightarrow$ (Frostman's Theorem) $\exists$ a Borel probability measure σ such that

$$\iint_{F\times F} \frac{d(\sigma(t))d(\sigma(s))}{|t-s|^{\alpha}} < \infty$$

We, now define the occupation measure, the time that the Brownian motion spends in a Borel set $A \in \mathcal{B}(\mathbb{R}^d)$:

$$\mu_{\sigma}^{B}(A) = \int_F \mathbb{1}_A(B_t)d(\sigma(t))$$

$$\mathbb{E}\left[\iint_{B(F)\times B(F)} \frac{d(\mu_\sigma^B(x))d(\mu_\sigma^B(y))}{|x-y|^\lambda}\right] = \mathbb{E}\left[\iint_{F\times F} \frac{d(\sigma(t))d(\sigma(s))}{|B_t - B_s|^\lambda}\right] = \iint_{F\times F} \mathbb{E}\left[\frac{1}{|B_t - B_s|^\lambda}\right] d\sigma(t)d\sigma(s)$$

The last step is due to Tonelli's Theorem, due to stationary increments and scaling, $B_t - B_s \sqrt{t-s}B_1$.

$$= \mathbb{E}\left[|B_1|^{-\lambda}\right] \iint_{F\times F} \frac{d\sigma(t)d\sigma(s)}{|t-s|^{\frac{\lambda}{2}}}$$

$\mathbb{E}\left[|B_1|^{-\lambda}\right]$ is finite as $\lambda < d$ and the integral is finite due to Frostman's Theorem. Thus the expectation is$< \infty$, hence the quantity inside the expectation is finite a.s.. Hence, by Energy Criterion: $\dim B(F) \geq \lambda$-a.s.(note that the null set obtained here may depend on F).

Making $\lambda \to$min$(d, 2\dim F)$ along a countable sequence gives $\dim B(F) \geq$min$(d, 2\dim F)$-a.s. ■

A6: Dimension of graph of Brownian Motion Let $(B_t)_{t\geq 0}$ be a BM^d and $F \subset [0,1]$, a closed set. Then

$$\dim \mathrm{Gr}B(F) = \min\{2\dim F, \dim F + \frac{1}{2}\} \text{ a.s.}$$

Proof: Here also we complete the proof in 2 steps, in a similar manner as the previous proof.

Upper bound: We require 2 different coverings of the graph for the 2 terms in the quantity. Pick $\alpha >\dim F$. Then there is some $\delta-$covering (F_j) of F by closed sets, with diameter $|F_j =: \delta_j \leq \delta$. $\mathrm{Gr}B(F_j) \subset F_j \times B(F_j)$, due to Hölder continuity of Brownian motion, we get $\forall \gamma \in (0, \frac{1}{2})$:

$$|\mathrm{Gr}B(F_j)| \leq |F_j| + |B(F_j)| \leq |F_j| + L_\gamma |F_j|^\gamma \leq (L_\gamma + 1)\delta^\gamma$$

Hence the closed sets $F_j \times B(F_j)$ are an $(L_\gamma + 1)\delta^\gamma-$covering of $\mathrm{Gr}B(F)$ and

$$\sum |\mathrm{Gr}B(F_j)|^{\frac{\alpha}{\gamma}} \leq (L_\gamma + 1)^{\frac{\alpha}{\gamma}} \sum |F_j|^\alpha < \infty$$

Thus $\dim \mathrm{Gr}B(F) \leq \gamma^{-1}\dim F$-a.s..

Letting $\gamma \to \frac{1}{2}$ via a countable sequence, gives $\dim \mathrm{Gr}B(F) \leq 2\dim F$-a.s.

On the other hand, using the Hölder continuity of $t \to B_t$, we can see that each $F_j \times B(F_j)$ is covered by $c\delta_j^{(\gamma-1)d}$ many cube of diameter δ_j. This gives a further $\delta-$ covering of $\mathrm{Gr}B(F)$ satisfying:

$$\sum (c\delta_j^{(\gamma-1)d})\delta_j^{\alpha+(1-\gamma)d} = c\sum \delta_j^\alpha < \infty$$

Thus $\mathcal{H}^{\alpha+(1-\gamma)d}(\mathrm{Gr}B(F)) < \infty \Rightarrow \dim \mathrm{Gr}B(F) \leq \alpha + (1-\gamma)d$-a.s.. Letting $\alpha \to\dim F$ and $\gamma \to \frac{1}{2}$ along a countable sequence gives $\dim \mathrm{Gr}B(F) \leq \alpha + \frac{1}{2}d$-a.s.. This proves the upper bound.

Lower bound: Assume first that $\dim F \leq \frac{1}{2}d$. Since $B(F)$ is the projection of $\mathrm{Gr}B(F)$ onto $\mathbb{R}^d$, and $\because$ projections are Hölder continuous with $\gamma = 1$, we see $\dim \mathrm{Gr}(B(F) \geq \dim B(F) \geq 2\dim F \geq$min$\{2\dim F, \dim F + \frac{1}{2}d\}$-a.s.

The situation dim$F > \frac{1}{2}d$ ca occur only if $d = 1$, then, dim Gr$B(F)$ $\geq$dim $F + \frac{1}{2}$-a.s.. This is because: fix $\gamma \in (1,\dim F + \frac{1}{2})$, $\because \gamma - \frac{1}{2}$ <dim$F$, we conclude from Frostman's Theorem, that $\exists$ a Borel probability measure on F such that

$$\iint_{F\times F} \frac{d\sigma(x)d\sigma(y)}{|x-y|^{\gamma-\frac{1}{2}}} < \infty$$

Consider the occupation measure

$$\mu_\sigma^{\mathrm{Gr}(B)}(A) = \int_F \mathbb{1}_A(t, B_t)d\sigma(t),\ A \in \mathcal{B}(\mathbb{R}^2)$$

$$\mathbb{E}\left[\iint_{\mathrm{Gr}B(F)\times \mathrm{Gr}B(F)} \frac{d\mu_\sigma^{\mathrm{Gr}B}(x)d\mu_\sigma^{\mathrm{Gr}B}(x)}{|x-y|^\gamma}\right] = \mathbb{E}\left[\iint_{F\times F} \frac{d\sigma(t)d\sigma(s)}{(|t-s|^2 + |B_t - B_s|^2)^{\frac{\gamma}{2}}}\right]$$

$$\leq \mathbb{E}\left[\iint_{F\times F} \frac{d\sigma(t)d\sigma(s)}{|B_t - B_s|^\gamma}\right]$$
$$= c_\gamma \iint_{F\times F} \frac{d\sigma(s)d\sigma(t)}{|t-s|^{\frac{\gamma}{2}}} \leq c_\gamma \iint_{F\times F} \frac{d\sigma(s)d\sigma(t)}{|t-s|^{\gamma-\frac{1}{2}}} < \infty,$$

hence, following similar argument, we can say dim Gr$B(F) \geq \gamma-$a.s. and making $\gamma \to$dim$F + \frac{1}{2}$ along a countable sequence, gives dim Gr$B(F)$ $\geq$dim$F + \frac{1}{2}$-a.s.. ■

A5: Exceptional set does not depend on the choice of F,i.e. If $d \geq 2$ in above result, then the null set, does not depend on F, i.e., if $(B_t)_{t\geq 0}$ be a BM^d and $E \subset [0,1]$, a Borel set. Then

$$\mathbb{P}\big(\dim B(E) = 2\dim E\ \forall E \in \mathcal{B}[0,1]\big) = 1$$

Proof: We present a proof, that is due to a lemma by Kaufman

Lemma: Let B_t be a BM^d, $d \geq 2$, define $I_n^k = [(k-1)4^{-n}, k4^{-n}]$, $k = 1, 2, \cdots, 4^n$, denote $A_n \subset \Omega$, the event

$$A_n := \{\omega \in \Omega \Big| \exists D = D(2^{-n}) : \#\{k : B^{-1}(D,\omega) \cap I_n^k \neq \Phi\} \geq n^{2d}\}$$

Then $N_0 := \limsup_{n\to\infty} A_n$ is a $\mathbb{P}-$ null set.

Proof of main theorem

The null set in the upper bound dim$B(E) \leq 2$dim$E$ does not depend on the set E $\forall E \in \mathcal{B}([0,1])$

For the lower bound, assume $E \in \mathcal{B}([0,1])$, and define I_n^k as before. Let (S_j) be a $\delta-$ covering of $B(E)$ with $2^{-n(j)+1} \geq |S_j| \geq 2^{-n(j)}$. Then S_j is contained in a ball D_j of radius $2^{-n(j)}$. WLOG: $n(1) \leq n(2) \leq \cdots$ and so $\delta \leq 2^{-n(1)+1}$.

Let N_0 be as previously defined and $\Omega_0 = \Omega - N_0$ and for sufficiently large $n(1)$, i.e. small δ, we can cover D_j, hence S_j by atmost $n(j)^{2d}$ sets of the the form $B(I_n^k, \omega)$. The corresponding intervals in $[0, 1]$,

$$I_n(j)^{k(1,j)}, \cdots I_n(j)^{k(i,j)}, \, i = i(j) \leq n(j)^{2d} \; j \geq 1$$

are a δ^2 cover of the set E. If $\beta < \alpha <$dim$E$, $\mathcal{H}^\alpha(E) = \infty, \therefore \exists c > 0$ such that

$$0 < c \leq \sum_{j=1}^{\infty} \sum_{l=1}^{j} |I_{n(j)}^{k(l,j)}|^\alpha \leq \sum_{j=1}^{\infty} n(j)^{2d} 4^{-\alpha n(j)} \leq C \sum_{j=1}^{\infty} 4^{-\beta n(j)} \leq C \sum_{j=1}^{\infty} |S_j|^{2\beta} \text{ on } \Omega_0$$

Since the $\delta-$ cover of $B(E, \omega)$ is arbitrary, we conclude that $\mathcal{H}^{2\beta}(B(E, \omega) \geq \frac{c}{C} > 0$, thus dim$B(E, \omega) \geq 2\beta \; \forall \omega \in \Omega_0$ and all Borel sets $E \subset [0, 1]$. Making $\beta \rightarrow$dim$E$ along a countable sequence completes the proof. ■

Notation:

1. Throughout the report, sets are considered to be subsets of $\mathbb{R}^n$, unless specified otherwise. U, V denote open subsets and K denotes compact subsets of $\mathbb{R}^n$
2. $L^p_{\text{loc}}(A) : \{f : \int_K |f|^p < \infty \; \forall$ compact $K \subseteq A\}$
3. $C(U) : \{f : U \rightarrow \mathbb{R} \big| f$ is continuous$\}$
4. $C^\infty(A) : \{f : U \rightarrow \mathbb{R} \big| f$ is infinitely differentiable$\}$
5. $V \subset\subset U$: V is compactly contained in U (V is an open set, such that $\exists$ compact K where $V \subseteq K \subseteq U$)
6. $C_c^k(U; \mathbb{R}^n) = \{f : U \rightarrow \mathbb{R} \big| f$ is compactly supported inside U, and is k times continuously differentiable.$\}$

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