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Boundary Value Problem in Electrostatics: Method of images

Theory

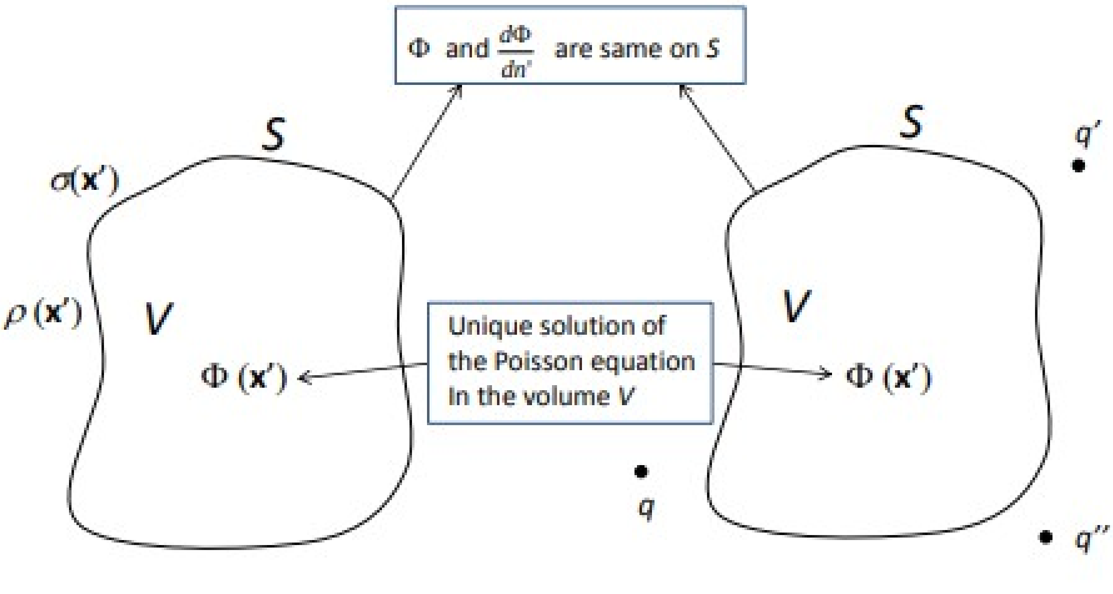
1. Method of images:

The method of images is based on the uniqueness theorem: for a given set of boundary conditions the solution to the Poisson's equation is unique. The following figure compares two different external charge distributions,

(a) ρ, Ω, and

(b) q, q', q'', ...

but the boundary conditions of two systems are identical. Then, the potentials are same inside the regions. The image must be external to the volume of the interest.

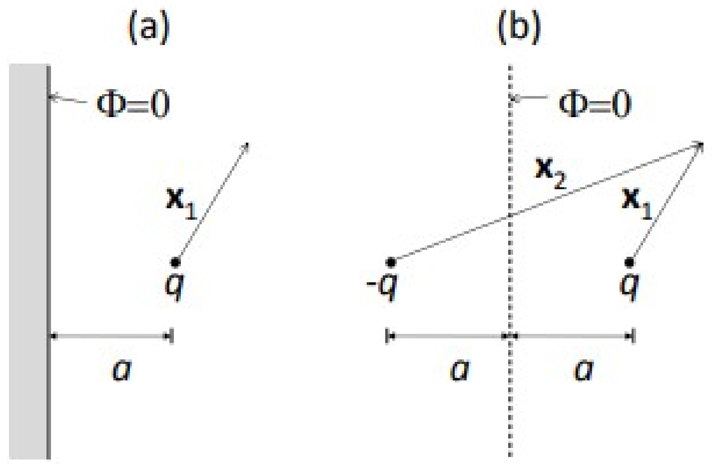


(a) External volume charge distribution ρ(x') and Surface charge distribution σ(x')

(b) Image charges q, q', q'', ...

2. Point charge and counducting plane:

A point charge q is placed near a counducting plane of infinite extent. The boundary condition is that Φ = 0, on the surface of the conducting plane. Let the conducting plane coincide with the yz – plane and the point charge line on the x – axis at x = a. Consider now a system of two point charges a distance 2a apart as shown in the following figure.

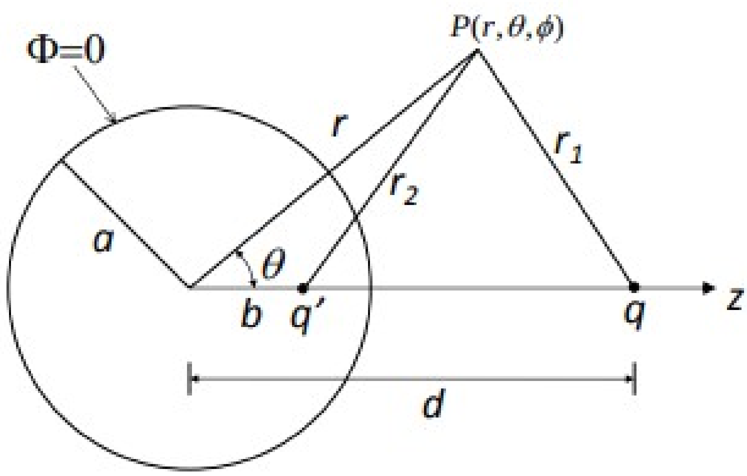


Potential: The potential of the two charges,

$$\Phi(x,y,z)=\frac{1}{(4\pi\varepsilon_o)}\frac{q}{r_1}-\frac{1}{(4\pi\varepsilon_o)}\frac{q}{r_2}=\frac{q}{(4\pi\varepsilon_o)}[\frac{1}{(\sqrt{(x-a)^2+y^2+z^2})}-\frac{1}{(\sqrt{(x+a)^2+y^2+z^2})}] \quad (1)$$

satisfies not only the poisson equation for x>0 and the boundary at all points exterior to the cahrges, but also the boundary condition of the original problem. Therefore, eq. (1) is the correct potential in the entire half-space exterior to the conducting plane(x>0).

3. Point charge and conducting sphere:



The above figure illustrates a point charge in the vicinity of a grounded conducitng sphere of radius a. It is convenient to formulate the problem by the means of spherical coordinates, with the origin of coordinates at the center of the sphere. Let the charge q at z = d on the z- axis. The boundary condition, Φ(r = a) = 0, can be satisfied by an image charge q' inside the sphere (z = b).

Potential: The potential due to the cahrges q and q'

$$\Phi(r,\theta,\phi)=\frac{1}{(4\pi\varepsilon_o)}\frac{q}{r_1}+\frac{1}{(4\pi\varepsilon_o)}\frac{(q')}{r_2}=\frac{1}{(4\pi\varepsilon_o)}\left[\frac{q}{(\sqrt{r^2+d^2-2rd\cos\theta})}+\frac{(q')}{(\sqrt{r^2+b^2-2rb\cos\theta})}\right]$$

Guide to the code of this problem

1. For point charge and conducting plane:

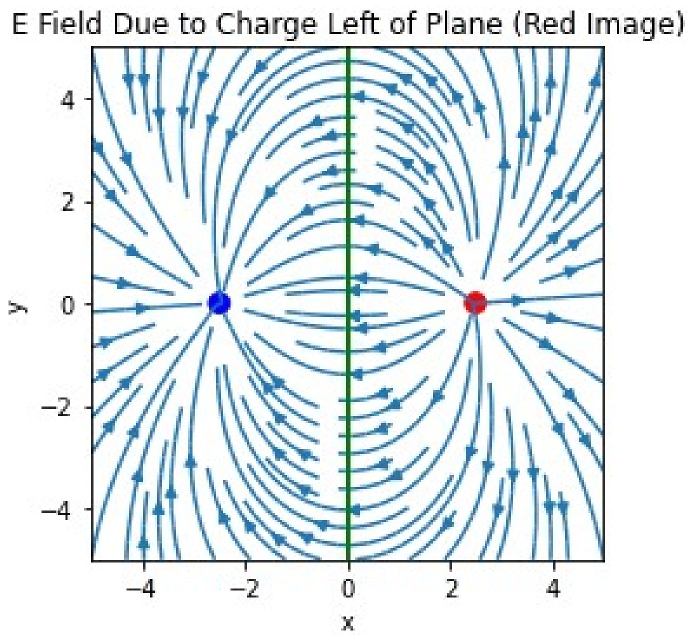
```
%matplotlib inline
import numpy as np,  matplotlib.pyplot as plt
from matplotlib.patches import Circle

Nx = 50;                   Ny = 50              # x,y 50 grid
x = np.linspace(-5,5,Nx);   y = np.linspace(-5,5,Ny)
X,Y = np.meshgrid(x,y)                    # Transform coordinates
Ex = np.zeros((Nx,Ny));     Ey = np.zeros((Nx,Ny))   # Ex,Ey(x,y)

def E(xx,x,y):                  # E  due to charge q at xx
   r = np.sqrt(x**2+y**2)                           # Distance
   dm = (x-xx)                          # Position q to xx
   d1 = np.sqrt((dm**2+y**2))                 # Position q to x
   dp = (x+xx)                            # x component q
   d2 = np.sqrt((dp**2+y**2))              # Distance -q to (x,y)
   Ex =  dm/d1**3-dp/d2**3
   Ey = y/d1**3 -y/d2**3
   return Ex,Ey

Ex,Ey = E(2.5,X,Y)
fig = plt.figure()
ax = fig.add_subplot(111)
circle1 = plt.Circle((2.5, 0),0.2, color='r')
circle2 = plt.Circle((-2.5, 0),0.2, color='b')
ax.add_artist(circle1)
ax.add_artist(circle2)
ax.streamplot(x,y,Ex,Ey)
ax.set_aspect('equal')
ax.set_title('E Field Due to Charge Left of Plane (Red Image)')
ax.set_xlabel('x')
ax.set_ylabel('y')
l = plt.axvline(x=0, linewidth=2, color='g')
plt.show()
```

Output: The output of the above code gives the image that is requied to calculate the potential of the point charge when it is place in the vicinity of a plane charge.



2. For point charge and conducting sphere:

```
%matplotlib inline
import numpy as np,   matplotlib.pyplot as plt
from matplotlib.patches import Circle

Nx = 50;   Ny = 50; q = 1          # x, y grids, charge strength
x = np.linspace(-5,5,Nx);     y = np.linspace(-5,5,Ny)
X,Y = np.meshgrid(x,y)                  # Transform coordinates
Ex = np.zeros((Nx,Ny));       Ey = np.zeros((Nx,Ny))    # Arrays
xx = 3.0;   yy = 3.0                    # Charge coordidates
dq = np.sqrt(xx*2+yy*2);      a = 1.    # Origin, sphere radius
qp = -a*q/dq                         # Magnitude image charge
xp =  a**2*xx/dq**2;          yp =  a**2*yy/dq**2

def E(xx,yy,x,y):                    # xx,yy coord image q
   r = np.sqrt(x**2+y**2)
   dx = (x-xx);              dy = (y-yy)
   d1 = np.sqrt((dx**2 + dy**2))          # Distance q to (x,y)
   dpx = x-xp;               dpy = y-yp
   d2 = np.sqrt((dpx**2 + dpy**2))      # Distance -q to (x,y)
   Ex =  dx/d1**3-dpx/d2**3;       Ey = dy/d1**3 -dpy/d2**3
   return Ex,Ey

Ex,Ey = E(xx,yy,X,Y)
fig = plt.figure()
ax = fig.add_subplot(111)
```

```
circle1 = plt.Circle((xx, yy),0.2, color='r')
circle2 = plt.Circle((xp,yp),0.2, color='b')
sphere = plt.Circle((0,0),a, color='y',alpha=0.5)
ax.add_artist(circle1);      ax.add_artist(circle2)
ax.add_artist(sphere);       ax.streamplot(x,y,Ex,Ey)
ax.set_aspect('equal')
ax.set_title('E Field for a Charge (in Sphere) & Image')
ax.set_xlabel('x');          ax.set_ylabel('y')
plt.show()
```

Output: The output gives us the image which helps us to calculate the potential of the system when a point charge is placed in the vicinity of the sphere.