Linear Algebra

Date
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Assignment -2

Problems:-
Are the following sets of vectors linearly independent or dependent ?

1) $[1\ 0\ 0], [1\ 1\ 0], [1\ 1\ 1]$

Solve:- Let $u_1 = (1,0,0)$, $u_2 = (1,1,0)$, $u_3 = (1,1,1)$

Now, Matrix (A) = $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$\therefore$ $Det(A) = 1(1-0) - 0 + 0 = 1 \neq 0$

$\therefore$ It is linearly independent. Ans

Note:- Different methods to find LI & LD :-

① Definition check:
- if the only sol^n is $c_1v_1 + c_2v_2 + c_3v_3 + \dots + c_nv_n = 0$ is $c_1 = c_2 = \dots = c_n = 0$, then LI.

② Matrix Determinant method: [vectors as columns]
if $det(A) \neq 0$, then LI.

③ Row Reduction (Gaussian Elimination)
$A:B \quad \left[\begin{array}{cc:c} a & b & 0 \\ c & d & 0 \end{array}\right] \quad \rho(A)$

④ Rank Nullity Theorem
If the rank of the matrix formed by the vectors equal no of vec[tors] they are linearly independent.

eg:- $v_1 = [1,2]$, $v_2 = [3,4]$, Form matrix & calc. its rank

$$\boxed{\underbrace{\rho(A)}_{\text{Rank}} + \underbrace{N}_{\text{Nullity}} = \underbrace{D}_{\text{Dimension/ no. of col}}}$$

$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$

$\rho(A) = 2$.

Dimension (no of col) = 2

$\therefore N = D - \rho(A)$
$= 2 - 2$
$= \boxed{0}$

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2. $[7\ -3\ 11\ -6], [-56\ 24\ -88\ 48]$

Solve:- Let $a[7\ -3\ 11\ -6] + b[-56\ 24\ -88\ 48] = 0$.

$\therefore\ 7a - 56b = 0 \rightarrow a = 8b$

$-3a + 24b = 0$

$11a - 88b = 0 \rightarrow$ Put $a = 8b \Rightarrow 11(8b) - 88b = 0 \rightarrow 0 = 0$

$-6a + 48b = 0$ (non-trivial solⁿ)

Let $b = k$, then $a = 8k$ $\therefore$ Eigen Space $= k\begin{bmatrix} 8 \\ 1 \end{bmatrix}$

- Also, the second vector is 8 times the First vector.

  $\therefore$ It is Linearly Dependent.

3. $[-1\ 5\ 0], [16\ 8\ -3], [-64, 56\ 9]$

Solve:- Now, $A = \begin{bmatrix} -1 & 16 & -64 \\ 5 & 8 & 56 \\ 0 & -3 & 9 \end{bmatrix}$

$\det(A) = -1(72 + 168) - 16(45 - 0) + (-64)(-15 - 0)$

$= -96 - 720 + 960$

$= -816 + 960$

$= \boxed{144} \neq 0$ $\therefore$ It is linearly Independent.

4. $[1\ -1\ 1], [1\ 1\ -1], [-1\ 1\ 1], [0\ 1\ 0]$

Solve:- Let $(A):B =$
$A:B$ $\begin{bmatrix} 1 & 1 & -1 & 0 & : 0 \\ -1 & 1 & 1 & 1 & : 0 \\ 1 & -1 & 1 & 0 & : 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 1 & 1 & -1 & 0 & : & 0 \\ -1 & 1 & 1 & 1 & : & 0 \\ 0 & 0 & 2 & 1 & : & 0 \end{bmatrix}$

$R_2 \longrightarrow R_2 + R_1$

$\Rightarrow A:B = \begin{bmatrix} 1 & 1 & -1 & 0 & : & 0 \\ 0 & 2 & 0 & 1 & : & 0 \\ 0 & 0 & 2 & 1 & : & 0 \end{bmatrix}$

Here, $\rho(A) = 3 = \rho(A:B)$

$\therefore$ It is Linearly Independent. [also vectors are not sc… multiple of each ot…

5) $[2\ -4], [1\ 9], [3\ 5]$

Solu:- Let $a[2\ -4] + b[1, 9] + c[3\ 5] = 0$

$\therefore 2a + b + 3c = 0$ — (i) $\Rightarrow b = -3c - 2a$ — (iii)

$\therefore -4a + 9b + 5c = 0$ — (ii).

Put b in (ii), $-4a + 9(-3c - 2a) + 5c = 0$

$\Rightarrow -4a - 27c - 18a + 5c = 0$

$\Rightarrow -22a + -22c = 0$

$\Rightarrow \boxed{a = -c}$

So, from (iii), $b = -3a - 2a = -5a$

if $a = k$, then

Now, here we have $X = \begin{bmatrix} k \\ -5k \\ -k \end{bmatrix} = k\begin{bmatrix} 1 \\ -5 \\ -1 \end{bmatrix}$ is a non-trivial solⁿ.

Hence, it's a linearly dependent (set of vectors)

6) $[3\ -2\ 0\ 4], [5\ 0\ 0\ 1], [-6\ 1\ 0\ 1], [2\ 0\ 0\ 3]$

Solu:- Here, $c_1v_1 + c_2v_2 + c_3v_3 = 0$

For scalars c_1, c_2, c_3 where v_1, v_2, v_3 are given vectors

$\therefore c_1\begin{bmatrix} 3 \\ 5 \\ -6 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_4\begin{bmatrix} 4 \\ 1 \\ 1 \\ 3 \end{bmatrix} = 0$

$\therefore 3c_1 - 2c_2 + 4c_4 = 0$ — (ii)

$5c_1 + c_4 = 0 \Rightarrow \boxed{c_4 = -5c_1}$

$-6c_1 + c_2 + c_4 = 0$ — (iii)

$2c_1 + 3c_4 = 0$ — (i)

Now, Put c_4 in (i), $2c_1 + 3(-5c_1) = 0 \Rightarrow 2c_1 - 15c_1 = 0 \Rightarrow \boxed{c_1 = 0}$

$\therefore \boxed{c_4 = 0}$

Now, from (ii) & (iii), $2(3c_1 - 2c_2 + 4c_4) = 0$

$-6c_1 + c_2 + c_4 = 0$

$-c_2 + 5c_4 = 0 \Rightarrow \boxed{c_2 = 5c_4}$

$\because c_4 = 0 \Rightarrow \boxed{c_2 = 0}$

Since, $c_1 = c_2 = c_3 = c_4 = 0$

$\therefore$ Linearly Independent

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7) $[3\ 4\ 7], [2\ 0\ 3], [8\ 2\ 3], [5\ 5\ 6]$

Soln:- let $A = \begin{bmatrix} 3 & 2 & 8 & 5 \\ 4 & 0 & 2 & 5 \\ 7 & 3 & 3 & 6 \end{bmatrix}$

Now, we perform elementary row operations, to convert into row echlon form

$$A = \begin{bmatrix} 3 & 2 & 8 & 5 \\ 4 & 0 & 2 & 5 \\ 7 & 3 & 3 & 6 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 3 & 2 & 8 & 5 \\ 4 & 0 & 2 & 5 \\ 3 & 3 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 3 & 2 & 8 & 5 \\ 1 & -2 & -6 & 0 \\ 0 & 1 & -7 & -4 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -2 & -6 & 0 \\ 3 & 2 & 8 & 5 \\ 0 & 1 & -7 & -4 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & -2 & -6 & 0 \\ 0 & 8 & 26 & 5 \\ 0 & 1 & -7 & -4 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -2 & -6 & 0 \\ 0 & 1 & -7 & -4 \\ 0 & 8 & 26 & 5 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 8R_2} \begin{bmatrix} 1 & -2 & -6 & 0 \\ 0 & 1 & -7 & -4 \\ 0 & 0 & 82 & 37 \end{bmatrix}$$

Here By Rank Nullity Theorem,

$$\rho(A) + N = D$$

$$\Rightarrow 3 + N = 4 \Rightarrow \boxed{N = 1}$$

Here, no. of unknown $= 4 = n$

But $\rho(A) = 3 \neq n$

$\therefore$ It is infinite soln. and linearly Independent.

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8) $[6\ 0\ 3\ 1\ 4\ 2]$, $[0\ -1\ 2\ 7\ 0\ 5]$, $[12\ 3\ 0\ -19\ 8\ -11]$

Solve:- Let $c_1[6\ 0\ 3\ 1\ 4\ 2] + c_2[0\ -1\ 2\ 7\ 0\ 5] + c_3[12\ 3\ 0\ -19\ 8\ -11] = 0$

(1) $\therefore\ 6c_1 + 12c_3 = 0 \Rightarrow \boxed{c_1 = -2c_3}$

(2) $-c_2 + 3c_3 = 0 \Rightarrow \boxed{c_2 = 3c_3}$

(3) $3c_1 + 2c_2 = 0 \Rightarrow \boxed{c_2 = \frac{-3c_1}{2}}$

(4) $c_1 + 7c_2 - 19c_3 = 0 \Rightarrow c_1 - \frac{21}{2}c_1 - 19c_3 = 0 \Rightarrow \frac{-19}{2}c_1 - 19c_3 = 0$

$\Rightarrow \frac{-19}{2}(-2c_3) - 19c_3 = 0$ $[\because c_1 = -2c_3]$

$\Rightarrow \frac{38}{2}c_3 - 19c_3 = 0 \Rightarrow 38c_3 - 38c_3 = 0 \Rightarrow \boxed{0 = 0}$

(5) $4c_1 + 8c_3 = 0 \Rightarrow 4(-2c_3) + 8c_3 = 0 \Rightarrow -8c_3 + 8c_3 = 0$

$\boxed{0 = 0}$

(6) $2c_1 + 5c_2 - 11c_3 = 0$

$\Rightarrow 2(-2c_3) + 5(3c_3) - 11c_3 = 0$

$\Rightarrow -4c_3 + 15c_3 - 11c_3 = 0$

$\Rightarrow -15c_3 + 15c_3 = 0$

$\Rightarrow \boxed{0 = 0}$

Let $c_3 = k$, then,

$c_1 = -2k,\ c_2 = 3k$

$$\therefore X = \begin{bmatrix} -2k \\ 3k \\ k \end{bmatrix} = k\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$, this is infinite solⁿ.

$\therefore$ It is linearly Dependent Ans.