Notes Assignment - 1

Algebra of Matrices

• Practice Examples :—

(a) Test for consistency and solve.

(i) $2x - 3y + 7z = 5$,
$3x + y - 3z = 13$,
$2x + 19y - 47z = 32$.

Here, $A = \begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$

Soln :- It can be written in form of $AX = B$. { Here, $AX = B$ ∴ non-homogeneous }

$\therefore [A:B] = \begin{bmatrix} 2 & -3 & 7 & : & 5 \\ 3 & 1 & -3 & : & 13 \\ 2 & 19 & -47 & : & 32 \end{bmatrix}$

$\xrightarrow{R_1 \leftrightarrow R_2}$ $\approx \begin{bmatrix} 3 & 1 & -3 & 13 \\ 2 & -3 & 7 & 5 \\ 2 & 19 & -47 & 32 \end{bmatrix}$

$R_1 \to \frac{1}{3} R_1$.

$\begin{bmatrix} 1 & \frac{1}{3} & -1 & 13 \\ 2 & -3 & 7 & 5 \\ 2 & 19 & -47 & 32 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_3} \begin{bmatrix} 1 & \frac{1}{3} & -1 & 13 \\ 0 & -22 & 54 & -27 \\ 2 & 19 & -47 & 32 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & \frac{1}{3} & -1 & 13 \\ 0 & -22 & 54 & -27 \\ 0 & \frac{55}{3} & -44 & -7 \end{bmatrix}$

$\xrightarrow{R_3 \to 6R_3 - 55R_1}$ $\begin{bmatrix} 1 & \frac{1}{3} & -1 & 13 \\ 0 & -22 & 54 & -27 \\ -55 & 0 & 11 & -708 \end{bmatrix}$

$\therefore [A:B] = \begin{bmatrix} 2 & -3 & 7 & : & 5 \\ 3 & 1 & -3 & : & 13 \\ 2 & 19 & -47 & : & 32 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2} R_1} \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{2} & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{bmatrix}$

$R_2 \to R_2 -$ [illegible], $R_3 \to R_3 -$ [illegible]

$\begin{bmatrix} 1 & \frac{3}{2} & \frac{7}{2} & 5 \\ 0 & \frac{1}{2} & -\frac{27}{2} & -2 \\ 0 & 22 & -54 & 22 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 4R_2} \begin{bmatrix} 1 & \frac{3}{2} & \frac{7}{2} & 5 \\ 0 & \frac{11}{2} & -\frac{27}{2} & -2 \\ 0 & 0 & [illegible] & 30 \end{bmatrix}$

Here, $\boxed{\rho(A:B) = 3}$ $\boxed{\rho(A) = 2}$ $\neq$ no. of unknown.

$\rho(A) \neq \rho(A:B)$

∴ The given system is inconsistent (No Solution)

(ii) $2x - y + 3z = 8$

$-x + 2y + z = 4$

$3x + y - 4z = 0$

Solve:- Given eq. can be written in form of $AX = B$.

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

Augmented matrix:

$$[A:B] = \begin{bmatrix} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & 4 & 0 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 4 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & 4 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 4 \\ 0 & \frac{3}{2} & \frac{1}{2} & 8 \\ 0 & \frac{5}{2} & -\frac{1}{2} & -12 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - \frac{5}{3}R_2} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 4 \\ 0 & \frac{3}{2} & \frac{1}{2} & 8 \\ 0 & 0 & -\frac{4}{3} & -\frac{76}{3} \end{bmatrix}$$

Here, Rank, $\rho(A:B) = \rho(A) = 3$

$\therefore$ System is consistent. (Unique sol^n)

Eq's are:- $-\frac{4}{3}z = -\frac{76}{3} \Rightarrow \boxed{z = 19}$

$\frac{3}{2}y + \frac{1}{2}z = 8 \Rightarrow \frac{3}{2}y + \frac{19}{2} = 8 \Rightarrow \frac{3}{2}y = -\frac{3}{2} \Rightarrow \boxed{y = -1}$

$x - \frac{1}{2}y + \frac{3}{2}z = 4 \Rightarrow x + \frac{1}{2} + \frac{57}{2} = 4 \Rightarrow 2x + 58 = 8 \Rightarrow 2x = -50 \Rightarrow \boxed{x = -25}$

(iii) $4x - y = 12$

$-x + 5y - 2z = 0$

$-2x + 4z = -8$

Solve:- Given eq. can be in form of $AX = B$, $\begin{bmatrix} 4 & -1 & 0 \\ -1 & 5 & -2 \\ -2 & 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 8 \end{bmatrix}$

$$\therefore [A:B] = \begin{bmatrix} 4 & -1 & 0 & : & 12 \\ -1 & 5 & -2 & : & 0 \\ 2 & 0 & 4 & : & 8 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{4}R_1} \begin{bmatrix} 1 & -\frac{1}{4} & 0 & 3 \\ -1 & 5 & -2 & 0 \\ 2 & 0 & 4 & 8 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & \frac{1}{4} & 0 & 3 \\ 0 & \frac{19}{4} & -2 & 3 \\ 0 & \frac{1}{2} & 4 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - \frac{2}{19}R_2} \begin{bmatrix} 1 & \frac{1}{4} & 0 & 3 \\ 0 & \frac{19}{4} & -2 & 3 \\ 0 & 0 & \frac{89}{19} & \frac{32}{19} \end{bmatrix}$$

r, $\rho(A:B) = \rho(A) = 3 = n$.

$\therefore$ Consistent & Unique

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Now, $\frac{80}{19}z = \frac{32}{19} \Rightarrow z = \frac{32}{80} \Rightarrow \boxed{z = \frac{2}{5}}$

$\frac{19}{4}y - 2z = 3 \Rightarrow \frac{19}{4}y - \frac{4}{5} = 3 \Rightarrow \frac{19}{4}y = \frac{19}{5} \Rightarrow \boxed{y = \frac{4}{5}}$

$x + \frac{1}{4}y = 3 \Rightarrow x + \frac{1}{5} = 3 \Rightarrow x = 3 - \frac{1}{5} \Rightarrow \boxed{x = \frac{14}{5}}$

(b) For what values of λ an μ the given sys of eq^n

$x+y+z=6$ has (i) no solution.

$x+2y-3z=10$ (ii) a unique sol^n.

$x+2y+\lambda z=\mu$ (iii) infinite no. of sol^n.

Solu. Given eq. in form of $AX=B$, $\begin{bmatrix}1&1&1\\1&2&-3\\1&2&\lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}6\\10\\\mu\end{bmatrix}$

$\therefore [A:B] = \left[\begin{array}{ccc:c}1&1&1&6\\1&2&-3&10\\1&2&\lambda&\mu\end{array}\right] \xrightarrow{R_3\to R_3-R_2} \begin{bmatrix}1&1&1&6\\1&2&-3&10\\0&0&(\lambda+3)&(\mu-10)\end{bmatrix}$

$\xrightarrow{R_2\to R_2 \ldots} \begin{bmatrix}1&1&1&6\\0&1&-4&4\\0&0&(\lambda-3)&(\mu-\ldots\end{bmatrix}$

Echlon form

Here, $\rho(A:B) = \rho(A) = 3 =$ no. of unknow.

$\therefore$ Consistent & Unique sol^n.

(i) No Sol^n, if $\lambda=3$, $\mu\neq 10$, rank $\rho(A)=2$, $\rho(A:B)=3$.

$\rho(A) \neq \rho(A:B)$.

$\therefore$ inconsistent and no sol^n

(ii) Unique Sol^n; if $a \neq 3$, then $\rho(A)=\rho(A:B)=n$.

$\therefore$ unique sol^n.

(iii) infinite Sol^n, if $\lambda=3$, $\mu=10$, then, $\rho(A)=\rho(A:B)=2 \neq n$

$\therefore$ infinite sol^n.

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$(\lambda^2-1)-3(\lambda-1)$
$\lambda^2-1-3\lambda+3 \quad \lambda^2-3\lambda+2.$

(c) Find for what value of λ the given eq.

$x+y+z=1$ have a sol^n and solve them completely in each case

$x+2y+4z=\lambda$

$x+4y+10z=\lambda^2$

Solu:- Given eq. in form of $AX=B$, $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & (\lambda-1) \\ 0 & 3 & 9 & (\lambda^2-1) \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_2}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & (\lambda-1) \\ 0 & 0 & 0 & (\lambda^2- \end{bmatrix}$$

Here, $\rho(A:B)=3$, $\rho(A)=2$.

$\rho(A:B) \neq \rho(A)$ $\therefore$ Inconsistent

In order that given eq. has sol^n, it should be consistent

$\rho(A:B)$ should $=$ $\rho(A)$

Rank-$A = 2$, Since rank of $[A:B]$ should also be 2, it is necessary

$\lambda^2-3\lambda+2=0$

$\Rightarrow \lambda^2-2\lambda-\lambda+2=0 \Rightarrow \lambda(\lambda-2)-1(\lambda-2)$

$\Rightarrow \boxed{\lambda=1,2}$

Case 1:- For $\lambda=1$, $\quad x+y+z=1.$ $\qquad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$y+3z=0$

No. of unknown (3) > Rank of A(2)

$\therefore$ Equations have infinite solution and we have to assign 3-2

Let $z=k$. $\therefore y+3k=0 \Rightarrow \boxed{y=-3k}$

& $x+y+z=1 \Rightarrow x-3k+k=1 \Rightarrow \boxed{x=2k+1}$

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Case:- ② For $\lambda = 2$ $\therefore \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\therefore x + y + z = 1.$

$y + 3z = 1.$

Here, also we have to assign $3 - 2 = 1$ Para. say k.

Let $z = k$.

$\therefore \boxed{y = 1 - 3k}$

$\therefore x + y + z = 1.$

$\Rightarrow x + \cancel{1} - 3k + k = \cancel{1} \Rightarrow \boxed{x = 2k}$

(d) Find the solⁿ. of system of eqⁿ $x + 3y - 2z = 0$, $2x - y + 4z = 0$, $x - 11y + 14z = 0$.

Solⁿ.- Here, given eq. are

$x + 3y - 2z = 0$

$2x - y + 4z = 0$

$x - 11y + 14z = 0$

it can be expressed in $AX = B$.

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here, $AX = 0$ $\therefore$ System is consistent (Unique / Infinite).

Now, [A [illegible]] $= \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 & -2 \\ 0 & -5 & 8 \\ 0 & -14 & 16 \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{14}{5}R_2} \begin{bmatrix} 1 & 3 \\ 0 & - \\ 0 & 0 \end{bmatrix}$

$\rho(A) = 3 =$ no. of unknown

$\therefore$ Consistent and has Unique Solⁿ. Ans.

(e) Find for what values of λ the given eqs $3x + y - \lambda z = 0$, $4x - 2y - 3z = 0$, $2\lambda x + 4y + \lambda z = 0$ may posses non-trivial sol and solve them completely in each case.

Soln:- Given eqs can be in form of $AX = B$

$3x - y - \lambda z = 0$

$4x - 2y - 3z = 0$

$2\lambda x + 4y + \lambda z = 0$

$$\begin{bmatrix} 3 & -1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here, $AX = 0$ ($\because$ System is consistent (Unique / Infinite))

Now, $A = \begin{bmatrix} 3 & -1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{3}R_1} \begin{bmatrix} 1 & -1/3 & -\lambda/3 \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix} \xrightarrow{R_2 \to R_2 - 4R_1}$

$\begin{bmatrix} 1 & -1/3 & -\lambda/3 \\ 0 & -10 & -3-2\lambda \\ 2 & 4 & \lambda \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2\lambda R_1} \begin{bmatrix} 1 & -1/3 & -\lambda/3 \\ 0 & -10 & -3-2\lambda \\ 0 & 14/3 & 5\lambda/3 \end{bmatrix} \xrightarrow{R_3 \to R_3 + \frac{7}{15}R_2} \begin{bmatrix} 1 & -1/3 & -\lambda/3 \\ 0 & -10 & -3-2\lambda \\ 0 & 0 & \left(\frac{11\lambda - 21}{15}\right) \end{bmatrix}$

Here, $\rho(A) =$ no. of unknow $= 3$

$\therefore$ Consistent & unique

For system to possess non-trivial solⁿ. (infinite solⁿ)

$\rho(A) <$ No. of unknow.

$\therefore \frac{11\lambda - 21}{15} = 0 \Rightarrow 11\lambda = 21 \Rightarrow \boxed{\lambda = \frac{21}{11}}$