Investigating the dispersion relation in a transmission line

Sara Capdevila Solé

***Abstract*—In this report, we investigated the properties of a lumped transmission line, by propagating pulses of various frequencies. A theoretical lossless approximation is proposed for low-frequency transmissions, however, for higher frequencies, an empirical model for the dispersion relation was derived and compared to a theorised sinusoidal, dispersive one. The cut-off frequency (ω_c) was measured in two ways, analysing the wave dispersion; $\omega_c = (0.945 \pm 0.033) rad/\mu s$, and examining the wave distortion; $\omega_c = (0.946 \pm 0.045) rad/\mu s$, at the end of the line, for different frequencies. These values are very similar to the cut-off frequency claimed by the manufacturer; $\omega_c = (0.9 \pm 0.1) rad/\mu s$.**

During the investigation, our transmission line was matched with its characteristic impedance, to model it as an infinite line - without reflections.

I. Introduction

In the 1850s, global communication was revolutionised when messages could be sent across the world in minutes, rather than weeks. This was due to the creation of a new transatlantic cable, that connected Europe and North America; although, at first, only transmitted weak and distorted signals[1]. Since then, Lord Kelvin and Oliver Heaviside, have mathematically modelled these lines. As a result of this and technology evolution, transmission lines now allow immense amounts of information to be exchanged around the world in a matter of seconds.

These transmission lines are designed to transmit electromagnetic waves over very long distances. Understanding how signals behave in such lines is crucial for modelling and optimising the signal distortion. To achieve this, one needs to take into account how waves of different frequencies behave and travel at different speeds, this is referred to as the line's dispersion, and directly depends on the material's dielectric function [2].

In this experiment, a lumped ladder network was modelled as an infinitely long transmission line, composed of a series of capacitors and inductors. This model was investigated to find the dispersion relation of propagating electromagnetic waves and as a result the corresponding cut-off frequency of the line. This analysis provides insight into square wave transmission, as these can be Fourier decomposed into its constituting frequencies.

II. Theory

A. Wave transmission

The propagation of a time-harmonic wave in the lumped transmission line, with voltage $V(x,t)$, can be described mathematically in 1-D by [3];

$$\frac{\partial^2 V}{\partial t^2} = v_p^2(\omega) \frac{\partial^2 V}{\partial x^2}, \tag{1}$$

Where $v_p(\omega)$ is the phase velocity of the wave.

B. Wave velocity

For any wave travelling in a spatial domain, it can be described in terms of its oscillatory frequency (ω), and wavenumber (k). Its phase velocity can be defined as the ratio of these two quantities [4];

$$v_p(\omega) = \frac{\omega}{k}. \tag{2}$$

In a non-dispersive system, this velocity is independent of frequency and can be found using a typical oscillatory wave solution, in the form of an exponential [4]. In this scenario, this value is equal to $V_p = 1/\sqrt{LC}$, L and C are the inductance and capacitance of the components in our transmission line respectively.

However, for a dispersive system, one which the $V_p = V_p(\omega)$, a secondary feature to the system arises; group velocity $V_g(\omega)$. This is because any individual wave can be represented as a sum of its individual constituting frequencies - Fourier components. Each wave will therefore propagate at a different speed, conspiring in such a way as to make their sum move with a speed, $V_g(\omega)$ [3]. The group velocity can be found to be,

$$v_g(\omega) = \frac{\partial \omega}{\partial k}. \tag{3}$$

In a non-dispersive system, these two values are uniquely defined as the wave velocity.

C. Dispersion relation

In a non-dispersive system, where we assume a lossless line, we find that k is linearly dependent on ω, meaning that the phase velocity is independent of frequency, preventing dispersion - this is represented in (2). Besides this, wave attenuation would also be independent of its frequency, preventing certain frequencies from being over-attenuated and distorting the signal [5].

Anyhow, in a dispersive medium, the propagation of a wave will depend on its frequency, thus a relationship can be usually derived expressing how k and ω vary in such system. This model should approximate to the non-dispersive model for low-loss and low-frequency waves, however, should increase the accuracy by taking into account dispersion at higher frequencies. Therefore, as one might expect, $v_p(\omega)$ is going to vary as the frequency of the wave changes.

Taking into account distortions for higher frequency waves, it can be theoretically derived, using Ohm's law that the dispersion relation will have a sinusoidal dependency[6];

$$\omega(k) = \omega_c \sin\left(\frac{k}{2}\right). \tag{4}$$

The cut-off frequency, ω_c, corresponding to the dispersive system, specifies the value at which the system will no longer propagate sinusoidal waves, thus signals can only exist below this certain threshold[3].

The purpose of this experiment was to derive a similar expression, empirically, knowing what the expected relationship was.

D. Characteristic impedance

At a boundary between two mediums, such as that encountered at the end of the transmission line, a wave won't propagate completely as some will be reflected into the line. The proportion of the reflected wave can be found using;

$$\Gamma = \frac{Z_T - Z_R}{Z_T + Z_R}, \tag{5}$$

where Z_T is the impedance of the transmission line, and Z_R is the impedance of the surroundings [4].

The impedance of the transmission line can be varied using a potentiometer. When $Z_T > Z_R$, the value of Γ will be between $0 \leq \Gamma \leq 1$, as a result, the wave will be reflected with the same polarity as the incident wave. In the limit where $Z_T \gg Z_R$, it can be treated as an open circuit, where there will be no output of energy - 'a wall'.

Similarly, when $Z_T < Z_R$, Γ will be in the range of $-1 \leq \Gamma < 0$, where the wave travelling into Z_R, encounters a 'denser' material, thus overpowering the waves leaving Z_T. As a result, reflecting an inverted pulse into the transmission line. In the case where $Z_T = 0$, mimics a short circuit; $\Gamma = -1$.

The final scenario, at which the impedance's are matched, stops reflection; $Z_T = Z_R$. The impedance of the line at this value is known as the characteristic impedance and is denoted as Z_0.

Z_0 can be calculated theoretically [4] using,

$$Z_0 = \sqrt{(L/C)\,(1/\,(1-\omega^2 LC/4))}. \tag{6}$$

Defining the theoretical cut-off frequency as;

$$\omega_c = 2/\sqrt{LC}, \tag{7}$$

allows us to approximate (6) for low frequencies; in the range where $\omega \ll \omega_c$ to,

$$Z_0 \approx \sqrt{L/C}. \tag{8}$$

III. Experimental method

The following diagram shows the setup of the lumped transmission line, composed of 40 sections, with a series of discrete inductors and capacitors, arranged in cylindrical symmetry.

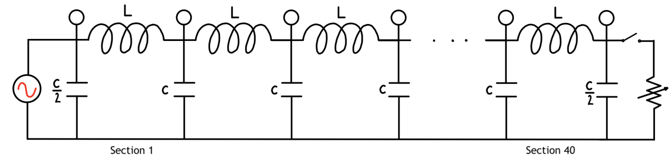

Fig. 1. Schematic of the complete set-up used. Includes the lumped transmission line connected to the oscilloscope (left) and the potentiometer (right).

A 2000 series PicoScope oscilloscope, was used as a signal generator and measurement device. The generated signal was divided into two coaxial BNC cables, and inputted into the transmission line and the oscilloscope. A BNC probe was connected to the secondary input in the oscilloscope, allowing us to measure the output voltage of the wave at different contact points along the line.

The impedance of the line was varied by adjusting the potentiometer, from $0 \leq R \leq 5\text{k}\Omega$. Its value was then measured using a multimeter.

The values of inductance and capacitance in each section as indicated by the manufacturer were; $L = (330 \pm 20\%)\mu$H and $C = (0.015 \pm 10\%)\mu$F.

IV. Results

A. Data representation

The frequency corresponding to each wavenumber was found, in distance units of circuit sections, using the oscilloscope. A dispersion relationship was then plotted for our lumped lossless transmission line. This can be seen in Fig. 2.

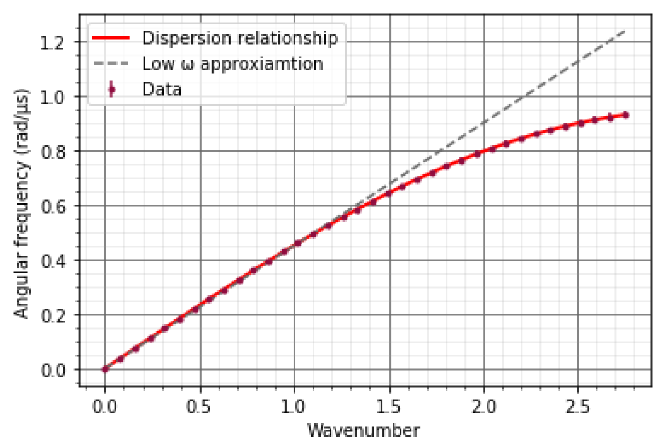

Fig. 2. Data obtained for the dispersion relationship with error bars, shown for both the low frequency approximation (2) (dashed line), and a fit to the experimental data, accounting for distortion at higher frequencies (red line).

Plotted in the graph is also the non-dispersive approximation ((2) section II.b), where the gradient is constant throughout; $V_p = (0.45 \pm 0.05) sections/\mu s$. As expected both models behave equally for low ω - small angle approximation from (4). The empirical dispersion relationship was found using a discrete Fourier transform algorithm in Python - fitting an expected sinusoidal curve [6];

$$\omega(k) = 0.92 \sin(0.51k - 0.04) + 0.03 \tag{9}$$

where its units are in μs. The associated errors of each value were calculated using error propagation formulas arising from the measured frequency, and the covariance of the fit. Most were around 0.01, hence making this model a very close fit to (4). Possible deviations of (9) from (4) will be discussed later.

Using (9), (2) and (3), expressions for $V_p(\omega)$ and $V_g(\omega)$ can be found and fitted to our data. Error propagation from (9), can be used to find the error associated with the fitted curves. This can all be seen in Fig. 3.

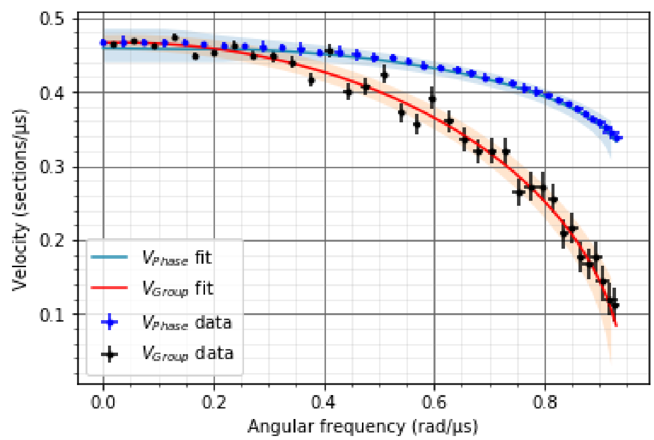

Fig. 3. Variation of the group and phase velocity with respect to the angular frequency of the wave with their corresponding error bars. The fit to V_p and V_g are $(9)/k$ and $\partial(9)/\partial k$ respectively. The error associated with each fitting function is shown shaded.

Besides these measurements, as we varied the frequency of the pulse along the line, the distortion of the wave at the end of the line changed. The output voltage reduction at this point was measured and compared relative to the input voltage. Results obtained are presented in Fig. 4.

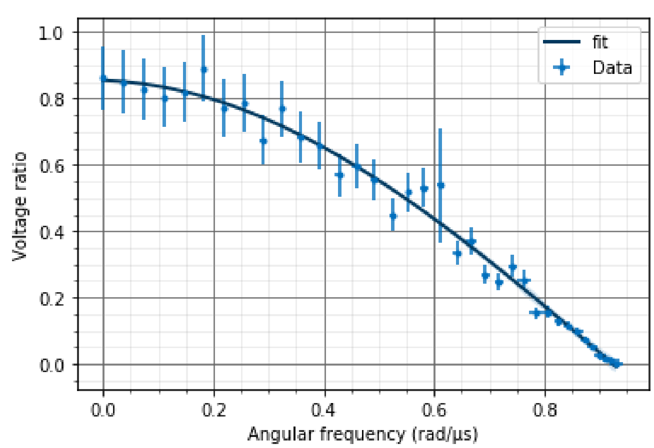

Fig. 4. The voltage ratio (V_{out}/V_{in}) variation with the angular frequency of the wave. The associated errors with each measurement are also shown.

B. Characteristic impedance

The characteristic impedance was found by setting the reflection in the circuit to $\Gamma \approx 0$, this was done at a fixed low frequency. The value was $Z_0 = (145 \pm 8)\Omega$, the associated error was found by resetting the apparatus and repeating the measurement multiple times.

Comparing this value to the theoretically derived using (8); $Z_0 = (148 \pm 17)\Omega$. Due to the errors associated with the values, they are compatible.

C. Cut-off frequency

The experiment was carried out by setting the resistance of the circuit at Z_0, this allowed the line to be approximately modelled as infinitely long, where we assumed no reflections occurred.

The cut-off frequency was extracted from our data in two ways. As one would expect, for the group velocity of the wave; $\lim_{\omega \to \omega_c} V_g(\omega) = 0$, the line is cut off. This would be the value of ω giving rise to $\partial\omega/\partial k = 0$, or in other words; the x-intercept of the group velocity fit in Fig. 3. This is also supported and can be seen in (4). This value was found to be $(0.945 \pm 0.33) rad/\mu s$.

A second value for the cut-off frequency can be calculated from the voltage attenuation (Fig. 4.), at the frequency at which the output voltage is completely attenuated, and a signal is no longer present. A cosine curve was used to fit our data, and the value was found to be $\omega_c = (0.946 \pm 0.45) rad/\mu s$.

An alternative approach to the fit was a linear relationship using the final portion of the data, however, this was discarded as it would require increasing the error associated with ω_c, as the data used would be less representative of the overall trend. Thus, the cosine fit seemed more appropriate. Nonetheless, the difference between each method seemed negligible, due to the number of measurements near the cut-off frequency.

The theoretical ω_c was calculated using (7) to be $(0.9 \pm 0.1) rad/\mu s$. All three values are reconcilable, as their errors allow for a margin of deviation.

V. Analysis and Conclusions

As seen in Fig. 3, the lower frequency waves propagate through the transmission line at the same speed with less distortion and dispersion, as a consequence sending a square pulse through the circuit would result in a more sinusoidal wave being present at the output, as the higher harmonics would be dispersed and distorted more, as they travel slower, and as a result 'lost'.

The voltage and associated error was measured using the peak to peak 'measure' tool in the Picoscope, as the amplitude was found to be asymmetric about the voltage-axis. This tool averages the amplitudes across 20 wave cycles and returns their standard deviation as their error. Frequency errors were measured similarly, taking into account also discrepancies between the measured value and the specified signal. Error propagation and covariance matrices were applied to calculate the errors associated with the remaining values, such as V_p and V_g.

The sensitivity of the potentiometer and possible dependency in physical parameters (including temperature), most likely contributed to errors we didn't take into account when measuring the characteristic impedance of the line; skewing our data, giving rise to systematic uncertainties, such as the offset and phase shift present in (9), in comparison to (4). This experiment should be repeated in another scenario to account for these.

The characteristic impedance of the line was set at a fixed frequency, ignoring its frequency dependency as ω increased. Hence, the error in Z_0 was underestimated at higher frequencies, as reflection start taking place, ie. $\Gamma \neq 0$. This explains the discrepancies in Figs. 3 and 4 at these frequencies, however, due to the wave distortion, it's not as evident (ie. the incoming wave amplitude is very small, hence even a fully reflected wave would have a very small amplitude). Z_0 should be adjusted at each frequency to prevent this from occurring.

Further discrepancies in our model could be from the assumption of a lossless line, ignoring any current leakage between conductors. Also, in practice, the capacitance and inductance of each section could've been slightly different, and are frequency-dependent (due to their dielectric properties), and as a result, the skin effect was ignored[2]. The skin effect adds some resistance and reactive impedance to the line, as it creates eddy magnetic fields, thus adding a frequency-dependent contribution to L and C.

The results of this experiment illustrate the limitations imposed by the electrical properties of a transmission line. At the same time, it provides relevant insight into the design of an improved electrical transmission circuit, with the goal of minimising dispersion and distortion.

References

[1] D. Cogan. (2010) History of the transatlantic cable. Whitehouse and the 1858 trans-Atlantic cable. Accessed:[08.12.2020]. [Online]. Available: https://atlantic-cable.com/Books/Whitehouse/DDC/]

[2] Z. Peterson. (2020) Transmission line dispersion and losses in your high speed pcb. Accessed:[08.12.2020]. [Online]. Available: https://www.nwengineeringllc.com/article/transmission-line-dispersion-and-losses-in-your-high-speed-pcb.php

[3] D. Morin, *Waves: Dispersion*. Harvard University, 2020.

[4] F. Suzuki-Vidal, "Thermal and electrical waves experiment," Imperial College London, 2020.

[5] D. Tougaw, *Applied Electromagnetic Field Theory*. Engineering Curricular Materials., 2018, vol. 1, ch. 28.

[6] (2012) Lumped-parameter delay line. Caltech. Accessed:[08.12.2020]. [Online]. Available: http://www.sophphx.caltech.edu/Physics$_6/Experiment_14.pdf$