Damping and stiffness in the wave equation

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***Abstract*—The resonant frequencies of a wound and unwound guitar string are studied under different tensions in the range of $(0-50)$N on a sonometer. The wave equation with stiffness and decay parameters is solved to find the Young Modulus (E) of both strings and the corresponding decay for various harmonics. The unwound string is found to overlap within the expected range of steel, at $E_{uw} = (214 \pm 50)G$Pa and the wound string $E_w = (19 \pm 12)G$Pa, with no significant variations at different tensions, with a p-value of 0.875 and 0.827 respectively. The decay constants of different harmonics are used to find the linear parameter in the Rayleigh damping model, which is found to be 2.46 times steeper for the wound string.**

I. Introduction

THE classical wave equation ($y_{tt} = v^2 y_{xx}$) models the displacement $y(x,t)$ of a perfectly elastic string with velocity v, with no damping or stiffness corrections. We can modify the equation to account for velocity-dependent damping ($\propto y_t$) [1] and a 4th-order stiffness correction ($\propto y_{xxxx}$) originating from the Euler–Bernoulli beam equation which models the deflection of rigid bars [2]. These are approximations to make the equation analytically solvable.

This report studies two types of guitar strings: wound and unwound, to find the damping constants for their fundamental frequencies and young modulus values.

II. Theory

A. The Physics of Guitar Strings

Guitar strings produce their sound when they're plucked. The velocity of the wave propagating transversely is given by $v = \sqrt{T/\mu}$, where T is the tension and μ is the linear mass density of the string. When the wavelength of the wave propagating is a multiple of the length of the string (L), it will vibrate at its natural frequency (f_0). This value is proportional to v, given by $f_0 = v/2L$, and higher harmonics will be multiples of the fundamental $f_n = nf_0$. As we increase the tension on the guitar string, the fundamental frequency increases.

B. Damping

Introducing the frequency-dependent decay constant $\gamma/2$, the wave equation with velocity damping dependence is given by

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} - \gamma \frac{\partial y}{\partial t}. \tag{1}$$

The linear variation of the decay of the initial harmonics can be represented in the Rayleigh damping model [3].

C. Boundary conditions and theoretical predictions

The boundary conditions in our investigation assumed perfectly pinned ends without any moments $y = y_{tt} = 0$ at $x = 0$ and $x = L$ for $t \geq 0$. This gives standing wave solutions where resonant frequencies will dominate and where the energy of the wave will be concentrated.

Using separation of variables, the expected displacement solutions are

$$y(x,t) = \sum_n A_n \exp\left(-\frac{\gamma}{2}t\right) \sin\left(\frac{nx\pi}{L}\right) \cos(w_n t), \tag{2}$$

where the angular frequency $w_n = \sqrt{(n\pi v/L)^2 - (\gamma/2)^2}$, and A_n are the amplitudes of the harmonics at $t = 0$, due to a pluck of height h, at position d along the string given by,

$$A_n = \frac{2hL^2}{\pi^2 n^2 d(L-d)} \sin\left(\frac{n\pi d}{L}\right). \tag{3}$$

In our investigation, these amplitudes are measured in the velocity spectrum (y_t) where $A_n \propto 1/n$. If the string is plucked at its centre, we only expect to odd harmonics.

D. String properties

Our wave equation still doesn't include the strings' impedance to bending. Hence, we can modify (1) to

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} - \frac{ESK^2}{\mu} \frac{\partial^4 y}{\partial x^4}, \tag{4}$$

where E is the young modulus, $S = \pi R^2$ is the cross-sectional area, and the radius of gyration $K = R/2$ in our experiment, with a cylindrical string of radius R [4].

These new parameters make the string no longer harmonised, where the higher harmonics are not multiples of the fundamental but instead follow

$$f_n = nf_\circ \sqrt{1 + \frac{n^2 \pi^2 ESK^2}{TL^2}} \tag{5}$$

for $n \geq 1$ [5]. Plotting $(f_n/n)^2$ against n^2 allows us to determine the value of E from the gradient of the linear fit.

III. Data Collection

The guitar strings containing Nickel, a ferromagnetic material, were mounted on a sonometer. To reduce the non-linearity of a magnetic driver, we placed the string between two drivers, one on top of each other, connected through a diode circuit with the a.c. signal generator. This reduced double-peaking, which happened when there was a superposition of half-integer multiples of the resonant frequencies. This had to be positioned very carefully to prevent asymmetric, elliptical oscillations.

As the string vibrates, by Faraday's law, the changing magnetic flux induced by the oscillatory motion of the string generates an oscillating voltage, which is picked up by the detector [6]. The signal measured is proportional to its velocity. These readings were recorded using an open-source platform, Audacity. The resonant frequencies were modified using the signal generator and read off an $rtb2004$ oscilloscope on its frequency recording setting.

All suspended loads were placed on a lever attached to the sonometer. This provided a multiplicity of 5 times the tension, assuming the weight hangs perpendicular to the lab bench.

IV. Results

A. Strings

The unwound string has $R = (0.14 \pm 0.05)mm$ and $\mu = (0.483 \pm 0.014) \cdot 10^{-3} Kgm^{-1}$. The wound string had $R = (0.45 \pm 0.05)mm$ and $\mu = (4.420 \pm 0.027) \cdot 10^{-3} Kgm^{-1}$. These values were measured using a micrometer and precision balance scale. In both cases, the length between both pinned ends was $L = (69.0 \pm 0.5)cm$.

B. Damping constants

Three 30s signals were recorded for each harmonic and analysed in 0.1s intervals. Both strings had a suspended mass of $0.8Kg$. Calculating the Fourier transform allowed us to measure the frequencies in the recordings. A time frame after the initial pluck is left before the decay is recorded, to prevent measuring the influence of the driver. Additionally, noisy peaks of $A_n < A_1 \cdot 5\%$ were rejected. The results showing the variation of the decay constant at different harmonics for both strings are shown in Fig. 1.

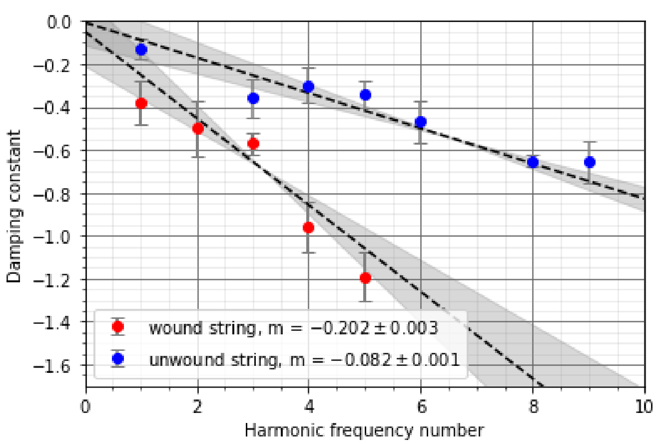

Fig. 1: **Damping constants:** decay of the different harmonics measured in both strings. The wound string has fewer harmonics present and decays faster with a steeper gradient (m) relative to the unwound string.

C. Young modulus

The tension in the string was varied and the different harmonics were recorded. The corresponding values of E were calculated for both strings and are presented in Table I and II.

V. Analysis and discussion

As we'd expect, Fig. 1 shows that all harmonics in the wound string decay much faster than in the unwound string and has fewer higher harmonics propagating, resulting in a 2.46 steeper linear parameter in the Rayleigh damping model relative to the unwound string.

TABLE I: **Wound string**

Mass (Kg)	E (GPa)	σ_E (GPa)
0.4	19.3	6.1
0.6	22.9	7.2
0.7	19.4	6.1
0.8	14.9	4.7

TABLE II: **Unwound string**

Mass (Kg)	E (GPa)	σ_E (GPa)
0.2	220	24
0.4	234	25
0.6	210	23
0.8	199	22
1.0	207	22

TABLE I, II: Values of the young modulus and its error at different tensions for both strings.

Previous experiments have measured values of Young's modulus for wound strings to be markedly lower and decreasing with frequency due to not relating the results to the properties of the core and winding [7]. This is hinted in Table I. On the other hand, [8] found that not until $n > 17$ there begin to be significant deviations from the predictions.

Performing a χ^2-test to determine if the difference in the values of E are significant for different tensions, gives $\chi^2_w = 0.894$ and a corresponding p-value $= 0.827$ for the wound string and $\chi^2_{uw} = 1.217$ and p-value$=$ 0.875 for the unwound string. From these values we conclude the difference is insignificant, hence calculating their mean and weighted error, our final values are $E_w = (19 \pm 12)G$Pa and $E_{uw} = (214 \pm 50)G$Pa. The E_{uw} is found to overlap within the expected range of $(200 - 220)G$Pa for steel [9].

Nonetheless, E_w is significantly lower than expected. This can be due to the winding. In [5], only the core radius is used in the calculations and the linear density of the string is modified to be a wrapping factor times the linear density of the core (steel). This wrapping factor is directly dependent on the ratio of the dimensions of the core to the winding, which was not known exactly in this investigation. Re-running the calculations with $(0.2 - 0.8)R$ and $(1 - 1.5)\mu$ would increase the values of E_w to $\mathcal{O}(100G$Pa), which is what is expected for a typical guitar string [10].

In future investigations, the non-linear velocity-squared damping wave equation could be explored numerically. Harmonic modes could be treated differently, as they may not be independent of each other (exchanging energy). More data could be recorded and analysed for more rigorous statistical tests, as well as reducing our errors with more precise equipment, such as detecting string vibration with a laser [3].

In addition, the non-linear region of the Rayleigh model could be explored, such as the mass $(0 < n < 1)$ and quadratic damping $(n \gg 1)$ [11].

VI. Conclusion

The decay rates of different harmonics and young modulus of a wound and unwound guitar string have been investigated, the wound string was found to have a 2.46 steeper linear parameter in the Rayleigh damping model relative to the unwound string. The unwound string is found to have a young modulus of $E_{uw} = (214 \pm 50)G$Pa overlapping within the expected range of steel, while the wound string was found to be less stiff with $E_w = (19 \pm 12)G$Pa, with no significant variations at different tensions.

APPENDIX A
ERROR PROPAGATION

Errors were propagated in quadrature, assuming each variable is uncorrelated;

$$\sigma_E^2 = \sum_i^{L,R,m,T} \left(\frac{\partial E}{\partial i}\right)^2 \sigma_i^2 \tag{6}$$

$$= (2\sigma_L/L)^2 + (4\sigma_R/R)^2 + (\sigma_m/m)^2 + (\sigma_T/T)^2.$$

The errors in the gradient (m) propagate the uncertainty in the measured frequencies, through a non-linear least squares fit to a linear function. The error in the tension was approximated by the tilt in the angle perpendicular to the direction of gravity, using an inclinometer. This value is ± 10º, while the error is the cosine of this value, $\sigma_T \approx 0.02T$.

APPENDIX B
χ^2 TEST

The χ^2 test is calculated using Williams correction, as the sample size is small (< 5),

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{\sigma_{O_i}^2} / \left(1 + \frac{N_m^2 - 1}{6N_{dof}N_m}\right). \tag{7}$$

Where E_i are the expected values, O_i are the observed and σ_{O_i} are its associated uncertainties [12], [13]. N_m is the number of masses analysed and the number of degrees of freedom given by $N_{dof} = (N_m - 1) \cdot (N_{variables} - 1)$, where $N_{variables} = 2$ in this investigation.

The p-value is then calculated using

$$p\text{−}value = 1\text{−}CDF\left(\chi^2\right) = 1\text{−}\frac{\int_{\chi^2_{\min}/2}^{\infty} t^{\frac{N_{dof}}{2}-1} e^{-t} dt}{\int_0^{\infty} t^{\frac{N_{dof}}{2}-1} e^{-t} dt}. \tag{8}$$

APPENDIX C
FEEDBACK FROM CYCLE 1

Experiment: Astronomical Imaging. Grade: 76%. Comments: Quality of the graphs and visuals on slide used in the presentation were very good and the presentation benefitted from their inclusion. Structure was good. Delivery occasionally halting and sometimes felt a bit rushed, though technical problems did not help. Good depth of description about their own algorithm, with images to help explain the steps used. Some details of the methods not covered e.g. how elliptical fits and fake sources were established.

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