A Current Review of the Implications of Locality, Unitarity and Symmetries in Cosmology

Abstract

The usual approach to deriving cosmological correlators is to follow a given theory and evolve its vacuum state forward in time. This can be quite computationally taxing due to the lack of time-translation invariance in cosmological spacetimes. This paper aims to review recent advances proposed to *bootstrap* correlators using principles such as locality, unitarity and symmetries. We summarise the key ideas behind this bootstrap method, focusing on de Sitter and quasi-de Sitter backgrounds. We discuss the perturbative implications of unitarity, including the *Cosmological Optical Theorem* - the curved spacetime equivalent of the Optical Theorem, and its associated cutting rules. For contact diagrams, this strongly dictates the analytic structure of the wavefunction coefficients, which are measured by cosmological surveys. Following this, we explore the non-perturbative implications of unitarity, including the positivity of the spectral densities of the two and four-point function, using the Källen-Lehmann decomposition. Next, we review the *Manifestly Local Test*, that any wavefunction of massless scalars or gravitons must satisfy if it arises from a manifestly local theory. Finally, we identify the Bootstrap Rules, which uses the MLT, the total energy pole of the correlator, the choice of vacuum and the symmetries of de Sitter to completely fix the form of the bispectra at the asymptotic boundary. As an illustration, we use these rules to bootstrap the form of the wavefunction for self-interactions of a massless scalar.

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Contents

1 Introduction

Inflationary cosmology is of great interest as it provides a unique opportunity to probe the earliest moments of the universe. The primordial density fluctuations created during the inflationary epoch are believed to be the seeds of all structures in the universe. During this period, both quantum mechanics and gravity played an essential role in creating small quantum fluctuations that were stretched to cosmological distances. The resulting correlations in the fluctuations provide valuable information about the universe in its infancy, offering insights into fundamental physics. By studying the statistical properties of these primordial fluctuations, we can gain a deeper understanding of the origins and evolution of the universe. For Minkowski and Anti-de Sitter spacetimes, these observables are scattering amplitudes and Conformal Field Theory (CFT) correlators, respectively. For an expanding, accelerated FLRW spacetime, as appropriate to model inflation, the relevant observables are equal-time correlators in the asymptotic future

We cannot directly observe the inflationary era but rather must infer its properties from spatial correlations in the initial conditions for the post-inflationary universe. These correlations, which exist on the future boundary of the inflationary spacetime or the past boundary of the hot Big Bang universe, contain valuable information about the dynamics and particle behaviour of inflation. Interestingly, time is not explicitly represented in the observed correlations, but instead manifests in their scale dependence, since different modes of varying wavelengths freeze out at different times during inflation. By studying the shape dependence of these cosmological correlations at late times, we can gain insight into the physics of the inflationary epoch.

There are various ways to compute and make predictions of these inflationary correlations. The *in-in formalism* and the *wavefunction formalism* are two standard approaches that require following the evolution of these correlators through the whole spacetime, from the origin to the end of inflation, where they are observed as reheating. However, given that we can't observe the time evolution directly during this epoch, we must infer it from the static boundary correlations. This approach is appealing in that it directly displays the consequences of *unitarity*, *locality* and *causality*. Nonetheless, it fails to take advantage of some powerful symmetries that all the correlators obey, in order to bypass some of the laborious time integrations over all inflationary evolution (see [1, 2]).

The implications of unitarity in Minkowski have been rigorously studied e.g. [3, 4, 5]. However, this field has been growing recently in cosmology. The *cosmological bootstrap* is an ongoing area of research that provides a more conservative approach to the above problems [6, 7, 8]. This strategy bypasses the need for a theory of inflation, which is a favourable feature to have given that we are highly uncertain of what the correct theory is. This is because inflation occurred at energies around the GUT (Grand Unified Theory) scale, which is approximately 10^{16} GeV [9]. This energy scale is far beyond the reach of current experimental probes. The bootstrap method allows us to directly reconstruct the correlation functions using *unitarity*, *locality* and *symmetries*. This opens a door to easily reproducing the bulk calculations and even to new, more complex predictions. Furthermore, it provides a map between the particle spectrum we observe, their quantum state, and the allowed interactions to the analytic structure of the correlators. The desired goal of this program is to continue to provide new connections between the fundamental principles of physics and our observations.

In this paper, we review some of the key tools required in bootstrapping cosmological correlators. In Section 2, we begin by summarising the basic properties of de Sitter spacetime, including the unitary irreducible representations of the de Sitter group, $SO(4, 1)$, which make up the symmetries of the spacetime [10]. We also introduce the notion of correlators in perturbation theory, by going through their definition, symmetries and analytic structure, in each formalism. In Section 3, we derive the Feynman rules for each formalism, by solving the non-linear equations of motion using the Green's functions. Next, in Section 4, we review the implications of unitarity in cosmology, going through the perturbative consequence of the Optical Theorem - the cosmological Optical Theorem with its associated cutting rules for any FLRW spacetimes [11, 12, 13], and in a non-perturbative context, through a Källén–Lehmann decomposition of cosmological spectral densities. Moreover, we introduce the *Manifestly Local Test* (MLT), which must be satisfied by the wavefunctions emerging from manifestly local interactions [7]. Finally, in Section 6, we identify and discuss the Bootstrap Rules, relying on all the de Sitter isometries (except boosts), the MLT, and the amplitude limit of correlators, to fix the form of the bispectra. As an example, we use these rules to bootstrap the form of the correlator for three scalar fields. In Section 7, we conclude by summarising some of the key results reviewed in this paper and we present an outlook for future work.

2 De Sitter, correlators and wavefunctions

In this Section, we begin by reviewing the fundamental properties of de Sitter (dS) spacetimes, including its different patches and its associated symmetries. In Subsection 2.2 we review the in-in and wavefunction formalism, both used to predict the form of correlators in dS.

2.1 De Sitter basics

De Sitter in $d = 3$ spatial dimensions and 1 of time, denoted as dS_{3+1}, can be thought of as a subset of four-dimensional flat Lorentzian signature space, i.e. Minkowski spacetime with convention $(-,+,+,+)$, consisting of a set of points a distance H^{-1} away from the origin $X^\mu = 0$ with $\mu \in [0,3]$ [14, 15]. X^μ are the *global inertial coordinates* satisfying,

$$dS_{3+1}: \qquad -(X^0)^2 + \sum_{i=1}^{3}(X_i)^2 = \frac{1}{H^2}, \tag{2.1}$$

where H is the Hubble constant i.e. $1/H$ is the Hubble radius, a measure of the size of dS. From this, it follows that the group of isometries of dS; $SO(4,1)$, can immediately be identified as the subgroup of $SO(4,2)$ that leaves the Hubble radius invariant. Topologically, dS is simply S^3, as they are simply related by a Wick Rotation $X_0 \to iX_0$ in (2.1). The induced spacetime metric on a hypersurface in *global coordinates*[1] takes the form,

$$ds^2 = \frac{-d\tau^2 + d\Omega_z^2}{H^2\cos^2\tau}, \qquad \tau \in (-\pi/2, \pi/2) \tag{2.3}$$

where $d\Omega_z^2$ is the induced metric of the unit 3-sphere. The metric in this form is conformally flat, and therefore its Penrose diagram is just a square, as shown in Fig. 1.

[1] *Global coordinates* are defined from *global inertial coordinates* through the following transformation

$$X^0 = \frac{\sin\tau}{H\cos\tau}, \qquad X^i = \frac{y^i}{H\cos\tau} \tag{2.2}$$

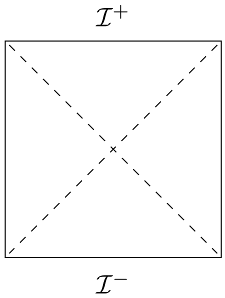


Figure 1: The Penrose diagram for de Sitter space. The horizontal lines $\mathcal{I}^+$ and $\mathcal{I}^-$ are the two spacelike boundaries, at $\tau = \pi/2$ and $\tau = -\pi/2$ respectively. The two vertical lines are not real boundaries, they are the north (left) and south (right) poles of the S^3 sphere. The dashed lines represent the past and future horizons for an observer at the south pole.

Finally, by picking a Poincaré patch covering half of our spacetime $X^0 + X^4 \geq 0$, and foliating it using conformally flat slices in *conformal* (*Poincaré*) *coordinates*[2],

$$ds^2 = a(\eta)^2(-d\eta^2 + d\mathbf{x}^2), \qquad \eta \in (-\infty, 0), \qquad \mathbf{x} \in \mathbb{R}^d. \tag{2.5}$$

We have introduced the conformal factor $a(\eta) = 1/(-H\eta)$. We will use conformal coordinates in the rest of this work. This metric (2.5) manifests the asymptotic boundary at $\eta = 0$, which we will denote as η_0.

2.1.1 De Sitter Symmetries

De Sitter spacetime is a maximally symmetric solution to the Einstein equation that describes a universe with a positive cosmological constant. According to *inflationary cosmology*, the background spacetime during the very early stages of our universe was well approximated by this spacetime [16]. Furthermore, observations suggest we are reentering another approximate dS epoch [17]. The symmetries of dS act like conformal symmetries on the late-time spacetime slice of the universe. We seek solutions that are constant on every η slice of our spacetime, which spontaneously breaks any possible time translation symmetry.

The spacetime is invariant under a group of transformations known as the de Sitter group, which includes rotations and translations in a five-dimensional space. These symmetries alone already provide powerful constraints on the statistical properties of primordial perturbations, which are thought to have been generated during inflation [2, 18].

A local scalar operator $\phi(\eta, \mathbf{x})$ transforms under the conformal generator Q as,

$$[Q, \phi(\eta, \mathbf{x}] = \hat{Q} \cdot \phi(\eta, \mathbf{x}). \tag{2.6}$$

[2]Notice this Poincaré patch is causally complete as it can't be connected by a causal (timelike or null) curve to the rest of the spacetime, outside the patch. For completeness, *global inertial coordinates* are related to *conformal coordinates* by,

$$X^0 = \frac{\eta^2 - 1 - x^2}{2\eta H}, \quad X^{i=1,2} = -\frac{x^i}{\eta H}, \quad X^3 = \frac{x^2 - 1 - \eta^2}{2\eta H}. \tag{2.4}$$

The 10 Killing vectors[3] associated with our spacetime, correspond to the generators associated with translations P_i, rotations M_{ij}, dilations D and special conformal K_i symmetries [19]. These generators are given by,

$$\begin{aligned} &P_i : \partial_i, && D : -\eta\partial_\eta - x^i\partial_i, \\ &M_{ij} : x_i\partial_j - x_j\partial_i, && K_i : 2x_i\eta\partial_\eta + 2x_ix^j\partial_j + \left(\eta^2 - x^2\right)\partial_i. \end{aligned} \tag{2.7}$$

The subscript i on the partial denotes a derivative with respect to conformal coordinate x^i. These generators are the differential operators $\hat{Q}$ given in (2.6), e.g. for spatial translations $[P_i, \phi(\eta, \mathbf{x})] = \partial_i\phi(\eta, \mathbf{x})$. This manifest invariance under spatial translations implies it will be convenient to work in Fourier space, where the same symmetry gives rise to momentum conservation,

$$P_i : \mathbf{x}_i \to \mathbf{x}_i + \boldsymbol{\epsilon}_i \implies \sum \mathbf{k}_i = 0. \tag{2.8}$$

Unitary representations Similarly, to how one-particle states transform under unitary irreducible representations of the Poincaré group in flat space, in dS_{3+1} particles are classified under unitary irreducible representations of the isometry group $SO(4,1)$ (2.7). In particular, the three representations that are of interest are the *principal series*, the *complementary series* and the *discrete series* [20]. It is useful to introduce a conformal dimension Δ, which is given by the quadratic Casimir,

$$\mathcal{C}_2 = -\frac{1}{2}\sum_{A,B}\mathcal{L}_{AB}\mathcal{L}^{AB} = \Delta(\Delta - d) + J(J + d - 2), \tag{2.9}$$

where J is the spin of the state and $d = 3$ is the space dimensions [21]. $\mathcal{L}_{AB}$ represents the isometries as differential operators, describing how an operator transforms in the bulk. In inertial coordinates it is given by [22],

$$\mathcal{L}_{AB} = X_A\frac{\partial}{\partial X^B} - X_B\frac{\partial}{\partial X^A}. \tag{2.10}$$

Acting on an irreducible field ϕ on a dS slice, one recovers the Killing vectors in (2.7). The conformal dimension is given by a quadratic relation to the mass of the fields, which can be solved for,

$$\frac{m^2}{H^2} = \Delta(d - \Delta) \implies \Delta_\pm = \frac{d}{2} \pm \nu, \qquad \nu = \sqrt{\frac{d^2}{4} - \frac{m^2}{H^2}}. \tag{2.11}$$

As a representation, the two solutions $\Delta_\pm$ are equivalent, which we will denote as Δ. The three representations are characterised depending on the mass. For our use, we will only be interested in states with zero spin, which the series and the corresponding masses in a free theory are,

$$\text{Principal Series:} \qquad \Delta = \frac{d}{2} + \nu, \qquad \frac{m^2}{H^2} > \frac{d^2}{4}, \tag{2.12a}$$

$$\text{Complementary Series:} \qquad 0 < \Delta < \frac{d}{2}, \qquad \frac{d^2}{4} > \frac{m^2}{H^2} > 0. \tag{2.12b}$$

[3] dS_{3+1} is invariant under SO(4,1) as we saw from its definition in (2.1). As a result, it will have $1/2(d+1)(d+2) = 10$ Killing vectors (for d=3).

In (A)dS the $(d+1)$ isometry group approaches the d conformal group as $\eta \to 0$. Therefore, boundary correlators satisfy the conformal symmetries presented above. This lies at the heart of the recently proposed *Cosmological Bootstrap* program [6, 7, 8, 23, 24].

2.2 Correlators in de Sitter

In flat spacetimes, the primordial fluctuations that characterise the formation of large-scale structures of the universe, are typically represented in the form of transition amplitudes - the S-matrix, which describe the probability of finding a certain configuration of fluctuations in a given patch of space. However, in dS, the existence of a cosmological horizon introduces a natural length scale that modifies the way in which these fluctuations propagate[4]. As a result, we can only specify the initial conditions and propagate these forwards in time, deriving cosmological correlations between different patches of the universe. In this section, we review two formalisms that can be used to calculate these cosmological correlators.

2.2.1 The in-in formalism

The in-in formalism has been extensively studied [1, 2, 25, 26, 27]. We can begin by introducing our main object of interest - a correlator, which is defined as the expectation of an operator $\mathcal{O}$ in the Heisenberg picture around the *in*itial vacuum state, of our theory $|in\rangle = |\Omega\rangle$,

$$\langle \mathcal{O} \rangle = \langle \Omega | \mathcal{O}(\eta) | \Omega \rangle. \tag{2.13}$$

For us, the operator will always be a product of field operators evaluated at equal times, $\mathcal{O}(\eta) \sim \prod_a \phi_a(\eta, \mathbf{k}_a)$. Translation invariance in (2.8), implies we can always write the correlator as

$$\langle \phi(\mathbf{k}_1) \cdots \phi(\mathbf{k}_n) \rangle = (2\pi)^3 \delta^{(3)} \left(\sum_{a=1}^{n} \mathbf{k}_a \right) \langle \phi(\mathbf{k}_1) \cdots \phi(\mathbf{k}_n) \rangle', \tag{2.14}$$

where the prime indicates we have extracted the momenta conserving delta function outside the correlator. Sometimes we will denote the primed correlator as $\langle \phi_{\mathbf{k}_1} \cdots \phi_{\mathbf{k}_n} \rangle' \equiv B_n(\mathbf{k}_1, \cdots, \mathbf{k}_n)$.

In any FLRW spacetime e.g. de Sitter and quasi-de Sitter, the vacuum state of our theory must be equivalent to the free theory vacuum in the asymptotic past of our universe; the Bunch-Davies vacuum $|0\rangle$,

$$\lim_{\eta \to -\infty} |\Omega\rangle = |0\rangle. \tag{2.15}$$

The *factorised form* for correlators in the in-in formalism is given by [25],

$$\langle \mathcal{O}(\eta) \rangle = \langle 0 | \left[\bar{T} e^{i \int_{-\infty(1+i\epsilon)}^{\eta} d\eta' \, H_{int}(\eta')} \right] \mathcal{O}_{int}(\eta) \left[T e^{-i \int_{-\infty(1-i\epsilon)}^{\eta} d\eta' \, H_{int}(\eta')} \right] | 0 \rangle, \tag{2.16}$$

[4]The presence of a cosmological horizon can be explicitly seen in Fig. 1. An observer at the north pole will only see half the spacetime once they reach $\mathcal{I}^+$. The boundary of this region is the cosmological horizon.

where the subscript *int* denotes the operators in the interacting picture, and H_{int} is the Hamiltonian. In (2.16) we have inserted the unitary time evolution operator in the interaction picture,

$$U(\eta,\eta_0) = T\left\{\exp\left[-i\int_{\eta}^{\eta_0} d\eta'\, H_{int}(\eta')\right]\right\} \tag{2.17}$$

where T denotes that the power series expansion of the exponential must be a time-ordered product of its arguments - written from left to right with consecutively smaller time arguments. $\bar{T}$ denotes an anti-time-ordering. We are interested in taking the limit as $\eta \to -\infty$, as this ensures that all the wavelengths are deep inside the horizon [26]. In this limit, the correlators are taken with respect to the free vacuum state from (2.15). This is simply because our evolution operators are unitary, i.e. $U^{\dagger}U = \mathbb{1}$, and hence any fluctuation automatically cancels in this formalism. However, the integral of (2.17) is usually not well defined in this limit, as it's composed of mode functions of the form $\propto e^{-ikt}$. For our choice of vacuum in the asymptotic past, we can create a well-defined integral by rotating our integration contour as we take limits into the past, $\eta \to -\infty(1-i\epsilon)$, or equivalently performing a wick rotation $\eta \to i\eta$. This, in turn, converts the oscillatory behaviour of the plane waves into exponential decay. We will also take the limit as $\eta_0 \to 0$. We will stick to the convention of limits and not explicitly evaluate the mode function or the scale factors at those values, as these are usually non-finite in these limits.

2.2.2 Wavefunction formulation

We are interested in the late-time inflationary wavefunction of our universe, which can be used to describe the quantum fluctuations produced in the early inflationary Universe [2, 28, 29]. In this section, we discuss the approach of wavefunctions to calculate cosmological correlators. The isotropic background is composed of fields denoted by $\bar{\phi}_a(\eta,\mathbf{x}) \equiv \bar{\phi}_a(\eta)$ with fluctuations $\phi_a(\eta,\mathbf{x})$, and field eigenstates $|\phi;\eta\rangle$. The state of the universe is encoded in the wavefunctional [8], which is simply a probability distribution for spacetime field configurations,

$$\Psi[\phi;\eta] = \langle\phi;\eta|\Omega\rangle. \tag{2.18}$$

As in the in-in formalism, the usual correlators of interest are momentum space correlators composed of products of the fields evaluated at the boundary. We can compute correlators from a given wavefunctional with (see Appendix A for a detailed derivation)[5],

$$\langle\Omega|\prod_i^n \hat{\phi}_{\mathbf{k}_i}|\Omega\rangle = \int d\bar{\phi}\prod_i^n \bar{\phi}_{\mathbf{k}_i}|\Psi[\bar{\phi};\eta_0]|^2. \tag{2.19}$$

The wavefunction is not an observable. To relate them to correlators we can solve the functional Schrödinger equation, but this becomes impracticable for a large number of fields n [30]. A more general approach is to use the path integral formalism, for which the late-time wavefunction is defined as [31][6],

[5]The normalisation factor in any FLRW spacetime with a bunch Davies vacuum is taken to be 1. This occurs because this vacuum state is homogeneous and therefore doesn't contribute to any vacuum bubbles.

[6]It's implicit the dependence of the action and the wavefunction, on the metric $g_{\mu v}$ and the induced three-metric h_{ij} respectively. We have also suppressed the wick rotation - the $i\epsilon$ prescription to rotate the integration contour and work with Euclidean time, which projects onto the vacuum state of our free theory, as $\eta \to -\infty$.

$$\Psi[\bar{\phi};\eta_0] = \int_{\phi(\eta\to-\infty)=\Omega}^{\phi(\eta_0)=\phi^{cl}} \mathcal{D}\phi \exp(iS[\phi]) \approx \exp(iS_{cl}[\phi^{cl}]). \tag{2.20}$$

The integral again is over a class of 4-geometries and fields observed in our universe, which, at tree level, is expected to be dominated by the extremum of the classical action. The on-shell action and Ψ can be constructed perturbatively in terms of the field configuration we expect at the boundary, $\psi_n = \psi_n(p_1, ..., p_n)$[7],

$$\begin{aligned}\Psi[\bar{\phi};\eta_0] &= \exp\left\{-\sum_{n=2}^{\infty}\frac{1}{n!}\left[\prod_{a=1}^{n}\int\frac{d^3\mathbf{k}_a}{(2\pi)^3}\bar{\phi}(\eta_0,\mathbf{k}_a)\right]\psi_n\right\}\\ &= \exp\left\{-\frac{1}{2}\int_{\mathbf{k}_1,\mathbf{k}_2}\bar{\phi}_{\mathbf{k}_1}\bar{\phi}_{\mathbf{k}_2}\psi_2(\mathbf{k}_1,\mathbf{k}_2) - \frac{1}{3!}\int_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}\bar{\phi}_{\mathbf{k}_1}\bar{\phi}_{\mathbf{k}_2}\bar{\phi}_{\mathbf{k}_3}\psi_3(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) - ...\right\}.\end{aligned} \tag{2.21}$$

The *wavefunction coefficients* ψ'_n are found once we decompose $\psi_n = (2\pi)^3\delta^3(\sum_i \mathbf{k}_i)\psi'_n$ into its momenta conserving delta function, following (2.14). Using this one can compute scalar field correlation functions to any order. It is common to call $n = 2$ the power spectrum, $n = 3$ is the bispectrum, $n = 4$ the trispectrum, which using standard Gaussian integrals and (2.19), we obtain correlation functions of the form,

$$\langle\phi^2\rangle \sim \frac{1}{\mathrm{Re}(\psi'_2)}, \qquad \langle\phi^3\rangle \sim \frac{\mathrm{Re}(\psi'_3)}{\mathrm{Re}(\psi'_2)^3}, \qquad \langle\phi^4\rangle \sim \frac{\mathrm{Re}(\psi'_4)}{\mathrm{Re}(\psi'_2)^4} + (\mathrm{p_s}, \mathrm{p_t}, \mathrm{p_u}) - \text{channels}. \tag{2.22}$$

We can make sense of the real components of the wavefunction coefficients appearing in (2.22), given the asymptotic past vacuum boundary condition from (2.15) is a complex state.

2.3 Preliminaries

Conventions We will work in natural units. Our convention for the Fourier transform is,

$$\begin{aligned}\phi(\eta,\mathbf{x}) &= \int_{\mathbb{R}^d}\frac{d^d\mathbf{k}}{(2\pi)^d}\phi(\eta,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} = \int_{\mathbf{k}}\phi(\eta,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}},\\ \phi(\eta,\mathbf{k}) &= \int_{\mathbb{R}^d}d^d\mathbf{x}\phi(\eta,\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}} = \int_{\mathbf{x}}\phi(\eta,\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}.\end{aligned}$$

We will use $\mathbf{k}$ to denote the vector quantity of the momentum, with corresponding energy $k = |\mathbf{k}|$[8]. For 4-point diagrams, we will use the following convention for $\{p_s, p_t, p_u\}$ momenta channels,

$$p_s \equiv |\mathbf{k}_1 + \mathbf{k}_2|, \quad p_t \equiv |\mathbf{k}_1 + \mathbf{k}_3|, \quad p_u \equiv |\mathbf{k}_1 + \mathbf{k}_4|, \tag{2.23}$$

satisfying,

$$p_s^2 + p_t^2 + p_u^2 = \sum_{i=1}^{4}\mathbf{k}_i^2. \tag{2.24}$$

[7] W.L.O.G. we have assumed all of our fields have the same spin, and hence our symmetry factor is just $n!$. This can be changed to include fields of arbitrary spin by changing $n! \to s_n[\alpha_i]$, where α_i is the spin of the field i.

[8] This is not entirely correct terminology, as energy is not strictly well-defined in cosmology. Nevertheless, $|\mathbf{k}|$ plays a similar role in mode functions that energies does in Minkowski.

2.3.1 Formulating an interacting theory

We denote the action as S, which is given as a time integral over the Lagrangian L, and as an additional spatial integral over the Lagrangian density, $\mathcal{L}$,

$$S = \int d\eta L[\phi;\eta] \equiv \int d\eta d^3x \mathcal{L}[\phi;\eta,\mathbf{x}]. \tag{2.25}$$

We are interested in the behaviour of the fluctuations, therefore we can consider a Lagrangian starting at second order. Considering a Lagrangian without higher order derivative couplings, it can only depend at most on 2nd order derivatives of the field, hence we can write,

$$\mathcal{L}_2 = \sum_a \frac{1}{2} a^2(\eta) \left[\phi_a'^2 - c_a^2(\nabla\phi_a)^2 - a^2(\eta) m_a^2 \phi_a^2\right], \tag{2.26}$$

where the $'$ represents a derivative with respect to conformal time. The constants c_a and m_a, denote the speed of sound and the mass of the scalar field ϕ_a. Without loss of generality, we will assume $c_a = 1$ for all massive fields, and $c_a \equiv c_s$ for the massless case.

The full Lagrangian can be written as the sum of (2.26) and the interaction Lagrangian,

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_{int}. \tag{2.27}$$

Considering interaction terms to be at most one coordinate (η or $\mathbf{x}$) derivative of our fields[9], and ensuring scale invariance of our action,

$$\begin{aligned}\mathcal{L}_{int} &= \lambda_n\, a(\eta)^{4-\sum d_a} \prod_{a=1}^{n} \partial^{d_a}\phi_a(\eta,\mathbf{x}) \\ &= \lambda_n\, a(\eta)^{4-\sum t_a} \prod_{a,b=1}^{n} \int_{\mathbf{k}_b} \mathcal{F}_n\left[\frac{\mathbf{k}_a\cdot\mathbf{k}_b}{a^2(\eta)}\right] e^{i\mathbf{k}_a\cdot\mathbf{x}}\partial_\eta^{t_a}\phi_a(\eta,\mathbf{k}_a),\end{aligned} \tag{2.28}$$

where we have moved into momentum space in the second equality. We have denoted the total number of derivatives acting on our fields as d_a - the sum of the total number of time t_a and spatial derivatives, and $\mathcal{F}_n(\mathbf{k}_a\cdot\mathbf{k}_b/a^2)$ is the sum of all the possible contractions of momenta $\mathbf{k}_a$ and $\mathbf{k}_b$, for all the fields $\phi_{a,b}(\eta,\mathbf{k}_{a,b})$ that have spatial derivatives acting on them.

The interacting action in Fourier space is

$$S_{int} = \lambda_n \int d\eta\, a(\eta)^{4-\sum t_a} \prod_{a,b=1}^{n} \int_{\mathbf{k}_b} \mathcal{F}_n\left[\frac{\mathbf{k}_a\cdot\mathbf{k}_b}{a^2(\eta)}\right] (2\pi)^3\delta^{(3)}\left(\sum_{i=1}^{n}\mathbf{k}_i\right) \partial_\eta^{t_a}\phi_a(\eta,\mathbf{k}_a). \tag{2.29}$$

We will now need the explicit expressions for the scalar fields $\phi_a(\eta,\mathbf{k})$ in terms of their mode functions. To do this we can decompose ϕ_a into positive and negative frequency parts, and then

[9]This is a valid assumption in perturbation theory, where one can use the equations of motion of the theory to replace every operator with a larger number of derivatives, with operators consisting of at most one derivative.

factorise these in terms of the mode functions, $f_a(\eta, k)$, using the creation and annihilation operators, $b_a^\dagger(k)$ and $b_a(k)$. They satisfy the usual commutation relations, with

$$[b_a(k), b_b^\dagger(k')] = (2\pi)^3\delta^{(3)}(k-k')\delta_{ab}, \tag{2.30}$$

being the only non-vanishing contribution. The mode functions are given by

$$\phi_a(\eta, \mathbf{k}) = \phi_a^+(\eta, \mathbf{k}) + \phi_a^-(\eta, \mathbf{k}) = f_a(\eta, k)b_a(\mathbf{k}) + f_a^*(\eta, -k)b_a^\dagger(-\mathbf{k}), \qquad k \in \mathbb{R}. \tag{2.31}$$

While the conjugate momenta,

$$\pi_a(\eta, \mathbf{k}) = a^2(\eta)\phi_a'(\eta, k) = a^2(\eta)\left[f_a'(\eta, k)b_a(\mathbf{k}) + f_a'^*(\eta, -k)b_a^\dagger(-\mathbf{k})\right]. \tag{2.32}$$

Deriving the equation of motion for (2.26), and writing it in the canonical form of the Bessel equation, we can find solutions corresponding to Hankel functions of the first and second kind, $H_{\nu_a}^{(1,2)}$ (up to an arbitrary phase, given a choice of η). Imposing the Bunch-Davies vacuum in the subhorizon limit, $k\eta \to -\infty$, we arrive at the solution using (2.11),

$$f_a(\eta, k) = -\frac{iH\sqrt{\pi}}{2}e^{i\pi(\nu_a/2+1/4)}\left(-\frac{\eta}{c_a}\right)^{3/2}\mathrm{H}_{\nu_a}^{(2)}(-c_a k\eta), \quad \nu_a = \sqrt{9/4 - m_a^2/H^2}. \tag{2.33}$$

The integral representation of $H_{\nu_a}^{(2)}$, over the lower half complex plane, is defined by [32],

$$\mathrm{H}_{\nu_a}^{(2)}(x) = -\frac{e^{i\nu_a\pi}}{i\pi}\int_{-\infty}^{\infty} d\eta\, e^{-ix\cosh(\eta) - \nu_a\eta}, \qquad \mathrm{Im}\,(x) < 0. \tag{2.34}$$

For inflationary scalar fields, where ν_a is real, we find the positive frequency mode functions satisfy,

$$\mathrm{H}_{\nu_a}^{(2)*}(x) = -\mathrm{H}_{\nu_a}^{(2)}(-x^*)\, e^{-i\pi\nu_a}, \qquad \nu_a \in \mathbb{R}. \tag{2.35}$$

If $\nu_a \in \mathbb{Im}$, it is easy to see (2.35) loses its phase. In the asymptotic past, their ratio obeys,

$$\lim_{\eta\to-\infty}\frac{\mathrm{H}^{(2)}(-c_a k\eta)}{\mathrm{H}^{(2)}(-c_a k\eta_0)} \to e^{i\mathrm{Re}(k)\eta}e^{-\mathrm{Im}\,(k)\eta}. \tag{2.36}$$

Inserting $m_a = \sqrt{2}H$, such that $\nu_a = 1/2$ and setting $c_a = 0$[10], we obtain conformally coupled scalar fields. The dS mode functions that create positive energy particles in the asymptotic past, are massless and are found by setting $\nu_a = 3/2$. In summary,

$$f_{\nu_a=1/2}(\eta, k) = \frac{iH\eta}{\sqrt{2k}}e^{-ik\eta}, \qquad f_{\nu_a=3/2}(\eta, k) = \frac{H}{\sqrt{2k^3c_s^3}}(1 + ikc_s\eta)e^{-ikc_s\eta}. \tag{2.37}$$

[10]Notice that the dS boosts are broken if $c_a \neq 1$. This is evident in the flat-space limit where a Lorentz boost leaves the light cone invariant, but in general displaces the sound cone, unless they coincide.

At this point, we can already write our first wavefunction coefficients. Using (2.20) and (2.21) and matching it with our expression for the action (2.29), we obtain,

$$\psi'_n(\mathbf{k}_1, ..., \mathbf{k}_n) = -in!\,\lambda_n \int_{-\infty(1+i\epsilon)}^{\eta_0} d\eta\, a(\eta)^{4-\sum t_a} \prod_{a=1}^{n} \mathcal{F}_n \left[\frac{\mathbf{k}_a \cdot \mathbf{k}_b}{a(\eta)^2}\right] \frac{\partial_\eta^{t_a} f_a^*(\eta, k_a)}{f_a^*(\eta_0, k_a)}. \tag{2.38}$$

Which already manifests itself with boundary observables at η_0. For $\mathrm{Im}\,(k) < 0$, the time integral in (2.38) is convergent due to the asymptotic behaviour of (2.36), and the property in (2.35).

3 Correlator diagrammatica

In this section, we want to derive rules to directly arrive at correlators from Feynman diagrams. In the in-in formalism (2.16), the rules follow very closely to ordinary Feynman rules in flat spacetime, but we need to account for some additional propagators (asides from the familiar Feynman propagator). We now have a notion of time ordering, which gives rise to right and left vertices, depending if the field came from a time-ordered or anti-time-ordered vertex respectively. However, the wavefunction formalism requires a more rigorous derivation to arrive at its rules.

3.1 In-in diagrammatica

Representing expectation values diagrammatically in an inflationary background, with a Bunch Davies vacuum, is similar to Minkowski spacetime, in that we can use Wicks theorem, to express the RHS of (2.16) as a sum over all possible pairwise contractions of our fields in S_{int}. In the in-in formalism, the Feynman rules are modified to [1, 4],

- **Order:** An nth order diagram will have V vertices each labelled as a point in space and time, with fields propagating in and/or out. Each vertex can be either "right" (R) or "left" (L) if it arises from the time-ordered (R) or anti-time (L) ordered H_{int} in (2.16)[11].
- **Vertices:** Performing Wicks theorem on (2.16). There are 2^V distinct ways of arranging these vertices to be R or L, we must sum over all possibilities. Each R and L vertex corresponds to $-i$ or $+i$ respectively, multiplied by the coupling parameter of our theory, and associated temporal and spatial derivatives, i.e. $\pm i\lambda\,\mathcal{F}[\mathbf{k}_a \cdot \mathbf{k}_b/a(\eta)^2]$.
- **Internal lines:** An internal line propagating from R to another R contributes a conventional Feynman propagator,

$$\langle T\{\phi(\eta, \mathbf{k})\phi(\eta', -\mathbf{k})\}\rangle' = \Delta_F(k, \eta; \eta') = \theta(\eta - \eta')f(\eta, k)f^*(\eta', k) + (\eta \leftrightarrow \eta'). \tag{3.1}$$

  Meanwhile, an internal line propagating from L to another L contributes,

$$\langle \bar{T}\{\phi(\eta, \mathbf{k})\phi(\eta', -\mathbf{k})\}\rangle' = \Delta_{\bar{F}}(k, \eta; \eta') = \theta(\eta - \eta')f^*(\eta, k)f(\eta', k) + (\eta \leftrightarrow \eta'). \tag{3.2}$$

- **External lines:** When one field is associated with an external line, i.e. it goes towards the boundary, the propagator changes. If the external line is joined to R it contributes $G_> = f(\eta, k)f^*(\eta', k)$, meanwhile if it is joined to L it contributes $G_< = f^*(\eta, k)f(\eta', k)$.

[11] In the literature, the convention of $+$ or $-$ is used to denote R and L vertices respectively.

- **Integration:** We must integrate over all times from $-\infty(1-i\epsilon)$ to the boundary $\eta_0 \to 0$, and all space at each vertex (this gives rise to momentum conservation)[12].
- **Symmetry factor:** Divide by the symmetric factor of the graph.
- **Result:** The final result once we sum over all diagrams is $B_n(\mathbf{k}_1, \cdots, \mathbf{k}_n)$.

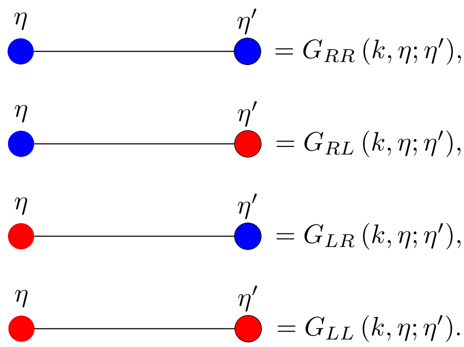
An alternative (and more compact) way of expressing the propagator rules is in matrix form,

$$G_{ab}\,(k,\eta;\eta') = \begin{pmatrix} G_{RR} & G_{RL} \\ G_{LR} & G_{LL} \end{pmatrix} \equiv \begin{pmatrix} \Delta_F & G_< \\ G_> & \Delta_{\bar{F}} \end{pmatrix}, \qquad G_> \equiv G_<^* = f(\eta,k)f^*(\eta',k). \tag{3.3}$$

The Feynman propagators can be written in terms of these propagators as,

$$\begin{aligned} \Delta_F\,(k,\eta;\eta') &= G_>\,(k,\eta;\eta')\,\theta(\eta-\eta') + G_<\,(k,\eta;\eta')\,\theta(\eta'-\eta), \\ \Delta_{\bar{F}}\,(k,\eta;\eta') &= G_<\,(k,\eta;\eta')\,\theta(\eta-\eta') + G_>\,(k,\eta;\eta')\,\theta(\eta'-\eta) = \Delta_F^*\,(k,\eta;\eta')\,. \end{aligned} \tag{3.4}$$

Diagrammatically, we use a blue and red dot to denote R and L vertices. The four types of propagators can be represented as:



3.2 Ψ propagators

In this Subsection, we want to derive the Feynman rules for the wavefunction coefficients. To do this, we need to find out what the propagators are. We'll solve this using the Greens method.

3.2.1 Green's functions

Now that we have an expression that relates the wavefunction coefficients to our action, we have to find a way to solve for the fields that satisfy the classical equation of motion ϕ^{cl}. These must obey the boundary conditions,

$$\lim_{\eta\to\eta_0} \phi^{\mathrm{cl}}(\eta) = \bar{\phi}, \qquad \lim_{\eta\to-\infty(1-i\epsilon)} \phi^{\mathrm{cl}}(\eta) = 0, \tag{3.5}$$

[12] Energy conservation is not a requirement, due to the spontaneous breaking of time translation symmetry. This is also evident by looking at the form of correlators, as they are evaluated at the same time (and not taking into account the time differences between the different fields).

which is simply the requirement that we must project onto the free vacuum in the asymptotic past. The differential operator $\mathcal{W}(\eta,k)$ that extremises the action is

$$\mathcal{W}(\eta,k)\phi(\eta,\mathbf{k}) = \frac{\delta S_2[\phi]}{\delta\phi(\eta,\mathbf{k})} = -\frac{\delta S_{int}[\phi]}{\delta\phi(\eta,\mathbf{k})}. \tag{3.6}$$

This is a non-linear equation, but we can proceed by assuming our interactions are weak, so that we can solve this perturbatively using the Greens method, as a series expansion construction in S_{int}. We introduce a bulk-to-boundary propagator $\mathcal{K}(k,\eta)$ and bulk-to-bulk propagator $\mathcal{G}(k,\eta;\eta')$, both symmetric in its time arguments and obeying,

$$\mathcal{W}(\eta,k)\mathcal{K}(k,\eta) = 0, \qquad \mathcal{W}(\eta,k)\mathcal{G}(k,\eta;\eta') = -\delta(\eta-\eta'), \tag{3.7}$$

with boundary conditions,

$$\lim_{\eta\to\eta_0}\mathcal{K}(k,\eta) = 1, \qquad \lim_{\eta\to-\infty(1-i\epsilon)}\mathcal{K}(k,\eta) = 0, \qquad \lim_{\eta\to\eta_0,-\infty(1-i\epsilon)}\mathcal{G}(k,\eta;\eta') = 0. \tag{3.8}$$

Using the above boundary constraints on the fields and the propagators, we can construct a solution to (3.6), with the following,

$$\phi^{cl}(\eta,\mathbf{k}) = \mathcal{K}(k,\eta)\bar{\phi}(\mathbf{k}) + \int d\eta'\mathcal{G}\left(k,\eta;\eta'\right)\left.\frac{\delta S_{int}[\phi]}{\delta\phi(\eta',\mathbf{k})}\right|_{\phi=\phi^{cl}(\eta',\mathbf{k})}. \tag{3.9}$$

The fields are composed of two independent frequency solutions defined in (2.31). As the positive frequencies lose support in the asymptotic past, this fixes the bulk-to-boundary propagator

$$\mathcal{K}(k,\eta) = \frac{f^*(\eta,k)}{f^*(\eta_0,k)}. \tag{3.10}$$

Solving for the bulk-to-bulk propagator requires solving a first-order differential equation defined from past infinity to the boundary of scattering $(-\infty,\eta_0)$. We proceed by defining a linearly independent basis of solutions $\{\phi_1,\phi_2\}$, where each obeys one of our homogeneous boundary conditions. Any solution in the region $(-\infty,\eta']$ obeying our asymptotic boundary condition (the right of (3.5)), must be $\propto\phi_1$, and similarly for ϕ_2 in the region $[\eta',\eta_0)$. These constants of proportionality are independent of η but may depend on k and η'. One can solve this to obtain

$$\phi_1(\eta,k) = f^*(\eta,k), \qquad \phi_2(\eta,k) = f^*(\eta,k) - \frac{f^*(\eta_0,k)}{f(\eta_0,k)}f(\eta,k). \tag{3.11}$$

To finally find the solution $\mathcal{G}$ in each region, we need to fix the constants of proportionality, which are calculated by ensuring continuity at $\eta=\eta'$ and using the canonical quantisation (2.30) for an appropriately normalised Wronskian $W(\phi_1,\phi_2;\eta,k)$ when acting on the mode functions,

$$W(f^*,f;\eta,k) \equiv a^2(\eta)\left(f^*(\eta,k)f'(\eta,k) - f(\eta,k)f^{*\prime}(\eta,k)\right) = -i. \tag{3.12}$$

Therefore, using (3.11) we find the normalised Wronskian to be,

$$W(\phi_1, \phi_2; \eta, k) \equiv a^2(\eta)\left(\phi_1(\eta,k)\phi_2'(\eta,k) - \phi_2(\eta,k)\phi_1^{*\prime}(\eta,k)\right) = i\frac{f^*(\eta_0,k)}{f(\eta_0,k)}. \tag{3.13}$$

We can write the solutions for the bulk-to-bulk propagator as

$$\mathcal{G}(k,\eta;\eta') = \begin{cases} \dfrac{\phi_2(k,\eta)\phi_1(k,\eta')}{W(\phi_1,\phi_2;\eta',k)}, & \eta = [\eta', \quad \eta_0) \\ \dfrac{\phi_1(k,\eta)\phi_2(k,\eta')}{W(\phi_1,\phi_2;\eta',k)}, & \eta = (-\infty, \eta']. \end{cases} \tag{3.14}$$

This can alternatively be written explicitly and in terms of the bulk-to-boundary propagator as,

$$\mathcal{G}(k,\eta;\eta') = iP_k[\theta(\eta-\eta')\mathcal{K}^*(k,\eta)\mathcal{K}(k,\eta') + \theta(\eta'-\eta)\mathcal{K}(k,\eta)\mathcal{K}^*(k,\eta') - \mathcal{K}(k,\eta)\mathcal{K}(k,\eta')], \tag{3.15}$$

where $P_k \equiv P_\phi(\eta_0,k) = \langle\bar{\phi}(\mathbf{k})\bar{\phi}(-\mathbf{k})\rangle' = |f(\eta_0,k)|^2$ is the power spectrum.

It is worth pointing out that for the massless and conformally coupled fields in dS, there is no branch cut and therefore we are free to assume all our energies are real $k_a^* = k_a$. Furthermore, the two propagators are *Hermitian analytic*[13],

$$\mathcal{K}^*(-k^*,\eta) = \mathcal{K}(k,\eta) \implies \mathcal{G}^*(-k^*,\eta;\eta') = \mathcal{G}(k,\eta;\eta'). \tag{3.16}$$

Having found the two propagators, we can now substitute them back into (3.9) and perform the η integral. This allows us to find the wavefunction coefficients ψ_n' at any order, which can be represented diagrammatically, as a sum over different diagrams. The propagators connect distinct vertices, where each vertex corresponds to an interaction in S_{int}. As the names imply, $\mathcal{K}$ acts as a propagator for external momentum lines (connects any vertex to the boundary at η_0), while $\mathcal{G}$ connects vertices of internal momenta (any two vertices at times $(\eta,\eta') < \eta_0$, with each other). They can be expressed diagrammatically as shown in Fig. 2.

Figure 2: A diagrammatic representation of the bulk-to-bulk and bulk-to-boundary propagators, used for the Feynman rules for the Wavefunction of the Universe.

Notice (3.15) can be rewritten in terms of the in-in propagators in (3.3) as,

$$\mathcal{G}(k,\eta;\eta') = i\Delta_F(k,\eta;\eta') - iP_k\mathcal{K}(k,\eta)\mathcal{K}(k,\eta'). \tag{3.17}$$

[13]We refer to a Hermitian analytic function, when the function is equivalent to its Hermitian analytic image, e.g. for a function F with momenta $\mathbf{k}$ and energy k, F is Hermitian analytic iff $F(k,\mathbf{k}) = F^*(-k^*,-\mathbf{k})$

This shows explicitly how both our external lines are related by just a boundary term[14]. This boundary term represents the difference between the bulk-to-bulk propagator and the Feynman propagator, as it vanishes when either η or $\eta' \to \eta_0$. It is this term that doesn't allow us to extend the well-known Cutting rules from Minkowski to those of cosmology, as a result, we'll need new cutting rules - the Cosmological Cutting Rules, which we'll review in the following section.

3.2.2 Ψ diagrammatica

Finally, we have the Feynman rules to express wavefunction coefficients diagrammatically:

- **Order:** There is no notion of time ordering now. A diagram with I internal lines and $V = I+1$ vertices (at tree level), has internal momenta running from $m \in [1, I]$, and external momenta $a \in [1, n]$.
- **Vertices:** At each vertex multiply by an overall factor of $i^{(V-I)}$, the coupling parameter of our theory and the associated temporal and spatial derivatives[15].
- **Internal and external lines:** Two vertices at conformal time η and η', connected by an internal line should be multiplied by a bulk-to-bulk propagator, $\mathcal{G}(k, \eta; \eta')$. If connected by an external line, it contributes a bulk-to-boundary propagator, $\mathcal{K}(k, \eta)$.
- **Integration:** We must integrate over all times from $-\infty(1-i\epsilon)$ to the boundary $\eta_0 \to 0$, and space at each vertex (this gives rise to momentum conservation).
- **Symmetry factor:** Divide by the symmetric factor of the graph, as usual.
- **Result:** The final result once we sum over all permutations of the diagrams, are the wavefunction coefficients, $\psi_n'(\{k\}; \{p\}; \{\mathbf{k}\})$.

In the final rule, we have introduced a new notation for the wavefunction coefficients. We have separated the set of all $\{k\} = k_1, ..., k_n$ into sets consisting of momenta $\{\mathbf{k}\}$, external $\{k\}$ and internal $\{p\}$ energies. This will be useful when we look at exchange diagrams, as we will need to distinguish between internal and external energies. For external diagrams, we write

$$\psi_n \equiv \psi_n(\{k\}; \{p\}; \{\mathbf{k}\}) = \psi_n(k_1, ...k_n; p_1, ..., p_m; \mathbf{k}_a \cdot \mathbf{k}_b, \mathbf{k}_a \times (\mathbf{k}_b \cdot \mathbf{k}_c)...). \tag{3.18}$$

Combining the above rules, at tree level, we can compute a general wavefunction coefficient with

$$\psi_n'(\{k\}; \{p\}; \{\mathbf{k}\}) = -i \left(\prod_\nu^V \int d\eta_\nu \, \mathcal{F}_\nu \left[\frac{\mathbf{k}_a \cdot \mathbf{k}_b}{a(\eta)^2} \right] \right) \left(\prod_a^n \mathcal{K}(k_a, \eta) \right) \left(\prod_a^n \mathcal{G}(p_a, \eta; \eta') \right). \tag{3.19}$$

4 Cosmological Unitarity

Unitarity lies at the heart of quantum field theory in flat spacetime, which in summary, sets the conservation of probability. The requirement of a theory to be unitarity imposes that the states

[14] This is a conformal boundary at η_0, which explicitly distinguishes between the future and past asymptotic boundaries of global dS, arising from the explicit definition of $\mathcal{G}(k, \eta; \eta') = \langle 0(\eta_0) | T\{\phi_a(\eta, \mathbf{k}) \phi_a(\eta', -\mathbf{k})\} | \Omega \rangle'$.

[15] This is a result of the factor of i arising from a wick rotation in evaluating momentum integrals in loops $i^{1-L} = i^{(V-I)}$ where L is the number of loops in our diagram. This is derived using the Betti number of the graph and for connected graphs $L = I - V + 1$ [33]. Equivalently, each vertex and propagator contributes a factor of i and $(-i)$ respectively.

in the Hilbert space must transform in unitary irreducible representations of the Poincaré group, $\mathrm{ISO}(1,3)$ [5]. This constrains the form of the S-matrix as well as the possible interactions of the theory. Given a hermitian interacting Hamiltonian H_{int}, the time evolution operator associated with this Hamiltonian will be unitary $U^\dagger U = \mathbb{1}$ and is given by (2.17). This is because any state $|\Psi;\eta\rangle$ must transform covariantly under a Poincaré transformation U, and hence its norm is a constant of motion. For an initially normalised state,

$$\langle\Psi;\eta|\Psi;\eta\rangle = \left\langle\Psi;\eta\left|U^\dagger U\right|\Psi;\eta\right\rangle = \mathbb{1}. \tag{4.1}$$

In this section we will see how unitarity constrains the analytic structure of perturbative observables in cosmology, fixing the structure of initial conditions as time evolves. We will start by reviewing the Optical Theorem and proceed to discuss the recently derived Cosmological Optical Theorem [11] and its associated cutting rules [12, 13] on any FLRW spacetime, with an initial Bunch-Davies vacuum in the asymptotic past[16]. These rules constrain the structure of the wavefunction under unitary time evolution. We will also cover a non-perturbative implication of unitarity, that leverages representation theory - the positivity of the Hilbert space norm. This is done following a spectral decomposition of the two and four-point functions and showing the positivity of their spectral densities [35, 36]. This sometimes is referred to as the cosmological Källén–Lehmann representation. Finally, we review the *Manifestly Local Test*, which selects local theories when the interactions of on-shell properties are manifestly local [7].

4.1 The S-matrix Optical Theorem

One of the most remarkable consequences of unitarity is the Optical Theorem for flat spacetime. It relates the scattering amplitudes to the cross-sections. Evaluated at each order, it predicts tree-level effects from loop cross-sections. In other words, it shows that classical physics is uniquely determined by quantum physics. To derive this theorem, we must expand the evolution operator into its non-hermitian part $U = \mathbb{1} + i\Delta U$, and substitute it back into (4.1). Simplifying, this implies

$$\begin{aligned}\langle\Psi;\eta_f|\Delta U^\dagger\Delta U|\Psi;\eta_i\rangle &= i\langle\Psi;\eta_f|(\Delta U^\dagger - \Delta U)|\Psi;\eta_i\rangle \\ &= i(2\pi)^4\delta^{(4)}(p_i - p_f)(\mathcal{A}^*_{f\to i} - \mathcal{A}_{i\to f}),\end{aligned} \tag{4.2}$$

for arbitrary initial and final times. The RHS measures its failure to be Hermitian, which we have rewritten in terms of scattering amplitudes $\mathcal{A}$, defined as

$$\langle\Psi;\eta_f|\Delta U|\Psi;\eta_i\rangle = (2\pi)^4\delta^{(4)}(p_i - p_f)\mathcal{A}_{i\to f}. \tag{4.3}$$

The LHS can be computed by introducing a Lorentz invariant complete set of states,

$$\mathbb{1} = \sum_{n=0}^{\infty}\sum_{\beta_1,\ldots,\beta_n}\prod_{a=1}^{m}\int_{\mathbf{p}_a}|\{\mathbf{k};\beta\}_n\rangle\langle\{\mathbf{k};\beta\}_n|. \tag{4.4}$$

Each β_a characterises the type of particle e.g. ϕ. Denoting the intermediate state $|\{\mathbf{k};\beta\}_n\rangle = |\psi;\eta\rangle$, $\sum_a \mathbf{k}_a = \mathbf{p}_\psi$, the LHS becomes

[16] This last assumption can be generalised to other asymptotic vacuum states (see e.g. [34]).

$$
\begin{aligned}
\langle\Psi;\eta_f|\Delta U^\dagger \Delta U|\Psi;\eta_i\rangle &= \sum_{n,\beta}\int d\Pi_{\mathbf{p}} \langle\Psi;\eta_f|\Delta U^\dagger|\{\mathbf{k};\beta\}_n\rangle\langle\{\mathbf{k};\beta\}_n|\Delta U|\Psi;\eta_i\rangle \\
&= (2\pi)^8 \sum_{n,\beta}\int d\Pi_{\mathbf{p}} \delta^{(4)}(p_i - p_\psi)\delta^{(4)}(p_f - p_\psi)\mathcal{A}^*_{f\to\psi}\mathcal{A}_{i\to\psi},
\end{aligned}
\tag{4.5}
$$

where $d\Pi_{\mathbf{p}}$ is a shorthand notation for the product, of all the momenta integrals in (4.4). Combining (4.2) and (4.5), we obtain the generalised Optical Theorem for amplitudes in flat spacetime [5],

$$
\mathcal{A}_{i\to f} - \mathcal{A}^*_{f\to i} = i(2\pi)^4 \sum_{n,\beta}\int d\Pi_{\mathbf{p}} \delta^{(4)}(p_i - p_\psi)\delta^{(4)}(p_f - p_\psi)\mathcal{A}_{i\to\psi}\mathcal{A}^*_{f\to\psi}. \tag{4.6}
$$

This result is non-perturbative, but it is useful to see how it is satisfied order by order. The LHS involves a product of amplitudes at $\mathcal{O}(\mathcal{A}^2)$ and is related to the RHS at $\mathcal{O}(\mathcal{A})$. This proves that any interacting theory must have loops, as classical (tree-level) interactions must be readily derived from these loop effects. To compute this at each order, one can use Cutkosky Rules [37]. These rules tell us how we can compute the discontinuity of each diagram. In general terms, the left of (4.6) corresponds to the imaginary part of the diagram, which can be calculated using the discontinuity of the integral, meanwhile, the right of (4.6) is a sum over all possible *cuttings* - each way of putting all the intermediate states ψ on-shell, e.g. $p^2 = m^2$. These are very analogous to the cosmological cutting rules we will review in the following section in more detail.

4.2 The Cosmological Optical Theorem

In this section, we use the perturbative expression for the Optical Theorem to derive the *Cosmological Optical Theorem* (COT). This is done at first and second order, and it still remains future work to find the non-perturbative expression for the COT, in a similar manner to (4.6). The COT expresses the unitary implications of quantum mechanics in a specific analytic structure of the wavefunction coefficients, to find a relation between ψ'_n and ψ'^*_n. This is a general result which is valid to any order in perturbation theory.

Contact Diagrams

The evolution operator up to linear order in H_{int} is, $U = \mathbb{1} + i\int d\eta\, H_{int}(\eta)$. Computing the Optical Theorem (4.2) at this order, leaves only the LHS,

$$
\langle\Psi;\eta_f|\int_{-\infty}^{\eta_0} d\eta\, H_{int}|\Psi;\eta_i\rangle - \langle\Psi;\eta_f|\int_{-\infty}^{\eta_0} d\eta\, H^\dagger_{int}|\Psi;\eta_i\rangle = 0. \tag{4.7}
$$

At linear order, $H_{int} = -\mathcal{L}_{int}$, where $\mathcal{L}_{int}$ is given in (2.28). We are interested in an initial vacuum state, so we set $|\Psi;\eta_i\rangle = |\Omega\rangle$. The final state is composed of n particles each of type β_a and momenta $\mathbf{k}_a$; $|\Psi;\eta_f\rangle = |\{\mathbf{k};\beta\}_n\rangle$. Taking the limit $\eta \to -\infty(1-i\epsilon)$ in the asymptotic past,

$$
\begin{aligned}
&\langle\{\mathbf{k};\beta\}_n|\int_{-\infty(1-i\epsilon)}^{\eta_0} d\eta\, H_{int}|0\rangle \\
&= -\lambda_n \int_{-\infty(1-i\epsilon)}^{\eta_0} d\eta\, a(\eta)^{4-\sum t_a} \prod_{a,b=1}^{n}\int_{\mathbf{k}_b} \mathcal{F}_n\left[\frac{\mathbf{k}_a\cdot\mathbf{k}_b}{a(\eta)^2}\right](2\pi)^3\delta^{(3)}\left(\sum_{i=1}^{n}\mathbf{k}_i\right)\langle\{\mathbf{k};\beta\}_n|\partial_\eta^{t_a}\phi_a(\eta,\mathbf{k}_a)|0\rangle.
\end{aligned}
\tag{4.8}
$$

Making a change of variables to absorb the part of our counter into the argument of the mode functions, we can relate this to the wavefunction coefficients in (2.38)

$$\langle\{\mathbf{k};\beta\}_n| \int_{-\infty(1-i\epsilon)}^{\eta_0} d\eta\, H_{int}|0\rangle = -\lambda_n \int_{-\infty}^{\eta_0} d\eta\, a(\eta)^{4-\sum t_a} \prod_{a=1}^{n} \mathcal{F}_n \left[\frac{\mathbf{k}_a\cdot\mathbf{k}_b}{a(\eta)^2}\right] \partial_\eta^{t_a} f_a^*(\eta, k_a - i\epsilon) \tag{4.9}$$
$$= -\frac{i}{n!}\psi_n'(k_1,...,k_n;\hat{k}_a\cdot\hat{k}_b)\prod_{a=1}^{n} f_a^*(\eta_0, k_a - i\epsilon).$$

We can follow a similar procedure for the hermitian conjugate, to obtain

$$\begin{aligned}\langle\{\mathbf{k};\beta\}_n| \int_{-\infty(1+i\epsilon)}^{\eta_0} d\eta\, H_{int}^\dagger|0\rangle &= -\lambda_n^* \int_{-\infty(1+i\epsilon)}^{\eta_0} d\eta\, a(\eta)^{4-\sum t_a} \prod_{a=1}^{n} \mathcal{F}_n \left[\frac{\mathbf{k}_a\cdot\mathbf{k}_b}{a(\eta)^2}\right] \partial_\eta^{t_a} f_a^*(\eta, k_a) \\ &= \frac{i}{n!}[\psi_n'(k_1,...,k_n;\hat{k}_a\cdot\hat{k}_b)]^*\prod_{a=1}^{n} f_a^*(\eta_0, k_a - i\epsilon).\end{aligned} \tag{4.10}$$

Plugging these into the Optical Theorem at first order in (4.7) and using the convergence properties of the integral in the asymptotic past given in (2.36), we can drop the $i\epsilon$ prescription for $\mathrm{Im}(k) < 0$. As reviewed above, for $\mathrm{Re}(k)$, the integral should be approached in the lower half-plane. Putting everything together, we arrive at the Cosmological Optical Theorem for contact diagrams,

$$\begin{aligned}\psi_n'\left(k_a,\hat{k}_a\cdot\hat{k}_b\right) &= -\left[\psi_n'\left(-k_a - i\epsilon,\hat{k}_a\cdot\hat{k}_b\right)\right]^* \qquad & k_a\in\mathbb{R}^+ \\ &= -\left[\psi_n'\left(-k_a^*,\hat{k}_a\cdot\hat{k}_b\right)\right]^* \qquad & k_a\in\mathbb{C}^{n-}.\end{aligned} \tag{4.11}$$

This is the cosmological equivalent of the Hermitian analyticity of amplitudes, $\mathcal{A}(k) = \mathcal{A}^*(k^*)$.

Exchange diagrams At next order in H_{int}, the COT finds that 4-point exchange diagrams, ψ_4, are related to a lower order 3-point correlator, ψ_3. This derivation follows very closely to that of contact diagrams except its more lengthy (see Appendix B for a detailed derivation). In this case, we have to sum over all the internal momentum channels $\psi_4 = \psi_4(p_s) + \psi_4(p_t) + \psi_4(p_u)$ to obtain the final form of the Cosmological Optical Theorem for exchange diagrams,

$$\begin{aligned}&\psi_4'(k_1,k_2,k_3,k_4,p_s,p_t,p_u) + [\psi_4'(-k_1,-k_2,-k_3,-k_4,p_s,p_t,p_u)]^* = \\ &\sum_{i=s,t,u} P_\sigma(\eta_0,p_i)\left[\psi_3'(k_1,k_2,p_i) - \psi_3'(k_1,k_2,-p_i)\right]\left[\psi_3'(k_3,k_4,p_i) - \psi_3'(k_3,k_4,-p_i)\right].\end{aligned} \tag{4.12}$$

Notice we have dropped the contractions term $\mathbf{k}_a\cdot\mathbf{k}_b$ in the wavefunction coefficients, as the inner products between the momenta can always be written as products of p_a's and k_b's. For the $a = s$ channel, the COT can be expressed diagrammatically, as shown in the Fig. 3.

4.3 Cutting rules for general FLRW spacetimes

In this section we review the Cosmological Cutting Rules. These rules, in analogy with their flat space counterpart [37], consist of a set of unitarity conditions to be satisfied order by order in

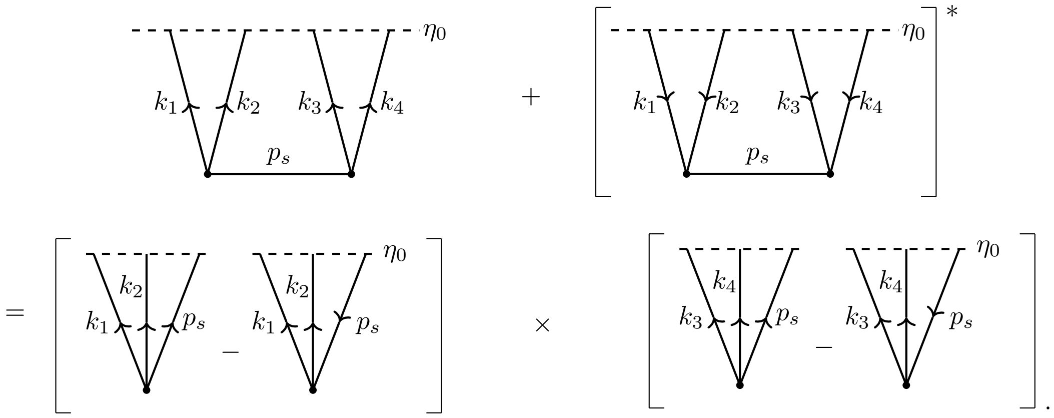


Figure 3: Diagrammatic representation of the COT for exchange diagrams from a bulk perspective, for the p_s channel. The ϕ fields carry momenta k_a while the exchange fields σ carry the internal momenta p_s.

perturbation theory, by the wavefunction coefficients. However, there are certain rules that cannot be extended to curved spacetime beyond tree-level diagrams. A priori, there isn't a natural expectation of what unitarity implies for correlators, due to the absence of time translation invariance. The main difference arises from the distinct propagators we have in our theory, specifically the presence of the boundary term we saw in the bulk-to-bulk propagator in (3.17). It is this term that makes the Cosmological Cutting Rules look quantitatively different from their flat spacetime analogue. Nonetheless, away from the boundary, i.e. in the so-called vanishing total energy limit, our cutting rules should reduce to the well-known ones for amplitudes. Cosmological cutting rules were first derived at tree level in [12], this was further extended to include loop interactions in [13].

4.3.1 Discontinuities

Let's first define what we mean by discontinuity. For an arbitrary function $f(x)$, we introduce the operator Disc that finds the discontinuity across the real axis,

$$\text{Disc}[f(x)] \coloneqq \lim_{\epsilon\to 0} f(x+i\epsilon) - f(x-i\epsilon) = f(x) + f^*(x) = 2\Re(f(x)). \tag{4.13}$$

The hypothesis of maximal analyticity assumes the function $f(x)$ is non-analytic *only* when the LHS of (4.13) is non-zero [38, 39]. Writing a subscript on the Disc allows us to indicate which energies are unaffected by the operator, while the rest are analytically continued to negative energies. The momentum are always mapped into the negative momenta[17]. The updated operator is,

$$\begin{aligned}&\underset{k_1,\ldots,k_j}{\text{Disc}}\,[i\psi'_n(k_1,\ldots k_n;\{p\};\{\mathbf{k}\})]\\ &\equiv i\psi'_n(k_1,\ldots k_n;\{p\};\{\mathbf{k}\}) + i\psi'^*_n(k_1,\ldots,k_j,-k^*_{j+1},\ldots,-k^*_n;\{p\};-\{\mathbf{k}\}).\end{aligned} \tag{4.14}$$

[17] This ensures that the complex conjugate of our wavefunction is well defined, given the asymptotic past and future states are flipped in complex conjugation. Also, notice the internal energies are never analytically continued.

With this notation, we can immediately rewrite the COT which must be satisfied by any contact n-point diagrams from (4.11), which ensures

$$\mathrm{Disc}[i\psi_n'(k_1, ...k_n; \{\mathbf{k}\})] = 0. \tag{4.15}$$

The Disc operator can be used to pick out the imaginary part of propagators, specifically internal propagators $\mathcal{G}$ will be of interest. The imaginary part of the internal propagator can be expressed in terms of the product of two external propagators. Through a direct calculation using (3.15) and $\theta(t_L - t_R) + \theta(t_R - t_L) = 1$ we arrive at

$$\mathrm{Im}\,\mathcal{G}(p, t_L; t_R) = 2P(t_0, p)\,\mathrm{Im}\,\mathcal{K}(p; t_L)\,\mathrm{Im}\,\mathcal{K}(p; t_R), \qquad p \in \mathbb{R}^+. \tag{4.16}$$

This will allow us to express a ψ_4' exchange diagram as a product of two ψ_3 contact diagrams, by replacing the internal propagators $\mathcal{G}$ with external propagators $\mathcal{K}$. As expected, this follows closely to the COT for exchange diagrams from Section 4.2.

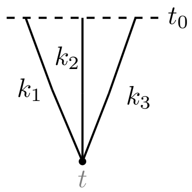
4.3.2 Single cut rules

To introduce how the cutting rules work, we begin with single cuts. Consider a simple action

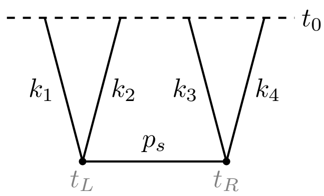
$$S_{int}[\phi; t] = \int dt\, d^3\mathbf{x}\, a^3(t)\lambda\phi^3(t, \mathbf{x}), \tag{4.17}$$

where λ is the coupling parameter. We can find the wavefunction coefficients with the Feynman rules derived in Section 3.2.2, specifically (3.19). In Minkowski i.e. $a = 1$, we obtain:

$$\begin{aligned} \psi_3'(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= i\lambda \int_{-\infty}^{t_0} dt\, \mathcal{K}(t, k_1)\mathcal{K}(t, k_2)\mathcal{K}(t, k_3) \\ \psi_4'(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4; p_s) &= i\lambda^2 \int_{-\infty}^{t_0} dt_L\, dt_R\, \mathcal{K}(t_L, k_1)\mathcal{K}(t_L, k_2)\mathcal{G}(t_L, p_s; t_R)\mathcal{K}(t_R, k_3)\mathcal{K}(t_R, k_4) \end{aligned} \tag{4.18}$$



$\psi_3'(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$

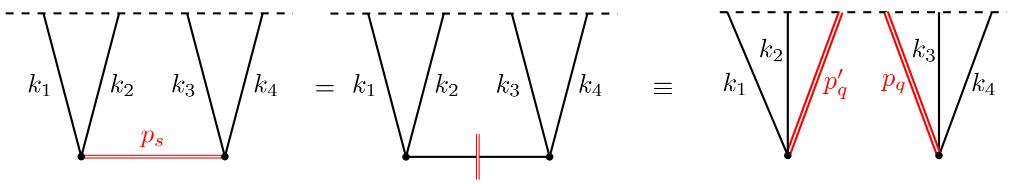


$\psi_4'(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4; p_s)$

In similar spirit to the COT, where we managed to express the ψ_4' in terms of a product of lower order ψ_3', we can now calculate and express the discontinuity of a tree-level diagram in terms of other tree-level diagrams with fewer external fields, using (4.14) and (4.16). Lets consider using the Disc to pick out the discontinuity of the internal momenta p_s, to rewrite ψ_4' as

$$
\begin{aligned}
&i\underset{p_s}{\text{Disc}}[i\psi_4'(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4; p_s)] \\
&= 2\lambda^2 \int_{-\infty}^{t_0} dt_L\, dt_R\, \mathcal{K}(t_L, k_1)\mathcal{K}(t_L, k_2)\text{Im}\,\mathcal{G}(t_L, p_s; t_R)\mathcal{K}(t_R, k_3)\mathcal{K}(t_R, k_4) \\
&= \int_{\mathbf{q}\mathbf{q}'} i\underset{q}{\text{Disc}}[i\psi_3'(\mathbf{k}_1, \mathbf{k}_2, \mathbf{q})]\, P(\mathbf{q}, \mathbf{q}')\, i\underset{q'}{\text{Disc}}[i\psi_3'(\mathbf{q}', \mathbf{k}_3, \mathbf{k}_4)]
\end{aligned}
\tag{4.19}
$$

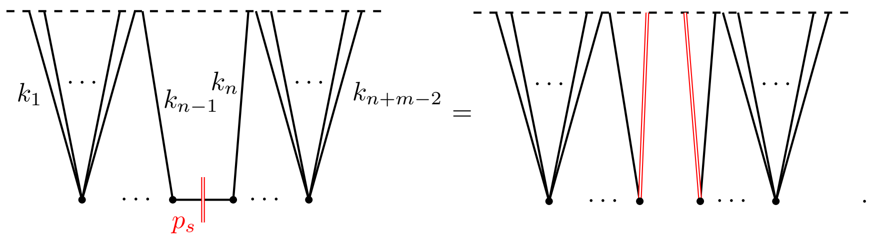
where $P(\mathbf{q}, \mathbf{q}') = (2\pi^3)\delta^{(3)}(\mathbf{q} + \mathbf{q}')P(t_0, q)$ is the unprimed power spectrum. Diagrammatically this represents a factorisation of ψ_4' into $\psi_3' \times \psi_3'$:



$i\underset{p_s}{\text{Disc}}[i\psi_4'(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4; p_s)]$ $\qquad$ $i\underset{q}{\text{Disc}}[i\psi_3'(\mathbf{k}_1, \mathbf{k}_2, \mathbf{q})]\, P(\mathbf{q}, \mathbf{q}')\, i\underset{q'}{\text{Disc}}[i\psi_3'(\mathbf{q}', \mathbf{k}_3, \mathbf{k}_4)]$

In the above, we have introduced diagrammatic notation to represent a discontinuity. If an energy argument of a propagator appears in the argument of the discontinuity (and hence is not analytically continued), we replace the propagator with a double red line (left figure). We "cut" this propagator by taking its imaginary part (middle figure) which by definition exchanges the internal propagator into two external ones, giving the final diagram (right figure). It is straight forwards to extend these single cuts to any wavefunction coefficient. If it has more than one internal momenta, we must make sure to analytically continue all internal lines that are not cut, i.e. that do not appear in the argument of the discontinuity operator. We can generalise the single-cut rules with,

$$
\begin{aligned}
\underset{p_s}{\text{Disc}}[i\psi_{n+m-2}'(\{k\}; \{p\}, p_s; \{\mathbf{k}\})] = -iP(p_s)&\underset{p_s}{\text{Disc}}[i\psi_n'(k_1, ..., k_{n-1}, p_s; \{p\}; \{\mathbf{k}\})] \\
&\times\underset{p_s}{\text{Disc}}[i\psi_m'(k_n, ..., k_{n+m-2}, p_s; \{p\}; \{\mathbf{k}\})],
\end{aligned}
\tag{4.20}
$$



4.4 The Cutting Rules

The previous Subsection introduced what 'cutting' a wavefunction means. We will now see that these rules can be extended further, to generic diagrams with any number of loops. However, we now don't have the freedom to pick which internal lines are to be cut, instead, we have to sum over all possible cuttings of our diagram. The rules used to perform these cuts are called the *Cosmological Cutting Rules*, and are given by the following:

- **The original diagram:** The original diagram D, with wavefunction coefficient ψ^D, contains I internal lines. We perform a series of 'cuts' C over all the internal lines, i.e. $C \subseteq I$. When we cut, we highlight the internal line with momentum p_a and replace its propagator with two external propagators.
- **The cut diagram:** The resulting cut diagrams we denote by D_C. The subset of connected cut diagrams are denoted by $D_C^{(n)} \subseteq D_C$, which contains $I^{(n)} \subseteq I$ internal lines. The wavefunction coefficient associated with the connected cut diagrams is $\psi^{D_C^{(n)}}$, satisfying,

$$D_C = \cup_n D_C^{(n)}. \tag{4.21}$$

- **The final diagram:** We denote the final diagram as $\tilde{D}_C[\psi]$, which we obtain by conducting the following steps. We take the Disc of $\psi^{D_C^{(n)}}$ with respect to $I^{(n)}$ and its cut lines. We replace the cut momenta p_a with a pair of momenta $\{\mathbf{q}_a, \mathbf{q}'_a\}$ (one for each external propagator) and multiply by the power spectrum $P(\mathbf{q}_a, \mathbf{q}'_a)$. In formulae,

$$\tilde{D}_C[\psi] = \left[\prod_{a\in C}\int_{\mathbf{q}_a\mathbf{q}'_a} P(\mathbf{q}_a, \mathbf{q}'_a)\right]\prod_n (-i) \underset{I_n\{q_a\}}{\mathrm{Disc}}\left[i\psi^{D_C^{(n)}}\right], \qquad C \subseteq I \tag{4.22}$$

In a similar spirit to the Cutkosky rules in Minkowski, summing over all possible cuts gives zero, as diagrams cancel two by two. The *general cutting rule* can be expressed as,

$$\sum_{C=0}^{2^I} \tilde{D}_C[\psi] = 0, \qquad C \subseteq I. \tag{4.23}$$

Taking the diagram without cuts ($C = 0$) out of the sum, we obtain the final expression

$$i\underset{I}{\mathrm{Disc}}\left[i\psi^D\right] = \sum_{C=1}^{2^{|I|}-1}\left[\prod_{a\in C}\int_{\mathbf{q}_a\mathbf{q}'_a} P_{\mathbf{q}_a\mathbf{q}'_a}\right]\prod_n (-i) \underset{I_n\{q_a\}}{\mathrm{Disc}}\left[i\psi^{D_C^{(n)}}\right], \qquad C \subseteq I. \tag{4.24}$$

This expresses the discontinuity of the original diagram of a wavefunction coefficient ψ^D as a sum over diagrams with cuts - a lower order diagram with fewer loops and/or lines. This relation shares many similarities with the equivalent Cutkosky cutting rules of flat spacetime, and in fact, we should expect the same relations to emerge on the total energy pole, $k_T \to 0$ [40]. We should think of the COT and its associated cutting rules as encoding the content of the Optical Theorem at all orders in perturbation theory, in a general FLRW spacetime.

4.5 Nonperturbative unitarity

In this section, we will proceed by computing the two and four-point correlator in terms of a spectral integral, also known as the Källén–Lehmann decomposition. We will find these spectral densities obey some definite positivity properties. This representation is a special case of a dispersion relation. In comparison to flat spacetime, dS is not strictly stationary, as we don't have a globally defined timelike killing vector. This already implies that any positivity constraint won't be able to be imposed by the mass alone. This will follow from Section 2.1.1, specifically unitary representations of the Euclidean conformal group $SO(d+1,1)$, isomorphic to dS [36, 35]. To do this, we will need to introduce hypergeometric functions as an alternative representation for the bulk-to-bulk propagators. We will keep our results general, to a spatial dimension d.

Hypergeometric functions Our results can be rewritten in terms of hypergeometric functions. The two-point de Sitter invariant distance ξ and the chordal distance ζ are defined by [8, 36],

$$\xi \equiv X_1 \cdot X_2 = \frac{4\eta\eta'}{|\mathbf{x}-\mathbf{x}'|^2 + |\eta-\eta'|^2}, \qquad \zeta = 2(1-\xi). \tag{4.25}$$

The four propagators from the in-in formalism in (3.3) can be rewritten in terms of a hypergeometric function. For free fields, as a function of the mass, these are,

$$G_{RR}(\xi;\Delta) = W(\xi + i\epsilon;\Delta), \quad G_{LL}(\xi;\Delta) = W(\xi - i\epsilon;\Delta), \tag{4.26a}$$
$$G_{LR}(\xi;\Delta) = G_{RL}(\xi;\Delta) = W(\xi - i\epsilon\,\mathrm{sgn}(\eta_L - \eta_R);\Delta). \tag{4.26b}$$

W is proportional to a hypergeometric function ${}_2F_1$ [41],

$$W(\xi;\Delta) = \frac{\Gamma(\Delta)\Gamma(d-\Delta)}{(4\pi)^{\frac{d+1}{2}}\Gamma(\frac{d+1}{2})}\,{}_2F_1\left(\Delta, d-\Delta, \frac{d+1}{2}, \frac{1+\xi(x,x')}{2}\right). \tag{4.27}$$

The limit as $\xi \to 0$ will be useful. In this limit, W takes the form,

$$\lim_{\zeta\to 0} W(\xi;\Delta) \approx \frac{\zeta^{\frac{1-d}{2}}}{2\pi^{\frac{1+d}{2}}(d-1)}. \tag{4.28}$$

This limit is precisely when the two conformal spacetime points $(\eta,\mathbf{x})$ and $(\eta',\mathbf{x}')$ approach the asymptotic boundary of dS at the same rate.

4.5.1 Two-point function

Let's consider the bulk two-point function in conformal coordinates, dependent on two scalar fields at $\mathbf{x}$ and $\mathbf{x}'$ that aren't causally connected,

$$\mathcal{G}(x,\eta;x') = \langle\Omega|\phi(\eta,\mathbf{x})\phi(\eta,\mathbf{x}')|\Omega\rangle. \tag{4.29}$$

This correlator can be defined either using the in-in or wavefunction contour. We can define the identity operator by summing over all possible unitary irreducible representations of the dS isometry. This involves taking an integral over the principal and complementary series given in (2.12),

$$\mathbb{1} = \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta'}{2\pi i}\mathcal{P}_{\Delta'} + (\text{complementary}). \tag{4.30}$$

Plugging in this resolution of identity between the two fields in (4.29),

$$\langle \phi(\eta,\mathbf{x})\phi(\eta,\mathbf{x}')\rangle = \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta'}{2\pi i} \langle \phi(\eta,\mathbf{x})\mathcal{P}_{\Delta'}\phi(\eta,\mathbf{x}')\rangle + (\text{complementary}). \tag{4.31}$$

To impose the symmetry constraints on the propagator we can act with the quadratic Casimir given in (2.9). Using the invariance of the Bunch-Davies vacuum under generators of the isometry group we arrive at

$$\langle (\mathcal{C}_2 \cdot \phi(\eta,\mathbf{x}))\mathcal{P}_{\Delta'}\phi(\eta,\mathbf{x}')\rangle = \Delta'(\Delta'-d)\langle \phi(\eta,\mathbf{x})\mathcal{P}_{\Delta'}\phi(\eta,\mathbf{x}')\rangle. \tag{4.32}$$

On the other hand, by acting with the differential operators from (2.10) twice, we obtain an alternative expression for the action of the Casimir, which is equivalent to the dS Laplacian operator,

$$\langle (\mathcal{C}_2 \cdot \phi(\eta,\mathbf{x}))\mathcal{P}_{\Delta'}\phi(\eta,\mathbf{x}')\rangle = -\Box_{\mathbf{x}}\langle \phi(\eta,\mathbf{x})\mathcal{P}_{\Delta'}\phi(\eta,\mathbf{x}')\rangle. \tag{4.33}$$

Both of these expressions are equivalent. Putting them together gives

$$(\Box_{\mathbf{x}} + \Delta'(\Delta'-d))\, g(\xi;\Delta') = 0, \qquad g(\xi;\Delta') = \langle \phi(\eta,\mathbf{x})\mathcal{P}_{\Delta'}\phi(\eta,\mathbf{x}')\rangle, \tag{4.34}$$

where ξ is given in (4.25). This is precisely the differential equation for the Greens function in dS, with $\Delta'(\Delta'-d) = m^2/H^2$ from (2.11). This has a non-singular solution given by

$$g_{\Delta'}(x,\eta;x') = \rho(\Delta')G(\xi;\Delta'), \tag{4.35}$$

where $\rho(\Delta')$ is a constant of proportionality. The Greens function $G(\xi;\Delta')$ is exactly (4.27),

$$G(\xi;\Delta') = \frac{\Gamma(\Delta')\Gamma(d-\Delta')}{(4\pi)^{\frac{d+1}{2}}\Gamma(\frac{d+1}{2})}\, {}_2F_1\left(\Delta', d-\Delta', \frac{d+1}{2}, \frac{1+\xi(x,x')}{2}\right). \tag{4.36}$$

Putting everything together, we can write the decomposed two-point function as,

$$\langle \phi(\eta,\mathbf{x})\phi(\eta,\mathbf{x}')\rangle = \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta'}{2\pi i} \rho(\Delta')G(\xi;\Delta') + (\text{complementary}). \tag{4.37}$$

Positivity of the spectral density Now that we have a decomposed two-point function, we have to examine the properties of the individual components.

The resolution of the identity in (4.31) is essentially glueing the two causally disconnected subsets of dS using the projector operator, which projects us onto the eigenspace of the Casimir operator. This allows us to analytically continue the two points of our states onto a sphere, with ζ balancing the symmetrical separation of these points from the equator. A pictorial explanation is shown in Fig. 4.

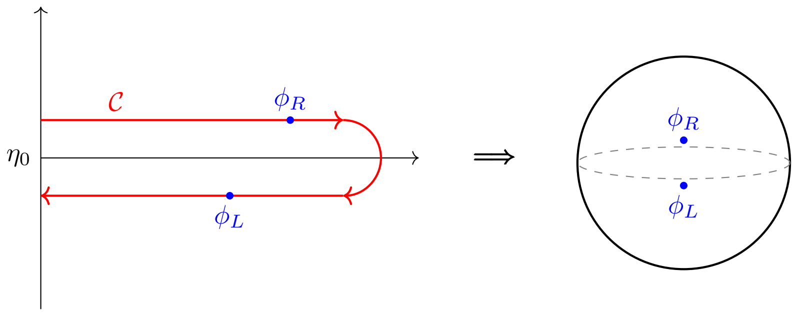


Figure 4: A diagrammatic representation of the insertion of the resolution identity. **Left:** An example of the in-in path integral contour C, in dS, for the G_{RL} bulk propagator. The insertion of the identity (4.30) corresponds to splitting the contour at η_0. **Right:** We analytically continue the points onto a sphere, and we place them a symmetrical distance, ζ, away from the equator.

Consider the projector onto the states ψ that form the principal series representation of Δ':

$$\mathcal{P}_{\Delta'} = \sum_{\psi \in \Delta'_{\rm P}} |\psi\rangle\langle\psi| \implies g(\xi;\Delta') = \sum_{\psi \in \Delta'_{\rm P}} |\langle\Omega|\phi(\mathbf{x})|\psi\rangle|^2 \geq 0, \tag{4.38}$$

therefore $g(\xi;\Delta')$ is non-negative. Inverting (4.35) to define $\rho(\Delta')$, and inserting $g(\xi;\Delta')$ from (4.38) and $G(\xi;\Delta')$ from (4.36). We then take the limit as the fields approach the equator symmetrically, $\zeta \to 0$, which from (4.28) we know is strictly positive

$$\rho(\Delta') = \lim_{\zeta\to 0}\left[\left(\frac{\zeta^{\frac{1-d}{2}}}{2\pi^{\frac{1+d}{2}}(d-1)}\right)^{-1} \sum_{\psi \in \Delta'_{\rm P}} |\langle\Omega|\phi(\mathbf{x})|\psi\rangle|^2\right] \geq 0. \tag{4.39}$$

By continuity, $\rho(\Delta')$ will still remain positive outside this limit. This result is very similar to the Källén–Lehmann decomposition of the two-point function in Minkowski, which can be written as an integral over the respective free two-point functions with a positive spectral density [42, 43].

4.5.2 Four-point function

We now consider the four-point function evaluated on the dS boundary $\eta \to 0$. But now, we'll need to apply insights from the Euclidean anti-de Sitter space (EAdS), which has the same singularity structure of late time correlators as dS with a Bunch-Davies vacuum [44, 45]. Consider

$$\mathcal{G}(x_1,x_2,x_3,x_4) = \langle\Omega|\mathcal{O}_1(\mathbf{x}_1)\mathcal{O}_2(\mathbf{x}_2)\mathcal{O}_3(\mathbf{x}_3)\mathcal{O}_4(\mathbf{x}_4)|\Omega\rangle, \tag{4.40}$$

and we assume the operators are pair-wise complex conjugate, $\mathcal{O}_1 = \mathcal{O}_3^\dagger$ and $\mathcal{O}_2 = \mathcal{O}_4^\dagger$ (with conformal dimension Δ_j)[18]. As before, we insert a resolution of the identity, which now involves a sum over the angular momenta of the states,

[18] This is assumed for simplicity, but this argument can be dropped and readily generalised to four different operators, see e.g. [46].

$$\mathbb{1} = \sum_{J=0}^{\infty} \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta'}{2\pi i} \mathcal{P}_{\Delta',J} + (\text{complementary} + \text{discrete}). \tag{4.41}$$

In EAdS, late-time correlators are single-valued functions, of conformally invariant cross-ratios. This implies that the four-point function can be expanded in terms of a complete, orthogonal basis of single-valued functions, called conformal partial waves $\hat{\mathcal{F}}_{\Delta,J}$ [47, 48][19]. These are eigenfunctions of the *Casimir differential equation* with eigenvalues given by (2.9),

$$\mathcal{D}_{12} \cdot \hat{\mathcal{F}}^{\{\Delta'_j\}}_{\Delta',J}(x_1, x_2, x_3, x_4;) = [\Delta'(\Delta' - d) + J(J + d - 2)] \cdot \hat{\mathcal{F}}^{\{\Delta'_j\}}_{\Delta',J}(x_1, x_2, x_3, x_4). \tag{4.42}$$

As before, we act with the quadratic Casimir on (4.40), and compare it with the Casimir acting as a differential operator. Combining both results, one obtains the differential equation satisfied by the conformal partial wave. Picking only the solutions that are single-valued in position at $\eta \to 0$ and have the correct behavior at $\eta \to -\infty$ when $x_1 \to x_2$ and $x_3 \to x_4$ [47], gives the conformal partial wave, which $g_{\Delta',J}(x_1, x_2, x_3, x_4)$ must be proportional to:

$$g_{\Delta',J}(x_1, x_2, x_3, x_4) = \rho_J(\Delta')\hat{\mathcal{F}}^{\{\Delta'_j\}}_{\Delta',J}(x_1, x_2, x_3, x_4), \tag{4.43}$$

where $\rho_J(\Delta')$ is the spectral density. This allows us to write the conformally invariant four-point function for the principal series and a specific channel, e.g. the (12)(34) channel,

$$\langle \mathcal{O}_1(\mathbf{x}_1)\mathcal{O}_2(\mathbf{x}_2)\mathcal{O}_3(\mathbf{x}_3)\mathcal{O}_4(\mathbf{x}_4)\rangle = \mathbb{1}_{12}\mathbb{1}_{34} + \sum_{J=0}^{\infty} \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta'}{2\pi i} d\Delta' \rho(\Delta')\hat{\mathcal{F}}^{\{\Delta'_j\}}_{\Delta',J}(x_1, x_2, x_3, x_4). \tag{4.44}$$

Positivity of the spectral density Explicit examples in [36, 50] show that $\rho(\Delta)$ is *generally* a meromorphic function of Δ in EAdS. This implies that generically it will also have isolated poles as its singularities in a static patch in dS, and hence is also a meromorphic function of Δ in dS. It is additionally shown that if (4.44) continues to be single-valued non-perturbatively, then it will always hold. Hence, with respect to the conformal wave expansion in (4.44), unitarity in the conformal group $SO(d + 1, 1)$, implies positivity of the spectral density, $\rho(\Delta) \geq 0$.

In contrast to the flat space implications of positivity, the consequences in dS are still not well known. Nonetheless, recent advancements in [36, 46, 50] propose a dS analogy of the numerical conformal bootstrap, which uses the positivity of the spectral density plus crossing symmetry to put bounds on the space of CFTs.

4.6 The Manifestly Local Test

The Manifestly Local Test (MLT) is another manifestation of unitarity in dS. This was first presented in [7] and it will provide a powerful bootstrap rule.

Massless MLT A *manifestly local* theory is one with a $\mathcal{L}_{int}$ that is absent of an inverse spatial Laplacian[20]. If the theory is manifestly local, no singularities in the external energies can occur.

[19] These expansions resemble harmonic analysis on the conformal group $SO(d + 1, 1)$ [49].

[20] As an example, some gravitational interactions during inflation violate manifest locality, as even after integrating out the non-dynamical lapse and the shift, interactions with inverse Laplacian appear [2].

However, when the sum of internal energies goes to zero, the power spectrum does indeed become singular. This comes from its explicit form, in terms of massless or graviton mode functions evaluated at the boundary; $\sim 1/p_s^3$. On the other hand, the bulk-to-bulk propagator (3.15), which contains a factor of the power spectrum, is finite in this limit as its divergences are suppressed by the product of bulk-to-boundary propagators (2.37). This also happens in the asymptotic past, as all the exponents contain a sum of energies that are finite in the limit $e^{ik\eta} \to 0$[21]. From a propagator's perspective

$$\frac{\partial}{\partial p_s}\left(\frac{d^N}{d\eta^N}\mathcal{K}(p_s,\eta)\right)\bigg|_{p_s=0} = 0, \qquad N \in \mathbb{Z}^+, \qquad \Delta = d = 3. \tag{4.45}$$

Consider instead the consequences of this limit for an $n-$point exchange correlator, from our single cut rules in (4.20). The LHS doesn't diverge in a manifestly local theory. However, the RHS contains the power spectrum, which as we described above, does. This means that the rest of the terms on the RHS must suppress this singularity and make it regular. To investigate this further, we can write out explicitly the RHS using the definition of the Disc from (4.14),

$$\begin{aligned} -iP(p_s)&[\psi'_n(k_1,...,k_{n-1},p_s;\{p\};\{\mathbf{k}\}) + \psi'^*_n(-k_1^*,...,-k_{n-1}^*,p_s;-\{p\};-\{\mathbf{k}\})] \\ &\times[\psi'_m(k_n,...,k_{n+m-2},p_s;\{p\};\{\mathbf{k}\}) + \psi'^*_m(-k_n^*,...,-k_{n+m-2}^*,p_s;-\{p\};-\{\mathbf{k}\})]. \end{aligned} \tag{4.46}$$

For IR finite wavefunctions, we can use the COT in (4.11) to re-express the above as,

$$\begin{aligned} -iP(p_s)&[\psi'_n(k_1,...,k_{n-1},p_s;\{p\};\{\mathbf{k}\}) - \psi'_n(k_1,...,k_{n-1},-p_s;\{p\};\{\mathbf{k}\})] \\ &\times[\psi'_m(k_n,...,k_{n+m-2},p_s;\{p\};\{\mathbf{k}\}) - \psi'_m(k_n,...,k_{n+m-2},-p_s;\{p\};\{\mathbf{k}\})]. \end{aligned} \tag{4.47}$$

This allows us to Taylor expand around $p_s = 0$, which is an odd function, to obtain

$$\begin{aligned} \sim \frac{2i}{p_s^3}&\left[\frac{\partial}{\partial p_s}\psi'_n(k_1,...,k_{n-1},p_s;\{p\};\{\mathbf{k}\})\bigg|_{p_s=0} p_s + \mathcal{O}(p_s^3)\right] \\ \times&\left[\frac{\partial}{\partial p_s}\psi'_m(k_1,...,k_{n+m-2},p_s;\{p\};\{\mathbf{k}\})\bigg|_{p_s=0} p_s + \mathcal{O}(p_s^3)\right], \end{aligned} \tag{4.48}$$

where we have inserted the dependence of the power spectra on p_s. As there are only integer powers of external momenta in exchange diagrams, the pole at p_s^{-3} can only be cancelled by,

$$\frac{\partial}{\partial p_s}\psi'_n(k_1,...,k_{n-1},p_s;\{p\};\{\mathbf{k}\})\bigg|_{p_s=0} = 0. \tag{4.49}$$

To emphasise that the cut internal energy becomes external, we denote it as k_c, to obtain,

$$\frac{\partial}{\partial k_c}\psi'_n(k_1,...,k_n;\{p\};\{\mathbf{k}\})\bigg|_{k_c=0} = 0, \qquad c \in [1,n], \qquad \Delta = d = 3. \tag{4.50}$$

This is the generalised MLT, which tells us that, as a consequence of unitarity, manifestly local theories with massless fields (and/or gravitons), with $\Delta = d$, must satisfy (4.50).

[21]This is non-physical, as we are assuming all other internal energies are kept fixed.

5 The Analytic Wavefunction

Singularities are non-physical points that don't belong to our spacetime, however, the study of singularities of physical systems is extremely useful and can be used to infer general properties of the system. Most observables are smoothly varying quantities that can be analytically extended to the complex plane. Considering singularities of flat spacetime amplitudes, the analyticity of the S-matrix gives rise to a map that connects singularities to the physical properties of the system when intermediate particles go on-shell e.g. from the spin and mass of the particles to their pole residues and branch cut discontinuities[22]. For theories involving massless particles, this information alone is enough to entirely reconstruct the amplitude, even ruling out possible interactions [52, 53].

Inspired by this, we can extend this insight into the study of correlators and consequently wavefunctions, which will be useful in our bootstrap program. This will provide a relation between their singularities (e.g. poles and branch cuts) and the properties of their fields. In the general case, this information is enough to uniquely reconstruct the entire correlator [54, 55].

In this section, we want to recover a relationship between amplitudes and correlators in the ubiquitous *total energy pole* as $k_T \equiv \sum_a^n k_a \to 0$[23]. We will do this by working directly with the formulation of Section 2.3, specifically the same interacting action (2.29).

5.1 Flat space amplitudes

Propagators: To focus on the essential features of the system we will only consider one type of field in our Lagrangian. This removes the constraints imposed by other fields and allows us to make certain simplifications. We can set (i) the speed of sound $c_s = 1$, (ii) the mass of the fields $m_a = 0$, which makes the fields scale-invariant, (iii) the scale factor $a = 1$[24]. We can write (2.29) as

$$S_{int}^{(n)} = \lambda_n \int dt \prod_{a,b=1}^{n} \int_{\mathbf{k}_b} \mathcal{F}_n \left[\mathbf{k}_a \cdot \mathbf{k}_b\right] (2\pi)^3 \delta^{(3)} \left(\sum_{i=1}^{n} \mathbf{k}_i\right) \partial_t^{t_a} \phi_a(t, \mathbf{k}_a). \tag{5.1}$$

Decomposing our fields as in (2.31), and assuming λ_n is small (the interaction is weak), we can expand in powers of H_{int} and solve for the free field operators in the asymptotic limits,

$$f(t,k) = \frac{1}{\sqrt{2k}} e^{ikt}. \tag{5.2}$$

Taking any number of time derivatives of the mode functions can be simply written as,

$$\partial_t^{t_a} f(t,k) = u(k)^{t_a} f(t,k), \qquad u(k) = ik. \tag{5.3}$$

In the limit the scalar fields are free, the bulk-to-bulk propagator reduces to the familiar Feynman propagator in Minkowski. The bulk-to-bulk propagators are defined using (3.1) as

[22] For example, the mass of the Higgs boson is determined by finding the location of an energy pole in the complex plane, using the Breit-Wigner formula [51].

[23] In this limit, the early time contribution of the integrals are not suppressed and leads to divergences.

[24] This reduces the de Sitter metric to the flat Minkowski metric. To emphasise this point we will work in Minkowski coordinates with the trivial relation $dt = d\eta$.

$$
\begin{aligned}
G_{\text{Mink.}}(k,t;t') \equiv \Delta_F(k,t;t') &= \langle 0|T\{\partial_t^{t_a}\phi(t,\mathbf{k})\partial_{t'}^{t'_a}\phi(t',-\mathbf{k})\}|0\rangle' \\
&= \theta(t-t')u(k)^{t_a}[u(k)^{t'_a}]^*\frac{e^{ik(t-t')}}{2k} + \theta(t'-t)[u(k)^{t_a}]^*u(k)^{t'_a}\frac{e^{-ik(t-t')}}{2k},
\end{aligned} \tag{5.4}
$$

and using (3.3) the bulk-to-boundary propagator is[25],

$$
K_{\text{Mink.}}(k,t) \equiv \lim_{t_0\to\infty} G_<(k,t;t_0) = \sqrt{2k}\,\langle 0|b_{\mathbf{k}}\partial_t^{t_a}\phi(t,-\mathbf{k})|0\rangle' = [u(k)^{t_a}]^*e^{-ikt}. \tag{5.5}
$$

The Feynman rules to construct amplitudes in flat spacetime are the same as those of the in-in formalism, without the notion of time ordering (no R or L vertices), therefore we only have two propagators, for external and internal lines. This greatly simplifies our rules from section 3.1,

- **Vertices:** Contribute a factor of $-i$, multiplied by the coupling parameter of our theory and associated temporal and spatial derivatives i.e. $-i\lambda\mathcal{F}(\mathbf{k})$.
- **Internal lines:** Contribute $G_{\text{Mink.}}(k,t;t')$.
- **External lines:** Contribute $K_{\text{Mink.}}(k,t)$.
- **Integration:** Integrate over all time $-\infty$ to ∞ and undetermined momenta, at each vertex. This ensures momenta is conserved at each vertex and over the whole diagram.
- **Result:** The final result is the amplitude $\mathcal{A}_n$.

Constructing amplitudes from propagators: For our action (5.1), the $n-$particle amplitude observed at the boundary is given by

$$
\mathcal{A}_n = -i\lim_{t,-t_0\to\infty}\left[\prod_a^n\sqrt{2k_a}\langle 0|b_{\mathbf{k}_a}U(t,t_0)|0\rangle\right], \tag{5.6}
$$

where $U(t,t_0)$ is the unitary operator given in (2.17). Using the Feynman rules derived in the previous subsection, we can express a generic amplitude for a diagram as,

$$
\mathcal{A}'_n = i^{-V-1}\prod_\alpha^L\prod_\beta^V\lambda_\beta\int_{\mathbf{k}_\alpha,\,t_\beta}\mathcal{F}_\beta(\mathbf{k})\prod_\gamma^I G_{\text{Mink.}}(K_\gamma,t;t')\prod_a^n K_{\text{Mink.}}(k_a,t), \tag{5.7}
$$

where a $\mathbf{K}$ denotes the momentum of an internal line $\mathbf{K}_I = -\sum_a^m\mathbf{k}_a$, and the sum is over all the lines coming into the diagram. The prime on the amplitude means we have factored out $(2\pi)^{(4)}\delta^{(4)}(\sum_a^n k_a)$. At tree level, we have no internal loop moment to integrate over i.e. $L=0$. We can perform the time integrals using the explicit expressions for the flat space propagators (5.4) and (5.5), to arrive at,

$$
\mathcal{A}'_n = -\prod_\alpha^V\lambda_\alpha\mathcal{F}_\alpha(\mathbf{k})\prod_\beta^I\tilde{G}_{\text{Mink.}}(K_\beta)\prod_a^n[u(k_a)^{t_a}]^*, \tag{5.8}
$$

[25] This is defined by our Feynman rules for external lines propagating to the boundary at $t_0 > t$. Taking the limit as $t_0\to\infty$ sets the exponential to 1 in the mode functions. Notice also that in this limit, our vacuum becomes the free theory vacuum (2.15).

where we have defined,

$$\tilde{G}_{\text{Mink.}}(K_I) = \frac{[u(K_I)^{t'_a}]^* u(K_I)^{t_a}}{2K_I(K_I - \sum_a^m k_a)} + \frac{u(K_I)^{t'_a}[u(K_I)^{t_a}]^*}{2K_I(K_I + \sum_a^m k_a)}. \tag{5.9}$$

5.2 Curved space correlators

We wish to achieve an equivalent result as above but for correlators in dS, so we reintroduce the scale factor a and return to conformal time. We can compute a general $n-$point correlator using the factorised form of the in-in formalism (2.16),

$$B_n = \langle \prod_a^n \phi(\eta_0, \mathbf{k}_a) \rangle' = \lim_{\eta \to -\infty} \langle 0 | U^\dagger(\eta, \eta_0) \prod_a^n \phi(\eta_0, \mathbf{k}_a) U(\eta, \eta_0) | 0 \rangle', \tag{5.10}$$

where we have left the Wick rotation implicit. As usual, we will also be interested in taking the limit as $\eta_0 \to 0$. Massless mode functions in dS are given in (2.37), with time derivatives

$$f(\eta, k) = \frac{H}{\sqrt{2k^3}}(1 - ik\eta)e^{ik\eta} \qquad \Longrightarrow \qquad \partial_\eta^{t_a} f(\eta, k) = \frac{u(k)^{t_a} H}{\sqrt{2k^3}}(1 - t_a - ik\eta)e^{ik\eta}. \tag{5.11}$$

We can proceed as before, but we have now recovered the notion of time ordering, so we need the full set of propagators in (3.3):

$$\begin{aligned} G_{RR}(k, \eta; \eta') &= G_{RL}(k, \eta; \eta')\, \theta(\eta - \eta') + G_{LR}(k, \eta; \eta')\, \theta(\eta' - \eta), \\ G_{RL}(k, \eta; \eta') &= \frac{H^2 [u(k)^{t_a}]^* u(k)^{t'_a}}{2k^3} (-H\eta)^{t_a} (-H\eta')^{t'_a} (1 - t_a - ik\eta)(1 - t'_a - ik\eta') e^{-ik(\eta - \eta')}. \end{aligned} \tag{5.12}$$

Note we can recover the final two by complex conjugation, $G_{LR} = G^*_{RL}$ and $G_{LL} = G^*_{RR}$. The corresponding bulk-to-boundary propagator is found by using the mode functions at the boundary,

$$K_R(k, \eta) = \frac{H^2 [u(k)^{t_a}]^*}{2k^3} (-H\eta)^{t_a} (1 - t_a - ik\eta)(1 - ik\eta_0) e^{-ik(\eta - \eta_0)}. \tag{5.13}$$

To factorise the time and momentum dependence of F_α, we can write it as

$$\mathcal{F}_\alpha \left[\frac{\mathbf{k}_a \cdot \mathbf{k}_b}{a(\eta)^2} \right] = a(\eta)^{-s_\alpha}\, \mathcal{F}_\alpha \left[\mathbf{k}_a \cdot \mathbf{k}_b \right] \equiv a(\eta)^{-s_\alpha}\, F(\mathbf{k}), \tag{5.14}$$

where s_α is the total number of spatial derivatives at each vertex α.

General Diagrams Following our Feynman rules from Section 3.1, a general diagram propagating from a time-ordered vertex ($\alpha = R$) without loops, is given by

$$B_n^R = i^{V_L - V_R} \prod_\alpha^{V_L + V_R} \int_{\eta_\alpha} \frac{\lambda_\alpha F_\alpha(\mathbf{k})}{a(\eta)^{s_\alpha - 4}} \prod_\beta^I G_{RR}(K_\beta, \eta; \eta') \prod_a^n K_R(k_a, \eta), \tag{5.15}$$

where we have introduced the valence of each vertex n_α, which is the total number of lines connected to the vertex α. The vertex dimension D_α, is the sum of the valence and derivatives at the vertex. With this, one can define the order of the total energy pole $p > 0$ as,

$$D_\alpha \equiv N_\alpha + n_\alpha \implies p \equiv 1 + \sum_\alpha^V (D_\alpha - 4). \tag{5.16}$$

Note that a general Bogoliubov transformation of the Bunch-Davies vacuum in (2.15), will modify the value of p, creating additional poles in the correlator [56]. With these definitions, we can evaluate the time integrals (see Appendix C), and find that the total energy pole of (5.15) as $\eta_0 \to 0$,

$$\lim_{k_T, \eta_0 \to 0} B_n = 2^{1-n}(-1)^n H^{p+n-1}\,(p-1)!\ \mathrm{Re}\left\{ \frac{i^{1+n+p}\mathcal{A}'_n}{(\prod_a^n k_a)^2 (k_T)^p} \right\}. \tag{5.17}$$

Understanding the behaviour of general diagrams in this limit will be very useful in attempting to bootstrap the form of correlators, specifically the bispectrum $n = 3$, as we do in the next section. Explicitly, we will see that for $\lambda\phi^3$ theory, for the pole at $p = 0$, we recover the IR-logarithmic divergence of the correlator (Rule 3 - Section 6.2.3). The existence of a singularity in cosmological correlators or the wavefunction, whose residue corresponds to a scattering amplitude, establishes a fascinating link between the investigations of cosmological correlators and the S-matrix. While the precise nature and order of the singularity rely on the specific fields and interactions involved, it is certain that some kind of singularity will be present [18].

6 A Boostless Bootstrap

We have looked at possible ways of deriving correlators, namely the in-in formalism and the wave-function method. The standard in-in formalism fails to take advantage of any of the isometries of dS space, as well as any of the commonalities these results share, to make it computationally less demanding. In this section, we consider an alternative way to calculate correlators in quasi-de Sitter spacetime. We can bypass the bulk calculation, by fixing some properties that we expect from correlators, using symmetries, locality and unitarity, with a Bunch Davies vacuum. These will be so constraining that it will completely fix our result. This is analogous to how amplitudes and CFT correlators are restricted by Poincaré and conformal symmetries [23, 24].

Tensor modes are the main targets of current and future cosmological surveys [57]. In this section, we focus only on bootstrapping the bispectra of massless scalars and spin-two fields, that become constant in the asymptotic future. We won't rely on the full set of de Sitter isometries, specifically not on dS boosts (see Section 2.1.1), hence the name, *Boostless Bootstrap*.

6.1 Tensor modes

Gravitons Gravitons are spin 2 particles which are usually denoted as $\gamma_{ij}(\eta, \mathbf{k})$. We can decompose the graviton into different polarisations h, with the polarisation tensor ϵ^h_{ij}[26]. For observational advantage, it's useful to study the plus and cross polarisation $h = \{+, \times\}$,

[26] At fine momenta $\mathbf{k} \neq 0$, we can work with fields $|\psi; h\rangle$ that are eigenvectors of the rotation operator $R(\theta)$ around $\mathbf{k}$, with eigenvalue $e^{ih\theta}$, i.e. $R(\theta)|\psi; h\rangle = e^{ih\theta}|\psi; h\rangle$, where for bosons, h is known as the helicity.

$$\gamma_{ij}(\eta, \mathbf{k}) = \sum_{h=+,\times} \epsilon^h_{ij}(\mathbf{k})\gamma^h(\eta, k). \tag{6.1}$$

Where ϵ^h_{ij} is transverse, traceless, real, symmetric and correctly normalised, i.e.

$$\epsilon^h_{ii}(\mathbf{k}) = k^i \epsilon^h_{ij}(\mathbf{k}) = 0, \qquad \epsilon^h_{ij}(\mathbf{k})\epsilon^{h'}_{ij}(\mathbf{k})^* = \epsilon^h_{ij}(\mathbf{k})\epsilon^{h'}_{ij}(-\mathbf{k}) = 2\delta^{hh'}. \tag{6.2}$$

Higher order tensor modes Following the decomposition of a graviton in terms of its polarisation modes, we can decompose a generic cosmological tensor, denoted by $X_{i_1,\dots i_h}$, as,

$$X_{i_1,\dots i_h}(\eta, \mathbf{k}) = \epsilon^h_{i_1,\dots i_h}(\mathbf{k})X^h(\eta, k) + \epsilon^{-h}_{i_1,\dots i_h}(\mathbf{k})X^{-h}(\eta, k). \tag{6.3}$$

These tensors have corresponding equal-time correlation functions given by,

$$\lim_{\eta_0 \to 0} \langle X^{h_1}(\mathbf{k}_1, \eta)Y^{h_2}(\mathbf{k}_2, \eta)Z^{h_3}(\mathbf{k}_3, \eta)\rangle' = B_{XYZ}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3; h_1, h_2, h_3), \tag{6.4}$$

where B denotes the bispectra, of the tensor modes X, Y, Z.

6.2 Bootstrap rules

We begin by covering the rules first proposed in [6], with additions later from [7] that don't invoke local amplitudes and soft theorems. We will use unitarity constraints to bootstrap the form of correlators, as well as the unquestionable discontinuity of all correlators in the total energy pole. The singularity that forms in this limit is non-physical, as it can't be probed through experiments when k is not real. However, we can access it by analytically continuing our energy into the complex plane, which *bootstraps* the correlator even in the physical domain, controlling observable kinematics. The exact way in which its bootstrapped in the physical energies, is given by the bootstrap rules.

6.2.1 Rule 1: dS symmetries

Statistical homogeneity and isotropy: From the definition of the bispectra, it depends on the three momentum vectors $\{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3\}$. However, because of momentum conservation, these three vectors must be related by $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$. This is a direct consequence of translational invariance (homogeneity). Furthermore, because of statistical isotropy, it should also be invariant under rotations and hence should only depend on the angle between the vectors. Because of this, the bispectra should only depend on the magnitude of the momenta[27]

$$B_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = B_3(k_1, k_2, k_3). \tag{6.5}$$

The above means that all momenta and polarisation tensors should not have any free indices. Ensuring that the polarisation tensor appears linearly, we can write[28]

[27] For a general $n-$point correlator, with $n \geq 3$, it should depend on $3n-6$ independent variables. For $n = 3$ we can pick these momenta scalings to be the three variables associated with the magnitude of the momentum, however, for $n > 3$, these variables depend on more than just the magnitudes of momentum but also the angles $k_a \cdot k_b$, $a, b \in [1, n]$, as $n < 3n - 6$.

[28] This equation also holds for scalar fields, by setting $\alpha_i = h_i = 0$. In this case $B = \tilde{B}$.

$$B_3(k_1,k_2,k_3) = \sum_{\text{contractions}} \left[\epsilon^{h_1}(\mathbf{k}_1)\epsilon^{h_2}(\mathbf{k}_2)\epsilon^{h_3}(\mathbf{k}_3)\mathbf{k}_1^{\alpha_1}\mathbf{k}_2^{\alpha_2}\mathbf{k}_3^{\alpha_3}\right] \tilde{B}_3(k_1,k_2,k_3), \tag{6.6}$$

where the sum is over all relevant contractions of the spin $\alpha_{1,2,3}$, and $\tilde{B}$ is the *trimmed bispectrum*, which should also be invariant under rotations and translations.

Parity invariance: As a direct consequence of (6.5) the B_3 is parity invariant i.e. $\mathbf{k} \to -\mathbf{k}$.

Scale invariance: This implies B must be a homogeneous function of degree -6, i.e.

$$B_3(\lambda k_1, \lambda k_2, \lambda k_3) = \frac{B_3(k_1,k_2,k_3)}{\lambda^6} \implies \tilde{B}_3(\lambda k_1, \lambda k_2, \lambda k_3) = \frac{\tilde{B}_3(k_1,k_2,k_3)}{\lambda^{6+\alpha_T}}, \tag{6.7}$$

where $\alpha_T = \sum_{i=1}^3 \alpha_i$. There exist slow-roll corrections, however, these are very difficult to detect in observations, and hence won't be considered here (see e.g. [58, 59, 60]).

6.2.2 Rule 2: Locality of bispectra

The next rule is a generalisation of locality in curved space. It states that the trimmed bispectrum must be a *rational function* of rotationally invariant contractions between the momenta and polarisation tensors. This means that after applying the symmetries of de Sitter space, the bispectrum should be expressible as a ratio of two polynomials of momenta. The reason for this is that all singular points are poles, and their behaviour away from their singularities is strongly constrained by unitarity and locality [55]. Therefore, under the assumption that we have a local effective field theory (involving only the product of fields and positive powers of the derivatives at the same spacetime point), that is weakly coupled and well approximated by massless dS mode functions in (2.37), the correlator can be expressed as rational functions of their energy variables

$$\tilde{B}_3 = \frac{\text{Poly}_\beta(k_1,k_2,k_3)}{\text{Poly}_{6+\alpha_T+\beta}(k_1,k_2,k_3)}, \tag{6.8}$$

where the subscript of the polynomial denotes the degree of the polynomials in the norm of momenta. Recall that when $c_s \neq 1$ these mode functions break de Sitter boosts[29]. This rule is generally violated when c_s is not constant, as well as when there is a logarithm arising from the sum of k [62].

6.2.3 Rule 3: Total energy pole of Amplitude

The bispectra is fixed by its UV limit when k_T vanishes. This is (5.17) for $n = 3$,

$$\lim_{k_T \to 0} B_3 = -\frac{H^{p+2}\,(p-1)!}{4} \,\text{Re}\left\{\frac{i^p \mathcal{A}_3'}{(k_1k_2k_3)^2 (k_T)^p}\right\}, \tag{6.9}$$

with a singularity of order p given by (5.16).

[29]In fact, it's proved in [61] that the only single-clock theory of curvature perturbations (i.e. satisfies all the soft theorems) that displays full de Sitter invariance in the slow-roll, decoupling limit, is a free theory.

The limiting case Equation (5.17) holds only for positive p, this is precisely when $N+n>3$. A limiting interaction that has $N+n\leq 3$ is ϕ^3. This is the case when $p=0$. Because we've evaluated the correlator at $\eta_0\to 0$, this gives rise to an IR - logarithmic divergence [6, 11][30],

$$\frac{(p-1)!}{(-ik_T)^p}\to \log(-\eta_0 k_T), \tag{6.10}$$

that replaces the Zeroth order pole. The bispectra at $p=0$ becomes,

$$\lim_{k_T\to 0} B_3 = -\frac{H^2\log(-\eta_0 k_T)}{(k_1k_2k_3)^2}\,\mathrm{Re}\mathcal{A}_3'. \tag{6.11}$$

An alternative derivation for p Let's consider an alternative way to derive the order of the singularity p from (6.7), using dimensional analysis and scale invariance. Under a theory with a Lagrangian containing interaction terms of dimensions D_α, we can introduce local operators compatible with the symmetries of our theory[31], which we collectively denote as $\mathcal{O}_\alpha$. A dimensionless action in $(3+1)$ dimensions has

$$[\mathcal{O}_\alpha]=D_\alpha \implies [\lambda_\alpha]=4-D_\alpha. \tag{6.12}$$

The Lagrangian takes the dimensional form,

$$\mathcal{L}_{int}\supset\sum_\alpha \lambda_\alpha^{4-D_\alpha}\,\mathcal{O}_\alpha. \tag{6.13}$$

The resulting amplitude generated from V interaction vertices in the Feynman diagram has dimensions,

$$\mathcal{A}_n\sim k^{4-n+\sum_\alpha^V(D_\alpha-4)}\prod_\beta^V\lambda_\beta^{-D_\beta}. \tag{6.14}$$

Additionally, scale invariance from (6.7) fixes correlators to scale with k as

$$B_n\sim\frac{1}{k^{3(n-1)}}. \tag{6.15}$$

On the other hand, (5.17) fixes the dimensions of B_n in terms of $\mathcal{A}_n$,

[30]In the limit as $\eta_0\to 0$ the early-time integration is not suppressed and forms a singularity. On the other hand, in flat space, when evaluating amplitudes, the integration limits run from $t\in(-\infty,\infty)$. This time integral gives rise to a delta function that ensures energy conservation, $\delta(E)$. This is an alternative way to show how energy is not conserved in a cosmological context, where $\eta\in(-\infty,0)$, which gives rise to an IR divergence, $1/E^p$ (where $E=k_T=\sum_a^n k_a$) coming from the exponentials with positive mode frequencies.

[31]The operators are built from products of fields and their derivatives evaluated at the same spacetime point. Each field contributes to a mass dimension of 1 to D_α. It is worth noting that some non-locality does emerge when distances are similar to the energy scale $1/\lambda$.

$$[B_n] = \frac{[\mathcal{A}_n]}{k^{2n}\, k_T^p} \implies p = 1 + \sum_{\alpha}^{V}(D_\alpha - 4). \tag{6.16}$$

This is particularly insightful, as it shows how any correlator is completely constrained by its corresponding amplitude in the total energy pole. For contact interactions near this pole, *a Laurent expansion in powers of k_T is equivalent to an effective field theory (EFT) expansion in powers of the operators $\mathcal{O}_\alpha$*[32]. This is because, in this limit, both expansions are describing the same low-energy dynamics - the light degrees of freedom still present in the theory[33].

6.2.4 Rule 4: Bose symmetry

Since our correlators are composed of bosonic fields, these must obey Bose statistics and as a consequence, *Bose symmetry* when all the fields are the same. It is convenient to introduce a basis of n Elementary Symmetric Polynomials (ESPs) to ensure the correlator is symmetric under permutations of its wavevectors $\{k_1, k_2, k_3\}$. Under the Fundamental Theorem of Symmetric Polynomials [65], any symmetric polynomial can be written uniquely as a combination of these ESPs[34]. For $n = 3$ they are,

$$e_1 \equiv k_1 + k_2 + k_3 = k_T, \tag{6.17a}$$
$$e_2 \equiv k_1 k_2 + k_2 k_3 + k_1 k_3, \tag{6.17b}$$
$$e_3 \equiv k_1 k_2 k_3. \tag{6.17c}$$

The bispectra can be expressed in terms of these ESPs, to give a more precise result for the form of the polynomials in rule 2 (Section 6.2.2). One could instead proceed by brute force and write down any polynomial and sum over all of its permutations, but this gives rise to a large degeneracy in the parameters that cannot be constrained. The form of these ESPs is particularly convenient as taking the limit as each ESP tends to zero, gives powerful constraints to the form of the correlator, e.g. $k_T = e_1 \to 0$ fixes the form of the correlator in terms of its amplitude (rule 3). Re-expressing the trimmed bispectrum (6.8) for three identical fields X, in terms of the ESPs,

$$\tilde{B}_{XXX} = \frac{\mathrm{Poly}_\beta(k_T, e_2, e_3)}{\mathrm{Poly}_{6+\alpha_T+\beta}(k_T, e_2, e_3)}, \tag{6.18}$$

In the presence of three identical spinning particles (6.18) is not always true. From (6.6), the above only holds when the polarisation factor is also invariant under the same permutations.

[32]In this limit, the singular behaviour of the amplitude at low energies is due to the exchange of massive particles, which corresponds to the presence of higher-dimensional operators in the EFT Lagrangian. As we integrate out these particles, the coupling constants of these operators depend on the energy scale and decrease as we move to lower energies. This results in a series of local operators with decreasing dimensions, which corresponds to the Laurent expansion in powers of k_T. As a result, the Wilson coefficients in the Laurent expansion can be interpreted as the coupling constants of the local operators in the EFT expansion in this limit, and the two expansions become equivalent

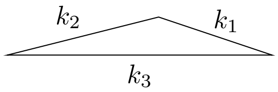
[33]In fact, the EFT framework was originally developed to describe the low-energy behaviour of scattering amplitudes in QCD, where it is used to systematically incorporate the effects of integrating out the heavy quarks and gluons into an effective theory that describes the dynamics of the light hadrons [63, 64].

[34]For $n > 3$, the ESPs have some momentum angular dependence. In this case, it is more convenient to compute the Hilbert series of the variety (solutions of the polynomial equations), and then decompose the variety into a finite number of smooth varieties with normal crossings, using a Hironaka decomposition. Each smooth variety can be used to extract a complete and mutually independent set of basis polynomials using the theory of Gröbner bases [66].

6.2.5 Rule 5: Locality and the Bunch-Davies vacuum

We can further constrain the form of the bispectra from Rule 2 by considering the effect of a different choice of vacuum and the implications on locality.

The Bunch-Davies vacuum In Section 5.16 we found the order p of the pole of a correlator, and we noticed that this value generically varies under a Bogoliubov transformation of the initial, Bunch-Davies, vacuum state. The addition of these poles imposes even physical singularities, that manifest themselves with flattened momenta configurations i.e. defined as those for which two momentum are collinear (see figure on the right). In [67], it's shown that removing these singularities is equivalent to restricting the theory to a Bunch-Davies vacuum.



This can be seen clearly by rewriting the in-in formalism (2.16) in its commutator form[35]. The interacting Hamiltonian is composed of mode functions, which are collapsed with the Bunch-Davies vacuum on the left. Performing a general Bogoliubov transformation does not preserve the symmetry properties of the original vacuum state, including the positive and negative frequency modes. As a result, the total energy pole at $1/(k_T)^p$ is now located at a different, flattened configuration $1/(k_a + k_b - k_c)^p$ for $\{a, b, c\}_{\neq} \in \{1, 2, 3\}$ [68][36].

6.2.6 Rule 6: Manifestly Local Test

The whole derivation of the above rules can be represented in terms of ψ_n, using simple algebraic relations such as those derived in (2.22). In this rule, it will be more convenient to work with the wavefunction form. One can use soft limits and locality to bootstrap the form of correlators, as it was first done in [6]. However, a more convenient way to completely fix all the free parameters we have left in our bootstrap ansatz is by using the *Manifestly Local Test*[37] (see Section 4.6). Applying the MLT from (4.50) to k_3, for the bispectra is

$$\frac{\partial}{\partial k_3} \psi_3'(k_1, k_2, k_3) \bigg|_{k_3=0} = 0. \tag{6.19}$$

This result is general, and it applies to an arbitrary leading k_T pole of order p, and to an arbitrary wavefunction form. The MLT allows us to bootstrap all contact 3-point correlators arising from scalar self-interactions as well as those arising from coupling the scalar to a massless graviton.

6.3 A Boostless Bootstrap

In this subsection, we go through some specific examples of how the Bootstrap rules can be used to arrive at the form of correlators for different p, by starting with the amplitude in flat space. We will show how the MLT is a powerful tool, that even fixes the residue of the subleading k_T poles, for a specific example of the bispectrum for scalar self-interactions; $\psi_3^{\phi\phi\phi}$. Nonetheless, this just

[35] This takes the form $\langle \mathcal{O}(t) \rangle = \sum_{n=0}^{\infty} i^n \int_{-\infty}^{t} \delta t_n \int_{-\infty}^{t_n} \delta t_{n-1} ... \int_{-\infty}^{t_2} \delta t_1 \langle \Omega | [H_{int}(t_1), ...[H_{int}(t_n), \mathcal{O}_I(t)]...]] | \Omega \rangle$, which can be proven by induction.

[36] Another example of this is the spontaneous symmetry breaking in gauge theories, such as the electroweak theory, where the Higgs field acquires a non-zero vacuum expectation value, which breaks the electroweak symmetry. The vacuum state is no longer invariant under the full electroweak symmetry group [69].

[37] This also makes the amplitude limit at the total energy pole redundant (rule 3).

provides an intuitive representation of the Bootstrap program, which can be extended to any sort of interaction by imposing the appropriate rules.

The three-scalar massless correlator

Combining all the bootstrap rules we can write a general ansatz for these wavefunction coefficients for scalar self-interactions. Dropping the superscript for notational clarity we obtain:

$$\psi_3' = \frac{1}{k_T^p} \sum_{n}^{\lfloor \frac{p+3}{3} \rfloor} \sum_{m}^{\lfloor \frac{p+3-3n}{2} \rfloor} C_{mn} k_T^{3+p-2m-3n} e_2^m e_3^n, \tag{6.20}$$

where e_i are the ESPs given in (6.17), $\lfloor \cdots \rfloor$ is the floor function, and C_{mn} are constant coefficients that are real because of unitarity, i.e. imposed by the COT in (4.11). The order p of the pole is fixed by (5.16) and coincides with the total number of derivatives in the interaction[38].

The number of free coefficients; N_{free}, in (6.20) is equivalent to the number of non-negative integer solutions to the k_T superscript, $3 + p \geq 2m + 3n$ [70], that ensures manifest locality. Imposing the MLT from (6.19) onto (6.20), allows us to derive a set of constraints. Specifically, the number of parameters that are fixed in the recursion relation from the MLT; $N_{\text{constraints}}$, and N_{free} are

$$N_{\text{free}} = \sum_{q=0}^{\lfloor \frac{p+3}{3} \rfloor} \left(1 + \left\lfloor \frac{p+3-2q}{3} \right\rfloor\right), \qquad N_{\text{constraints}} = 1 + \left\lfloor \frac{p+3}{2} \right\rfloor. \tag{6.21}$$

The total number of IR-finite wavefunctions that will arise from this interaction are $N_{\text{total}} = N_{\text{free}} - N_{\text{constraints}}$. As $k_T \to 0$, the amplitude fixes the bispectra, which for massless fields is (6.9)

$$\lim_{k_T \to 0} \text{Re}(\psi_3) \sim e_3 \frac{i^p \mathcal{A}_3'}{k_T^p}, \tag{6.22}$$

which has at least one power of e_3 appearing in the nominator, to be a manifestly local theory. Furthermore, by finding the highest powers of $\partial_{k_3} e_3$ from the constraint recursion relations imposed by the MLT we find

$$C_{\frac{p+3}{2}0} = 0; \qquad \text{odd p,} \tag{6.23a}$$

$$C_{\frac{p}{2}1} = (p-1)C_{\frac{p+2}{2}0}; \qquad \text{even p.} \tag{6.23b}$$

This fixes the subleading k_T poles for even p[39].

$\mathbf{p = 0}$: The case when $p = 0$, leads to a pole of the form in (6.10). Imposing the constraints from the MLT we obtain

$$\psi_3' = C_{00}(k_1^3 + k_2^3 + k_3^3) + (-3C_{00} + C_{10})[4e_3 - e_2 k_T + (k_1^3 + k_2^3 + k_3^3)\log(-k_T \eta_0)]. \tag{6.24}$$

[38] Which is equal to the largest number of derivatives in the EFT expansion [6].

[39] If this wasn't the case, and the subleading pole weren't constrained, there would exist a freedom to cancel and remove the leading k_T pole, turning the subleading pole into the leading pole, and hence this amplitude would not come from a manifestly local theory.

The above ansatz is also another solution without the first term. The first term proportional to C_{00} coincides with the local non-gaussianity from [71]. If instead one used soft limits to fix the free parameters of our bootstrap ansatz from (6.20), one would miss one of these constraints and hence would need to resort to conformal invariance [6, 7].

p = 1: Applying $p = 1$ to the ansatz from (6.20), gives

$$\psi_3' = \frac{1}{k_T}[C_{00}k_T^4 + C_{10}k_T^2 e_2 + C_{01}k_T e_3 + C_{20}e_2^2]. \tag{6.25}$$

Using the MLT to fix the coefficients, we find

$$\psi_3' = \frac{C_{00}}{k_T}[k_T^4 - 3k_T^2 e_2 + 3k_T e_3]. \tag{6.26}$$

p = 2: Doing the same for $p = 2$, we find the wavefunction takes the form,

$$\psi_3' = \frac{C_{11}}{4}\left[-k_T^3 + 3k_T e_2 - 11e_3 + \frac{4e_2^2}{k_T} + \frac{4e_2 e_3}{k_T^2}\right] + \mathcal{O}\left(\frac{1}{k_T}\right), \tag{6.27}$$

where now subleading k_T poles also appear.

The imaginary part The above shows some examples of how one can use the bootstrap rules to derive the wavefunction coefficients in a very computationally efficient way. Which can be extended to further orders p, to bootstrap the EFT of inflation up to any order in derivatives. However, we haven't yet considered the imaginary part of ψ_3. This is done in the same manner but by ensuring the amplitude doesn't have any time dependence in the late time limit $\eta_0 \to 0$. The ansatz is

$$\lim_{\eta_0 \to 0} \psi_3' = \frac{\mathrm{Poly}_{3+p}}{k_T^p} + \log(-\eta_0 k_T)\mathrm{Poly}_3 + \frac{\mathrm{Poly}_2}{\eta_0} + \frac{\mathrm{Poly}_1}{\eta_0^2} + \frac{\mathrm{Poly}_0}{\eta_0^3}. \tag{6.28}$$

Imposing our bootstrap rules, the COT from (4.11), and the MLT from (6.19), gives us constraints on the form of the polynomials, which results in,

$$\lim_{\eta_0 \to 0} \psi_3' \supset ib\frac{(k_T^2 - e_2)}{\eta_0} + i\frac{a}{\eta_0^3}, \qquad (a, b) \in \mathbb{R}. \tag{6.29}$$

This is purely imaginary, and hence the divergences at $\eta_0 = 0$ are not observed. This is because the correlators only depend on the real part of the wavefunction coefficients, as we saw in (2.22).

7 Conclusions

In this paper, we have reviewed different ways of deriving cosmological correlators, namely the in-in and wavefunction formalism. We have derived the Feynman rules for each, by perturbatively solving the non-linear equations of motion using the Greens function. In section 4 we have reviewed the implications of unitarity in cosmology, including the Cosmological Optical Theorem (COT) for contact and exchange diagrams, as well as the associated cutting rules for any FLRW spacetime, which are valid for fields of any mass and spin with arbitrary local interactions [11, 12, 13]. We have proceeded by computing the non-perturbative consequences of unitarity through the Källen-Lehmann decomposition, which allows us to decompose the two and four-point functions in terms of a spectral integral, which obeys some positivity properties. Furthermore, we review the *Manifestly Local Test* (MLT), a test that must be satisfied by any $n-$point contact and exchange function, emerging from manifestly local interactions [7]. This does not assume any invariance under dS boosts, which therefore can be applied to more generic and large interactions describing inflation. Finally, in Section 6, we summarise the Bootstrap rules [6], which uses the MLT, the total energy pole of correlators and the symmetries of dS to completely fix the form of the bispectra. We find that the COT can be used to show that the late time divergences as $\eta_0 \to 0$ are imaginary and hence do not contribute to observables. As an illustration, we use these rules to bootstrap the form of the wavefunction for self-interactions of a massless scalar. It is expected that the tools reviewed in this work will play a crucial role in the context of cosmological collider physics [67, 72].

This area of research is relatively new and there are still very interesting areas to be worked on. To name a few:

- Currently the COT and the MLT work in perturbation theory. Finding a generalisation of the recursion relations of the MLT and the $n-$point COT, that is more general than on a diagram-by-diagram basis, would be useful.
- The bootstrap rules, have also been used to constrain the trispectrum, assuming it's associated with a meromorphic function [7]. Further work could consider its non-analytic behaviour, such as branch cuts, that arise in the cosmological particle production in dS [67].
- The analyticity of the S-matrix program has been extremely successful in deriving a variety of UV/IR *sum rules* that relate low-energy observables to certain integrals of discontinuities over its corresponding high-energy theory [73]. This provides a direct link between EFT Wilson coefficients and its UV completion. These sum rules can be combined with unitarity and locality which lead to *positivity bounds* on low-energy EFTs [74, 75]. Positivity bounds for the S-matrix program have successfully constrained low energy EFT couplings [76, 77, 78], and these bounds have been applied to boost breaking amplitudes [79]. In contrast to the bootstrap approach, these bounds in principle, can constrain the size of the coupling of the interaction. It would be interesting to see what these consequences are for cosmological correlators. Recent work in [73] has been on the analyticity of the Minkowski wavefunction of interacting scalar fields. A natural next step is to consider the extension of their analysis to wavefunctions in quasi-de sitter space. Constructing these UV/IR relations would establish a set of *positivity bounds* for the wavefunction coefficients. In comparison to the sum rules for scattering amplitudes [79, 80], the wavefunction sum rules could also fix total-derivative interactions.

Acknowledgments

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A The wavefunction of the Universe

The fields $\phi_a(\eta, \mathbf{x})$ represent the fluctuations of the classical background, with eigenstates $|\phi, \eta\rangle$. We will only consider a homogeneous, isotropic background $\bar{\phi}_a(\eta, \mathbf{x}) \equiv \bar{\phi}_a(\eta)$.

A.1 A review of Ψ

The state of the universe is simply the amplitude for finding the fields in a specific configuration, at a given time. It's called the wavefunction and is given by,[40]

$$\Psi[\phi; \eta_0] = \langle \phi, \eta_0 | \Psi, \eta \rangle, \qquad |\Psi, \eta\rangle = \int d\phi |\phi, \eta\rangle \Psi[\phi; \eta], \tag{A.1}$$

for a state $|\Psi, \eta\rangle$ at conformal time η. We wish to find a relation to compare the value of the wavefunction to different η. The orthogonality relations between distinct conjugate momenta and the corresponding fields of our theory are

$$\langle \bar{\phi}, \eta | \phi, \eta \rangle = \delta\left(\bar{\phi} - \phi\right), \quad \langle \bar{\pi}, \eta | \pi, \eta \rangle = \delta\left(\bar{\pi} - \pi\right), \quad \langle \phi, \eta | \pi, \eta \rangle = \frac{1}{\sqrt{2\pi}} \exp(i\phi\pi). \tag{A.2}$$

We wish to find the orthogonality relations for fields at different conformal times. We do this by considering two fields with an infinitesimal time difference $\eta_0 \to \eta = \eta_0 - d\eta$. The time evolution of the field eigenstates can be decomposed using the unitary operator in the Schrödinger picture,

$$|\phi, \eta\rangle = |\phi, \eta_0 - d\eta\rangle = U^\dagger\left(\eta_\mathrm{x}, \eta_0 - d\eta\right) U(\eta_\mathrm{x}, \eta_0) \left|\phi, \eta_0\right\rangle. \tag{A.3}$$

Using this, and simplifying the expression, we can express the unitary operator in the Heisenberg picture, and we can expand it in small powers of $d\eta$ i.e. an infinitesimal change in time,

$$\langle \bar{\phi}, \eta_0 | \phi, \eta \rangle = \langle \bar{\phi}, \eta_0 | T\{\exp(-i \int_{\eta_0 - d\eta}^{\eta_0} H_H d\eta\} |\phi, \eta_0\rangle \approx \langle \bar{\phi}, \eta_0 | 1 - iH_H d\eta | \phi, \eta_0 \rangle. \tag{A.4}$$

Inserting an infinite number of complete sets of field and conjugate momenta, and using the orthogonality relation given in (A.2), and inserting this into (A.1), we can finally compare the wavefunctions at different times, using the path integral formalism,

$$\Psi[\bar{\phi}; \eta_0] = \left\langle \bar{\phi}, \eta_0 | \Omega \right\rangle = \int_{\phi(\eta_0) = \bar{\phi}} \mathcal{D}\phi \int \frac{\mathcal{D}\pi}{2\pi} \exp\left[i \int d\eta \left\{\phi'(\eta)\pi(\eta) - H[\phi, \pi; \eta]\right\}\right] \Psi[\phi; \eta]. \tag{A.5}$$

We have rewritten $|\Psi, \eta\rangle = |\Omega\rangle$, given our initial conditions during inflation, the wavefunction of interest is the Bunch-Davis vacuum, which remains constant in the Heisenberg picture. In this limit, we are at the infinite past of our universe, $\lim \eta \to -\infty$. In dS, all the pictures have equivalent representations, so we expect the wavefunction to be constant in the past. Therefore,

$$\Psi[\bar{\phi}; \eta_0] = \mathcal{N} \lim_{\eta \to -\infty} \int_\eta^{\eta_0} \mathcal{D}\phi \int \frac{\mathcal{D}\pi}{2\pi} \exp\left[i \int_\eta^{\eta_0} d\eta \left\{\phi'(\eta)\pi(\eta) - H[\phi, \pi; \eta]\right\}\right] \tag{A.6}$$

[40] As these states form a complete basis and obey the usual orthogonality relations.

A.2 Ψ form for correlators

In this section, we attempt to cover how we can derive predictions for our statistical correlation functions produced during inflation, from a given action $S[g_{\mu v}]$, with coupled matter fields ϕ to a metric $g_{\mu v}$ associated with the spacetime. Given a primordial operator $\mathcal{O}(\hat{\phi}, \hat{\pi})$, with possible dependence on both the field and its conjugate momenta, we can introduce a complete basis into our wavefunction,

$$\langle\Psi, \eta_0|\mathcal{O}(\hat{\phi}, \hat{\pi})|\Psi, \eta_0\rangle = \int d\bar{\phi} \int d\phi \int d\pi \Psi^*[\phi; \eta_0]\Psi[\bar{\phi}; \eta_0]\langle\phi, \eta_0|\mathcal{O}(\hat{\phi}, \hat{\pi})|\pi, \eta_0\rangle\langle\pi, \eta_0|\bar{\phi}, \eta_0\rangle. \quad \text{(A.7)}$$

Without loss of generality but for simplicity in our derivation, we assume the operator is defined by $\hat{\phi}$'s on the left and $\hat{\pi}$ on the right. This makes the π and ϕ integrals trivial,

$$\begin{aligned}\langle\Psi, \eta_0|\mathcal{O}(\hat{\phi}, \hat{\pi})|\Psi, \eta_0\rangle &= \int d\bar{\phi} \int d\phi \Psi^*[\phi; \eta_0]\Psi[\bar{\phi}; \eta_0]\mathcal{O}(\phi, \pi)\delta(\phi - \bar{\phi}) \\ &= \int d\bar{\phi} \Psi^*[\bar{\phi}; \eta_0]\mathcal{O}(\bar{\phi}, -i\partial_{\bar{\phi}})\Psi[\bar{\phi}; \eta_0].\end{aligned} \quad \text{(A.8)}$$

Our usual operators of interest are composed of products of the fields, $\mathcal{O} = \prod_i^n \bar{\phi}_{\mathbf{k}_i}$, which gives the expression in (2.19). A more practicable approach for large n is the path integral formalism, which we go through in the main text, in Section 2.2.2.

B Cosmological Optical Theorem for Exchange diagrams

To derive the COT for exchange diagrams, we will need to use the Optical Theorem at second order. We, therefore, need to consider cubic interactions S_3. Let's take two Lagrangians consisting of interactions between ϕ and σ, of the form[41]

$$\begin{aligned}\mathcal{L}_3^A &= -\lambda_A a(\eta)^{4-d_A-\bar{d}_A}\partial^{d_A}\phi(\eta, \mathbf{x})\partial^{\bar{d}_A}\phi(\eta, \mathbf{x})\,\sigma(\eta, \mathbf{x}), \\ \mathcal{L}_3^B &= -\lambda_B a(\eta)^{4-d_B-\bar{d}_B}\partial^{d_B}\phi(\eta, \mathbf{x})\partial^{\bar{d}_B}\phi(\eta, \mathbf{x})\,\sigma(\eta, \mathbf{x}).\end{aligned} \quad \text{(B.1)}$$

As for the COT for contact diagrams, $d_{A,B}$ and $\bar{d}_{A,B}$ are either 0 (no derivative) or 1, which could be a spatial or temporal derivative acting on ϕ. By adding the two Lagrangians in (B.1), we can include two different cubic interactions between σ and ϕ, into the total cubic Lagrangian and action:

$$S_3 \equiv \int d\eta d^3\mathbf{x}\mathcal{L}_3, \qquad \mathcal{L}_3 \equiv \mathcal{L}_3^A + \mathcal{L}_3^B. \quad \text{(B.2)}$$

The corresponding Hamiltonian can be expressed in Fourier space as[42],

[41] This is a valid assumption for the form of the exchange field σ in a four-point exchange diagram, even if we replaced it with another of our fields $\sigma \to \phi$ or by including derivatives $\sigma \to \partial\sigma$, as we can always rewrite it using the equations of motion, such that we obtain a Lagrangian of the form (B.1). See Appendix C in [11] for a discussion.

[42] Notice we are neglecting the addition of any possible quartic Hamiltonian to H_{int}, which consists of 4th order field interactions because even if these interactions exist, it would also be at quartic order in σ and hence will not contribute to our final result.

$$
\begin{aligned}
&H_{int} = H^A + H^B \\
&H^A = -\lambda_A \int d\eta a(\eta)^{4-t_A-\bar{t}_A} \left(\prod_{a=1}^{3} \int_{\mathbf{k}_a}\right) \mathcal{F}_A \left[\frac{\mathbf{k}_a \cdot \mathbf{k}_b}{a^2(\eta)}\right] (2\pi)^3 \delta^{(3)} \left(\sum_{a=1}^{3} \mathbf{k}_a\right) \\
&\qquad \times \partial_\eta^{t_A} \phi(\eta, \mathbf{k}_1) \partial_\eta^{\bar{t}_A} \phi(\eta, \mathbf{k}_2)\, \sigma(\eta, \mathbf{k}_3),
\end{aligned} \tag{B.3}
$$

as before, we have denoted the number of temporal derivatives (1 or 0) as $t_A, \bar{t}_A$, and H^B is equivalent to H^A with $A \to B$. Expanding the Optical Theorem in (4.2) to second order in H_{int}, and inserting (B.3) for our interacting Hamiltonian, we obtain,

$$
\begin{aligned}
&\langle \{\mathbf{k}, \phi\}_4 | \Delta U_{AB} | 0 \rangle + \langle \{\mathbf{k}, \phi\}_4 | \Delta U_{AB}^\dagger | 0 \rangle = \\
&- \prod_{a=1}^{3} \int_{\mathbf{p}_a} \langle \{\mathbf{k}, \phi\}_4 | \Delta U_A | \mathbf{p}_1, \phi, \mathbf{p}_2, \phi; \mathbf{p}_3, \sigma \rangle \langle \mathbf{p}_1, \phi, \mathbf{p}_2, \phi; \mathbf{p}_3, \sigma | \Delta U_B^\dagger | 0 \rangle + (A \leftrightarrow B),
\end{aligned} \tag{B.4}
$$

where we have included only the contributing order of the identity operator in (4.4). The first and second-order terms of the unitary matrix are given by,

$$
\begin{aligned}
\Delta U_{AB} &= -\int_{-\infty(1-i\epsilon)}^{\eta_0} d\eta\, d\eta' H^A(\eta) H^B(\eta') \theta(\eta - \eta') + (A \leftrightarrow B), \\
\Delta U_A &= -i \int_{-\infty(1-i\epsilon)}^{\eta_0} d\eta H^A(\eta).
\end{aligned} \tag{B.5}
$$

To evaluate explicitly the expression (B.4) we plug in (B.5) and (B.3). We'll make use of the mode function decomposition of the exchange field $\sigma_{\mathbf{k}} = g_k b_{\mathbf{k}} + g_{-k}^* b_{-\mathbf{k}}^\dagger$, and work term by term,

$$
\begin{aligned}
\langle \{\mathbf{k}, \phi\}_4 | \Delta U_{AB} | 0 \rangle' = &i\lambda_A \lambda_B \int d\eta\, d\eta' \Delta_F^\sigma(p_s, \eta; \eta')\, a(\eta)^{4-t_A-\bar{t}_A}\, a(\eta')^{4-t_B-\bar{t}_B} \\
&\times \left(\mathcal{F}_A \left[\frac{\mathbf{k}_a \cdot \mathbf{k}_b}{a^2(\eta)}\right] \partial_\eta^{t_A} f^*(\eta, k_1) \partial_\eta^{\bar{t}_A} f^*(\eta, k_2) + (k_1 \leftrightarrow k_2) \right) \\
&\times \left(\mathcal{F}_B \left[\frac{\mathbf{k}_a \cdot \mathbf{k}_b}{a^2(\eta')}\right] \partial_\eta^{t_B} f^*(\eta', k_3) \partial_\eta^{\bar{t}_B} f^*(\eta', k_4) + (k_3 \leftrightarrow k_4) \right) \\
&+ (A \leftrightarrow B) + (p_t, p_u \text{ channels}),
\end{aligned} \tag{B.6}
$$

and similarly,

$$
\begin{aligned}
\langle \{\mathbf{k}, \phi\}_4 | \Delta U_{AB}^\dagger | 0 \rangle' = &- i\lambda_A \lambda_B \int d\eta\, d\eta' \Delta_{\bar{F}}^\sigma(p_s, \eta; \eta')\, a(\eta)^{4-t_A-\bar{t}_A}\, a(\eta')^{4-t_B-\bar{t}_B} \\
&\times \left(\mathcal{F}_A \left[\frac{\mathbf{k}_a \cdot \mathbf{k}_b}{a^2(\eta)}\right] \partial_\eta^{t_A} f^*(\eta, k_1) \partial_\eta^{\bar{t}_A} f^*(\eta, k_2) + (k_1 \leftrightarrow k_2) \right) \\
&\times \left(\mathcal{F}_B \left[\frac{\mathbf{k}_a \cdot \mathbf{k}_b}{a^2(\eta')}\right] \partial_\eta^{t_B} f^*(\eta', k_3) \partial_\eta^{\bar{t}_B} f^*(\eta', k_4) + (k_3 \leftrightarrow k_4) \right) \\
&+ (A \leftrightarrow B) + (p_t, p_u \text{ channels}).
\end{aligned} \tag{B.7}
$$

In the last equation, we have simplified the expression using the (anti) Feynman propagators from (3.1) and (3.2) for the field σ. We can further simplify this using the cubic and quartic wavefunction coefficient for our action (B.2); $\psi_3^{\phi\phi\sigma} = \psi_3$ and $\psi_4^{\phi\phi\phi\phi} = \psi_4$ respectively. We can separate out the A and B contribution linearly, i.e. $\psi'_3 = \psi_3'^A + \psi_3'^B$. We multiply the terms out in (B.6) and (B.7), and denote the $\mathcal{O}(\lambda_A\lambda_B)$ contributions to ψ_4 as ψ_4^{AB}. Converting the Feynman propagator into a wavefunction propagator using (3.17), the primed LHS of (B.4) becomes,

$$
\begin{aligned}
&\langle\{\mathbf{k},\phi\}_4|\Delta U_{AB}|0\rangle' + \langle\{\mathbf{k},\phi\}_4|\Delta U_{AB}^\dagger|0\rangle' = \\
&\left(\prod_{a=1}^{4} f^*(\eta_0,k_a)\right)\{-\psi_4'^{AB}(k_a,p_s,k_a\cdot k_b) - \left[\psi_4'^{AB}(-k_a,p_s,k_a\cdot k_b)\right]^* \\
&+ P_\sigma(\eta_0,p_s)\left(\psi_3'^A\,(k_1,k_2,p_s)\,\psi_3'^B\,(k_3,k_4,p_s) + (p_s\to -p_s) + (A\leftrightarrow B)\right)\} \\
&+ (p_t,p_u\,\text{channels}).
\end{aligned}
\tag{B.8}
$$

Computing the primed RHS of (B.4) gives,

$$
\begin{aligned}
&-\prod_{a=1}^{3}\int_{\mathbf{p}_a}\langle\{\mathbf{k},\phi\}_4|\Delta U_A|\mathbf{p}_1,\phi,\mathbf{p}_2,\phi;\mathbf{p}_3,\sigma\rangle'\langle\mathbf{p}_1,\phi,\mathbf{p}_2,\phi;\mathbf{p}_3,\sigma|\Delta U_B^\dagger|0\rangle' + (A\leftrightarrow B) \\
&= -\lambda_A\lambda_B\int d\eta\, d\eta'\, a(\eta)^{4-t_A-\bar{t}_A}\left(\mathcal{F}_A\left[\frac{\mathbf{k}_a\cdot\mathbf{k}_b}{a^2(\eta)}\right]\partial_\eta^{t_A}f^*(\eta,k_1)\partial_\eta^{\bar{t}_A}f^*(\eta,k_2) + (k_1\leftrightarrow k_2)\right)g(\eta,-p_s) \\
&\quad\times a(\eta')^{4-t_B-\bar{t}_B}\left(\mathcal{F}_A\left[\frac{\mathbf{k}_a\cdot\mathbf{k}_b}{a^2(\eta')}\right]\partial_\eta^{t_A}f^*(\eta',k_3)\partial_\eta^{\bar{t}_A}f^*(\eta',k_4) + (k_3\leftrightarrow k_4)\right)g^*(\eta',p_s) \\
&\quad + (A\leftrightarrow B) + (p_t,p_u\,\text{channels}), \\
&= \left(\prod_{a=1}^{4} f^*(\eta_0,k_a)\right)P_\sigma(\eta_0,p_s)\left[\psi_3'^A(k_1,k_2,-p_s)\psi_3'^B(k_3,k_4,p_s) + \psi_3'^A(k_3,k_4,-p_s)\psi_3'^B(k_1,k_2,p_s)\right] \\
&\quad + (A\leftrightarrow B) + (p_t,p_u\,\text{channels}).
\end{aligned}
\tag{B.9}
$$

Finally, equating (B.8) and (B.9), and summing over all the wavefunction for each channel $\psi_4 = \psi_4(p_s)+\psi_4(p_t)+\psi_4(p_u)$, we obtain the Cosmological Optical Theorem for exchange diagrams, which is included in the main text (4.12).

C The Amplitude limit

Contact diagrams We begin by taking the case of a diagram without loops, with n external legs propagating from a right, time-ordered vertex ($\alpha = R$). The correlator is given by,

$$
\begin{aligned}
B_n^R &= -i\,\lambda\int_{-\infty}^{\eta_0} d\eta\, a(\eta)^{4-s_R}\,F(\mathbf{k})\prod_a^n K_R(k_a,\eta), \\
&= -i\,\lambda\int_{-\infty}^{\eta_0} d\eta\,(-\eta H)^{s_R-4}\,F(\mathbf{k})\prod_a^n\frac{H^2[u(k_a)^{t_a}]^*}{2k_a^3}(-H\eta)^{t_a}(1-t_a-ik_a\eta)(1-ik_a\eta_0)e^{-ik_a(\eta-\eta_0)} \\
&= -i\,\lambda\int_{-\infty}^{\eta_0} d\eta\, H^{N+2n-4}(-\eta)^{N-4}\,F(\mathbf{k})\prod_a^n\frac{[u(k_a)^{t_a}]^*}{2k_a^3}(1-t_a-ik_a\eta)(1-ik_a\eta_0)e^{-ik_a(\eta-\eta_0)},
\end{aligned}
$$

$$(C.1)$$

where s_R denotes the total number of spatial derivatives at the right vertex R, and in the last line, we have rewritten it in terms of the total number of derivatives $N \equiv \sum^n d_a = s_R + \sum^n t_a$. With this, we can separate the η dependency,

$$B_n^R = B_n^R(\mathbf{k}) \int_{-\infty}^{\eta_0} d\eta\, B_n^R(\eta, k), \qquad B_n^R(\mathbf{k}) = -i\,\lambda H^{N+2n-4}\, F(\mathbf{k}) \prod_a^n \frac{[u(k_a)^{t_a}]^*}{2k_a^3}(1 - ik_a\eta_0). \quad (C.2)$$

Evaluating the η integral explicitly,

$$\begin{aligned}\int_{-\infty}^{\eta_0} d\eta\, B_n^R(\eta, k) &= \int_{-\infty}^{\eta_0} d\eta\, (-\eta)^{N-4} \prod_a^n (1 - t_a - ik_a\eta) e^{-ik_a(\eta-\eta_0)} \\ &= \sum_b^n P_b(k_a) \int_{-\infty}^{\eta_0} d\eta\, \eta^{N-4+b} e^{-ik_T(\eta-\eta_0)} \\ &= \sum_b^n P_b(k_a) \sum_{c=0}^{N+b-4} \frac{(-1)^{c+b+N}(\eta_0)^c (N+b-4)!}{c!\,(-ik_T)^{N+b-3-c}}. \end{aligned} \quad (C.3)$$

We have defined the total momenta, $k_T = \sum_a^n k_a$, and $P_b(k_a)$ is a polynomial of k_a. The *total energy pole* is when $k_T \to 0$, this corresponds to the UV limit of the correlator. In this limit, assuming $N + n > 3$, the worst energy pole is the one with the highest power of energy in the denominator, which dominates the behaviour of the B_n. This occurs at $b = n$ and $c = 0$[43]. In this limit we expect the polynomial of k_a to be dominated by the highest powers of η, which only appear as $-ik_a\eta$ in (C). Taking only linear terms in k_a, we can write the polynomial under this limit as $P_b(k_a) = P_n(k_a) = (-1)^N \prod_a^n (ik_a)$[44]. Bringing everything together,

$$\begin{aligned}\lim_{k_T\to 0} B_n^R &= (-1)^{n+1} i\,\lambda H^{N+2n-4}\, F(\mathbf{k}) \prod_a^n \frac{(i + k_a\eta_0)[u(k_a)^{t_a}]^*}{2k_a^2} \frac{(N+n-4)!}{(-ik_T)^{N+n-3}} \\ &= (-1)^n i\, H^{N+2n-4}\, \mathcal{A}_n' \frac{(N+n-4)!}{(-ik_T)^{N+n-3}} \prod_a^n \frac{i + k_a\eta_0}{2k_a^2}, \end{aligned} \quad (C.4)$$

where in the second equality we have substituted the explicit expression for the amplitude from (5.8) with no internal lines, $I = 0$. This expression still holds for a generic contact diagram with more than one exchange vertex $I \neq 0$, as we will now see.

If we had considered a left vertex, switching all the $R \to L$, the result would've been the complex conjugate of (C), as all our propagators are (5.2). We can therefore write the total energy pole of an $n-$point correlator as,

[43]This is equivalent to the pole with the largest residue, which happens when we have a maximum number of derivatives at the vertex (the maximum number of momenta in the denominator). This makes the correlator the most singular.

[44]The factor of $(-1)^N$ comes from the initial $(-\eta)^{N-4}$ in (C) which we factored into $P_b(k_a)$.

$$\lim_{k_T\to 0} B_n \equiv \lim_{k_T\to 0}(B_n^R + B_n^L). \tag{C.5}$$

$$\lim_{k_T\to 0} B_n = 2(-1)^n H^{N+2n-4}(N+n-4)!\ \mathrm{Re}\left\{\frac{i\mathcal{A}'_n}{(-ik_T)^{N+n-3}}\prod_a^n \frac{i+k_a\eta_0}{2k_a^2}\right\}. \tag{C.6}$$

General Diagrams Following our Feynman rules from section 3.1, a general diagram propagating from a right (time-ordered) vertices without loops is given by,

$$B_n^R = i^{V_L - V_R}\prod_\alpha^{V_L+V_R}\int_{\eta_\alpha}\frac{\lambda_\alpha F_\alpha(\mathbf{k})}{a(\eta)^{s_\alpha-4}}\prod_\beta^I G_{RR}(K_\beta,\eta;\eta')\prod_a^n K_R(k_a,\eta). \tag{C.7}$$

The hand-shacking lemma ensures that we can always find a vertex V, connected to another vertex $V-1$, such that it connects two tree-level contact diagrams. Factorising out a right vertex V from (C.7) and using the explicit expressions for the propagators in (5.2) and (5.13),

$$\begin{aligned}
B_n^R &= i^{-V}(-1)^{t_V+t_{V-1}+\sum_a^m t_a+s_V} H^{2I+2n+\sum_\alpha^V(N_\alpha-4)}\prod_\alpha^{V-1}\int_{\eta_\alpha}\frac{\lambda_\alpha F_\alpha(\mathbf{k})}{a(\eta)^{s_\alpha-4}}\prod_\beta^{I-1}G_{RR}(K_\beta,\eta;\eta')\\
&\prod_a^{n-m}K_R(k_a,\eta)\lambda_V F_V(\mathbf{k})\int_{-\infty}^{\eta_0}d\eta_V\prod_b^m\left[\frac{[u(k)^{t_b}]^*}{2k_b^3}(1-ik_b\eta_0)(\eta_V)^{t_b}(1-t_b+ik_b\eta_V)(\eta_{V-1})^{t_{V-1}}\right]\\
&\left\{\frac{[u(k_I)^{t_{V-1}}]^*u(k_I)^{t_V}}{2k_I^3}(\eta_V)^{t_V}(1-t_{V-1}+ik_I\eta_{V-1})(1-t_V-ik_I\eta_V)e^{-ik_I(\eta_{V-1}-\eta_V)}\theta(\eta_V-\eta_{V-1})\right.\\
&\left.+(\eta_V\leftrightarrow\eta_{V-1})\right\}(\eta_V)^{s_V-4}e^{-i\sum_a^m k_a(\eta_V-\eta_0)}.
\end{aligned} \tag{C.8}$$

As $k_T\to 0$, we only take the highest powers of η_V and η_{V-1}, when we perform the η_V integral

$$\begin{aligned}
\lim_{k_T\to 0} B_n^R &= \lim_{k_T\to 0} i^{-V+1}(-1)^{N_V+t_{V-1}}H^{2I+2n+\sum_\alpha^V(N_\alpha-4)}\prod_\alpha^{V-1}\int_{\eta_\alpha}\frac{\lambda_\alpha F_\alpha(\mathbf{k})}{a(\eta)^{s_\alpha-4}}\prod_\beta^{I-1}G_{RR}(K_\beta,\eta;\eta')\\
&\prod_a^{n-m}K_R(k_a,\eta)\lambda_V F_V(\mathbf{k})\prod_b^m\frac{[u(k)^{t_b}]^*}{2k_b^2}(i+k_b\eta_0)e^{-ik_b(\eta_{V-1}-\eta_0)}\\
&(\eta_{V-1})^{m+N_V-2+t_{V-1}}\left[\frac{[u(K_I)^{t_{V-1}}]^*u(K_I)^{t_V}}{2K_I(K_I-\sum_a^m k_a)}+\frac{u(K_I)^{t_{V-1}}[u(K_I)^{t_V}]^*}{2K_I(K_I+\sum_a^m k_a)}\right].
\end{aligned} \tag{C.9}$$

The final expression is equivalent to $\tilde{G}_{Mink.}(K_I)$ from (5.9) so,

$$\begin{aligned}
\lim_{k_T\to 0} B_n^R &= \lim_{k_T\to 0} i^{-V+1}\lambda_V F_V(\mathbf{k})\,H^{2I+2n+\sum_\alpha^V(N_\alpha-4)}\tilde{G}_{Mink.}(K_I)\prod_\alpha^{V-1}\int_{\eta_\alpha}\frac{\lambda_\alpha F_\alpha(\mathbf{k})}{(-\eta)^{4-s_\alpha}}\\
&\prod_\beta^{I-1}G_{RR}(K_\beta,\eta;\eta')\prod_a^{n-m}K_R(k_a,\eta)\prod_b^m K_R(k_b,\eta_{V-1})(-\eta_{V-1})^{N_V+t_{V-1}-2}.
\end{aligned} \tag{C.10}$$

Instead of this derivation, we could have done it the other way around and factorised all the vertices out of the product:

$$\lim_{k_T\to 0} B_n^R = \lim_{k_T\to 0} -iH^{2I+2n+\sum_\alpha^V(N_\alpha-4)}$$
$$\prod_\alpha^V \lambda_\alpha F_\alpha(\mathbf{k}) \prod_\beta^I \tilde{G}_{Mink.}(K_\beta) \int_{-\infty}^{\eta_0} d\eta \prod_a^N K_R(k_a,\eta)(-\eta)^{\sum_\alpha^V s_\alpha + \sum_\beta^I (t_\beta + t'_\beta - 2) - 4}. \tag{C.11}$$

This looks the same as our expression for contact terms (C), but we now have a different value for the total number of derivatives N,

$$N \to \sum_\alpha^V s_\alpha + \sum_\beta^I (t_\beta + t'_\beta - 2) + \sum_a^n t_a = \sum_\alpha^V (N_\alpha - 2). \tag{C.12}$$

The sum of the total number of derivatives and the order of the diagram becomes,

$$N + n \to \sum_\alpha^V (N_\alpha - 2) + n = \sum_\alpha^V (N_\alpha + n_\alpha - 4) + 4. \tag{C.13}$$

In the above, we have introduced the valence of the vertex n_α, which is the total number of lines connected to the vertex α. This allows us to rewrite the total energy pole of a general correlator in the same way as (C.6),

$$\lim_{k_T\to 0} B_n = 2(-1)^n H^{2I+2n+\sum_\alpha^V(N_\alpha-4)} (\sum_\alpha D_\alpha - 4)! \,\mathrm{Re}\left\{ \frac{i\mathcal{A}'_n}{(-ik_T)^p} \prod_a^n \frac{i + k_a\eta_0}{2k_a^2} \right\}. \tag{C.14}$$

D_α and p are the vertex dimension and the order of the total energy pole, respectively. Explicit expressions for them are given in the main text in (5.16). Taking the limit as $\eta_0 \to 0$, and rewriting it all in terms of p, we obtain the simplified expression in the main text in (5.17).

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