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# Probabilistic and Statistical

# Chapter 5 (part 3)
# Probability Distribution

## Binomial Probability Distributions

# *Key Concept*

**This section presents a basic definition of a binomial distribution along with notation, and it presents methods for finding probability values.**

**Binomial probability distributions allow us to deal with circumstances in which the outcomes belong to two relevant categories such as acceptable/defective or survived/died.**

# Definitions

**A binomial probability distribution results from a procedure that meets all the following requirements:**

1. **The procedure has a <u>fixed number of trials</u>.**

2. **The outcomes of each trial must be independent of one another.**

3. **Each trial can have only two outcomes or outcomes that can be reduced to two outcomes. These outcomes can be considered as either success or failure.**

4. **The probability of a success remains the same in all trials.**

## Notation for the Binomial Distribution

| | |
|---|---|
| $P(S)$ | The symbol for the probability of success |
| $P(F)$ | The symbol for the probability of failure |
| $p$ | The numerical probability of a success |
| $q$ | The numerical probability of a failure |

$$P(S) = p \qquad \text{and} \qquad P(F) = 1 - p = q$$

| | |
|---|---|
| $n$ | The number of trials |
| $X$ | The number of successes in $n$ trials |

Note that $0 \le X \le n$ and $X = 0, 1, 2, 3, \ldots, n$.

## Binomial Probability Formula

In a binomial experiment, the probability of exactly $X$ successes in $n$ trials is

$$P(X) = \frac{n!}{(n - X)!X!} \cdot p^X \cdot q^{n-X}$$

# Methods for Finding Probabilities
## Using the Binomial Probability Formula

$$P(x) = {}_nC_x \cdot p^x \cdot q^{n-x}$$

$$x = 0,1,2,\ldots,n$$

***where***

**n = number of trials**

**x = number of successes among n trials**

**p = probability of success in any one trial**

**q = probability of failure in any one trial (q = 1 – p)**

# Binomial distribution
# Mean, Variance & Standard deviation

**Mean, Variance, and Standard Deviation for the Binomial Distribution**

The mean, variance, and standard deviation of a variable that has the *binomial distribution* can be found by using the following formulas.

Mean: $\mu = n \cdot p$ Variance: $\sigma^2 = n \cdot p \cdot q$ Standard deviation: $\sigma = \sqrt{n \cdot p \cdot q}$

**Ex.1 Use the binomial probability formula to find the probability of getting exactly 3 correct responses among 5 different requests from directory assistance. Assume that in general the responses is correct 90% of the time.**

**Solution:**

**That is Find P(3) given that n=5, x=3, *p=0.9* & *q=0.1***

$$P(x) = {}_nC_x \cdot p^x \cdot q^{n-x}$$

$$P(3) = {}_5C_3 \cdot P^3 \cdot q^{5-3}$$

$$P(3) = {}_5C_3\,(0.9)^3 \cdot (0.1)^{5-3} = 10 \times 0.729 \times 0.01 = 0.0729$$

**Ex.2 Consider the experiment of flipping a coin 3 times. If we let the event of getting tails on a flip be considered "success", and if the random variable T represents the number of tails obtained, then T will be binomially distributed with** $n=3$ **,** $p=0.5$ **, and** $q=0.5$ **. Calculate the probability of exactly 2 tails.**

**Solution:**

$$P(2) = {}_3C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 = 3\times\frac{1}{2}^3 = \frac{3}{8} = .375$$

**Ex.3 Each sample of water has a 10% chance of containing a particular organic pollutant. Assume that the samples are independent with regard to the presence of the pollutant. Find the probability that in the next 18 samples, exactly 2 contain the pollutant.**

**Solution:**

**Let n=18, x=2, p=0.1 & q=0.9**

$$P(x) = {}_{n}C_{x} \cdot P^{x} \cdot q^{n-x}$$

$$P(3) = {}_{18}C_{2} \cdot P^{2} \cdot q^{16}$$

$$P(3) = {}_{18}C_{2} \cdot (0.1)^{2} \cdot (0.9)^{16}$$

# Example 3 (cont.)

**Also,**
**Find the probability that in the next 18 samples, that 3 or 4 contain the pollutant.**
**Let n=18, p=0.1 & q=0.9**

$$P(3) + P(4) = {}_{18}C_3 \cdot P^3 \cdot q^{(18-3)} + {}_{18}C_4 \cdot P^4 \cdot q^{(18-4)}$$

**Find mean & standard deviation**

$$\mu = np = 18(0.1) = 1.8$$
$$\sigma = \sqrt{np(1-p)} = \sqrt{1.8(1-0.1)} = 1.27$$

Ex.4 A survey found that one out of five Americans says he or she has visited a doctor in any given month. If 10 people are selected at random, find the probability that exactly 3 will have visited a doctor last month.

Solution:

In this case, $n = 10$, $X = 3$, $p = \frac{1}{5}$, and $q = \frac{4}{5}$. Hence,

$$P(3) = \frac{10!}{(10-3)!3!}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^7 \approx 0.201$$

Ex.5 A survey from Teenage Research Unlimited (Northbrook, Illinois) found that 30% of teenage consumers receive their spending money from part-time jobs. If 5 teenagers are selected at random, find the probability that at least 3 of them will have part-time jobs.

Solution:

$$P(3) = \frac{5!}{(5-3)!3!}(0.3)^3(0.7)^2 \approx 0.132$$

$$P(4) = \frac{5!}{(5-4)!4!}(0.3)^4(0.7)^1 \approx 0.028$$

$$P(5) = \frac{5!}{(5-5)!5!}(0.3)^5(0.7)^0 \approx 0.002$$

$$P(\text{at least three teenagers have part-time jobs})$$
$$= 0.132 + 0.028 + 0.002 = 0.162$$

# Using table to Calculate the Binomial Probability
# Table B in Appendix A

$p = 0.5$

| | | | | | | $p$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $X$ | 0.05 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 0.95 |
| 2 | 0 | | | | | | | | | | | |
| | 1 | | | | | | | | | | | |
| | 2 | | | | | | | | | | | |
| 3 | 0 | | | | | | 0.125 | | | | | |
| | 1 | | | | | | 0.375 | | | | | |
| | 2 | | | | | | 0.375 | | | | | |
| | 3 | | | | | | 0.125 | | | | | |

$n = 3$

$X = 2$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 2 | 0.002 | 0.010 | 0.040 | 0.090 | 0.160 | 0.250 | 0.360 | 0.490 | 0.640 | 0.810 | 0.902 |
| 3 | 0 | 0.857 | 0.729 | 0.512 | 0.343 | 0.216 | 0.125 | 0.064 | 0.027 | 0.008 | 0.001 | |
| | 1 | 0.135 | 0.243 | 0.384 | 0.441 | 0.432 | 0.375 | 0.288 | 0.189 | 0.096 | 0.027 | 0.007 |
| | 2 | 0.007 | 0.027 | 0.096 | 0.189 | 0.288 | 0.375 | 0.432 | 0.441 | 0.384 | 0.243 | 0.135 |
| | 3 | | 0.001 | 0.008 | 0.027 | 0.064 | 0.125 | 0.216 | 0.343 | 0.512 | 0.729 | 0.857 |
| 4 | 0 | 0.815 | 0.656 | 0.410 | 0.240 | 0.130 | 0.062 | 0.026 | 0.008 | 0.002 | | |
| | 1 | 0.171 | 0.292 | 0.410 | 0.412 | 0.346 | 0.250 | 0.154 | 0.076 | 0.026 | 0.004 | |
| | 2 | 0.014 | 0.049 | 0.154 | 0.265 | 0.346 | 0.375 | 0.346 | 0.265 | 0.154 | 0.049 | 0.014 |
| | 3 | | 0.004 | 0.026 | 0.076 | 0.154 | 0.250 | 0.346 | 0.412 | 0.410 | 0.292 | 0.171 |
| | 4 | | | 0.002 | 0.008 | 0.026 | 0.062 | 0.130 | 0.240 | 0.410 | 0.656 | 0.815 |
| 5 | 0 | 0.774 | 0.590 | 0.328 | 0.168 | 0.078 | 0.031 | 0.010 | 0.002 | | | |
| | 1 | 0.204 | 0.328 | 0.410 | 0.360 | 0.259 | 0.156 | 0.077 | 0.028 | 0.006 | | |
| | 2 | 0.021 | 0.073 | 0.205 | 0.309 | 0.346 | 0.312 | 0.230 | 0.132 | 0.051 | 0.008 | 0.001 |
| | 3 | 0.001 | 0.008 | 0.051 | 0.132 | 0.230 | 0.312 | 0.346 | 0.309 | 0.205 | 0.073 | 0.021 |
| | 4 | | | 0.006 | 0.028 | 0.077 | 0.156 | 0.259 | 0.360 | 0.410 | 0.328 | 0.204 |
| | 5 | | | | 0.002 | 0.010 | 0.031 | 0.078 | 0.168 | 0.328 | 0.590 | 0.774 |
| 6 | 0 | 0.735 | 0.531 | 0.262 | 0.118 | 0.047 | 0.016 | 0.004 | 0.001 | | | |
| | 1 | 0.232 | 0.354 | 0.393 | 0.303 | 0.187 | 0.094 | 0.037 | 0.010 | 0.002 | | |
| | 2 | 0.031 | 0.098 | 0.246 | 0.324 | 0.311 | 0.234 | 0.138 | 0.060 | 0.015 | 0.001 | |
| | 3 | 0.002 | 0.015 | 0.082 | 0.185 | 0.276 | 0.312 | 0.276 | 0.185 | 0.082 | 0.015 | 0.002 |
| | 4 | | 0.001 | 0.015 | 0.060 | 0.138 | 0.234 | 0.311 | 0.324 | 0.246 | 0.098 | 0.031 |
| | 5 | | | 0.002 | 0.010 | 0.037 | 0.094 | 0.187 | 0.303 | 0.393 | 0.354 | 0.232 |
| | 6 | | | | 0.001 | 0.004 | 0.016 | 0.047 | 0.118 | 0.262 | 0.531 | 0.735 |
| 7 | 0 | 0.698 | 0.478 | 0.210 | 0.082 | 0.028 | 0.008 | 0.002 | | | | |
| | 1 | 0.257 | 0.372 | 0.367 | 0.247 | 0.131 | 0.055 | 0.017 | 0.004 | | | |
| | 2 | 0.041 | 0.124 | 0.275 | 0.318 | 0.261 | 0.164 | 0.077 | 0.025 | 0.004 | | |
| | 3 | 0.004 | 0.023 | 0.115 | 0.227 | 0.290 | 0.273 | 0.194 | 0.097 | 0.029 | 0.003 | |
| | 4 | | 0.003 | 0.029 | 0.097 | 0.194 | 0.273 | 0.290 | 0.227 | 0.115 | 0.023 | 0.004 |
| | 5 | | | 0.004 | 0.025 | 0.077 | 0.164 | 0.261 | 0.318 | 0.275 | 0.124 | 0.041 |
| | 6 | | | | 0.004 | 0.017 | 0.055 | 0.131 | 0.247 | 0.367 | 0.372 | 0.257 |
| | 7 | | | | | 0.002 | 0.008 | 0.028 | 0.082 | 0.210 | 0.478 | 0.698 |
| 8 | 0 | 0.663 | 0.430 | 0.168 | 0.058 | 0.017 | 0.004 | 0.001 | | | | |
| | 1 | 0.279 | 0.383 | 0.336 | 0.198 | 0.090 | 0.031 | 0.008 | 0.001 | | | |
| | 2 | 0.051 | 0.149 | 0.294 | 0.296 | 0.209 | 0.109 | 0.041 | 0.010 | 0.001 | | |
| | 3 | 0.005 | 0.033 | 0.147 | 0.254 | 0.279 | 0.219 | 0.124 | 0.047 | 0.009 | | |
| | 4 | | 0.005 | 0.046 | 0.136 | 0.232 | 0.273 | 0.232 | 0.136 | 0.046 | 0.005 | |
| | 5 | | | 0.009 | 0.047 | 0.124 | 0.219 | 0.279 | 0.254 | 0.147 | 0.033 | 0.005 |
| | 6 | | | 0.001 | 0.010 | 0.041 | 0.109 | 0.209 | 0.296 | 0.294 | 0.149 | 0.051 |
| | 7 | | | | 0.001 | 0.008 | 0.031 | 0.090 | 0.198 | 0.336 | 0.383 | 0.279 |
| | 8 | | | | | 0.001 | 0.004 | 0.017 | 0.058 | 0.168 | 0.430 | 0.663 |

Ex.6 There is a **0.54** probability that a randomly selected freshman at a two-year college will return the second year. In each case, assume that **5** freshmen at a two-year college are randomly selected and find the probability indicated

- Find the probability that **at least four** of the freshmen **return for the second year**
- Find the probability that **at most two** of the freshmen **return for the second year**
- Find the probability that **more than one** of the freshmen **return for the second year**

**Solution:**

$$P(4) + P(5)$$

$$= {}_5C_4 \cdot 0.54^4 \cdot 0.46^1 + {}_5C_5 \cdot 0.54^5 \cdot 0.46^0$$

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$$P(0) + P(1) + P(2)$$

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$$P(2) + P(3) + P(4) + P(5)$$

Ex.7 The rate of on-time flights for commercial jets are continuously tracked by the U.S. Department of Transportation. Recently, Southwest Air had the best rate with **80% of its flights arriving on time**. A test is conducted by randomly selecting **15 southwest flights** and observing whether they arrive on time.

- Find the probability that **exactly 10 flights** arrive on time
- Find the probability that **at least 10 flights** arrive late

# Chapter 5 (part 4)
# Probability Distributions

# The Multinomial Distribution

A **multinomial experiment** is a probability experiment that satisfies the following four requirements:

1. There must be a fixed number of trials.
2. Each trial has a specific—but not necessarily the same—number of outcomes.
3. The trials are independent.
4. The probability of a particular outcome remains the same.

## Formula for the Multinomial Distribution

If $X$ consists of events $E_1, E_2, E_3, \ldots, E_k$, which have corresponding probabilities $p_1, p_2, p_3, \ldots, p_k$ of occurring, and $X_1$ is the number of times $E_1$ will occur, $X_2$ is the number of times $E_2$ will occur, $X_3$ is the number of times $E_3$ will occur, etc., then the probability that $X$ will occur is

$$P(X) = \frac{n!}{X_1! \cdot X_2! \cdot X_3! \cdots X_k!} \cdot p_1^{X_1} \cdot p_2^{X_2} \cdots p_k^{X_k}$$

where $X_1 + X_2 + X_3 + \cdots + X_k = n$ and $p_1 + p_2 + p_3 + \cdots + p_k = 1$.

**Ex.1** In a large city, 50% of the people choose a movie, 30% choose dinner and a play, and 20% choose shopping as a leisure activity. If a sample of 5 people is randomly selected, find the probability that 3 are planning to go to a movie, 1 to a play, and 1 to a shopping mall.

**Solution:**

$$n = 5, X_1 = 3, X_2 = 1, X_3 = 1, p_1 = 0.50, p_2 = 0.30, \text{ and } p_3 = 0.20.$$

$$P(X) = \frac{5!}{3! \cdot 1! \cdot 1!} \cdot (0.50)^3(0.30)^1(0.20)^1 = 0.15$$

**Ex.2** A box contains 4 white balls, 3 red balls, and 3 blue balls. A ball is selected at random, and its color is written down. It is replaced each time. Find the probability that if 5 balls are selected, 2 are white, 2 are red, and 1 is blue.

**Solution**:

$n = 5, X_1 = 2, X_2 = 2, X_3 = 1; p_1 = \frac{4}{10}, p_2 = \frac{3}{10}$, and $p_3 = \frac{3}{10}$;

$$P(X) = \frac{5!}{2!2!1!} \cdot \left(\frac{4}{10}\right)^2 \left(\frac{3}{10}\right)^2 \left(\frac{3}{10}\right)^1 = \frac{81}{625} = 0.1296$$