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Minimum Variance Control

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62.1 Introduction

Historically, Minimum-Variance Control (MVC) has been a practical control strategy for applying of linear stochastic control theory. The control algorithm was first formulated in [1]. Åström used the MVC technique to minimize the variance of the output signal for control of paper thickness in a paper machine. The control objective was to achieve the lowest possible variation of paper thickness with stochastic disturbances acting on the process. Since then, the MVC technique has attracted many practical applications (e.g. [8], [12]) and has gained significant theoretical development leading to a wide range of control algorithms known as Model-Based Predictive Control techniques.

The reasons for MVC's popularity lie in its simplicity, ease of interpretation and implementation, and its relatively low requirements for model accuracy and complexity. In fact, the models can be obtained from straightforward identification schemes. The MVC technique was also used for self-tuning algorithms. When MVC was developed, its low computational overhead was important. MVC is still used as a simple and efficient algorithm in certain types of control problems [10]. It is particularly well suited for situations where [1]

- the control task is to keep variables close to the operating points,
- the process can be modeled as linear, time invariant but with significant time delay, and
- disturbances can be described by their stochastic characteristics.

MVC, as it was originally presented, provides a stable control action for only a limited range of processes. One of its main assumptions is so-called *"minimum-phase"* behavior of the system. Later, extensions to MVC were introduced in [3], [5], [7] to relax the most restrictive assumptions. The penalty for this extension was increased complexity of the algorithms, eventually approaching or sometimes even exceeding the complexity of the standard *Linear Quadratic Gaussian (LQG)* solution. However, the formulation of MVC was a cornerstone in stimulating development of the more general class of Model-Based Predictive Control, of which MVC is an example.

This article is organized as follows. The model and control design requirements are described in Section 62.2.1. The formulation of an MV predictor is given in Section 62.3. The basic MVC and its variations are developed in Section 62.4. The state-space formulation is given in Section 62.5. An example of the application of MVC is given in Section 62.6. Finally, conclusions are drawn in Section 62.7.

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62.2 Basic Requirements

62.2.1 The Process Model

In its basic form, the minimum-variance control assumes a single-input, single-output (SISO), linear, time-invariant stationary stochastic process described by

$$A\left(z^{-1}\right)\cdot y(t) = z^{-d}\cdot B\left(z^{-1}\right)\cdot u(t) + C\left(z^{-1}\right)\cdot w(t) \quad (62.1)$$

where $y(t)$ represents variation of the output signal around a given steady-state operating value, $u(t)$ is the control signal, and $w(t)$ denotes a disturbance assumed to be a zero mean, Gaussian *white noise* of variance σ. $A\left(z^{-1}\right)$, $B\left(z^{-1}, C\right)\left(z^{-1}\right)$ are nth order polynomials,

$$\begin{aligned} A\left(z^{-1}\right) &= 1 + a_1\cdot z^{-1} + a_2\cdot z^{-2} + \cdots \\ &\quad + a_n\cdot z^{-n}, & (62.2) \\ B\left(z^{-1}\right) &= b_0 + b_1\cdot z^{-1} + b_2\cdot z^{-2} + \cdots \\ &\quad + b_n\cdot z^{-n}, \quad b_0 \neq 0, & (62.3) \\ C\left(z^{-1}\right) &= 1 + c_1\cdot z^{-1} + c_2\cdot z^{-2} + \cdots \\ &\quad + c_n\cdot z^{-n}, & (62.4) \end{aligned}$$

and z^{-1} is the backward shift operator, i.e.,

$$z^{-1}\cdot x(t) = x(t-1). \quad (62.5)$$

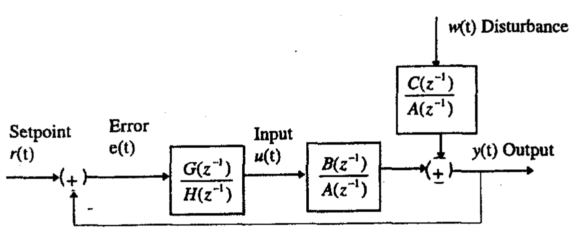

Figure 62.1 The process and the feedback system structure.

Thus z^{-d} in Equation 62.1 represents a d-step delay in the control signal. This means that the control signal starts to act on the system after d time increments. The coefficients of polynomials $A\left(z^{-1}\right)$, $B\left(z^{-1}\right)$, and $C\left(z^{-1}\right)$ are selected so that the linear dynamics of the system are accurately represented about an operating point. The coefficient values may result from considering physical laws governing the process or, more likely, from system identification schemes. Some of these coefficients may be set to zero. In particular, if certain coefficients with index n (e.g., c_n) are set to zero then the polynomials will have different orders. Also, note that the first coefficients associated with zero power of z^{-1} in $A\left(z^{-1}\right)$ and $C\left(z^{-1}\right)$ are unity. This does not impose any restrictions on the system description. For example, if the coefficient in polynomial $A\left(z^{-1}\right)$ is a_0 then both sides of Equation 62.1 may be divided by a_0 and the remaining coefficients adjusted accordingly. Similarly, if the coefficient in polynomial $C\left(z^{-1}\right)$ is c_0 then its value may be included in the disturbance description,

$$w(t) = c_0\cdot w(t). \quad (62.6)$$

The new disturbance is still a zero mean, Gaussian white noise with variance $\sigma' = c_0^2\cdot\sigma$.

62.2.2 The Disturbance Model

Consider a discrete, time-variant system of transfer function $G\left(z^{-1}\right) : y(t) = G\left(z^{-1}\right)w(t)$. Let the input signal be white noise of zero mean and unity variance. The spectral density of the signal may be written as

$$\phi_{yy}(\omega) = G(e^{-j\omega})G(e^{j\omega})\phi_{ww}(\omega) = |G(e^{-j\omega})|^2\phi_{ww}(\omega). \quad (62.7)$$

For the special class of a stationary stochastic process with rational spectral density, the process can be represented by the rational transfer function, $G\left(z^{-1}\right) = C\left(z^{-1}\right)/A\left(z^{-1}\right)$ driven by white noise so that G has all its poles outside the unit circle and zeros outside or on the unit circle. This process is known as spectral factorization and may be used to model the disturbance part of the model given in Equation 62.1. This implies that the output spectrum should first be calculated using the output measurement. The transfer function $G\left(z^{-1}\right)$ is then fitted to the spectrum using a least-squares algorithm.

62.2.3 The Performance Index

The performance index to be minimized is the variance of the output at $t+d$, given all the information up to time t, expressed by

$$J(t) = E\left\{y(t+d)^2\right\}, \quad (62.8)$$

where $E\{\cdot\}$ denotes the *expectation operator*. It is assumed that the control selected must be a function of information available at time t, i.e., all past control signals, all past outputs, and the present output is

$$Y(t) = [u(t-d-1), u(t-d-2), \cdots, y(t), y(t-1)\cdots]. \quad (62.9)$$

Therefore, the feasibility of the solution requires rewriting the performance index Equation 62.8 in the form ,

$$J(t) = E\left\{E\left\{y(t+k)^2 \middle| Y(t)\right\}\right\}, \quad (62.10)$$

where $E\{\cdot|\cdot\}$ is the conditional expectation. Then, the optimization task is choosing control signal $u(t)$ so that $J(t)$ is minimized with respect to $u(t)$,

$$J(t) = E\left\{\min_{u(t)}\left(E\left\{y(t+d)^2 \middle| Y(t)\right\}\right)\right\}. \quad (62.11)$$

62.2.4 The Control Design Specification

From a practical point of view, after applying the MVC it is important that the closed-loop system meets the following requirements :

- The closed-loop system is stable.
- The *steady-state offset* is zero.
- The system is robust, i.e., the behavior of the system does not change with respect to parameter changes.

It will be shown in later sections how the minimum-variance design fulfills these requirements.

62.3 Optimal Prediction

The control objective of the feedback system shown in Figure 62.1 is to compensate for the effect of the disturbance $w(t)$ at any time t. Because of the process time delay, z^{-d}, the control input, $u(t)$, at time t, influences the output, $y(t)$, at time $t+d$. This implies that the value of the output at time $t+d$ is needed to calculate the control input $u(t)$ at time t. Because the output is not known at the future time $t+d$, we predict its value using the process model given in Equation 62.1. Thus, we first develop a predictor in the next section.

62.3.1 The d-Step Ahead Predictor

Our objective is to predict the value of the output at time $t+d$, defined by $\hat{y}(t+d|t)$, given all the information, $Y(t) = [u(t-d-1), u(t-d-2)\cdots y(t), y(t-1), \cdots]$ up to time t. We also want to predict the output so that the variance of the prediction error, $e_p(t+d) = [y(t+d) - \hat{y}(t+d|t)]$, is minimized, i.e., find $\hat{y}(t+d|t)$ so that

$$J(t+d) = E\left[y(t+d) - \hat{y}(t+d|t)\right]^2 \tag{62.12}$$

is minimized subject to the process dynamics given in Equation 62.1.

62.3.2 Predictor Solution

We first calculate the output at time $t+d$ using the process model,

$$y(t+d) = \frac{B}{A}u(t) + \frac{C}{A}w(t+d), \tag{62.13}$$

where the argument z^{-1} is deleted from the polynomials A, B, and C for convenience.

The control input is known up to time $t-1$. The disturbance input can be split into two sets of signals, namely, $[w(t), w(t-1)\cdots,]$ and $[w(t+d-1), w(t+d-2), \cdots, w(t+1)]$. The latter are future disturbances, and so their values are unknown. The former are past disturbances and their values may be calculated from the known past inputs and outputs by

$$w(t) = \frac{A}{C}y(t) - \frac{z^{-d}B}{C}u(t). \tag{62.14}$$

The random variables $[w(t+d-1), w(t+d-2), \ldots]$ are assumed independent of the process output $[y(t), y(t-1), \ldots]$, and all future control inputs are assumed to be zero. We can therefore separate the disturbance signal into casual and noncasual parts [3].

$$y(t+d) = \frac{B}{A}u(t) + \left[E + z^{-d}\frac{F}{A}\right]w(t+d), \tag{62.15}$$

where E and F are polynomials defined as

$$\begin{aligned} E(z^{-1}) &= 1 + \sum_{i=1}^{d} e_i z^{-i}, \\ F(z^{-1}) &= f_0 + \sum_{i=1}^{n-1} f_i z^{-i}, \end{aligned} \tag{62.16}$$

and the following polynomial relationship should be satisfied:

$$A(z^{-1})E(z^{-1}) + z^{-d}F(z^{-1}) = C(z^{-1}). \tag{62.17}$$

Substituting for $w(t)$ in Equation 62.13 using Equation 62.14 gives

$$\begin{aligned} y(t+d) &= \frac{B}{A}u(t) + \left(\frac{F}{A}\right)\left(\frac{A}{C}y(t) - z^{-d}\frac{B}{C}u(t)\right) \\ &\quad + Ew(t+d) \end{aligned} \tag{62.18}$$

which can be simplified to

$$\begin{aligned} y(t+d) &= \frac{B}{A}u(t) + \left(\frac{F}{C}\right)y(t) - \left(z^{-d}\frac{FB}{AC}u(t)\right) \\ &\quad + Ew(t+d). \end{aligned} \tag{62.19}$$

Replacing $(z^{-d}F)$ from Equation 62.17,

$$y(t+d) = \frac{BE}{C}u(t) + \frac{F}{C}y(t) + Ew(t+d). \tag{62.20}$$

Substituting this equation in the minimum prediction cost $J(t+d)$ leads to

$$\begin{aligned} J(t+d) &= E[y(t+d) - \hat{y}(t+d|t)]^2 \\ &= E\left\{\left[Ew(t+d) + \frac{BE}{C}u(t)\frac{F}{C}y(t) - \hat{y}\right]^2\right\}, \\ &= E\left\{\left[\frac{BE}{C}u(t) + \frac{F}{C}y(t) - \hat{y}\right]^2\right\} \\ &\quad + E\left\{[Ew(t+d)]^2\right\} \\ &\quad + 2E\left\{\left[\frac{BE}{C}u(t) + \frac{F}{C}y(t) - \hat{y}\right][Ew(t+d)]\right\} \end{aligned} \tag{62.21}$$

The last term on the right-hand side is zero because $[w(t+d), w(t+d-1), \ldots, w(t+1)]$ are independent of $[y(t), y(t-1), \ldots]$ and $E\{w(t)\} = 0$. The variance of the error is therefore the sum of two positive terms. The second term represents the future disturbance input at $t+d$ and can take any value. To minimize $J(t+d)$, we make the first term zero to find the minimum variance error predictor,

$$\hat{y}(t+d|t) = \frac{BE}{C}u(t) + \frac{F}{C}y(t). \tag{62.22}$$

This can be implemented in time domain as

$$C\hat{y}(t+d|t) = EBu(t) + Fy(t). \qquad (62.23)$$

For the predictor to be stable, the polynomials C and B should have all of their roots outside the unit circle. The implementation of the above optimal predictors requires solving the polynomial Equation 62.16 to find $E(z^{-1})$ and $F(z^{-1})$. There are a number of efficient techniques to solve this equation. A simple solution is to equate the coefficients of different power of z^{-1} and solve the resulting set of linear algebraic equations. The minimum variance of the error can also be calculated as

$$\begin{aligned} J(t+d) &= E[y(t+d) - \hat{y}]^2 \\ &\geq 1 + \sum_{i=1}^{k-1} e_i^2. \end{aligned} \qquad (62.24)$$

62.4 Minimum-Variance Control

Given the dynamical process model in Equation 62.1, the minimum-variance control problem can be stated as finding a control law which minimizes the variance of the output described in the performance index of Equation 62.11.

62.4.1 Minimum-Variance Control Law

Assuming that all of the outputs $[y(t), y(t-1), \ldots]$ and the past control inputs $[u(t-1), u(t-2), \ldots]$ are known at time t, the problem is to determine $u(t)$ so that the variance of the output is as small as possible. The control signal $u(t)$ will affect only $y(t+d)$ but not any earlier outputs. Use Equation 62.17 for the output,

$$y(t+d) = \frac{EB}{C}u(t) + \frac{F}{C}y(t) + Ew(t+d). \qquad (62.25)$$

Substitute this equation in the minimum variance performance index to obtain

$$\begin{aligned} E\left\{y^2(t+d)|Y(t)\right\} &= E\left\{\left[\frac{EB}{C}u(t) + \frac{F}{C}y(t)\right]^2\right\} \\ &+ E\left\{[Ew(t+d)]^2\right\} \end{aligned} \qquad (62.26)$$

The expected value of the cross-product term is zero because $[w(t+d), w(t+d-1), \ldots, w(t+1)]$ are assumed to be independent of $\{y(t), y(t-1), \ldots\}$. The performance index is minimum if the first term is zero:

$$\tfrac{BE}{C}u(t) + \tfrac{F}{C}y(t) = 0. \qquad (62.27)$$

The minimum variance control law is then given by

$$u(t) = \tfrac{G}{H}y(t) = -\tfrac{F}{BE}y(t). \qquad (62.28)$$

Note that the controller is stable only if $B(z^{-1})$ has all of its roots outside the unit circle. This implies that the process should be minimum phase for the closed-loop system to be stable.

62.4.2 Closed-Loop Stability

The closed-loop characteristic equation for the system of Figure 62.1 may be written as

$$HA + z^{-d}BG = 0 \qquad (62.29)$$

Replacing for G and H from Equation 62.27 and using the polynomial identity Equation 62.17 gives

$$CB = 0. \qquad (62.30)$$

The closed loop is asymptotically stable if the polynomials C and B have all of their roots outside the unit circle. This implies that the basic version of MVC is applicable to minimum-phase systems with disturbances which have a stable and rational power spectrum.

62.4.3 The Tracking Problem

The basic MVC regulates the variance of the output about the operating point. It is often necessary in process control to change the level of the operating point while the system is under closed-loop control. The control will then be required to minimize the variance of the error between the desired process set point and the actual output. Assuming the set point , $r(t+d)$, is known, the error at time $t+d$ is defined as

$$e(t+d) = r(t+d) - y(t+d). \qquad (62.31)$$

The performance index to be minimized may now be written

$$J(t) = E\left\{\min_{u(t)}\left(E\left\{e(t+d)^2 \middle| Y(t)\right\}\right)\right\} \qquad (62.32)$$

Following the same procedure as in the previous section, the control law may be written

$$u(t) = -\frac{F}{BE}y(t) + \frac{C}{BE}r(t+d) \qquad (62.33)$$

The feedback controller is similar to the basic MVC and hence the stability properties are the same. The set point is introduced directly in the controller and variance of the error rather than the variance of the output signal is minimized.

62.4.4 The Weighted Minimum Variance

The performance index given in Equation 62.10 does not penalize the control input $u(t)$. This can lead to excessive input changes and hence actuator saturation. Moreover, the original form of MVC developed by Åström [1] stabilizes only the minimum-phase systems. Clarke and Hastings-James [7] proposed that a weighting, R, of the manipulated variable be included in the performance index,

$$J(t) = E\left\{\min_{u(t)}\left(E\left\{e(t+d)^2 \mid Y(t) + Ru^2(t)\right\}\right)\right\}. \qquad (62.34)$$

Minimizing this performance index with respect to the control input and subject to the process dynamic will lead to the weighted MV control law,

$$u(t) = -\frac{F}{BE + \frac{R}{b_0}C} y(t) + \frac{C}{BE + \frac{R}{b_0}C} r(k+d). \quad (62.35)$$

The closed-loop characteristic equation for the weighted minimum variance is calculated from Equation 62.24 as

$$C\left[B + \frac{R}{b_0}A\right] = 0. \quad (62.36)$$

The closed-loop system is now asymptotically stable only if the roots of C are outside the unit circle. The system can now be nonminimum phase if an appropriate value of R is chosen to move the unstable zero outside the unit circle. The disadvantage is that the variance of error is not directly minimized. It should be pointed out that MVC without the control input weighting can also be designed for nonminimum-phase systems as shown by Åström and Wittenmark [3].

62.4.5 The Generalized Minimum Variance Control

Clarke and Gawthrop [5], [6] developed the Generalized Minimum-Variance Controller (GMVC) for self-tuning control application by introducing the reference signal and auxiliary variables (weighting functions) into the performance index. GMVC minimizes the variance of an auxiliary output of the form,

$$\begin{aligned} \phi(t+d) &= Q(z^{-1})y(t+d) \\ &\quad + R(z^{-1})u(t) - P(z^{-1})r(t+d), \end{aligned} \quad (62.37)$$

where $Q(z^{-1}) = Q_n(z^{-1})/Q_d(z^{-1})$, $R(z^{-1}) = R_n(z^{-1})/R_d(z^{-1})$, $P(z^{-1}) = P_n(z^{-1})/P_d(z^{-1})$ are stable weighting functions. The performance index minimized subject to the process dynamics is

$$J(t) = E[\phi^2(t+d)|t]. \quad (62.38)$$

The signals $u(t)$ and $r(t+d)$ are known at time t. The prediction of $\phi(t+d|t)$ will reduce to the prediction of $Qy(t+d)$. Multiplying the process model by Q,

$$Q_dA \cdot [Qy(t)] = z^{-d} \cdot Q_nBu(t) + Q_nCw(t). \quad (62.39)$$

Replacing A, B, and C by Q_dA, Q_nB, and Q_nC in Equation 62.23 leads to the predictor equation for GMVC,

$$Q_dC[Q\hat{y}(t+d|t)] = Q_dEBu(t) + Fy(t), \quad (62.40)$$

and the polynomial identity,

$$\begin{aligned} &Q_d(z^{-1})A(z^{-1}) + z^{-d}F(z^{-1}) \\ &= Q_n(z^{-1})C(z^{-1}). \end{aligned} \quad (62.41)$$

Adding the contribution from $u(t)$ and $r(t)$ to Equation 62.41, the predictor for the auxiliary output $\phi(t)$ is

$$\phi(t+d/t) = \frac{F}{Q_dC}y(t) + \left(\frac{EB}{C} + R\right)u(t) - Pr(t+d). \quad (62.42)$$

The control law is chosen so that the d-step ahead prediction $\phi(t+d|t)$ is zero:

$$u(t) = \frac{R_dCPr(t) - R_dFy(t)}{Q_d(R_dEB + CR_n)}. \quad (62.43)$$

Note that by setting $P_d = Q_d = 1$ and $R = P_n = 0$, the basic MVC is obtained. Q_d may be used to influence the magnitude of control input signal. In particular, choosing $Q_d = (1-z^{-1})Q_d$ introduces integral action in the controller. The weighting $R(z^{-1})$ adjusts the speed of response of the controller and hence prevents actuator saturation.

The closed-loop transfer function, derived by substituting for $u(t)$ in Equation 62.1 is

$$y(t) = \frac{EB + RC}{QB + RA}w(t) + \frac{z^{-d}BR}{QB + RA}r(t). \quad (62.44)$$

Solving the characteristic equation,

$$R_dQ_nB + R_nQ_dA = 0 \quad (62.45)$$

gives the closed loop poles. The weightings Q and R may be selected so that the poles are located in a desired position on the complex plane.

62.5 State-Space Minimum-Variance Controller

The state-space description of a linear, time-invariant discrete-time system has the form,

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + Gw(t); \\ y(t) &= Cx(t) + v(t), \quad x(t_0) = x_0, \end{aligned} \quad (62.46)$$

where $x(t)$, $u(t)$, $y(t)$ are state, input, and output vectors of size n, m, and r, respectively, and A, B, and C are system matrices of appropriate dimensions. Also $[w(t)]$ and $[v(t)]$ are white noise sequences. The states and outputs of the system may be estimated using a Kalman filter as discussed in other chapters of the Handbook,

$$\begin{aligned} \hat{x}(t+1) &= A\hat{x}(t) + Bu(t) + K(t)e(t), \\ \hat{y}(t) &= C\hat{x}(t), \\ e(t) &= [y(t) - C\hat{x}(t)], \end{aligned} \quad (62.47)$$

where $e(t)$ is the innovation signal with property, $E[e(t)|t] = 0$ and $K(t)$ is the Kalman filter gain.

The optimal d-step ahead predictor of $[\hat{x}(t)]$ is the conditional mean of $x(t+d)$. From Equation 62.47,

$$\begin{aligned} \hat{y}(t+d|t) &= CA^{d-1}\hat{x}(t+1) \\ &\quad + \sum_{j=t+1}^{t+d-1} CA^{t+d-j-1}B(j)u(j) \\ &\quad + \sum_{j=t+1}^{t+d-1} CA^{t+d-j-1}K(j)e(j) \end{aligned} \quad (62.48)$$

Taking the conditional expectation and using the property of innovation signal,

$$\begin{aligned}\hat{y}(t+d|t) &= CA^{d-1}\hat{x}(t+1) \qquad (62.49)\\ &= CA^{d-1}\{[A-K(t)C]\,\hat{x}(t)\\ &\quad + CBu(t) + K(t)y(t)\}.\end{aligned}$$

For the performance index given by Equation 62.11, the optimum control input will be achieved if the d-step prediction of output signal is zero. Therefore, the state-space version of minimum-variance controller takes the form,

$$u(t) = \left[CA^{d-1}B\right]^{-1} CA^{d-1}\{[A-K(t)C]\,\hat{x}(t) + K(t)y(t)\}. \qquad (62.50)$$

It can be shown [4], [13] that this controller is equivalent to the minimum-variance regulator described by Equation 62.28.

The multivariable formulation of the MVC algorithm in state-space form is similar to the one described above. The polynomial MVC for multi-input, multioutput is given in [9].

62.6 Example

As a simple example illustrating some properties of minimum-variance control, consider a plant described by the following input-output relationship:

$$\begin{aligned}y(t) &= \frac{z^{-2}(1.0-0.995z^{-1})}{1-1.81z^{-1}+0.819z^{-2}}u(t)\\ &\quad + \frac{1-1.8z^{-1}+0.85z^{-2}}{1-1.81z^{-1}+0.819z^{-2}}w(t).\end{aligned}$$

Note that the delay of control signal is 2. The open-loop response of the system is shown in Figure 62.2 where the variance of the output is 2.8. To illustrate the performance of the predictor, the polynomial identity is solved to give $E(z^{-1}) = (1-0.01z^{-1})$ and $F(z^{-1}) = (0.049-0.0082z^{-1})$. A sine wave, $u(t) = \sin(0.2t)$, is applied to the input and the prediction of the output is shown in Figure 62.3. The theoretical minimum prediction error variance of 1 should be obtained but the simulated error is calculated as 1.49.

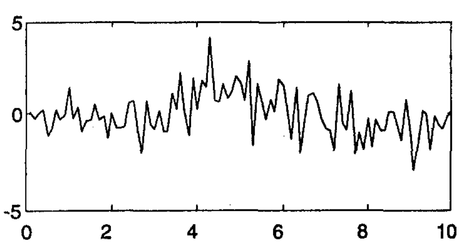

Figure 62.2 The open-loop response.

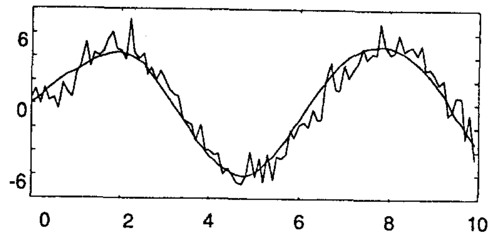

Figure 62.3 The prediction of a sine wave.

The basic MV controller Equation 62.28 is then applied to the system. The input and output are shown in Figure 62.4. The variance is now reduced to 1.26.

To examine the set point tracking of the basic MVC, a reference of magnitude 10 is used.

The results are shown in Figure 62.5. The tracking error is small, but the magnitude of the control signal is large and this is not realistic in many practical applications. To reduce the size of the control input, WMV may be used.

The result of introducing a weighting of $R = 0.8$ is shown in Figure 62.6. The set point is changed to 20 to illustrate the effect more clearly. The WMVC will reduce the control activity but it will also increase the tracking error. To keep the control input within a realistic range and to obtain zero tracking error, the GMVC may be used.

Selecting $Q_d = (1-z^{-1})$ will introduce integral action in the controller. The solution to the identity polynomial may then be obtained as $E(z^{-1}) = (1-0.01z^{-1})$ and $F(z^{-1}) = (1.06 - 1.84z^{-1} + 0.083z^{-2})$. Note that the order of the controller has increased by one due to integral action. The response of the system in Figure 62.7 shows the reduction in the tracking error and the control activity.

If the plant description changes slightly,

$$\begin{aligned}y'(t) &= \frac{z^{-2}(1.0-1.001z^{-1})}{1-1.81z^{-1}+0.819z^{-2}}u(t)\\ &\quad + \frac{1-1.8z^{-1}+0.85z^{-2}}{1-1.81z^{-1}+0.819z^{-2}}w(t).\end{aligned}$$

The basic MVC produces an unstable closed-loop system due to an unstable zero. The GMVC will however stabilize the system as shown in Figure 62.8.

62.7 Defining Terms

62.7.1 Expectation Operator

If $g(\zeta)$ is a function of a random variable ζ described by the probability density $\varphi(\zeta)$, the expectation operator of g is defined

$$E[g(\zeta)] = \int_{-\infty}^{\infty}[g(\zeta)\varphi(\zeta)]d\zeta.$$

The conditional expectation will be obtained if the probability density function $\varphi(\zeta)$ is replaced by conditional probability

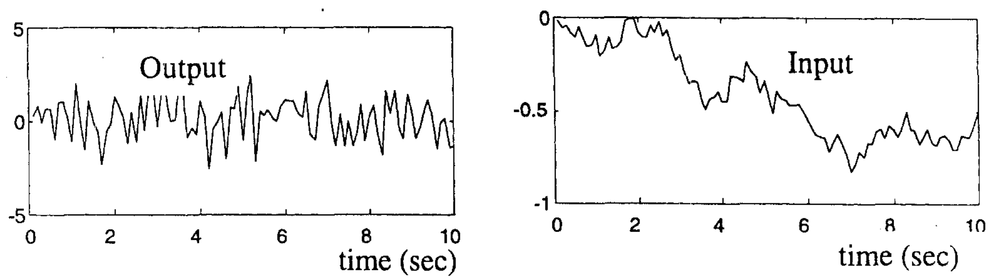

Figure 62.4 Basic MVC. (a) (output), (b) (input).

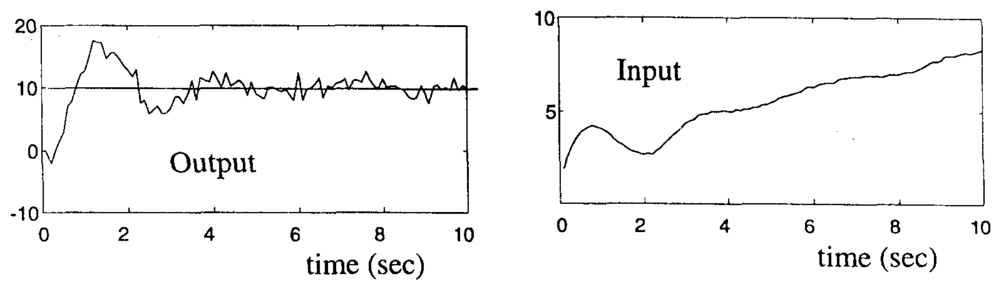

Figure 62.5 Basic MVC with set point.

$\varphi(\zeta\,|\eta)$, where η is another random variable,

$$E\left[g(\zeta\,|\eta)\right] = \int_{-\infty}^{\infty} \left[g(\zeta)\varphi(\zeta\,|\eta)\right] d\zeta .$$

The conditional probability can be obtained from Bayes equation,

$$\varphi(\zeta\,|\eta) = \frac{\varphi(\zeta, \eta)}{\varphi(\eta)} .$$

62.7.2 Linear Quadratic Gaussian (LQG)

A dynamic optimization problem can be described by a linear model of the system (either continuous-time or discrete-time) with additive disturbances which are stochastic processes with Gaussian probability density functions. The performance index to be minimized is a finite- or infinite-horizon sum involving quadratic terms on the state of the system and/or on the control.

62.7.3 Minimum Phase

A system is minimum phase if the open-loop transfer function does not include right half-plane zeros (in the continuous-time case). In the discrete-time case this corresponds to all zeros lying outside the unit circle. Practically, minimum-phase behavior of the system implies that the step response of the system starts in a "right direction". The right-hand side limit of the time derivative of the system's step response has the same sign as the steady-state gain.

62.7.4 Steady-State Offset

The difference between the set point (desired) value of the output and the actual steady-state value of the output is called steady-state offset. It is desirable to reduce the steady-state offset (to zero, if possible). It is well-known that the systems with integral action have zero steady-state offset.

62.7.5 White Noise

A stochastic process $v(t)$ is called white noise if its realizations in time are uncorrelated, i.e., the autocorrellation function, $E[v(t)v(t+\delta)]$ has nonzero value only for $\delta = 0$. The process is called Gaussian white noise if, in addition to the above assumption, the probability density function is Gaussian at each instant.

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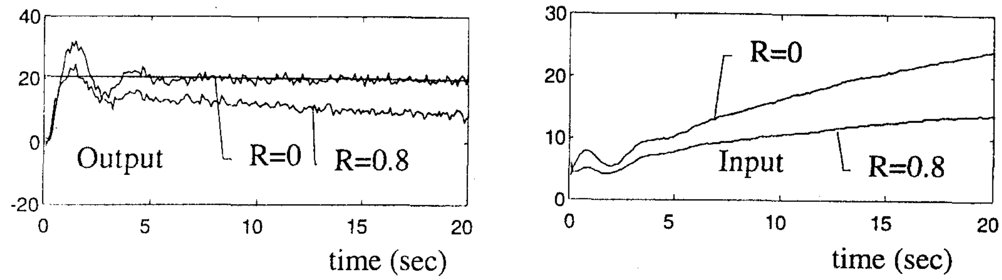

Figure 62.6 WMVC with set point.

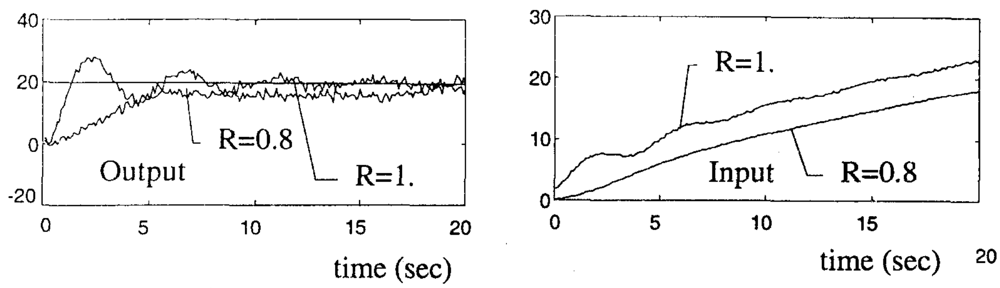

Figure 62.7 GMVC.

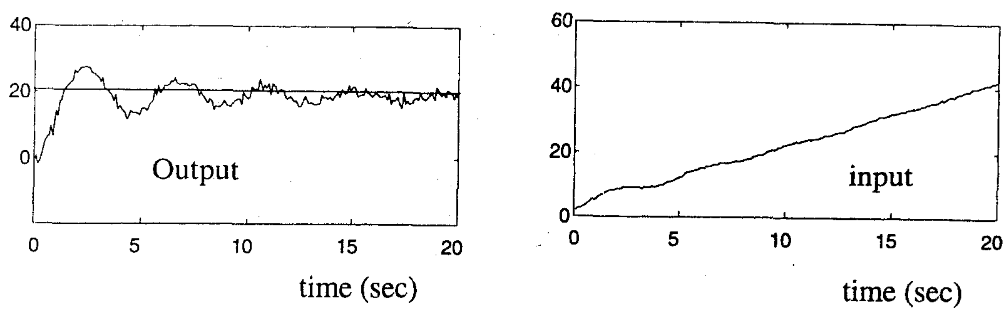

Figure 62.8 Nonminimum-phase system.

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Further Reading

The standard minimum-variance controller was introduced in [1] where a detailed derivation can be found. The weighted minimum variance and the generalized minimum variance are discussed in [7] and [6], respectively. More general discussion of MVC and its relations to the LQG problem can be found in excellent books [2] and [9]. Also the book [11] surveys all minimum-variance control techniques.

63

Dynamic Programming

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63.1 Introduction

Dynamic programming is a recursive method for obtaining the optimal control as a function of the state in multistage systems. The procedure first determines the optimal control when there is only one stage left in the life of the system. Then it determines the optimal control where there are two stages left, etc. The recursion proceeds backward in time.

This procedure can be generalized to continuous-time systems, stochastic systems, and infinite horizon control. The cost criterion can be a total cost over several stages, a discounted sum of costs, or the average cost over an infinite horizon.

A simple example illustrates the main idea.

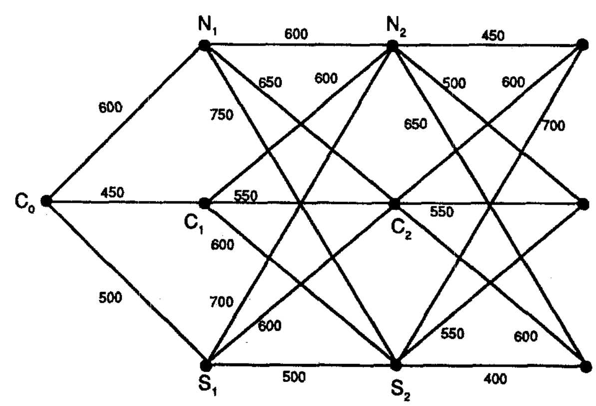

Figure 63.1 A shortest path problem.

63.1.1 Example: The Shortest Path Problem

A bicyclist wishes to determine the shortest path from Bombay to the Indian East Coast. The journey can end at any one of the cities N_3, C_3, or S_3. The journey is to be done in 3 stages.

Stage zero is the starting stage. The bicyclist is in Bombay, labelled C_0 in Figure 63.1. Three stages of travel remain. The bicyclist has to decide whether to go north, center, or south. If the bicyclist goes north, she reaches the city N_1, after travelling 600 kms. If the bicyclist goes to the center, she reaches C_1, after travelling 450 kms. If she goes south, she reaches S_1 after travelling 500 kms.

At stage one, she will therefore be in one of the cities N_1, C_1, or S_1. From wherever she is, she has to decide which city from among N_2, C_2, or S_2 she will travel to next. The distances between cities N_1, C_1, and S_1 and N_2, C_2, S_2 are shown in Figure 63.1. At stage two, she will be in one of the cities N_2, C_2, or S_2, and she will then have to decide which of the cities N_3, C_3, or S_3 to travel to. The journey ends in stage three with the bicyclist in one of the cities N_3, C_3 or S_3.

63.1.2 The Dynamic Programming Method

We first determine the optimal decision when there is only *one* stage remaining.

If the bicyclist is in N_2, she has three choices—north, center, or south, leading respectively to N_3, C_3 or S_3. The corresponding distances are 450, 500 and 650 kms. The best choice is to go

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