In the name of God

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Pattern Recognition

Project1

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Contents

Abstract

In this exercise the Cauchy function is considered as a normal distribution function for describing data of 2 classes. It is assumed that one feature is extracted from the data so, there is a one-dimensional classification problem.

One of the important issues for classification is the decision boundary of classes that must be calculated. In this problem the decision boundaries are one-dimensional points. In this exercise it is checked that under what conditions the decision boundary is in the middle of the distance between the two means. Also, the minimum probability of error is obtained parametrically and it is determined that the maximum value of this error probability is under what conditions. After that, the Bayes minimum error classifier is designed. In this case, one decision boundary is obtained for the classification. In the next case, the Bayes minimum risk classifier for a given lost function is designed. In this case, because of the penalties considered for the incorrect choice of classes, the decision boundary is different from the previous case. In this case, 2 decision boundaries are obtained. The different regions for the classes in two cases results in different probability of error. At the end of the problem, the results of error probabilities from the 2 cases are compared.

Conclusions of each section are written at the end of that section. The execution of MATLAB codes is given in the Appendix section.

Problem1

Consider Cauchy distributions in a two-class one-dimensional classification problem:

$$P(x|w_i) = \frac{1}{\pi b}\frac{1}{1+(\frac{x-a_i}{b})^2} \quad \text{i} = 1{,}2 \qquad a_2 > a_1$$

a) By explicit integration, show that the distributions are normalized.

$$\int_{-\infty}^{+\infty} p(x|\omega_i)\,dx = \int_{-\infty}^{+\infty} \frac{1}{\pi b}\frac{1}{1+(\frac{x-a_i}{b})^2}\,dx$$

change of variable : $\frac{x-a_i}{b} = Y \Rightarrow \frac{1}{b}dx = dY \Rightarrow$

$$\int_{-\infty}^{+\infty} \frac{1}{\pi \cancel{b}}\frac{1}{1+Y^2}\cancel{b}\,dY = \frac{1}{\pi} Arc\tan Y \Big[_{-\infty}^{+\infty} = \frac{1}{\pi} Arc\tan\left(\frac{x-a_i}{b}\right)\Big[_{-\infty}^{+\infty}$$

$$\begin{cases} Arc\tan(+\infty) = \frac{\pi}{2} \\ Arc\tan(-\infty) = -\frac{\pi}{2} \end{cases}$$

$$\Rightarrow \int_{-\infty}^{+\infty} p(x|\omega_i)\,dx = \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1$$

As shown in above, by integration from $x = -\infty$ to $x = +\infty$ on the conditional probability distribution of x given ω_i, we get to the number 1, which means that the area below the distribution function is one and is equivalent to the normality of the distribution function.

b) Assume $P(\omega_1) = P(\omega_2)$, show that $P(\omega_1|x) = P(\omega_2|x)$ if x=$(a_2 + a_1)$/2.

$P(\omega_1) = P(\omega_2)$

we want the two probabilities $p(\omega_1|x)$ and $p(\omega_2|x)$ to be equal.

$$p(\omega_1|x) = \frac{p(x|\omega_1)\,p(\omega_1)}{p(x)} \quad , \quad p(\omega_2|x) = \frac{p(x|\omega_2)\,p(\omega_2)}{p(x)}$$

$$\Rightarrow \quad p(\omega_1|x) = p(\omega_2|x) \Rightarrow \frac{p(x|\omega_1)\,\cancel{p(\omega_1)}}{\cancel{p(x)}} = \frac{p(x|\omega_2)\,\cancel{p(\omega_2)}}{\cancel{p(x)}} \Rightarrow$$

$$p(x|\omega_1) = p(x|\omega_2) \quad (*)$$

$$p(x|\omega_1) = \frac{1}{\pi b}\,\frac{1}{1+\left(\frac{x-a_1}{b}\right)^2} \quad , \quad p(x|\omega_2) = \frac{1}{\pi b}\,\frac{1}{1+\left(\frac{x-a_2}{b}\right)^2}$$

$$\overset{(*)}{\Rightarrow} \quad \cancel{\frac{1}{\pi b}}\,\frac{1}{1+\left(\frac{x-a_1}{b}\right)^2} = \cancel{\frac{1}{\pi b}}\,\frac{1}{1+\left(\frac{x-a_2}{b}\right)^2} \Rightarrow$$

$$\cancel{1}+\left(\frac{x-a_2}{b}\right)^2 = \cancel{1}+\left(\frac{x-a_1}{b}\right)^2 \Rightarrow \frac{(x-a_2)^2}{\cancel{b^2}} = \frac{(x-a_1)^2}{\cancel{b^2}} \Rightarrow$$

$$|x-a_2| = |x-a_1| \Rightarrow x-a_1 = \pm(x-a_2) \Rightarrow x-a_1 = \begin{cases} x-a_2 & \times \\ -x+a_2 & \checkmark \end{cases}$$

$$\Rightarrow \quad x-a_1 = -x+a_2 \Rightarrow 2x = a_2+a_1 \Rightarrow \boxed{x^* = \frac{a_2+a_1}{2}}$$

The decision boundary in the case of $P(\omega_1) = P(\omega_2)$ is in the middle of the means because of the equal conditions for two classes. Because of the one-dimension of the feature, the decision boundary is one-dimension.

Plot $P(\omega_1|x)$ and $P(\omega_2|x)$ on one axis for the case a_1 = 3, a_2 = 5 and b=1.

$$p(\omega_i|x) = \frac{p(x|\omega_i)\,p(\omega_i)}{p(x)}$$

if we have 2 categories:

$$p(x) = \sum_{j=1}^{2} p(x|\omega_j)\,p(\omega_j) \Rightarrow$$

$$p(\omega_1|x) = \frac{p(x|\omega_1)\,p(\omega_1)}{p(x|\omega_1)\,p(\omega_1) + p(x|\omega_2)\,p(\omega_2)}$$

$$p(\omega_2|x) = \frac{p(x|\omega_2)\,p(\omega_2)}{p(x|\omega_1)\,p(\omega_1) + p(x|x_2)\,p(\omega_2)}$$

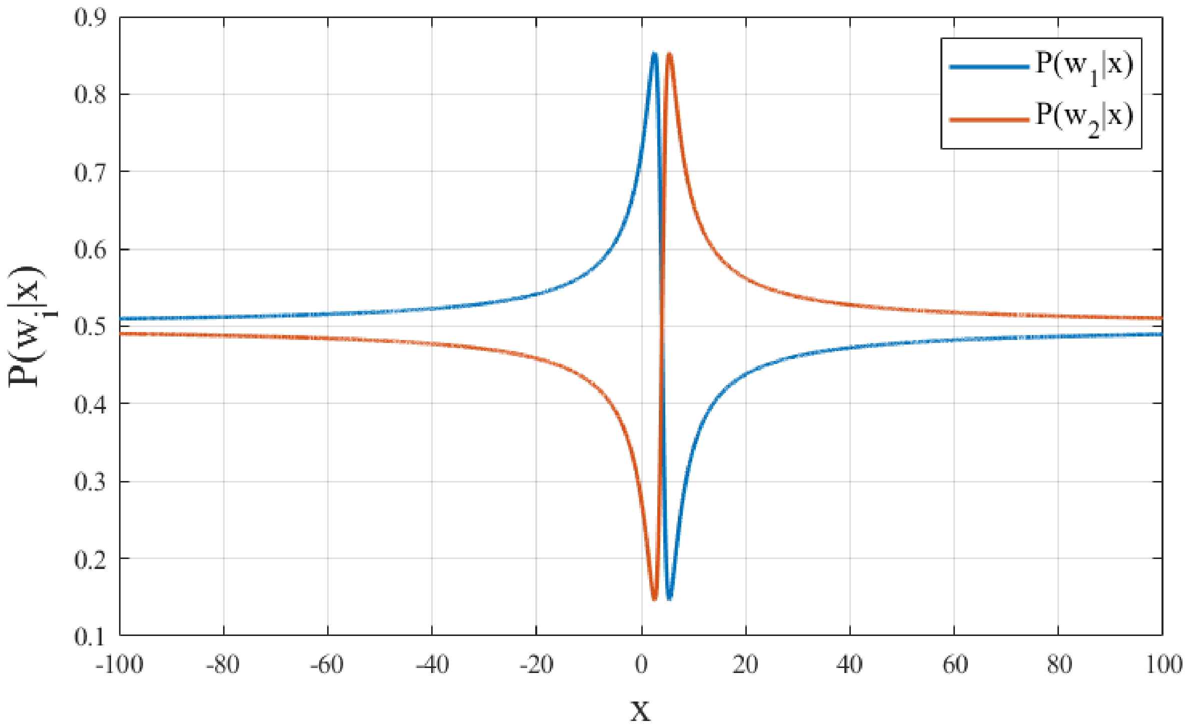


Figure 1. Plot of $P(\omega_1|x)$ and $P(\omega_2|x)$ on one axis for the case a_1 = 3, a_2 = 5 and b=1.

c) Show that the minimum probability of error is given by:

$$P(\text{error}) = \frac{1}{2} - \frac{1}{\pi}\tan^{-1}\left|\frac{a_2 - a_1}{2b}\right|$$

$$\left.\begin{aligned} p(error) &= \int_{-\infty}^{+\infty} p(error|x)\,dx \\ p(error|x) &= \min\left[p(\omega_1|x), p(\omega_2|x)\right] \end{aligned}\right\} \Rightarrow$$

$$p(error) = \int_{R} p(\omega_2|x)\,dx + \int_{R_1} p(\omega_1|x)\,dx = \int_{-\infty}^{x^*} \frac{p(x|\omega_2)\,p(\omega_1)}{p(x)}\,dx + \int_{x^*}^{\infty} \frac{p(x|\omega_1)\,p(\omega_2)}{p(x)}\,dx \Rightarrow$$

$$p(error) = p(\omega_2)\int_{-\infty}^{x^*} p(x|\omega_2)\,dx + p(\omega_1)\int_{x^*}^{\infty} p(x|\omega_1)\,dx \Rightarrow$$

$$p(error) = p(\omega_2)\left[\frac{1}{\pi}\operatorname{Arctan}\left(\frac{x-a_2}{b}\right)\Big|_{-\infty}^{x^*}\right] + p(\omega_1)\left[\frac{1}{\pi}\operatorname{Arctan}\left(\frac{x-a_1}{b}\right)\Big|_{x^*}^{\infty}\right] =$$

$$\frac{p(\omega_2)}{\pi}\left[\operatorname{Arctan}\left(\frac{x^*-a_2}{b}\right) - \left(-\frac{\pi}{2}\right)\right] + \frac{p(\omega_1)}{\pi}\left[\frac{\pi}{2} - \operatorname{Arctan}\left(\frac{x^*-a_1}{b}\right)\right]$$

Minimum value of P(error) is occurred when $P(\omega_1) = P(\omega_2)$. So,

$$\text{if } p(\omega_1) = p(\omega_2) \Rightarrow x^* = \frac{a_1+a_2}{2}$$

$$p(error) = \frac{1}{4} + \frac{1}{2\pi}\operatorname{Arctan}\left(\frac{\frac{a_1+a_2}{2} - a_2}{b}\right) + \frac{1}{4} - \frac{1}{2\pi}\operatorname{Arctan}\left(\frac{\frac{a_1+a_2}{2} - a_1}{b}\right)$$

$$= \frac{1}{2} + \frac{1}{2\pi}\left[\operatorname{Arctan}\left(\frac{a_1-a_2}{2b}\right) - \operatorname{Arctan}\left(\frac{a_2-a_1}{2b}\right)\right]$$

$$= \frac{1}{2} - \frac{1}{2\pi}\left[\operatorname{Arctan}\left(\frac{a_2-a_1}{2b}\right) + \operatorname{Arctan}\left(\frac{a_2-a_1}{2b}\right)\right]$$

$$= \frac{1}{2} - \frac{1}{\pi}\operatorname{Arctan}\left|\frac{a_2-a_1}{2b}\right|$$

d) What is the maximum value of P(error) and under which conditions can this occur?

$$P(error) = \frac{1}{2} - \frac{1}{\pi}\tan^{-1}\left|\frac{a_2-a_1}{2b}\right|$$

$$Max\left\{P(error)\right\} \equiv \min\left\{\frac{1}{\pi}\tan^{-1}\left|\frac{a_2-a_1}{2b}\right|\right\}$$

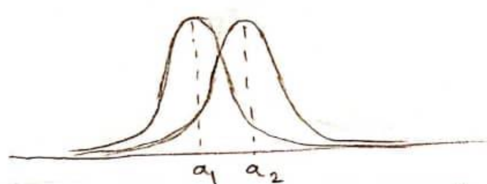
if $\left|\frac{a_2-a_1}{2b}\right| \longrightarrow 0 \Rightarrow \tan^{-1}\left|\frac{a_2-a_1}{2b}\right| \longrightarrow 0$

Maximum value is occurred when:

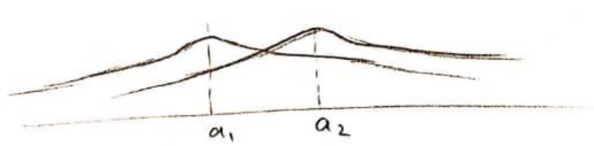
1- a_2 and a_1 are so close to each other

2- b is so big

part 1 means that the mean values of two probabilities are close to each other;



part 2 means that the standard deviation of two probabilities are so big:



The more overlapping the two probabilities, the greater the error.

e) Design the Bayes minimum error classifier in terms of a_i and b if $P(\omega_1)$ =$P(\omega_2)$. Plot the decision boundaries in this case.

$P(\omega_1) = P(\omega_2)$

$P(\text{error}|x) = \min\left[P(\omega_1|x), P(\omega_2|x)\right] = \min\left[p(x|\omega_1)P(\omega_1), p(x|\omega_2)P(\omega_2)\right]$

$\xrightarrow{P(\omega_1)=P(\omega_2)} \quad P(\text{error}|x) = \min\left[p(x|\omega_1), p(x|\omega_2)\right]$

if $p(x|\omega_1) > p(x|\omega_2) \Rightarrow$

$$\cancel{\frac{1}{\pi b}} \frac{1}{1+\left(\frac{x-a_1}{b}\right)^2} > \frac{1}{\cancel{\pi b}\left(1+\left(\frac{x-a_2}{b}\right)^2\right)} \Rightarrow$$

$$\cancel{1} + \left(\frac{x-a_2}{b}\right)^2 > \cancel{1} + \left(\frac{x-a_1}{b}\right)^2 \Rightarrow (x-a_2)^2 > (x-a_1)^2 \Rightarrow$$

$$x - a_2 < x + a_1 \Rightarrow x < \boxed{\frac{a_1+a_2}{2}} \rightarrow \text{decision boundary}$$

if $x > \frac{a_1+a_2}{2} \rightarrow \omega_2 \Rightarrow$ category 2

if $x < \frac{a_1+a_2}{2} \rightarrow \omega_1 \Rightarrow$ category 1

In this case, the decision boundary is a one dimension point at $x = 4$.
If x<4 decision is the choice of category 1. If x>4 decision is the choice of category 2.

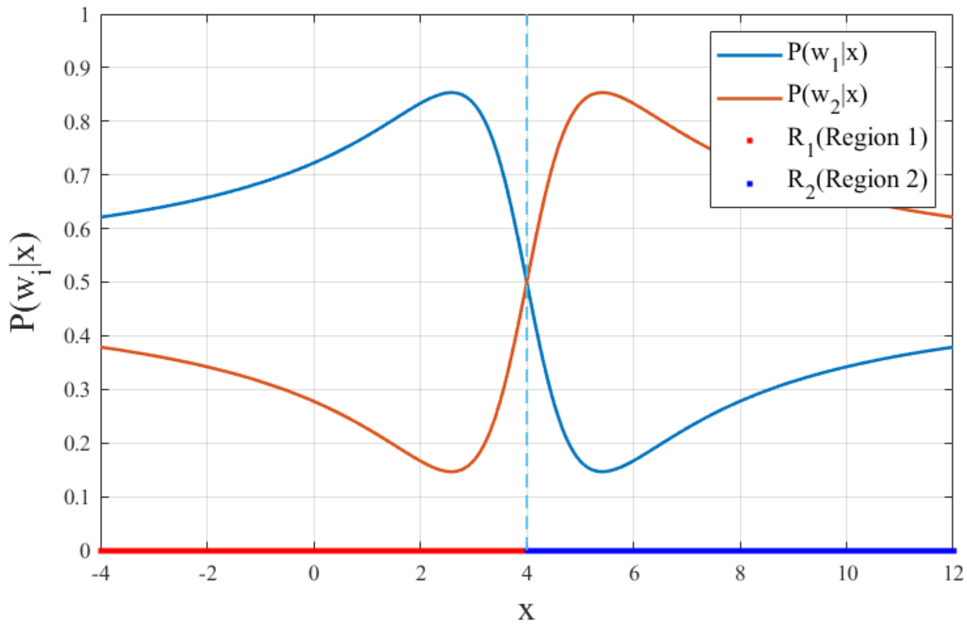


Figure 2. The decision boundary of Bayes minimum error classifier

Fig. 2 shows that $P(\omega_1|x)$ and $P(\omega_2|x)$ on one axis. It is easily visible that if $P(\omega_1)$ = $P(\omega_2)$ and b is the same, the decision boundary of Bayes minimum error classifier is in the middle of the probabilities.

What is the probability of error?

$$P(\omega_1)=P(\omega_2) \Longrightarrow P(\omega_1)=\frac{1}{2}, \quad P(\omega_2)=\frac{1}{2}$$

$$P(\text{error}) = \int_{-\infty}^{\frac{a_1+a_2}{2}} p(x|\omega_2)\times\frac{1}{2}\,dx + \int_{\frac{a_1+a_2}{2}}^{+\infty} p(x|\omega_1)\times\frac{1}{2}\,dx$$

$$= \frac{1}{2\pi}\,\mathrm{Arctan}\left(\frac{x-a_2}{b}\right)\Big[_{-\infty}^{\frac{a_1+a_2}{2}} + \frac{1}{2\pi}\,\mathrm{Arctan}\left(\frac{x-a_1}{b}\right)\Big[_{\frac{a_2+a_1}{2}}^{\infty}$$

$$\frac{1}{2\pi}\left(\mathrm{Arctan}\left(\frac{a_1-a_2}{2b}\right)+\frac{\pi}{2}\right)+\frac{1}{2\pi}\left(\frac{\pi}{2}-\mathrm{Arctan}\left(\frac{a_2-a_1}{2b}\right)\right)=$$

$$\frac{1}{2}-\frac{1}{\pi}\mathrm{Arctan}\left(\frac{a_2-a_1}{2b}\right)=\frac{1}{2}-\frac{1}{\pi}\mathrm{Arctan}\left(\frac{5-3}{2}\right)$$

$$=\frac{1}{2}-\frac{\pi}{4\pi}=0.25$$

The minimum probability of error is obtained for this section.

f) Design the Bayes minimum risk classifier with the following error weights

$$\begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

Plot the decision boundaries in this case.

Two decision boundaries are x=4.3542 and x=9.6458.

$$\theta_1 = \frac{\lambda_{12} - \lambda_{22}}{\lambda_{21} - \lambda_{11}} \frac{P(\omega_2)}{P(\omega_1)} = 0.5$$

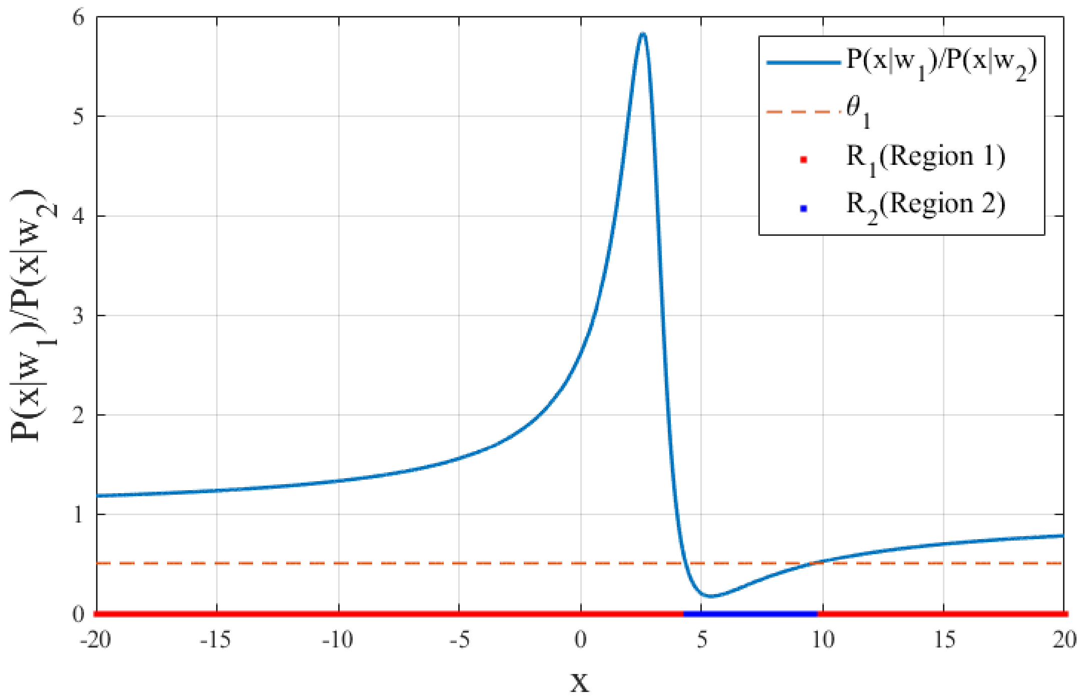


Figure 3. $P(x|\omega_1)/P(x|\omega_2)$ in the design of Bayes minimum risk classifier.

What is the probability of error? Compare the result with that of (e).

$$\text{if } \frac{p(x|\omega_1)}{p(x|\omega_2)} > \frac{\lambda_{12}-\lambda_{22}}{\lambda_{21}-\lambda_{11}} \cdot \frac{P(\omega_2)}{P(\omega_1)} \Rightarrow \text{action } \alpha_1 \text{ (decide } \omega_1)$$

$$\Rightarrow \frac{p(x|\omega_1)}{p(x|\omega_2)} > \frac{1}{2} \times \frac{\frac{1}{2}}{\frac{1}{2}} = \frac{1}{2}$$

$$\text{so, if } \frac{p(x|\omega_1)}{p(x|\omega_2)} > \frac{1}{2} \Rightarrow \text{action } \alpha_1 \text{ (decide } \omega_1)$$

$$\text{The decision boundary is on } x^* \text{ when } \frac{p(x^*|\omega_1)}{p(x^*|\omega_2)} = \frac{1}{2}$$

$$\Rightarrow \frac{1}{\pi b}\frac{1}{1+\left(\frac{x-a_1}{b}\right)^2} = \frac{1}{2}\frac{1}{\pi b}\frac{1}{1+\left(\frac{x-a_2}{b}\right)^2} \Rightarrow$$

$$2 + 2\frac{(x-a_2)^2}{b^2} = 1 + \frac{(x-a_1)^2}{b^2} \Rightarrow \frac{(x-a_1)^2}{b^2} - 2\frac{(x-a_2)^2}{b^2} = 1$$

$$\left.\begin{array}{l} a_1 = 3 \\ a_2 = 5 \\ b = 1 \end{array}\right\} \Rightarrow (x-3)^2 - 2(x-5)^2 = 1 \Rightarrow x^2 + 9 - 6x - 2x^2 - 50 + 20x = 1$$

$$\Rightarrow x^2 - 14x + 42 = 0 \Rightarrow \begin{cases} x_2^* = 9.6458 \\ x_1^* = 4.3542 \end{cases} \Rightarrow \begin{cases} x < x_1^* \rightarrow \omega_1 \\ x_1^* < x < x_2^* \rightarrow \omega_2 \\ x > x_2^* \rightarrow \omega_1 \end{cases}$$

$$p(\text{error}) = \int_{-\infty}^{x_1^*} p(x|\omega_2)\times\frac{1}{2}\,dx + \int_{x_1^*}^{x_2^*} p(x|\omega_1)\times\frac{1}{2}\,dx + \int_{x_2^*}^{\infty} p(x|\omega_2)\times\frac{1}{2}\,dx$$

$$= \frac{1}{2\pi}\left(\text{Arctan}\left(\frac{x_1^*-5}{1}\right) + \frac{\pi}{2}\right) + \frac{1}{2\pi}\left(\text{Arctan}(x_2^*-3) - \text{Arctan}(x_1^*-3)\right)$$

$$+ \frac{1}{2\pi}\left(\frac{\pi}{2} - \text{Arctan}(x_2^*-5)\right) = \frac{1}{2\pi}(0.997 + 0.4867 + 0.2120) = 0.2699$$

By applying the lost function to the problem, the region of category 1 is increased while the region of category 2 is decreased because the penalty of not choosing category 1 is more than not choosing category 2. So in some regions, the smaller posterior is selected. In this case, the error of choosing the incorrect class for the data is a grower. So, P(error) in this part is bigger than P(error) in part e.

Appendix

All codes are located in folder Codes.

The MATLAB codes of Problem 1part b is located in folder 9930264-E1b. In this folder there is a function for Cauchy distributions which name is Cauchy_function.m and the main m-file is Project1_b.m.

The MATLAB codes of Problem 1part e is located in folder 9930264-E1e. In this folder there is a function for Cauchy distributions which name is Cauchy_function.m and the main m-file is Project1_e.m.

The MATLAB codes of Problem 1part f is located in folder 9930264-E1f. In this folder there is a function for Cauchy distributions which name is Cauchy_function.m and the main m-file is Project1_f.m.