Intermediate Data Structures Algorithms
CS 141 - Spring 2020
Discussion 01 - Basics-Proof, Asymptotic Notation and Execution Time.

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Outline

- **Proposition:** is a statement that is either true or false.
  - EX1. 2+4=6 true
  - Ex2. 3+2=7 false
  - Ex3. “it’s five o’clock” is not a proposition.
- **Predicate:** is a proposition whose truth depends on the value of one or more variables.
  - Ex. P(n)= “n is a perfect square” n=4 T, n=5 F

Basics

- **Axiom:** Propositions that are simply accepted as true.
  - Ex1. “There is a straight line segment between every pair of points”
  - Ex2. Reflective axiom (e.g. a = a).
- **Theorem:** is a mathematical statement that is true and can be verified as true.
  - Ex1. if $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$

Basics

- Proof: Is a sequence of logical deductions from axioms and previously-proved statements that concludes with the proposition in question.
  - Contradiction
  - Induction
  - Proving an implication

  …

Logical deduction: used to prove new propositions using previously proved ones.

- Rule 1.Modus ponens: $\frac{p, p \implies q}{q}$
- Rule 2. $\frac{p \implies q, q \implies r}{p \implies r}$

Outline

Proof Examples

- **Induction:**
  Basic outline:
  Proof: we want to show that $P(n)$ is true for all n.
  Base case: show that $P(1)$ is true.
  Induction: Assume that $P(k)$ is true for all $1 \cdots k$, we need to show that $P(k+1)$ is true.

Induction

Example: for every positive integer $n \geq 4, 2^n < n!$
Solution: "We use induction approach"
Base case: when $n = 4$ $2^4 < 4! \implies 16 < 24$
Induction: we assume the claim holds for $1 \cdots k$, We need to show that the claim holds for $k+1$.
By the induction hypothesis we have: $2^k < k! \implies 2^k \times 2 < k! \times 2$
And we know: $2 < k+1 \implies 2^{k+1} < (k+1)!$

Direct Proof

Begin by assuming that P is true (We don't need to be worry about P being false) and show this forces Q to be true.
Outline of direct proof:
Proposition if P, then Q Proof. Suppose P. . . . Therefore Q

Direct Proof

If x and y are integers and $x^2 + y^2$ is even, prove that $x + y$ is even as well.

$$x^2 + y^2 = x^2 + y^2 + 2xy - 2xy =$$
$$(x+2)^2 - 2xy$$

we know that this is even, so:

$$(x+2)^2 - 2xy = 2m$$

$$(x+2)^2 = 2(m + xy)$$

thus $(x + 2)$ is even.

Proof by Contradiction

Basic outline:
In order to prove a proposition P by a contradiction:
Write: "We use proof by contradiction"
Write: "Suppose P is false" Deduce something known to be false (a logical contradiction)
Write: "This is a contradiction. Therefore, P must be true."

Proof by Contradiction

Example: prove that $\sqrt{2}$ is irrational
Solution: Suppose that$\sqrt{2}$ is rational, thus suppose: $\sqrt{2} = \frac{n}{d}$ where $\frac{n}{d}$ is the **lowest term**.
$2 = \frac{n^2}{d^2}$ or $2d^2 = n^2$ thus n is a multiple of 2.
$2d^2$ is a multiple of 4, it implies that d is a multiple of 2 So, numerator and denominator have **2 as a common factor**.

Outline

Big-$\mathcal{O}$ notation

Definition 3.1

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $\mathcal{O}(g(x))$ if there are constants C and k such that,

$$|f(x)| \leq C|g(x)|$$

whenever $x > k$. [This is read as "$f(x)$ is big-oh of $g(x)$."]

Big-Ω notation

Definition 3.2

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $\Omega(g(x))$ if there are positive constants C and k such that,

$$|f(x)| \geq C|g(x)|$$

whenever $x > k$. [This is read as "$f(x)$ is big-omega of $g(x)$."]

Big-Θ notation

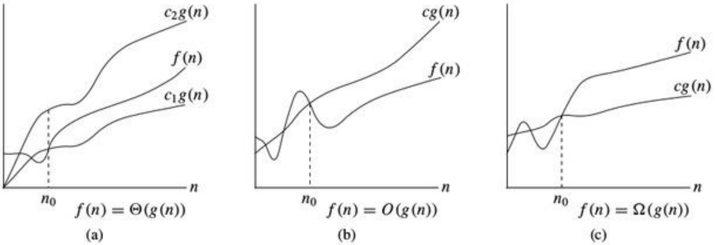
Definition 3.3

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $\Theta(g(x))$ if $f(x)$ is $\mathcal{O}(g(x))$ and $f(x)$ is $\Omega(g(x))$.
Also note that $f(x)$ is $\Theta(g(x))$ iif there are real numbers C_1 and C_2 and a positive real number k such that,

$$C_1|g(x)| \leq |f(x)| \leq C_2|g(x)|$$

whenever $x > k$. [This is read as "$f(x)$ is big-theta of $g(x)$."]

Asymptotic notation [1]



[1] Great additional resources at `https://tinyurl.com/o5lwvgp`

Examples

- $T(n) = n^2 + 2n + 1$
- Prove $T(n) = O(n^2)$

Examples

Solution 1:

To prove: $T(n) \leq Cn^2$

$n^2 \leq n^2$

$n^2 + 2n \leq n^2 + 2n^2$

$n^2 + 2n + 1 <= n^2 + 2n^2 + n^2 = 4n^2 = O(n^2)$ where c = 4 and $n_0 = 1$

Examples

- $T(n) = \frac{n^2}{2} - 2n$
- prove: $T(n) = \Theta(n^2)$

Examples

Solution 2:
To prove: $T(n) \leq (C_1 n^2)$ and $n^2 \geq (C_2 n^2)$
Part 1:
To prove $T(n) \leq (C_1 n^2)$
$\frac{n^2}{2} \leq n^2 \implies \frac{n^2}{2} - 2n$
Thus $T(n) \leq (C_1 n^2)$ where $C_1 = 1$ and $n_0 = 1$
$T(n) = O(n^2)$
Part 2:
To prove $T(n) \geq (C_2 n^2)$
$\frac{n^2}{2} \geq \frac{n^2}{4}$
$\frac{n^2}{2} - 2n \geq \frac{n^2}{4} - \frac{n^2}{8}$
Thus $T(n) \geq (C_2 n^2)$ where $C_2 = \frac{1}{8}$ and $n_0 = 8$
$T(n) = \Omega(n^2)$
To combine part 1 and part 2, we have $C_1 = 1$ and $C_2 = 1/8$ and $n_0 = max(1, 8) = 8$ and $T(n) = \Theta(n^2)$

Order Growth

$$n^5 < logn < n^n < n! < \sqrt{2}^{logn} < 4^n$$

Reference

- Mathematics for Computer Science” by Eric Lehman, F. Thomson Leighton, Albert R Meyer