EGIM07 - Dynamics and Earthquake Analysis of Structures Individual Project 1

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I. INTRODUCTION

The Newmark's method is a numerical integration method that is widely used in many fields of engineering where structures are subjected to dynamic loadings. It is a second-order accurate implicit method (explicit for β=0), that is parameterized by two parameters, γ and β, that control the accuracy and stability of the method. The values of γ and β can be chosen to optimize the performance of the method for a particular problem. This method allows for the direct solution of second-order differential equations without the need of transforming the equation to a pair of simultaneous first order differential equations [1].

The second order differential equation (1) is transformed to the set of equations (1), (2) and (3) when time is discretized, and linear acceleration is assumed in each time interval upon applying Newmark's method.

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = f(t) \tag{1}$$

$$u_{n+1} = u_n + \frac{1}{2}\Delta t^2\big((1-2\beta)\ddot{u}_n + 2\beta\ddot{u}_{n+1}\big) \tag{2}$$

$$\dot{u}_{n+1} = \dot{u}_n + \Delta t\big((1-\gamma)\ddot{u}_n + \gamma\ddot{u}_{n+1}\big) \tag{3}$$

$$\ddot{u}_{n+1} = \frac{F_{n+1} - c\dot{u}^0{}_{n+1} - ku^0{}_{n+1}}{m + \gamma c\Delta t + \beta k\Delta t^2} \tag{4}$$

where:

$$u^0{}_{n+1} = u_n + \Delta t\dot{u}_n + \frac{1}{2}\Delta t^2(1-2\beta)\ddot{u}_n \tag{5}$$

$$\dot{u}^0{}_{n+1} = \dot{u}_n + \Delta t(1-\gamma)\ddot{u}_n \tag{6}$$

With the equations discretized, a simple code using any programming language can be written to find a solution, MATLAB was the program of choice for this project. This report covers the procedures and results obtained from applying Newmark's method for different problems using MATLAB. In part 1, accuracy and stability of Newmark's method's implementation is assessed by solving a simple undamped one degree of freedom system using a fixed γ value of 0.5, with 5 different values of β for a wide range of time steps Δt.

Part 2 of the report covers the procedures and solutions obtained from applying Newmark's method to solve a problem of a damped dynamic system under ground motion. This procedure was repeated, varying the natural frequency of the system each time in order to form a response spectrum for the given ground motion data of the El Centro earthquake.

In part 3, a simple three-story building is considered, where an analysis is conducted to compute the maximum shear forces and displacements the El Centro earthquake will induce on the building at each floor. An investigation is then conducted to find a way of reducing the maximum displacement at the top floor by 10%.

II. PART ONE

The undamped one degree of freedom system to be solved in order to assess the stability of Newmark's technique is modelled by equation (7) below.

$$\ddot{u} + \omega u = 0 \tag{7}$$

$$KE_{max} = \frac{1}{2}mv_0^2 \tag{8}$$

$$T = {}^{2\pi}/_{\omega} \tag{9}$$

where ω, $\ddot{u}$ and u denote the natural frequency, acceleration, and displacement respectively. In this study, values of 4π rad/s and 5 kg were chosen for ω and m respectively, and the initial displacement u_0 and initial velocity v_0 were set to be 0 m and 3 m/s respectively.

A. Stability Criteria for Different β Values

First, the maximum kinetic energy of the system KE_{max} and period T are calculated analytically with equations (8) and (9), giving 22.5 J and 0.5s respectively. These are the values to be utilized to assess the stability and accuracy of the solutions obtained numerically, since they should be constant throughout the entire motion for a free undamped, such as the one considered.

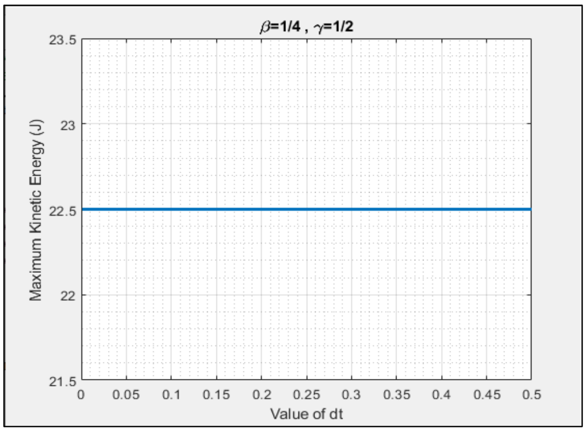

FIGURE 1
MAXIMUM KE THROUGHOUT MOTION AGAINST Δt
(β=1/4 , γ = 1/2)

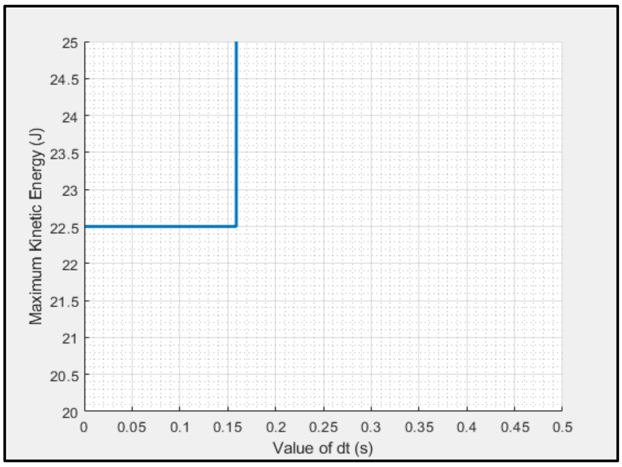

FIGURE 2
MAXIMUM KE THROUGHOUT MOTION AGAINST Δt
(β= 0 , γ = 1/2)

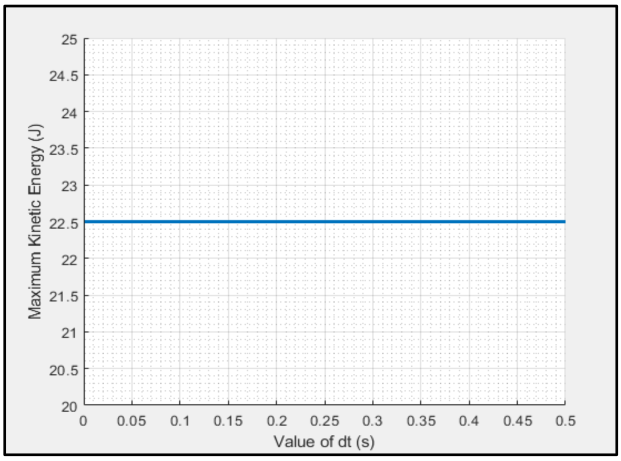

FIGURE 3
MAXIMUM KE THROUGHOUT MOTION AGAINST Δt
(β=1/12 , γ = 1/2)

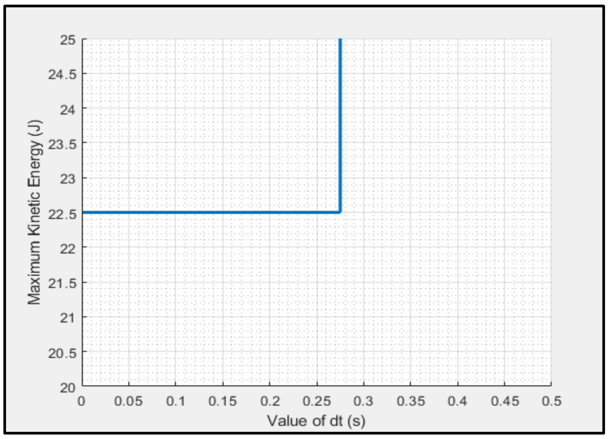

FIGURE 4
MAXIMUM KE THROUGHOUT MOTION AGAINST Δt
(β=1/6 , γ = 1/2)

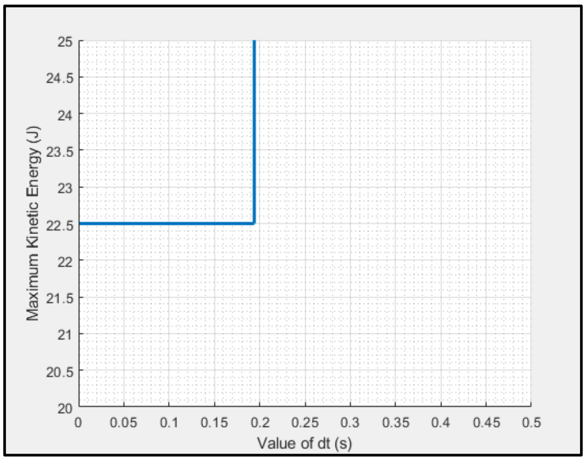

FIGURE 5
MAXIMUM KE THROUGHOUT MOTION AGAINST Δt
(β=1/12 , γ = 1/2)

An undamped system where $\gamma \geq 0.5$ and $\beta \geq 0.25(\gamma + 0.5)^2$, is unconditionally stable and guaranteed to converge for any time-step size [1], therefore our case here, where β is equal to 0.25 and γ equal to 0.5, should be stable for all values of Δt. The solution for 100 cycles was computed for using a range of Δt values, and the maximum value of kinetic energy in the motion corresponding to each Δt was saved. Figure 1 shows the plot of the recorded KE_{max} of the mass through the entire motion of 100 cycles for Δt ranging from 0.001s to 0.5s. The figure shows that for all values of Δt, the maximum kinetic energy is equal to KE_{max} even when Δt is equal to T.

The same procedure was repeated for β values of 0, 1/3, 1/6 and 1/12, however, the code was modified to terminate solving once Δt is sufficiently large to cause an error of more than 10% in the calculated maximum kinetic energy, indicating instability. Figures 2, 3, 4 and 5 show plots of maximum kinetic energy against Δt for different β values (Note that the x-coordinate of the vertical line shown on each graph denotes the maximum stable value of Δt). Figure 6 shows a table summarizing the stability criterion for each β value.

The figures clearly show that as β increases, the undamped system seems to become more stable, and the stability criteria becomes less strict. The reason behind this can be deduced by observing the discretized set of equations (2), (3) and (4). In an undamped system, numerical damping is caused through the stiffness coefficient, a larger beta causes the (1-2β) term to become smaller thus reducing the effect of the stiffness matrix on the solution [2].

γ	β	Stability Criteria
1/2	1/3	**Unconditionally stable**
	1/4	**Unconditionally stable**
	1/6	Δt < 0.275s
	1/12	Δt < 0.194s
	0	Δt < 0.159s

FIGURE 6
CRITICAL TIME STEP FOR EACH VALUE OF β

B. Accuracy

Changing the value of β can also have a significant effect on the accuracy of the solution obtained for a given problem. For this undamped system, the accuracy of the solution calculated for each β was assessed by finding the difference ΔT between the numerically calculated period T^* and the analytically computed period T, for different time step values.

Figure 7 below shows the non-dimensional plot of $\Delta T/T$ against $\Delta t/T$. The figure clearly shows that as Δt decreases, accuracy of the results increases. It can also be deduced that for this problem, β equal to 1/12 provides the highest accuracy, given that the Newmark's scheme is stable for the chosen Δt.

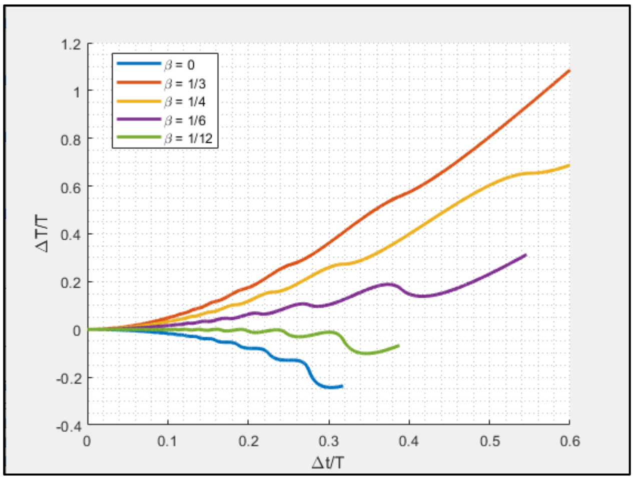

FIGURE 7
ACCURACY PLOT FOR DIFFERENT β VALUES

III. PART TWO

In part 2, we consider a general damped system under ground motion with equation (10) presenting the motion.

$$\ddot{u} + 2\zeta\omega\dot{u} + \omega^2 u = -a_g(t) \quad (10)$$

where ζ denotes the damping ratio, and $a_g(t)$ denotes the ground acceleration caused by the El Centro earthquake in 1940. The aim here is to compute the dynamic response of different bodies with different periods T ranging from 0.1s to 10s, over the duration of the earthquake. Following this, the maximum values of displacement, velocity, and acceleration for each response are found and used to produce a plot of maximum response against period.

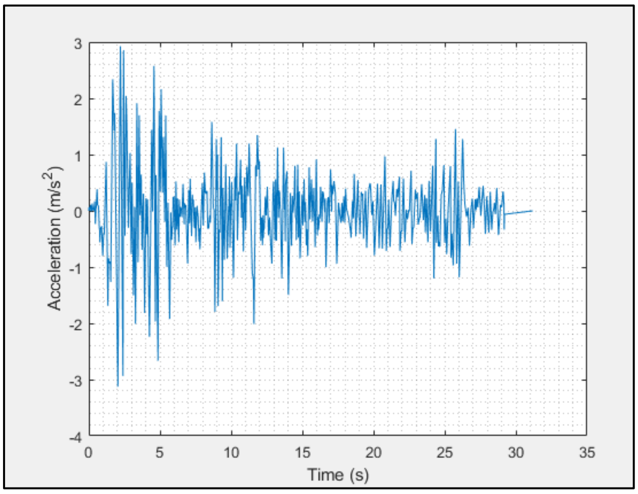

FIGURE 8
EL CENTRO EARTHQUAKE GROUND ACCELERATION DATA

The data file available of the El Centro earthquake contains values of the ground acceleration caused by the earthquake taken at 0.02s timestep, therefore, in this part we will take a fixed Δt value of 0.02s. Figure 8 shows the ground acceleration data taken from the El Centro earthquake.

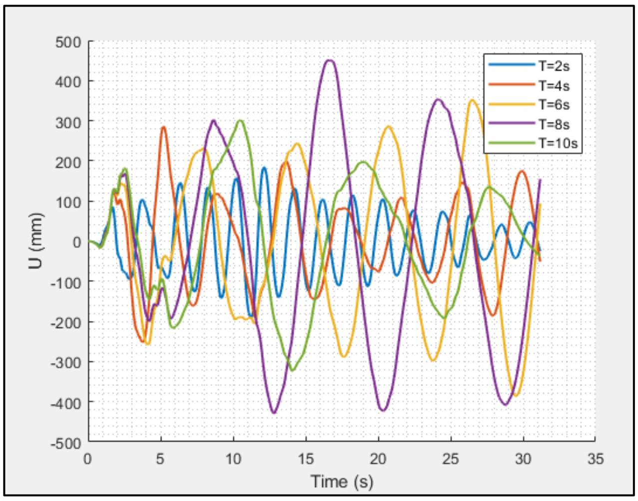

FIGURE 9
DISPLACEMENT OF DIFFERENT BODIES OVER THE DURATION OF THE EARTHQUAKE

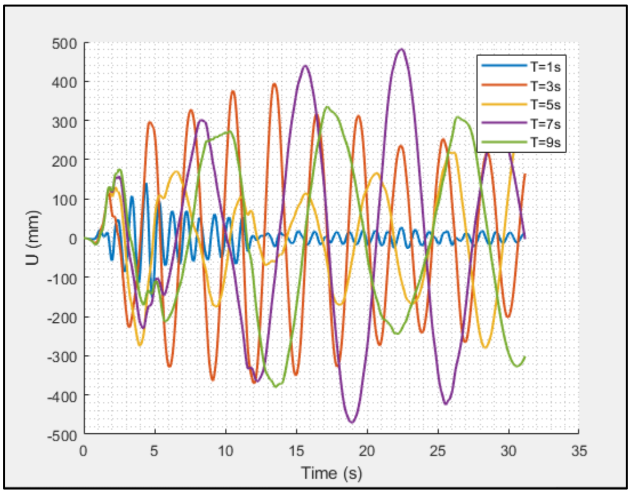

FIGURE 10
DISPLACEMENT OF DIFFERENT BODIES OVER THE DURATION OF THE EARTHQUAKE

A fixed Newmark's scheme will be used for all calculations where β and γ are 0.25 and 0.5 respectively and the mass of all bodies was set to 100,000 kg. Figures 9 and 10 show the relative displacement response of different bodies over the duration of the earthquake, illustrating how buildings with different natural frequencies have different responses to the same ground motion.

Response spectra play a crucial role in the dynamics analysis of structures as they provide insightful representation of a structure's behavior under dynamic loads, which helps identify critical modes of vibration. The maximum displacement, acceleration and velocity recorded throughout the motion was recorded for all values of *T*. Figures 11, 12 and 13 show the plots of the displacement, velocity, and acceleration response spectra.

From these figures we can find that the maximum displacement response to this ground motion occurs to structure with a period of 7.09s, while the maximum velocity and total acceleration occur for structures with periods 0.88s and 0.47s respectively. From such plots, engineers can make educated decisions to reduce the maximum responses by adding/removing masses or changing the stiffness to alter the natural frequency to a more favorable value.

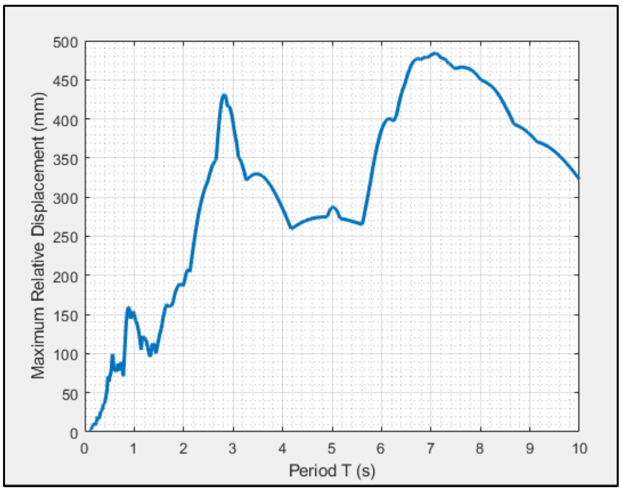

FIGURE 11
DISPLACEMENT SPECTRUM PLOT

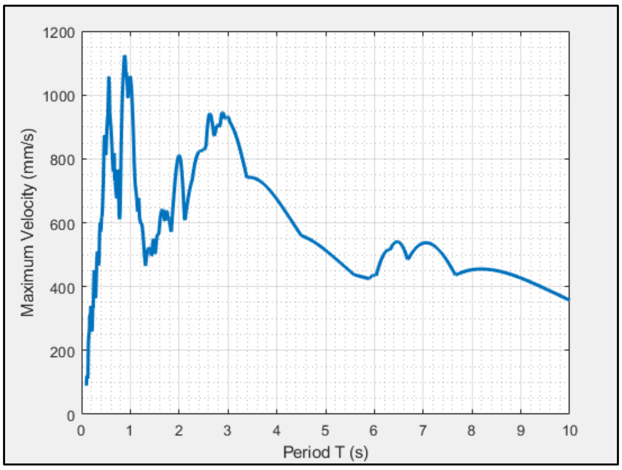

FIGURE 12
VELOCITY SPECTRUM PLOT

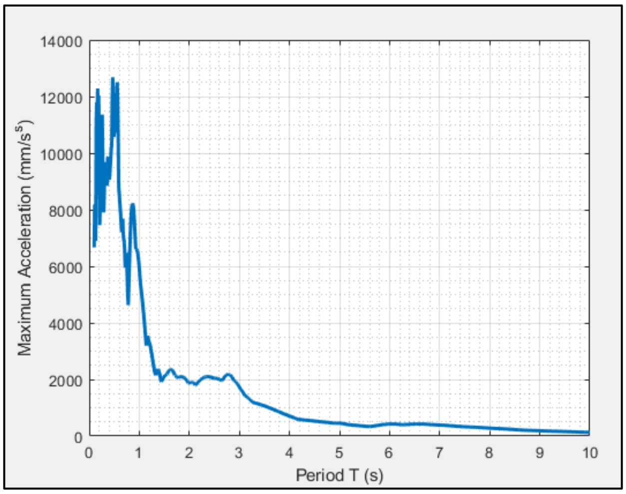

FIGURE 13
TOTAL ACCELERATION SPECTRUM PLOT

IV. PART THREE

In this section, the simple three-story frame building shown in figure 14 below is considered to be under ground motion caused by the El Centro earthquake. The building is supported by six 0.45x0.35 m^2 concrete columns with a Youngs modulus E equal to 30 GN/m^2, and the masses of the first, second and third floor are 400,000kg, 300,000 kg and 200,000 kg respectively. Using the response spectrums obtained from part two, the aim of this section is to calculate the maximum displacement and shear forces each floor of such a building may be subjected to underground motion.

To simplify such a structure so an analysis can be conducted, each floor is only assumed to have a single lateral degree of freedom and the supporting columns are assumed to be massless.

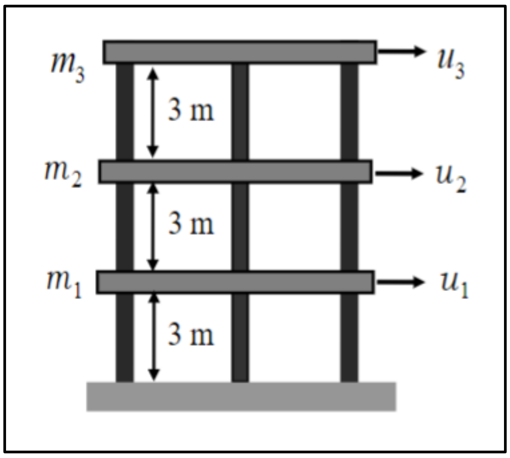

FIGURE 14
SKETCH OF A SIMPLE STRUCTURE

The first step of conducting the analysis is computing and assembling the global stiffness **K** and global mass matrix **M**. The lateral stiffness of a single column k_p can be calculated with equation (11). The lateral stiffness of each floor in such a simplified structure can be approximated as the sum of lateral stiffness of all columns, and since all columns are identical, equation (12) can be used to calculate the stiffness of each floor k_f, where N_{col} is the number of columns per floor. The number of columns and column properties are identical for all three floor, therefore global stiffness matrix **K** is given by equation (13).

$$k_p = \frac{12EI}{h^3} \tag{11}$$

$$k_f = N_{col} k_p \tag{12}$$

$$\mathbf{K} = \begin{bmatrix} 2k_f & -k_f & 0 \\ -k_f & 2k_f & -k_f \\ 0 & -k_f & k_f \end{bmatrix} \tag{13}$$

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \tag{14}$$

A. Approximation with One Ritz Vector

First, we conduct an analysis using a single ritz vector r_1, that is linearly increasing with height to approximate the first mode of vibration v_1. With this assumption, the modal mass $\hat{m}$ and modal stiffness $\hat{k}$ can then be found with equations (16) and (17). The approximate period of the first mode $\hat{T}$ can then be computed with equation (19), which is around 0.5898 s. Using the computed modal mass, the load participation factor l_i for the mode is then computed with equation (18), where **d** is the earthquake input direction vector equal to $[1 \quad 1 \quad 1]^T$.

$$r_1 = [1 \quad 2 \quad 3]^T \tag{15}$$

$$\hat{m}_i = r_i^T M r_i \tag{16}$$

$$\hat{k}_i = r_i^T K r_i \tag{17}$$

$$l_i = {v_i M d}/{\hat{m}_i} \tag{18}$$

$$\hat{T}_i = 2\pi \left(\sqrt{\widehat{k_i}/\hat{m}_i} \right)^{-1} \tag{19}$$

$$U_{i\,max} = l_i S_u(\hat{T}_i) v_i \tag{20}$$

$$F_{i\,max} = l_i S_a(\hat{T}_i) M v_i \tag{21}$$

Peak displacement S_u and peak acceleration S_a were then interpolated from figures 11 and 13 obtained in part two and are equal to 0.0776 m and 8.7969 m/s^2 respectively. The maximum shear force and maximum displacement vector can then be computed with equations (20) and (21) respectively. The results are shown in figure 15 below.

```
Approximations with Linear Mode Shape
Max Shear Force
Floor 1: 2633387.060142 N
Floor 2: 3950080.590214 N
Floor 3: 3950080.590214 N

Displacement
Floor 1: 0.029424 m
Floor 2: 0.058849 m
Floor 3: 0.088273 m
```

FIGURE 15
CALCULATED MAXIMUM DISPLACEMENT AND SHEAR FORCE AT EACH USING 1 RITZ VECTOR

B. Approximation with Two Ritz Vector

Equations (23) and (24) show the reduced stiffness and mass matrices can be obtained using ritz vectors r_1 and r_2. These are then used to solve the 2 DOF eigenvalue problem given by equation (25), from which approximations of mode shape vectors and natural frequency are calculated.

$$r_2 = [1 \quad 4 \quad 9]^T \tag{22}$$

$$\hat{\mathbf{K}} = [r_1 \quad r_2]^T \mathbf{K} [r_1 \quad r_2] \tag{23}$$

$$\hat{\mathbf{M}} = [r_1 \quad r_2]^T \mathbf{M} [r_1 \quad r_2] \tag{24}$$

$$\hat{\mathbf{K}} \mathrm{x_i} = \hat{\omega}_i^2 \hat{\mathbf{M}} \mathrm{x}_i \tag{25}$$

The first two mode vectors v_1 and v_2 are equal to $[1 \quad 1.657 \quad 1.971]^T$ and $[1 \quad 0.4659 \quad -1.602]^T$,where corresponding natural frequencies ω_1 and ω_2 are 9.92 rad/s^2 and 27.04 rad/s^2 respectively. The modal mass, modal stiffness and load participation factor are then computed for each mode to find the maximum displacement and shear force vectors associated with each mode. Since the peaks of each modal component take place at different times and the frequencies are different, SRSS modal combination can be used to approximate the maximum responses of shear forces and displacement. Figure 16 shows the computed maximum shear forces and displacements for each floor.

```
Max Response Approximations with 2nd Ritz Vector
and SRSS modal combination
Max Shear Force
Floor 1: 2947925.336312 N
Floor 2: 3428624.312496 N
Floor 3: 2837769.315473 N

Displacement
Floor 1: 0.042056 m
Floor 2: 0.069593 m
Floor 3: 0.082843 m
```

FIGURE 16
CALCULATED MAXIMUM DISPLACEMENT AND SHEAR FORCE AT EACH USING 2 RITZ VECTORS

C. Reducing Maximum Displacement of Top Floor

An efficient way of reducing the maximum displacement at all the floors would be to increase the lateral stiffness of the structure. For such a simple structure, the only way to do so would be to increase the lateral stiffness of the columns.

The lateral stiffness of a column is given by equation (11), from which it can be deduced that for a column with a given material, the only way to increase its lateral stiffness would be to increase the moment of inertia I, or decrease the height. The moment of inertia of the column is given by equation (26).

$$I = \frac{bh^3}{12} \tag{26}$$

where b and h are the dimensions of the column's cross section. h here is the dimension with greatest effect on the column's moment of inertia, and therefore has the greatest effect on the lateral stiffness of the column.

To find a 'close to exact' value of *h* that would decrease the displacement at the top floor by 10%, the bi-section method was employed. The initial minimum and maximum values of *h* in the bisection algorithm were set to 0.4 and 0.7m respectively, and the bi-section algorithm was repeated 50 times. The results are shown in figure 17.

The required value of *h* found using the bisection algorithm that would decrease the displacement at the top floor by 10% is 0.45278 m. This is off the target displacement value by 0.914% which is and error of less than 1% and therefore negligible. Therefore, we can conclude that increasing the width *h* of all 6 columns by about 2.8mm is more than enough to decrease the maximum displacement at the top floor by the required 10%. This is arguably the easiest/cheapest way to achieve this goal, rather than changing the height of the columns which would impact the size and available space inside the building.

```
Displacement
Target Floor 3 displacement Value: 0.079446 m
Floor 3: 0.078719 m
Value of h to achieve 10 percent decrease = 0.452784 m
Error = 0.914841 percent
```

FIGURE 17
BI-SECTION ALGORITHM RESULTS

V. REFERENCES

[1] Lindfield GR, Penny JET. Numerical Methods : Using MATLAB. Burlington: Elsevier Science; 2012.

[2] Chen C, Ricles JM. Stability Analysis of Direct Integration Algorithms Applied to Nonlinear Structural Dynamics. Journal of Engineering Mechanics-asce. 2008 Sep 1;134(9):703–11.