GUC German University in Cairo

CSEN 703 - Analysis and Design of Algorithms

Lecture 4 - Divide and Conquer II

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In the Previous Lecture

- We looked into designing problems using the D&C strategy.
- We learned how to write recurrences to represent the running time of D&C algorithms.
- We learned about solving recurrences using the recursion tree method.

Outline

Solving Recurrences

The Master Method

The Master Theorem

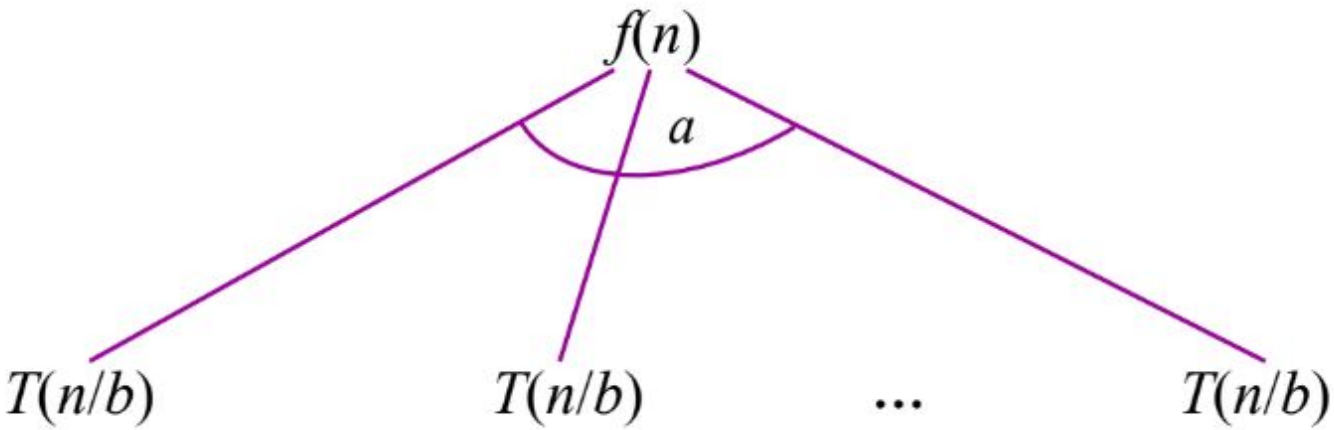
A cookbook for solving recurrences of the form

$$T(n) = aT(\frac{n}{b}) + f(n)$$

where $a \geq 1$, $b > 1$, and $f(n)$ is an asymptotically positive function.

Intuition

$$T(n) = aT(\tfrac{n}{b}) + f(n)$$



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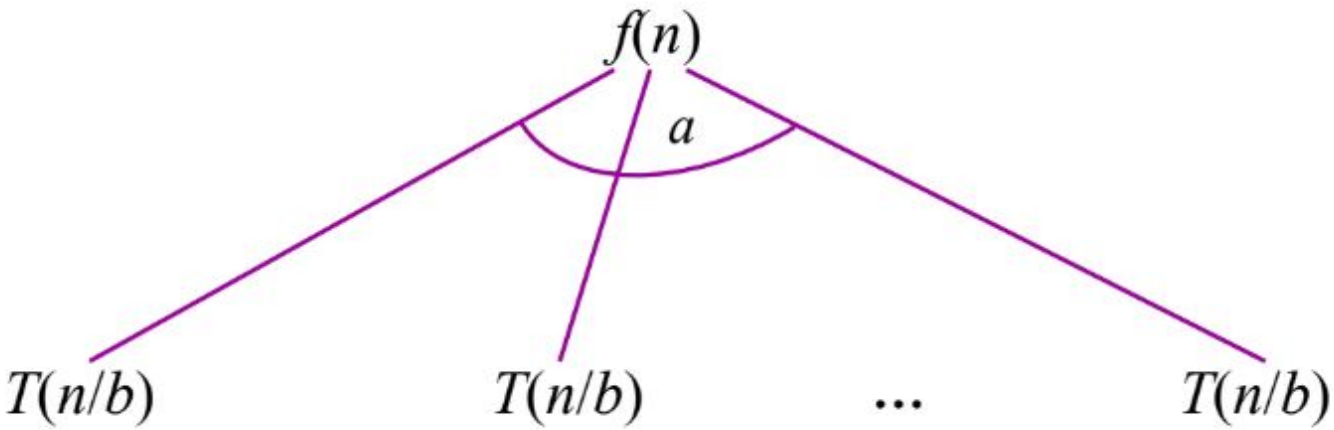

- To open up the next round of recurrences, we substitute $\frac{n}{b}$ in our recurrence.

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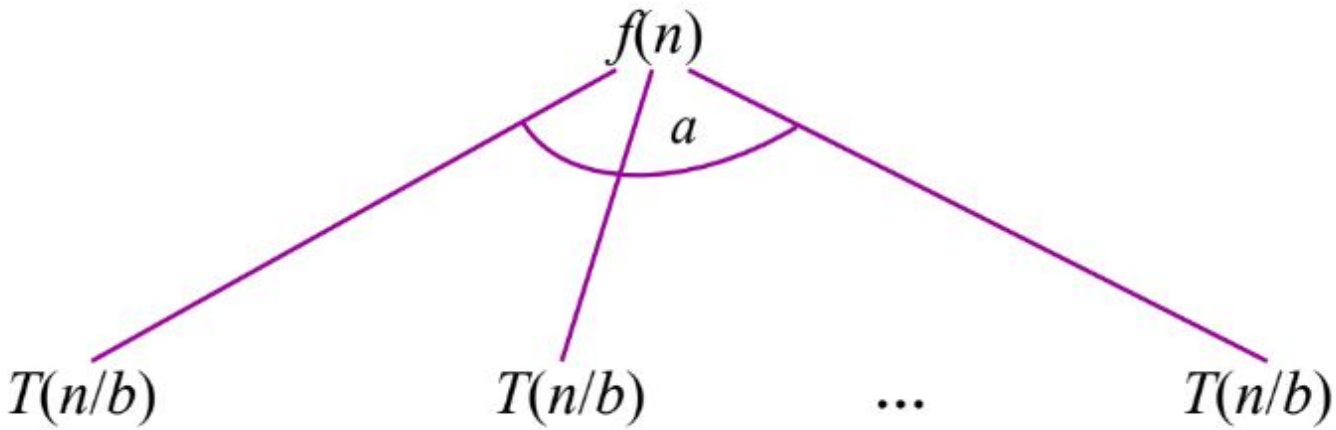

- To open up the next round of recurrences, we substitute $\frac{n}{b}$ in our recurrence.
- We get $T(\frac{n}{b}) = aT(\frac{\frac{n}{b}}{b}) + f(\frac{n}{b}) = aT(\frac{n}{b^2}) + f(\frac{n}{b})$

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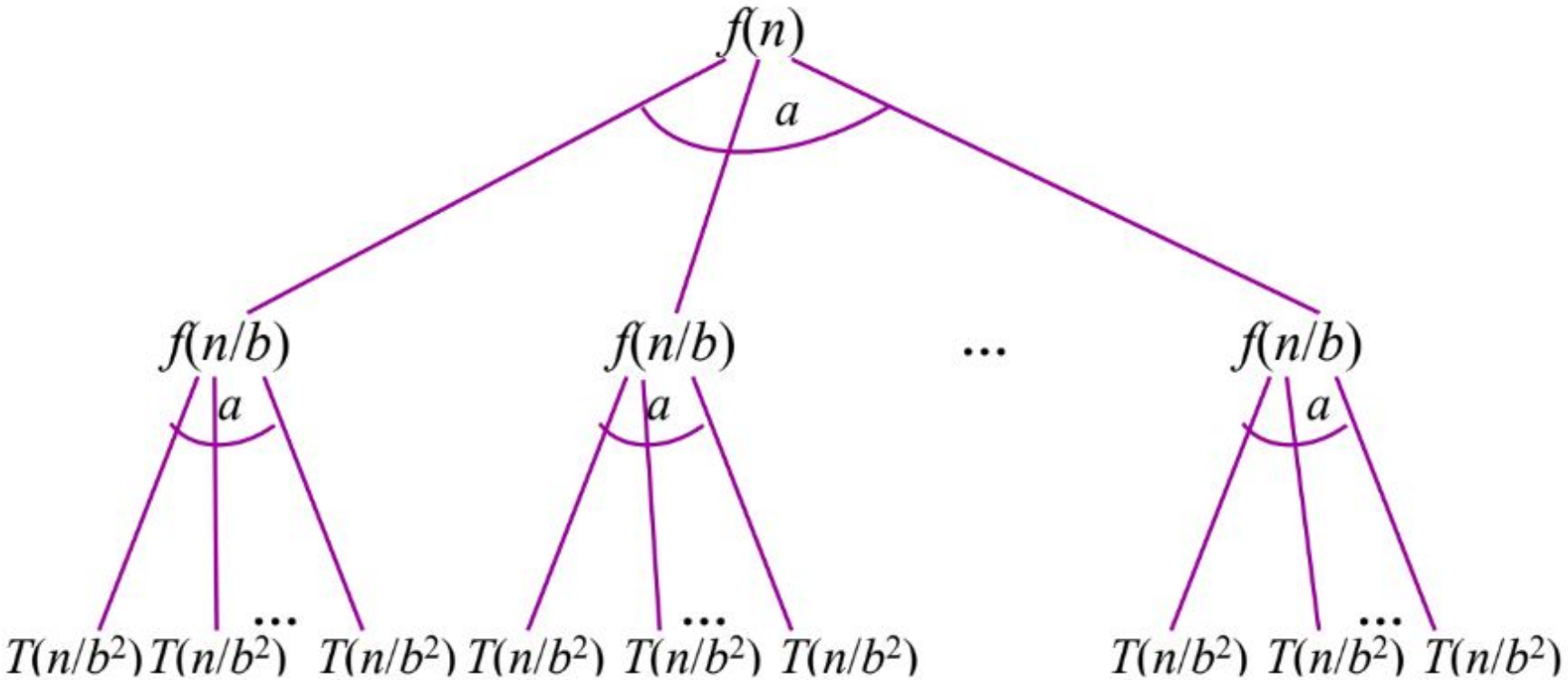

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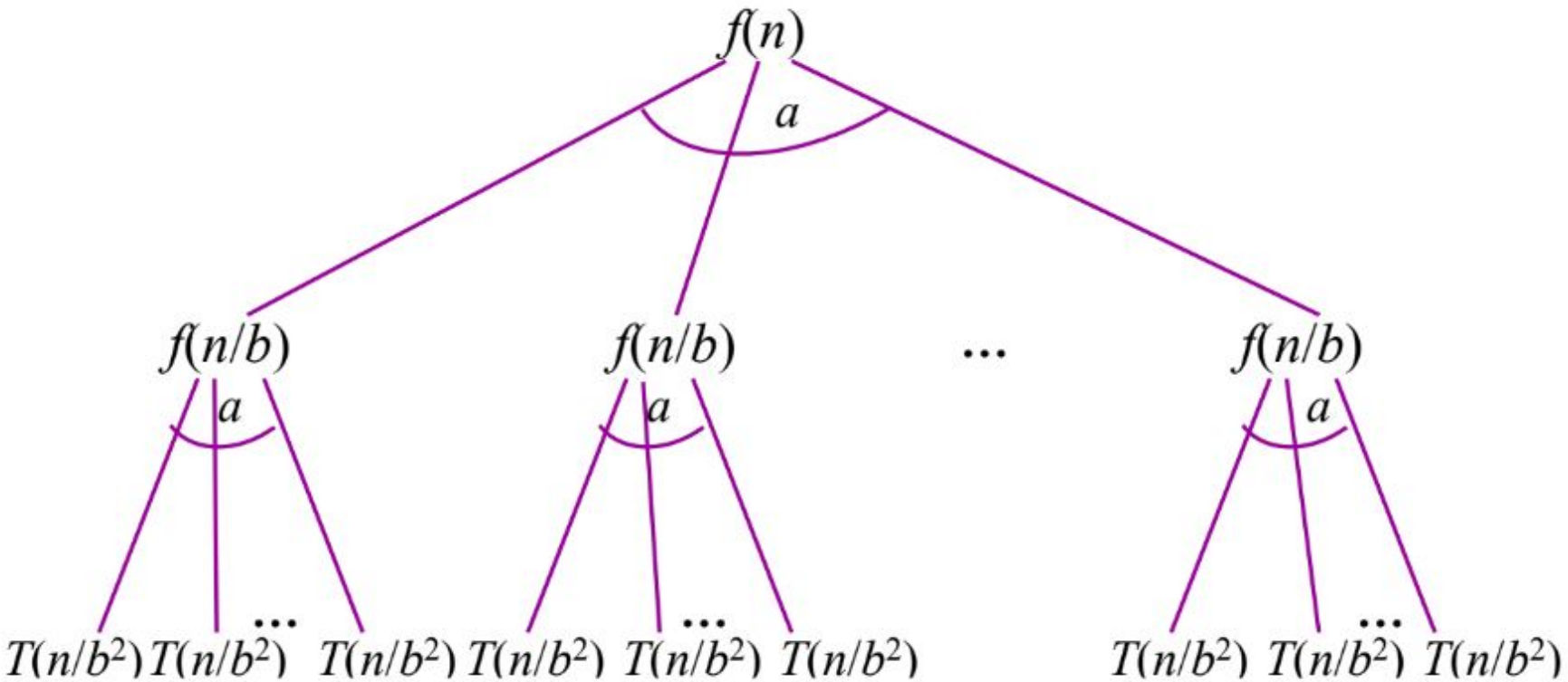

We can continue expanding the recursion tree until the sub-problems bottom out, that is, they reduce to a size of 1.

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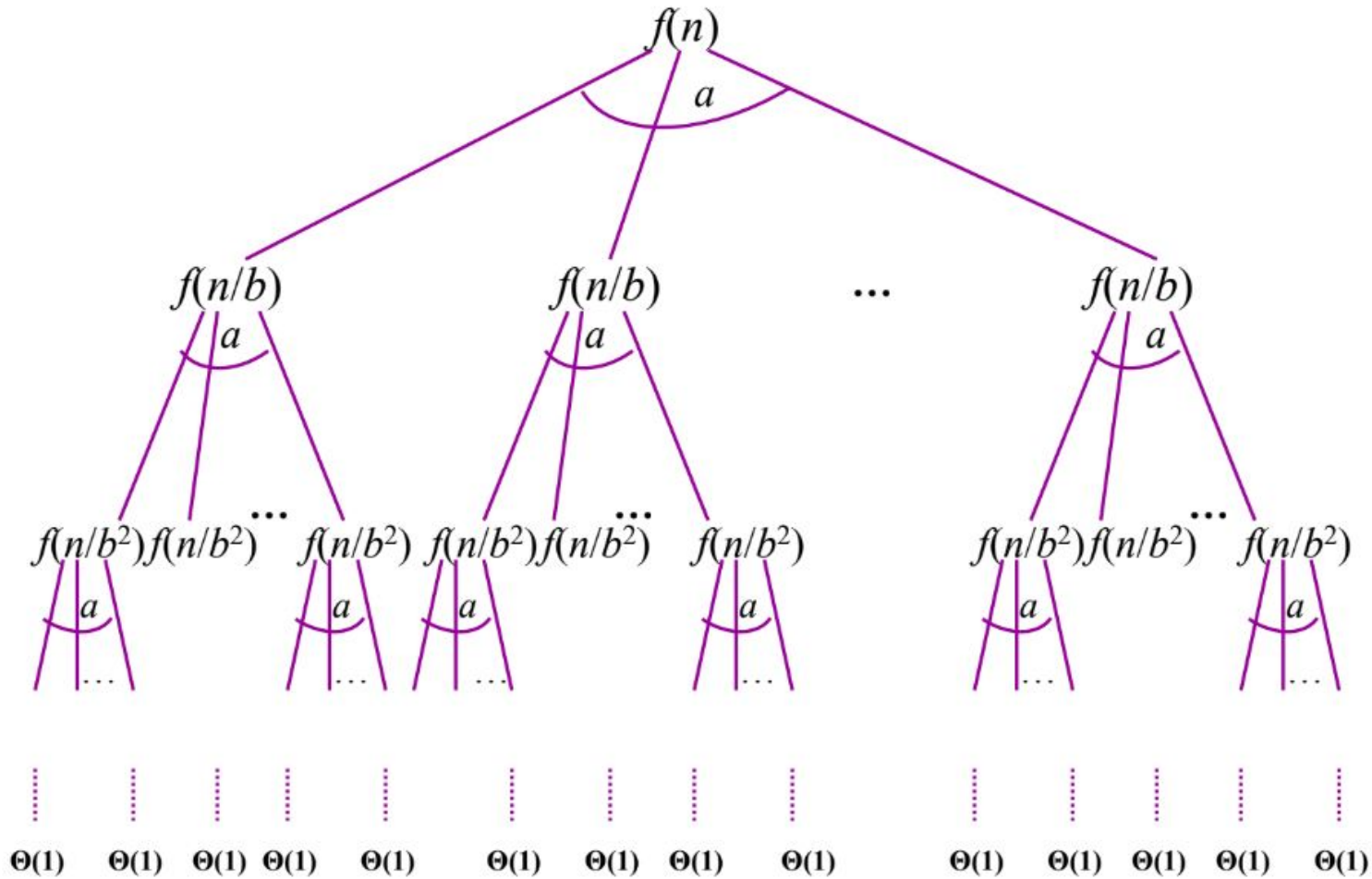

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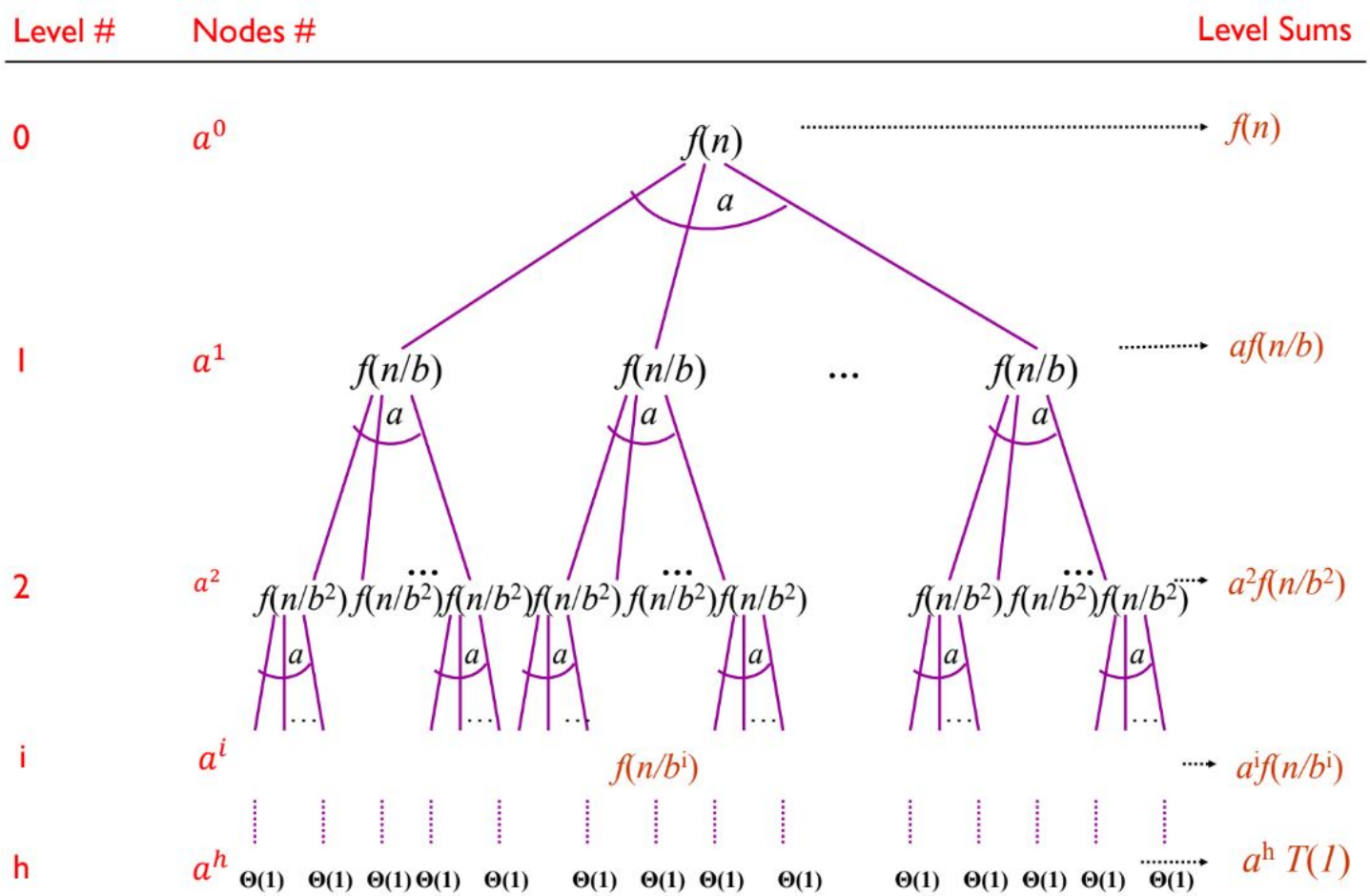

$$T(n) = \Sigma_{i=0}^{h-1} a^i f(\frac{n}{b^i}) + a^h \text{ with } h = log_b(n)$$

Intuition

$$T(n) = \Sigma_{i=0}^{log_b(n)-1} a^i f(\frac{n}{b^i}) + n^{log_b(a)}$$

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1. The summation is an increasing geometric series.
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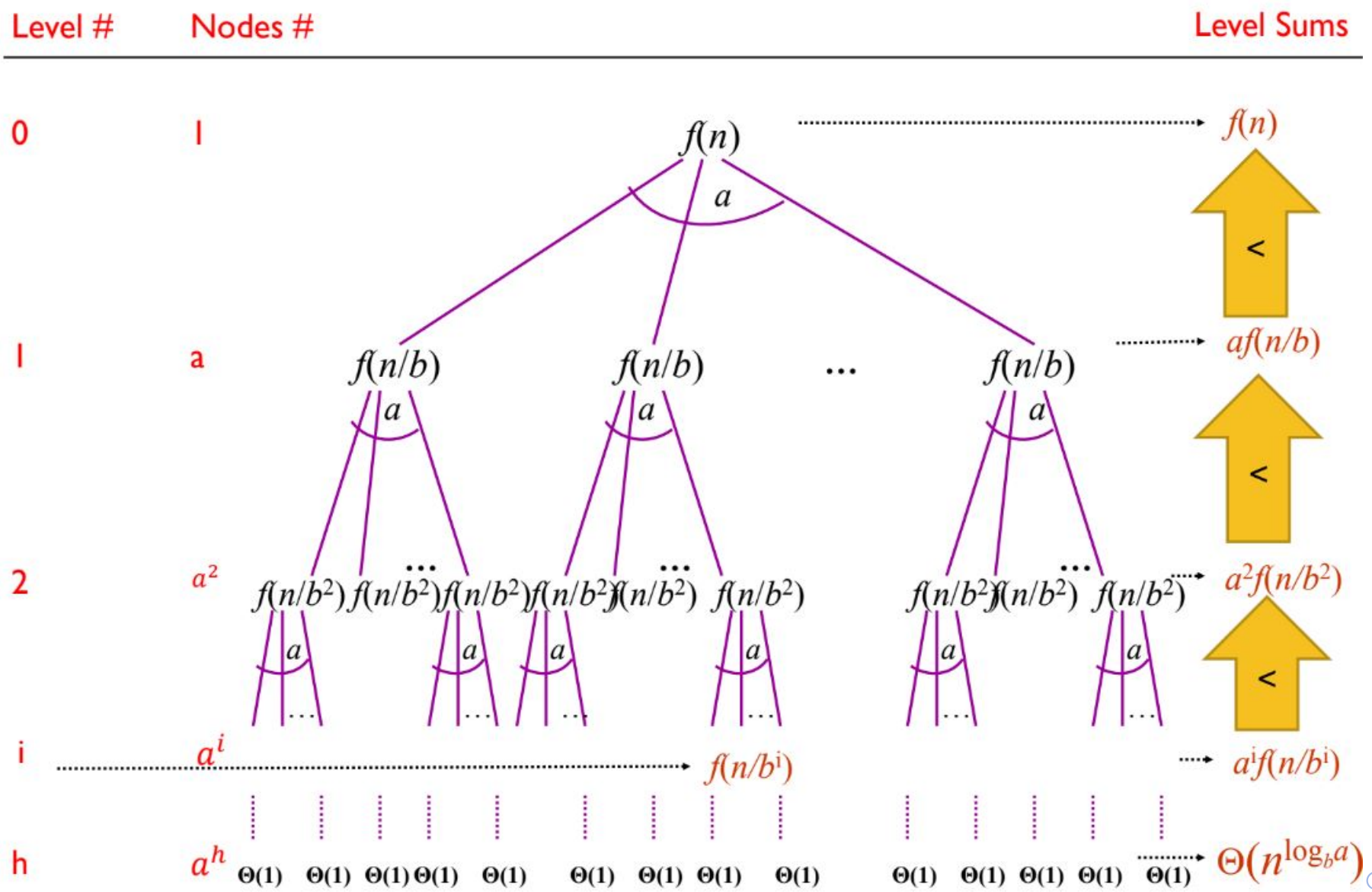
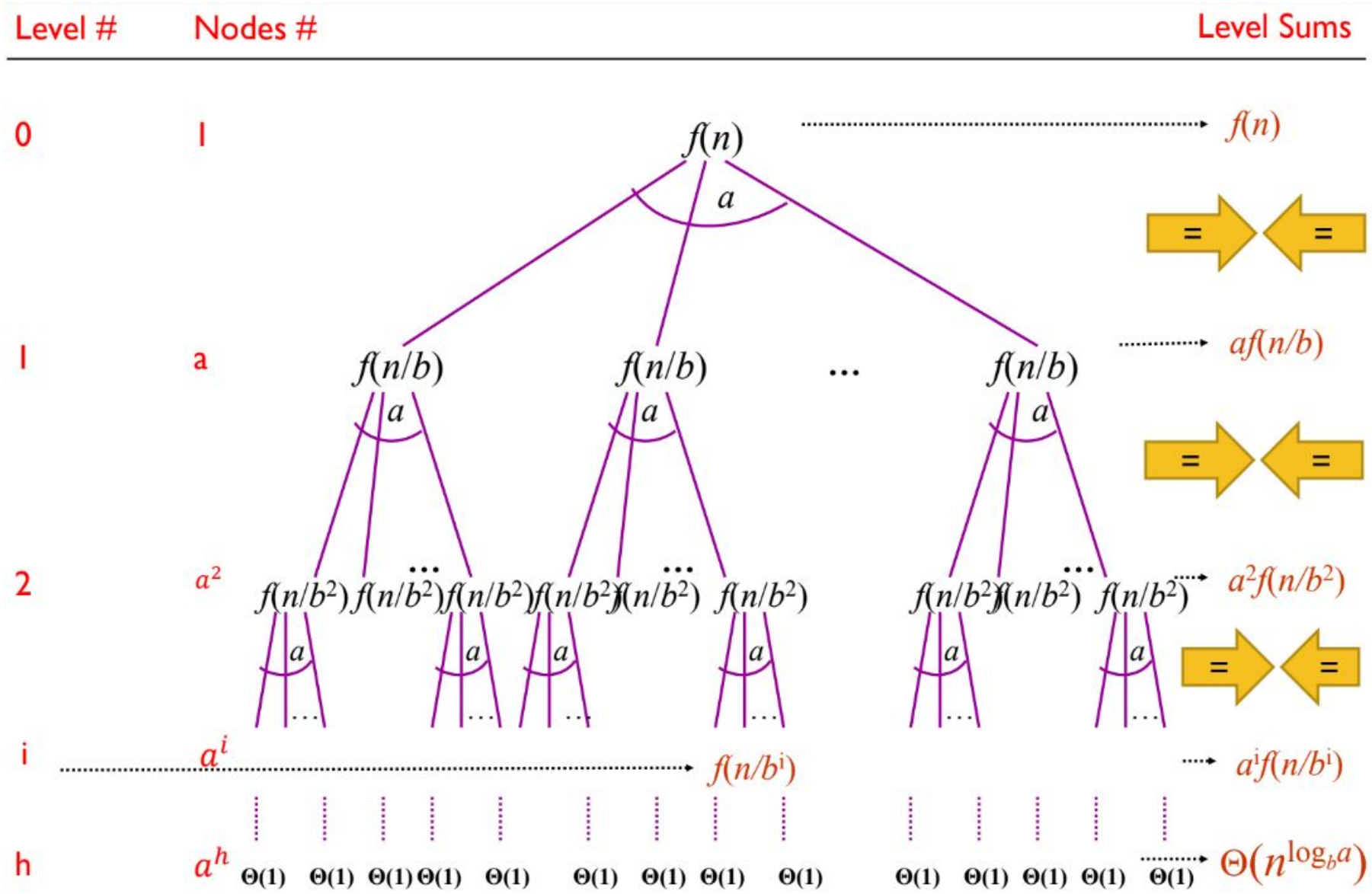
3. The summation is a decreasing geometric series.
   - $T(n) = 2T(\frac{n}{2}) + n^2 \Rightarrow$ Cost is dominated by the root.

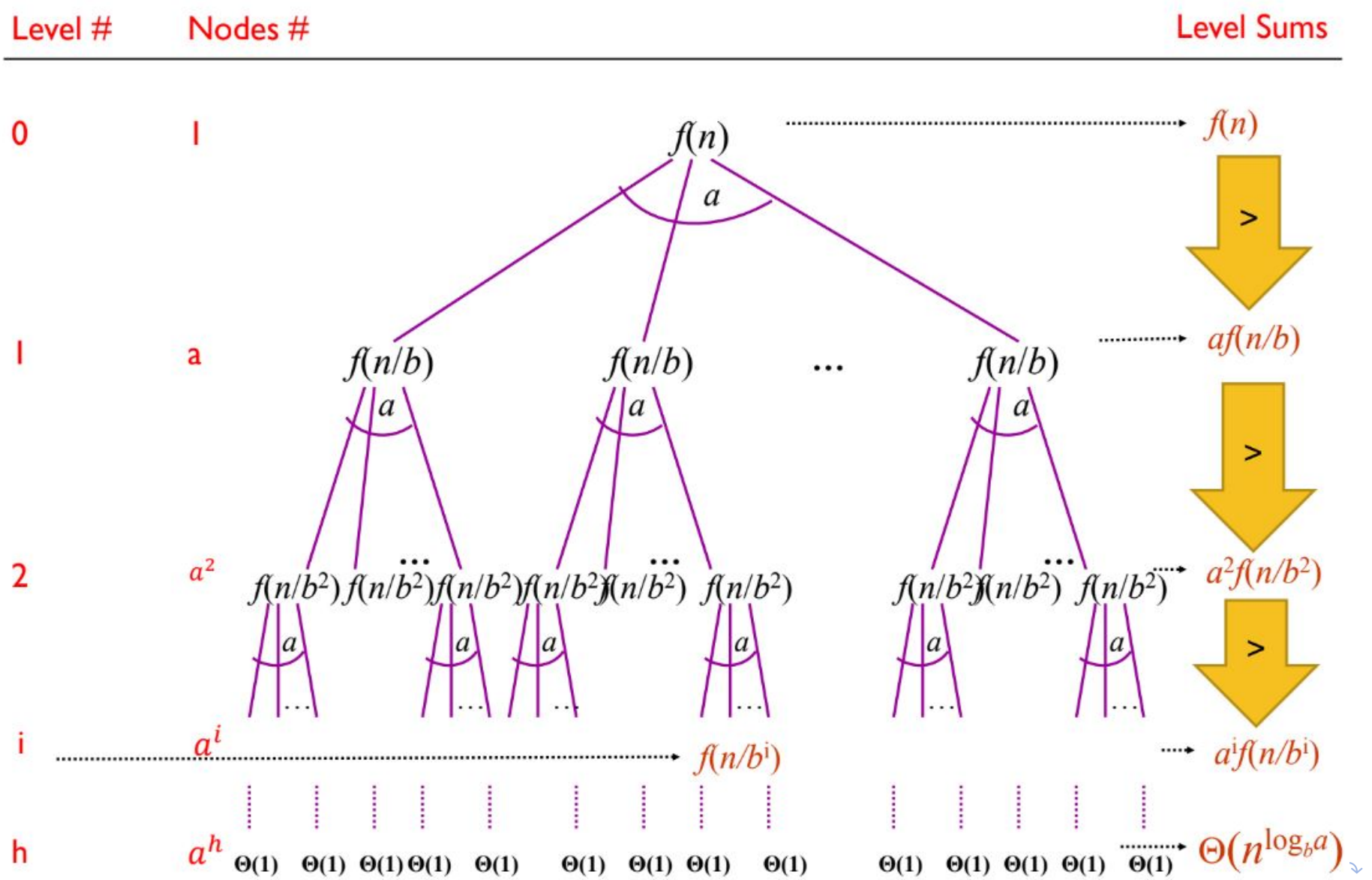
Case 1 - Cost Dominated by Leaves



Case 2 - Cost Distributed along the Tree



Case 3 - Cost Dominated by Root



The Master Method

The Master Theorem

Let $a \geq 1$ and $b > 1$ be constants, $f(n)$ be an asymptotically positive function, and $T(n) = aT(\frac{n}{b}) + f(n)$.
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2. If $f(n) = \Theta(n^{log_b(a)})$, then $T(n) = \Theta(n^{log_b(a)} log(n))$.
3. If $f(n) = \Omega(n^{log_b(a)+\varepsilon})$ for some $\varepsilon > 0$ and $af(\frac{n}{b}) \leq cf(n)$ for $0 < c < 1$, then $T(n) = \Theta(f(n))$.

The Master Theorem - Examples

Example

Use the master theorem to obtain a bound on $T(n)$.

1. $T(n) = T(\frac{n}{2}) + \Theta(1)$.

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   $a = 1, b = 2, n^{log_b(a)} = n^{log_2(1)} = 1$
   Cost of root $= c$, Cost of leaves $= c$.
   This is Case 2 of the master theorem since $c = \Theta(1)$.
   Hence, $T(n) = \Theta(log(n))$.
2. $T(n) = 2T(\frac{n}{2}) + \Theta(n)$.

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   Hence, $T(n) = \Theta(log(n))$.
2. $T(n) = 2T(\frac{n}{2}) + \Theta(n)$.
   $a = 2, b = 2, n^{log_b(a)=n}$
   Cost of root $= n$, Cost of leaves $= n$
   This is Case 2 of the master theorem since $n = \Theta(n)$.
   Hence, $T(n) = \Theta(nlog(n))$.

The Master Theorem - Examples

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Example

Use the master theorem to obtain a bound on $T(n)$.

3. $T(n) = 16T(\frac{n}{4}) + \Theta(n)$.

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3. $T(n) = 16T(\frac{n}{4}) + \Theta(n)$.
   $a = 16, b = 4, n^{log_b(a)} = n^{log_4(16)} = n^2$
   Cost of root $= cn$, Cost of leaves $= cn^2$.
   Looks like Case 1 of the master theorem.
   $n \leq \frac{n^2}{n^\varepsilon}$ for $\varepsilon = 1$. Case 1 proved. Hence, $T(n) = \Theta(n^2)$.
4. $T(n) = 2T(\frac{n}{2}) + \Theta(n^4)$.

The Master Theorem - Examples

Example

Use the master theorem to obtain a bound on $T(n)$.

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4. $T(n) = 2T(\frac{n}{2}) + \Theta(n^4)$.
   $a = 2, b = 2, n^{log_b(a)} = n$
   Cost of root $= cn^4$, Cost of leaves $= cn$
   Looks like Case 3 of the master theorem.
   $n^4 \geq n.n^\varepsilon$ for $\varepsilon = 1$.
   Regularity condition: $2(\frac{n}{2})^4 \leq cn^4$, $\frac{n^4}{2^3} \leq cn^4$ for $c = \frac{1}{8}$.
   Case 3 proved. Hence, $T(n) = \Theta(n^4)$.

The Master Theorem - Examples

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Example

Use the master theorem to obtain a bound on $T(n)$.

5. $T(n) = 2T(\frac{n}{2}) + \Theta(\frac{n}{log(n)})$.

The Master Theorem - Examples

Example

Use the master theorem to obtain a bound on $T(n)$.

5. $T(n) = 2T(\frac{n}{2}) + \Theta(\frac{n}{log(n)})$.

   $a = 2, b = 2, n^{log_b(a)} = n$

   Cost of root $= c\frac{n}{log(n)}$, Cost of leaves $= cn$

   Looks like Case 1 of the master theorem.

   $\frac{n}{log(n)} \leq \frac{n}{n^{\varepsilon}}$, $log(n) \geq n^{\varepsilon} \Rightarrow$ does not hold!

   This is a gap between cases 1 and 2 and can not be solved using the master theorem.

The Master Theorem - Examples

Example

Use the master theorem to obtain a bound on $T(n)$.

6. $T(n) = 2T(\frac{n}{2}) + \Theta(nlog(n))$.

The Master Theorem - Examples

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Example

Use the master theorem to obtain a bound on $T(n)$.

6. $T(n) = 2T(\frac{n}{2}) + \Theta(nlog(n))$.
   $a = 2, b = 2, n^{log_b(a)} = n^{log_2(2)} = n$
   Cost of root $= cnlog(n)$, Cost of leaves $= cn$.
   Looks like Case 3 of the master theorem.
   $nlog(n) \geq n.n^{\varepsilon}$, $log(n) \geq n^{\varepsilon} \Rightarrow$ does not hold!
   This is a gap between cases 2 and 3 and can not be solved using the master theorem.
7. $T(n) = T(n-1) + \Theta(n)$.

The Master Theorem - Examples

Example

Use the master theorem to obtain a bound on $T(n)$.

6. $T(n) = 2T(\frac{n}{2}) + \Theta(nlog(n))$.
   $a = 2, b = 2, n^{log_b(a)} = n^{log_2(2)} = n$
   Cost of root $= cnlog(n)$, Cost of leaves $= cn$.
   Looks like Case 3 of the master theorem.
   $nlog(n) \geq n.n^{\varepsilon}$, $log(n) \geq n^{\varepsilon} \Rightarrow$ does not hold!
   This is a gap between cases 2 and 3 and can not be solved using the master theorem.
7. $T(n) = T(n-1) + \Theta(n)$.
   Not in the form of the master theorem.

Outline

Quick Sort - D&C Approach

- Divide: Partition $A[p, \ldots, r]$ into two (possibly empty) arrays $A[p, \ldots, q-1]$ and $A[q+1, \ldots, r]$:
  - each element in $A[p, \ldots, q-1] \leq A[q]$
  - each element in $A[q+1, \ldots, r] \geq A[q]$
- Conquer: The subarrays by sorting them.
- Combine: No work needed! Array already sorted.

Quick sort - Pseudo Code

```
1 QuickSort(A, p, r)
2 if p < r then
3 |   q = partition(A, p, r);
4 |   QuickSort(A, p, q − 1) ;
5 |   QuickSort(A, q + 1, r) ;
6 end
```

The complexity lies in the partition procedure which will rearrange the array such that all elements before the pivot has smaller values and all elements after the pivot has bigger values.

Ross was Probably Good at Quick Sort!

Quick sort - Partition

```
1  Partition(A, p, r)
2  pivot = A[r] ;
3  i = p − 1 ;
4  for j = p to r − 1 do
5  |   if A[j] ≤ pivot then
6  |   |   i++ ;
7  |   |   Exchange A[i] and A[j] ;
8  |   end
9  end
10 Exchange A[i + 1] and A[r] ;
11 return i+1;
```

Trace it on $[8, 1, 6, 4, 0, 3, 9, 5]$.

Quick Sort

Quick Sort - Correctness

- Correctness is based on the correctness of partition.

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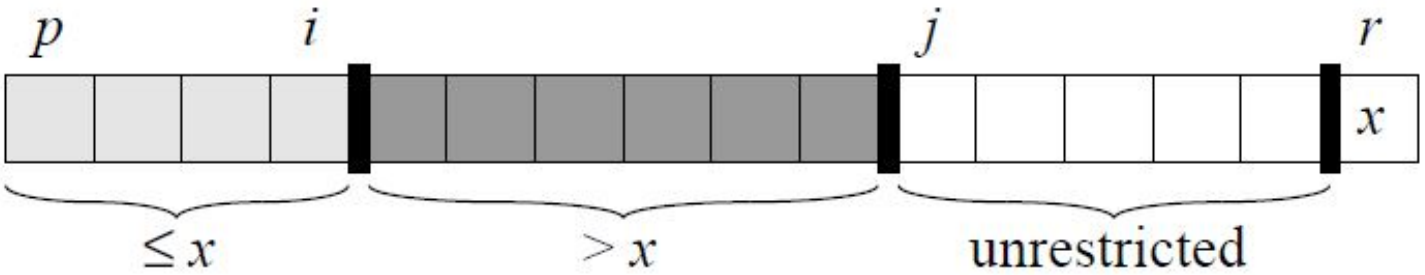
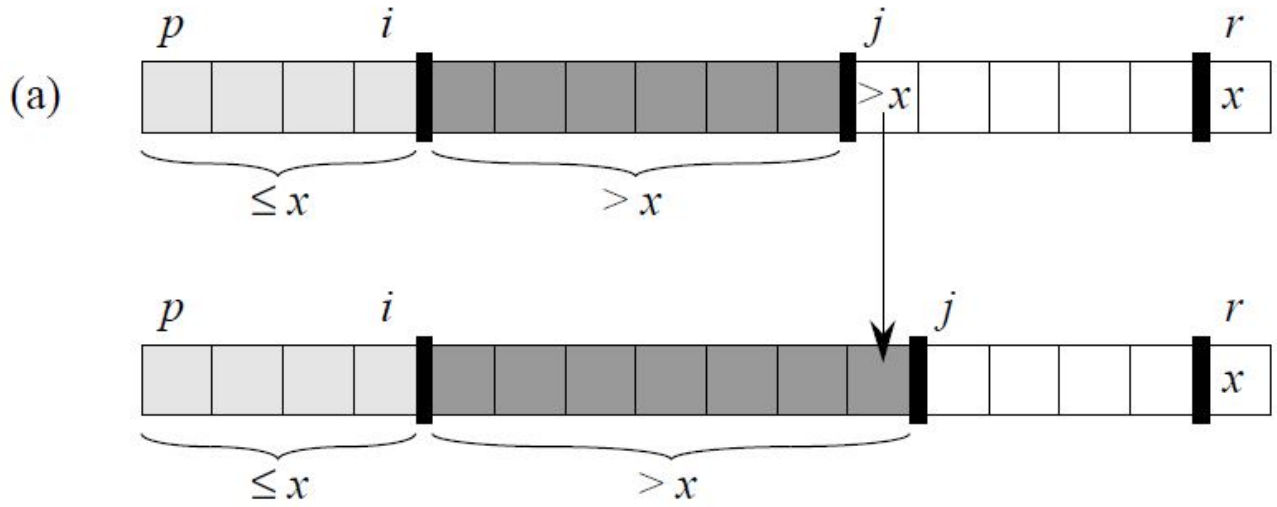
- Correctness is based on the correctness of partition.
- Loop Invariant:
  At the beginning of each iteration of the loop of lines 4–9, for any array index k:
  - If $p \leq k \leq i$, $then A[k] \leq pivot$.
  - If $i+1 \leq k \leq j-1$, then $A[k] > pivot$.
  - If $k = pivot$, then $A[k] = pivot$.
- The indices between j and $r-1$ are not covered by any of the three cases, and the values in these entries have no particular relationship to the pivot.

Quick Sort - Correctness Initialization



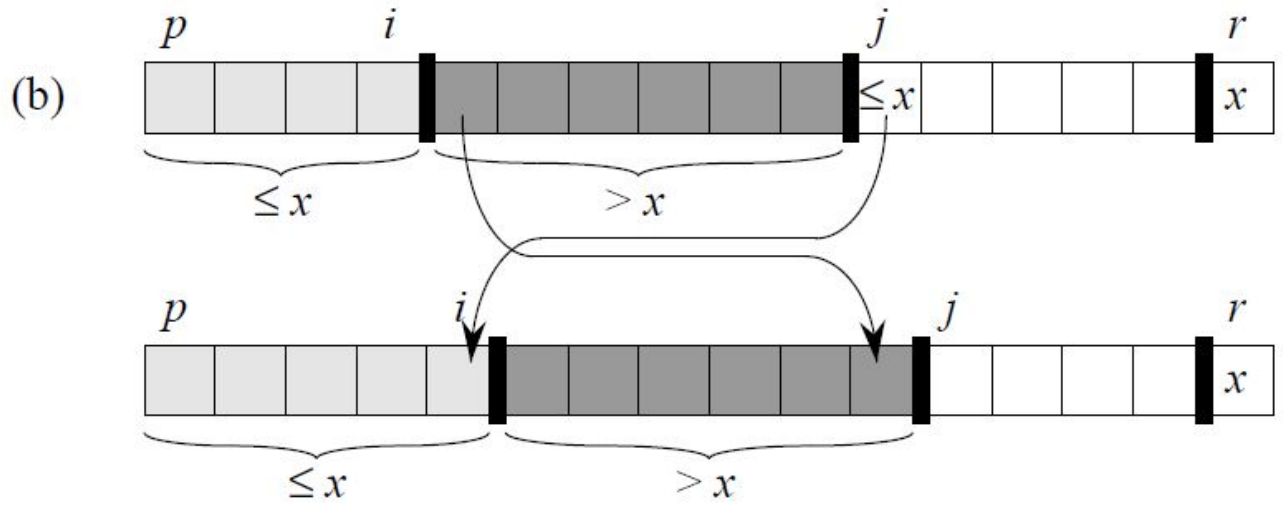
1. Initialization: Prior to the first iteration, $i = p - 1$ and $j = p$. Because no values lie between p and i and no values lie between $i + 1$ and $j - 1$, the first two conditions of the loop invariant are trivially satisfied. The assignment in line 2 satisfies the third condition.

Quick Sort - Correctness Maintenance 1



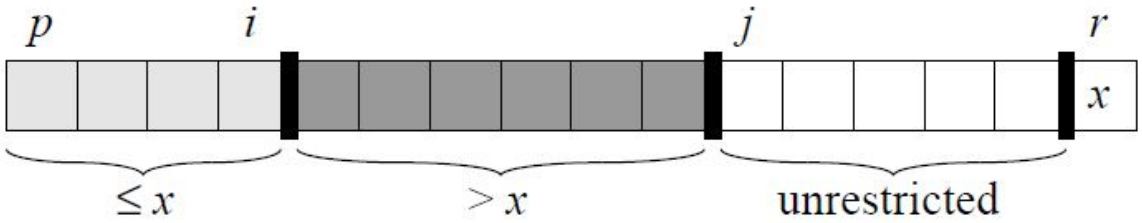
2. Maintenance: We have two cases.
   a. If $A[j] > x$, the only action in the loop is to increment j . After j is incremented, condition 2 holds for $A[j-1]$ and all other entries remain unchanged.

Quick Sort - Correctness Maintenance 2



2. **Maintenance:** We have two cases.
   b. If $A[j] \leq x$, the loop increments i , swaps $A[i]$ and $A[j]$, then and then increments j . Because of the swap, we now have that $A[i] \leq x$, and Condition 1 is satisfied. Similarly, we also have that $A[j-1] > x$, since the item that was swapped into $A[j-1]$ is greater than x.

Quick Sort - Termination



2. Termination: At termination, $j = r$. Therefore, every entry in the array is in one of the three sets described by the invariant, and we have partitioned the values in the array into three sets: those less than or equal to x, those greater than x, and a singleton set containing x.

Quick Sort - Analysis

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  - If the resulting subarrays are balanced, then $T(n) = 2T(\frac{n}{2}) + \Theta(n)$.
    This is $\Theta(nlog(n))$ by Case 2 of the master theorem.
  - If the resulting subarrays are not balanced, then $T(n) = T(n-1) + \Theta(n)$. This is in $\Theta(n^2)$.

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  - If the resulting subarrays are not balanced, then $T(n) = T(n-1) + \Theta(n)$. This is in $\Theta(n^2)$.
- The best case is as fast as merge sort. The worst case is as bad as insertion sort, but happens when the array is already sorted. The average case is as good as the best case.

Outline

Points to Take Home

1. The Master Theorem.
2. Quick Sort.
3. Reading Material:
   - Introduction to Algorithms, Chapter 4: Section 4.5 and Chapter 7: Sections 7.1 and 7.2.

Next Lecture: Greedy Algorithms!

Due Credits

The presented material is based on:

1. Previous editions of the course at the GUC due to Dr. Wael Aboulsaadat, Dr. Haythem Ismail, Dr. Amr Desouky, and Dr. Carmen Gervet.
2. Stony Brook University's Analysis of Algorithms Course.
3. MIT's Introduction to Algorithms Course.
4. Stanford's Design and Analysis of Algorithms Course.