German University in Cairo
Department of Computer Science
Dr. Nourhan Ehab

CSEN 703 Analysis and Design of Algorithms, Winter Term 2022
Practice Assignment 2

Exercise 2-1 From CLRS (©MIT Press 2001)

Asymptotically rank the following functions:
$n, n^{1/2}, log(n), log(log(n)), log^2(n), (\frac{1}{3})^n, 4, (\frac{3}{2})^n, n!$

Solution:

$(\frac{1}{3})^n < 4 < log(log(n)) < log(n) < log^2(n) < \sqrt(n) < n < (\frac{3}{2})^n < n!$

Exercise 2-2 From CLRS (©MIT Press 2001)

Explain why the statement: The running time of algorithm A is at least $O(n^2)$ is meaningless.

Solution:

The statement states that the upper bound for the lower bound of Algorithm A is $O(n^2)$.This does not make any sense as if $f(n)$ belong to a set of functions $g(n)$, one cannot describe the lower bound provided by $g(n)$ in terms of an upper bound. In fact, the correct notation to use to describe a lower bound of an algorithm is the Ω notation.

Exercise 2-3 From CLRS (©MIT Press 2001)

Prove that the running time of an algorithm is $\Theta(g(n))$ if and only if its worst-case running time is $O(g(n))$ and its best-case running time is $\Omega(g(n))$.

Solution:

Let the running time be $f(n)$. If $f(n) = \Theta(g(n))$, then $0 < c_1g(n) \leq f(n) \leq c_2g(n)$ where c_1 and c_2 are positive constants. Since $c_1g(n) \leq f(n)$, therefore $f(n) = \Omega(g(n))$. Moreover, since $f(n) \leq c_2g(n)$, then $f(n) = O(g(n))$.

Proving the other direction, if $f(n) = O(g(n)$, and $f(n) = \Omega(g(n))$, this implies that there are two positive constants c_1 and c_2 such that $f(n) \leq c_2g(n)$ and $c_1g(n) \leq f(n)$. Therefore, $c_1g(n) \leq f(n) \leq c_2g(n)$ satisfying the definition of the Θ notation. Hence, $f(n) = \Theta(g(n)$.

Exercise 2-4 From CLRS (©MIT Press 2001)

For every given $f(n)$ and $g(n)$ prove that $f(n) = \Theta(g(n))$

a) $g(n) = n^3$, $f(n) = 3n^3 + n^2 + n$

b) $g(n) = 2^n$, $f(n) = 2^{n+1}$

c) $g(n) = \ln(n)$, $f(n) = \log_{10}(n) + \log_{10}(\log_{10} n)$

Solution:

For all the given $f(n)$ and $g(n)$ we can prove that $f(n) = \Theta(g(n))$ using the limit test and/or proving the following statement:

$$0 \leq c_1g(n) \leq f(n) \leq c_2g(n) \, \forall \, n \geq n_0 \tag{1}$$

a) $g(n) = n^3,\ f(n) = 3n^3 + n^2 + n$

Solution 1 Using equation 1 we get

$$\begin{array}{lllllllll} & 0 & \le & c_1(g(n)) & \le & f(n) & \le & c_2(g(n)) & \forall n \ge n_0 \\ = & 0 & \le & c_1 n^3 & \le & 3n^3 + n^2 + n & \le & c_2 n^3 & \forall n \ge n_0 \end{array}$$

$$\textit{dividing by } n^3$$

$$\begin{array}{lllllllll} = & 0 & \le & c_1 & \le & 3 + \frac{1}{n} + \frac{1}{n^2} & \le & c_2 & \forall n \ge n_0 \end{array}$$

Choosing $c_1 = 3$, $c_2 = 5$, and $n_0 = 1$ helps us in proving the relations of the equation

$$\begin{array}{lllllll} = & 0 & \le & 3 & \le & 3+1+1 & \le & 5 \\ = & 0 & \le & 3 & \le & 5 & \le & 5 \end{array}$$

Therefore, $f(n) = \Theta(g(n))$

Solution 2

$$\begin{aligned} \lim_{n\to\infty} \frac{f(n)}{g(n)} &= \tfrac{3n^3+n^2+n}{n^3} \\ &= 3 + \tfrac{1}{n} + \tfrac{1}{n^2} \\ &= 3+0+0 \\ &= 3 \in R^+ \end{aligned}$$

Therefore, $f(n) = \Theta(g(n))$

b) $g(n) = 2^n,\ f(n) = 2^{n+1}$

Solution 1 Using equation 1 we get

$$\begin{array}{lllllllll} & 0 & \le & c_1(g(n)) & \le & f(n) & \le & c_2(g(n)) & \forall n \ge n_0 \\ = & 0 & \le & c_1 2^n & \le & 2^{n+1} & \le & c_2 2^n & \forall n \ge n_0 \end{array}$$

$$\textit{dividing by } 2^n$$

$$\begin{array}{lllllllll} = & 0 & \le & c_1 & \le & 2 & \le & c_2 & \forall n \ge n_0 \end{array}$$

Choosing $c_1 = 2$, $c_2 = 2$, and $n_0 = 1$ helps us in proving the relations of the equation

$$= \quad 0 \quad \le \quad 2 \quad \le \quad 2 \quad \le \quad 2$$

Therefore, $f(n) = \Theta(g(n))$.

Solution 2

$$\begin{aligned} \lim_{n\to\infty} \frac{f(n)}{g(n)} &= \tfrac{2^{n+1}}{2^n} \\ &= \tfrac{2.2^n}{2^n} \\ &= 2 \in R^+ \end{aligned}$$

Therefore, $f(n) = \Theta(g(n))$

c) $g(n) = \ln(n),\ f(n) = \log_{10}(n) + \log_{10}(\log_{10} n)$

Solution 1

$$\begin{aligned}\lim_{n\to\infty}\frac{f(n)}{g(n)} &= \tfrac{\log_{10}(n)+\log_{10}(\log_{10}(n))}{\ln(n)} \\ &= \tfrac{\infty}{\infty}{}^{1}\end{aligned}$$

$$Using\ L'H\hat{o}pital's\ rule$$

$$\begin{aligned}\lim_{n\to\infty}\frac{f'(n)}{g'(n)} &= \tfrac{\frac{1}{n\ln(10)}}{\frac{1}{n}} + \tfrac{\frac{1}{\log_{10}(n)\ln(10)}\cdot\frac{1}{n\ln(10)}}{\frac{1}{n}} \\ &= \tfrac{n}{n\ln(10)} + \tfrac{n}{\ln^2(10)n\log_{10}(n)} \\ &= \tfrac{1}{\ln(10)} + \tfrac{1}{\ln^2(10)\log_{10}(n)} \\ &= 0.434(3d.p) + 0 \in R^+\end{aligned}$$

Therefore, $f(n) = \Theta(g(n))$

Solution 2

$$\begin{array}{cccccccc} & 0 & \leq & c_1(g(n)) & \leq & f(n) & \leq & c_2(g(n)) & \forall n \geq n_0 \\ = & 0 & \leq & c_1\ln(n) & \leq & \log_{10}(n)+\log_{10}(\log_{10}(n)) & \leq & c_2\ln(n) & \forall n \geq n_0 \end{array}$$

$$dividing\ by\ \ln(n)$$

$$\begin{array}{cccccccc} = & 0 & \leq & c_1 & \leq & \log_{10}(e) + \frac{\log_{10}(\log_{10}(n))}{\ln(n)} & \leq & c_2 & \forall n \geq n_0 \end{array}$$

Choosing$c_1 = 0.434, c_2 = 0.5,$ and $n_0 = 10$ helps us in proving the relations of the equation

$$= \ 0 \ \leq \ 0.434 \ \leq \ 0.434 \ \leq \ 0.5$$

Therefore, $f(n) = \Theta(g(n))$

Exercise 2-5

For every given $f(n)$ and $g(n)$ prove that $f(n) = o(g(n))$ or $f(n) = \omega(g(n))$

a) $f(n) = n^3,\ g(n) = n^2$

b) $f(n) = \log(n),\ g(n) = \log^2(n)$

Solution:

a) $f(n) = n^3,\ g(n) = n^2$

$$\begin{aligned}\lim_{n\to\infty}\frac{f(n)}{g(n)} &= \tfrac{n^3}{n^2} \\ &= n \\ &= \infty\end{aligned}$$

[1]Note that $\frac{\log_{10}(n)}{\ln(n)}$ is the same as $\log_{10}(e)$ which is 0.434(3 d.p). The ∞ comes from the second part of the equation.

Therefore, $f(n) = \omega(g(n))$

b) $f(n) = \log(n)$, $g(n) = \log^2(n)$

$$\begin{aligned}\lim_{n\to\infty}\frac{f(n)}{g(n)} &= \tfrac{\log(n)}{\log^2(n)}\\ &= \tfrac{1}{\log(n)}\\ &= 0\end{aligned}$$

Therefore, $f(n) = o(g(n))$

Exercise 2-6 From CLRS (©MIT Press 2001)

Let $f(n)$ and $g(n)$ be asymptotically non-negative functions. Using the basic definition of Θ-notation, prove that $max(f(n), g(n)) = \Theta(f(n) + g(n))$.

Solution:

First let us define the function $h(n) = max(f(n), g(n))$ as follows:

$$h(n) = \begin{cases} f(n) & if\ \ f(n) \geq g(n), \\ g(n) & if\ \ g(n) > f(n). \end{cases}$$

We need to show that $0 \leq c_1(f(n)+g(n)) \leq h(n) \leq c_2(f(n)+g(n))$. Since $f(n)$ and $g(n)$ are asymptotically non-negative, there exists $n_0 > 0$ such that $f(n) \geq 0$ and $g(n) \geq 0$ for all $n \geq n_0$. Thus, $\forall n \geq n_0$, $f(n) + g(n) \geq f(n) \geq 0$ and $f(n) + g(n) \geq g(n) \geq 0$. Since for any n, $h(n)$ is either $f(n)$ or $g(n)$, then $f(n) + g(n) \geq h(n) \geq 0$ as well. Therefore, $h(n) = max(f(n), g(n)) \leq c_2(f(n) + g(n))$ for all $n \geq n_0$ (taking $c_2 = 1$ in the definition of Θ notation).

Similarly, for all n, $h(n)$ is the bigger value of $f(n)$ and $g(n)$. Therefore, $0 \leq f(n) \leq h(n)$ and $0 \leq g(n) \leq h(n)$. Adding both equations we get $0 \leq f(n) + g(n) \leq 2h(n)$, or equivalently $0 \leq \frac{1}{2}(f(n) + g(n)) \leq h(n)$ (taking $c_1 = \frac{1}{2}$ in the definition of Θ notation).

Therefore, taking $c_1 = \frac{1}{2}$, $c_2 = 1$ and $n_0 = 1$, the definition of the Θ notation is satisfied and $h(n) = max(f(n), g(n)) = \Theta(f(n) + g(n))$.

Exercise 2-7 From CLRS (©MIT Press 2001)

Show that for any real constants a and b, where b>0, $(n + a)^b = \Theta(n^b)$.

Solution:

We need to find $c_1, c_2, n_0 > 0$ such that $0 \leq c_1 n^b \leq (n + a)^b \leq c_2 n^b \ \ \forall n \geq n_0$. We start by trying to find $c_1, c_2, n_0 > 0$ such that :

$$0 \leq c_1 n \leq n + a \leq c_2 n \ \ \forall n \geq n_0.$$

Choosing $n_0 = 2a$, we get:

$$0 \leq c_1(2a) \leq 3a \leq c_2(2a) \ \ \forall n \geq n_0 \quad (dividing\ \ by\ \ 2a)$$

$$0 \leq c_1 \leq \frac{3}{2} \leq c_2 \ \ \forall n \geq n_0$$

Therefore, choosing $c_1 = \frac{1}{2}$ and $c_2 = 1$ the above equation holds. Since $b > 0$ the inequality still holds when all parts are raised to power b:

$$0 \leq (\frac{1}{2}n)^b \leq (n + a)^b \leq (2n)^b \ \ \forall n \geq n_0$$

$$0 \leq (\frac{1}{2})^b n^b \leq (n + a)^b \leq 2^b n^b \ \ \forall n \geq n_0$$

The definition of the Θ notation is satisfied choosing $c_1 = (\frac{1}{2})^b$, $c_2 = 2^b$, $n_0 = 2a$. Therefore, $(n+a)^b = \Theta(n^b)$ where $b > 0$.

Exercise 2-8

Prove that, for $a, b \in \mathbb{R}$, $b > a \rightarrow a^n = o(b^n)$.

Solution:

Using the limit test: $\lim_{n\to\infty} \frac{a^n}{b^n} = \lim_{n\to\infty} (\frac{a}{b})^n = 0$ since b>a.
Therefore, $a^n = o(b^n)$.