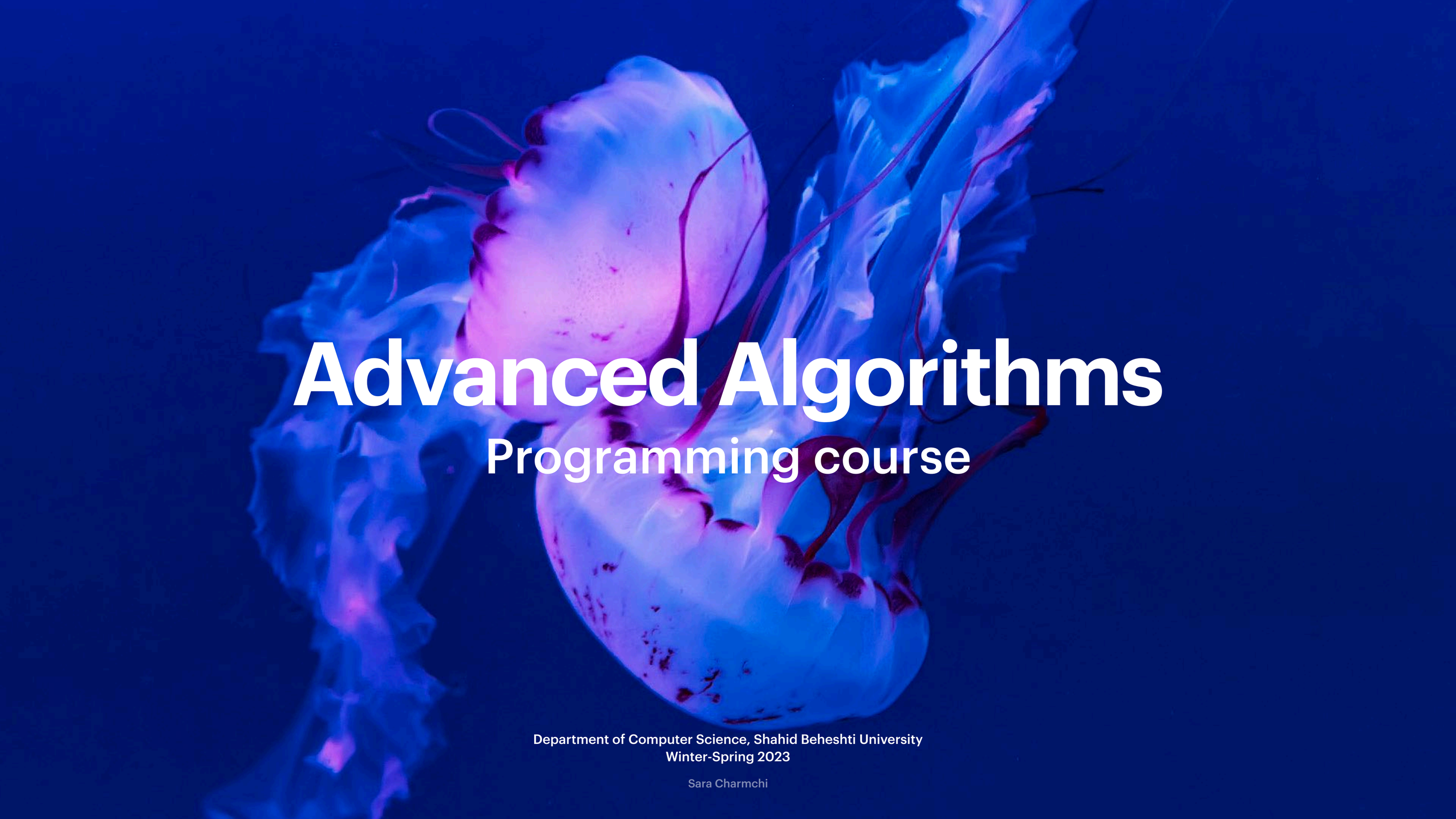


Advanced Algorithms

Programming course

Department of Computer Science, Shahid Beheshti University
Winter-Spring 2023

Sara Charmchi

Strategies for Algorithm Design

Dynamic Programming

Dynamic Programming
Definition

- General, powerful algorithm design technique
- Optimization problems : max/min result

DP≈ “Careful Exhaustive Search”

DP≈ Subproblems + re-use

re-use the solutions of subproblems

main problem —> split main problem subproblems —>slove subproblems

Dynamic Programming

Definition; Fibonacci numbers

0,1,1,2,3,5,8,13,...

$$\begin{cases} F_1 = F_2 = 1 \\ F_n = F_{n-1} + F_{n-2} \end{cases}$$

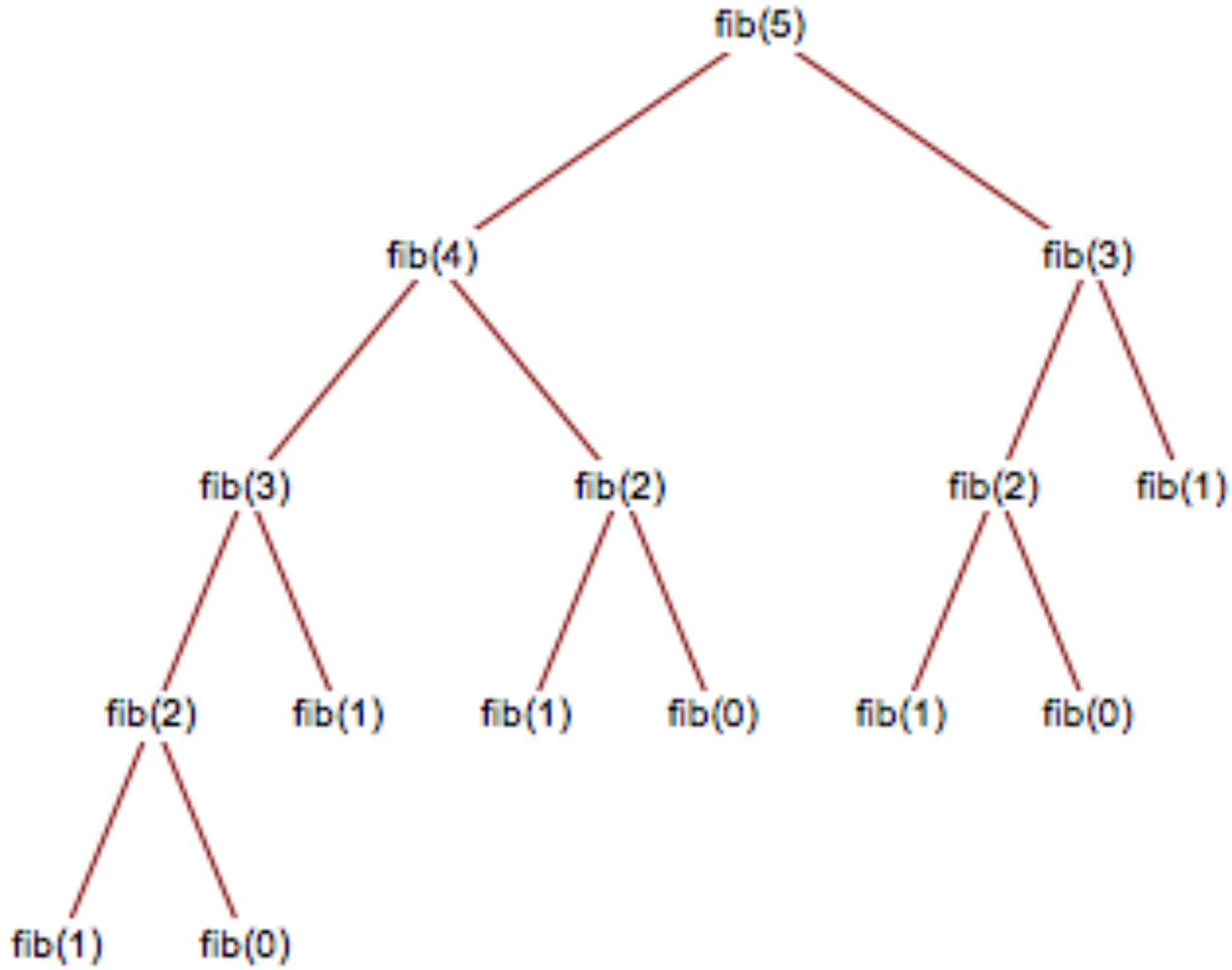
- Goal : compute F_n

Dynamic Programming

Fibonacci numbers; Naive Approach

```
fib(n):
    if n≤1 : f=n
    else : f = fib(n-1) + fib(n-2)
 return f
```

compute fib(n=5) :recursion tree



Dynamic Programming

Fibonacci numbers; Naive Approach

```
fib(n):
    if n≤1 : f=n
 else : f = fib(n−1) + fib(n−2)
 return f
```

Time complexity ? exponential time

consider running time as recurrence :

$T(n) = T(n-1) + T(n-2) + \Theta(1)$

$T(n) \geq 2T(n-2)$ ⟶ $T(n) = O(2^{n/2})$

Dynamic Programming

Fibonacci numbers; DP : Memoization (Top-down)

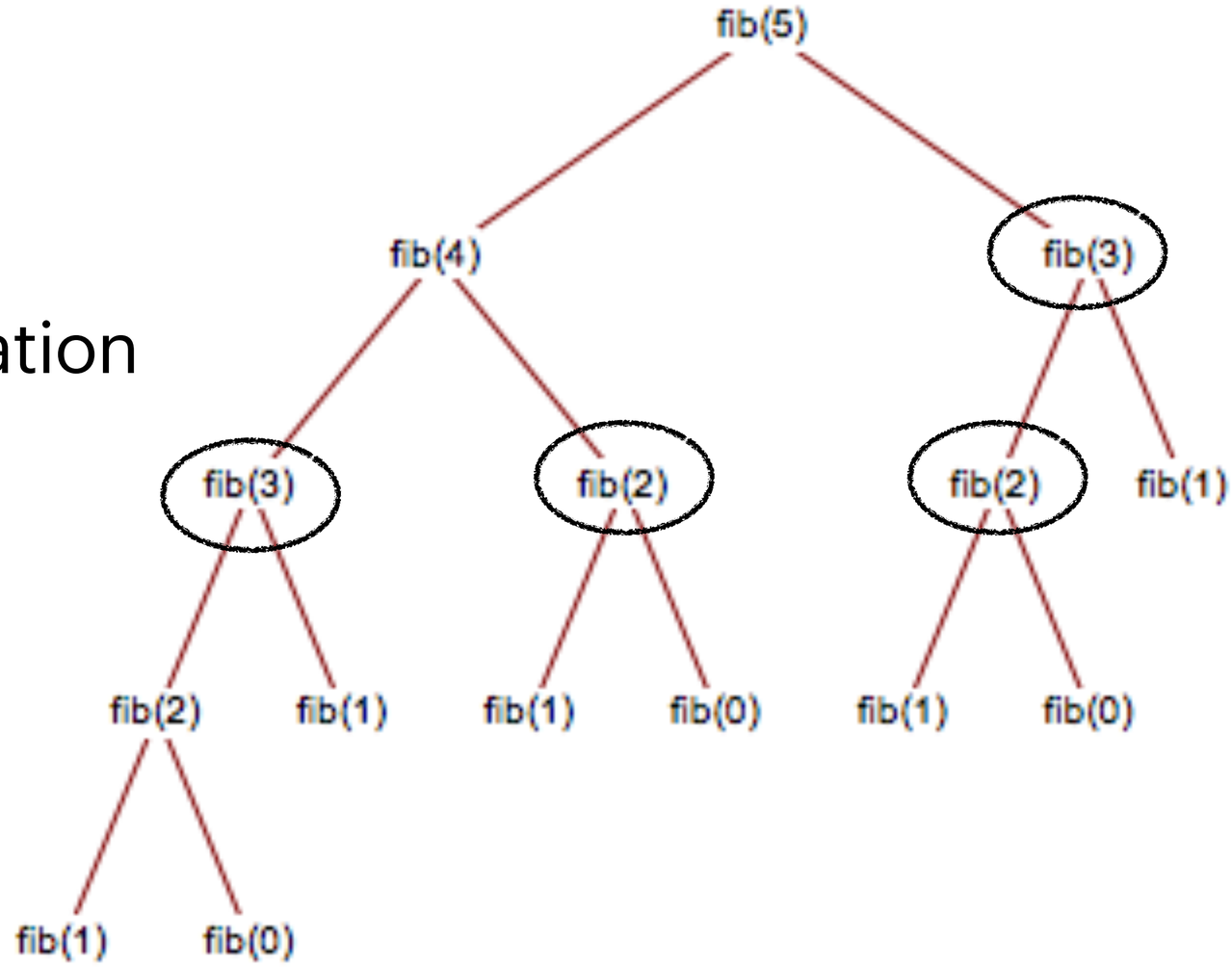
Any better idea?

store the computed result
somewhere, to avoid re-computation

Fib(n=5) ; take one global array

initially fill with -1 :

-1	-1	-1	-1	-1	-1
0	1	2	3	4	5



Dynamic Programming

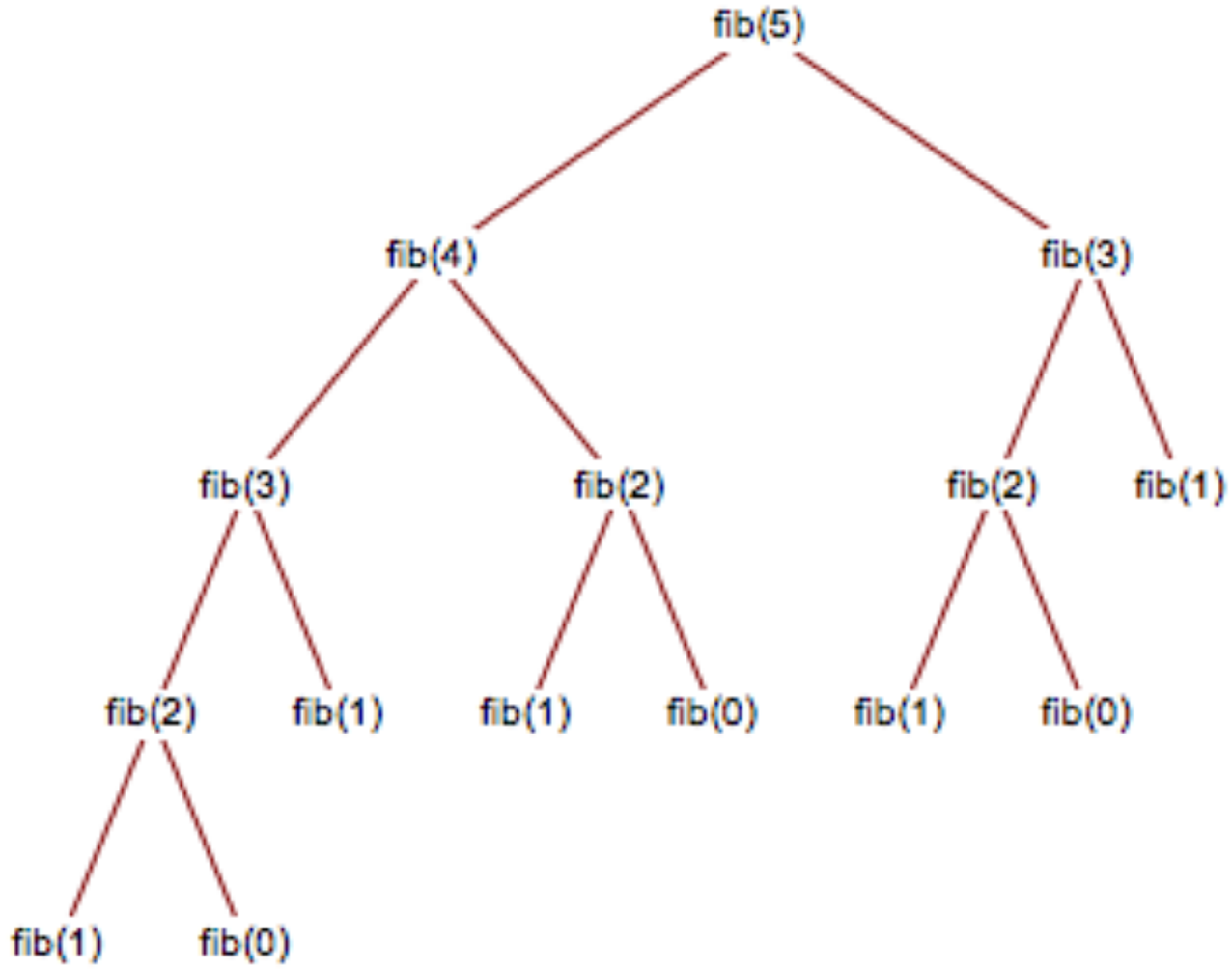
Fibonacci numbers; DP : Memoization

```
fib(n):
    if n≤1 : f=n
 else : f = fib(n-1) + fib(n-2)
 return f
```

Fib(n=5) ; take one global array

Call Fib(n=5) —> follow the tree

-1	-1	-1	-1	-1	-1
0	1	2	3	4	5



Dynamic Programming

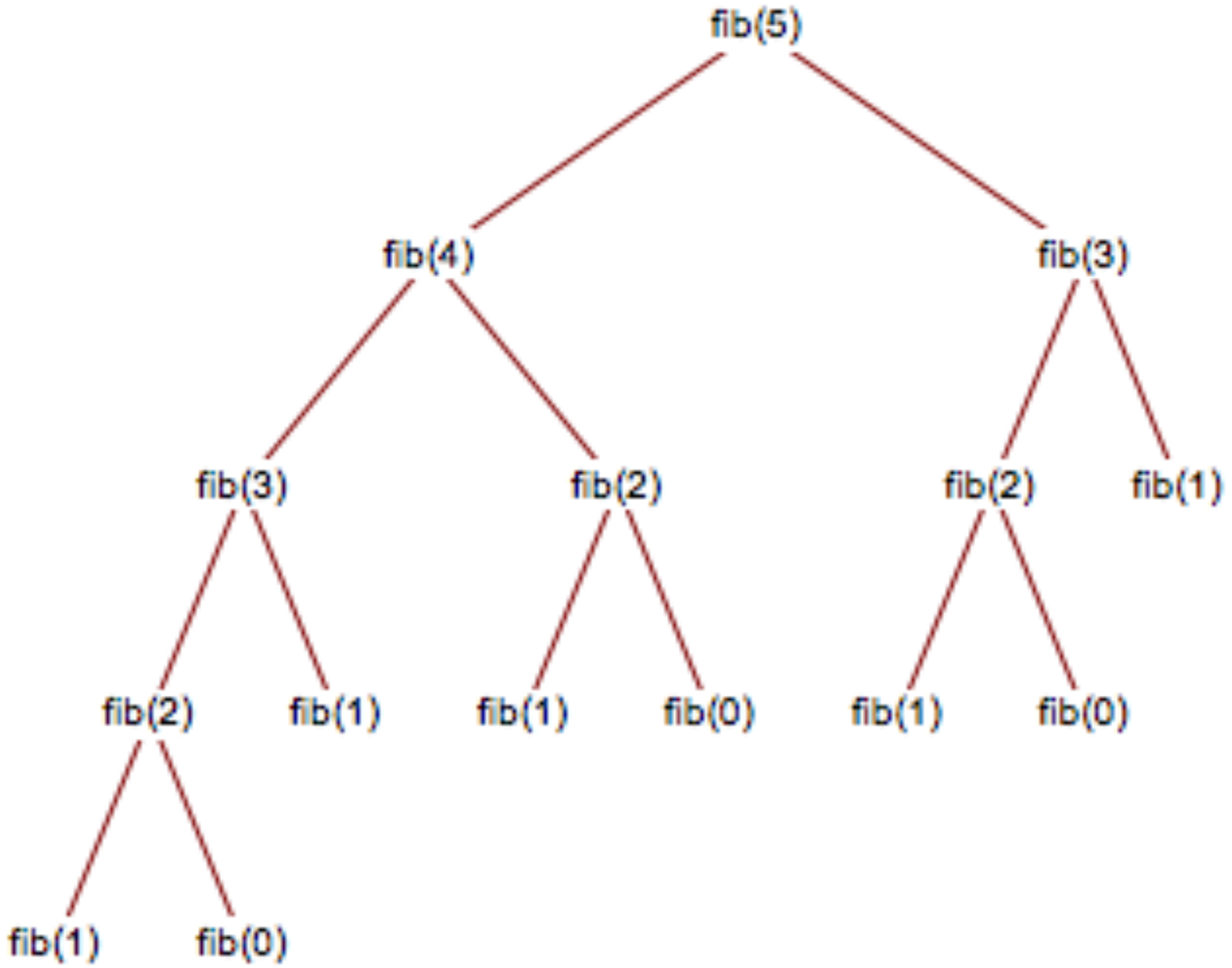
Fibonacci numbers; DP : Memoization

```
fib(n):
    if n≤2 : f=1
 else : f = fib(n-1) + fib(n-2)
 return f
```

Fib(n=5) ; take one global array

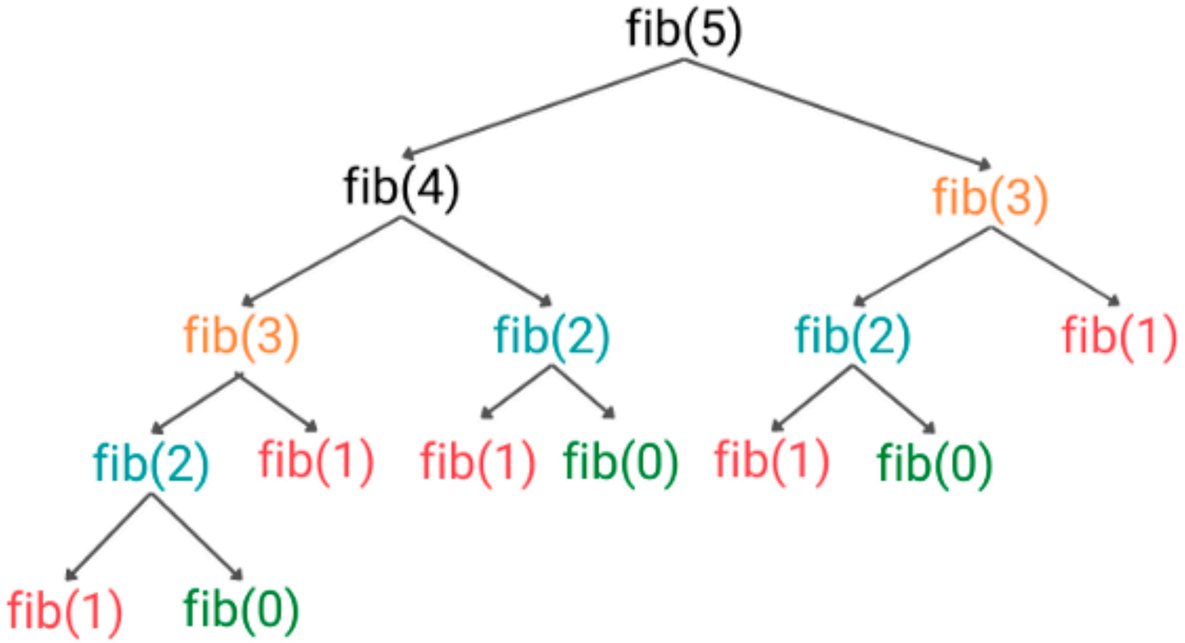
Call Fib(n=5) —> follow the tree

0	1	1	2	3	5
0	1	2	3	4	5

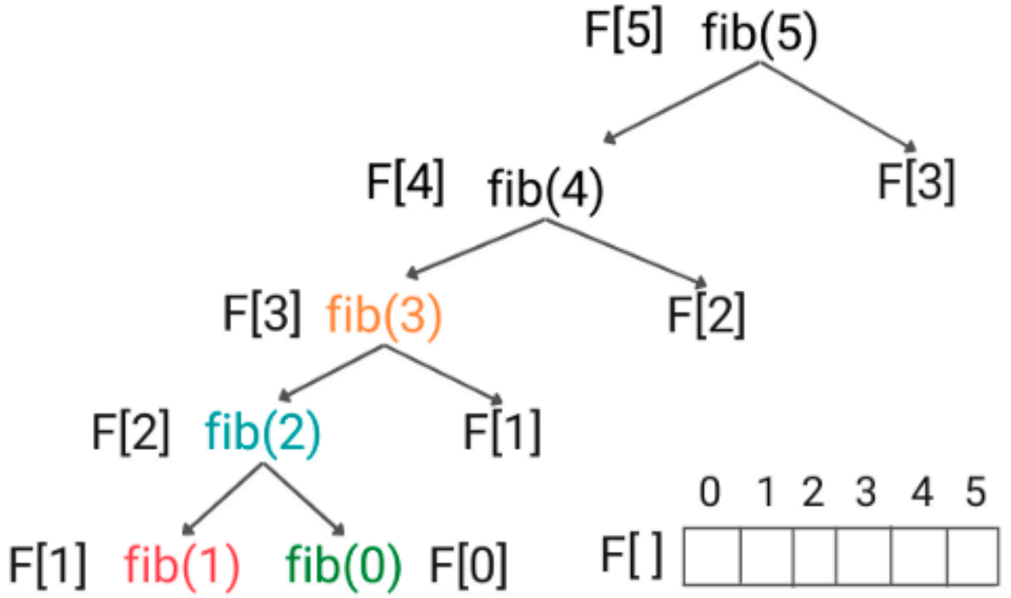


Dynamic Programming

Fibonacci numbers; DP : Memoization



Calculating 5th Fibonacci recursively.



Calculating 5th Fibonacci recursively but storing sub-problems solution in extra memory.

Dynamic Programming

Fibonacci numbers; DP : Memoization

```
memo = {}
fib(n):
    if n in memo : return memo [n]
 if n≤2 : f=1
 else : f = fib(n-1) + fib(n-2)
 memo[n] = f

 return f
```

f(k) only recurses the first time it's called, $\forall k$

two versions : memoized calls, non-memoized calls

Dynamic Programming

Fibonacci numbers; DP : Memoization

- Memoized calls cost constant time : $\Theta(1)$
- Number of non-memoized calls is n

  fib(1),fib(2),fib(3),...,fib(n)
- non recursive work per call is constant

Total time complexity : linear : $\Theta(n)$

```
memo = {}
fib(n):
    if n in memo : return memo [n]
  if n≤2: f=1
  else : f = fib(n-1) + fib(n-2)
  memo[n] = f

  return f
```

Naive Approach : Exponential ⟶ Dynamic Programming : Linear

Dynamic Programming

DP : Memoization

- Memoize (remember) & reuse solutions to subproblems that help solve the problem

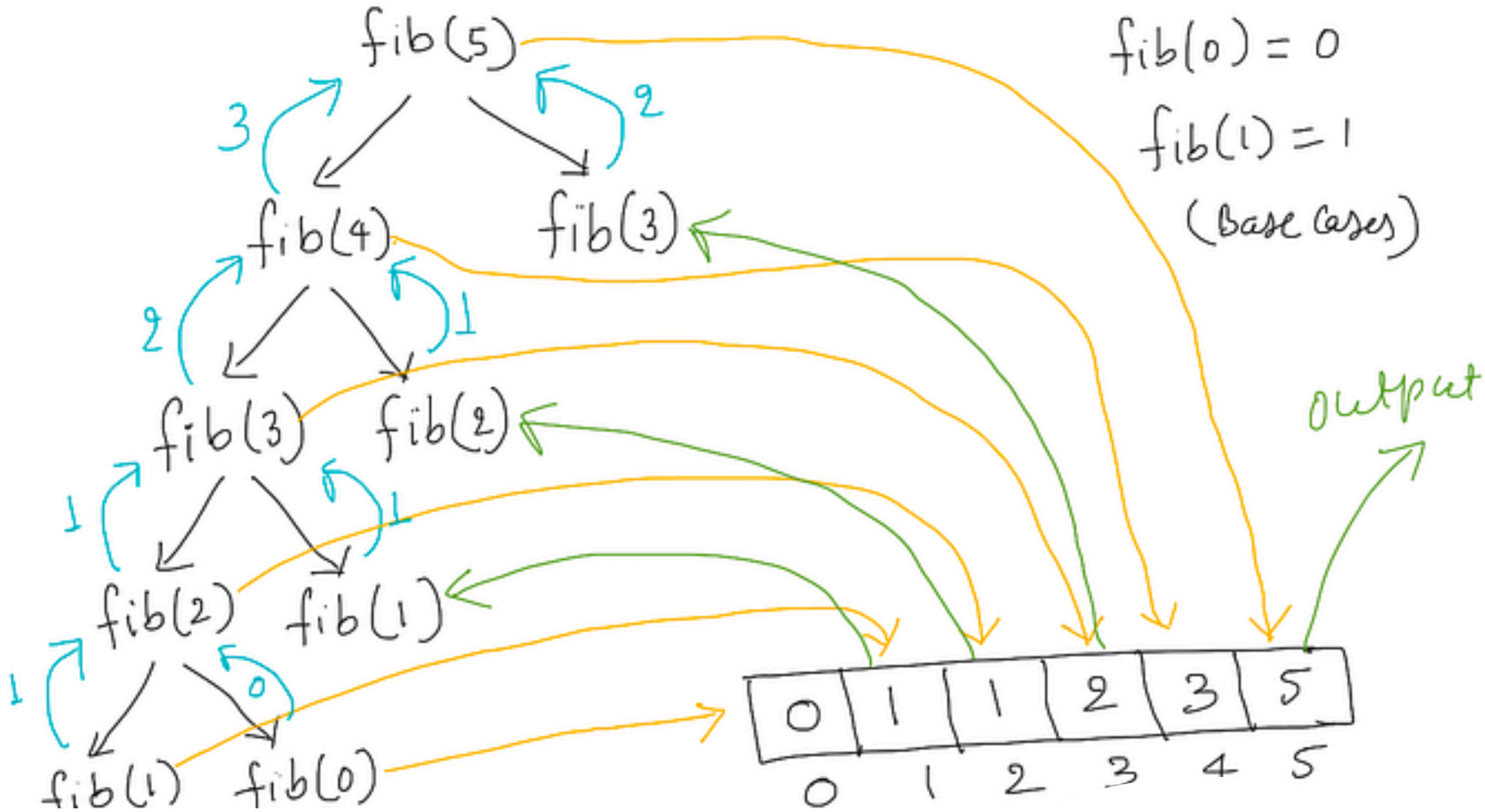
DP≈ recursion + memoization

- for any DP algorithm the time complexity would be :

time = number of subproblems * time/subproblem

Dynamic Programming

Fibonacci numbers; DP : Tabulation (Bottom-up)

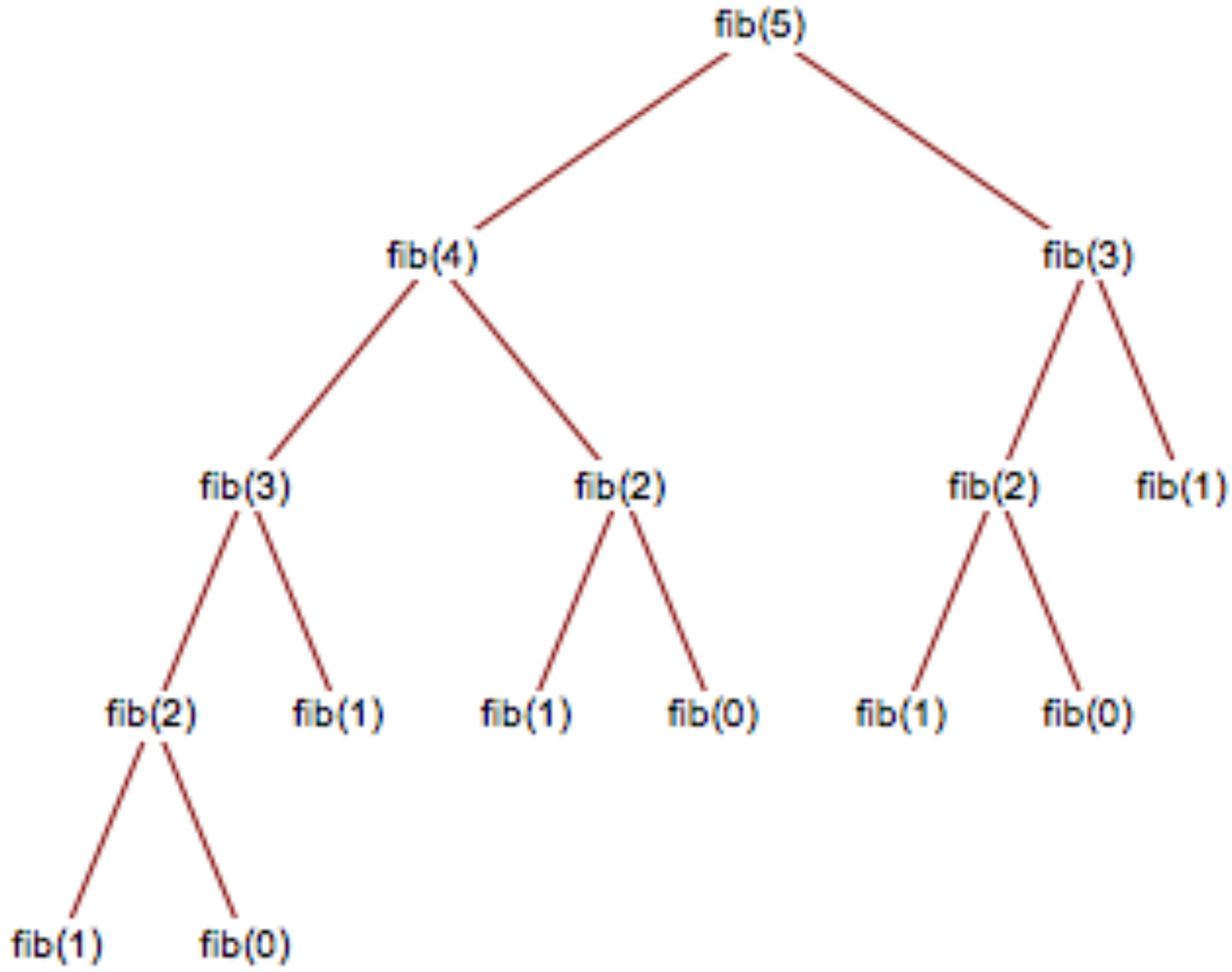


Dynamic Programming

Fibonacci numbers; DP : Tabulation (Bottom-up)

```
fib = {}
for k in range(1,n+1):
    if k≤2 f=1
    else f=fib[k-1]+fib[k-2]
    fib[k]=f
return fib[n]
```

0	1	1	2	3	5
0	1	2	3	4	5



Dynamic Programming

Fibonacci numbers; DP : Tabulation (Bottom-up)

```
fib = {}
for k in range(1,n+1):
    if k≤2 f=1
    else f=fib[k-1]+fib[k-2]
    fib[k]=f
return fib[n]
```

same method, different perspective

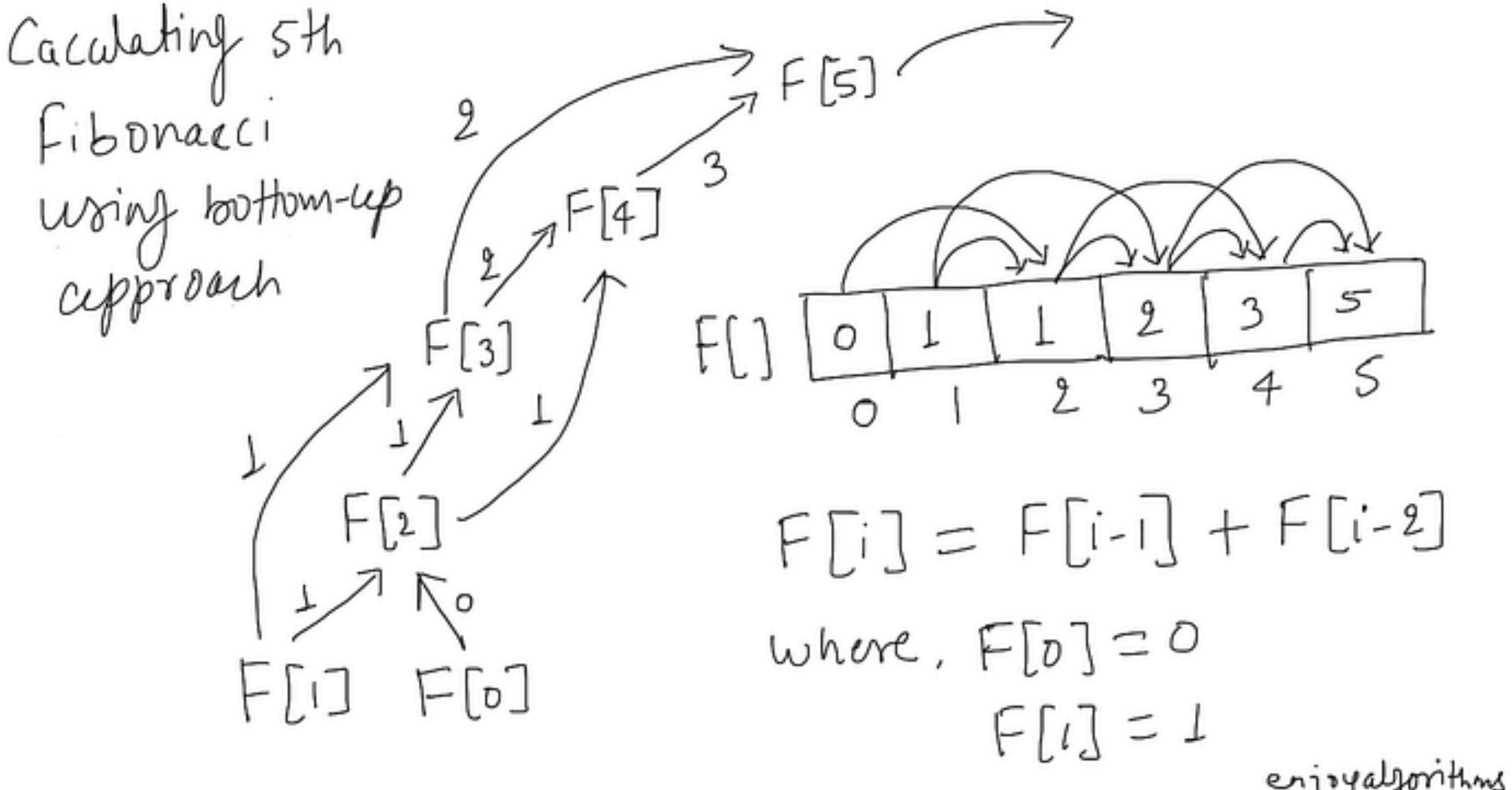
main difference : use loop instead of recursion

which one is better in practice?

Tabulation ⟶ not recursive calls , just look in a table. ⟶ save space

Dynamic Programming

DP : Tabulation



Dynamic Programming
DP : Tabulation

- Topological sort of subproblem dependency **directed acyclic graph(DAG)** :
  - vertices (the subproblems) and edges between the subproblems (dependencies).
  - The edges have a direction, since it matters which subproblem for every connected pair is the dependent one.
  - The graph has no cycles, meaning there must not be a way to start at one subproblem, then end up at that same subproblem just by following the arrows. Otherwise, we would end up in a situation where, to calculate one subproblem, we'd need to first calculate itself!

Rod-Cutting Problem
Definition

- you are in a company that buys long steel rods and cuts them into shorter rods, and then sells them. cutting is free
- they hired you to find the best way to cut up the rods

- the company charges p_i $ for a length of rod i. Rods are always integral in length

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

- Rod-cutting problem ; Given a rod of length n inches and a table of pieces p_i (i is in the range ($1,2,\ldots,n$), determine the maximum revenue r_n obtained by cutting the rod and selling the pieces. if no cutting gives the best price, we don't cut at all.

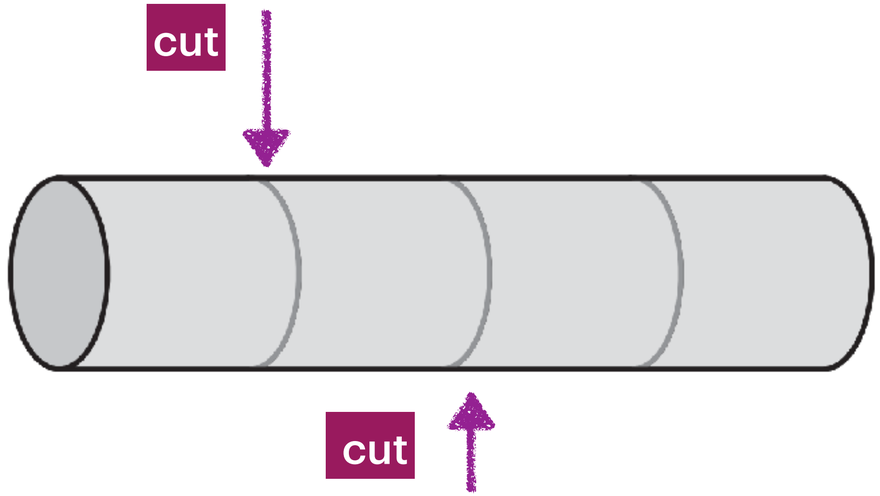
Rod-Cutting Problem
Definition

- you are given a rod of 4 inches and a price table

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

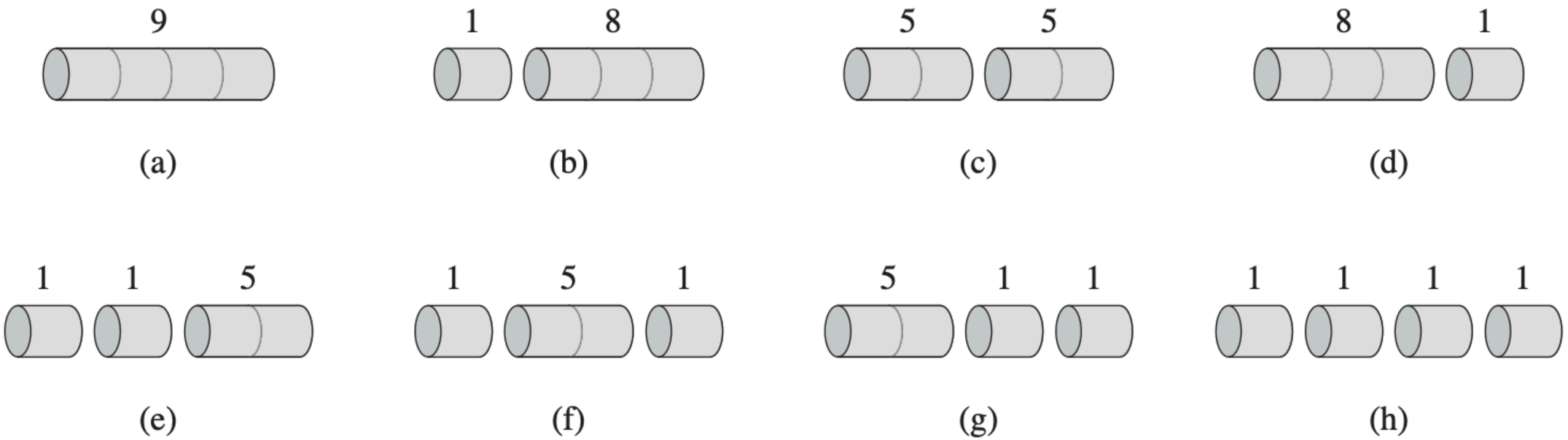
- should you cut the rod at all ?

- no cut : l=4 → 9
- 1 cut :1,3 : l= 1+3 → 1+8=9
- 1 cuts :2,2 l=2+2 → 5+5=10 ✓



Rod-Cutting Problem
Definition

- what are all possible cuts for length=4 ?



- we cut up a rod of length n in 2^{n-1} different ways

Rod-Cutting Problem

Definition

Length(n)	Price(p_n)	Cut/Don't Cut	Revenue(r_n)
1	1	N	1
2	5	N	5
3	8	N	8
4	9	2,2	10
5	10	2,3	13

$$
\begin{aligned}
r_1 &= 1 \quad \text{from solution } 1 = 1 \quad \text{(no cuts)}, \\
r_2 &= 5 \quad \text{from solution } 2 = 2 \quad \text{(no cuts)}, \\
r_3 &= 8 \quad \text{from solution } 3 = 3 \quad \text{(no cuts)}, \\
r_4 &= 10 \quad \text{from solution } 4 = 2 + 2, \\
r_5 &= 13 \quad \text{from solution } 5 = 2 + 3, \\
r_6 &= 17 \quad \text{from solution } 6 = 6 \quad \text{(no cuts)}, \\
r_7 &= 18 \quad \text{from solution } 7 = 1 + 6 \text{ or } 7 = 2 + 2 + 3, \\
r_8 &= 22 \quad \text{from solution } 8 = 2 + 6, \\
r_9 &= 25 \quad \text{from solution } 9 = 3 + 6, \\
r_{10} &= 30 \quad \text{from solution } 10 = 10 \quad \text{(no cuts)}.
\end{aligned}
$$

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

Rod-Cutting Problem
Formal Definition

- The optimal revenue from a rod of length n , r_n :

$$r_n = \max(p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots, r_{n-1} + r_1) \ .$$

p_n ⟶ making no cut at all, $r_1 + r_{n-1}$ ⟶ a cut so that we have rods of length *1* and *n-1*

- In other words :

$$r_n = \max_{1<i<n} (p_i + r_{n-i}) \ .$$

p_i ⟶ piece of rod length *i*, r_{n-1} ⟶ revenue from rod of length *n-1*

- Is this a recursive approach?

Rod-Cutting Problem

Recursive top-down Implementation

```
CUT-ROD(p, n)
1  if n == 0
2      return 0
3  q = -∞
4  for i = 1 to n
5      q = max(q, p[i] + CUT-ROD(p, n - i))
6  return q
```

Rod-Cutting Problem

Recursive top-down Implementation

```
price = [-100000,1, 5, 8, 9, 10, 17, 17, 20]
# Returns the best obtainable price for a rod of length n
# and price[] as prices of different pieces
def cutRod(price, n):
    if(n <= 0):
        return 0
    max_val = -100000

    # Recursively cut the rod in different pieces
    # and compare different configurations
    for i in range(1, n+1):
        max_val = max(max_val, price[i] +
                      cutRod(price, n - i))
    return max_val
```

Rod-Cutting Problem

Recursive top-down Implementation

- what is one limitation of the solution?

  gives the optimal revenue, but no answer to where to cut the rod!
- why is the recursive program so inefficient(slow) ?

Rod-Cutting Problem

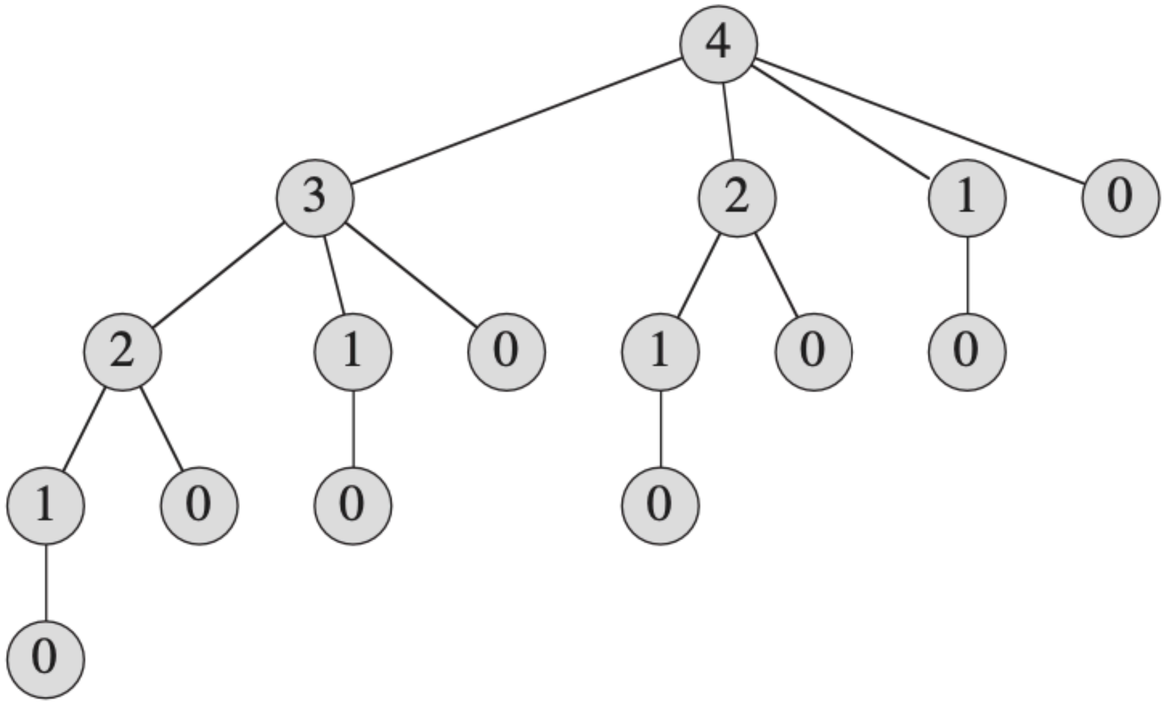
Analyze Recursive solution : recursion tree

- the rod-cut algorithm calls itself recursively over and over again with the same parameter values; it solves the same problems repeatedly

- the running time of the algorithm is exponential in n :

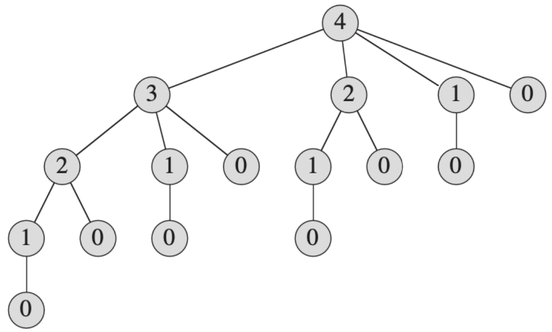
$$T(n) = O(2^n)$$

$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i}) .$$



Rod-Cutting Problem
Dynamic Programming

- We arrange for each subproblem to be solved only **once**, saving it's solution
- During the computation if we need the solution to a subproblem again, we simply look it up rather than recomputing it.
- Two Dynamic Programming approaches:
    - Top-Down Dynamic Programming approach with memoization
    - Bottom-Up Dynamic Programming approach



Rod-Cutting Problem

Dynamic Programming : Memoization

```
MEMOIZED-CUT-ROD(p, n)
1  let r[0..n] be a new array
2  for i = 0 to n
3      r[i] = -∞
4  return MEMOIZED-CUT-ROD-AUX(p, n, r)
```

initialize new auxilary array with -inf

```
MEMOIZED-CUT-ROD-AUX(p, n, r)
1  if r[n] ≥ 0
2      return r[n]
3  if n == 0
4      q = 0
5  else q = -∞
6      for i = 1 to n
7          q = max(q, p[i] + MEMOIZED-CUT-ROD-AUX(p, n - i, r))
8  r[n] = q
9  return q
```

if the solution is already in the table

r : presolved table of solutions;an array that maintains the best known solution for subproblem

memoized version of previous procedure

once found the solution, store it in the table

Rod-Cutting Problem

Dynamic Programming (Memoization) vs Recursive

```
CUT-ROD(p, n)
1  if n == 0
2      return 0
3  q = -∞
4  for i = 1 to n
5      q = max(q, p[i] + CUT-ROD(p, n - i))
6  return q
```

each time you increase n by 1, your program's running time would approximately double

```
MEMOIZED-CUT-ROD-AUX(p, n, r)
1  if r[n] ≥ 0
2      return r[n]
3  if n == 0
4      q = 0
5  else q = -∞
6      for i = 1 to n
7          q = max(q, p[i] + MEMOIZED-CUT-ROD-AUX(p, n - i, r))
8  r[n] = q
9  return q
```

check whether it is already in the table

update the table

Rod-Cutting Problem

Dynamic Programming : Tabulation

```
BOTTOM-UP-CUT-ROD(p, n)
1  let r[0..n] be a new array
2  r[0] = 0
3  for j = 1 to n
4      q = -∞
5      for i = 1 to j
6          q = max(q, p[i] + r[j - i])
7      r[j] = q
8  return r[n]
```

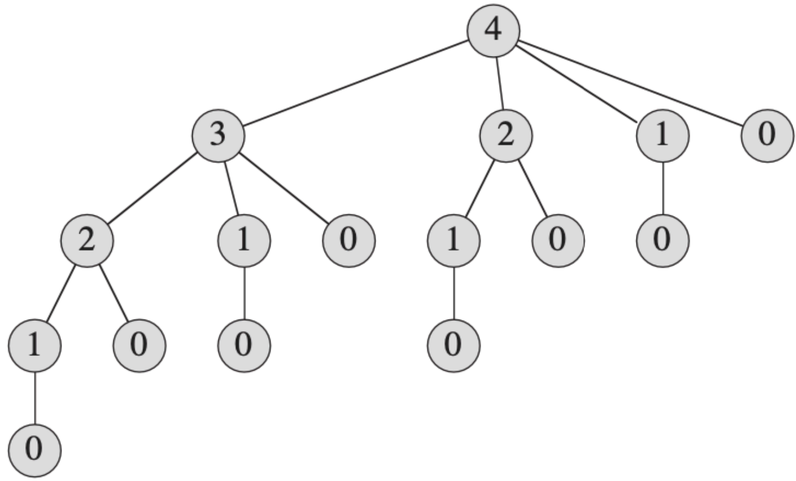
creates a new array

initialize r[0]

calculate the revenue for the rod of all the length :n

looks for all the possible ways to cut the rod

no recursive call



- a problem of size i is “smaller” than a subproblem of size j if i < j
- Thus, the procedure solves subproblems of sizes j=0,1,…,n, in that order

Rod-Cutting Problem

Dynamic Programming (Memoization) vs Recursive

- the bottom-up and top-down versions have same asymptotic running time : $\Theta(n^2)$, because of the double for loops
- the bottom-up approach usually outperforms the top-down approach by a small constant factor

```
MEMOIZED-CUT-ROD-AUX(p, n, r)
1  if r[n] ≥ 0
2      return r[n]
3  if n == 0
4      q = 0
5  else q = −∞
6      for i = 1 to n
7          q = max(q, p[i] + MEMOIZED-CUT-ROD-AUX(p, n − i, r))
8  r[n] = q
9  return q
```

```
BOTTOM-UP-CUT-ROD(p, n)
1  let r[0..n] be a new array
2  r[0] = 0
3  for j = 1 to n
4      q = −∞
5      for i = 1 to j
6          q = max(q, p[i] + r[j − i])
7      r[j] = q
8  return r[n]
```

Rod-Cutting Problem

Returning actual solution

- For each rod of size j, we would like to compute not only the maximum revenue r_j, but also s_j, the optimal size of the first piece to cut off.

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

i	0	1	2	3	4	5	6	7	8	9	10
$r[i]$	0	1	5	8	10	13	17	18	22	25	30
$s[i]$	0	1	2	3	2	2	6	1	2	3	10

Rod-Cutting Problem

Returning actual solution

```
EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
 1  let r[0..n] and s[0..n] be new arrays
 2  r[0] = 0
 3  for j = 1 to n
 4      q = -∞
 5      for i = 1 to j
 6          if q < p[i] + r[j - i]
 7              q = p[i] + r[j - i]
 8              s[j] = i
 9      r[j] = q
10  return r and s
```

list of indexes to make the optimal cut

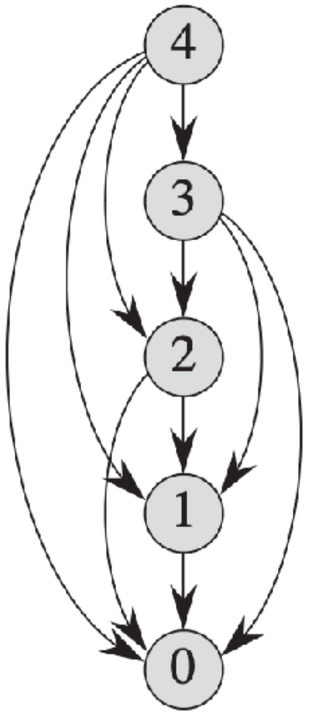
max —> if

```
PRINT-CUT-ROD-SOLUTION(p, n)
1  (r, s) = EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
2  while n > 0
3      print s[n]
4      n = n - s[n]
```

Rod-Cutting Problem

Subproblem Graphs

- When applying dynamic programming to a problem we should understand the set of subproblems involved.
- How do the subproblems depend on each other?
- A subproblem graph for the problem embodies this information
- It is like a ‘reduced’ or ‘collapsed’ version of the recursion tree



- subproblem graph for n=4 for the rod -cutting problem
- the vertex labels give the size of the corresponding subproblems
- a directed edge x->y indicates that we need a solution to subproblem y when solving subproblem x

Elements of Dynamic Programming

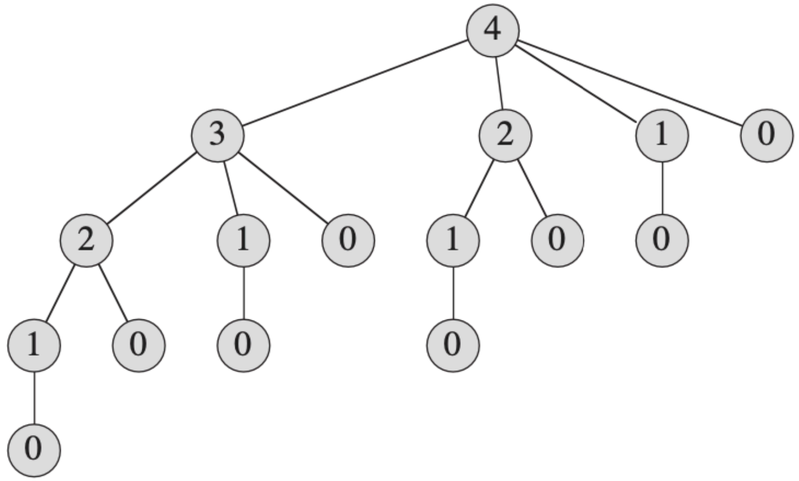
Two key ingredients that an optimization problem must have in order for dynamic programming to apply :

- **Optimal Substructure**

  a problem exhibits Optimal Substructure if an optimal solution to the problem containns within it optimal solutions to subproblems

- **Overlapping subproblems**

  the space of subproblems must be ‘small’ in the sense that the recursive algorithm for the problem solves the same subproblems over and over (not generating new subproblems)



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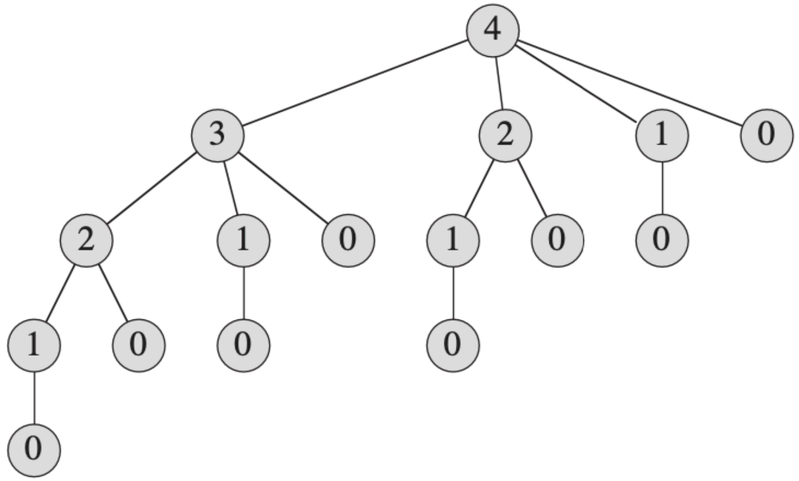
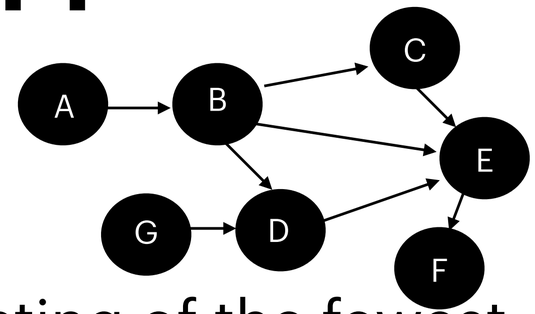


# of unique subproblems << # of total subproblems

When is Dynamic Programming not applicable?



Given a directed graph G= (V,E) and vertices u,v ∈ V

✓ **(a)Unweighted shortest path** : Find a path from u to v consisting of the fewest edges. Such a path must be simple (no cycles).

Any path p from u to v must contain w (say). Then we can decompose p into p1 and p2 so that p1 is the path from u to w and p2 from w to v

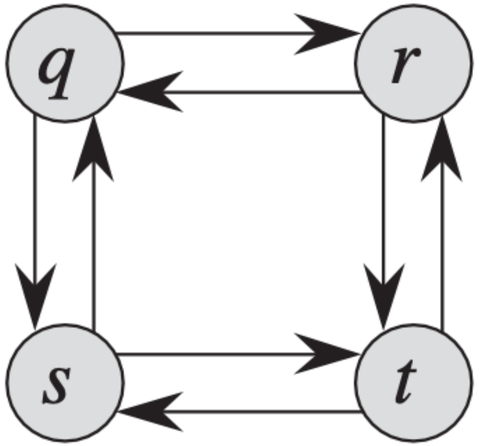
Can we argue that p1 is the shortest path from u to w, and p2 from w to v?

✗ **(a)Unweighted longest path** : Find a simple path from u to v consisting of the most edges

Unweighted longest path

Say we have a path p as the longest path from u to v. we have intermediate vertex w that lies in the path such that path p is composed of p1 and p2.

Does this imply that p1 is the longest path from u to w and p2 is the longest path from w to v?

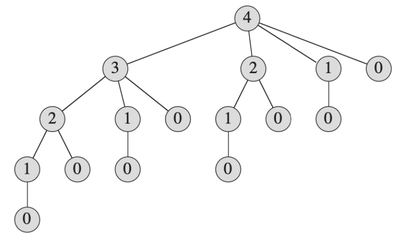


consider path q—>r—>t → the longest path from q to t

Is q—>r the longest path from q to r?

Is r—>t the longest path from r to t?

Steps for developing a dynamic programming algorithm



1. Characterize the structure of n optimal solution
2. Recursively define the value of an optimal solution

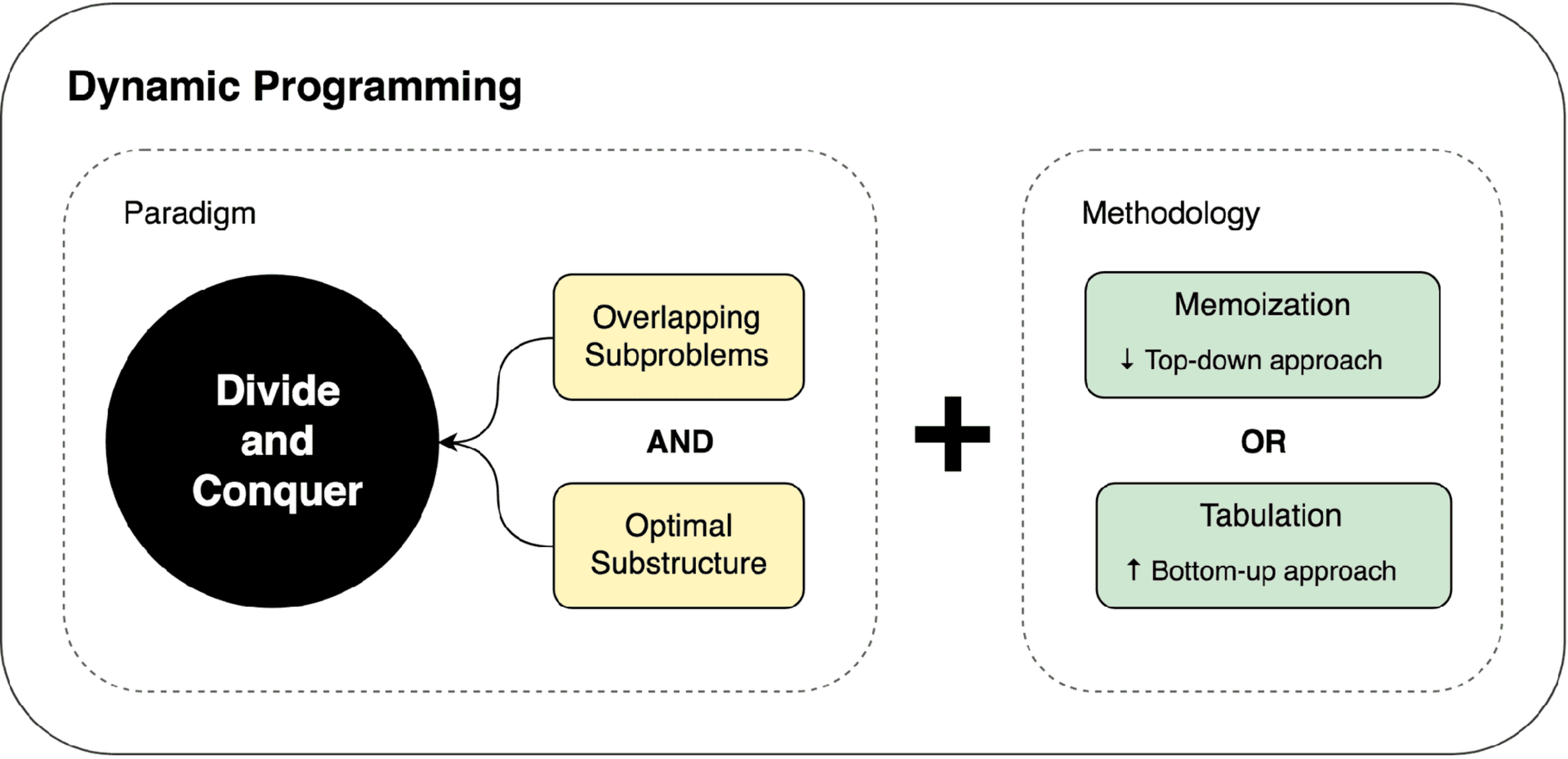
$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i}) \ .$$

3. Compute the value of an optimal solution, typically in a bottom-up fashion
4. Construct an optimal solution. (where to make a cut in the rod-cutting problem

```
BOTTOM-UP-CUT-ROD(p, n)
1  let r[0..n] be a new array
2  r[0] = 0
3  for j = 1 to n
4      q = -∞
5      for i = 1 to j
6          q = max(q, p[i] + r[j - i])
7      r[j] = q
8  return r[n]
```

```
EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
1  let r[0..n] and s[0..n] be new arrays
2  r[0] = 0
3  for j = 1 to n
4      q = -∞
5      for i = 1 to j
6          if q < p[i] + r[j - i]
7              q = p[i] + r[j - i]
8              s[j] = i
9      r[j] = q
10 return r and s
```

What is the difference between Divide and Conquer Algorithms and Dynamic Programming Algorithms?


Dynamic Programming
Paradigm
Divide and Conquer
Overlapping Subproblems
AND
Optimal Substructure
Methodology
Memoization
↓ Top-down approach
OR
Tabulation
↑ Bottom-up approach

Longest Common Subsequence

Finding DNA similarity

- A strand of DNA consists of a string of molecules called bases, where the possible bases are adenine, guanine, cytosine, and thymine.
- we can express a strand of DNA as a string over the finite set :{A; C; G; T}
- compare two strands of DNA to determine how “similar” the two strands are : some measure of how closely related the two organisms are.

S_1 =ACCGGTCGAGTGCGCGGAAGCCGGCCGAA

S_2=GTCGTTCGGAATGCCGTTGCTCTGTAAA

how to measure similarity? find a third strand S_3 in which the bases in S_3 appear in each of S_1 and S_2; these bases must appear in the same order, but not necessarily consecutively.

S_3 =GTCGTCGGAAGCCGGCCGAA.

Longest Common Subsequence

What is a subsequence

Given a sequence $X = <x_1, x_2, ..., x_m>$

A sequence $Z = <z_1, z_2, ..., z_k>$ is a subsequence of X if there exists a strictly increasing sequence $i_1, i_2, ..., i_k$ of indices of X such that for all j=1, 2, .., k we have $x_{ij}=z_j$

Example: $X= <A, B, C, B, D, A, B>$ then $Z =<B, C, D, B>$ is a subsequence of X with corresponding index sequence $<2, 3, 5, 7>$

Longest Common Subsequence

What is a longest common subsequence

Z is a **common subsequence** of X and Y if it is sequence of both X and Y.

Example: X= $<$A, B, C, B, D, A, B$>$ then Y =$<$B, D, C, A, B, A$>$ then the sequence $<$B, C, A$>$ is a common subsequence of X and Y.

$<$B, C, A$>$ is not the longest common subsequence.

$<$B, C, A, B$>$, $<$B, D, A, B$>$ and $<$B, C, B, A$>$ are the three longest common subsequences.

The LCS problem: Given two sequences X = $<x_1, x_2, \ldots, x_m>$ and Y = $<y_1, y_2, \ldots, y_n>$, find the maximum length common subsequence of X and Y

Longest Common Subsequence

The brute-force approach

1. Enumerate all subsequences of X and Y

   takes 2^m time, because X has that many subsequences

2. Then for each subsequence in X check if it matches each subsequence in Y

3. Find the longest matching subsequence

Exponential time

Longest Common Subsequence

Optimal Substructure of LCS problem

Theorem 15.1 (Optimal substructure of an LCS)
Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ be sequences, and let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be any LCS of X and Y.

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}.

   X= <A, B, C, B, D, A, A> then Y =<B, D, C, A, B, A> , Z =<B, C, B, A>

2. If $x_m \neq y_n$, then $z_k \neq x_m$ implies that Z is an LCS of X_{m-1} and Y.

   X= <A, B, C, B, D, A> then Y =<B, D, C, A, B> , Z =<B, C, B>

3. If $x_m \neq y_n$, then $z_k \neq y_n$ implies that Z is an LCS of X and Y_{n-1}.

   X= <A, B, C, B, D, A, B> then Y =<B, D, C, A, B, A> , Z =<B, D, A, B>

Longest Common Subsequence

Optimal Substructure of LCS problem

Theorem 15.1 (Optimal substructure of an LCS)
Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ be sequences, and let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be any LCS of X and Y.

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}.
2. If $x_m \neq y_n$, then $z_k \neq x_m$ implies that Z is an LCS of X_{m-1} and Y.
3. If $x_m \neq y_n$, then $z_k \neq y_n$ implies that Z is an LCS of X and Y_{n-1}.

Sketch of proofs:

(1) can be proven by contradiction: if the last characters of X and Y are not included in Z, then a longer LCS can be constructed by adding this character to Z, a contradiction.

(2) and (3) have symmetric proofs: Suppose there exists a subsequence W of X_{m-1} and Y (or of X and Y_{n-1}) with length > k. Then W is a common subsequence of X and Y, contradicting Z being an LCS.

Therefore, an LCS of two sequences contains as prefix an LCS of prefixes of the sequences. We can now use this fact construct a recursive formula for the value of an LCS.

Longest Common Subsequence

A recursive solution

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0\,, \\ c[i-1,j-1]+1 & \text{if } i,j > 0 \text{ and } x_i = y_j\,, \\ \max(c[i,j-1],c[i-1,j]) & \text{if } i,j > 0 \text{ and } x_i \neq y_j\,. \end{cases}$$

S_1 =ACCGGTCGAGTGCGCGGAAGCCGGCCGAA

S_2=GTCGTTCGGAATGCCGTTGCTCTGTAAA

Longest Common Subsequence

A recursive solution

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0\,, \\ c[i-1,j-1]+1 & \text{if } i,j > 0 \text{ and } x_i = y_j\,, \\ \max(c[i,j-1], c[i-1,j]) & \text{if } i,j > 0 \text{ and } x_i \neq y_j\,. \end{cases}$$

x_i

S_1 =ACCGGTCGAGTGCGCGGAAGCCGGCCGAA

S_2=GTCGTTCGGAATGCCGTTGCTCTGTAAA

y_j

Longest Common Subsequence

A recursive solution

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0\,, \\ c[i-1,j-1]+1 & \text{if } i,j > 0 \text{ and } x_i = y_j\,, \\ \max(c[i,j-1], c[i-1,j]) & \text{if } i,j > 0 \text{ and } x_i \neq y_j\,. \end{cases}$$

x_i

S_1 =ACCGGTCGAGTGCGCGGAAGCCGGCCG

S_2=GTCGTTCGGAATGCCGTTGCTCTGTAAA

y_j

Longest Common Subsequence

A recursive solution

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0\,, \\ c[i-1,j-1]+1 & \text{if } i,j > 0 \text{ and } x_i = y_j\,, \\ \max(c[i,j-1], c[i-1,j]) & \text{if } i,j > 0 \text{ and } x_i \neq y_j\,. \end{cases}$$

x_i

S_1 =ACCGGTCGAGTGCGCGGAAGCCGGCCG

S_2=GTCGTTCGGAATGCCGTTGCTCTGTAAA

y_j

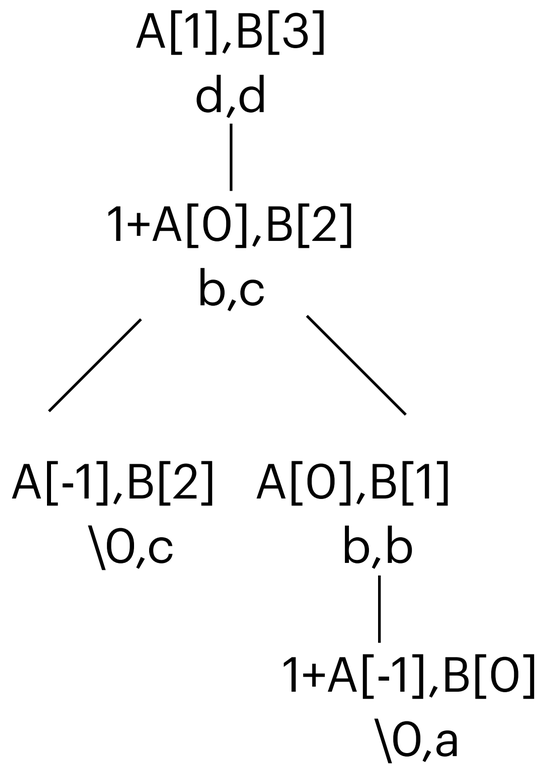
A

b	d
0	1

B

a	b	c	d
0	1	2	3

```
int lcs(A,B,i,j):
{
    if (i==-1 & j==-1):
      return 0;
  else if (A[i]==B[j]):
      return 1+lcs(A,B,i-1,j-1);
  else
      return max(lcs(A,B,i-1,j),lcs(A,B,i,j-1));
}
```



A

b	d
0	1

B

a	b	c	d
0	1	2	3

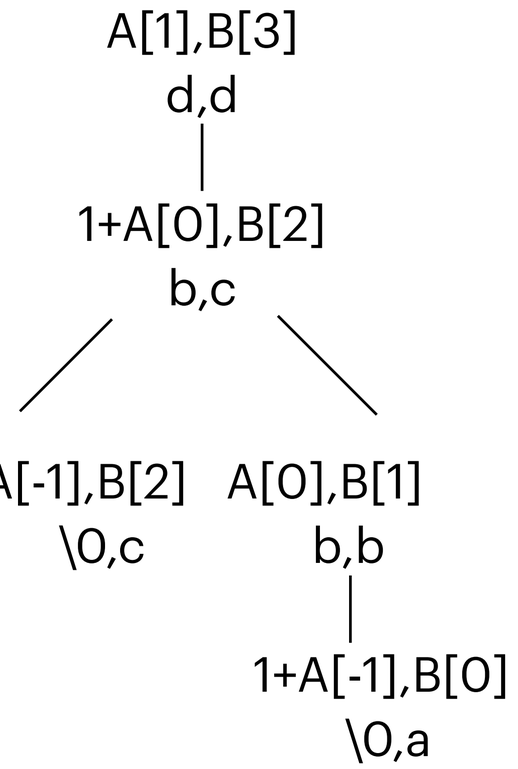
```
int lcs(A,B,i,j):
{
    if (i==-1 & j==-1):
      return 0;
  else if (A[i]==B[j]):
      return 1+lcs(A,B,i-1,j-1);
  else
      return max(lcs(A,B,i-1,j),lcs(A,B,i,j-1));
}
```

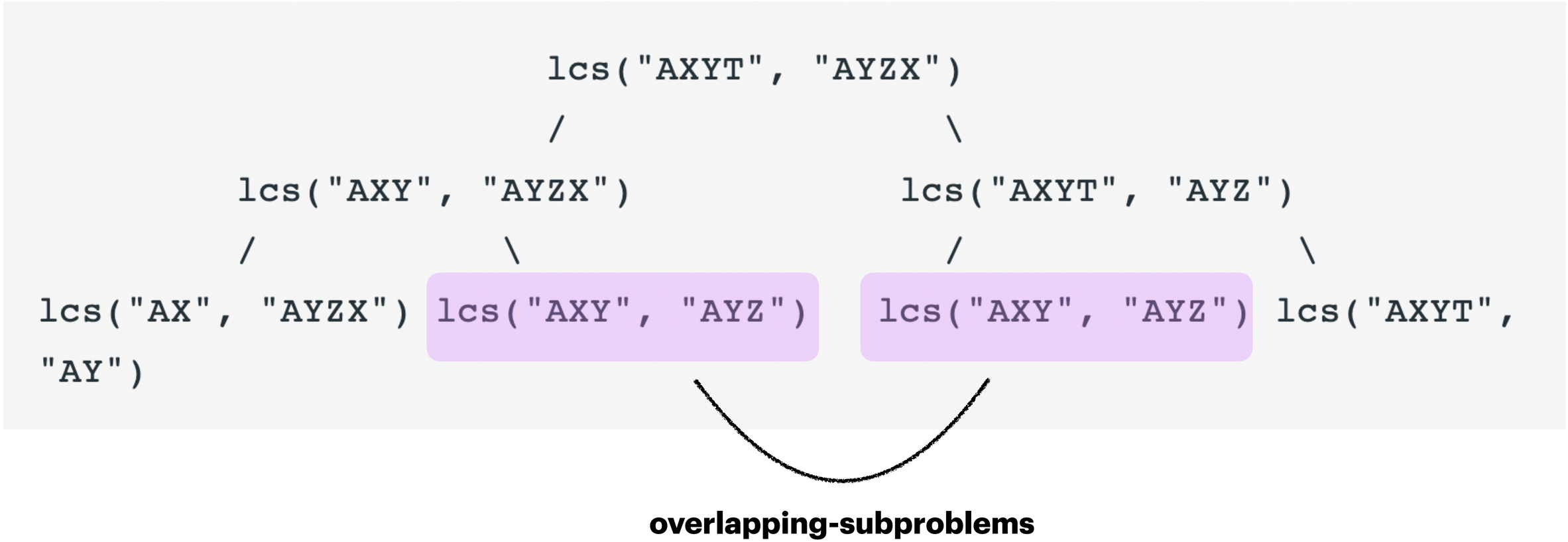
2

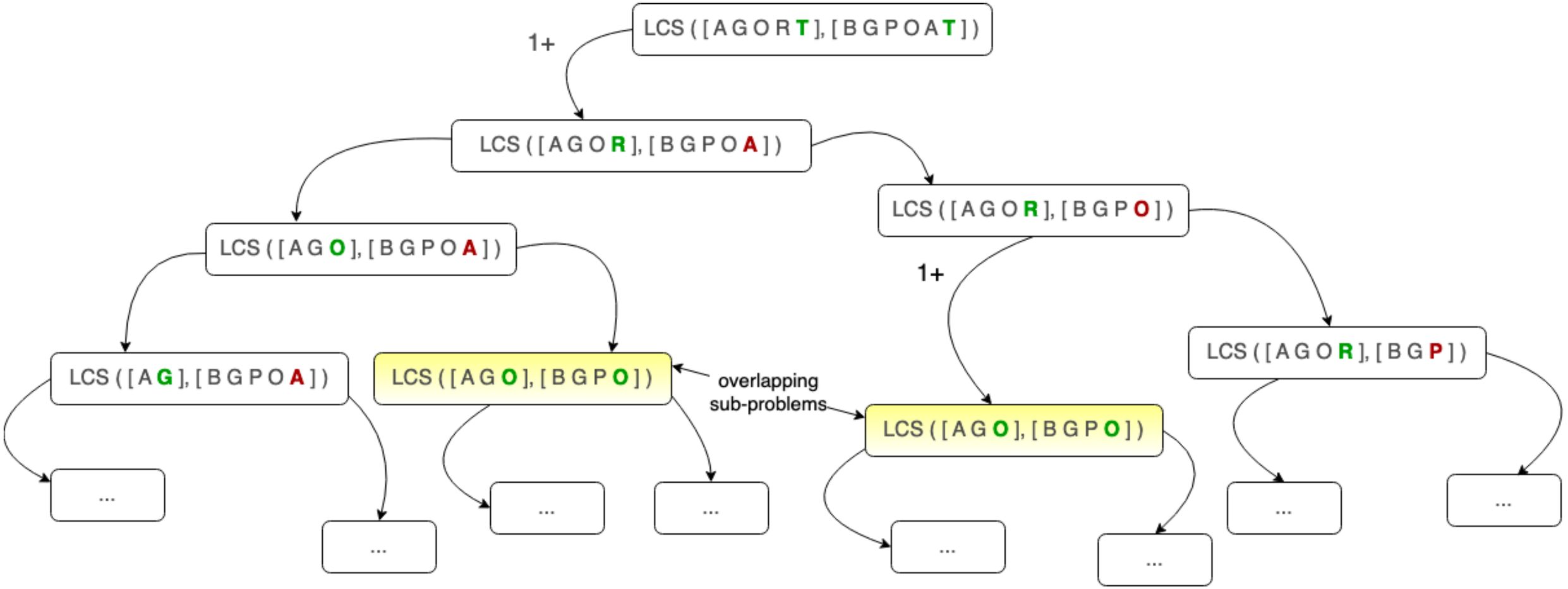
1+1=2

MAX(0,1)=1

1+0=1




lcs("AXYT", "AYZX")
lcs("AXY", "AYZX")
lcs("AXYT", "AYZ")
lcs("AX", "AYZX")
lcs("AXY", "AYZ")
lcs("AXY", "AYZ")
lcs("AXYT", "AY")
overlapping-subproblems



Recursion tree for finding the longest common sub-sequence

Longest Common Subsequence

Dynamic Programming

- Procedure LCS-LENGTH takes two sequences :

$$X = \langle x_1, x_2, \ldots, x_m \rangle \qquad Y = \langle y_1, y_2, \ldots, y_n \rangle$$

- Returns : b and c tables; c[m][n] contains the length of LCS of length X and Y
  - ❖store c[i,j] values in a table c[0...m][0...n]

    compute entries in row-major order (i.e. fill in the first row of c from left to right, then second row, and so on.)
  - ❖Also maintain table b[0...m][0...n] to help constuct an optimal solution

    b[i][j] points to the table entry corresponding to optimal subproblem chosen when computing c[i][j]

Time Complexity : T(n) = O (m * n)

```
LCS-LENGTH(X, Y)
 1  m = X.length
 2  n = Y.length
 3  let b[1..m, 1..n] and c[0..m, 0..n] be new tables
 4  for i = 1 to m
 5      c[i, 0] = 0
 6  for j = 0 to n
 7      c[0, j] = 0
 8  for i = 1 to m
 9      for j = 1 to n
10          if x_i == y_j
11              c[i, j] = c[i − 1, j − 1] + 1
12              b[i, j] = "↖"
13          elseif c[i − 1, j] ≥ c[i, j − 1]
14              c[i, j] = c[i − 1, j]
15              b[i, j] = "↑"
16          else c[i, j] = c[i, j − 1]
17              b[i, j] = "←"
18  return c and b
```

Longest Common Subsequence

Dynamic Programming

```
LCS-LENGTH(X, Y)
 1  m = X.length
 2  n = Y.length
 3  let b[1..m, 1..n] and c[0..m, 0..n] be new tables
 4  for i = 1 to m
 5      c[i, 0] = 0
 6  for j = 0 to n
 7      c[0, j] = 0
 8  for i = 1 to m
 9      for j = 1 to n
10          if x_i == y_j
11              c[i, j] = c[i − 1, j − 1] + 1
12              b[i, j] = "↖"
13          elseif c[i − 1, j] ≥ c[i, j − 1]
14              c[i, j] = c[i − 1, j]
15              b[i, j] = "↑"
16          else c[i, j] = c[i, j − 1]
17              b[i, j] = "←"
18  return c and b
```

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2	B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3	C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4	B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	← 3
5	D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4

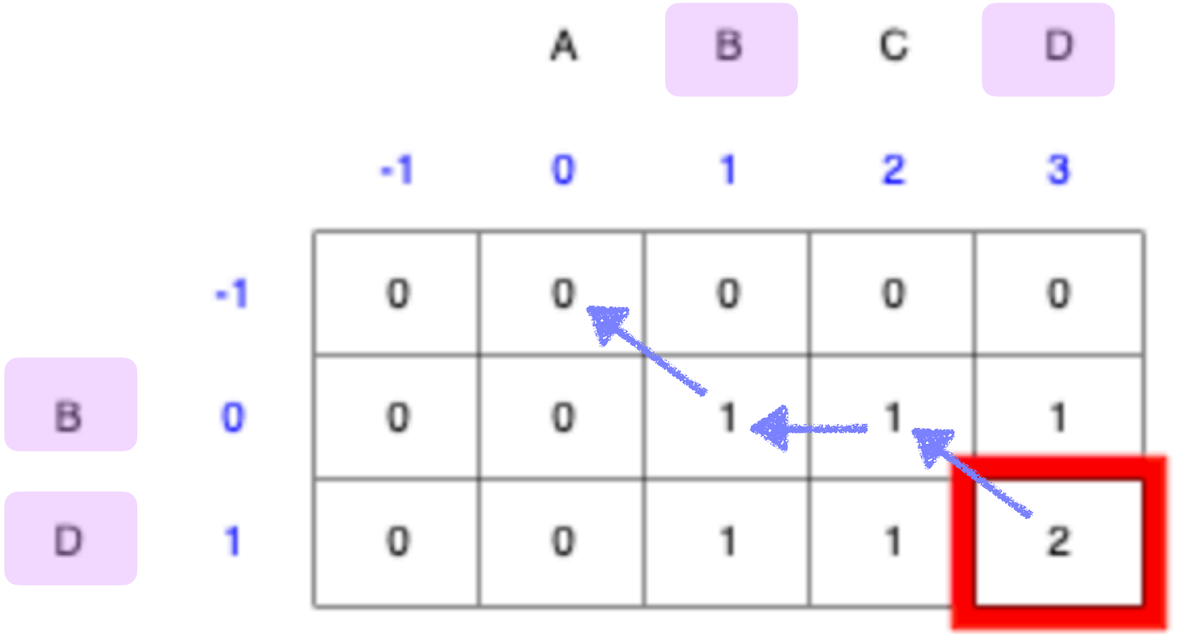
Longest Common Subsequence

Dynamic Programming; Constructing LCS

- Begin at b[m][n] and trace through the table by following the arrows.
- whenever we encounter a "↖" in the entry b[i][j] it implies that xi=yi is an element of the LCS
- we encounter the elements of LCS in reverse order

```
PRINT-LCS(b, X, i, j)
1  if i == 0 or j == 0
2       return
3  if b[i, j] == "↖"
4       PRINT-LCS(b, X, i − 1, j − 1)
5       print x_i
6  elseif b[i, j] == "↑"
7       PRINT-LCS(b, X, i − 1, j)
8  else PRINT-LCS(b, X, i, j − 1)
```

		j	0	1	2	3	4	5	6
i			y_j	B	D	C	A	B	A
0	x_i		0	0	0	0	0	0	0
1	A		0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2	B		0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3	C		0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4	B		0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	← 3
5	D		0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	A		0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7	B		0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4


A
B
C
D
-1
0
1
2
3
B
D
-1
0
1
0 0 0 0 0
0 0 1 1 1
0 0 1 1 2