Due: Tuesday, 13th October, 12 noon (AEDST)

Submission is through WebCMS/give and should be a single pdf file, maximum size 2Mb. Prose should be typed, not handwritten. Use of LaTeX is encouraged, but not required.

Discussion of assignment material with others is permitted, but the work submitted *must* be your own in line with the University's plagiarism policy.

Problem 1 (22 marks)
For $x, y \in \mathbb{Z}$ we define the set:

$$S_{x,y} = \{mx + ny : m, n \in \mathbb{Z}\}.$$

(a) Give five elements of $S_{2,-4}$. (5 marks)

(b) Give five elements of $S_{12,18}$. (5 marks)

For the following questions, let $d = \gcd(x, y)$ and z be the smallest positive number in $S_{x,y}$, or 0 if there are no positive numbers in $S_{x,y}$.

(c) (i) Show that $S_{x,y} \subseteq \{n : n \in \mathbb{Z} \text{ and } d|n\}$. (4 marks)

(ii) Show that $d \leq z$. (2 marks)

(d) (i) Show that $z|x$ and $z|y$ (*Hint: consider* $(x \,\%\, z)$ *and* $(y \,\%\, z)$). (4 marks)

(ii) Show that $z \leq d$. (2 marks)

Remark

The result that there exists $m, n \in \mathbb{Z}$ such that $mx + ny = \gcd(x, y)$ is known as Bézout's Identity.

Problem 2 (12 marks)

(a) Prove that if $\gcd(x, y) = 1$ then there is at least one $w \in [0, y) \cap \mathbb{N}$ such that $wx \stackrel{\text{mod } y}{=\!=} 1$.
(*Hint: Use Bézout's identity*) (4 marks)

(b) Prove that if $\gcd(x, y) = 1$ and $y|kx$ then $y|k$. (4 marks)

(c) Prove that if $\gcd(x, y) = 1$ then there is at most one $w \in [0, y) \cap \mathbb{N}$ such that $wx \stackrel{\text{mod } y}{=\!=} 1$. (4 marks)

Problem 3* (4 marks)
Prove that for all $m, n \in \mathbb{N}_{>0}$ with $n \leq m$:

$$\frac{3}{2}\big(n + (m \,\%\, n)\big) < m + n.$$

Problem 4 (16 marks)

Use the laws of set operations to prove the following identities:

(a) $(A \oplus A) = \emptyset$ (4 marks)

(b) (Annihilation): $A \cup \mathcal{U} = \mathcal{U}$ (4 marks)

(c) $A \oplus B = (A \cup B) \cap (A^c \cup B^c)$ (4 marks)

(d) (De Morgan's law): $(A \cup B)^c = A^c \cap B^c$ (4 marks)

Problem 5 (12 marks)

Let $\Sigma = \{0, 1\}$. For each of the following, prove that the result holds for all sets $X, Y, Z \subseteq \Sigma^*$, or provide a counterexample to disprove:

(a) $(X \cup Y)^* = X^* \cup Y^*$ (4 marks)

(b) $X(Y \cup Z) = (XY) \cup (XZ)$ (4 marks)

(c) $X(X^*) = X^*$ (4 marks)

Problem 6 (12 marks)

(a) List all possible functions $f : \{a, b, c\} \to \{0, 1\}$, that is, all elements of $\{0, 1\}^{\{a,b,c\}}$. (4 marks)

(b) Describe a connection between your answer for (a) and $\text{Pow}(\{a, b, c\})$. (4 marks)

(c) Describe a connection between your answer for (a) and $\{w \in \{0, 1\}^* : \text{length}(w) = 3\}$. (4 marks)

Problem 7* (6 marks)

Show that for any sets A, B, C there is a bijection between $A^{(B \times C)}$ and $(A^B)^C$.

Problem 8 (16 marks)

Recall the relation composition operator ; defined as:

$$R_1; R_2 = \{(a, c) : \text{there is a } b \text{ with } (a, b) \in R_1 \text{ and } (b, c) \in R_2\}$$

Let S be an arbitrary set. For each of the following, prove it holds for any binary relations $R_1, R_2, R_3 \subseteq S \times S$, or give a counterexample to disprove:

(a) $(R_1; R_2); R_3 = R_1; (R_2; R_3)$ (4 marks)

(b) $I; R_1 = R_1; I = R_1$ where $I = \{(x, x) : x \in S\}$ (4 marks)

(c) $(R_1 \cup R_2); R_3 = (R_1; R_3) \cup (R_2; R_3)$ (4 marks)

(d) $R_1; (R_2 \cap R_3) = (R_1; R_2) \cap (R_1; R_3)$ (4 marks)

Advice on how to do the assignment

Collaboration is encouraged, but all submitted work must be done individually without consulting someone else's solutions in accordance with the University's "Academic Dishonesty and Plagiarism" policies.

- Assignments are to be submitted via WebCMS (or give) as a single pdf file.
- When giving answers to questions, we always would like you to prove/explain/motivate your answers. You are being assessed on your understanding and ability.
- Be careful with giving multiple or alternative answers. If you give multiple answers, then we will give you marks only for your worst answer, as this indicates how well you understood the question.
- Some of the questions are very easy (with the help of external resources). You may make use of external material provided it is properly referenced[1] – however, answers that depend too heavily on external resources may not receive full marks if you have not adequately demonstrated ability/understanding.

[1] Proper referencing means sufficient information for a marker to access the material. Results from the lectures or textbook can be used without proof, but should still be referenced.