

$$C_1: w^3 = I \quad C_2: H^4 = I \quad C_3: S^3 = I \quad C_5: SS' = S'S \quad \text{Def 4: } :=$$

$$R_{23}^4: \begin{array}{c} \text{Diagram: } S \oplus \text{ (top wire)} \\ \text{Diagram: } S \oplus S' \text{ (top wire)} \end{array} \cdot w^2 \quad C_8: \begin{array}{c} \text{Diagram: } S \text{ (top wire)} \\ \text{Diagram: } H^2 \text{ (top wire)} \end{array} = \begin{array}{c} \text{Diagram: } H^2 \text{ (top wire)} \\ \text{Diagram: } S \text{ (top wire)} \end{array}$$

$$C_7: \begin{array}{c} \text{Diagram: } S \text{ (top wire)} \\ \text{Diagram: } S \text{ (top wire)} \end{array} = \begin{array}{c} \text{Diagram: } S \text{ (top wire)} \\ \text{Diagram: } S \text{ (top wire)} \end{array} \quad C_8^*: \begin{array}{c} \text{Diagram: } H^2 \text{ (top wire)} \\ \text{Diagram: } H^2 \text{ (top wire)} \end{array} = \begin{array}{c} \text{Diagram: } H^2 \text{ (top wire)} \\ \text{Diagram: } H^2 \text{ (top wire)} \end{array} \quad C_8^{6*}: \begin{array}{c} \text{Diagram: } H^2 \text{ (top wire)} \\ \text{Diagram: } H^2 \text{ (top wire)} \end{array} = \begin{array}{c} \text{Diagram: } \oplus \text{ (top wire)} \\ \text{Diagram: } \oplus \text{ (top wire)} \end{array}$$

$$R_{15}: \begin{array}{c} \text{Diagram: } S' \text{ (top wire)} \\ \text{Diagram: } S' \text{ (top wire)} \end{array} = \begin{array}{c} \text{Diagram: } S' \text{ (top wire)} \\ \text{Diagram: } H^2 \oplus \text{ (top wire)} \end{array} \quad C_8^*: \begin{array}{c} \text{Diagram: } H^2 \oplus \text{ (top wire)} \\ \text{Diagram: } H^2 \oplus \text{ (top wire)} \end{array} = \begin{array}{c} \text{Diagram: } \oplus \text{ (top wire)} \\ \text{Diagram: } \oplus \text{ (top wire)} \end{array}$$

Lem W

$$R_{56}: \begin{array}{c} \text{Diagram: } \oplus \text{ (top wire)} \\ \text{Diagram: } H \text{ (top wire)} \end{array} = \begin{array}{c} \text{Diagram: } \oplus \text{ (top wire)} \\ \text{Diagram: } H \text{ (top wire)} \\ \text{Diagram: } S \text{ (top wire)} \\ \text{Diagram: } H^2 \text{ (top wire)} \\ \text{Diagram: } S \text{ (top wire)} \\ \text{Diagram: } H \text{ (top wire)} \\ \text{Diagram: } S^2 \text{ (top wire)} \\ \text{Diagram: } S^{12} \text{ (top wire)} \end{array}$$

Proof cont. $R_{56} \cdot \text{RHS} =$

$$\begin{array}{c} \text{Diagram: } \oplus \text{ (top wire)} \\ \text{Diagram: } H \text{ (top wire)} \\ \text{Diagram: } S \text{ (top wire)} \\ \text{Diagram: } H^2 \text{ (top wire)} \\ \text{Diagram: } H^2 \text{ (top wire)} \\ \text{Diagram: } S^2 \text{ (top wire)} \end{array} \cdot w^2$$

$$\frac{R_{23}^4}{C_2} \begin{array}{c} \text{Diagram: } \oplus \text{ (top wire)} \\ \text{Diagram: } H \text{ (top wire)} \\ \text{Diagram: } S' \text{ (top wire)} \\ \text{Diagram: } S \text{ (top wire)} \\ \text{Diagram: } S^2 \text{ (top wire)} \end{array} \cdot w^2 \cdot w^2$$

$$\frac{C_1, R_{15}}{C_1, C_3} \begin{array}{c} \text{Diagram: } \oplus \text{ (top wire)} \\ \text{Diagram: } H \text{ (top wire)} \\ \text{Diagram: } S' \text{ (top wire)} \\ \text{Diagram: } S \text{ (top wire)} \end{array} \cdot w$$

Hence $R_{56}:$

$$\begin{array}{c} \text{Diagram: } \oplus \text{ (top wire)} \\ \text{Diagram: } H \text{ (top wire)} \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} \text{Diagram: } \oplus \text{ (top wire)} \\ \text{Diagram: } H \text{ (top wire)} \\ \text{Diagram: } S' \text{ (top wire)} \\ \text{Diagram: } S \text{ (top wire)} \end{array} \cdot w$$

$C_2 \parallel$

$$R_{56}: \begin{array}{c} \text{Diagram: } H \text{ (top wire)} \\ \text{Diagram: } H^3 \text{ (top wire)} \\ \text{Diagram: } H^2 \text{ (top wire)} \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} \text{Diagram: } H^2 \text{ (top wire)} \\ \text{Diagram: } S' \text{ (top wire)} \\ \text{Diagram: } S \text{ (top wire)} \end{array} \cdot w$$

Def 4 $\parallel C_8^*, C_8^{6*}, C_8^8, C_8$

$$R_{56}: \begin{array}{c} \text{Diagram: } \oplus \text{ (top wire)} \\ \text{Diagram: } \oplus \text{ (top wire)} \\ \text{Diagram: } H^2 \text{ (top wire)} \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} \text{Diagram: } \oplus \text{ (top wire)} \\ \text{Diagram: } \oplus \text{ (top wire)} \\ \text{Diagram: } S' \text{ (top wire)} \\ \text{Diagram: } S \text{ (top wire)} \\ \text{Diagram: } H^2 \text{ (top wire)} \end{array} \cdot w$$

$$C_1: w^3 = I \quad C_2: H^4 = I \quad C_3: S^3 = I \quad C_5: SS' = S'S \quad C_6^*: \text{Diagram} = \text{Diagram}$$

$$C_{16}^{22}: \text{Diagram} = \text{Diagram}$$

$$C_{16}^{23}: \text{Diagram} = \text{Diagram}$$

$$C_{16}^{24}: \text{Diagram} = \text{Diagram}$$

$$C_6^{2*}: \text{Diagram} = \text{Diagram}$$

Lem W

$$R_{56}: \text{Diagram} = \text{Diagram} \quad \text{Diagram} = \text{Diagram} [H] \text{ } [S] \text{ } [H^2] \text{ } [S] \text{ } [H] \text{ } [H] \text{ } [S^2] \text{ } [S^2]$$

$$\text{Proof cont. } R_{56}: \text{Diagram} \xrightarrow{\text{WTS}} \text{Diagram} \cdot w$$

$C_2 \parallel C_{16}^{22}$

$$R_{56}: \text{Diagram} \xrightarrow{\text{WTS}} \text{Diagram} \cdot w$$

$$R_{56}: \text{Diagram} \xrightarrow{\text{WTS}} \text{Diagram} \cdot w$$

$C_6^* \parallel$

$$R_{56}: \text{Diagram} \xrightarrow{\text{WTS}} \text{Diagram} \cdot w$$

$C_{16}^{23} \parallel$

$$R_{56}: \text{Diagram} \xrightarrow{\text{WTS}} \text{Diagram} \cdot w$$

$C_{16}^{24} \parallel$

$$R_{56}: \text{Diagram} \xrightarrow{\text{WTS}} \text{Diagram} \cdot w$$

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$$\begin{aligned}
C_1: w^3 = I & \quad C_2: H^4 = I & \quad C_3: S^3 = I & \quad C_5: SS' = S'S & \quad C_7: \text{Diagram} = \text{Diagram} \\
C_{13}: \text{Diagram} & = \text{Diagram} & R_{25}^5: \text{Diagram} & = \text{Diagram} \cdot w^2 \\
C_{13}^4: \text{Diagram} & = \text{Diagram} & C_{13}^9: \text{Diagram} & = \text{Diagram} & R_{15}: \text{Diagram} = \text{Diagram}
\end{aligned}$$

Lem W

$$R_{56}: \text{Diagram} = \text{Diagram}$$

$$\text{Proof cont. } R_{56}: \text{Diagram} \xrightarrow{\text{WTS}} \text{Diagram} \cdot w$$

$$\text{Then } R_{56} \cdot \text{LHS} := \text{Diagram} \xrightarrow{C_{13}} \text{Diagram}$$

$$\begin{array}{c}
\text{Diagram} \xrightarrow{R_{25}^5, C_1, C_2, C_3} \text{Diagram} \cdot w \\
\xparallel{C_{13}^4, G_7} \text{Diagram} \cdot w
\end{array}$$

$$\begin{array}{c}
\text{Diagram} \xrightarrow{C_5} \text{Diagram} \cdot w =: R_{56} \cdot \text{RHS} \\
\text{Diagram}
\end{array}$$

□

$$C_1: w^3 = I \quad C_2: H^4 = I \quad C_3: S^3 = I \quad C_5: SS' = S'S$$

$$C_8^3: \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} H^2 \\ \text{---} \\ H^2 \end{array} = \begin{array}{c} H^2 \\ \text{---} \\ H^2 \end{array} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array}$$

$$\text{Def 1: } [S'] := [H] [H] [S] [H] [H]$$

$$R_{24}^2: \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} S \\ \text{---} \\ \circledast \end{array} = \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} S \\ \text{---} \\ \circledast \end{array}$$

$$C_8^{6*}: \begin{array}{c} \oplus \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} H^2 \\ \text{---} \\ \bullet \end{array} = \begin{array}{c} \oplus \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \\ H^2 \end{array}$$

$$C_8^{14}: \begin{array}{c} \oplus \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} H^2 \\ \text{---} \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \text{---} \\ H^2 \end{array} \begin{array}{c} \oplus \\ \text{---} \\ \text{---} \end{array}$$

$$\text{Def 7: } \begin{array}{c} \oplus \\ \text{---} \\ \text{---} \end{array} := \begin{array}{c} \times \\ \text{---} \\ \times \end{array} \begin{array}{c} \oplus \\ \text{---} \\ \text{---} \end{array}$$

$$\text{Def 5: } \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} := \begin{array}{c} \times \\ \text{---} \\ \times \end{array} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array}$$

$$\text{Def 2: } \begin{array}{c} \oplus \\ \text{---} \\ \bullet \end{array} := \begin{array}{c} H \\ \text{---} \\ H \end{array}$$

$$\text{Def 4: } \begin{array}{c} \bullet \\ \text{---} \\ \circledast \end{array} := \begin{array}{c} H \\ \text{---} \\ H^2 \end{array}$$

$$R_{23-25}^1: \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} S^2 \\ \text{---} \\ \circledast \end{array} = \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} S^2 \\ \text{---} \\ \circledast \end{array} \cdot w$$

$$C_8^7: \begin{array}{c} \bullet \\ \text{---} \\ \circledast \end{array} = \begin{array}{c} H^2 \\ \text{---} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{---} \\ \circledast \end{array}$$

$$C_8^1: \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} H^2 \\ \text{---} \\ \bullet \end{array}$$

Lem X Def 1-2, Def 4-5, Def 7, $C_1, C_2, C_3, C_5, C_6, C_7, C_8, C_{13}, C_{16}, R_{15}, R_{23-25}, R_{24}$ imply

$$R_{57}: \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} \oplus \\ \text{---} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} \oplus \\ \text{---} \\ \bullet \end{array} = \begin{array}{c} H^2 \\ \text{---} \\ H \end{array} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} H^2 \\ \text{---} \\ H \end{array} \begin{array}{c} S \\ \text{---} \\ S \end{array} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} H \\ \text{---} \\ H \end{array} \begin{array}{c} S' \\ \text{---} \\ S' \end{array} \cdot w^2$$

$$\text{Proof: } R_{57}.\text{RHS} := \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} H^2 \\ \text{---} \\ H \end{array} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} H^2 \\ \text{---} \\ H \end{array} \begin{array}{c} S \\ \text{---} \\ S^2 \end{array} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} H \\ \text{---} \\ H \end{array} \begin{array}{c} S' \\ \text{---} \\ S' \end{array} \cdot w^2$$

$$\text{Def 1: } \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} H^2 \\ \text{---} \\ H \end{array} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} H^2 \\ \text{---} \\ H \end{array} \boxed{\begin{array}{c} S \\ \text{---} \\ S^2 \end{array}} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} H \\ \text{---} \\ H \end{array} \boxed{S'} \begin{array}{c} \bullet \\ \text{---} \\ H^2 \end{array} \cdot w^2$$

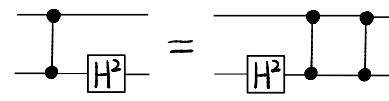
$$\text{Def 4: } \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} H^2 \\ \text{---} \\ H \end{array} \boxed{\begin{array}{c} S \\ \text{---} \\ S^2 \end{array}} \boxed{\begin{array}{c} H^2 \\ \text{---} \\ H \end{array}} \boxed{\begin{array}{c} S \\ \text{---} \\ S^2 \end{array}} \boxed{\begin{array}{c} H^2 \\ \text{---} \\ H \end{array}} \boxed{\begin{array}{c} S \\ \text{---} \\ H^2 \end{array}} \cdot w^2$$

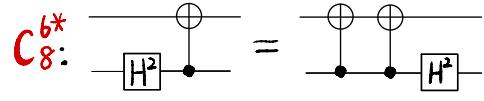
$$\begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} H^2 \\ \text{---} \\ H \end{array} \boxed{\begin{array}{c} H^2 \\ \text{---} \\ H \end{array}} \boxed{\begin{array}{c} S^2 \\ \text{---} \\ S^2 \end{array}} \boxed{\begin{array}{c} S \\ \text{---} \\ S \end{array}} \boxed{\begin{array}{c} H^2 \\ \text{---} \\ H^2 \end{array}} \cdot w^2 \cdot w$$

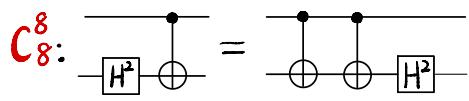
$$\begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} H^2 \\ \text{---} \\ H \end{array} \boxed{\begin{array}{c} H^2 \\ \text{---} \\ H \end{array}} \boxed{\begin{array}{c} H^2 \\ \text{---} \\ H^2 \end{array}}$$

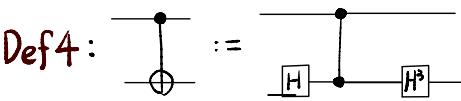
$$\begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} H^2 \\ \text{---} \\ H \end{array} \boxed{\begin{array}{c} \oplus \\ \text{---} \\ \bullet \end{array}} \boxed{\begin{array}{c} \oplus \\ \text{---} \\ \bullet \end{array}} \boxed{\begin{array}{c} H^2 \\ \text{---} \\ H^2 \end{array}}$$

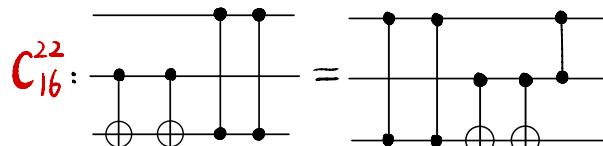
$$C_1: w^3 = I \quad C_2: H^4 = I \quad C_3: S^3 = I \quad C_5: SS' = S'S \quad C_8^{1*}:$$

$C_8:$ 

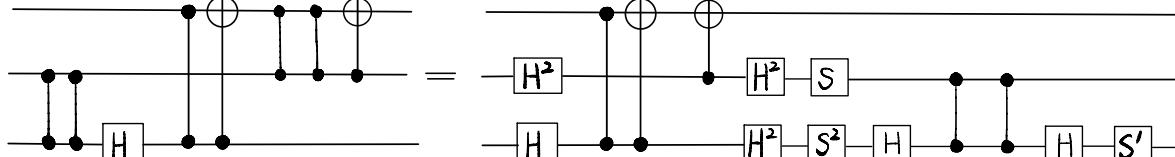
$C_8^{6*}:$ 

$C_8^8:$ 

Def4: 

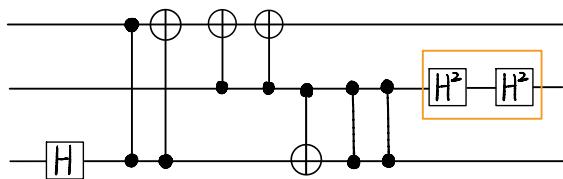
$C_{16}^{22}:$ 

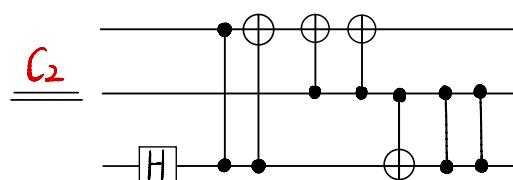
Lem X

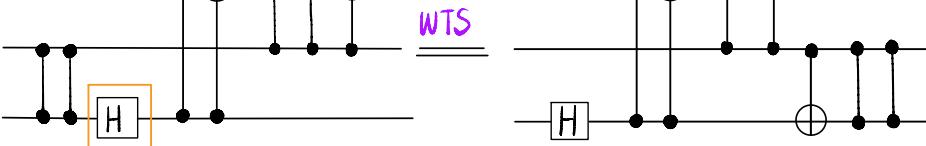
$R_{57}:$ 

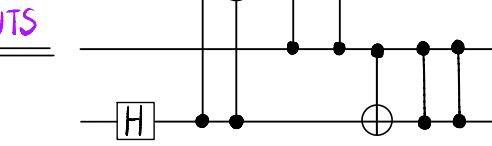
$\bullet w^2$

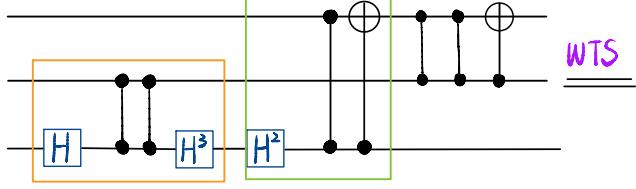
Proof cont.

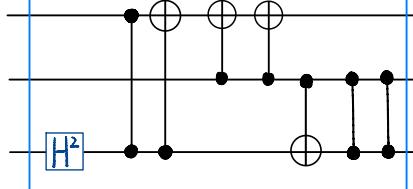
$R_{57} \cdot \text{RHS} =$ 

C_2 

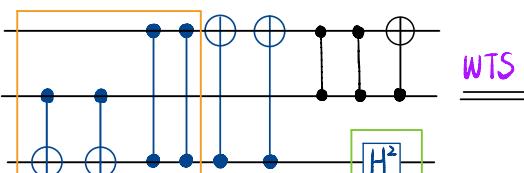
Hence $R_{57}:$ 

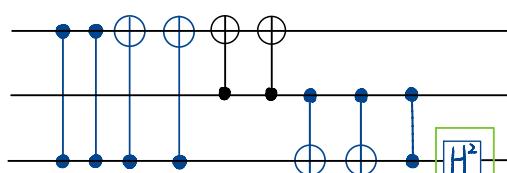
\equiv 

$R_{57}:$ 

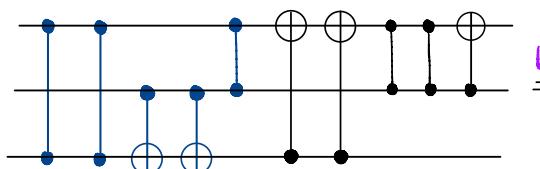
\equiv 

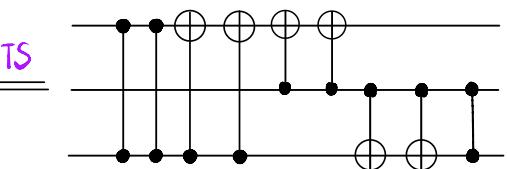
Def 4 ||| $C_8^{1*}, C_8^{6*}, C_8^8, C_8$

$R_{57}:$ 

\equiv 

$C_2 ||| C_{16}^{22}$

$R_{57}:$ 

\equiv 

$$C_1: w^3 = I \quad C_2: H^4 = I \quad C_3: S^3 = I \quad C_5: SS' = S'S$$

$$C_{16}^{23}: \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 4 horizontal wires and 4 vertical controls} \end{array} = \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 4 horizontal wires and 4 vertical controls} \end{array}$$

$$C_{16}^{24}: \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 4 horizontal wires and 4 vertical controls} \end{array} = \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 4 horizontal wires and 4 vertical controls} \end{array}$$

$$C_6^*: \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 4 horizontal wires and 4 vertical controls} \end{array} = \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 4 horizontal wires and 4 vertical controls} \end{array}$$

$$C_6^*: \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 4 horizontal wires and 4 vertical controls} \end{array} = \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 4 horizontal wires and 4 vertical controls} \end{array}$$

$$C_{13}^9: \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 4 horizontal wires and 4 vertical controls} \end{array} = \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 4 horizontal wires and 4 vertical controls} \end{array}$$

$$C_{13}^3: \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 4 horizontal wires and 4 vertical controls} \end{array} = \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 4 horizontal wires and 4 vertical controls} \end{array}$$

Lem X

$$R_{57}: \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 7 horizontal wires and 4 vertical controls} \end{array} = \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 7 horizontal wires and 4 vertical controls} \end{array} \cdot w^2$$

$$\begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 7 horizontal wires and 4 vertical controls} \end{array} = \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 7 horizontal wires and 4 vertical controls} \end{array} \cdot w^2$$

Proof cont.

$$R_{57}: \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 7 horizontal wires and 4 vertical controls} \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 7 horizontal wires and 4 vertical controls} \end{array}$$

$$C_2 \equiv C_{16}^{23}$$

$$R_{57}: \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 7 horizontal wires and 4 vertical controls} \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 7 horizontal wires and 4 vertical controls} \end{array}$$

$$C_6 \equiv C_{16}^{24}$$

$$R_{57}: \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 7 horizontal wires and 4 vertical controls} \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 7 horizontal wires and 4 vertical controls} \end{array}$$

$$\text{Then } R_{57} \cdot \text{LHS} := \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 7 horizontal wires and 4 vertical controls} \end{array} \stackrel{C_{13}^9}{=} \begin{array}{c} \text{Quantum circuit diagram} \\ \text{with 7 horizontal wires and 4 vertical controls} \end{array} =: R_{57} \cdot \text{RHS}$$

□

$$C_0 : (-1)^2 = 1$$

$$C1: w^3 = 1$$

$$C_2: H^4 = I$$

$$C_3 : S^3 = I$$

Def 2: :=

Def 1 : $\boxed{s'}$:= $\boxed{H} \quad \boxed{H} \quad \boxed{s} \quad \boxed{H} \quad \boxed{H}$ **C5** : $ss' = s's$

$$R_{23}^1 : \quad \begin{array}{c} \text{---} \\ | \end{array} \boxed{S} \text{---} \oplus \text{---} \quad = \quad \begin{array}{c} \text{---} \\ | \end{array} \oplus \text{---} \bullet \text{---} \boxed{S} \text{---} \quad \cdot w^2$$

$$C_8^{6*} = \begin{array}{c} \text{Diagram showing } C_8^{6*} \text{ as a } H^2 \text{ box with two nodes connected to it.} \\ \text{Diagram showing } C_8^{6*} \text{ as two nodes connected to two } H^2 \text{ boxes.} \end{array}$$

$$\text{Def 4: } \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} := \begin{array}{c} \bullet \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array}$$

$$\text{Def 7 : } \begin{array}{c} \oplus \\ \parallel \\ \parallel \\ \bullet \end{array} := \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \oplus \\ \bullet \end{array}$$

$$\text{Def 5: } \begin{array}{c} \bullet \\ \parallel \\ \bullet \end{array} := \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \times \quad \times \\ \diagdown \quad \diagup \end{array}$$

$$C_8: \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \quad \boxed{H^2}$$

$$C_4^2: \quad \boxed{S} - \boxed{H^+} - \boxed{S} = \quad \boxed{H} - \boxed{S^2} - \boxed{H} \cdot (-w^2)$$

LemY Def0-2, Def4-5, Def7, C₀₋₈, C₁₃, C₁₆, R₁₅, R₂₃, R₂₄, R₂₅ imply

$$R_{51} : \text{Circuit Diagram} = \text{Circuit Diagram} \cdot (-\omega)$$

Proof: $R_{51} \cdot RHS :=$

$$SHS'HS \stackrel{\text{Defl}}{=} SHH^2S H^2H \stackrel{C_4^2}{=} HS^2 HH^3 \cdot (-w^2) \stackrel{C_2}{=} HS^2 \cdot (-w^2)$$

The diagram illustrates two equivalent quantum circuit representations for G_1 and G_2 . The top circuit, labeled 'Def 1', consists of three horizontal wires. The leftmost wire starts with a Hadamard gate (H). The middle wire starts with a H^3 gate. The rightmost wire starts with a S gate. All three wires then pass through a sequence of gates: H , S^2 , S , H^2 , S^2 , and H . The bottom circuit shows an equivalent sequence: H^2 , S^2 , H^2 , and S . The entire circuit is enclosed in a box with a green border and the label $\bullet (-w) \bullet (-w^2)$.

The diagram illustrates a quantum circuit with two horizontal wires. The top wire starts with a CNOT gate (Control H, Target \oplus) followed by a sequence of gates: H^3 , S , H , H , H^2 , S^2 , and H . The bottom wire starts with a H^2 gate followed by a sequence of gates: S^2 , H^2 , and S . Two orange boxes highlight groups of gates: one box encloses the sequence H^3 , S , and H , and another box encloses the sequence S^2 , H^2 , and S . Red annotations C_3 and C_0, C_1 are placed near the top left and bottom left respectively.

$$\text{S}'\text{S}^2\text{H}^2\text{S} \xrightleftharpoons{\text{C}_5} \text{S}^2\text{S}'\text{H}^2\text{S} \xrightleftharpoons{\text{DefI}} \text{S}^2\text{H}^2\text{S} \quad \boxed{\text{H}^2\text{H}^2\text{S}} \xrightleftharpoons{\text{C}_2} \text{S}^2\text{H}^2\text{S}^2$$

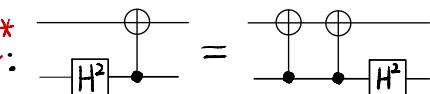
$$C_0: (-1)^2 = 1$$

$$C_1: w^3 = 1$$

$$C_2: H^4 = I$$

$$C_3: S^3 = I$$

$$C_8^{6*}$$



$$\text{Def 1: } \boxed{s'} := \boxed{H} \boxed{H} \boxed{S} \boxed{H} \boxed{H} \quad C_5: ss' = s's$$

$$\text{Def 2: } \boxed{\oplus} := \boxed{H} \cdot \boxed{H^3}$$

$$C_{13}^7:$$

$$C_{13}^{11}: \boxed{\oplus} = \boxed{\oplus} \cdot \boxed{\oplus}$$

Lem Y

$$R_{51}: \quad \begin{array}{c} \text{Quantum circuit diagram} \\ \text{Left: } \dots \oplus \dots H \dots \oplus \dots \\ \text{Right: } \dots H \dots H^3 \dots S \dots H \dots S \dots H \dots S' \dots H \dots S \dots H^2 \dots S^2 \dots H \dots \cdot (-w) \end{array}$$

Proof cont.

$$R_{51} \cdot \text{RHS} = \begin{array}{c} \text{Quantum circuit diagram} \\ \text{Left: } \dots \oplus \dots H \dots H^3 \dots \oplus \dots H \dots \cdot w^2 \\ \text{Right: } \dots H^2 \dots S^2 \dots H^2 \dots S^2 \dots \end{array}$$

$$\frac{\text{Def 1}}{C_2} \quad \begin{array}{c} \text{Quantum circuit diagram} \\ \text{Left: } \dots \oplus \dots H \dots H^3 \dots \oplus \dots H \dots \cdot w^2 \\ \text{Right: } \dots S'^2 \dots S^2 \dots \end{array}$$

$$\text{Hence } R_{51}: \quad \begin{array}{c} \text{Quantum circuit diagram} \\ \text{Left: } \dots \oplus \dots H \dots \text{WTS} \dots H \dots \cdot w^2 \\ \text{Right: } \dots H \dots H^3 \dots \oplus \dots H \dots \cdot w^2 \\ \dots S'^2 \dots S^2 \dots \end{array}$$

$$R_{51}: \quad \begin{array}{c} \text{Quantum circuit diagram} \\ \text{Left: } \boxed{H} \dots \boxed{H^3} \dots \text{WTS} \dots \boxed{H^2} \dots \cdot w^2 \\ \text{Right: } \dots \boxed{H^2} \dots \boxed{H^3} \dots \cdot w^2 \\ \dots S'^2 \dots S^2 \dots \end{array}$$

Def 2 ||| C_8^{1*}, C_8^{6*}

$$R_{51}: \quad \begin{array}{c} \text{Quantum circuit diagram} \\ \text{Left: } \boxed{\oplus} \dots \boxed{\oplus} \dots \text{WTS} \dots \boxed{H^2} \dots \cdot w^2 \\ \text{Right: } \dots \boxed{H^2} \dots \boxed{H^3} \dots \cdot w^2 \\ \dots S'^2 \dots S^2 \dots \end{array}$$

$C_2 ||| C_{13}^7, C_{13}^{11}$

$$R_{51}: \quad \begin{array}{c} \text{Quantum circuit diagram} \\ \text{Left: } \dots \boxed{\oplus} \dots \text{WTS} \dots \boxed{H^2} \dots \cdot w^2 \\ \text{Right: } \dots \boxed{H} \dots \boxed{H} \dots \cdot w^2 \\ \dots S'^2 \dots S^2 \dots \end{array}$$