

$$C_8^7: \quad \begin{array}{c} \bullet \\ \text{---} \\ \bigoplus \end{array} \quad \boxed{H^2} = \quad \begin{array}{c} \bullet \\ \text{---} \\ \bigoplus \\ \boxed{H^2} \end{array} \quad \begin{array}{c} \bullet \\ \text{---} \\ \bigoplus \end{array}$$

$$C_8^1: \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \quad \boxed{H^2}$$

$$C_2: H^4 = I \quad C_3: S^3 = I$$

$$C_8^3: \quad \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \quad \boxed{H^2} \quad = \quad \boxed{H^2} \quad \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \end{array} \quad \boxed{H^2}$$

$$C_6: \quad \begin{array}{c} \bullet & \bullet & \bullet \\ | & | & | \\ \bullet & \bullet & \bullet \end{array} = \quad \underline{\hspace{2cm}} \quad \underline{\hspace{2cm}}$$

$$R36: \quad \begin{array}{c} \bullet \\ \bullet \\ \oplus \\ \oplus \end{array} = \begin{array}{c} \oplus \\ \bullet \\ H^2 \\ H^2 \end{array}$$

Lem V

$$\begin{array}{c} \text{R}_{30}: \\ \text{Circuit Diagram} \end{array} = \text{Circuit Diagram}$$

Proof cont.

$$R_{30} : \quad \text{WTS} = \quad \begin{array}{c} \text{Diagram showing } R_{30} \text{ as a sequence of nodes: } \\ \text{WTS} = \quad \text{Diagram showing WTS as a sequence of nodes:} \end{array}$$

$$R_{30} \cdot \text{RHS} =$$

Circuit diagram for C_8^7 showing a sequence of operations: a Hadamard (H), followed by two S gates, then another H gate, and finally a CNOT gate between the second and fourth wires.

Hence

$$R_{30} : \quad \text{WTS} = \quad C_2 \parallel C_3$$

The diagram illustrates the decomposition of the R_{30} gate into a sequence of operations. On the left, the R_{30} gate is shown as a sequence of three horizontal lines representing qubits. The first two qubits pass through identity gates (I). The third qubit passes through a sequence of three gates: a single-qubit rotation S , followed by a two-qubit control operation H^2 , and another single-qubit rotation S . This sequence is highlighted with an orange box. To the right of an equals sign, the decomposition is shown as the WTS gate followed by the parallel combination (\parallel) of two two-qubit operations, C_2 and C_3 . Each C_i consists of two horizontal lines representing qubits. The top qubit passes through a two-qubit control operation H^2 . The bottom qubit passes through a sequence of three gates: a single-qubit rotation S , followed by a two-qubit control operation H^2 , and another single-qubit rotation S . These sequences are also highlighted with orange boxes.

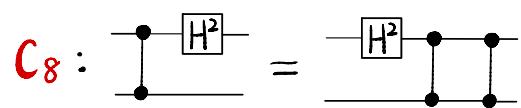
$$R_{30} \cdot \text{RHS} = \begin{array}{c} \text{Circuit Diagram} \\ \equiv \\ \text{Circuit Diagram} \\ \equiv \\ \text{Circuit Diagram} \end{array}$$

Hence R_{30} :

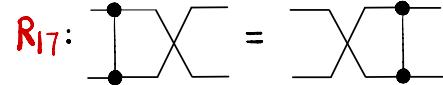
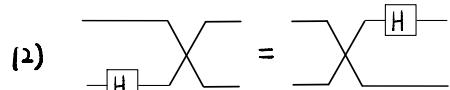
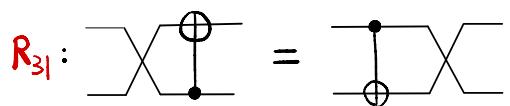
The diagram illustrates the decomposition of a C_6 gate into a sequence of quantum operations. On the left, a horizontal line of six qubits starts with two Hadamard (H) gates. The third qubit passes through a CNOT gate with the second qubit as control. The fourth qubit passes through a CNOT gate with the second qubit as control. The fifth qubit passes through a CNOT gate with the second qubit as control. The sixth qubit passes through a CNOT gate with the second qubit as control. A vertical line connects the second and third qubits. This sequence is enclosed in an orange box. To the right of the equals sign, the circuit is shown again, but the first four qubits now pass through H^2 gates. The fifth and sixth qubits remain as they were. A second orange box encloses the last four qubits. Below the circuit, the label C_6 is written in red.

$$R_{30} : \quad \begin{array}{c} \text{Diagram of } R_{30} \end{array} \quad = \quad \begin{array}{c} \text{Diagram of } R_{36} \end{array} : R_{36}$$





$$C_2 : H^4 = I$$



Lem V'

R_{30} :

implies

R_{30}^1 :

&

R_{30}^2 :

Proof:

R_{30} :

$C_8 \parallel$

R_{30} :

Appending both sides by an H^2 to the left

R_{30} :

$C_2 \parallel$

R_{30} :

Appending both sides by a SWAP to the left

R_{30} :

$R_{17}, R_{19} \parallel R_{31}$

R_{30} :

R_{30}^1

C2: $H^4 = I$

$$R_{31}: (1) \quad \text{Diagram} = \text{Diagram} \quad (2) \quad \text{Diagram} = \text{Diagram}$$

Def 2: $\begin{array}{c} \oplus \\ \parallel \end{array} := \begin{array}{c} H \quad H \quad H \quad H \\ \parallel \end{array} \quad \begin{array}{c} \parallel \\ \oplus \end{array} := \begin{array}{c} H \quad H \quad H \quad H \\ \parallel \end{array}$

Lem V'

$$R_{30}: \quad \text{Diagram} = \text{Diagram} \quad \text{implies}$$

$$R_{30}^1: \quad \text{Diagram} = \text{Diagram} \quad \&$$

$$R_{30}^2: \quad \text{Diagram} = \text{Diagram}$$

Proof cont.

$$R_{30}^1: \quad \text{Diagram} = \text{Diagram}$$

Def 2 ||| R₃₁, H₂

$$R_{30}^1: \quad \text{Diagram} =$$

$$\text{Diagram} = \text{Diagram}$$

Def 2 |||

$$R_{30}^1: \quad \text{Diagram} =$$

$$\text{Diagram} = \text{Diagram}$$

: R₃₀²



$$\text{Def 4: } \begin{array}{c} \diagup \quad \diagdown \\ \text{---} \end{array} = \begin{array}{c} \bullet \quad \bullet \quad \oplus \quad \bullet \quad \bullet \\ \bigcirc \quad \bigcirc \quad \bigcirc \quad \bullet \quad \boxed{\mathbb{H}^2} \\ \text{---} \end{array}$$

$$\text{Def 7: } \begin{array}{c} \diagup \quad \diagdown \\ \text{---} \end{array} = \begin{array}{c} \oplus \quad \oplus \quad \bullet \quad \bullet \quad \bullet \\ \boxed{\mathbb{H}^2} \quad \bullet \quad \bigcirc \quad \bigcirc \quad \bigcirc \\ \text{---} \end{array}$$

$$R_{36}: \begin{array}{c} \bullet \quad \bullet \quad \oplus \quad \oplus \\ \bigcirc \quad \bigcirc \quad \bigcirc \quad \bullet \\ \text{---} \end{array} = \begin{array}{c} \oplus \quad \bullet \quad \boxed{\mathbb{H}^2} \\ \bigcirc \quad \bullet \quad \boxed{\mathbb{H}^2} \\ \text{---} \end{array}$$

$$C_2: \mathbb{H}^4 = I$$

$$C_6^1: \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \bigcirc \quad \bigcirc \quad \bigcirc \\ \text{---} \end{array} = \text{---}$$

$$C_6^2: \begin{array}{c} \oplus \quad \oplus \quad \oplus \quad \oplus \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \text{---} \end{array} = \text{---}$$

$$C_8^{14}: \begin{array}{c} \bullet \quad \mathbb{H}^2 \\ \bigcirc \quad \boxed{\mathbb{H}^2} \\ \text{---} \end{array} = \begin{array}{c} \mathbb{H}^2 \quad \bullet \\ \boxed{\mathbb{H}^2} \quad \bigcirc \\ \text{---} \end{array}$$

Lem W Def 4, Def 7, C_2, C_6, C_8 & R_{36} imply

$$R_{16}: \begin{array}{c} \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \text{---} \end{array} = \text{---}$$

$$\text{Proof: } R_{16}.LHS := \begin{array}{c} \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \text{---} \end{array}$$

$$\frac{\text{Def 4}}{\text{Def 7}} \begin{array}{c} \bullet \quad \bullet \quad \oplus \quad \bullet \quad \bullet \quad \oplus \quad \bullet \quad \bullet \\ \bigcirc \quad \bigcirc \quad \bigcirc \quad \bullet \quad \bigcirc \quad \bigcirc \quad \bullet \quad \bigcirc \\ \boxed{\mathbb{H}^2 \quad \mathbb{H}^2} \quad \text{---} \end{array}$$

$$\stackrel{C_2}{=} \begin{array}{c} \bullet \quad \bullet \quad \oplus \quad \bullet \quad \bullet \quad \oplus \quad \bullet \quad \bullet \\ \bigcirc \quad \bigcirc \quad \bigcirc \quad \bullet \quad \bigcirc \quad \bigcirc \quad \bullet \quad \bigcirc \\ \boxed{\text{---}} \quad \text{---} \end{array}$$

$$\stackrel{R_{36}}{=} \begin{array}{c} \bullet \quad \bullet \quad \oplus \quad \bullet \quad \bullet \quad \oplus \quad \bullet \quad \bullet \\ \bigcirc \quad \bigcirc \quad \bigcirc \quad \bullet \quad \bigcirc \quad \bigcirc \quad \bullet \quad \bigcirc \\ \boxed{\text{---}} \quad \boxed{\mathbb{H}^2} \quad \boxed{\mathbb{H}^2} \quad \text{---} \end{array}$$

$$\stackrel{R_{36}}{=} \begin{array}{c} \oplus \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \oplus \quad \bullet \quad \bullet \\ \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bullet \quad \bigcirc \\ \boxed{\mathbb{H}^2} \quad \boxed{\mathbb{H}^2} \quad \boxed{\mathbb{H}^2} \quad \boxed{\mathbb{H}^2} \quad \text{---} \end{array}$$

$$\stackrel{C_8^{14}}{=} \begin{array}{c} \oplus \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \oplus \quad \bullet \quad \bullet \\ \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bullet \quad \bigcirc \\ \boxed{\mathbb{H}^2 \quad \mathbb{H}^2} \quad \boxed{\mathbb{H}^2 \quad \mathbb{H}^2} \quad \text{---} \end{array}$$

$$\stackrel{C_2}{=} \begin{array}{c} \oplus \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \oplus \quad \bullet \quad \bullet \\ \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bigcirc \quad \bullet \quad \bigcirc \\ \boxed{\text{---}} \quad \text{---} \end{array}$$

$$\stackrel{C_6^2}{=} \begin{array}{c} \oplus \quad \oplus \quad \oplus \quad \oplus \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \boxed{\text{---}} \end{array}$$

$$\stackrel{C_6^1}{=} \text{---} =: R_{16}.RHS$$



$$R_{25}^1: \text{Diagram} = \text{Diagram} \cdot w \quad C_1: \omega^3 = I \quad C_2: H^4 = I \quad C_3: S^3 = I \quad C_5: SS' = S'S$$

$$C_7: \text{Diagram} = \text{Diagram} \quad R_{15}: \text{Diagram} = \text{Diagram} \quad C_6: \text{Diagram} = \text{Diagram}$$

$$\text{Def 1: } [S'] := [H][H][S][H][H]$$

$$\text{Def 2: } [\oplus] := [H][H][H][H] \quad C_8^2: \text{Diagram} = \text{Diagram} = \text{Diagram}$$

$$R_{32}^4: \text{Diagram} = \text{Diagram}$$

Lem X Def 1-2, C1-3, C5-8, R15, R25 & R32 imply

$$R_{33}: \text{Diagram} = \text{Diagram} \cdot w$$

$$\text{Proof: } R_{33}. \text{RHS} := \text{Diagram} \cdot w$$

$$R_{25}^1: \text{Diagram} = \text{Diagram} \cdot w^2$$

$$C_1: \text{Diagram} \equiv \text{Diagram}$$

$$C_5, C_7: \text{Diagram} \equiv \text{Diagram} \quad C_6: \text{Diagram} \equiv \text{Diagram}$$

$$\text{Def 1: } \text{Diagram} \equiv \text{Diagram} \quad C_2: \text{Diagram} \equiv \text{Diagram} \quad \text{Def 2: } \text{Diagram} \equiv \text{Diagram}$$

$$C_8^2: \text{Diagram} \equiv \text{Diagram}$$

$$C_2: \text{Diagram} \equiv \text{Diagram}$$

$$R_{32}^4: \text{Diagram} \equiv \text{Diagram} =: R_{33}. \text{LHS.}$$

$$R_{23}^1: \quad \begin{array}{c} S \\ \oplus \\ \bullet \end{array} = \begin{array}{c} \oplus \\ \bullet \\ \bullet \end{array} \cdot w^2$$

$$R_{23}^3: \quad \begin{array}{c} S \\ \oplus \\ S \\ S' \end{array} \cdot w = \begin{array}{c} \oplus \\ \bullet \\ \bullet \end{array}$$

$$\text{Def 1: } \boxed{S'} := H \cdot H \cdot S \cdot H \cdot H$$

$$C_2: H^4 = I \quad C_5: \quad \begin{array}{c} \oplus \\ \bullet \\ H^2 \end{array} = \begin{array}{c} \oplus \\ \bullet \\ H^2 \end{array}$$

$$\text{Def 7: } \begin{array}{c} \times \\ \text{H}^4 \end{array} = \begin{array}{c} \oplus \\ \bullet \\ \oplus \\ \bullet \\ \oplus \\ \bullet \end{array}$$

$$C_8: \quad \begin{array}{c} \oplus \\ \bullet \\ H^2 \end{array} = \begin{array}{c} \bullet \\ \oplus \\ \bullet \\ \oplus \\ \bullet \end{array}$$

$$C_6^2: \quad \begin{array}{c} \oplus \\ \bullet \\ \oplus \\ \bullet \\ \bullet \end{array} = \quad \quad \quad$$

Lem A1 R_{23} & R_{25} imply

$$R_{37}: \quad \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} = \begin{array}{c} S \\ \oplus \\ S^2 \\ S \end{array} \cdot w^2$$

Proof:

$$R_{25}^1: \quad \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} = \quad \quad \quad \boxed{\begin{array}{c} \oplus \\ \bullet \\ \bullet \end{array}} \cdot w$$

$\equiv R_{23}^3$

$$R_{37}: \quad \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} = \quad \quad \quad \boxed{\begin{array}{c} S \\ \oplus \\ S^2 \end{array}} \cdot w \cdot w$$

Def 1 $\equiv C_2$

$$R_{37}: \quad \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} = \begin{array}{c} S \\ \oplus \\ S^2 \\ S \end{array} \cdot w^2$$

□

Lem A2 C_6, C_8 & Def 7 imply R_{38} :

Proof:

$$\text{Def 7: } \begin{array}{c} \times \\ \text{H}^4 \end{array} = \quad \quad \quad \boxed{\begin{array}{c} \oplus \\ \bullet \\ \oplus \\ \bullet \\ \oplus \\ \bullet \end{array}}$$

$C_8^5 \equiv C_8^8$

$$\text{Def 7: } \begin{array}{c} \times \\ \text{H}^4 \end{array} = \quad \quad \quad \begin{array}{c} \oplus \\ \bullet \\ \oplus \\ \bullet \\ \oplus \\ \bullet \end{array} \quad \boxed{H^4}$$

$C_6^2 \equiv C_2$

$$R_{38}: \quad \begin{array}{c} \times \\ \text{H}^4 \end{array} \quad \begin{array}{c} \oplus \\ \bullet \\ \oplus \\ \bullet \end{array} = \quad \quad \quad \begin{array}{c} \oplus \\ \bullet \\ \oplus \\ \bullet \end{array}$$

□ 62

$$R_{37}: \text{Diagram} = S \oplus S^2 \cdot w^2 \quad R7: \boxed{x^2} = H S S H H S H$$

$$C1: w^3 = I \quad C2: H^4 = I \quad C3: S^3 = I \quad C5: SS' = S'S \quad R11: \boxed{z^2} = S' S S$$

$$R_{24}: \text{Diagram} = \text{Diagram} \quad \text{Def 4: } \text{Diagram} := H H H H$$

$$C7: \text{Diagram} = \text{Diagram} \quad R_{38}: \text{Diagram} = \text{Diagram}$$

$$\text{Def 3: } \text{Diagram} := H H H H \quad C8: \text{Diagram} = \text{Diagram}$$

Lem Y Def 1-4, C0-8, R7, R11, R15, R18, R19, R23, R24, R25, R32, R37 & R38 imply

$$R_{34}: \text{Diagram} = \text{Diagram} \cdot (-w^2)$$

$$\text{Proof: } R_{34}. \text{RHS} := \text{Diagram} \cdot (-w) \quad \text{HSHS } \boxed{SS^2} \xrightarrow[G]{C_5} \text{HSHS}'$$

$$\underline{C_3, C_5, R_{37}} \quad \text{R7, R11} \quad \text{Diagram} \cdot (-w^2) \cdot w^2$$

$$\underline{C_1, C_2} \quad \underline{R_{24}} \quad \text{Diagram} \cdot (-w)$$

$$\underline{C_2, C_3, C_7} \quad \text{Def 4} \quad \text{Diagram} \cdot (-w)$$

$$\underline{R_{38}} \quad \underline{\text{Def 3}} \quad \text{Diagram} \cdot (-w)$$

$$\underline{C_8} \quad \text{Diagram} \cdot (-w)$$

$$\underline{C_2} \quad \text{Diagram} \cdot (-w)$$

$$C_4: \quad \boxed{S} - \boxed{S} - \boxed{H} - \boxed{S} - \boxed{S} = \quad \boxed{H} - \boxed{H} - \boxed{H} - \boxed{S} - \boxed{H} - \boxed{H} - \boxed{H} \cdot (-\omega) \quad C_0: (-1)^2 = 1$$

$$C_7: \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \quad \begin{array}{c} \boxed{S} \\ | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \boxed{S} \end{array} \quad \begin{array}{c} | \\ \bullet \\ \text{---} \end{array}$$

$$R_{24} : \quad \begin{array}{c} \text{---} \\ | \\ \oplus \\ | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \oplus \\ | \\ \text{---} \\ \bullet \\ | \\ S \\ | \\ \bullet \\ | \\ \text{---} \end{array}$$

$$C_2 : H^4 = I \quad C_3 : S^3 = I$$

$$\text{Def 2: } \bigoplus := \begin{array}{c} \text{---} \\ \textcircled{+} \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}$$

Def 1: $\text{S}' := \text{HHSHHS}$

$$R_{18} : (1) \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad \boxed{S} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad \boxed{S}$$

$$(2) \quad \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad \boxed{S}$$

$$R_{19}: \quad (1) \quad \text{Diagram} = \text{Diagram}$$

$$(2) \quad \text{Diagram showing two equivalent circuit representations for a series RLC circuit. The left side shows a series combination of a resistor (R), an inductor (L), and a capacitor (C). The right side shows the same components connected in parallel. An equals sign between them indicates they are equivalent. A small 'H' is placed near the top of the rightmost diagram.}$$

Lem Y

$$R4 : \quad \begin{array}{c} \text{Circuit Diagram} \\ \text{Left: } H \text{ and } H \text{ gates followed by a CNOT gate.} \\ \text{Right: } S^2 \text{ and } H \text{ gates followed by a CNOT gate.} \end{array} = \quad \begin{array}{c} \text{Circuit Diagram} \\ \text{Left: } S^2 \text{ and } H \text{ gates followed by a CNOT gate.} \\ \text{Right: } S^2 \text{ and } H \text{ gates followed by a CNOT gate.} \end{array} \bullet (-w^*)$$

Proof cont.

C_4, R_{18}
 R_{24}

• $(-w)$ • $(-w)$

A quantum circuit diagram illustrating two different paths for the transformation $C_7 \equiv R_{18}$. The circuit consists of six horizontal wires representing qubits. The top wire starts with a S^2 gate, followed by a H gate. The second wire starts with a H^\dagger gate. The third wire is a control wire that interacts with both the first and second wires. The fourth wire starts with a S^3 gate, followed by a H^\dagger gate. The fifth wire starts with an S gate. The bottom wire ends with an H gate. The circuit is annotated with labels: $C_7 \equiv$ is on the far left; R_{18} is below the second wire; and $\cdot w^2$ is on the far right.

The diagram shows a quantum circuit with two horizontal wires. The top wire starts with a box labeled S^2 , followed by a box labeled H . A vertical line connects the output of H to a point where it meets another vertical line from a box labeled H^2 . From this junction, the line continues as the top wire. The bottom wire starts with a box labeled H^t , followed by a box labeled H . A vertical line connects the output of H to a point where it meets another vertical line from a box labeled H^2 . From this junction, the line continues as the bottom wire. The top wire then passes through a box labeled S and a box labeled H . The bottom wire then passes through a box labeled H^2 , a box labeled S , and a box labeled H^2 .

The diagram shows a ladder network with two vertical legs and three horizontal rungs. The left vertical leg contains a resistor R_{18} in series with an inductor L_1 . The right vertical leg contains a resistor R_{19} in series with an inductor L_2 . The top rung contains a resistor S in series with an inductor L_3 . The middle rung contains a resistor H . The bottom rung contains a resistor H . A central node is connected to the ground plane. The entire circuit is symmetric about a vertical axis.