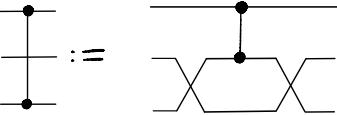
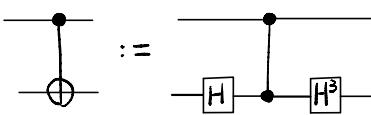
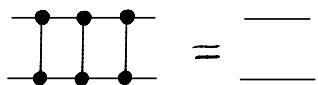
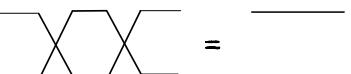
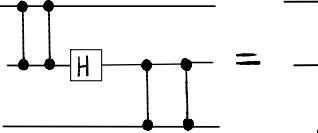
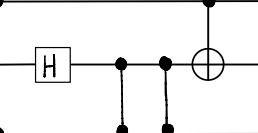


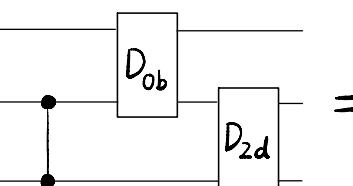
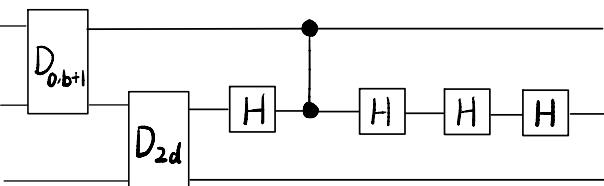
Def 5: 

Def 4: 

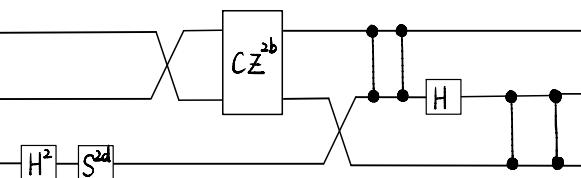
C_6 :  = $C_2: H^4 = I$ $C_3: S^3 = I$ R_{16} :  = 

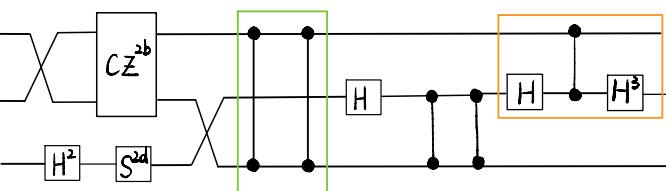
R_{40} :  = 

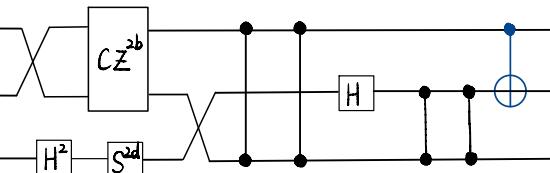
Lem3

10. (c)  =  , $b, d \in \mathbb{Z}_3$.

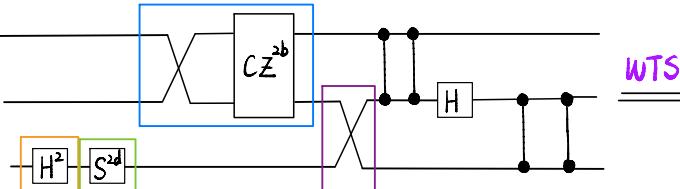
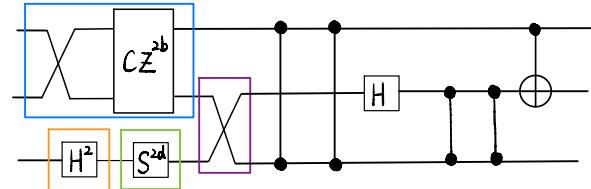
Proof cont:

10.(c).LHS = 

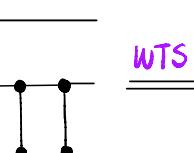
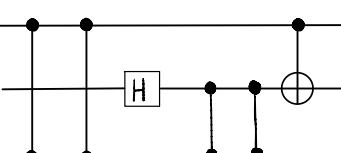
10.(c).RHS = 

Def 4
Def 5 

Hence

 \equiv 

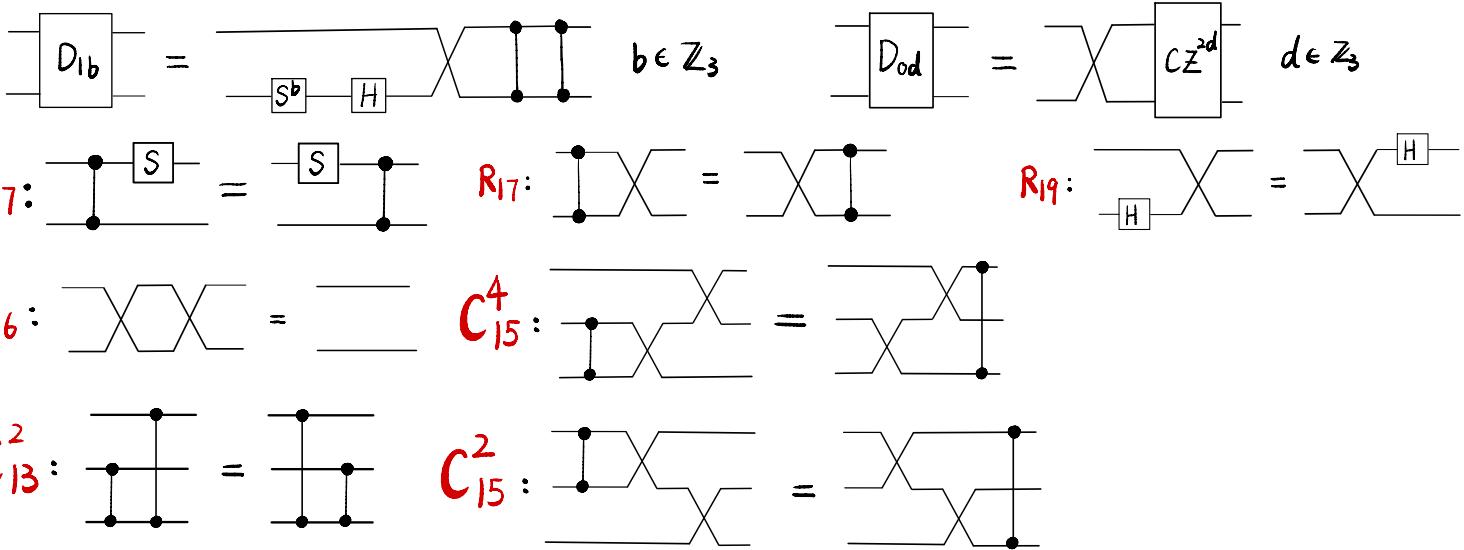
$C_3, C_6 \equiv R_{16}$

 \equiv 

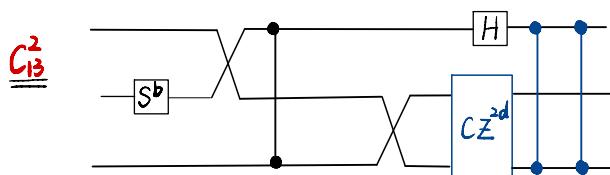
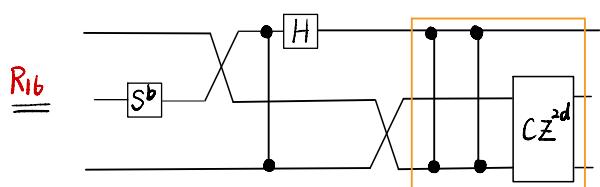
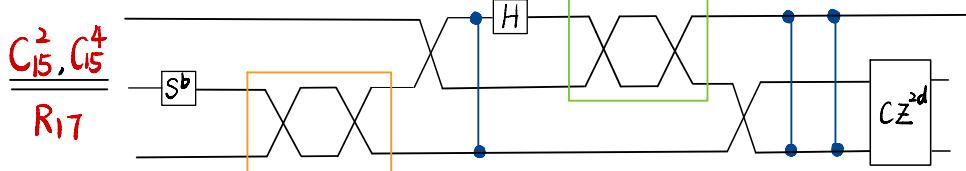
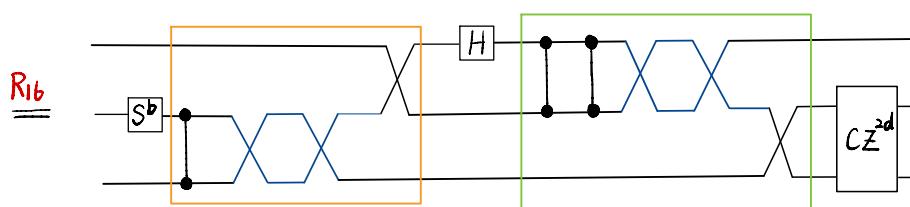
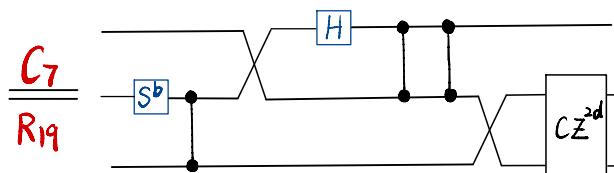
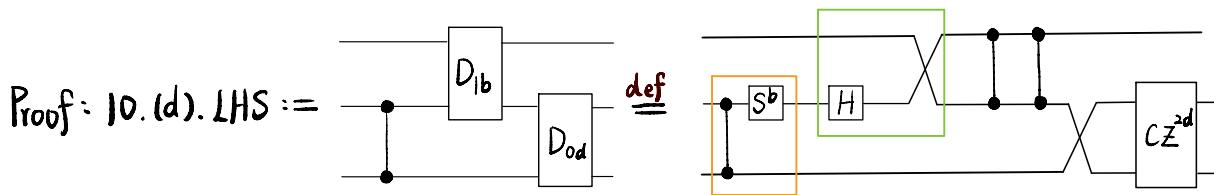
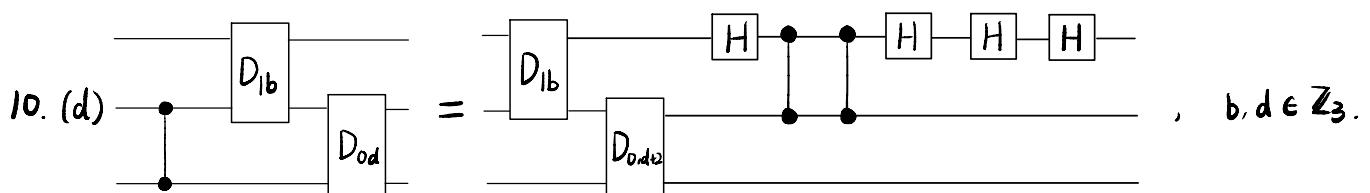
By R_{40} , this completes the proof.

III

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Lem 4 Def2, Def 5, C₂₋₃, C₆₋₇, R₁₆, R₁₇, R₁₉, R₄₁, C₁₃ & C₁₅ imply



$$D_{1b} = \begin{array}{c} \text{---} \\ | \quad | \\ \boxed{D_{1b}} \quad | \quad | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \quad | \\ | \quad | \\ \text{---} \end{array} \quad b \in \mathbb{Z}_3$$

$$R_{16} : \text{Diagram } 1 = \text{Diagram } 2 \quad R_{17} : \text{Diagram } 3 = \text{Diagram } 4 \quad C_2 : H^4 = I$$

$$C_{15}^5 : \text{Diagram 1} = \text{Diagram 2} \quad C_{13} : \text{Diagram 3} = \text{Diagram 4} \quad \text{Def 2} : \text{Diagram 5} := \text{Diagram 6}$$

Lem 4

$$10. (d) \quad \text{Diagram showing } D_{1b} \text{ and } D_{od} \text{ connected in series, equal to } D_{odd} \text{ followed by three } H \text{ gates. The condition is } b, d \in \mathbb{Z}_3.$$

Proof cont :

10.(d). LHS = 

The diagram illustrates a quantum circuit for the R_{16} gate. It features two horizontal lines representing qubits. The top line starts with a box labeled S^b , followed by a sequence of controlled operations. The first operation is a CNOT gate with its control on the top line and target on the bottom line. This is followed by a sequence of three controlled operations: a CNOT gate with control on the top line and target on the bottom line, a CNOT gate with control on the bottom line and target on the top line, and another CNOT gate with control on the top line and target on the bottom line. These three operations are grouped by an orange vertical bracket. After this sequence, there is another CNOT gate with control on the top line and target on the bottom line. The circuit then continues with a sequence of three controlled operations: a CNOT gate with control on the bottom line and target on the top line, a CNOT gate with control on the top line and target on the bottom line, and another CNOT gate with control on the bottom line and target on the top line. This sequence is grouped by a green vertical bracket. Finally, the circuit ends with a H gate on the top line and a CZ^{ad} gate between the two lines.

A quantum circuit diagram showing a sequence of operations. The circuit starts with two horizontal lines representing qubits. The top line has a red label C_{13} at its left end. A blue box labeled S^\dagger is positioned on the top line. A blue box labeled CZ^{2d} is positioned on the bottom line. A blue box labeled H is positioned on the top line. The circuit consists of several controlled operations: a CNOT gate from the top line to the bottom line, followed by a CNOT gate from the bottom line to the top line, then another CNOT gate from the top line to the bottom line, and finally a CNOT gate from the bottom line to the top line. The circuit concludes with a measurement operation on both lines.

10.(d).RHS :=

The diagram shows a quantum circuit with four horizontal lines representing qubits. The circuit starts with a S^\dagger gate on the first qubit, followed by an H gate on both the first and second qubits. A CNOT gate connects the first and second qubits. The third qubit has two vertical control lines from the second qubit, and a CZ^{id} gate connects the second and third qubits. The fourth qubit has two vertical control lines from the third qubit, and a $H^{\otimes 3}$ gate acts on all three control lines. The total depth of the circuit is labeled as $2(d+2) = 2d+1$.

$$R_{16}: \text{Diagram} = \text{Diagram} \quad R_{17}: \text{Diagram} = \text{Diagram} \quad C_2: H^4 = I \quad C_3: S^3 = I$$

$$C_{15}^2: \text{Diagram} = \text{Diagram} \quad C_{13}^2: \text{Diagram} = \text{Diagram} \quad C_6: \text{Diagram} = \text{Diagram}$$

$$R_{19}: \text{Diagram} = \text{Diagram} \quad R_{41}: \text{Diagram} = \text{Diagram}$$

Lem 4

$$10. (d) \quad \text{Diagram} = \text{Diagram}, \quad b, d \in \mathbb{Z}_3.$$

Proof cont:

$$10. (d). \text{RHS} = \text{Diagram}$$

$$\begin{aligned} & C_{15}^2 \\ & C_{13}^2, R_{17} \end{aligned}$$

$$R_{16}$$

$$R_{19}$$

Hence

$$\text{Diagram} \xrightarrow{\text{WTS}} \text{Diagram}$$

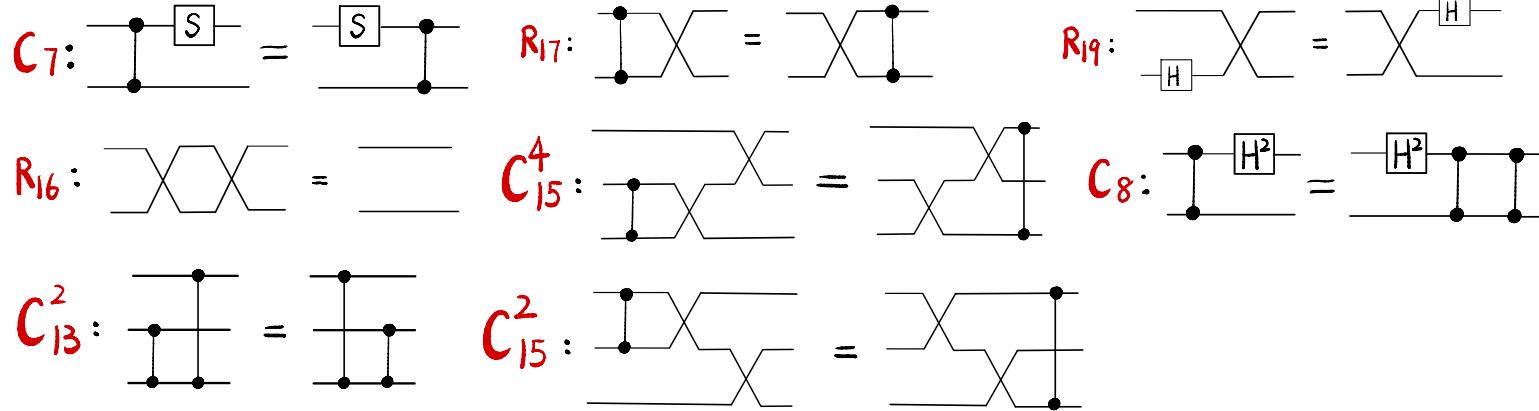
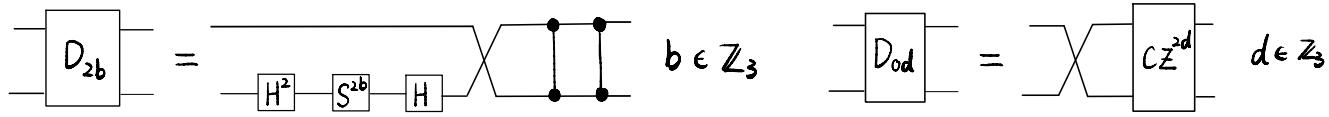
$C_3, C_6 \parallel R_{16}$

$$\text{Diagram} \xrightarrow{\text{WTS}}$$

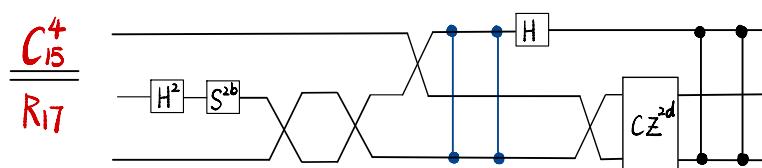
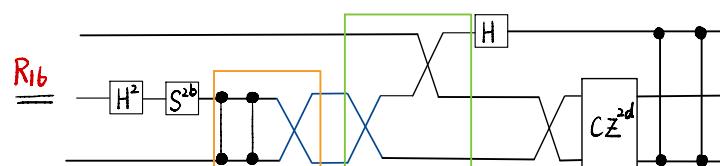
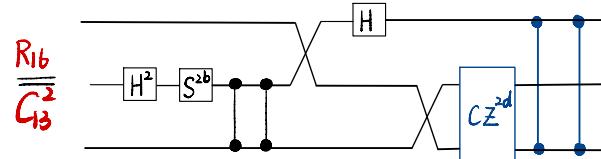
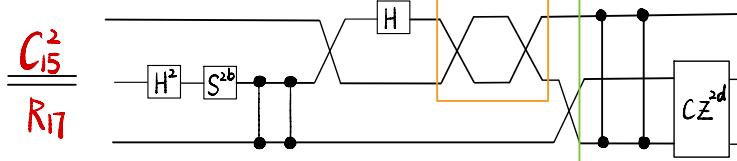
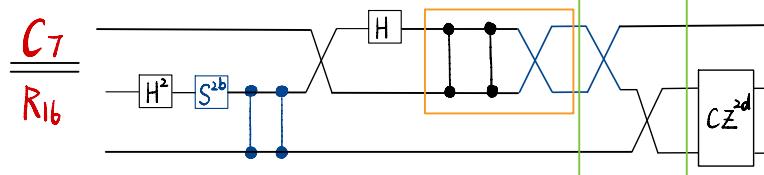
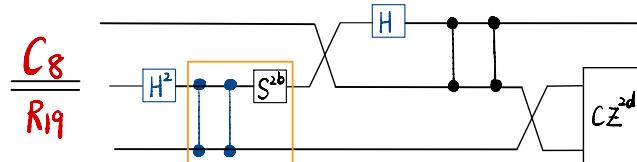
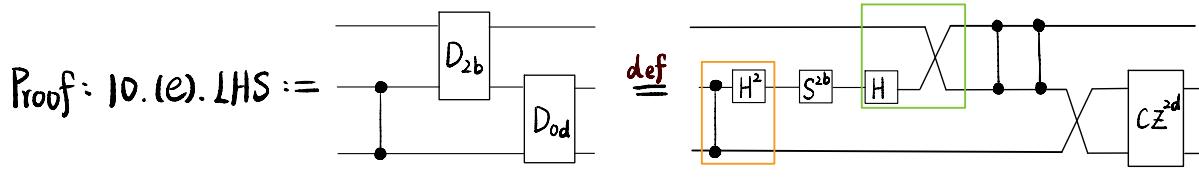
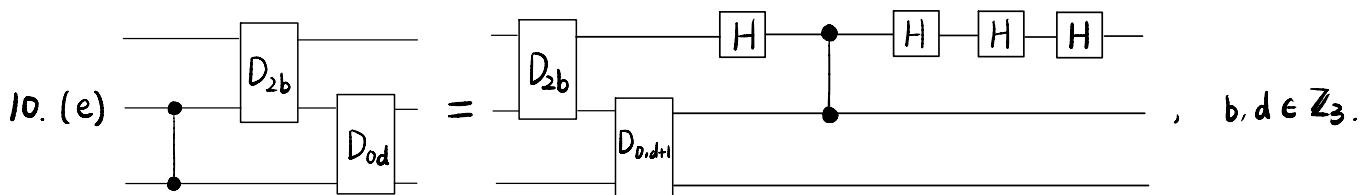
$$\text{Diagram}$$

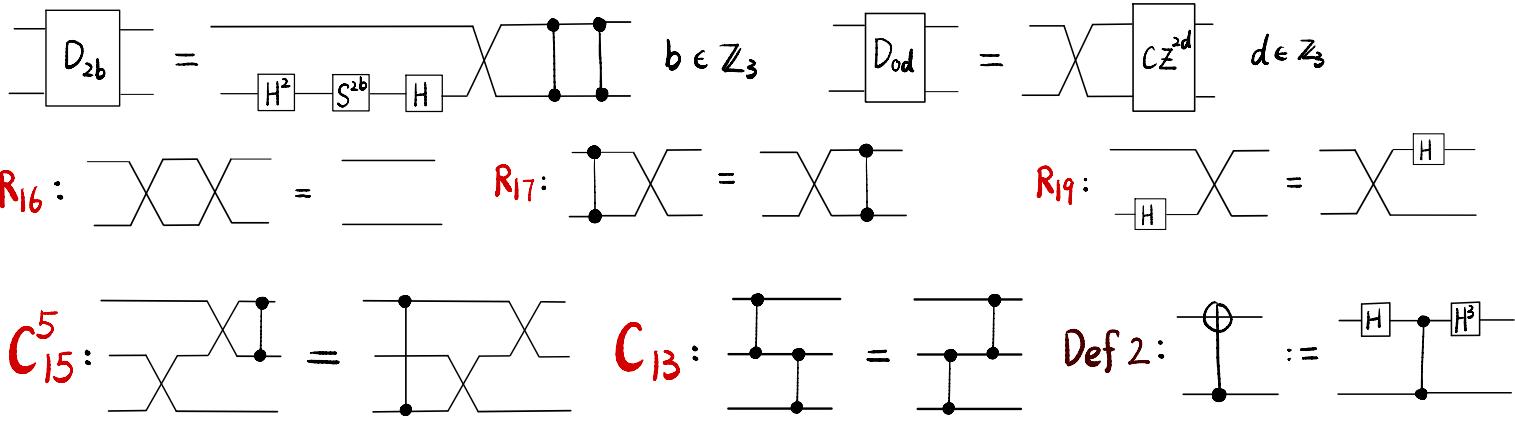
By R_{41} , this completes the proof.





Lem 5 Def2, Def 5, C_{2-3} , C_{6-8} , R_{16} , R_{17} , R_{19} , R_{42} , C_{13} & C_{15} imply





Lem 5

10. (e)
 $, b, d \in \mathbb{Z}_3.$

Proof cont:

10. (e). LHS =

$\stackrel{R_{16}}{=} \begin{array}{c} \text{H}^2 \text{ S}^{2b} \end{array} \boxed{\text{CNOT sequence}} \text{ CZ}^{2d}$

$\stackrel{C_{15}^5}{=} \begin{array}{c} \text{H}^2 \text{ S}^{2b} \end{array} \boxed{\text{CNOT sequence}} \text{ CZ}^{2d}$

$\stackrel{R_{17}}{=} \begin{array}{c} \text{H}^2 \text{ S}^{2b} \end{array} \boxed{\text{CNOT sequence}} \text{ CZ}^{2d}$

$\stackrel{C_{13}}{=} \begin{array}{c} \text{H}^2 \text{ S}^{2b} \end{array} \boxed{\text{CNOT sequence}} \text{ CZ}^{2d}$

10. (e). RHS :=

$\stackrel{\text{Def 2}}{=} \begin{array}{c} \text{H}^2 \text{ S}^{2b} \end{array} \boxed{\text{H}} \text{ CZ}^{2d} \oplus$
 $2(d+1) = 2d+2$

$\stackrel{R_{16}}{=} \begin{array}{c} \text{H}^2 \text{ S}^{2b} \end{array} \boxed{\text{H}} \text{ CZ}^{2d} \oplus$

$$R_{16}: \text{Diagram} = \text{Diagram}$$

$$R_{17}: \text{Diagram} = \text{Diagram}$$

$$R_{19}: \text{Diagram} = \text{Diagram}$$

$$C_{13}^2: \text{Diagram} = \text{Diagram}$$

$$C_{15}^2: \text{Diagram} = \text{Diagram}$$

$$C_6: \text{Diagram} = \text{Diagram}$$

$$C_2: H^4 = I \quad C_3: S^3 = I$$

$$R_{42}: \text{Diagram} = \text{Diagram}$$

Lem 5

$$10. (e) \text{Diagram} = \text{Diagram}, \quad b, d \in \mathbb{Z}_3.$$

Proof cont:

$$10. (e). LHS = \text{Diagram}$$

$$10. (e). RHS = \text{Diagram}$$

$$\frac{C_5^2}{R_{17}} \text{Diagram}$$

$$\frac{C_3^2}{R_{16}} \text{Diagram}$$

Hence

$$\text{Diagram} \xrightarrow{\text{WTS}} \text{Diagram}$$

$C_3, C_6 \parallel R_{16}$

$$\text{Diagram} \xrightarrow{\text{WTS}} \text{Diagram}$$

By R_{42} , this completes the proof.

□

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