

$C_0 : (-1)^2 = 1$

$C_1 : w^3 = 1$

$C_2 : H^4 = I$

$C_3 : S^3 = I$

$\text{Def 2: } \begin{array}{c} \oplus \\ \parallel \end{array} := \begin{array}{c} H \quad \cdot \quad H^3 \\ \parallel \quad \parallel \end{array}$

$\text{Def 1: } [S'] := [H] [H] [S] [H] [H] \quad C_5 : SS' = S'S$

$\text{Def 7: } \begin{array}{c} \oplus \\ \parallel \end{array} := \begin{array}{c} X \quad X \\ \parallel \quad \parallel \end{array}$

$\text{Def 5: } \begin{array}{c} \bullet \\ \parallel \end{array} := \begin{array}{c} X \quad X \\ \parallel \quad \parallel \end{array}$

$\text{Def 4: } \begin{array}{c} \bullet \\ \parallel \end{array} := \begin{array}{c} H \quad \cdot \quad H^3 \\ \parallel \quad \parallel \end{array}$

$R_{11}: [Z^2] = [S'] [S] [S]$

$R_7: [X^2] = [H] [S] [S] [H] [H] [S] [H]$

$R_{23}^8: [S^2] \begin{array}{c} \oplus \\ \parallel \end{array} = \begin{array}{c} \oplus \\ \parallel \end{array} \cdot w$

Lem D1 Def 0-2, Def 4-5, Def 7, C_0 - 8 , C_{13} , C_{16} , R_{15} , R_{23} , R_{24} , R_{25} imply

$R_{59}: \begin{array}{c} \oplus \quad \oplus \\ \parallel \quad \parallel \end{array} = \begin{array}{c} S^2 \quad H \quad \cdot \quad H^3 \quad S^2 \quad H \quad S' \quad S^2 \\ \parallel \quad \parallel \quad \parallel \quad \parallel \quad \parallel \quad \parallel \end{array} \cdot w$

$\text{Proof: } R_{59} \cdot \text{RHS} := \begin{array}{c} \oplus \quad \oplus \quad H \quad S^2 \quad H^3 \quad S \quad H \quad S' \quad S^2 \\ \parallel \quad \parallel \end{array} \cdot w$

$HS^2 H^2 SH \boxed{S'S^2} \xrightarrow{R_{11}} HS^2 H^2 SH \boxed{Z^2} \xrightarrow{R_7} X^2 Z^2$

$\frac{R_1}{R_7} \begin{array}{c} \oplus \quad \oplus \quad X^2 \quad Z^2 \\ \parallel \quad \parallel \quad \parallel \quad \parallel \end{array} = \begin{array}{c} S^2 \quad H \quad \cdot \quad H^3 \quad S^2 \quad H \quad S'^2 \quad H \quad S^2 \quad S \\ \parallel \quad \parallel \end{array} \cdot w$

$\underline{\text{Def 1}} \begin{array}{c} \oplus \quad \oplus \quad X^2 \quad Z^2 \\ \parallel \quad \parallel \quad \parallel \quad \parallel \end{array} = \begin{array}{c} S^2 \quad H \quad \cdot \quad H^2 \quad S^2 \quad H^2 \quad H \quad S'^2 \quad H \quad S^2 \quad S \\ \parallel \quad \parallel \end{array} \cdot w$

$\underline{\text{Def 2}} \begin{array}{c} \oplus \quad \oplus \quad X^2 \quad Z^2 \\ \parallel \quad \parallel \quad \parallel \quad \parallel \end{array} = \begin{array}{c} S^2 \quad H \quad \cdot \quad S^2 \quad \oplus \quad S^2 \quad H^2 \quad H \quad S'^2 \quad H \quad S^2 \quad S \\ \parallel \quad \parallel \end{array} \cdot w$

$\underline{R_{23}^8} \begin{array}{c} \oplus \quad \oplus \quad X^2 \quad Z^2 \\ \parallel \quad \parallel \quad \parallel \quad \parallel \end{array} = \begin{array}{c} S^2 \quad H \quad \cdot \quad S^2 \quad H^3 \quad S^2 \quad H \quad S'^2 \quad H \quad S^2 \quad S \\ \parallel \quad \parallel \end{array} \cdot w \cdot w$

$$C_0 : (-1)^2 = 1 \quad C_1 : w^3 = 1 \quad C_2 : H^4 = I \quad C_3 : S^3 = I \quad C_5 : SS' = S'S \quad C_6^* : \text{Diagram} = \text{Diagram}$$

$$R_{25}^{1*} : \text{Diagram} = \text{Diagram} \cdot w \quad R_{23-25}^2 : \text{Diagram} = \text{Diagram} \cdot w$$

$$C_{16}^4 : \text{Diagram} = \text{Diagram} \quad C_{13}^{11} : \text{Diagram} = \text{Diagram}$$

$$C_4^3 : \text{Diagram} = \text{Diagram} \cdot (-w^2)$$

Lem D1

$$R_{59} : \text{Diagram} = \text{Diagram} \cdot w$$

Proof cont.

$$R_{59} \cdot \text{RHS} = \text{Diagram} \cdot w \cdot w$$

$$S^2 H S^2 H S^2 S \xrightarrow[C_0, C_1]{C_4^3} S^2 S' H^+ S' S^2 S \cdot (-w) \xrightarrow[C_2]{C_3} H^+ S \cdot (-w)$$

$$\frac{R_{25}^{1*}, C_0, C_1}{C_2, C_3, C_4^3} \text{Diagram} \cdot w^2 \cdot w \cdot (-w)$$

$$\frac{C_1}{C_3} \text{Diagram} \cdot (-w)$$

$$\frac{C_4^4}{C_{16}} \text{Diagram} \cdot (-w)$$

$$\frac{C_6}{C_{13}^{11}} \text{Diagram} \cdot (-w)$$

$$\frac{R_{23-25}^2}{C_6^*} \text{Diagram} \cdot (-w) \cdot w$$

$$C_0 : (-1)^2 = 1 \quad C_1 : w^3 = 1 \quad C_2 : H^4 = I \quad C_3 : S^3 = I \quad C_5 : SS' = S'S \quad C_6^*: \text{Diagram} = \text{Diagram}$$

$$R_{25}^{6'}: \text{Diagram} = \text{Diagram} \cdot w^2 \quad C_{16}^7: \text{Diagram} = \text{Diagram}$$

$$C_{13}^{11}: \text{Diagram} = \text{Diagram}$$

Lem D1

$$R_{59}: \text{Diagram} = \text{Diagram} \cdot w$$

Proof cont.

$$R_{59} \cdot \text{RHS} = \text{Diagram} \cdot (-w^2)$$

$$R_{25}^{b'} \text{ (C}_1, C_2, C_3\text{)}: \text{Diagram} \cdot (-w^2) \cdot w$$

$$\frac{C_5}{C_1}: \text{Diagram} \cdot (-1)$$

Hence

$$R_{59}: \text{Diagram} \stackrel{\text{WTS}}{=} \text{Diagram} \cdot (-1)$$

$\parallel C_6$

$$R_{59}: \text{Diagram} \stackrel{\text{WTS}}{=} \text{Diagram} \cdot (-1)$$

$$\text{Then } R_{59} \cdot \text{RHS} = \text{Diagram} \cdot (-1)$$

$$\frac{C_{13}^{11}}{\parallel} \text{Diagram} \cdot (-1)$$

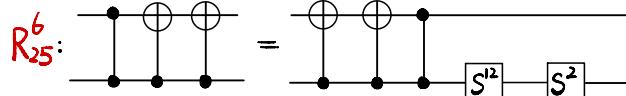
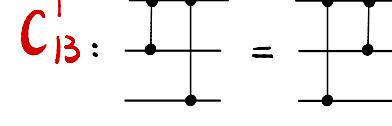
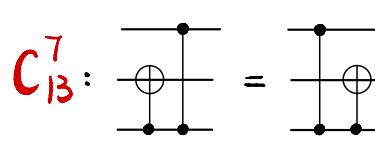
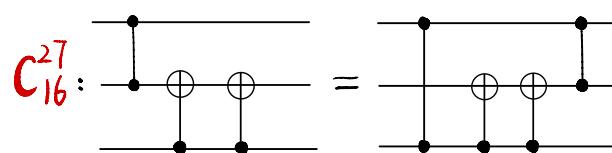
$C_0 : (-1)^2 = 1$

$C_1 : w^3 = 1$

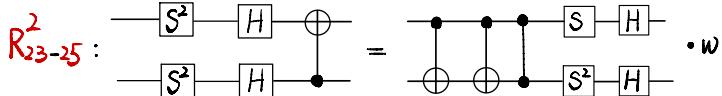
$C_2 : H^4 = I$

$C_3 : S^3 = I$

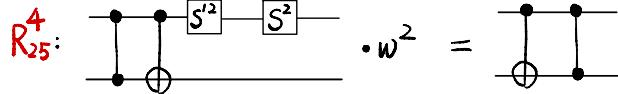
$C_5 : SS' = S'S$



$\cdot w^2$

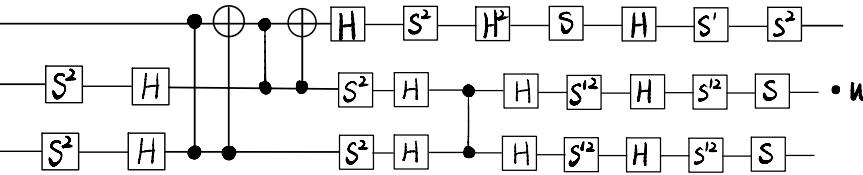
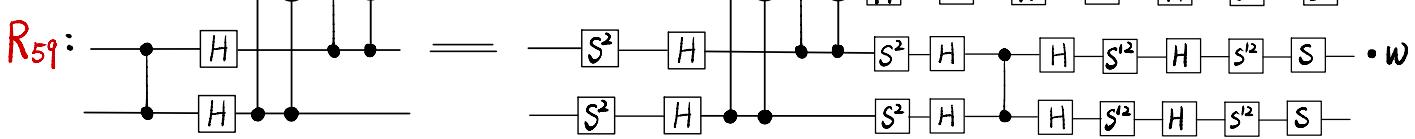


$\cdot w$



$\cdot w^2 =$

Lem D1



Proof cont.

$R_{59} \cdot \text{RHS} =$

C_{16}^{27}

$\frac{C_1^1}{C_1^7}$

R_{25}^6

$\cdot (-1) \cdot w^2$

R_{23-25}^2

$\cdot (-1) \cdot w^2 \cdot w$

$\cdot (-1)$

$\cdot (-1) \cdot w^2$

$$C_0 : (-1)^2 = 1$$

$$C_1 : w^3 = 1$$

$$C_2 : H^4 = I$$

$$C_3 : S^3 = I$$

$$C_5 : SS' = S'S$$

$$R_{23-25}^1 : \begin{array}{c} \text{Diagram showing } S^2 \text{ and } H \text{ operations on } w \end{array} = \begin{array}{c} \text{Diagram showing } S^2 \text{ and } H \text{ operations on } w \end{array} \cdot w$$

$R_{23}^{8'}$:

Lem D1

$$R_{59} : \begin{array}{c} \text{Diagram showing } H \text{ and } S^2 \text{ operations on } w \end{array} = \begin{array}{c} \text{Diagram showing } H, S^2, H, S^2, H, S^2, H, S^2, H, S^2 \text{ operations on } w \end{array} \cdot w$$

Proof cont.

$$R_{59} \cdot \text{RHS} = \begin{array}{c} \text{Diagram showing } S^2, S^2, S', S \text{ operations on } w \end{array} \cdot (-1) \cdot w^2$$

$$= \begin{array}{c} \text{Diagram showing } S^2, H, S^2, H, S^2, H, S^2, H, S^2, H, S^2 \text{ operations on } w \end{array} \cdot (-1) \cdot w^2$$

$$\text{Hence } R_{59} : \begin{array}{c} \text{Diagram showing } H \text{ and } S^2 \text{ operations on } w \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} \text{Diagram showing } H, S^2, H, S^2, H, S^2, H, S^2, H, S^2 \text{ operations on } w \end{array} \cdot (-1) \cdot w^2$$

$$R_{59} \cdot \text{RHS} = \begin{array}{c} \text{Diagram showing } H, H, H^3 \text{ operations on } w \end{array} \cdot (-1) \cdot w^2$$

$$= \begin{array}{c} \text{Diagram showing } S^2, H, S^2, H, S^2, H, S^2, H, S^2, H, S^2 \text{ operations on } w \end{array} \cdot (-1) \cdot w^2$$

$$= \begin{array}{c} \text{Diagram showing } S^2, H, H, S^2, H, S^2, H, S^2, H, S^2 \text{ operations on } w \end{array} \cdot (-1) \cdot w^2$$

$$= \begin{array}{c} \text{Diagram showing } S^2, H, H, H, S^2, H, S^2, H, S^2, H, S^2 \text{ operations on } w \end{array} \cdot (-1) \cdot w^2$$

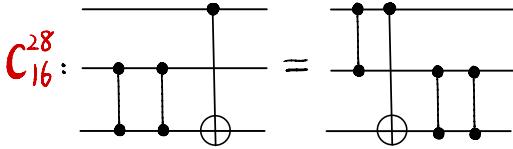
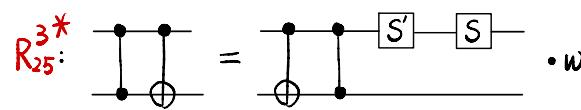
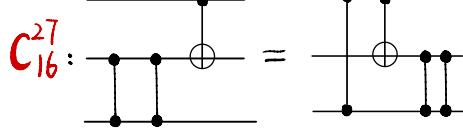
$$= \begin{array}{c} \text{Diagram showing } S^2, H, H, H, S^2, H, H, S^2, H, S^2, H, S^2 \text{ operations on } w \end{array} \cdot (-1) \cdot w^2$$

$$= \begin{array}{c} \text{Diagram showing } S^2, H, H, H, S^2, H, H, S^2, H, S^2, H, S^2 \text{ operations on } w \end{array} \cdot (-1) \cdot w^2 \cdot w$$

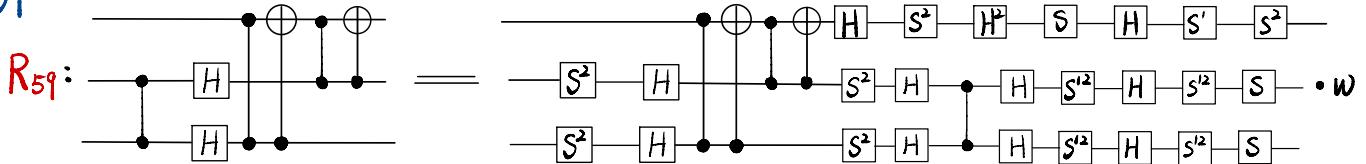
$$= \begin{array}{c} \text{Diagram showing } S^2, H, H, H, S^2, H, H, S^2, H, S^2, H, S^2 \text{ operations on } w \end{array} \cdot (-1) \cdot w^2 \cdot w$$

$$= \begin{array}{c} \text{Diagram showing } S^2, H, H, H, S^2, H, H, S^2, H, S^2, H, S^2 \text{ operations on } w \end{array} \cdot (-1)$$

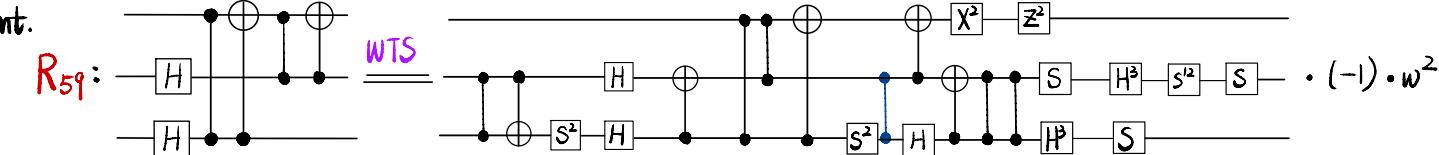
$$= \begin{array}{c} \text{Diagram showing } S^2, H, H, H, S^2, H, H, S^2, H, S^2, H, S^2 \text{ operations on } w \end{array} \cdot (-1)$$



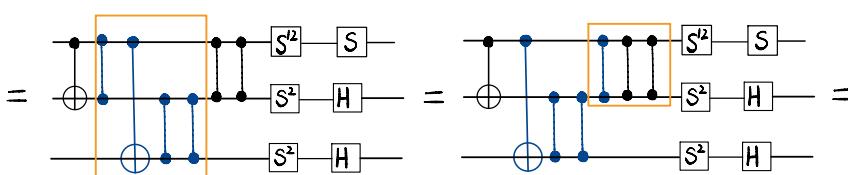
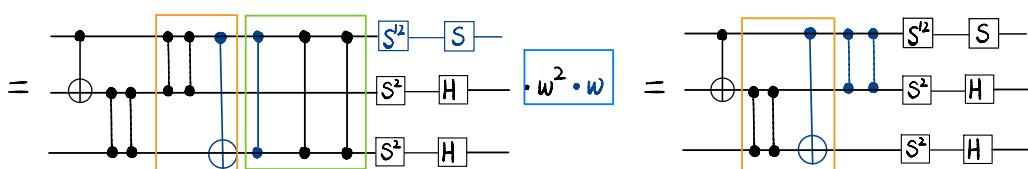
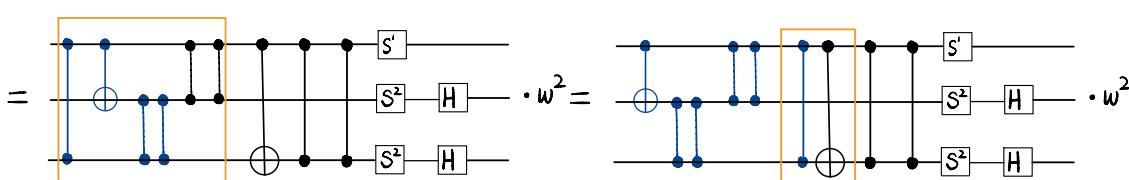
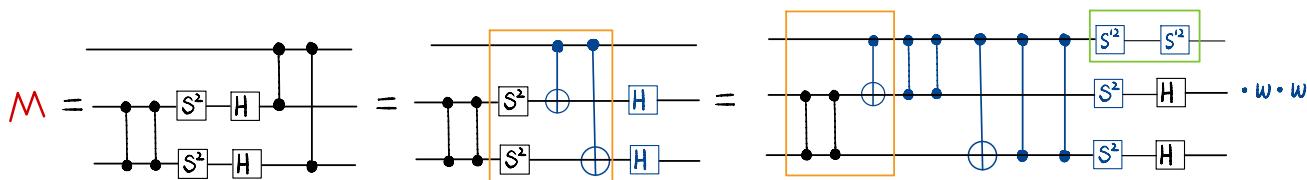
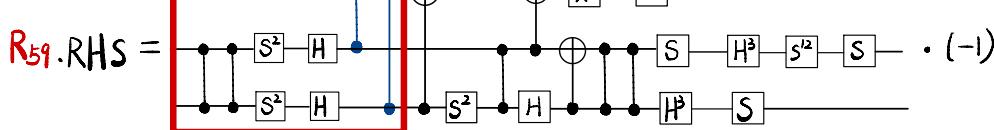
Lem D1



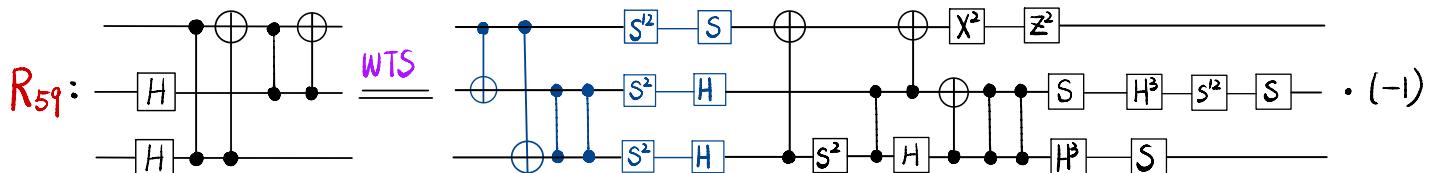
Proof cont.



M



Hence



$$C_{16}^7 : \quad = \quad$$

Lem D1

$$R_{59} : \text{Diagram} = \text{Diagram} \cdot w$$

Proof cont.

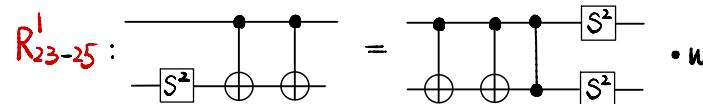
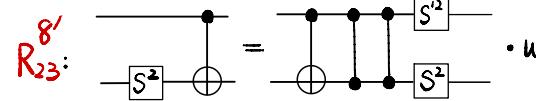
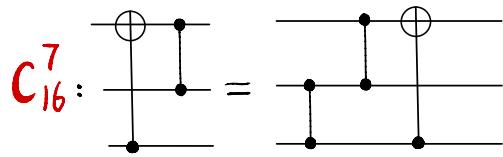
$$R_{59} : \begin{array}{c} \text{Circuit Diagram} \\ \text{WTS} \end{array} = \begin{array}{c} \text{Circuit Diagram} \\ \text{WTS} \end{array} \cdot (-1)$$

$R_{5^q}:$  The circuit consists of two horizontal lines representing qubits. The top line starts with a Hadamard gate (H), followed by a control dot, another control dot, and then a CNOT gate with the control on the top line and the target on the bottom line. The bottom line starts with a Hadamard gate (H), followed by a control dot, another control dot, and then a CNOT gate with the control on the bottom line and the target on the top line. A purple label "WTS" is placed above the circuit.

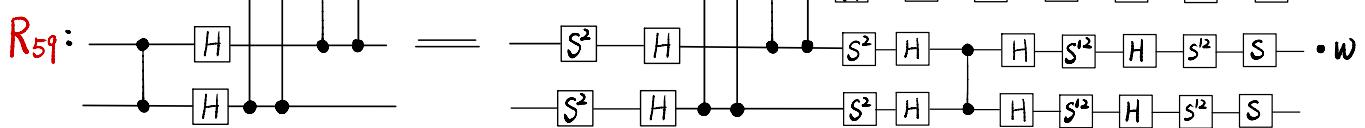
$R_{59}:$ 

III

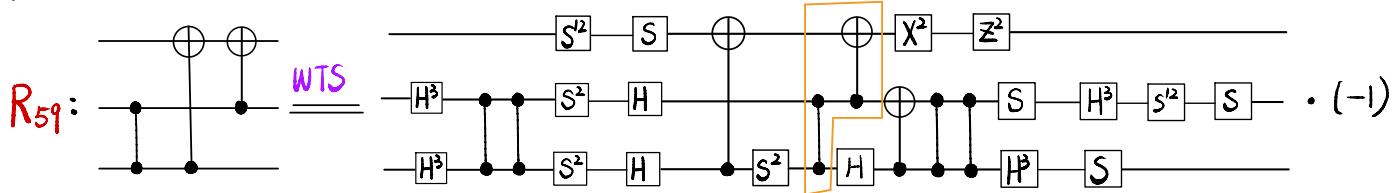
$R_{59}:$  The circuit starts with a WTS gate (indicated by a box with orange dashed lines) followed by a sequence of operations: H^3 , S^2 , H , \oplus , S , H^3 , S^{12} , and S . The circuit ends with a vertical ellipsis $\cdots (-1)$.



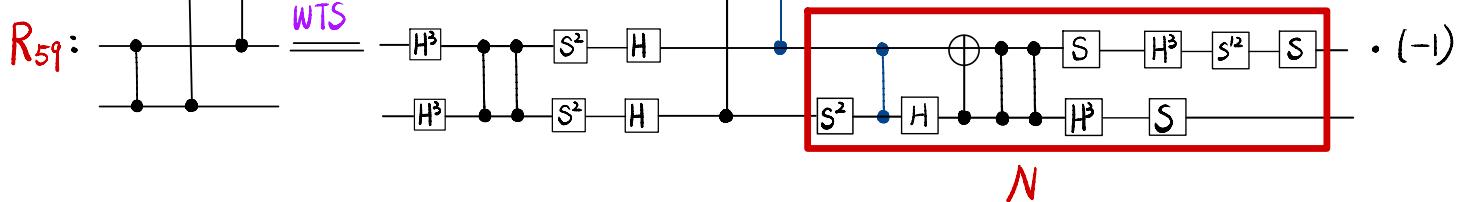
Lem D1



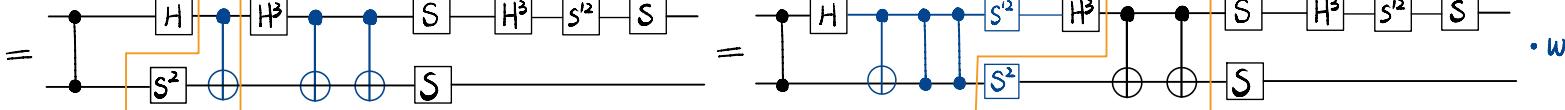
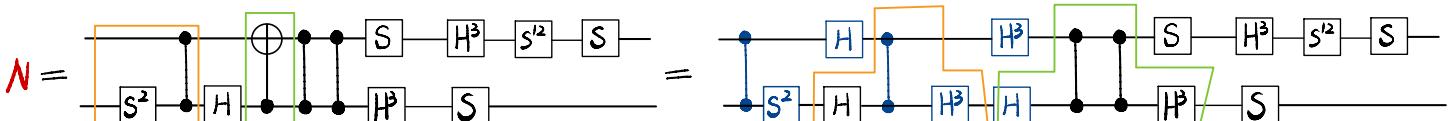
Proof cont.



|||



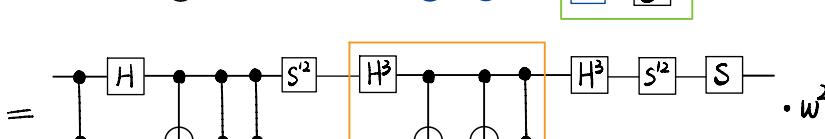
N



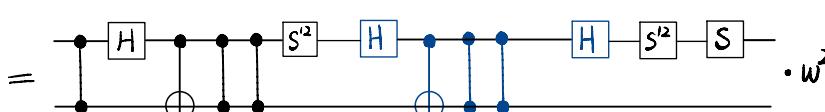
$\cdot w$



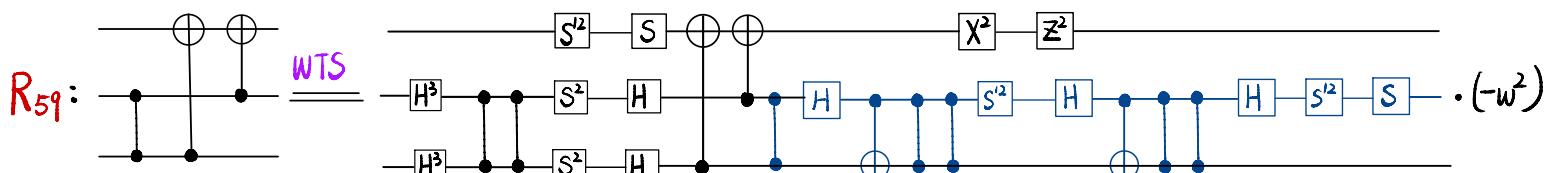
$\cdot w \cdot w$



$\cdot w^2$



$\cdot w^2$



$$\begin{array}{l}
R_{23}^*: \quad \text{Diagram} = \text{Diagram} \cdot w^2 \quad R_{23}: \quad \text{Diagram} = \text{Diagram} \cdot w \\
R_{10}: \quad \text{Diagram} = \text{Diagram} \cdot w^2 \quad R_8: \quad XZ = w^2 ZX \quad \text{Diagram} = \text{Diagram} \cdot w^2
\end{array}$$

Lem D1

$$\begin{array}{l}
R_{59}: \quad \text{Diagram} = \text{Diagram} \cdot w \\
\text{Diagram: } \text{Diagram} = \text{Diagram} \cdot w
\end{array}$$

Proof cont.

$$\begin{array}{l}
R_{59}: \quad \text{Diagram} = \text{Diagram} \cdot (-w^2) \\
\text{WTS: } \text{Diagram} = \text{Diagram} \cdot (-w^2)
\end{array}$$

$$\begin{array}{l}
0 = \text{Diagram} = \text{Diagram} \cdot w^2 = \text{Diagram} \cdot w^2 \cdot w^2 \\
= \text{Diagram} \cdot w \cdot w = \text{Diagram} \cdot w^2 \cdot w = \text{Diagram}
\end{array}$$

Hence

$$\begin{array}{l}
R_{59} \cdot \text{RHS} = \text{Diagram} \cdot (-w^2) \\
\text{Diagram: } \text{Diagram} = \text{Diagram} \cdot (-1)
\end{array}$$

$\boxed{ZX^2 Z^2} \xrightarrow{R_8} X \boxed{ZXZ^2} \cdot w^2 \xrightarrow{R_8} X^2 \boxed{ZZ^2} \cdot w^2 \xrightarrow{R_{12}} \boxed{X^2 \cdot w^2} \xrightarrow{C} X^2 \cdot w$