

$$\begin{array}{c}
 C_{16}: \quad \text{Diagram} = \text{Diagram} \\
 \text{Def 4:} \quad \text{Diagram} := \text{Diagram} \\
 C_8: \quad \text{Diagram} = \text{Diagram} \\
 C_8^7: \quad \text{Diagram} = \text{Diagram} \\
 \text{Def 5:} \quad \text{Diagram} := \text{Diagram} \\
 C_2: \quad H^2 = I
 \end{array}$$

Lem G Def 4-5, C_2, C_8 & C_{16} imply

$$R_{39}: \quad \text{Diagram} = \text{Diagram}$$

$$\begin{array}{l}
 \text{Proof: } R_{39}: \quad \text{Diagram} \stackrel{\text{WTS}}{=} \text{Diagram} \\
 \qquad\qquad\qquad C_2 \parallel
 \end{array}$$

$$R_{39}: \quad \text{Diagram} \stackrel{\text{WTS}}{=} \text{Diagram}$$

$\text{Def 4} \parallel C_8, C_8^7$

$$R_{39}: \quad \text{Diagram} \stackrel{\text{WTS}}{=} \text{Diagram}$$

$C_2 \parallel$

$$R_{39}: \quad \text{Diagram} \stackrel{\text{WTS}}{=} \text{Diagram}$$

By C_{16} , this completes the proof.



$$C_{16}: \begin{array}{c} \text{Diagram 1} \\ \oplus \end{array} = \begin{array}{c} \text{Diagram 2} \\ \oplus \end{array}$$

$$\text{Def 5: } \begin{array}{c} \text{Diagram 1} \\ \oplus \end{array} := \begin{array}{c} \text{Diagram 2} \\ \times \end{array}$$

$$C_2: H^2 = I$$

$$\text{Def 4: } \begin{array}{c} \text{Diagram 1} \\ \oplus \end{array} := \begin{array}{c} \text{Diagram 2} \\ H \quad H^3 \end{array}$$

$$C_8: \begin{array}{c} \text{Diagram 1} \\ H^2 \end{array} = \begin{array}{c} \text{Diagram 2} \\ H^2 \end{array}$$

$$C_{13}^5: \begin{array}{c} \text{Diagram 1} \\ \oplus \end{array} = \begin{array}{c} \text{Diagram 2} \\ \oplus \end{array}$$

$$C_8^8: \begin{array}{c} \text{Diagram 1} \\ H^2 \quad \oplus \end{array} = \begin{array}{c} \text{Diagram 2} \\ \oplus \quad \oplus \quad H^2 \end{array}$$

$$C_{16}: \begin{array}{c} \text{Diagram 1} \\ \oplus \end{array} = \begin{array}{c} \text{Diagram 2} \\ \oplus \end{array}$$

Lem H Def 4-5, C_2, C_8, C_{13} & C_{16} imply

$$R_{40}: \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array}$$

$$\text{Proof: } R_{40}: \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array}$$

$C_2 \parallel$

$$R_{40}: \begin{array}{c} \text{Diagram 1} \\ \boxed{H} \quad \boxed{H^3} \quad \boxed{H^2} \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} \text{Diagram 2} \\ \boxed{H^2} \end{array}$$

$\text{Def 4} \parallel C_8, C_8^8$

$$R_{40}: \begin{array}{c} \text{Diagram 1} \\ \oplus \quad \oplus \quad H^2 \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} \text{Diagram 2} \\ \text{Diagram 3} \end{array}$$

$C_2 \parallel$

$$R_{40}: \begin{array}{c} \text{Diagram 1} \\ \oplus \quad \oplus \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} \text{Diagram 2} \\ \text{Diagram 3} \end{array}$$

$$R_{40}. \text{LHS} := \begin{array}{c} \text{Diagram 1} \\ \oplus \quad \oplus \end{array} \stackrel{C_{16}}{=} \begin{array}{c} \text{Diagram 2} \\ \oplus \quad \oplus \end{array} \stackrel{C_5^3}{=} \begin{array}{c} \text{Diagram 3} \\ \oplus \quad \oplus \end{array} \stackrel{C_{16}}{=} \begin{array}{c} \text{Diagram 4} \\ \oplus \quad \oplus \end{array} =: R_{40}. \text{RHS}$$

□

$$R_{39} : \quad \text{Circuit Diagram} = \text{Circuit Diagram}$$

$$R_{40} : \quad \text{Circuit Diagram} = \text{Circuit Diagram}$$

$$\text{Circuit Diagram} = \text{Circuit Diagram}$$

$$R_{16} : \quad \text{Circuit Diagram} = \text{Circuit Diagram}$$

$$C_{15}^3 : \quad \text{Circuit Diagram} = \text{Circuit Diagram}$$

$$R_{17} : \quad \text{Circuit Diagram} = \text{Circuit Diagram}$$

$$C_{15}^4 : \quad \text{Circuit Diagram} = \text{Circuit Diagram}$$

$$R_{19} : \quad \text{Circuit Diagram} = \text{Circuit Diagram}$$

$$R_{31} : \quad \text{Circuit Diagram} = \text{Circuit Diagram}$$

Lem I Def 2, Def 4-5, $R_{16}, R_{17}, R_{19}, R_{31}, R_{39}, R_{40}, C_{13}$ & C_{15} imply

$$R_{41} : \quad \text{Circuit Diagram} = \text{Circuit Diagram}$$

$$R_{42} : \quad \text{Circuit Diagram} = \text{Circuit Diagram}$$

Proof:

$$R_{41}.LHS := \text{Circuit Diagram} \underset{R_{16}}{\equiv} \text{Circuit Diagram}$$

$$\underset{R_{16}}{\equiv} \text{Circuit Diagram}$$

$$\underset{C_{15}^3}{\equiv} \text{Circuit Diagram}$$

$$\underset{R_{16}}{\equiv} \text{Circuit Diagram}$$

$$\underset{R_7}{\equiv} \text{Circuit Diagram}$$

$$\underset{R_{39}}{\equiv} \text{Circuit Diagram}$$

$$\underset{R_{31}}{\equiv} \text{Circuit Diagram}$$

$$\underset{R_{16}}{\equiv} \text{Circuit Diagram}$$

$$\underset{C_{15}^4}{\equiv} \text{Circuit Diagram}$$

$$R_{39} : \quad \begin{array}{c} \text{---} \\ | \\ \bullet \end{array} \quad \boxed{H} \quad \begin{array}{c} \text{---} \\ | \\ \bullet \end{array} \quad =$$

$$R_{40} : \quad \text{Diagram showing a resistor labeled H connected in series with a horizontal line. The line has two vertical branches on either side of the resistor, each with two black dots at the ends. The rightmost branch has two black dots at its ends. An equals sign follows the line.}$$

$$R_{16} : \begin{array}{c} \text{Diagram of } R_{16} \text{ (a hexagon with internal diagonal)} \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \text{Diagram of } R_{16} \text{ result} \\ \diagup \quad \diagdown \end{array}$$

C_{15}^3 :

$$R_{1q} : \quad \begin{array}{c} \text{---} \\ | \quad | \\ \text{H} \end{array} \quad \begin{array}{c} \text{---} \\ | \quad | \\ \diagup \quad \diagdown \end{array} = \quad \begin{array}{c} \text{---} \\ | \quad | \\ \diagup \quad \diagdown \end{array} \quad \boxed{\text{H}}$$

$$R_{17}: \quad \begin{array}{c} \text{Diagram of } R_{17} \\ \text{Left: } \text{Diagram of } R_{17} \\ \text{Right: } \text{Diagram of } R_{17} \end{array} = \quad \begin{array}{c} \text{Diagram of } R_{17} \\ \text{Left: } \text{Diagram of } R_{17} \\ \text{Right: } \text{Diagram of } R_{17} \end{array}$$

$$C_{15}^4 : \quad \text{Diagram} = \quad \text{Diagram}$$

$$R_{31} : \quad \text{Diagram} = \quad \text{Diagram}$$

$$C_{13}^2 : \quad \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array}$$

Lem I

$$R_{41} : \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \xrightarrow{H} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} = \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \xrightarrow{\oplus} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array}$$

$$R_{42} : \quad \begin{array}{c} \text{Circuit Diagram} \\ \text{Left: } \text{H} \text{ gate on wire 4, wire 2 is controlled by wire 4} \\ \text{Right: } \text{H} \text{ gate on wire 4, wire 2 is controlled by wire 4, followed by a CNOT gate with control on wire 4 and target on wire 2} \end{array} =$$

Proof cont.

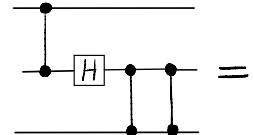
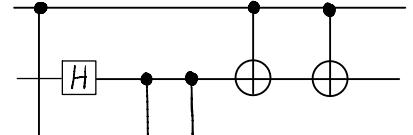
$$R_{4\cdot LHS} = \text{Diagram A} \underset{\text{K1b}}{=} \text{Diagram B}$$

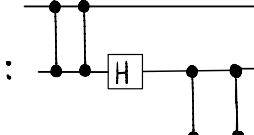
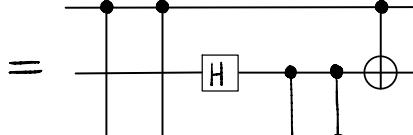
The diagram illustrates the decomposition of a CNOT gate (labeled C_{15}) into a sequence of quantum operations. The circuit starts with a R_{17} gate (represented by a red box) on the bottom wire. This is followed by a H gate (represented by a blue box) on the top wire. A CNOT gate (represented by a green box) is then applied between the two wires. Finally, a R_{1b} gate (represented by a red box) is applied to the bottom wire. The circuit concludes with three measurement operations (represented by circles with a plus sign) on both wires.

C_{13}^2

$=: R_{41} \cdot \text{RHS}$

$$R_{42}.LHS := \text{Diagram A} \quad R_{1b} = \text{Diagram B}$$

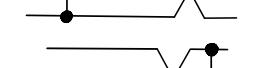
$R_{29}:$  = 

$R_{40}:$  = 

$R_{16}:$  = 

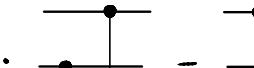
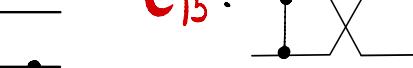
$C_{15}^3:$  = 

$R_{17}:$  = 

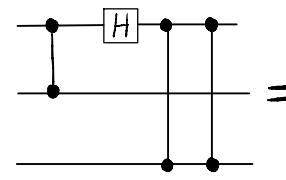
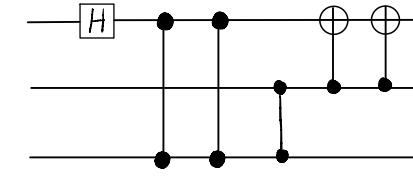
$C_{15}^4:$  = 

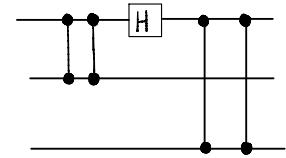
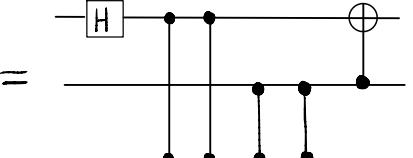
$R_{19}:$  = 

$R_{31}:$  = 

$C_{13}^2:$  = 

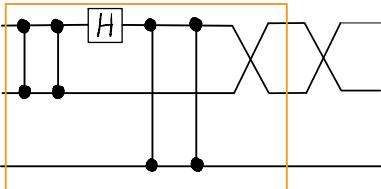
Lem I

$R_{41}:$  = 

$R_{42}:$  = 

Proof cont.

$R_{42}.LHS =$



$$\dots \equiv \text{[Circuit diagram with a blue box highlighting the CNOTs between wires 1 and 2, 2 and 3, and 3 and 4.]}$$

$$\underline{\underline{R_{40}}} \equiv \text{[Circuit diagram with a blue box highlighting the CNOTs between wires 1 and 2, 2 and 3, and 3 and 4.]}$$

$$\dots \equiv \text{[Circuit diagram with orange and green boxes highlighting different parts of the circuit.]}$$

$$\underline{\underline{C_{13}^2}} \quad \underline{\underline{R_{16}}} \quad =: R_{42}.RHS$$



$$\text{Def 5: } \begin{array}{c} \bullet \\ \parallel \\ \bullet \end{array} := \begin{array}{c} \bullet \\ \parallel \\ \bullet \end{array} \quad \text{Def 4: } \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} := \begin{array}{c} \bullet \\ \boxed{H} \\ \bullet \end{array}$$

$$\text{C}_8 : \begin{array}{c} \bullet \\ \boxed{H^2} \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \boxed{H^2} \\ \bullet \end{array}$$

$$\text{C}_8^8 : \begin{array}{c} \bullet \\ \boxed{H^2} \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} \quad \text{C}_8^8 : \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \boxed{H^2} \\ \bullet \end{array}$$

$$\text{C}_{16}^4 : \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} \quad = \quad \begin{array}{c} \bullet \\ \parallel \\ \bullet \end{array} \quad = \quad \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array}$$

$$\text{C}_2 : H^4 = I$$

Lem K Def 4-5, C₂, C₈ & C₁₆ imply

$$\text{R}_{45}: \begin{array}{c} \bullet \\ \parallel \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \boxed{H} \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \parallel \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \boxed{H} \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \parallel \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \boxed{H} \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \parallel \\ \bullet \end{array}$$

Proof:

$$\text{R}_{45}: \begin{array}{c} \bullet \\ \parallel \\ \bullet \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} \bullet \\ \boxed{H} \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \parallel \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \boxed{H} \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \parallel \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \boxed{H} \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \parallel \\ \bullet \end{array}$$

$\text{C}_2 \parallel$

$$\text{R}_{45}: \begin{array}{c} \bullet \\ \boxed{H} \quad \boxed{H^3} \\ \parallel \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} \bullet \\ \boxed{H} \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \boxed{H} \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \parallel \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \boxed{H} \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \boxed{H} \quad \boxed{H^3} \\ \parallel \end{array}$$

$\text{C}_2 \parallel$

$$\text{R}_{45}: \begin{array}{c} \bullet \\ \parallel \\ \bullet \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} \bullet \\ \boxed{H} \quad \boxed{H^3} \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \parallel \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \boxed{H} \quad \boxed{H^3} \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \parallel \\ \bullet \end{array}$$

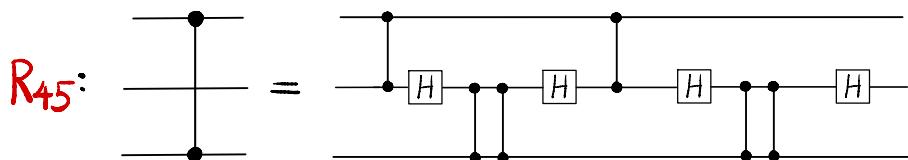
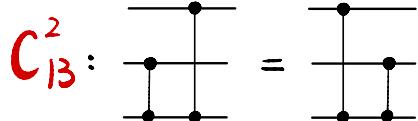
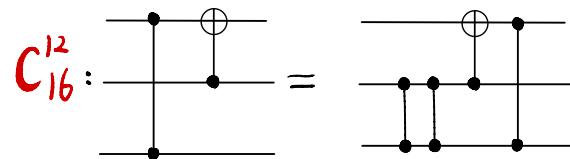
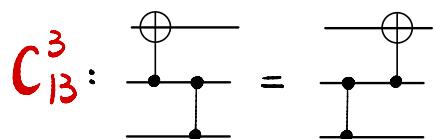
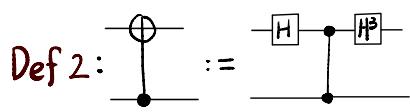
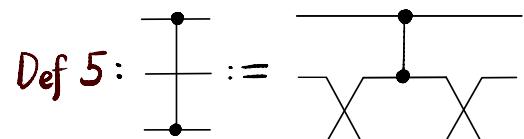
$\parallel \text{Def 4}$

$$\text{R}_{45}: \begin{array}{c} \bullet \\ \parallel \\ \bullet \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \parallel \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \parallel \\ \bullet \end{array}$$

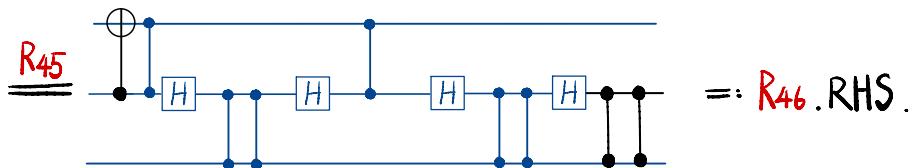
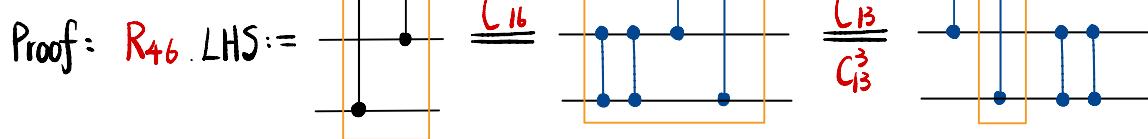
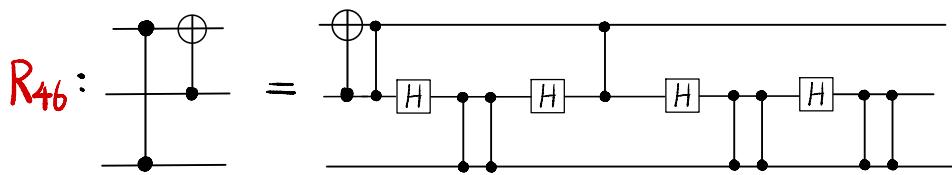
$$\text{Then R}_{45} \text{ RHS} := \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \boxed{H^2} \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \boxed{H^2} \\ \bullet \end{array} \quad \stackrel{\text{C}_8}{=} \quad \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \boxed{H^2} \quad \boxed{H^2} \\ \bullet \end{array} \quad \stackrel{\text{C}_8^8}{=}$$

$$\stackrel{\text{C}_2}{=} \quad \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} \quad \stackrel{\text{C}_{16}^4}{=} \quad \begin{array}{c} \bullet \\ \parallel \\ \bullet \end{array} \quad =: \text{R}_{45} \text{ LHS.}$$

\parallel



Lem L Def 2, Def 5, C_B , C_{16} & R_{45} imply



$$\begin{array}{ll}
R_{23}^*: \quad \text{Diagram} = \text{Diagram} \cdot w^2 & R_{25}^*: \quad \text{Diagram} = \text{Diagram} \cdot w \\
C_7: \quad \text{Diagram} = \text{Diagram} & R_{24}: \quad \text{Diagram} = \text{Diagram} \\
C_{16}: \quad \text{Diagram} = \text{Diagram} & C_{13}^7: \quad \text{Diagram} = \text{Diagram} \\
\text{Def 7:} \quad \text{Diagram} := \text{Diagram} & \text{Def 5:} \quad \text{Diagram} := \text{Diagram} \\
\text{Def 2:} \quad \text{Diagram} := \text{Diagram} &
\end{array}$$

Lem M Def 2, Def 5, Def 7, $C_3, C_6, C_7, C_{13}, C_{16}, R_{15}, R_{23}^*, R_{25}^*, R_{23-25}$ & R_{24} imply

$$R_{43}: \quad \text{Diagram} = \text{Diagram}$$

$$\text{Proof: } R_{43} \cdot \text{RHS} := \text{Diagram}$$

$$\underline{\underline{R_{23-25}^*}} = \text{Diagram} \cdot w$$

$$\underline{\underline{R_{23-25}^*}} = \text{Diagram} \cdot w \cdot w$$

$$\underline{\underline{C_3, C_7}} \quad R_{24} = \text{Diagram} \cdot w^2$$

$$\underline{\underline{C_{13}^7}} = \text{Diagram} \cdot w^2$$