A Boxes	Construction	Matrix Evaluation	Generic Expression
$-\boxed{A_{01}}-$	$:= \; -$	I	
$-\boxed{A_{02}}-$	$:= \; -\boxed{H}-\boxed{H}-$	H^2	
$-\boxed{A_{10}}-$	$:= \; -\boxed{H}-$	H	$[\![-\boxed{A_{1b}}-]\!] = HS^b$, $b \in \mathbb{Z}_3$
$-\boxed{A_{11}}-$	$:= \; -\boxed{S}-\boxed{H}-$	HS	
$-\boxed{A_{12}}-$	$:= \; -\boxed{S}-\boxed{S}-\boxed{H}-$	HS^2	
$-\boxed{A_{20}}-$	$:= \; -\boxed{H}-\boxed{H}-\boxed{H}-$	$H^3 = HH^2$	$[\![-\boxed{A_{2b}}-]\!] = HS^{2b}H^2$, $b \in \mathbb{Z}_3$
$-\boxed{A_{21}}-$	$:= \; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-$	HS^2H^2	
$-\boxed{A_{22}}-$	$:= \; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-$	HSH^2	

(1) $-\boxed{H}-\boxed{A_{01}}- \; = \; -\boxed{A_{10}}-$

(2) $-\boxed{H}-\boxed{A_{02}}- \; = \; -\boxed{A_{20}}-$

(3) $-\boxed{H}-\boxed{A_{10}}- \; = \; -\boxed{A_{02}}-$

trivial

1. $-\boxed{H}-\boxed{A_{ab}}-$ $\quad (a,b) \in \mathbb{Z}_3 \times \mathbb{Z}_3 \setminus \{(0,0)\}$

(4) $-\boxed{H}-\boxed{A_{11}}- \; = \; -\boxed{A_{12}}-\boxed{S^2}-\boxed{X^2}- \cdot (-w^2)$

(5) $-\boxed{H}-\boxed{A_{12}}- \; = \; -\boxed{A_{22}}-\boxed{X}-\boxed{S}- \cdot (-w)$

(6) $-\boxed{H}-\boxed{A_{20}}- \; = \; -\boxed{A_{01}}-$

C4

(7) $-\boxed{H}-\boxed{A_{21}}- \; = \; -\boxed{A_{11}}-\boxed{X}-\boxed{S}- \cdot (-w)$

(8) $-\boxed{H}-\boxed{A_{22}}- \; = \; -\boxed{A_{21}}-\boxed{S^2}-\boxed{X^2}- \cdot (-w^2)$

$$-\boxed{A_{01}}- \;:=\; ——$$

$$-\boxed{A_{02}}- \;:=\; -\boxed{H}-\boxed{H}-$$

$$-\boxed{A_{10}}- \;:=\; -\boxed{H}-$$

$$-\boxed{A_{20}}- \;:=\; -\boxed{H}-\boxed{H}-\boxed{H}-$$

C2 : $H^4 = I$

Lem 1 By definition and C_2, we have

1\. (1) $-\boxed{H}-\boxed{A_{01}}- \;=\; -\boxed{A_{10}}-$

(2) $-\boxed{H}-\boxed{A_{02}}- \;=\; -\boxed{A_{20}}-$

(3) $-\boxed{H}-\boxed{A_{10}}- \;=\; -\boxed{A_{02}}-$

(6) $-\boxed{H}-\boxed{A_{20}}- \;=\; -\boxed{A_{01}}-$

Proof 1. (1). LHS $:= -\boxed{H}- \;=:\; -\boxed{A_{10}}- \;=$ 1. (1). RHS

1\. (2). LHS $:= -\boxed{H}-\boxed{H}-\boxed{H}- \;=:\; -\boxed{A_{20}}- \;=$ 1. (2). RHS

1\. (3). LHS $:= -\boxed{H}-\boxed{H}- \;=:\; -\boxed{A_{02}}- \;=$ 1. (3). RHS

1\. (6). LHS $:= -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}- \;\overset{C_2}{=}\; —— \;=:\; -\boxed{A_{01}}- \;=$ 1. (6). RHS.

$\boxed{/\!/}$

$$-\boxed{A_{11}}- \;:=\; -\boxed{S}-\boxed{H}-$$

$$-\boxed{A_{21}}- \;:=\; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-$$

$$-\boxed{A_{12}}- \;:=\; -\boxed{S}-\boxed{S}-\boxed{H}-$$

$$-\boxed{A_{22}}- \;:=\; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-$$

$C2$: $H^4 = I$

$R1$: $$-\boxed{H}-\boxed{S}-\boxed{H}- \;=\; -\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S^2}-\boxed{X^2}- \;\cdot\; (-w^2)$$

R_1': $$-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \;=\; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{X}-\boxed{S}- \cdot (-w)$$

Lem 2 By definition, C_2 & R_1, we have 1.

(4) $$-\boxed{H}-\boxed{A_{11}}- \;=\; -\boxed{A_{12}}-\boxed{S^2}-\boxed{X^2}- \cdot (-w^2)$$

(5) $$-\boxed{H}-\boxed{A_{12}}- \;=\; -\boxed{A_{22}}-\boxed{X}-\boxed{S}- \cdot (-w)$$

(7) $$-\boxed{H}-\boxed{A_{21}}- \;=\; -\boxed{A_{11}}-\boxed{X}-\boxed{S}- \cdot (-w)$$

(8) $$-\boxed{H}-\boxed{A_{22}}- \;=\; -\boxed{A_{21}}-\boxed{S^2}-\boxed{X^2}- \cdot (-w^2)$$

Proof: 1. (4). LHS $$:= -\boxed{H}-\boxed{S}-\boxed{H}- \;\stackrel{R_1}{=}\; -\boxed{-\boxed{S}-\boxed{S}-\boxed{H}-}-\boxed{S^2}-\boxed{X^2}- \cdot (-w^2)$$

$$=: -\boxed{A_{12}}-\boxed{S^2}-\boxed{X^2}- \cdot (-w^2) = 1.(4).RHS$$

1. (5). LHS $$:= -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \;\stackrel{R_1'}{=}\; -\boxed{-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-}-\boxed{X}-\boxed{S}- \cdot (-w)$$

$$=: -\boxed{A_{22}}-\boxed{X}-\boxed{S}- \cdot (-w) = 1.(5).RHS$$

1. (7). LHS $$:= -\boxed{H}-\boxed{H}-\boxed{-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-}-$$

$$\stackrel{R_1'}{=} -\boxed{-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}-}-\boxed{S}-\boxed{H}-\boxed{X}-\boxed{S}- \cdot (-w)$$

$$\stackrel{C_2}{=} -\boxed{-\boxed{S}-\boxed{H}-}-\boxed{X}-\boxed{S}- \cdot (-w)$$

$$=: -\boxed{A_{11}}-\boxed{X}-\boxed{S}- \cdot (-w) = 1.(7).RHS$$

1. (8). LHS $$:= -\boxed{H}-\boxed{H}-\boxed{-\boxed{H}-\boxed{S}-\boxed{H}-}-$$

$$\stackrel{R_1}{=} -\boxed{-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-}-\boxed{S^2}-\boxed{X^2}- \cdot (-w^2)$$

$$=: -\boxed{A_{21}}-\boxed{S^2}-\boxed{X^2}- \cdot (-w^2) = 1.(8).RHS \qquad \blacksquare$$

C_0: $(-1)^2 = 1$ $C1$: $w^3 = 1$

C_3: $S^3 = I$

$R9$: $X^3 = I$

$C2$: $H^4 = I$

Lemma A R_1': $-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \;=\; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{X}-\boxed{S}- \cdot (-w)$

is a consequence of $R1$: $-\boxed{H}-\boxed{S}-\boxed{H}- \;=\; -\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S^2}-\boxed{X^2}- \cdot (-w^2)$

Proof: Concatenating both sides of R_1 by H to the left:

$$-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}- \;=\; -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S^2}-\boxed{X^2}- \cdot (-w^2)$$

Concatenating both sides of R_1 by $X;S$ to the right:

$$-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{X}-\boxed{S}- \;=\; -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S^2}-\boxed{X^2}-\boxed{X}-\boxed{S}- \cdot (-w^2)$$

By C_3 & $R9$, $-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{X}-\boxed{S}- \;=\; -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \cdot (-w^2)$

Multiplying both sides of R_1 by $-w$:

$$-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{X}-\boxed{S}- \cdot (-w) \;=\; -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \cdot (-w^2)(-w)$$

By C_0 & C_1, $-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{X}-\boxed{S}- \cdot (-w) \;=\; -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \equiv R_1'$. ☑

Lemma B C_5': $-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- \;=\; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$ is a consequence of C_5: $-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- \;=\; -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$

Proof: Concatenating both sides of C_5 by H^2 to the left:

$$-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- \;=\; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$$

By C_2, $-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- \;=\; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}- \equiv C_5'$. ☑

A Boxes	Construction	Matrix Evaluation	Generic Expression
$-\boxed{A_{01}}-$	$:= \;-\!-\!-$	I	
$-\boxed{A_{02}}-$	$:= -\boxed{H}-\boxed{H}-$	H^2	
$-\boxed{A_{10}}-$	$:= -\boxed{H}-$	H	$[\![-\boxed{A_{1b}}-]\!] = HS^b$, $b \in \mathbb{Z}_3$
$-\boxed{A_{11}}-$	$:= -\boxed{S}-\boxed{H}-$	HS	
$-\boxed{A_{12}}-$	$:= -\boxed{S}-\boxed{S}-\boxed{H}-$	HS^2	
$-\boxed{A_{20}}-$	$:= -\boxed{H}-\boxed{H}-\boxed{H}-$	$H^3 = HH^2$	$[\![-\boxed{A_{2b}}-]\!] = HS^{2b}H^2$, $b \in \mathbb{Z}_3$
$-\boxed{A_{21}}-$	$:= -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-$	HS^2H^2	
$-\boxed{A_{22}}-$	$:= -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-$	HSH^2	

2. $-\boxed{S}-\boxed{A_{ab}}-$

(1) $-\boxed{S}-\boxed{A_{01}}- \;=\; -\boxed{A_{01}}-\boxed{S}-$ trivial

(2) $-\boxed{S}-\boxed{A_{02}}- \;=\; -\boxed{A_{02}}-\boxed{S}-\boxed{Z^2}-$

(3) $-\boxed{S}-\boxed{A_{10}}- \;=\; -\boxed{A_{11}}-$

(4) $-\boxed{S}-\boxed{A_{11}}- \;=\; -\boxed{A_{12}}-$

(5) $-\boxed{S}-\boxed{A_{12}}- \;=\; -\boxed{A_{10}}-$

(6) $-\boxed{S}-\boxed{A_{20}}- \;=\; -\boxed{A_{22}}-\boxed{X}-$

(7) $-\boxed{S}-\boxed{A_{21}}- \;=\; -\boxed{A_{20}}-\boxed{X}-$

(8) $-\boxed{S}-\boxed{A_{22}}- \;=\; -\boxed{A_{21}}-\boxed{X}-$

$$-\boxed{A_{01}}- \;:=\; -\!-\!-$$

$$-\boxed{A_{10}}- \;:=\; -\boxed{H}-$$

$$-\boxed{A_{11}}- \;:=\; -\boxed{S}-\boxed{H}-$$

$$-\boxed{A_{12}}- \;:=\; -\boxed{S}-\boxed{S}-\boxed{H}-$$

$$\left.\begin{matrix}A_{10}\\A_{11}\\A_{12}\end{matrix}\right\} \quad -\boxed{A_{1b}}- \;:=\; -\boxed{S^b}-\boxed{H}- \qquad b \in \mathbb{Z}_3$$

C_3 : $S^3 = I$

Lem 3 By definition and C_3, we have

2.(1) $-\boxed{S}-\boxed{A_{01}}- \;=\; -\boxed{A_{01}}-\boxed{S}-$

(3) $-\boxed{S}-\boxed{A_{10}}- \;=\; -\boxed{A_{11}}-$

(4) $-\boxed{S}-\boxed{A_{11}}- \;=\; -\boxed{A_{12}}-$

(5) $-\boxed{S}-\boxed{A_{12}}- \;=\; -\boxed{A_{10}}-$

Proof: 2.(1). LHS $:= -\boxed{S}- \;=:\; -\boxed{A_{01}}-\boxed{S}- \;=$ 2.(1). RHS

2.(3)/(4)/(5). LHS $= -\boxed{S}-\boxed{A_{1b}}- \;:=\; -\boxed{S}-\boxed{S^b}-\boxed{H}-$

$$= -\boxed{S^{b+1}}-\boxed{H}-$$

$$=: -\boxed{A_{1,b+1}}- \;=$$ 2.(3)/(4)/(5). RHS

Note that 2.(5). LHS $:= -\boxed{-\boxed{S}-\boxed{S}-\boxed{S}-}-\boxed{H}- \;\overset{C_3}{=}\; -\boxed{H}- \;=:\; -\boxed{A_{10}}- \;=$ 2.(5). RHS.

$\blacksquare$

$$A_{02} := H\,H$$

$$A_{20} := H\,H\,H$$

$$A_{21} := H\,H\,S\,S\,H$$

$$A_{22} := H\,H\,S\,H$$

$$\left.\begin{matrix} A_{02} \\ A_{20} \\ A_{21} \\ A_{22} \end{matrix}\right\} \quad A_{2b} := H^2\,S^{2b}\,H$$

$b \in \mathbb{Z}_3$

C3 : $S^3 = I$

R2 : $S\,H\,H = H\,H\,S\,Z^2$

R3 : $SZ = ZS$

R4 : $HZ^2H^\dagger = X \qquad Z^2\,H = H\,X$

Lem 4 By definition, R2, R3 & R4, we have

2.(2) $S\,A_{02} = A_{02}\,S\,Z^2$

(6) $S\,A_{20} = A_{22}\,X$

(7) $S\,A_{21} = A_{20}\,X$

(8) $S\,A_{22} = A_{21}\,X$

Proof: 2.(2). LHS $:= S\,H\,H \overset{R2}{=} \boxed{H\,H}\,S\,Z^2$

$=: A_{02}\,S\,Z^2 =$ 2.(2). RHS.

2.(6)/(7)/(8). LHS $= S\,A_{2b} := \boxed{S\,H^2}\,S^{2b}\,H \overset{R2}{=} H\,H\,S\,\boxed{Z^2\,S^{2b}}\,H$

$\overset{R3}{=} H\,H\,S\,S^{2b}\,Z^2\,H = H\,H\,S^{2b+1}\,Z^2\,H$

$= H\,H\,S^{2(b+2)}\,\boxed{Z^2\,H} \overset{R4}{=} \overbrace{\boxed{H\,H\,S^{2(b+2)}\,H}}^{A_{2,b+2}}\,X$

$=: A_{2,b+2}\,X =$ 2.(6)/(7)/(8).RHS

Note that 2.(7).LHS $:= \boxed{S\,H^2}\,S^2\,H \overset{R2}{=} H\,H\,S\,\boxed{Z^2\,S^2}\,H$

$\overset{R3}{=} H\,H\,\boxed{S\,S^2}\,Z^2\,H \overset{C3}{=} H\,H\,Z^2\,H$

$\overset{R4}{=} H\,H\,H\,X =: A_{20}\,X =$ 2.(7). RHS.

$-\boxed{C_0}- = -$

$-\boxed{C_1}- = -\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-$

$-\boxed{C_2}- = -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-$

C1: $w^3 = 1$

R5 : $-\boxed{X}- = -\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- =: -\boxed{C_1}-$

R6 : $-\boxed{S}-\boxed{X}- = -\boxed{X}-\boxed{S}-\boxed{Z}- \cdot w^2$

R7 : $-\boxed{X^2}- = -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}- =: -\boxed{C_2}-$

R8 : $XZ = w^2 ZX$ $\quad -\boxed{Z}-\boxed{X}- = -\boxed{X}-\boxed{Z}- \cdot w^2$

Lem5 By definition, C1, R5, R6, R7 & R8, we have 15. (1) $-\boxed{S}-\boxed{C_0}- = -\boxed{C_0}-\boxed{S}-$

(2) $-\boxed{S}-\boxed{C_1}- = -\boxed{C_1}-\boxed{S}-\boxed{Z}- \cdot w^2$

(3) $-\boxed{S}-\boxed{C_2}- = -\boxed{C_2}-\boxed{S}-\boxed{Z}-\boxed{Z}-$

Proof: 15.(1). LHS $:= -\boxed{S}- =:$ 15.(1). RHS

15.(2). LHS $:= -\boxed{S}-\boxed{H-S-H-H-S-S-H}- \overset{R5}{=} -\boxed{S}-\boxed{X}- \overset{R6}{=}$

$-\boxed{X}-\boxed{S}-\boxed{Z}- \cdot w^2 \overset{R5}{=:} -\boxed{C_1}-\boxed{S}-\boxed{Z}- \cdot w^2 =$ 15.(2). RHS

15.(3). LHS $:= -\boxed{S}-\boxed{H-S-S-H-H-S-H}- \overset{R7}{=} -\boxed{S-X}-\boxed{X}-$

$\overset{R6}{=} -\boxed{X}-\boxed{S}-\boxed{Z-X}- \cdot w^2 \overset{R8}{=} -\boxed{X}-\boxed{S}-\boxed{X}-\boxed{Z}- \cdot w^2 \cdot w^2$

$\overset{C1}{=} -\boxed{X}-\boxed{S-X}-\boxed{Z}- \cdot w \overset{R6}{=} -\boxed{X}-\boxed{X}-\boxed{S}-\boxed{Z}-\boxed{Z}- \cdot w \cdot w^2$

$\overset{C1}{=} -\boxed{X}-\boxed{X}-\boxed{S}-\boxed{Z}-\boxed{Z}- \overset{R7}{=:} -\boxed{C_2}-\boxed{S}-\boxed{Z}-\boxed{Z}-$

= 15.(3). RHS

$-\boxed{C_0}- \;=\; ——$

$-\boxed{C_1}- \;=\; -\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-$

$-\boxed{C_2}- \;=\; -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-$

C1: $\omega^3 = 1$

R5 : $-\boxed{X}- \;=\; -\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \;=:\; -\boxed{C_1}-$

R7: $-\boxed{X^2}- \;=\; -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}- \;=:\; -\boxed{C_2}-$

R8 : $XZ = \omega^2 ZX \qquad -\boxed{Z}-\boxed{X}- \;=\; -\boxed{X}-\boxed{Z}- \cdot \omega^2$

Lem 6 By definition, R5, R7 & R8 , we have 16. $-\boxed{Z}-\boxed{C_c}-$

(1) $-\boxed{Z}-\boxed{C_0}- \;=\; -\boxed{C_0}-\boxed{Z}-$

(2) $-\boxed{Z}-\boxed{C_1}- \;=\; -\boxed{C_1}-\boxed{Z}- \cdot \omega^2$

(3) $-\boxed{Z}-\boxed{C_2}- \;=\; -\boxed{C_2}-\boxed{Z}- \cdot \omega$

Proof : 16.(1). LHS $:= -\boxed{Z}- =:$ 16.(1). RHS

16.(2). LHS $:= -\boxed{Z}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \overset{R5}{=} -\boxed{Z}-\boxed{X}-$

$\overset{R8}{=} -\boxed{X}-\boxed{Z}- \cdot \omega^2 \overset{R5}{=:} -\boxed{C_1}-\boxed{Z}- \cdot \omega^2 =$ 16.(2). RHS

16.(3). LHS $:= -\boxed{Z}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}- \overset{R7}{=} -\boxed{\boxed{Z}-\boxed{X}}-\boxed{X}-$

$\overset{R8}{=} -\boxed{X}-\boxed{\boxed{Z}-\boxed{X}}- \cdot \omega^2 \overset{R8}{=} -\boxed{X}-\boxed{X}-\boxed{Z}- \cdot \omega^2 \cdot \omega^2$

$\overset{C_1}{=} -\boxed{X}-\boxed{X}-\boxed{Z}- \cdot \omega \overset{R7}{=:} -\boxed{C_2}-\boxed{Z}- \cdot \omega =$ 16.(3). RHS

$\square$

$$-\boxed{C_0}- \;=\; -$$

$$-\boxed{C_1}- \;=\; -\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-$$

$$\boxed{C_2}- \;=\; -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-$$

R5 : $-\boxed{X}- \;=\; -\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \;=:\; -\boxed{C_1}-$

R7: $-\boxed{X^2}- \;=\; -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}- \;=:\; -\boxed{C_2}-$

R9: $X^3 = I$

Lem 7 By definition, R5, R7 & R9, we have 17. $-\boxed{X}-\boxed{C_c}-$

(1) $-\boxed{X}-\boxed{C_0}- \;=\; -\boxed{C_1}-$

(2) $-\boxed{X}-\boxed{C_1}- \;=\; -\boxed{C_2}-$

(3) $-\boxed{X}-\boxed{C_2}- \;=\; -\boxed{C_0}-$

Proof: 17.(1). LHS $:= -\boxed{X}- \overset{R5}{=:} -\boxed{C_1}- =$ 17.(1). RHS

17.(2). LHS $:= -\boxed{X}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-$

$\overset{R5}{=} -\boxed{X}-\boxed{X}- \overset{R7}{=:} -\boxed{C_2}- =$ 17.(2). RHS

17.(3). LHS $:= -\boxed{X}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-$

$\overset{R7}{=} -\boxed{X}-\boxed{X}-\boxed{X}- \overset{R9}{=} - \;=:\; -\boxed{C_0}- =$ 17.(3). RHS

$\square$

$$\left.\begin{array}{l} -\boxed{E_0}- \;=\; ——— \\ -\boxed{E_1}- \;=\; -\boxed{S}- \\ -\boxed{E_2}- \;=\; -\boxed{S}-\boxed{S}- \end{array}\right\} \quad -\boxed{E_h}- \;=\; -\boxed{S^h}- \quad h \in \mathbb{Z}_3$$

C_3 : $S^3 = I$

R_3 : $SZ = ZS$

Lem 8 By definition & C_3, we have 24 (1) $-\boxed{S}-\boxed{E_0}- \;=\; -\boxed{E_1}-$

(2) $-\boxed{S}-\boxed{E_1}- \;=\; -\boxed{E_2}-$

(3) $-\boxed{S}-\boxed{E_2}- \;=\; -\boxed{E_0}-$

Proof : 24. (1)/(2)/(3). LHS $= -\boxed{S}-\boxed{E_h}- \;:=\; -\boxed{S}-\boxed{S^h}-$

$= -\boxed{S^{h+1}}- \;=:\; -\boxed{E_{h+1}}- \;=$ RHS.

Note that 24. (3). LHS $:= -\boxed{S}-\boxed{S}-\boxed{S}- \overset{C_3}{=} ——— =: -\boxed{E_0}- =$ 24. (3). RHS.

$\square$

Lem 9 By definition & R_3, we have 25. (1) $-\boxed{Z}-\boxed{E_0}- \;=\; -\boxed{E_0}-\boxed{Z}-$

(2) $-\boxed{Z}-\boxed{E_1}- \;=\; -\boxed{E_1}-\boxed{Z}-$

(3) $-\boxed{Z}-\boxed{E_2}- \;=\; -\boxed{E_2}-\boxed{Z}-$

Proof : 25. (1)/(2)/(3). LHS $= -\boxed{Z}-\boxed{E_h}- \;:=\; -\boxed{Z}-\boxed{S^h}- \overset{R_3}{=} -\boxed{S^h}-\boxed{Z}-$

$=: -\boxed{E_h}-\boxed{Z}- \;=$ RHS.

$\square$

$$-\boxed{F_0}- \;=\; ——$$

$$-\boxed{F_1}- \;=\; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-$$

$$-\boxed{F_2}- \;=\; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-$$

R10 : $-\boxed{Z}- = -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- \;=:\; -\boxed{F_2}-$

R11 : $-\boxed{Z^2}- = -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}- \;=:\; -\boxed{F_1}-$

R12 : $Z^3 = I$

Lem 10 By definition R10, R11 & R12, we have 26. (1) $-\boxed{Z}-\boxed{F_0}- = -\boxed{F_2}-$

(2) $-\boxed{Z}-\boxed{F_1}- = -\boxed{F_0}-$

(3) $-\boxed{Z}-\boxed{F_2}- = -\boxed{F_1}-$

Proof: 26.(1). LHS $:= -\boxed{Z}- \overset{R10}{=} -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-$

$=: -\boxed{F_2}- = $ 26.(1). RHS.

26.(2). LHS $:= -\boxed{Z}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}- \overset{R11}{=}$

$-\boxed{Z}-\boxed{Z}-\boxed{Z}- \overset{R12}{=} I := -\boxed{F_0}- = $ 26.(2). RHS.

26.(3). LHS $:= -\boxed{Z}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- \overset{R10}{=}$

$-\boxed{Z}-\boxed{Z}- \overset{R11}{=:} -\boxed{F_1}- = $ 26.(3). RHS. $\square$