

$$C_2: H^4 = I$$

$$C_3: S^3 = I$$

$$C_5: SH^2SH^2 = H^2SH^2S \quad S' := H^2SH^2$$

$$\text{---} [H] [H] [S] [H] [H] [S] \text{ ---} = \text{---} [S] [H] [H] [S] [H] [H] \text{ ---}$$

$$R_{10}: [Z] = \text{---} [H] [H] [S] [S] [H] [H] [S]$$

$$R_{11}: [Z^2] = \text{---} [H] [H] [S] [H] [H] [S] [S]$$

Lem 11 By C_2, C_3 , & C_5 , R_{10} implies R_{11} .

Proof: By R_{10} , $R_{11} \cdot \text{LHS} =$

$$\text{---} [H] [H] [S] [S] \boxed{[H] [H] [S]} [H] [H] [S] [S] [H] [H] [S] \text{ ---}$$

$$\stackrel{C_5}{=} \text{---} [H] [H] \boxed{[S] [S] [S]} [H] [H] [S] [H] [H] [S] [H] [H] [S]$$

$$\stackrel{C_3}{=} \boxed{[H] [H] [H] [H]} [S] [H] [H] [S] [H] [H] [S]$$

$$\stackrel{C_2}{=} \boxed{[S] [H] [H] [S] [H] [H]} [S]$$

$$\stackrel{C_5}{=} \text{---} [H] [H] [S] [H] [H] [S] [S]$$

$\text{---} : R_{11} \cdot \text{RHS}$.

$$C_2: H^4 = I$$

$$C_3: S^3 = I$$

$$C_5: SH^2SH^2 = H^2SH^2S \quad S' := H^2SH^2$$

$$\text{---} [H] \text{---} [H] \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S] \text{---} = \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [H] \text{---} [H] \text{---}$$

$$R_{10}: \text{---} [\bar{z}] \text{---} = \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S] \text{---}$$

$$R_{11}: \text{---} [\bar{z}^2] \text{---} = \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [S] \text{---}$$

$$R_{12}: z^3 = I$$

Lem 12 By C_2, C_3 , & C_5 , R_{10} and R_{11} imply R_{12} .

Proof: By R_{10} and R_{11} , $R_{12} \cdot LHS = \text{---} [\bar{z}] \text{---} [\bar{z}^2] \text{---} :=$

$$\text{---} [H] \text{---} [H] \text{---} [S] \text{---} [S] \text{---} \boxed{\text{---} [H] \text{---} [H] \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S] \text{---}} \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [S] \text{---}$$

$$\stackrel{C_5}{=} \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [S] \text{---} [S] \text{---} \boxed{\text{---} [H] \text{---} [H] \text{---} [H] \text{---} [H] \text{---}} \text{---} [S] \text{---} [S] \text{---}$$

$$\stackrel{C_2}{=} \text{---} [H] \text{---} [H] \text{---} \boxed{\text{---} [S] \text{---} [S] \text{---} [S] \text{---}} \text{---} [H] \text{---} [H] \text{---} \boxed{\text{---} [S] \text{---} [S] \text{---} [S] \text{---}}$$

$$\stackrel{C_3}{=} \text{---} [H] \text{---} [H] \text{---} [H] \text{---} [H] \text{---}$$

$$\stackrel{C_2}{=} I = R_{12} \cdot LHS$$

$$C_2: H^4 = I$$

$$C_3: S^3 = I$$

$$C_5: SH^2SH^2 = H^2SH^2S \quad S' := H^2SH^2$$

$$\text{---} [H] [H] [S] [H] [H] [S] \text{ ---} = \text{---} [S] [H] [H] [S] [H] [H] \text{ ---}$$

$$R_5: [x] = \text{---} [H] [S] [H] [H] [S] [S] [H] \text{ ---}$$

$$R_7: [x^2] = \text{---} [H] [S] [S] [H] [H] [S] [H] \text{ ---}$$

Lem 13 By C_2, C_3 , & C_5 , R_5 implies R_7 .

Proof: By R_5 , R_7 . LHS =

$$\text{---} [H] [S] [H] [H] [S] [S] \boxed{[H] [H] [S] [H] [H] [S]} \text{ ---} [S] [H]$$

$$\stackrel{C_5}{=} \text{---} [H] [S] [H] [H] \boxed{[S] [S] [S]} [H] [H] [S] [H] [H] [S] [H] \text{ ---}$$

$$\stackrel{C_3}{=} \text{---} [H] [S] \boxed{[H] [H] [H] [H]} [S] [H] [H] [S] [H] \text{ ---}$$

$$\stackrel{C_2}{=} \text{---} [H] [S] [S] [H] [H] [S] [H] \text{ ---} =: R_7 \cdot \text{RHS}$$

□

$$C_2: H^4 = I$$

$$C_3: S^3 = I$$

$$C_5: SH^2SH^2 = H^2SH^2S \quad S' := H^2SH^2$$

$$\text{---} [H] [H] [S] [H] [H] [S] \text{ ---} = \text{---} [S] [H] [H] [S] [H] [H] \text{ ---}$$

$$R_5: [X] = [H] [S] [H] [H] [S] [S] [H]$$

$$R_7: [X^2] = [H] [S] [S] [H] [H] [S] [H]$$

$$R_9: X^3 = I$$

Lem 14 By C_2, C_3 , & C_5 , R_5 and R_7 imply R_9 .

Proof: By R_5 and R_7 , $R_9 \cdot \text{LHS} = [X^2] [X] =$

$$\text{---} [H] [S] [S] [H] [H] \boxed{[S] [H]} \text{ ---} [H] [S] [H] [H] \boxed{[S] [S] [H]} \text{ ---}$$

$$\stackrel{C_5}{=} \text{---} [H] [S] [S] \boxed{[H] [H] [H] [H]} \text{ ---} [S] [H] [H] \boxed{[S] [S] [S]} [H]$$

$$\stackrel{C_2}{=} \text{---} [H] \boxed{[S] [S] [S]} [H] [H] [H]$$

$$\stackrel{C_3}{=} \text{---} [H] [H] [H] [H]$$

$$\stackrel{C_2}{=} I = R_9 \cdot \text{LHS}$$

□

$$C_2: H^4 = I$$

$$C_5: SH^2SH^2 = H^2SH^2S \quad S' := H^2SH^2$$

$$\overbrace{H-H-S-H-H-S} = \overbrace{S-H-H-S-H-H}$$

$$C_5': S-H-H-S = H-H-S-H-H-S-H-H$$

$$R_3: ZS = SZ \quad \overbrace{S-Z} = \overbrace{Z-S}$$

$$R_{10}: \overbrace{Z} = \overbrace{H-H-S-S-H-H-S}$$

Lem 15 By C_2 & C_5 , R_{10} imply R_3 .

Proof: R_3 . LHS := $\overbrace{S-Z}$

$$\stackrel{R_{10}}{=} \overbrace{S-H-H-S} \overbrace{S-H-H-S}$$

$$\stackrel{C_5'}{=} \overbrace{H-H-S} \overbrace{H-H-S-H-H-S} \overbrace{H-H-S}$$

$$\stackrel{C_5}{=} \overbrace{H-H-S-S-H-H-S} \overbrace{H-H-H-H-S}$$

$$\stackrel{C_2}{=} \overbrace{H-H-S-S-H-H-S} \overbrace{S} \stackrel{R_{10}}{=} \overbrace{Z-S}$$

$$C_2: H^4 = I$$

$$C_3: S^3 = I$$

$$C_5: SH^2SH^2 = H^2SH^2S \quad S' := H^2SH^2$$

$$\text{---} [H] \text{---} [H] \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S] \text{---} = \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [H] \text{---} [H] \text{---}$$

$$C'_5: \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S] \text{---} = \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [H] \text{---} [H]$$

$$R_4: Hz^2H^+ = X \quad \text{---} [z^2] \text{---} [H] \text{---} = \text{---} [H] \text{---} [X] \text{---}$$

$$R_5: [X] = \text{---} [H] \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [S] \text{---} [H]$$

$$R_{10}: [z] = \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S]$$

$$\downarrow \\ R_{11}: [z^2] = \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [S]$$

Lem 16 By C_2, C_3 , & C_5 , R_5 and R_{10} imply R_4 .

Proof: $R_4.$ LHS := $\text{---} [z^2] \text{---} [H] \text{---}$

$$\stackrel{R_{11}}{=} \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [S] \text{---} [H]$$

$$\stackrel{C_5}{=} \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [H]$$

$$\stackrel{C_2}{=} \text{---} [H] \text{---} [H] \text{---} \boxed{\text{---} [H] \text{---} [H]} \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [H]$$

$$\stackrel{C'_5}{=} \text{---} [H] \text{---} [H] \text{---} [S] \text{---} \underline{\text{---} [H] \text{---} [H]} \text{---} [S] \text{---} [S] \text{---} [H]$$

$$\stackrel{R_5}{=} \text{---} [H] \text{---} [X] \text{---} = R_4.\text{RHS}$$

$$C_0: (-1)^2 = 1$$

$$C_1: w^3 = 1$$

$$C_2: H^4 = I$$

$$C_3: S^3 = I$$

$$C_4: (HS^2)^3 = -wI$$

$$\text{---} \boxed{S} \text{---} \boxed{S} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{S} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{S} \text{---} \boxed{H} \text{---} = -wI$$

$$C_4': \text{---} \boxed{S} \text{---} \boxed{S} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{S} = \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \cdot (-w)$$

$$C_5: SH^2SH^2 = H^2SH^2S \quad S' := H^2SH^2$$

$$\text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{S} = \text{---} \boxed{S} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{H} \text{---} \boxed{H}$$

$$C_5': \text{---} \boxed{S} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{S} = \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{H} \text{---} \boxed{H}$$

$$R1: \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{H} = \text{---} \boxed{S} \text{---} \boxed{S} \text{---} \boxed{H} \text{---} \boxed{S^2} \text{---} \boxed{x^2} \cdot (-w^2)$$

$$R5: \text{---} \boxed{x} = \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{S} \text{---} \boxed{H}$$

$$R7: \text{---} \boxed{x^2} = \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{S} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{H}$$

Lem 17 By C_0, C_1, C_2, C_3 & C_4 , R_5 implies R_1 .

Proof: By Lem 13, R_5 implies R_7 . Then $R1.$ RHS

$$\stackrel{R7}{=} \text{---} \boxed{S} \text{---} \boxed{S} \text{---} \boxed{H} \text{---} \boxed{S^2} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{S} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{H} \cdot (-w^2)$$

$$\stackrel{C_4'}{=} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{S} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{H} \cdot (-w^2)(-w)$$

$$\stackrel{C_1}{=} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{S} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{H}$$

$$\stackrel{C_2}{=} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{S} \text{---} \boxed{S} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{H}$$

$$\stackrel{C_3}{=} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{H}$$

$$\stackrel{C_2}{=} \text{---} \boxed{H} \text{---} \boxed{S} \text{---} \boxed{H} = R1. LHS$$

□

$$C_2: H^4 = I$$

$$C_3: S^3 = I$$

$$C_5: SH^2SH^2 = H^2SH^2S \quad S' := H^2SH^2$$

$$\text{---} [H] \text{---} [H] \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S] \text{---} = \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [H] \text{---} [H] \text{---}$$

$$R_2: [S] \text{---} [H] \text{---} [H] = [H] \text{---} [H] \text{---} [S] \text{---} [Z^2]$$

$$R_{10}: [Z] = [H] \text{---} [H] \text{---} [S] \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S]$$

$$R_{11}: [Z^2] = [H] \text{---} [H] \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [S]$$

Lem 18 By C_2, C_3 & C_5 , R_{10} implies R_2 .

Proof: By Lem 11, R_{10} implies R_{11} . Then R_2 . RHS

$$\stackrel{R_{11}}{=} \boxed{[H] \text{---} [H] \text{---} [S] \text{---} [H] \text{---} [H] \text{---} [S]} \text{---} [H] \text{---} [H] \text{---} [S] \text{---} [S]$$

$$\stackrel{C_5}{=} [S] \text{---} [H] \text{---} [H] \text{---} [S] \text{---} \boxed{[H] \text{---} [H] \text{---} [H] \text{---} [H]} \text{---} [S] \text{---} [S]$$

$$\stackrel{C_2}{=} [S] \text{---} [H] \text{---} [H] \text{---} \boxed{[S] \text{---} [S] \text{---} [S]}$$

$$\stackrel{C_3}{=} [S] \text{---} [H] \text{---} [H] = R_2 \text{. LHS.}$$

□

$$[S'] = [H] - [H] - [S] - [H] - [H]$$

$$S' := H^2 S H^2$$

$$[S'] - [S'] = [H] - [H] - [S] - [S] - [H] - [H]$$

$$S'^2 = (H^2 S H^2)(H^2 S H^2) = H^2 S^2 H^2$$

$$C_5: SH^2 SH^2 = H^2 S H^2 S \quad S' := H^2 S H^2 \quad SS' = S'S$$

$$[H] - [H] - [S] - [H] - [H] - [S] = [S] - [H] - [H] - [S] - [H] - [H]$$

$$R_4 : HZ^2 H^+ = X \quad Z^2 - [H] = [H] - X$$

$$R_4 : H^+ Z H = X \quad [H] - Z - [H^+] = X \equiv X - [H] = [H] - Z$$

$$R_4^2 : H^+ Z^2 H = X^2 \quad [H] - Z - Z - [H^+] = X^2$$

$$R_5 : X = [H] - [S] - [H] - [H] - [S] - [S] - [H]$$

$$R_{10} : Z = [H] - [H] - [S] - [S] - [H] - [H] - [S] = S' - S' - S$$

Lem 19 By C_5 , R_5 and R_{10} imply R_4^1 .

Proof: R_4^1 . LHS := $X - [H]$

$$\stackrel{R_5}{=} [H] - [S] - \boxed{[H] - [H] - [S] - [S] - [H] - [H]}$$

$$\stackrel{\text{def}}{=} [H] - [S] - S' - S'$$

$$\stackrel{C_5}{=} [H] - \boxed{S' - S' - S}$$

$$\stackrel{\text{def}}{=} [H] - Z = R_4^1 \cdot \text{RHS}$$

IV

Cor 1 By C_5 , R_5 and R_{10} imply R_4^2 .

$$\text{Proof: } R_4^2 \cdot \text{RHS} = X^2 = [H] - Z - \boxed{H^+ - H} - Z - H^+$$

$$= [H] - Z - Z - H^+ = R_4^2 \cdot \text{LHS.}$$

V

$C_0: (-1)^2 = 1$

$C_1: w^3 = 1$

$C_2: H^4 = I$

$C_3: S^3 = I$

$C_4: (HS^2)^3 = -wI \quad \text{--- } [S][S][H][S][S][H][S][H][S][H] = -wI$

$C_4': [S][S][H][S][S] = [H][H][H][S][H][H][H] \cdot (-w)$

$C_4^2: [S][H^+][S] = [H][S^2][H] \cdot (-w^2)$

$C_5: SH^2SH^2 = H^2SH^2S \quad SS' = S'S$

$\text{--- } [H][H][S][H][H][S] = [S][H][H][S][H][H]$

$\text{Def: } [S'] = [H][H][S][H][H] \quad [S'][S'] = [H][H][S][S][H][H]$

$S' := H^2SH^2$

$S'^2 = (H^2SH^2)(H^2SH^2) = H^2S^2H^2$

$C_4^3: [S'][H^+][S'] = [H][S'][S'][H] \cdot (-w^2)$

$R8: XZ = w^2 Z X \quad [Z][X] = [X][Z] \cdot w^2 \quad R9: X^3 = I \quad R12: Z^3 = I$

$R4: [H][Z][H^+] = [X] \quad R4^2: [H][Z][Z][H^+] = [X^2]$

$R10: [Z] = [H][H][S][S][H][H][S] = [S'][S][S]$

$R11: [Z^2] = [H][H][S][H][H][S][S] = [S'][S][S]$

$R5: [X] = [H][S][H][H][S][S][H]$

$R7: [X^2] = [H][S][S][H][H][S][H]$

Lem 20 By C_0, C_1, C_2, C_3, C_4 & C_5 , R_5 and R_{10} imply R_8 .

Proof: To show R_8 , by R_9 and R_{12} , it is sufficient to show

$R_8: [X][Z][X^2][Z^2] \cdot w^2 = I$

$R_8 \text{ LHS} \xrightarrow[R_4^2]{R_4} [H][Z][H^+][Z][H][Z][\boxed{H^+}[H][Z][H^+][Z^2] \cdot w^2$
 $= [H][\boxed{Z}[H^+][\boxed{Z}[H][\boxed{Z}[Z][H^+][\boxed{Z^2}] \cdot w^2$

$\stackrel{\text{def}}{=} [H][S'][S'][S][H^+][\boxed{S'}[S'][S]] [H][S'][S][S][H^+][S'][S][S] \cdot w^2$

$\stackrel{C_5}{=} [H][S'][S'][\boxed{S[H^+][S]}] [S'][S][H][S'][S][S][H^+][S'][S][S] \cdot w^2$

$\stackrel{C_4^2}{=} [H][S'][S'][H][S^2][H][S'][S'][H][\boxed{S'[S][S]}][H^+][S'][S][S] \cdot (-w)$

$\stackrel{C_5}{=} [H][S'][S'][H][S^2][H][S'][S'][H][S][S][\boxed{S'[H^+][S']}][S][S] \cdot (-w)$

$\stackrel{C_4^3}{=} [H][S'][S'][H][S^2][H][S'][S'][H][S][S][H][S'][S'][H][S^2]$

$$C_4: \quad \boxed{S} - \boxed{S} - \boxed{H} - \boxed{S} - \boxed{S} = \boxed{H} - \boxed{H} - \boxed{H} - \boxed{S} - \boxed{H} - \boxed{H} - \boxed{H} \cdot (-w) \quad (S^2 H)^3 = -w I$$

$$\text{C4: } \boxed{S} - \boxed{H^+} - \boxed{S} = \boxed{H} - \boxed{S^2} - \boxed{H} \cdot (-w^2)$$

$$C_4 : \text{---} [S] \text{---} [S] \text{---} [H] \text{---} [S] \text{---} [S] \text{---} [H] \text{---} [S] \text{---} [S] \text{---} [H] \text{---} = -wI$$

$$\text{C}_4^2 : \text{---} [S] \text{---} [\text{H}^+] \text{---} [S] \text{---} \cdot (-w) = \text{---} [S] \text{---} [\text{H}^+] \text{---} [S] \text{---}$$

$$C_2: H^4 = I$$

Def: $S' S' = H H S S H H$

$$C5: ss' = s's$$

$$S'^2 = (H^2 S H^2)(H^2 S H^2) = H^2 S^2 H^2$$

Hence, it is sufficient to show $(HS^2 HS^2)^3 = I$, compositions in diagrammatic order.

The sequence is: $H - S' - S - H - S' - S' - H - S - S - H - S' - S' - H - S^2$. A double underline is placed under the last two groups of symbols (S' , H , S^2), and the label "WTS" is written above the sequence.

Conjugating both sides of (1) by H yields: (1). LHS = $H^+(HS^{12}HS^2 HS^{12}HS^2 HS^{12}HS^2)$

$$= H^+ H S^{12} H S^2 H H^+ H S^{12} H S^2 H \boxed{H^+ H S^{12} H S^2 H}$$

$$= (S^{12} \boxed{H S^2 H})^3 = I =: (1). \text{ RHS.}$$

$$\underline{\underline{C_4}} (S^2 S H^+ S)^3 = (-w)^3 I = -I$$

$$\text{Hence } (S^T S H^T S)^3 \stackrel{\text{WTS}}{=} -I \quad (2)$$

Conjugating both sides of (2) by S yields: (2). LHS = $S(S^2SH^+S)^3S^+$

$$= S S'^2 S H^+ \boxed{S S^+} S S'^2 S H^+ \boxed{S S^+} S S'^2 S H^+ \boxed{S S^+}$$

$$= (\boxed{SS'^2}SH^+)^3 \stackrel{\textcolor{red}{C_5}}{=} (S'^2S^2H^+)^3$$

$$\stackrel{\text{def}}{=} \left(H^2 S^2 H^2 S^2 H^+ \right)^3 = -I =: (2). \text{ RHS.}$$

$$\text{Hence } \left(H^2 S^2 H^2 S^2 H^+ \right)^3 \stackrel{\text{WTS}}{=} -I \quad (3)$$

Concatenating both sides of (3) by H^2 yields: (3). LHS = $H^2(H^2S^2H^2S^2H^+)^3H^2$

$$= \boxed{H^2 H^2} S^2 H^2 S^2 H^+ H^2 \boxed{H^2 H^2} S^2 H^2 S^2 H^+ H^2 \boxed{H^2 H^2} S^2 H^2 S^2 H^+ H^2$$

$$= (S^2 H^2 S^2 H)^3 = -I \quad \text{Hence} \quad - (S^2 H^2 S^2 H)^3 \stackrel{\text{WTS}}{=} I \quad (4)$$

$$(4). LHS = -S^2 H^3 [S^2 H S^2 H^3] [S^2 H S^2 H^3 S^2 H]$$

$$C_4 = -S^2 H^2 H^3 S H^3 H^2 H^3 S H^3 H^2 S^2 H \cdot (-\omega) \cdot (-\omega)$$

$$C_2 = -S^2 H S^2 H S^2 H \cdot w^2$$

$$\underline{C_4} = (-w^2)(-uI) = I = \text{RHS.}$$

C₀

$C_0: (-1)^2 = 1$

$C_1: w^3 = 1$

$C_2: H^4 = I$

$C_3: S^3 = I$

$C_4: (HS^2)^3 = -wI \quad \text{Diagram: } S-S-H-S-S-H-S-S-H = -wI$

$C_4': \text{Diagram: } S-S-H-S-S = H-H-H-S-H-H-H \cdot (-w)$

$C_4^2: \text{Diagram: } S-H^+ S = H-S^2 H \cdot (-w^2)$

$C_5: SH^2 SH^2 = H^2 S H^2 S \quad SS' = S'S$

$\text{Diagram: } H-H-S-H-H-S = S-H-H-S-H-H$

$\text{Def: } S' = H-H-S-H-H \quad S'S' = H-H-S-S-H-H$

$S' := H^2 S H^2$

$S'^2 = (H^2 S H^2)(H^2 S H^2) = H^2 S^2 H^2$

$C_4^3: S' H^+ S' = H-S'-S'H \cdot (-w^2)$

$R8: XZ = w^2 ZX \quad \text{Diagram: } Z-X = X-Z \cdot w^2 \quad R9: X^3 = I \quad R12: Z^3 = I$

$R4: H-Z-H^+ = X \quad R4^2: H-Z-Z-H^+ = X^2$

$R10: Z = H-H-S-S-H-H-S = S'S'S$

$R11: Z^2 = H-H-S-H-H-S-S = S'SS$

$R5: X = H-S-H-S-S-H$

$R7: X^2 = H-S-S-H-H-S-H$

Lem 21 By C_0, C_1, C_2, C_3, C_4 & C_5 , R_5 and R_{10} imply $R_8: S-X = X-S-Z \cdot w^2$

Proof: By C_1, C_3, R_9 & R_{12} , it is sufficient to show

$\text{Diagram: } S-X-Z-Z-S-S-X-X \cdot w = I \quad (1) \quad \checkmark$

$(1). \text{LHS} \stackrel{R4'}{=} S-H-Z-H^+ Z-Z-S-S-H-Z-Z-H^+ \cdot w \quad \checkmark$

$\frac{R_{10}}{R_{11}} \text{Diagram: } S-H-S'-S'-S-H^+ S'-S \boxed{S-S-S} H-S'-S-S-H^+ \cdot w \quad \checkmark$

$\frac{C_3}{C_5} \text{Diagram: } S-H-S'-S' \boxed{S-H^+ S} S'-H-S'-S-S-H^+ \cdot w \quad \checkmark$

$\frac{C_4^2}{C_0, C_1} \text{Diagram: } S-H \boxed{S' S'} H-S^2 H \boxed{S' H} S'-H \boxed{S' S'} S-S-H^+ \cdot (-1) \quad \checkmark$

def $\equiv SH(H^2 S^2 H^2) HS^2 H(H^2 SH^2) H(H^2 SH^2) S^2 H^3 \cdot (-1)$, compositions in diagrammatic order.

Hence, $SH(H^2 S^2 H^2) HS^2 H(H^2 SH^2) H(H^2 SH^2) S^2 H^3 \cdot (-1) \stackrel{\text{WTS}}{=} I \quad (2)$

$$C_0: (-1)^2 = 1$$

$$C_1: w^3 = 1$$

$$C_2: H^4 = I$$

$$C_3: S^3 = I$$

$$C_4: (HS^2)^3 = -wI \quad \text{---} \quad \boxed{S} \quad \boxed{S} \quad \boxed{H} \quad \boxed{S} \quad \boxed{S} \quad \boxed{H} \quad \boxed{S} \quad \boxed{S} \quad \boxed{H} \quad = -wI$$

$$C_4': \quad \boxed{S} \quad \boxed{S} \quad \boxed{H} \quad \boxed{S} \quad \boxed{S} \quad = \quad \boxed{H} \quad \boxed{H} \quad \boxed{H} \quad \boxed{S} \quad \boxed{H} \quad \boxed{H} \quad \boxed{H} \quad \cdot (-w)$$

$$C_4^4: \quad \boxed{S} \quad \boxed{S} \quad \boxed{H} \quad \boxed{S} \quad \boxed{S} \quad \boxed{H} \quad \cdot (-w^2) \quad = \quad \boxed{H} \quad \boxed{H} \quad \boxed{H} \quad \boxed{S}$$

$$C_4^2: \quad \boxed{S} \quad \boxed{H^+} \quad \boxed{S} \quad = \quad \boxed{H} \quad \boxed{S^2} \quad \boxed{H} \quad \cdot (-w)$$

$$C_5: SH^2SH^2 = H^2SH^2S \quad SS' = S'S$$

$$\boxed{H} \quad \boxed{H} \quad \boxed{S} \quad \boxed{H} \quad \boxed{H} \quad \boxed{S} \quad = \quad \boxed{S} \quad \boxed{H} \quad \boxed{H} \quad \boxed{S} \quad \boxed{H} \quad \boxed{H}$$

$$\text{Def: } \boxed{S'} = \boxed{H} \quad \boxed{H} \quad \boxed{S} \quad \boxed{H} \quad \boxed{H} \quad \quad \boxed{S'} \quad \boxed{S'} = \boxed{H} \quad \boxed{H} \quad \boxed{S} \quad \boxed{S} \quad \boxed{H} \quad \boxed{H}$$

$S' := H^2SH^2$

$$S'^2 = (H^2SH^2)(H^2SH^2) = H^2S^2H^2$$

$$C_4^3: \quad \boxed{S'} \quad \boxed{H^+} \quad \boxed{S'} \quad = \quad \boxed{H} \quad \boxed{S'} \quad \boxed{S'} \quad \boxed{H} \quad \cdot (-w^2)$$

$$(2). LHS = SH(H^2S^2H^2)HS^2H(H^2SH^2)H(H^2SH^2)S^2H^3 \cdot (-1) \quad \star$$

$$= SH^3S^2H^3S^2H^3SHSH^2S^2H^3 \cdot (-1)$$

$$= S(\underline{H^3S})S(\underline{H^3S})S(\underline{H^3S})HSH^2S^2H^3 \cdot (-1)$$

$$\frac{C_4^4}{C_0} \underbrace{S(S^2HS^2H)}_{C_1} \underbrace{S(S^2HS^2H)}_{C_3} \underbrace{S(S^2HS^2H)}_{\text{Lem G}} HSH^2S^2H^3 \cdot (-1) \cdot (-w^2)^3$$

$$C_1 = HS^2 \underbrace{H}_{H^2} \underbrace{H}_{H^2} \underbrace{H}_{H^2} \underbrace{H}_{H^2} \underbrace{H}_{H^2} \underbrace{H}_{H^2} S^2H^3$$

C_3

$$\stackrel{\text{def}}{=} HS^2S^2S^2S^2S^2S^2H^3$$

$$\stackrel{C_3}{=} H \boxed{S^2S^2S^2} S^2S^2S^2H^3$$

$$\stackrel{\text{Lem G}}{=} H^4$$

$$\stackrel{C_2}{=} I =: (2). \text{RHS.}$$

VI