

$$\text{Def 7: } \begin{array}{c} \times \\ \diagup \quad \diagdown \\ \text{H}^* \end{array} = \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ \text{H}^* \end{array} \quad \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ \text{H}^* \end{array} \quad \dots$$

$$R_{24}^1: \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ \text{S}' \end{array} = \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ \text{S}' \end{array}$$

$$C_1: w^3 = 1$$

$$R_{23}^1: \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{S}' \end{array} = \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ \text{S}' \end{array} \cdot w^2$$

$$R_{24}: \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{S}' \end{array} = \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{S}' \end{array}$$

$$R_{15}: \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{S}' \end{array} = \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{S}' \end{array}$$

$$R_{25}^1: \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ \text{S}' \end{array} = \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ \text{S}' \end{array} \cdot w$$

$$R_{25}^2: \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ \text{S}'^2 \end{array} = \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ \text{S}'^2 \end{array} \cdot w^2 = \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ \text{S}' \end{array}$$

$$C_7: \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{S}' \end{array} = \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{S}' \end{array}$$

$$\text{Def 1: } \begin{array}{c} \text{S}' \\ \oplus \\ \diagup \quad \diagdown \\ \text{H} \end{array} = \begin{array}{c} \text{H} \\ \oplus \\ \diagup \quad \diagdown \\ \text{H} \end{array} \quad \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{H} \end{array} = \begin{array}{c} \text{H} \\ \oplus \\ \diagup \quad \diagdown \\ \text{H} \end{array}$$

$$C_2: H^4 = I \quad C_3: S^3 = I \quad C_5: SH^2SH^2 = H^2SH^2S \quad SS' = S'S$$

Lem S Def 1-3, C1-7, R15, R23, R24 & R25 imply

$$R_{18}: (1) \quad \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{S}' \end{array} = \begin{array}{c} \times \\ \diagup \quad \diagdown \\ \text{S}' \end{array}$$

$$\text{Proof: } R_{18}.(1). \text{LHS} := \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{S}' \end{array} \quad \text{Def 7} \quad \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{H}^* \end{array} \quad \dots$$

$$R_{23}^1 = \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{H}^2 \end{array} \cdot w^2$$

$$\frac{R_{15}, R_{24}^1}{R_{23}^1} \quad \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{H}^2 \end{array} \cdot \begin{array}{c} \text{S}' \\ \oplus \\ \diagup \quad \diagdown \\ \text{S}' \end{array} \cdot w^2 \cdot w^2$$

$$\frac{R_{15}, R_{24}^1}{C_1, R_{24}} \quad \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{H}^2 \end{array} \cdot \begin{array}{c} \text{S}'^2 \\ \oplus \\ \diagup \quad \diagdown \\ \text{S}'^2 \end{array} \cdot w$$

$$\frac{R_{23}^1}{R_{15}, R_{24}^1} \quad \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{H}^2 \end{array} \cdot \begin{array}{c} \text{S}'^2 \\ \oplus \\ \diagup \quad \diagdown \\ \text{S}'^2 \end{array} \cdot w \cdot w^2$$

$$\frac{R_{15}, R_{24}^1}{C_1, R_{23}^1} \quad \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{H}^2 \end{array} \cdot \begin{array}{c} \text{S}'^2 \\ \oplus \\ \diagup \quad \diagdown \\ \text{S}'^2 \end{array} \cdot w^2$$

$$\frac{R_{25}^1}{R_{15}, R_{24}^1} \quad \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{H}^2 \end{array} \cdot \begin{array}{c} \text{S}'^2 \\ \oplus \\ \diagup \quad \diagdown \\ \text{S}'^2 \end{array} \cdot \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{S}'^2 \end{array} \cdot w^2 \cdot w$$

$$\frac{\text{Def 1, } C_1, C_2}{C_7, R_{15}, C_3, C_5} \quad \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{H}^2 \end{array} \cdot \begin{array}{c} \text{S}'^2 \\ \oplus \\ \diagup \quad \diagdown \\ \text{S}'^2 \end{array} \cdot \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{S}' \end{array}$$

$$\frac{R_{25}^1}{R_{15}, R_{24}^1} \quad \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{H}^2 \end{array} \cdot \begin{array}{c} \text{S}'^2 \\ \oplus \\ \diagup \quad \diagdown \\ \text{S}'^2 \end{array} \cdot \begin{array}{c} \text{S} \\ \oplus \\ \diagup \quad \diagdown \\ \text{S}' \end{array} \cdot w$$

$$\text{Def 7: } \begin{array}{c} \times \\ \diagup \quad \diagdown \\ \text{H} \end{array} = \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ \text{H} \end{array} \quad \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ \text{H} \end{array} \quad \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ \text{H} \end{array} \quad \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ \text{H} \end{array}$$

$$R_{24}^1: \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ S \end{array} = \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ S' \end{array}$$

$$C_1: w^3 = 1$$

$$R_{23}^1: \begin{array}{c} S \\ \oplus \\ \diagup \quad \diagdown \end{array} = \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ S \end{array} \cdot w^2 \quad R_{24}: \begin{array}{c} S \\ \oplus \\ \diagup \quad \diagdown \end{array} = \begin{array}{c} S \\ \oplus \\ \diagup \quad \diagdown \end{array}$$

$$R_{15}: \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ S' \end{array} = \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ S' \end{array}$$

$$C_7: (1) \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ S \end{array} = \begin{array}{c} S \\ \bullet \\ \diagup \quad \diagdown \end{array} \quad (2) \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ S \end{array} = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ S \end{array}$$

$$C_2: H^4 = I \quad C_3: S^3 = I \quad C_5: SH^2SH^2 = H^2SH^2S \quad SS' = S'S$$

$$\text{Def 1: } \begin{array}{c} S' \\ \diagup \quad \diagdown \\ H \end{array} = \begin{array}{c} H \\ \diagup \quad \diagdown \\ H \end{array} \quad \begin{array}{c} S \\ \diagup \quad \diagdown \\ H \end{array} = \begin{array}{c} H \\ \diagup \quad \diagdown \\ S \end{array}$$

$$R_{23}^4: \begin{array}{c} S \\ \oplus \\ \diagup \quad \diagdown \end{array} = \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ S' \end{array} \cdot w^2$$

$$R_{25}^3: \begin{array}{c} \bullet \\ \oplus \\ \diagup \quad \diagdown \end{array} = \begin{array}{c} \bullet \\ \oplus \\ \diagup \quad \diagdown \\ S' \end{array} \cdot w$$

$$R_{24}: \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ S \end{array} = \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ S \end{array}$$

$$R_{25}^4: \begin{array}{c} \bullet \\ \oplus \\ \diagup \quad \diagdown \\ S'^2 \end{array} \cdot w^2 = \begin{array}{c} \bullet \\ \oplus \\ \diagup \quad \diagdown \end{array}$$

$$C_6: \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}$$

Lem S

$$R_{18}: (1) \begin{array}{c} S \\ \diagup \quad \diagdown \\ \times \end{array} = \begin{array}{c} \times \\ \diagup \quad \diagdown \\ S \end{array}$$

Proof cont.

$$\begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ H^2 \end{array} \begin{array}{c} S'^2 \\ \oplus \\ \diagup \quad \diagdown \end{array} \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ S' \end{array} \dots \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ S \end{array} \dots \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ S \end{array} \cdot w$$

$$\frac{R_{15}, R_{24}^1}{C_7} \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ H^2 \end{array} \begin{array}{c} S^3 \\ \oplus \\ \diagup \quad \diagdown \end{array} \dots \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ S \end{array} \dots \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ S \end{array} \cdot w$$

$$\frac{\text{Def 1, } C_2}{C_3, R_{23}^4} \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ H^2 \end{array} \dots \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ S' \end{array} \dots \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ S \end{array} \dots \boxed{\cdot w \cdot w^2}$$

$$\frac{C_1, C_7}{R_{24}, R_{25}^4} \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ H^2 \end{array} \dots \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \dots \begin{array}{c} S'^2 \\ \oplus \\ \diagup \quad \diagdown \\ S^2 \end{array} \dots \boxed{\cdot w^2}$$

$$\frac{\text{Def 1, } C_2}{C_3, C_5, C_6} \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ H^2 \end{array} \dots \begin{array}{c} S \\ \oplus \\ \diagup \quad \diagdown \end{array} \dots \begin{array}{c} S \\ \oplus \\ \diagup \quad \diagdown \\ S^2 \end{array} \dots \boxed{\cdot w^2}$$

$$\frac{R_{23}^1}{C_1} \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ H^2 \end{array} \dots \begin{array}{c} S \\ \oplus \\ \diagup \quad \diagdown \end{array} \dots \begin{array}{c} S \\ \oplus \\ \diagup \quad \diagdown \\ S' \end{array} \dots \boxed{\cdot w^2 \cdot w^2}$$

$$\frac{C_1, R_{23}^1}{R_{24}^1, R_{15}} \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ H^2 \end{array} \dots \begin{array}{c} S \\ \oplus \\ \diagup \quad \diagdown \end{array} \dots \begin{array}{c} S \\ \oplus \\ \diagup \quad \diagdown \\ S' \end{array} \dots \begin{array}{c} S' \\ \oplus \\ \diagup \quad \diagdown \\ S^2 \end{array} \dots \boxed{\cdot w \cdot w^2}$$

$$\frac{C_1, C_6}{C_7, R_{15}} \begin{array}{c} \oplus \\ \diagup \quad \diagdown \\ H^2 \end{array} \dots \begin{array}{c} S \\ \oplus \\ \diagup \quad \diagdown \end{array} \dots \begin{array}{c} S^2 \\ \oplus \\ \diagup \quad \diagdown \\ S'^2 \end{array} \dots \begin{array}{c} S^2 \\ \oplus \\ \diagup \quad \diagdown \\ S^2 \end{array}$$

$$\text{Def 7: } \begin{array}{c} \diagup \diagdown \\ \square \end{array} = \begin{array}{c} \oplus \\ | \\ \square \end{array} \quad \begin{array}{c} \oplus \\ | \\ \square \end{array} \quad \begin{array}{c} \bullet \\ | \\ \square \end{array} \quad \begin{array}{c} \oplus \\ | \\ \square \end{array} \quad \begin{array}{c} \oplus \\ | \\ \square \end{array}$$

$$R_{24}^1: \begin{array}{c} \oplus \\ | \\ \square \end{array} = \begin{array}{c} \oplus \\ | \\ \square \end{array}$$

$$C_1: w^3 = 1$$

$$R_{23}^1: \begin{array}{c} \square \\ | \\ \oplus \end{array} = \begin{array}{c} \oplus \\ | \\ \square \end{array} \cdot w^2 \quad R_{24}: \begin{array}{c} \square \\ | \\ \bullet \end{array} = \begin{array}{c} \bullet \\ | \\ \oplus \end{array}$$

$$R_{15}: \begin{array}{c} \bullet \\ | \\ \square \end{array} = \begin{array}{c} \bullet \\ | \\ \square \end{array}$$

$$C_7: (1) \begin{array}{c} \bullet \\ | \\ \square \end{array} = \begin{array}{c} \square \\ | \\ \bullet \end{array} \quad (2) \begin{array}{c} \bullet \\ | \\ \square \end{array} = \begin{array}{c} \bullet \\ | \\ \square \end{array}$$

$$C_2: H^4 = I \quad C_3: S^3 = I \quad C_5: SH^2SH^2 = H^2SH^2S \quad SS' = S'S$$

$$\text{Def 1: } \begin{array}{c} \square \\ | \\ \square \end{array} = \begin{array}{c} H \\ | \\ H \end{array} \quad \begin{array}{c} \square \\ | \\ \square \end{array} = \begin{array}{c} H \\ | \\ S \end{array} \quad \begin{array}{c} \square \\ | \\ \square \end{array} = \begin{array}{c} S \\ | \\ H \end{array} \quad \begin{array}{c} \square \\ | \\ \square \end{array} = \begin{array}{c} H \\ | \\ H \end{array}$$

$$R_{25}^3: \begin{array}{c} \bullet \\ | \\ \oplus \end{array} = \begin{array}{c} \bullet \\ | \\ \square \end{array} \cdot w \quad R_{24}: \begin{array}{c} \oplus \\ | \\ \square \end{array} = \begin{array}{c} \oplus \\ | \\ \bullet \end{array}$$

$$R_{25}^1: \begin{array}{c} \bullet \\ | \\ \oplus \end{array} = \begin{array}{c} \oplus \\ | \\ \square \end{array} \cdot w \quad C_6: \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}$$

Lem S

$$R_{18}: (1) \begin{array}{c} \square \\ | \\ \begin{array}{c} \diagup \diagdown \\ \square \end{array} \end{array} = \begin{array}{c} \diagup \diagdown \\ \square \end{array}$$

Proof cont.

$$\begin{array}{c} \oplus \\ | \\ \square \\ | \\ \oplus \\ | \\ \square \\ | \\ \oplus \\ | \\ \square \\ | \\ \square^2 \\ | \\ \square^2 \end{array}$$

$$R_3^1 = \begin{array}{c} \oplus \\ | \\ \square \\ | \\ \oplus \\ | \\ \square \\ | \\ \oplus \\ | \\ \square \\ | \\ \square^2 \\ | \\ \square^2 \end{array} \cdot w^2$$

$$\frac{C_7, R_5}{R_4^1} \begin{array}{c} \oplus \\ | \\ \square \\ | \\ \oplus \\ | \\ \square \\ | \\ \oplus \\ | \\ \square \\ | \\ \square^3 \\ | \\ \square^2 \end{array} \cdot w^2$$

$$\frac{\text{Def 1}}{C_2, C_3} \begin{array}{c} \oplus \\ | \\ \square \\ | \\ \oplus \\ | \\ \square \\ | \\ \oplus \\ | \\ \square \\ | \\ \square^2 \\ | \\ \square^2 \end{array} \cdot w^2$$

$$\frac{R_3^1}{\text{Def 1}} \begin{array}{c} \oplus \\ | \\ \square \\ | \\ \oplus \\ | \\ \square \\ | \\ \oplus \\ | \\ \square \\ | \\ \square^2 \\ | \\ \square^2 \end{array} \cdot w^2 \cdot w^2$$

$$\frac{C_1}{C_3} \begin{array}{c} \oplus \\ | \\ \square \\ | \\ \oplus \\ | \\ \square \\ | \\ \oplus \\ | \\ \square \\ | \\ \square' \\ | \\ \square \end{array} \cdot w$$

$$\frac{R_5^1}{C_6} \begin{array}{c} \oplus \\ | \\ \square \\ | \\ \oplus \\ | \\ \square \\ | \\ \oplus \\ | \\ \square \\ | \\ \square' \\ | \\ \square \end{array} \quad C_6: \begin{array}{c} \oplus \\ | \\ \square \\ | \\ \oplus \\ | \\ \square \\ | \\ \oplus \\ | \\ \square \\ | \\ \square' \\ | \\ \square \end{array}$$

$$\frac{\text{Def 1}}{\text{Def 1}} \begin{array}{c} \diagup \diagdown \\ \square \end{array} =: R_{18}.(1).RHS.$$

**Def5:** = **R<sub>25</sub>**: = • w

**R<sub>24</sub>:** = **R<sub>24</sub>**: = **C1:**  $w^3 = 1$

**R<sub>23</sub>**: = • w<sup>2</sup> **R<sub>25</sub>**: = • w

**C7:** (1) = (2) = **C5:**  $SH^2SH^2 = H^2SH^2S$   $SS' = S'S$

**R<sub>15</sub>:** (1) = (2) =

Lem T Def 1-3, C1-8, R<sub>15</sub>, R<sub>23</sub>, R<sub>24</sub> & R<sub>25</sub> imply

**R<sub>17</sub>:** =

**Proof:** **R<sub>17</sub>**. LHS := **Def5** **R<sub>25</sub>**: = • w

**R<sub>24</sub>**: = • w

**R<sub>25</sub>**: = • w<sup>2</sup>

**R<sub>23</sub>**: = • w<sup>2</sup> • w<sup>2</sup>

**R<sub>23</sub>**: = • w<sup>2</sup> • w<sup>2</sup> • w<sup>2</sup>

**R<sub>15</sub>**: = **C1**

**R<sub>25</sub>**: = • w

**R<sub>15</sub>**: = • w

$$C_1: w^3 = I \quad C_2: H^4 = I \quad C_3: S^3 = I \quad C_5: SH^2SH^2 = H^2SH^2S \quad SS' = S'S$$

$$C_6: \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \end{array} = \text{---}$$

$$R_{24}: \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \end{array} = \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \end{array}$$

$$\text{Def 1: } [S'] := [H] [H] [S] [H] [H] \quad [S'^2] := [H] [H] [S^2] [H] [H]$$

$$C_8^8: \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ H^2 \end{array} = \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ H^2 \end{array} \quad R_{24}^3: \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ S \end{array} = \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ S \end{array}$$

$$C_8^5: (1) \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ H^2 \end{array} = \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ H^2 \end{array} \quad (2) \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ H^2 \end{array} = \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ H^2 \end{array}$$

$$R_{23}^4: \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ S \end{array} = \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ S' \end{array} \cdot w^2$$

Lem T

$$R_{17}: \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \text{---} \end{array}$$

Proof cont.

$$\begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ S'^2 \end{array} \cdot w$$

$$C_2, C_3: \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ S' \end{array} \cdot w$$

$$C_6, R_{24}: \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ S^2 \end{array} \cdot w$$

$$C_8^8: \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ H^2 \end{array} \cdot w$$

$$C_8^5: \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ S^2 \end{array} \cdot w$$

$$R_{23}^4: \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ H^2 \end{array} \cdot w \cdot w^2$$

$$R_{24}^3: \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ S \end{array}$$

$$C_1: \begin{array}{c} \text{---} \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ \bullet \bullet \bullet \bullet \\ | \\ H^2 \end{array}$$

$$R_{24}^3: \quad \begin{array}{c} \text{---} \\ | \\ S' \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ S' \\ | \\ \text{---} \end{array} \quad R_{15}: \quad (1) \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ S' \\ | \\ \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ S' \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \quad (2) \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ S' \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ S' \\ | \\ \text{---} \\ | \\ \text{---} \end{array}$$

$$C_7: (1) \quad \text{Diagram} = \quad \text{Diagram} \quad (2) \quad \text{Diagram} = \quad \text{Diagram} \quad R_{24}: \quad \text{Diagram} = \quad \text{Diagram}$$

$$C_8^8: \quad \text{Diagram} = \quad \text{Def 1: } [S'] := [H] - [H] - [S] - [H] - [H]$$

$$C1: \omega^3 = I \quad C2: H^4 = I \quad C3: S^3 = I \quad C5: SS' = S'S \quad R_{24}^2: \begin{array}{c} S \\ \square \\ \bullet \\ \hline \end{array} = \begin{array}{c} \bullet \\ \square \\ S \\ \hline \end{array}$$

$$C_8 : (1) \quad \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \quad \boxed{\mathbb{H}^2} \quad = \quad \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \end{array} \quad (2) \quad \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \end{array} \quad \boxed{\mathbb{H}^2} \quad = \quad \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \end{array}$$

$$R_{23}^4: \quad \text{Diagram} = \quad \text{Diagram} \cdot w^2 \quad R_{23}^1: \quad \text{Diagram} = \quad \text{Diagram} \cdot w^2$$

$$R_{25}^3: \quad \text{Diagram} = S' \text{ } S \cdot w$$

## Lem T

$$R_{17}: \quad \text{Diagram} = \text{Diagram}$$

Proof cont.

•  $w^2 \cdot w^2$

$$\frac{C_1, R^3_{24}}{C_7, R_{15}}$$

A quantum circuit diagram consisting of two horizontal lines representing qubits. The top line starts with two control circles, followed by a sequence of operations:  $H^\dagger$  (orange box),  $S$ ,  $H^\dagger$  (blue box),  $S^3$  (green box),  $S$ ,  $H^\dagger$  (blue box), and  $S^2$ . The bottom line has a sequence of operations:  $H^\dagger$  (orange box),  $S$ ,  $H^\dagger$  (blue box), and  $S^2$ . Control circles are placed above the first two operations on each line.

$C_8^8, R_{24}, C_2$

Def 1,  $C_3$

A quantum circuit diagram with three horizontal lines representing qubits. The first qubit has two identity gates (circles). The second qubit has two Hadamard gates ( $H^z$ ). The third qubit starts with a CNOT gate targeting it from the second qubit. It then receives two single-qubit operations: an S gate followed by another  $H^z$  gate. Finally, a three-qubit operation (S,  $H^z$ ,  $S^2$ ) acts on the second, third, and fourth qubits respectively.

$C_8$ ,  $R_{23}^4$

R<sub>1</sub>

$C_8, C_1$

R<sub>5</sub>, G<sub>7</sub>

$C_5, C_7, C_8$

$$\underline{\underline{R_{24}}}, \underline{\underline{R_{25}^3}}$$

A quantum circuit diagram illustrating a sequence of operations. The circuit consists of six horizontal wires. From left to right, the operations are: CNOT (ctrl=1, target=2), CNOT (ctrl=1, target=2), S gate, H gate, S gate, CNOT (ctrl=1, target=2), S gate, H gate, S gate, CNOT (ctrl=1, target=2), S gate, H gate, S gate. The first two CNOT gates are grouped by an orange box, and the last three CNOT gates are grouped by a green box.

C<sub>6</sub>, G<sub>7</sub>, R<sub>34</sub><sup>3</sup>

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$$R_{25}^3: \text{Diagram} = \text{Diagram} \cdot w \quad R_{23}^1: \text{Diagram} = \text{Diagram} \cdot w^2 \quad C_1: w^3 = 1$$

$$C_8^6: (1) \text{Diagram} = \text{Diagram} \quad (2) \text{Diagram} = \text{Diagram}$$

$$\text{Def 1: } [S'] := [H][H][S][H][H] \quad C_5: SS' = S'S \quad C_2: H^4 = I \quad C_3: S^3 = I$$

$$R_{15}: (1) \text{Diagram} = \text{Diagram} \quad (2) \text{Diagram} = \text{Diagram}$$

$$C_7: (1) \text{Diagram} = \text{Diagram} \quad (2) \text{Diagram} = \text{Diagram}$$

$$C_8^1: (1) \text{Diagram} = \text{Diagram} \quad (2) \text{Diagram} = \text{Diagram}$$

Lem T

$$R_{17}: \text{Diagram} = \text{Diagram}$$

$$\text{Proof cont.} \quad \text{Diagram} \xrightarrow{\text{R}_{25}^3, C_5, C_6} \text{Diagram} \cdot w$$

$$\text{Diagram} \xrightarrow{\text{R}_{25}^3, C_5, C_6} \text{Diagram} \cdot w$$

$$\text{Diagram} \xrightarrow{\text{Def 1, } C_2} \text{Diagram} \cdot w$$

$$\text{Diagram} \xrightarrow{\text{Def 1, } C_2} \text{Diagram} \cdot w$$

$$H^2(H^2SH^2)S^2H^2S^2 \stackrel{C_2}{=} SH^2S^2H^2S^2 \stackrel{\text{Def 1}}{=} SS^2S^2 \stackrel{C_5}{=} S^2SS^2 \stackrel{C_3}{=} S^2$$

$$\text{Diagram} \xrightarrow{\text{Def 1, } C_5} \text{Diagram} \cdot w$$

$$\text{Diagram} \xrightarrow{\text{R}_{23}^1} \text{Diagram} \cdot w \cdot w^2$$

$$\text{Diagram} \xrightarrow{\text{R}_{23}^1, C_1} \text{Diagram} \cdot w^2 \cdot w^2$$

**Def 1:**  $S' := H \otimes H \otimes S \otimes H \otimes H$        $R_{25}^2:$        $\cdot w^2 =$

**C1:**  $w^3 = I$       **C2:**  $H^4 = I$       **C3:**  $S^3 = I$       **C5:**  $SS' = S'S$

**C6:**      **C7:** (1) =      (2) =

**R<sub>24</sub><sup>1</sup>:** =      **R<sub>15</sub>:** (1) =      (2) =

**R<sub>24</sub>:** =      **R<sub>23</sub><sup>1</sup>:** =       $\cdot w^2$       **C<sub>8</sub><sup>1</sup>:** =

**Def 5:**

**Lem T**

**R<sub>17</sub>:**

**Proof cont.**

**R<sub>25</sub><sup>2</sup>**      **C<sub>1</sub>**

**C<sub>1</sub>, C<sub>2</sub>, C<sub>3</sub>**      **C<sub>5</sub>, C<sub>6</sub>**

**G<sub>1</sub>, R<sub>24</sub>**      **R<sub>15</sub>, R<sub>24</sub><sup>1</sup>**

**C<sub>2</sub>, C<sub>3</sub>, C<sub>6</sub>**      **Def 1, R<sub>23</sub><sup>1</sup>**

**C<sub>3</sub>, G<sub>7</sub>**      **C<sub>5</sub>, R<sub>15</sub>**

**R<sub>25</sub><sup>2</sup>**

**C<sub>8</sub><sup>1</sup>**

**Def 5**      = **R<sub>17</sub>.RHS**

□