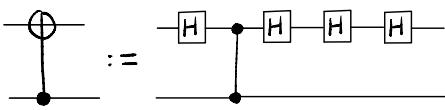
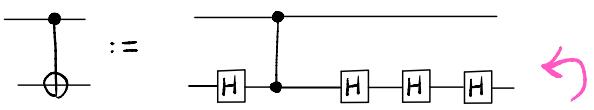
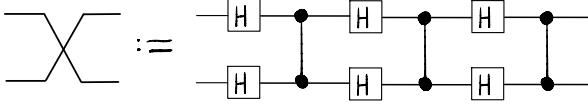


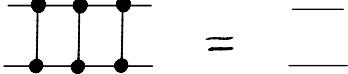
Two - Qutrit Derived Relations : T₃

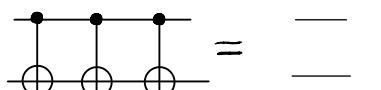
Def 1 : $S' := H \otimes H \otimes S \otimes H \otimes H$ $S'^2 := H \otimes H \otimes S^2 \otimes H \otimes H$

Def 2 :  

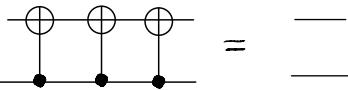
Def 3 : 

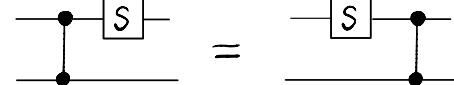
We chose this definition because it shows up in the microscopic picture of many box relations

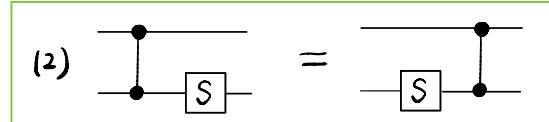
C₆: 

C₆¹:  = 

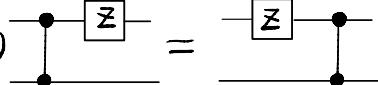
Lem G

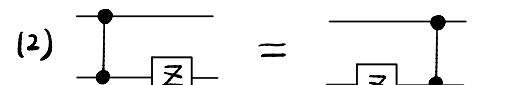
C₆²: 

C₇: (1) 

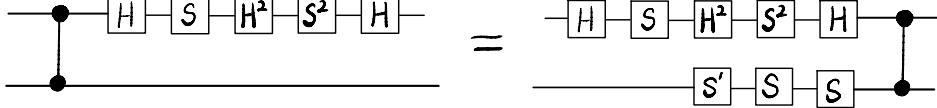
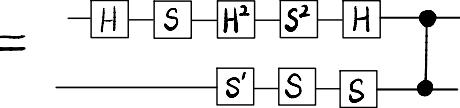
(2) 

Lem C

R₁₃: (1) 

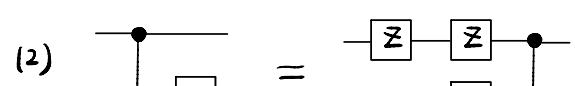
(2) 

Lem F

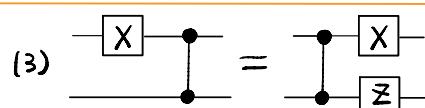
R₁₄:  = 

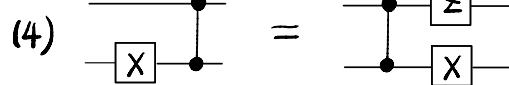
Lem A3

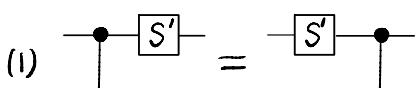
$\equiv R_{14}^1:$ (1) 

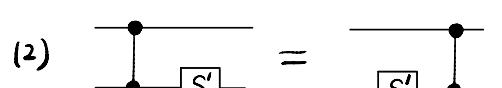
(2) 

Lem H

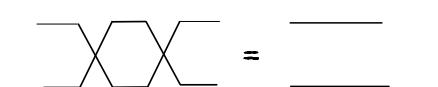
(3) 

(4) 

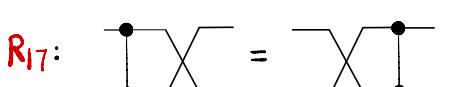
R₁₅: (1) 

(2) 

Lem E

R₁₆: 

Lem W

R₁₇: 

Lem T

$$C_8 : (1) \quad \text{Diagram} = \text{Diagram}$$

$$(2) \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array}$$

H^2

$$C_8^1: (i) \quad \begin{array}{c} \text{---} \\ | \end{array} \boxed{H^2} \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \boxed{H^2} = \bullet \text{---} \bullet \text{---} \bullet \text{---} \boxed{H^2}$$

$$(2) \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \quad \boxed{\mathbb{H}^2}$$

$$C_8^2: \quad \begin{array}{c} \text{---} \\ | \quad | \\ \boxed{H^2} \quad \bullet \quad \bullet \quad \boxed{H^2} \end{array} \quad = \quad \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \text{---} \\ | \quad | \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ | \quad | \quad | \quad | \\ \boxed{H^2} \quad \bullet \quad \bullet \quad \boxed{H^2} \end{array}$$

$$C_8^3: \quad \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{|c|} \hline H^2 \\ \hline \end{array} \quad = \quad \begin{array}{|c|} \hline H^2 \\ \hline \end{array} \quad \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \quad = \quad \begin{array}{|c|} \hline H^2 \\ \hline \end{array} \quad \begin{array}{c} \bullet \\ \bullet \end{array}$$

$$C_8^4: \quad \begin{array}{c} \text{---} \boxed{H^2} \text{---} \bullet \text{---} \boxed{H^2} \text{---} \\ | \qquad \qquad | \\ \text{---} \bullet \text{---} \bullet \text{---} \\ | \qquad \qquad | \\ \text{---} \bullet \text{---} \bullet \text{---} \end{array} = \quad \begin{array}{c} \bullet \text{---} \bullet \text{---} \\ | \qquad | \\ \bullet \text{---} \bullet \text{---} \\ | \qquad | \\ \bullet \text{---} \bullet \text{---} \end{array} = \quad \begin{array}{c} \text{---} \bullet \text{---} \\ | \\ \text{---} \boxed{H^2} \text{---} \bullet \text{---} \boxed{H^2} \text{---} \end{array}$$

Lem D

$$C_8^5: \quad (1) \quad \begin{array}{c} \text{---} \\ \oplus \\ \text{---} \\ | \\ \bullet \\ \text{---} \end{array} \boxed{H^2} = \begin{array}{c} \text{---} \\ \boxed{H^2} \\ \text{---} \\ | \\ \bullet \\ \text{---} \\ | \\ \bullet \end{array}$$

$$(2) \quad \begin{array}{c} \oplus \\ \hline \end{array} \quad = \quad \begin{array}{c} \oplus \\ \hline \end{array} \quad \begin{array}{c} \oplus \\ \hline \end{array}$$

$$C_8^6: (1) \quad \begin{array}{c} \text{---} \\ | \end{array} \boxed{H^2} \text{---} \bigoplus \text{---} = \text{---} \bigoplus \text{---} \bigoplus \text{---} \boxed{H^2} \text{---}$$

$$(2) \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \oplus \text{---} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \oplus \text{---} \quad \boxed{H^2}$$

$$C_8^7: \quad (1) \quad \begin{array}{c} \text{---} \\ | \\ \bullet \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \textcircled{1} \end{array} \quad \boxed{\textcircled{2}} = \quad \begin{array}{c} \text{---} \\ | \\ \bullet \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \textcircled{1} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \textcircled{1} \end{array}$$

$$(2) \quad \begin{array}{c} \bullet \\ \circ \end{array} - \boxed{H^2} = \boxed{H^2} - \begin{array}{c} \bullet \\ \circ \end{array} \quad \bullet$$

$$C_8^8: \quad (1) \quad \begin{array}{c} \text{---} \\ | \\ \boxed{12} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \\ | \\ \bullet \\ | \\ \text{---} \\ | \\ \boxed{12} \end{array}$$

$$(2) \quad \begin{array}{c} \text{---} \\ | \quad \square \quad | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \text{---} \\ | \quad | \quad | \\ \text{---} \end{array} \quad \begin{array}{c} \square \quad | \\ \text{---} \end{array}$$

$$C_8^9: \quad \text{---} \boxed{H^2} \oplus \text{---} \oplus \text{---} \boxed{H^2} \text{---} = \quad \text{---} \oplus \text{---}$$

$$C_8^{10} = \dots$$

$$= \quad \begin{array}{c} \text{H}^2 \\ \text{H}^2 \end{array}$$

$$C_8^{'''} = \begin{array}{c} \text{---} \\ | \\ \text{H}^2 \end{array} \oplus \begin{array}{c} \text{---} \\ | \\ \text{H}^2 \end{array} = \begin{array}{c} \text{---} \\ | \\ \oplus \end{array} \oplus \begin{array}{c} \text{---} \\ | \\ \oplus \end{array}$$

$$= \begin{array}{c} \text{---} \\ | \quad | \\ \boxed{112} \quad \bullet \quad \boxed{112} \end{array}$$

$$C_8^{12} : \quad \text{Diagram} \quad = \quad \text{Diagram}$$

$$= \begin{array}{c} \text{H}^2 \\ \text{H}^2 \end{array}$$

$$C_8^{13} : \quad \begin{array}{c} \oplus \\ \parallel \\ \square H^2 \\ \parallel \\ \square H^2 \end{array} = \quad \begin{array}{c} \square H^2 \\ \parallel \\ \square H^2 \\ \parallel \\ \oplus \end{array}$$

$$C_8^{14} : \quad \begin{array}{c} \bullet \\ \square \\ \circ \end{array} \quad = \quad \begin{array}{c} \square \\ \bullet \\ \circ \end{array}$$

Lem 0

$$R_{18}: (1) \quad \text{Diagram} = \text{Diagram}$$

$$R_{18} : (1) \quad \times \quad = \quad \times \quad s$$

$$(2) \quad \text{Diagram showing } S = S$$

Lem A

$$R_{19}: \quad (1) \quad \text{Diagram} = \text{Diagram}$$

$$(2) \quad \text{Diagram showing } H \otimes H = H$$

Lem B

$$\text{Def 4: } \text{X} = \text{---} \oplus \text{---}$$

$$\text{Def 5: } \text{X} = \text{H}^{\otimes 5}$$

$$\text{Def 6: } \text{---} \times \text{---} = \text{---} \boxed{H^2} \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \circ \text{---} \bullet \text{---} \bullet \text{---}$$

$$\text{Def 7: } \text{---} \times \text{---} = \text{---} \circ \text{---}$$

$$\text{Def 8: } \text{Diagram} = \text{Diagram with labels } H^3$$

$$\text{Def 10: } \text{X} = \text{H} \otimes \text{H}$$

$$\text{Def 11: } \text{X} = \begin{array}{c} \text{H} \\ \text{CNOT} \end{array}$$

Lem P

$$R_{20} : (1) \quad \text{Diagram showing } R_{20} \text{ as a box } X \text{ connected between two nodes. The output node is connected to the left input node through a switch. The right input node is connected to the right output node through a switch. The two switches are connected in series. This is equivalent to a box } X \text{ connected between the two nodes.}$$

$$(2) \quad \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad \boxed{X}$$

$$R_{21} : \quad (1) \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad \boxed{Z}$$

$$(2) \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \times \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \times \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}$$

$$R_{22}: (1) \quad \begin{array}{c} S' \\ \square \end{array} \quad = \quad \begin{array}{c} \square \\ S' \end{array}$$

$$(2) \quad \text{Diagram showing two hexagonal loops connected by a central crossing point. The left loop has a box labeled } S' \text{ at its bottom-left corner. The right loop has a box labeled } S' \text{ at its top-right corner. An equals sign follows the diagram.}$$

Prop I & Cor I

$$R_{3|} : (1) \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \bullet \\ | \\ \text{---} \end{array}$$

$$(2) \quad \text{Diagram} = \text{Diagram}$$

Lem K

$$R_{23}: \quad \begin{array}{c} S \\ \oplus \\ \bullet \end{array} = \begin{array}{c} \oplus \\ \bullet \\ \bullet \end{array} \begin{array}{c} S \\ \bullet \\ \bullet \end{array} \cdot w^2$$

$$R_{23}^8: \quad \begin{array}{c} S^2 \\ \oplus \\ \bullet \end{array} = \begin{array}{c} \oplus \\ \bullet \\ \bullet \end{array} \begin{array}{c} S^2 \\ \bullet \\ \bullet \end{array} \cdot w$$

$$\equiv R_{23}^1: \quad \begin{array}{c} S \\ \oplus \\ \bullet \end{array} = \begin{array}{c} \oplus \\ \bullet \\ \bullet \end{array} \begin{array}{c} S \\ \bullet \\ S' \end{array} \cdot w^2$$

Lem B1

$$\equiv R_{23}^2: \quad \begin{array}{c} \oplus \\ S \end{array} = \begin{array}{c} S \\ S \\ \oplus \\ \bullet \\ S' \end{array} \cdot w^2$$

$$R_{23}^{8'}: \quad \begin{array}{c} S^2 \\ \oplus \\ \bullet \end{array} = \begin{array}{c} \oplus \\ \bullet \\ \bullet \end{array} \begin{array}{c} S^2 \\ \bullet \\ S^2 \end{array} \cdot w$$

$$\equiv R_{23}^3: \quad \begin{array}{c} S \\ \oplus \\ S \\ S \end{array} \cdot w = \begin{array}{c} \oplus \\ \bullet \\ \bullet \end{array}$$

Lem B31

$$\equiv R_{23}^7: \quad \begin{array}{c} \bullet \\ \oplus \\ \bullet \\ S \\ \oplus \\ S^2 \end{array} \cdot w = \begin{array}{c} \bullet \\ \oplus \\ \oplus \\ S \\ \oplus \\ S^2 \end{array} = \begin{array}{c} \oplus \\ \bullet \\ \bullet \\ S \\ \oplus \\ S^2 \end{array} \cdot w$$

$$15 \quad R_{23}: \quad \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \oplus \\ \bullet \\ S^2 \\ \oplus \\ S \end{array} \cdot w^2 = \begin{array}{c} \oplus \\ \bullet \\ \bullet \\ S^2 \\ \oplus \\ S' \end{array} \cdot w^2$$

Lem B36

$$16 \quad R_{23}: \quad \begin{array}{c} S^2 \\ \oplus \\ \oplus \\ \bullet \end{array} = \begin{array}{c} \oplus \\ \bullet \\ \bullet \\ S^2 \\ \oplus \\ S^2 \end{array} \cdot w$$

Lem B42

$$R_{23}^4: \quad \begin{array}{c} S \\ \oplus \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \oplus \\ \bullet \\ S' \\ \bullet \\ S \end{array} \cdot w^2$$

$$R_{23}^{10}: \quad \begin{array}{c} S' \\ \oplus \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \oplus \\ \bullet \\ S \\ \bullet \\ S' \end{array} \cdot w^2$$

$$\equiv R_{23}^5: \quad \begin{array}{c} \bullet \\ \oplus \\ S \end{array} = \begin{array}{c} \bullet \\ S \\ S \\ \oplus \\ \bullet \\ S \\ S \end{array} \cdot w^2$$

Lem B28

$$\equiv R_{23}^6: \quad \begin{array}{c} S \\ \oplus \\ S' \\ S' \end{array} \cdot w = \begin{array}{c} \oplus \\ \bullet \\ \bullet \end{array}$$

Lem L

$$R_{23}^9: \quad \begin{array}{c} S \\ \oplus \\ \oplus \\ \bullet \end{array} = \begin{array}{c} \oplus \\ \bullet \\ \bullet \\ S \\ \bullet \\ S \end{array} \cdot w^2$$

Lem B5

$$R_{23}^{11}: \quad \begin{array}{c} S^2 \\ \oplus \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \oplus \\ \bullet \\ S^2 \\ \bullet \\ S^2 \end{array} \cdot w$$

$$R_{23}^{12}: \quad \begin{array}{c} S' \\ \oplus \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \oplus \\ \bullet \\ S \\ \bullet \\ S' \end{array} \cdot w^2$$

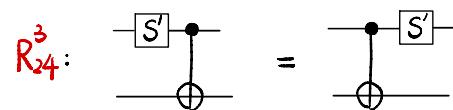
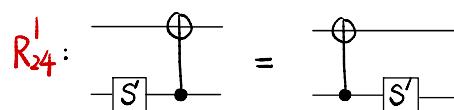
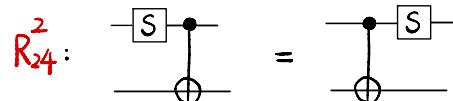
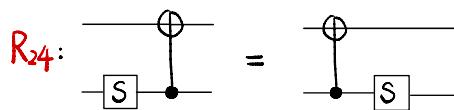
$$R_{23}^{13}: \quad \begin{array}{c} S^2 \\ \oplus \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \oplus \\ \bullet \\ S^2 \\ \bullet \\ S^2 \end{array} \cdot w$$

$$R_{23}^{14}: \quad \begin{array}{c} S^2 \\ \oplus \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \oplus \\ \bullet \\ S^2 \\ \bullet \\ S^2 \end{array} \cdot w$$

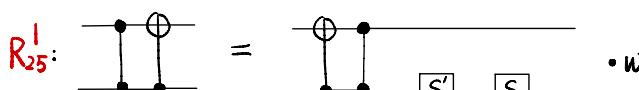
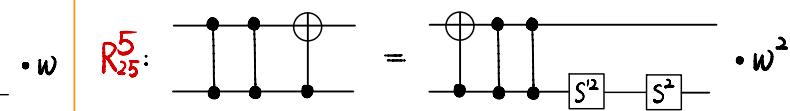
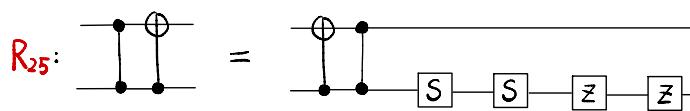
$$R_{23}^{9'}: \quad \begin{array}{c} S \\ \oplus \\ \oplus \\ \bullet \end{array} = \begin{array}{c} \oplus \\ \bullet \\ \bullet \\ S \\ \bullet \\ S \end{array} \cdot w^2$$

Lem B25

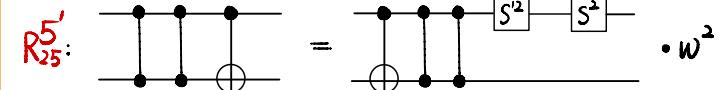
Lem B35



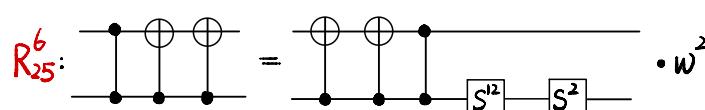
Lem I



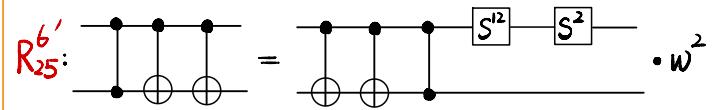
Lem B2



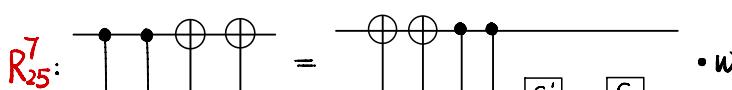
Lem B21



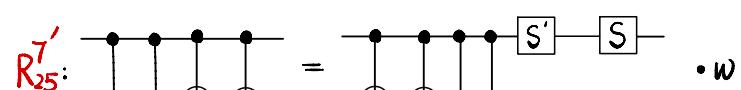
Lem B4



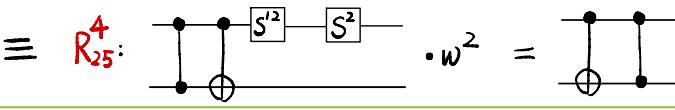
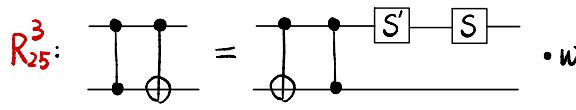
Lem B30



Lem B7



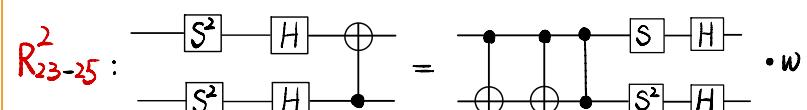
Lem B40



Lem M



Lem B3

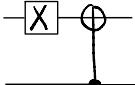
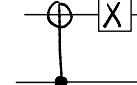
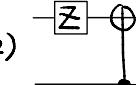
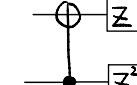
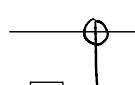
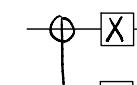
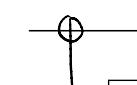
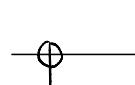
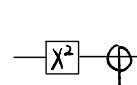
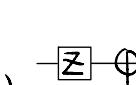
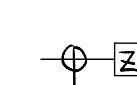
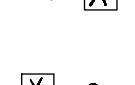
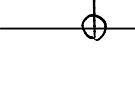
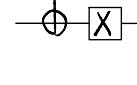
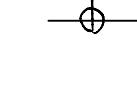
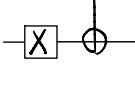
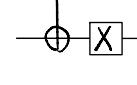
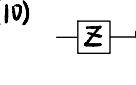


Lem B29



Lem B17

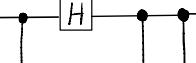
R₂₆:

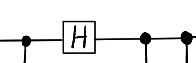
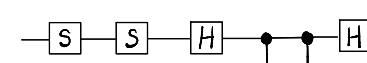
(1)			(2)		
(3)			(4)		
(5)			(6)		
(7)			(8)		
(9)			(10)		
(11)			(12)		

Lem J

$$\begin{aligned}
 R_{27}: & \quad \text{Circuit Diagram} = \text{Circuit Diagram} \\
 & \quad \text{Diagram shows } H, CNOT, \oplus, S, \bar{S}, \bar{Z} \text{ gates.} \\
 R_{27}^1: & \quad \text{Circuit Diagram} = \text{Circuit Diagram} \\
 & \quad \text{Diagram shows } H, CNOT, \oplus, S, \bar{S}, H, H, H \text{ gates.} \\
 R_{27}^2: & \quad \text{Circuit Diagram} = \text{Circuit Diagram} \\
 & \quad \text{Diagram shows } H, CNOT, \oplus, S, \bar{S}, H, S, S \text{ gates.}
 \end{aligned}$$

Lem Q

R_{28} :  = 

R'_{28} :  = 

R''_{28} :  = 

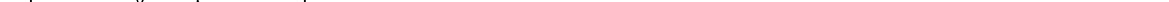
Lem R

$$R_{29} = \text{Circuit Diagram} = \text{Circuit Diagram} \cdot w^2$$

Lem 4

$$R_{30} : \quad \begin{array}{c} \text{Quantum circuit diagram showing two qubits. The left qubit starts with an H gate, followed by a CNOT gate targeting the right qubit. The right qubit starts with an H gate, followed by a CNOT gate targeting the left qubit.} \\ \hline \end{array} = \quad \begin{array}{c} \text{Quantum circuit diagram showing two qubits. The left qubit starts with an H gate, followed by an S gate, then another H gate, followed by an S gate, and finally a Z gate. The right qubit starts with an H gate, followed by an S gate, then another H gate, followed by an S gate, and finally a Z gate.} \\ \hline \end{array}$$

$$R_{30} : \text{Circuit Diagram} = \text{Circuit Diagram}$$

$R_{3D}^2:$  =

$$R_{33} : \quad \begin{array}{c} \text{---} \\ | \quad | \quad | \\ \oplus \quad \oplus \quad \oplus \\ | \quad | \quad | \\ \bullet \quad \bullet \quad H \quad H \end{array} = \quad \begin{array}{c} \text{---} \\ | \quad | \quad | \quad | \\ \bullet \quad \bullet \quad \oplus \quad \bullet \\ | \quad | \quad | \quad | \\ \bullet \quad \bullet \quad S \quad H \quad H \quad S \end{array} \cdot w \quad \text{Lem X}$$

$$\text{R4 : } \begin{array}{c} \text{Circuit Diagram} \\ \text{Two vertical lines connected by a horizontal line. The top line has a box labeled } H \text{ and a circle with a plus sign. The bottom line has a box labeled } H \text{ and a circle with a minus sign.} \end{array} = \begin{array}{c} \text{Circuit Diagram} \\ \text{A sequence of boxes: } S^2, H, \text{ followed by a vertical line with a circle plus sign, then } S, X^2. \\ \text{Below it is another sequence: } S^2, H, S, H, H, S^2, H, S, Z^2. \end{array} \cdot (-w^2)$$

$$R_{35} : \begin{array}{c} \text{Quantum Circuit Diagram} \\ \text{with } H \text{ and } \oplus \text{ gates} \end{array} = \begin{array}{c} \text{Quantum Circuit Diagram} \\ \text{with } S, H, \oplus, S, X, S \text{ gates} \\ \text{and } H^2, S, H, S^2, H, S^2, Z^2 \text{ gates} \end{array} \cdot w$$

$$R_{36}: \quad \text{Diagram} = \quad \text{Diagram with } H^2 \text{ blocks}$$

$$R_{37} : \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \oplus \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \boxed{S} \\ \text{---} \\ | \\ \text{---} \end{array} \oplus \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \cdot \begin{array}{c} S^2 \\ \text{---} \\ | \\ \text{---} \end{array} \cdot w^2 \quad \text{Lem A1}$$

$$R_{38} : \quad \text{Diagram showing two ways to connect nodes 3 and 8. The left side shows a complex connection involving node 1, while the right side shows a simplified direct connection.} = \quad \text{Diagram showing a simplified direct connection between nodes 3 and 8.}$$

$$R_{32} : \begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} H^2 \text{---} \bullet \\ \text{---} \bullet \text{---} \bullet \text{---} H \text{---} \oplus \end{array} = \begin{array}{c} \text{---} H^2 \text{---} \bullet \\ \text{---} H \text{---} \oplus \end{array}$$

$$R_{32}^1 : \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \quad \boxed{H} \quad \bigoplus = \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \quad \boxed{H}$$

$$R_{32}^2 : \quad \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \quad \boxed{H} \quad \begin{array}{c} \text{---} \\ | \\ \circ \\ | \\ \text{---} \end{array} = \quad \begin{array}{c} \text{---} \\ | \\ \boxed{H} \\ | \\ \text{---} \end{array}$$

$$R_{32}^3 : \quad \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array}$$

$$R_{32}^4 : \quad \begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{H}^3} \text{---} \bullet \text{---} \boxed{\text{H}} \text{---} \begin{array}{c} \text{---} \\ | \end{array} = \quad \begin{array}{c} \text{---} \\ | \end{array} \bigcirc \text{---} \bullet \text{---} \begin{array}{c} \text{---} \\ | \end{array}$$

$$R_{32}^5 : \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} H \\ | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} H \\ | \\ \text{---} \\ | \\ \text{---} \end{array}$$

$$R_{32}^6 : \quad \begin{array}{c} \text{---} \bullet \text{---} \boxed{H} \text{---} \\ | \qquad \qquad \qquad | \\ \text{---} \bullet \text{---} \end{array} = \quad \begin{array}{c} \boxed{H} \text{---} \bigcirc \text{---} \bigcirc \\ | \qquad | \qquad | \\ \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \end{array}$$

$$R_{32}^7 : \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ H \end{array} = \quad \begin{array}{c} \boxed{H^2} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ H \end{array}$$

$$\begin{array}{c} 8 \\ R_{32}: \end{array} \quad \begin{array}{c} \text{---} \bullet \text{---} [H] \text{---} \bullet \text{---} \text{---} \\ | \qquad \qquad | \\ \text{---} \bullet \text{---} \bullet \text{---} \text{---} \end{array} = \begin{array}{c} \text{---} \bullet \text{---} [H] \text{---} \bullet \text{---} \text{---} \\ | \qquad \qquad | \\ \text{---} \bullet \text{---} [H^2] \text{---} \bullet \text{---} \text{---} \end{array}$$

$$R_{32}^q : \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} = \begin{array}{c} \boxed{H^2} \\ | \\ \bullet \\ | \\ \boxed{H^2} \end{array}$$

$$R_{32}^{10} : \quad \begin{array}{c} \text{---} \\ | \end{array} \boxed{H^3} \text{---} \bullet \text{---} \boxed{H} \text{---} \quad = \quad \begin{array}{c} \text{---} \\ | \end{array} \bullet \text{---} \quad \bigcirc \quad \begin{array}{c} \text{---} \\ | \end{array} \quad \boxed{H^2} \text{---} \bullet \text{---} \boxed{H^2} \text{---} \quad \begin{array}{c} \text{---} \\ | \end{array}$$

$$R_{32}^{11} : \quad \begin{array}{c} \text{Diagram showing two horizontal wires. The top wire has a square box labeled 'H' and a circle with a plus sign. The bottom wire has a circle with a minus sign. Vertical dashed lines connect the two wires at three points. The first and third connections are on the 'H' box, and the second connection is on the circle with a plus sign.} \\ = \end{array} \quad \begin{array}{c} \text{Diagram showing two horizontal wires. The top wire has a square box labeled 'H' and a circle with a plus sign. The bottom wire has a square box labeled 'H^2' and a circle with a minus sign. Vertical dashed lines connect the two wires at three points. The first and third connections are on the 'H^2' box, and the second connection is on the circle with a minus sign.} \end{array}$$

$$R_{32}^{12} : \quad \begin{array}{c} \text{Diagram showing } R_{32}^{12} \text{ as a sequence of operations: } \\ \text{1. Horizontal line with three dots.} \\ \text{2. Vertical line connecting the first two dots.} \\ \text{3. Horizontal line with three dots.} \\ \text{4. Vertical line connecting the second and third dots.} \\ \text{5. Box labeled } H \text{ below the middle horizontal line.} \\ \text{6. Box labeled } \bigcirc \text{ below the bottom horizontal line.} \end{array} = \quad \begin{array}{c} \text{Diagram showing the result:} \\ \text{1. Horizontal line with three dots.} \\ \text{2. Box labeled } H^2 \text{ above the first dot.} \\ \text{3. Box labeled } H^2 \text{ above the third dot.} \\ \text{4. Box labeled } H \text{ below the middle horizontal line.} \\ \text{5. Box labeled } \bigcirc \text{ below the bottom horizontal line.} \end{array}$$

Lem N