Single-Qutrit Derived Relations : T_2

C0: $(-1)^2 = 1$

C1: $\omega^3 = 1$

C2: $H^4 = I$

C3: $S^3 = I$

R1: $-\boxed{H}-\boxed{S}-\boxed{H}- \; = \; -\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S^2}-\boxed{X^2}- \cdot (-\omega^2)$

R2: $-\boxed{S}-\boxed{H}-\boxed{H}- \; = \; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{Z^2}-$

R3: $SZ = ZS$

R4: $HZ^2H^\dagger = X$ $\quad -\boxed{Z^2}-\boxed{H}- \; = \; -\boxed{H}-\boxed{X}-$

R5: $-\boxed{X}- \; = \; -\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-$

R6: $-\boxed{S}-\boxed{X}- \; = \; -\boxed{X}-\boxed{S}-\boxed{Z}- \cdot \omega^2$

R7: $-\boxed{X^2}- \; = \; -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-$

R8: $XZ = \omega^2 ZX$ $\quad -\boxed{Z}-\boxed{X}- \; = \; -\boxed{X}-\boxed{Z}- \cdot \omega^2$

R9: $X^3 = I$

R10: $-\boxed{Z}- \; = \; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-$

R11: $-\boxed{Z^2}- \; = \; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-$

R12: $Z^3 = I$

C2 : $H^4 = I$

C3 : $S^3 = I$

C5 : $SH^2SH^2 = H^2SH^2S$ $S' := H^2SH^2$

$$-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- \;=\; -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$$

R10 : $-\boxed{Z}- \;=\; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-$

R11 : $-\boxed{Z^2}- \;=\; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-$

Lem 11 By $C_2, C_3,$ & C_5, R10 implies R11.

Proof : By R10, R11. LHS =

$$-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-$$

$$\overset{C_5}{=} -\boxed{H}-\boxed{H}-\boxed{-\boxed{S}-\boxed{S}-\boxed{S}-}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-$$

$$\overset{C_3}{=} \boxed{-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}-}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-$$

$$\overset{C_2}{=} \boxed{-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-}-\boxed{S}-$$

$$\overset{C_5}{=} -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-$$

=: R11. RHS.

∎

$C2:\ H^4 = I$

$C3:\ S^3 = I$

$C5:\ SH^2SH^2 = H^2SH^2S \qquad S' := H^2SH^2$

$$-H-H-S-H-H-S- \;=\; -S-H-H-S-H-H-$$

$R10:\ -Z- \;=\; -H-H-S-S-H-H-S-$

$R11:\ -Z^2- \;=\; -H-H-S-H-H-S-S-$

$R12:\ Z^3 = I$

Lem 12 By $C_2, C_3, \&\ C_5$, $R10$ and $R11$ imply $R12$.

Proof: By $R10$ and $R11$, $R_{12}.LHS = -Z-Z^2- :=$

$$-H-H-S-S-\boxed{H-H-S-H-H-S}-H-H-S-S-$$

$$\overset{C_5}{=} -H-H-S-S-S-H-H-S-\boxed{H-H-H-H}-S-S-$$

$$\overset{C_2}{=} -H-H-\boxed{S-S-S}-H-H-\boxed{S-S-S}-$$

$$\overset{C_3}{=} -H-H-H-H-$$

$$\overset{C_2}{=} I = R_{12}.LHS.$$

$\blacksquare$

$C2: \; H^4 = I$

$C3: \; S^3 = I$

$C5: \; SH^2SH^2 = H^2SH^2S \qquad S' := H^2SH^2$

$$-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- \;=\; -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$$

$R5: \; -\boxed{X}- = -\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-$

$R7: \; -\boxed{X^2}- = -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-$

Lem 13 By C_2, C_3, & C_5, R5 implies R7.

Proof: By R_5, R_7. LHS =

$$-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-}-\boxed{S}-\boxed{H}-$$

$$\overset{C_5}{=\!=} -\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{-\boxed{S}-\boxed{S}-\boxed{S}-}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-$$

$$\overset{C_3}{=\!=} -\boxed{H}-\boxed{S}-\boxed{-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}-}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-$$

$$\overset{C_2}{=\!=} -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}- \;=:_{R_7}\; \text{RHS}$$

$\blacksquare$

C2 : $H^4 = I$

C3 : $S^3 = I$

C5 : $SH^2SH^2 = H^2SH^2S \qquad S' := H^2SH^2$

$$-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- \;=\; -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$$

R5 : $-\boxed{X}- \;=\; -\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-$

R7 : $-\boxed{X^2}- \;=\; -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-$

R9 : $X^3 = I$

Lem 14 By C2, C3, & C5, R5 and R7 imply R9.

Proof : By R5 and R7, R9. LHS $= -\boxed{X^2}-\boxed{X}- =$

$$-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-$$

$$\overset{C_5}{=} -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{S}-\boxed{H}-$$

$$\underset{C_3}{\overset{C_2}{=}} -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-$$

$$\overset{C_3}{=} -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}-$$

$$\overset{C_2}{=} I = R_9.\ LHS.$$

$\square$

$C2: \ H^4 = I$

$C5: \ SH^2SH^2 = H^2SH^2S \qquad S' := H^2SH^2$

$$-H-H-S-H-H-S- \;=\; -S-H-H-S-H-H-$$

$$C5': \ -S-H-H-S- \;=\; -H-H-S-H-H-S-H-H-$$

$R3: \ ZS = SZ \qquad -S-Z- \;=\; -Z-S-$

$$R10: \ -Z- \;=\; -H-H-S-S-H-H-S-$$

Lem 15 By C_2 & C_5, $R10$ imply $R3$.

Proof: $R3.LHS := -S-Z-$

$$\overset{R10}{=} \ -\boxed{S-H-H-S}-S-H-H-S-$$

$$\overset{C_5'}{=} \ -H-H-S-\boxed{H-H-S-H-H-S}-H-H-S-$$

$$\overset{C_5}{=} \ -H-H-S-S-H-H-S-\boxed{H-H-H-H}-S-$$

$$\overset{C_2}{=} \ -\boxed{H-H-S-S-H-H-S}-S- \ \overset{R10}{=} \ -Z-S- .$$

∎

$C2: \; H^4 = I$

$C3: \; S^3 = I$

$C5: \; SH^2SH^2 = H^2SH^2S \qquad S' := H^2SH^2$

$$-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- \;=\; -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$$

$$C_5': \; -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- \;=\; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$$

$R4: \; HZ^2H^\dagger = X \qquad -\boxed{Z^2}-\boxed{H}- \;=\; -\boxed{H}-\boxed{X}-$

$$R5: \; -\boxed{X}- \;=\; -\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-$$

$$R10: \; -\boxed{Z}- \;=\; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-$$

$\downarrow$

$$R11: \; -\boxed{Z^2}- \;=\; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-$$

Lem 16 By $C_2, C_3,$ & C_5, R_5 and R_{10} imply R_4.

Proof: $R4.\text{LHS} := -\boxed{Z^2}-\boxed{H}-$

$$\overset{R_{11}}{=} -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-$$

$$\overset{C_5}{=} -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-$$

$$\overset{C_2}{=} -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-$$

$$\overset{C_5'}{=} -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-$$

$$\overset{R_5}{=} -\boxed{H}-\boxed{X}- \;=\; R4.\text{RHS}$$

$\square$

C_0: $(-1)^2 = 1$ C_1: $w^3 = 1$ C_2: $H^4 = I$ C_3: $S^3 = I$

C_4: $(HS^2)^3 = -wI$

$$-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -wI$$

C_4': $$-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}- = -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}- \cdot (-w)$$

C_5: $SH^2SH^2 = H^2SH^2S$ $S' := H^2SH^2$

$$-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- = -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$$

C_5': $$-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- = -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$$

R1 : $$-\boxed{H}-\boxed{S}-\boxed{H}- = -\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S^2}-\boxed{X^2}- \cdot (-w^2)$$

R5 : $$-\boxed{X}- = -\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-$$

R7: $$-\boxed{X^2}- = -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-$$

Lem17 By C_0, C_1, C_2, C_3 & C_4, R_5 implies R_1.

Proof: By Lem13, R_5 implies R_7. Then R_1.RHS

$$\overset{R_7}{=} \boxed{-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S^2}-}\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}- \cdot (-w^2)$$

$$\overset{C_4'}{=} -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}- \boxed{\cdot (-w^2)(-w)}$$

$$\overset{C_1}{\underset{C_0}{=}} -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-$$

$$\overset{C_2}{=} -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{\boxed{S}-\boxed{S}-\boxed{S}}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-$$

$$\overset{C_3}{=} -\boxed{\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}}-\boxed{H}-\boxed{S}-\boxed{H}-$$

$$\overset{C_2}{=} -\boxed{H}-\boxed{S}-\boxed{H}- = R_1.\text{LHS}$$

∎

$C2: \; H^4 = I$

$C3: \; S^3 = I$

Lemma C $\; C4' : \; -S-S-H-S-S- \;=\; -H-H-H-S-H-H-H- \cdot (-\omega)$

is a consequence of $C4 : \; -S-S-H-S-S-H-S-S-H- = -\omega I$

Proof: Concatenating both sides of $C4$ by $-H-H-H-S-H-H-H-$ to the right.

$$C4.\mathrm{LHS} = -S-S-H-S-S-H-S-S-\boxed{H-H-H-H}-S-H-H-H-$$

$$\overset{C_2}{\equiv} -S-S-H-S-S-H-\boxed{S-S-S}-H-H-H-$$

$$\overset{C_3}{\equiv} -S-S-H-S-S-\boxed{H-H-H-H}-$$

$$\overset{C_2}{\equiv} -S-S-H-S-S-$$

$$C4.\mathrm{RHS} = -H-H-H-S-H-H-H- \cdot (-\omega)$$

$\Big\} \equiv C4'$

$\boxtimes$

$C2: \; H^4 = I$

$C3: \; S^3 = I$

$C5: \; SH^2SH^2 = H^2SH^2S \qquad S' := H^2SH^2$

$$-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- \;=\; -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$$

$R2: \; -\boxed{S}-\boxed{H}-\boxed{H}- \;=\; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{Z^2}-$

$R10: \; -\boxed{Z}- \;=\; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-$

$R11: \; -\boxed{Z^2}- \;=\; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-$

Lem 18 By C_2, C_3 & C_5, R_{10} implies R_2.

Proof: By Lem 11, R_{10} implies R_{11}. Then R_2.RHS

$$\overset{R_{11}}{=} -\boxed{-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-$$

$$\overset{C_5}{=} -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}-}-\boxed{S}-\boxed{S}-$$

$$\overset{C_2}{=} -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{-\boxed{S}-\boxed{S}-\boxed{S}-}-$$

$$\overset{C_3}{=} -\boxed{S}-\boxed{H}-\boxed{H}- = R_2.\text{LHS}.$$

$\blacksquare$

$\boxed{S'} = \boxed{H}\boxed{H}\boxed{S}\boxed{H}\boxed{H}$ $\qquad$ $\boxed{S'}\boxed{S'} = \boxed{H}\boxed{H}\boxed{S}\boxed{S}\boxed{H}\boxed{H}$

$S' := H^2 S H^2$ $\qquad$ $S'^2 = (H^2SH^2)(H^2SH^2) = H^2 S^2 H^2$

C5 : $SH^2SH^2 = H^2SH^2S$ $\qquad$ $S' := H^2SH^2$ $\qquad$ $SS' = S'S$

$$\boxed{H}\boxed{H}\boxed{S}\boxed{H}\boxed{H}\boxed{S} = \boxed{S}\boxed{H}\boxed{H}\boxed{S}\boxed{H}\boxed{H}$$

R4 : $HZ^2H^+ = X$ $\qquad$ $\boxed{Z^2}\boxed{H} = \boxed{H}\boxed{X}$

R_4^1 : $H^+ZH = X$ $\qquad$ $\boxed{H}\boxed{Z}\boxed{H^+} = \boxed{X} \equiv \boxed{X}\boxed{H} = \boxed{H}\boxed{Z}$

R_4^2 : $H^+Z^2H = X^2$ $\qquad$ $\boxed{H}\boxed{Z}\boxed{Z}\boxed{H^+} = \boxed{X^2}$

R5 : $\boxed{X} = \boxed{H}\boxed{S}\boxed{H}\boxed{H}\boxed{S}\boxed{S}\boxed{H}$

R10 : $\boxed{Z} = \boxed{H}\boxed{H}\boxed{S}\boxed{S}\boxed{H}\boxed{H}\boxed{S} = \boxed{S'}\boxed{S'}\boxed{S}$

Lem 19 By C_5, R_5 and R_{10} imply R_4^1.

Proof: R_4^1. LHS $:= \boxed{X}\boxed{H}$

$$\overset{R_5}{=} \boxed{H}\boxed{S}\boxed{\boxed{H}\boxed{H}\boxed{S}\boxed{S}\boxed{H}\boxed{H}}$$

$$\overset{\text{def}}{=} \boxed{H}\boxed{S}\boxed{S'}\boxed{S'}$$

$$\overset{C_5}{=} \boxed{H}\boxed{\boxed{S'}\boxed{S'}\boxed{S}}$$

$$\overset{\text{def}}{=} \boxed{H}\boxed{Z} = R_4^1.\ \text{RHS}$$

$\square$

Cor 1 By C_5, R_5 and R_{10} imply R_4^2.

Proof: R_4^2.RHS $= \boxed{X^2} = \boxed{H}\boxed{Z}\boxed{\boxed{H^+}\boxed{H}}\boxed{Z}\boxed{H^+}$

$$= \boxed{H}\boxed{Z}\boxed{Z}\boxed{H^+} = R_4^2.\ \text{LHS}.$$

$\square$

C_0: $(-1)^2 = 1$ $C1$: $\omega^3 = 1$ $C2$: $H^4 = I$ $C3$: $S^3 = I$

C_4: $(HS^2)^3 = -\omega I$

$$-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -\omega I$$

C_4^1: $$-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}- = -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}- \cdot (-\omega)$$

Lem D By C_0, C_1 & C_3, C_4^2: $-\boxed{S}-\boxed{H^+}-\boxed{S}- = -\boxed{H}-\boxed{S^2}-\boxed{H}- \cdot (-\omega^2)$

is a consequence of C_4: $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -\omega I$

Proof: By Lem C, C_4^1: $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}- = -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}- \cdot (-\omega)$

is a consequence of C_4: $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -\omega I$.

By C_3, concatenating both sides of C_4^1 by $-\boxed{S}-$ to the right yields

C_4^1: $$-\boxed{S}-\boxed{S}-\boxed{H}- = -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}- \cdot (-\omega)$$

By C_3, concatenating both sides of C_4^1 by $-\boxed{S}-$ to the left yields

C_4^1: $$-\boxed{H}- = -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}- \cdot (-\omega)$$

By C_2 & C_3, concatenating both sides of C_4^1 by $-\boxed{S}-\boxed{S}-\boxed{H}-$ to the right yields

$$-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{S}-\boxed{H}-$$

$$= -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}- \cdot (-\omega) \quad (1)$$

By C_0 & C_1, multiplying both sides of (1) by $(-\omega^2)$ yields

$$-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \cdot (-\omega^2) = -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}- \cdot (-\omega) \cdot (-\omega^2)$$

It follows that C_4^2: $-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \cdot (-\omega^2) = -\boxed{S}-\boxed{H^+}-\boxed{S}-$

∎

C_0 : $(-1)^2 = 1$ C_1 : $\omega^3 = 1$ C_2 : $H^4 = I$ C_3 : $S^3 = I$

C_4 : $(HS^2)^3 = -\omega I$

$$-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -\omega I$$

C_4^1 : $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}- = -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}- \cdot (-\omega)$

C_4^2 : $-\boxed{S}-\boxed{H^+}-\boxed{S}- = -\boxed{H}-\boxed{S^2}-\boxed{H}- \cdot (-\omega^2)$

Def : $-\boxed{S'}- = -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$ $\quad -\boxed{S'}-\boxed{S'}- = -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-$

$S' := H^2 S H^2$ $\quad S'^2 = (H^2SH^2)(H^2SH^2) = H^2 S^2 H^2$

Lem E By C_0, C_1, C_2 & C_3, C_4^3 : $-\boxed{S'}-\boxed{H^+}-\boxed{S'}- = -\boxed{H}-\boxed{S'}-\boxed{S'}-\boxed{H}- \cdot (-\omega^2)$

is a consequence of C_4 : $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -\omega I$

Proof : C_4^3. LHS $= -\boxed{S'}-\boxed{H^+}-\boxed{S'}-$

$$\overset{\text{def}}{=\!=} -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$$

$$\overset{C_2}{=\!=} -\boxed{H}-\boxed{H}-\boxed{\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}}-\boxed{H}-\boxed{H}-$$

$$\overset{C_4^2}{=\!=} -\boxed{H}-\boxed{\boxed{H}-\boxed{H}-\boxed{S^2}-\boxed{H}-\boxed{H}}-\boxed{H}- \cdot (-\omega^2)$$

$$\overset{\text{def}}{=\!=} -\boxed{H}-\boxed{S'}-\boxed{S'}-\boxed{H}- \cdot (-\omega^2) = C_4^3 . \text{RHS}$$

$\square$

Lem F By C_0, C_1 & C_2, C_4^4 : $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \cdot (-\omega^2) = -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-$

is a consequence of C_4 : $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -\omega I$

Proof : By **Lem C**, C_4^1 : $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}- = -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}- \cdot (-\omega)$

By C_2, concatenating both sides of the equation by $-\boxed{H}-$ yields:

$$-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}- \cdot (-\omega)$$

By C_0 & C_1, multiplying both sides of the equation by $(-\omega^2)$ yields:

$$-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \cdot (-\omega^2) = -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}- \equiv C_4^4$$

$\square$

C_0: $(-1)^2 = 1$ $C1$: $\omega^3 = 1$ $C2$: $H^4 = I$ C_3: $S^3 = I$

C_4: $(HS^2)^3 = -\omega I$ $\boxed{S}\boxed{S}\boxed{H}\boxed{S}\boxed{S}\boxed{H}\boxed{S}\boxed{S}\boxed{H} = -\omega I$

C_4^1: $\boxed{S}\boxed{S}\boxed{H}\boxed{S}\boxed{S} = \boxed{H}\boxed{H}\boxed{H}\boxed{S}\boxed{H}\boxed{H}\boxed{H} \cdot (-\omega)$

C_4^2: $\boxed{S}\boxed{H^+}\boxed{S} = \boxed{H}\boxed{S^2}\boxed{H} \cdot (-\omega^2)$

$C5$: $SH^2SH^2 = H^2SH^2S$ $SS' = S'S$

$$\boxed{H}\boxed{H}\boxed{S}\boxed{H}\boxed{H}\boxed{S} = \boxed{S}\boxed{H}\boxed{H}\boxed{S}\boxed{H}\boxed{H}$$

Def: $\boxed{S'} = \boxed{H}\boxed{H}\boxed{S}\boxed{H}\boxed{H}$ $\boxed{S'}\boxed{S'} = \boxed{H}\boxed{H}\boxed{S}\boxed{S}\boxed{H}\boxed{H}$

$S' := H^2SH^2$ $S'^2 = (H^2SH^2)(H^2SH^2) = H^2S^2H^2$

C_4^3: $\boxed{S'}\boxed{H^+}\boxed{S'} = \boxed{H}\boxed{S'}\boxed{S'}\boxed{H} \cdot (-\omega^2)$

$R8$: $XZ = \omega^2 ZX$ $\boxed{Z}\boxed{X} = \boxed{X}\boxed{Z} \cdot \omega^2$ $R9$: $X^3 = I$ $R12$: $Z^3 = I$

R_4^1: $\boxed{H}\boxed{Z}\boxed{H^+} = \boxed{X}$ R_4^2: $\boxed{H}\boxed{Z}\boxed{Z}\boxed{H^+} = \boxed{X^2}$

$R10$: $\boxed{Z} = \boxed{H}\boxed{H}\boxed{S}\boxed{S}\boxed{H}\boxed{H}\boxed{S} = \boxed{S'}\boxed{S'}\boxed{S}$

$R11$: $\boxed{Z^2} = \boxed{H}\boxed{H}\boxed{S}\boxed{H}\boxed{H}\boxed{S}\boxed{S} = \boxed{S'}\boxed{S}\boxed{S}$

$R5$: $\boxed{X} = \boxed{H}\boxed{S}\boxed{H}\boxed{H}\boxed{S}\boxed{S}\boxed{H}$

$R7$: $\boxed{X^2} = \boxed{H}\boxed{S}\boxed{S}\boxed{H}\boxed{H}\boxed{S}\boxed{H}$

Lem 20 By C_0, C_1, C_2, C_3, C_4 & C_5, R_5 and R_{10} imply R_8.

Proof: To show R_8, by R_9 and R_{12}, it is sufficient to show

R_8: $\boxed{X}\boxed{Z}\boxed{X^2}\boxed{Z^2} \cdot \omega^2 = I$

R_8. LHS $\overset{R_4^1}{\underset{R_4^2}{=}} \boxed{H}\boxed{Z}\boxed{H^+}\boxed{Z}\boxed{H}\boxed{Z}\boxed{H^+}\boxed{H}\boxed{Z}\boxed{H^+}\boxed{Z^2} \cdot \omega^2$

$= \boxed{H}\boxed{Z}\boxed{H^+}\boxed{Z}\boxed{H}\boxed{Z}\boxed{Z}\boxed{H^+}\boxed{Z^2} \cdot \omega^2$

$\overset{def}{=} \boxed{H}\boxed{S'}\boxed{S'}\boxed{S}\boxed{H^+}\boxed{S'}\boxed{S'}\boxed{S}\boxed{H}\boxed{S'}\boxed{S}\boxed{S}\boxed{H^+}\boxed{S'}\boxed{S}\boxed{S} \cdot \omega^2$

$\overset{C_5}{=} \boxed{H}\boxed{S'}\boxed{S'}\boxed{S}\boxed{H^+}\boxed{S}\boxed{S'}\boxed{S'}\boxed{H}\boxed{S'}\boxed{S}\boxed{S}\boxed{H^+}\boxed{S'}\boxed{S}\boxed{S} \cdot \omega^2$

$\overset{C_4^2}{\underset{C_1}{=}} \boxed{H}\boxed{S'}\boxed{S'}\boxed{H}\boxed{S^2}\boxed{H}\boxed{S'}\boxed{S'}\boxed{H}\boxed{S'}\boxed{S}\boxed{S}\boxed{H^+}\boxed{S'}\boxed{S}\boxed{S} \cdot (-\omega)$

$\overset{C_5}{=} \boxed{H}\boxed{S'}\boxed{S'}\boxed{H}\boxed{S^2}\boxed{H}\boxed{S'}\boxed{S'}\boxed{H}\boxed{S}\boxed{S}\boxed{S'}\boxed{H^+}\boxed{S'}\boxed{S}\boxed{S} \cdot (-\omega)$

$\overset{C_4^3}{\underset{C_0, C_1}{=}} \boxed{H}\boxed{S'}\boxed{S'}\boxed{H}\boxed{S^2}\boxed{H}\boxed{S'}\boxed{S'}\boxed{H}\boxed{S}\boxed{S}\boxed{H}\boxed{S'}\boxed{S'}\boxed{H}\boxed{S^2}$

C_4^1: $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}- \;=\; -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}- \cdot (-w)$

$(S^2H)^3 = -wI$

C_4^2: $-\boxed{S}-\boxed{H^+}-\boxed{S}- \;=\; -\boxed{H}-\boxed{S^2}-\boxed{H}- \cdot (-w^2)$

C_4 : $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -wI$

$C_4^{2'}$: $-\boxed{S}-\boxed{H^+}-\boxed{S}- \cdot (-w) = -\boxed{S}-\boxed{H^+}-\boxed{S}-$

$C2$: $H^4 = I$

Def: $-\boxed{S'}-\boxed{S'}- = -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-$

$C5$: $SS' = S'S$

$S'^2 = (H^2SH^2)(H^2SH^2) = H^2S^2H^2$ $\quad -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- \;=\; -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$

Hence, it is sufficient to show $(HS'^2HS^2)^3 = I$, compositions in diagrammatic order.

$$-\boxed{H}-\boxed{S'}-\boxed{S'}-\boxed{H}-\boxed{S^2}-\boxed{H}-\boxed{S'}-\boxed{S'}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S'}-\boxed{S'}-\boxed{H}-\boxed{S^2}- \overset{\text{WTS}}{=\!=} I \quad (1)$$

Conjugating both sides of (1) by H yields: (1). LHS $= H^+(HS'^2HS^2\ HS'^2HS^2\ HS'^2HS^2)H$

$$= \boxed{H^+H}S'^2HS^2H\ H^+HS'^2HS^2H\boxed{H^+H}S'^2HS^2H$$

$$= (S'^2\boxed{HS^2H})^3 = I =: (1).\ \text{RHS}.$$

$$\overset{C_4^2}{\underset{C_0,\,C_1}{=\!=}} (S'^2SH^+S)^3 = (-w)^3 I = -I$$

Hence $(S'^2SH^+S)^3 \overset{\text{WTS}}{=\!=} -I \quad (2)$

Conjugating both sides of (2) by S yields: (2). LHS $= S(S'^2SH^+S)^3S^+$

$$= SS'^2SH^+\boxed{SS^+}SS'^2SH^+\boxed{SS^+}SS'^2SH^+\boxed{SS^+}$$

$$= (\boxed{SS'^2}SH^+)^3 \overset{C_5}{=\!=} (S'^2S^2H^+)^3$$

$$\overset{\text{def}}{=\!=} (H^2S^2H^2S^2H^+)^3 = -I =: (2).\ \text{RHS}.$$

Hence $(H^2S^2H^2S^2H^+)^3 \overset{\text{WTS}}{=\!=} -I \quad (3)$

Concatenating both sides of (3) by H^2 yields: (3). LHS $= H^2(H^2S^2H^2S^2H^+)^3H^2$

$$= \boxed{H^2H^2}S^2H^2S^2H^+H^2\boxed{H^2H^2}S^2H^2S^2H^+H^2\boxed{H^2H^2}S^2H^2S^2H^+H^2$$

$$= (S^2H^2S^2H)^3 = -I$$

Hence $-(S^2H^2S^2H)^3 \overset{\text{WTS}}{=\!=} I \quad (4)$

(4). LHS $= -S^2H^2\boxed{S^2HS^2}H^2\boxed{S^2HS^2}H^2S^2H$

$$\overset{C_4^1}{\underset{C_0}{=\!=}} -S^2H^2\boxed{H^3S}\boxed{H^3H^2H^3}S\boxed{H^3H^2}S^2H \cdot (-w) \cdot (-w)$$

$$\overset{C_2}{=\!=} -\boxed{S^2HS^2HS^2H} \cdot w^2$$

$$\overset{C_4}{\underset{C_0}{=\!=}} (-w^2)(-wI) = I = (4).\ \text{RHS}.$$

$\square$

C_0: $(-1)^2=1$ C_1: $\omega^3=1$ C_2: $H^4=I$ C_3: $S^3=I$

C_4: $(HS^2)^3=-\omega I$ $\quad -\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -\omega I$

C_4^1: $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}- = -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}- \cdot(-\omega)$

C_4^2: $-\boxed{S}-\boxed{H^+}-\boxed{S}- = -\boxed{H}-\boxed{S^2}-\boxed{H}- \cdot(-\omega^2)$

C_5: $SH^2SH^2=H^2SH^2S$ $\quad SS'=S'S$

$$-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- = -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$$

Def: $-\boxed{S'}- = -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$ $\quad -\boxed{S'}-\boxed{S'}- = -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-$

$S':=H^2SH^2$ $\quad S'^2=(H^2SH^2)(H^2SH^2)=H^2S^2H^2$

C_4^3: $-\boxed{S'}-\boxed{H^+}-\boxed{S'}- = -\boxed{H}-\boxed{S'}-\boxed{S'}-\boxed{H}- \cdot(-\omega^2)$

$R8$: $XZ=\omega^2 ZX$ $\quad -\boxed{Z}-\boxed{X}- = -\boxed{X}-\boxed{Z}- \cdot\omega^2$ $\quad R9$: $X^3=I$ $\quad R12$: $Z^3=I$

R_4^1: $-\boxed{H}-\boxed{Z}-\boxed{H^+}- = -\boxed{X}-$ $\quad R_4^2$: $-\boxed{H}-\boxed{Z}-\boxed{Z}-\boxed{H^+}- = -\boxed{X^2}-$

$R10$: $-\boxed{Z}- = -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- = -\boxed{S'}-\boxed{S'}-\boxed{S}-$

$R11$: $-\boxed{Z^2}- = -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}- = -\boxed{S'}-\boxed{S}-\boxed{S}-$

$R5$: $-\boxed{X}- = -\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-$

$R7$: $-\boxed{X^2}- = -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-$

Lem 21 By C_0, C_1, C_2, C_3, C_4 & C_5, R_5 and R_{10} imply R_6: $-\boxed{S}-\boxed{X}- = -\boxed{X}-\boxed{S}-\boxed{Z}- \cdot\omega^2$

Proof: By C_1, C_3, R_9 & R_{12}, it is sufficient to show

$$-\boxed{S}-\boxed{X}-\boxed{Z}-\boxed{Z}-\boxed{S}-\boxed{S}-\boxed{X}-\boxed{X}- \cdot\omega = I \quad (1) \quad ✓$$

(1). LHS $\overset{R_4^1}{\underset{R_4^2}{=}} -\boxed{S}-\boxed{H}-\boxed{Z}-\boxed{H^+}-\boxed{Z}-\boxed{Z}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{Z}-\boxed{Z}-\boxed{H^+}- \cdot\omega$ ✓

$\overset{R_{10}}{\underset{R_{11}}{=}} -\boxed{S}-\boxed{H}-\boxed{S'}-\boxed{S'}-\boxed{S}-\boxed{H^+}-\boxed{S'}-\boxed{S}-\boxed{S}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S'}-\boxed{S}-\boxed{S}-\boxed{H^+}- \cdot\omega$ ✓

$\overset{C_3}{\underset{C_5}{=}} -\boxed{S}-\boxed{H}-\boxed{S'}-\boxed{S'}-\boxed{S}-\boxed{H^+}-\boxed{S}-\boxed{S'}-\boxed{H}-\boxed{S'}-\boxed{S}-\boxed{S}-\boxed{H^+}- \cdot\omega$ ✓

$\overset{C_4^2}{\underset{C_0, C_1}{=}} -\boxed{S}-\boxed{H}-\boxed{S'}-\boxed{S'}-\boxed{H}-\boxed{S^2}-\boxed{H}-\boxed{S'}-\boxed{H}-\boxed{S'}-\boxed{S}-\boxed{S}-\boxed{H^+}- \cdot(-1)$ ✓

$\overset{def}{=} SH(H^2S^2H^2)HS^2H(H^2SH^2)H(H^2SH^2)S^2H^3\cdot(-1)$, compositions in diagrammatic order.

Hence, $SH(H^2S^2H^2)HS^2H(H^2SH^2)H(H^2SH^2)S^2H^3\cdot(-1) \overset{WTS}{=} I$ (2)

C_0: $(-1)^2 = 1$ C_1: $\omega^3 = 1$ C_2: $H^4 = I$ C_3: $S^3 = I$

C_4: $(HS^2)^3 = -\omega I$ $\quad -\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -\omega I$

C_4^1: $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}- = -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}- \cdot (-\omega)$

C_4^4: $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \cdot (-\omega^2) = -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-$

C_4^2: $-\boxed{S}-\boxed{H^+}-\boxed{S}- = -\boxed{H}-\boxed{S^2}-\boxed{H}- \cdot (-\omega^2)$

C_5: $SH^2SH^2 = H^2SH^2S$ $\quad SS' = S'S$

$-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- = -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$

Def: $-\boxed{S'}- = -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$ $\quad -\boxed{S'}-\boxed{S'}- = -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-$

$S' := H^2SH^2$ $\quad S'^2 = (H^2SH^2)(H^2SH^2) = H^2S^2H^2$

C_4^3: $-\boxed{S'}-\boxed{H^+}-\boxed{S'}- = -\boxed{H}-\boxed{S'}-\boxed{S'}-\boxed{H}- \cdot (-\omega^2)$

---

(2). LHS $= SH(H^2S^2H^2)HS^2H(H^2SH^2)H(H^2SH^2)S^2H^3 \cdot (-1)$ ★

$= SH^3S^2H^3S^2H^3SHSH^2S^2H^3 \cdot (-1)$

$= S(H^3S)S(H^3S)S(H^3S)HSH^2S^2H^3 \cdot (-1)$

$\overset{C_4^4}{\underset{C_0}{=}} S(S^2HS^2H)S(S^2HS^2H)S(S^2HS^2H)HSH^2S^2H^3 \cdot (-1) \cdot (-\omega^2)^3$

$\overset{C_1}{\underset{C_3}{=}} HS^2H\,HS^2H\,HS^2H\,HSH^2S^2H^3$

$\overset{\text{def}}{=} HS^2S'S'S^2S'S^2H^3$

$\overset{C_5}{=} H\boxed{S^2S^2S^2}S'S'S'H^3$

$\overset{C_3}{=} H\boxed{S'S'S'}H^3$

$\overset{\text{Lem G}}{=} H^4$

$\overset{C_2}{=} I =:$ (2). RHS

$\square$

$C2$: $H^4 = I$ $\qquad$ $C3$: $S^3 = I$

Def: $-\boxed{S'}- \;=\; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$

$S' := H^2 S H^2$

Lem G By C_2 & C_3, $S'^3 = I$.

Proof: $S'^3 = H^2 S \boxed{H^2 H^2} S \boxed{H^2 H^2} S H^2 \overset{C_2}{=\!=} H^2 S^3 H^2 \overset{C_3}{=\!=} H^2 H^2 \overset{C_2}{=\!=} I$ $\blacksquare$