

$$\text{Def 3: } \text{Diagram} := \text{Diagram} \quad C_2: H^4 = I \quad C_6: \text{Diagram} = \text{Diagram}$$

$$C_8: (1) \text{Diagram} = \text{Diagram} \quad R_{16}: \text{Diagram} = \text{Diagram} \quad R_{17}: \text{Diagram} = \text{Diagram}$$

$$R_{19}: (1) \text{Diagram} = \text{Diagram} \quad (2) \text{Diagram} = \text{Diagram} \quad R_{31}: (1) \text{Diagram} = \text{Diagram}$$

$$\text{Lem V} \quad R_{32}: \text{Diagram} = \text{Diagram} \quad (1) \text{ implies } R'_{32}: \text{Diagram} = \text{Diagram} \quad (4),$$

$$R_{32}^2: \text{Diagram} = \text{Diagram}, \quad R_{32}^3: \text{Diagram} = \text{Diagram}, \quad R_{32}^4: \text{Diagram} = \text{Diagram},$$

$$R_{32}^5: \text{Diagram} = \text{Diagram}, \quad R_{32}^6: \text{Diagram} = \text{Diagram},$$

$$R_{32}^7: \text{Diagram} = \text{Diagram} \quad \& \quad R_{32}^8: \text{Diagram} = \text{Diagram}$$

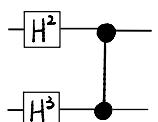
$$\text{Proof: } R'_{32}. \text{RHS} := \text{Diagram} \underset{\text{C}_2}{=} \text{Diagram} \underset{R_{32}}{=} \text{Diagram}$$

$$\underset{\text{C}_8}{=} \text{Diagram} \underset{\text{C}_2}{=} \text{Diagram} =: R'_{32}. \text{LHS}$$

$$R_{32}^2. \text{RHS} := \text{Diagram} \underset{\text{R}_{16}}{=} \text{Diagram} \underset{\text{R}_{19}}{=} \text{Diagram}$$

$$\underset{\text{R}_{32}^1}{=} \text{Diagram} \underset{\text{R}_{17}, \text{R}_{19}}{=} \text{Diagram} \underset{\text{R}_{16}}{=} \text{Diagram} =: R_{32}^2. \text{LHS}$$

Left-appending both sides of (1) by yields



$$\text{Diagram} = \text{Diagram} \underset{(2)}{=} \text{Diagram} \underset{C_6}{=} \text{Diagram} = \text{Diagram} \quad (3)$$

$$\underset{C_2}{=} \text{Diagram} = \text{Diagram} : R_{32}$$

$$R_{17}: \quad \text{Diagram} = \text{Diagram} \quad R_{16}: \quad \text{Diagram} = \text{Diagram} \quad C_6^1: \quad \text{Diagram} = \text{Diagram}$$

$$R_{19}: (1) \quad \text{Diagram} = \text{Diagram} \quad (2) \quad \text{Diagram} = \text{Diagram} \quad C_2: H^4 = I$$

$$R_{31}: (1) \quad \text{Diagram} = \text{Diagram} \quad (2) \quad \text{Diagram} = \text{Diagram}$$

Lem V $R_{32}^1:$ $\text{Diagram} = \text{Diagram}$ (1) implies $R_{32}^1:$ $\text{Diagram} = \text{Diagram}$ (4),

$$R_{32}^2: \quad \text{Diagram} = \text{Diagram}, \quad R_{32}^3: \quad \text{Diagram} = \text{Diagram}, \quad R_{32}^4: \quad \text{Diagram} = \text{Diagram},$$

$$R_{32}^5: \quad \text{Diagram} = \text{Diagram}, \quad R_{32}^6: \quad \text{Diagram} = \text{Diagram},$$

$$R_{32}^7: \quad \text{Diagram} = \text{Diagram} \quad \& \quad R_{32}^8: \quad \text{Diagram} = \text{Diagram}$$

Proof cont. $R_{32}^4 \cdot LHS :=$ $\text{Diagram} \xrightarrow{R_{16}} \text{Diagram} \xrightarrow{R_{17}} \text{Diagram}$

$$\text{Diagram} \xrightarrow{R_{32}^3} \text{Diagram} \xrightarrow{R_{31}} \text{Diagram} \xrightarrow{R_{16}} \text{Diagram} =: R_{32}^4 \cdot RHS$$

Right-appending both sides of (4) by Diagram yields

$$\text{Diagram} = \text{Diagram} \xrightleftharpoons{C_6^1} \text{Diagram} = \text{Diagram} : R_{32}^5$$

$$R_{32}^6 \cdot RHS := \text{Diagram} \xrightleftharpoons{R_{32}^4} \text{Diagram}$$

$$\xrightleftharpoons{C_2} \text{Diagram} \xrightleftharpoons{R_{32}^2} \text{Diagram} =: R_{32}^6 \cdot LHS$$

$$C_8^2: \quad \begin{array}{c} \text{H}^2 \\ \square \end{array} \quad \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \quad \begin{array}{c} \text{H}^1 \\ \square \end{array} = \quad \begin{array}{c} \bullet \\ \bullet \end{array} \quad C_2: \text{H}^4 = I$$

$$\text{Lem V } R_{32} : \begin{array}{c} \text{---} \\ | \quad | \quad | \end{array} \begin{array}{c} \bullet \quad \bullet \quad \square H^2 \\ \square H \quad \bigoplus \end{array} = \begin{array}{c} \text{---} \\ | \quad | \quad | \end{array} \begin{array}{c} \square H^2 \\ \square H \end{array} \quad (1) \text{ implies } R'_{32} : \begin{array}{c} \text{---} \\ | \quad | \quad | \end{array} \begin{array}{c} \bullet \quad \bullet \quad \bigoplus \\ \square H \end{array} = \begin{array}{c} \text{---} \\ | \quad | \quad | \end{array} \begin{array}{c} \square H \\ \square H \end{array} \quad (4),$$

$$R_{32}^2 : \quad \text{Circuit Diagram} = \quad \boxed{H} \quad , \quad R_{32}^3 : \quad \text{Circuit Diagram} = \quad \text{Circuit Diagram} \quad , \quad R_{32}^4 : \quad \text{Circuit Diagram} = \quad \text{Circuit Diagram}$$

$$R_{32}^5 : \quad \text{Diagram showing } R_{32}^5 \text{ as a sequence of operations: } H \text{ (vertical), } H \text{ (horizontal), } \oplus \text{ (circle), } H \text{ (horizontal), } H \text{ (vertical).} \\ = \quad \text{Diagram showing the result of } R_{32}^5 \text{ as a sequence of operations: } H \text{ (horizontal), } \oplus \text{ (circle), } H^2 \text{ (square), } H^2 \text{ (square).} \\ R_{32}^6 : \quad \text{Diagram showing } R_{32}^6 \text{ as a sequence of operations: } H \text{ (vertical), } H \text{ (horizontal), } \oplus \text{ (circle), } H \text{ (horizontal), } H \text{ (vertical).} \\ = \quad \text{Diagram showing the result of } R_{32}^6 \text{ as a sequence of operations: } H \text{ (horizontal), } \oplus \text{ (circle), } H^2 \text{ (square), } H^2 \text{ (square).}$$

$$R_{32}^7 : \quad \begin{array}{c} \text{Diagram showing two qubits 3 and 2 coupled by an } H \text{ gate.} \\ \text{Qubit 3 is on top, Qubit 2 is on bottom.} \\ \text{The } H \text{ gate is placed between them.} \end{array} = \quad \begin{array}{c} \text{Diagram showing two qubits 3 and 2 coupled by an } H^2 \text{ gate.} \\ \text{Qubit 3 is on top, Qubit 2 is on bottom.} \\ \text{The } H^2 \text{ gate is placed between them.} \end{array}$$

Proof cont. R_{32}^7 . RHS :=

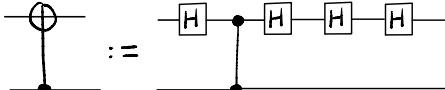
$$\stackrel{C_2}{=} \begin{array}{c} \text{Diagram showing two horizontal lines with nodes. The top line has three nodes connected by two edges. The bottom line has two nodes connected by one edge. A blue box encloses the top line's nodes. Two boxes labeled } H^2 \text{ are placed above the top line's nodes. One box labeled } H \text{ is placed below the bottom line's left node.} \end{array} \stackrel{C_8^2}{=} \begin{array}{c} \text{Diagram showing two horizontal lines with nodes. The top line has two nodes connected by one edge. The bottom line has two nodes connected by one edge. A box labeled } H \text{ is placed below the bottom line's left node.} \end{array} =: R_{32}^7 \cdot LHS$$

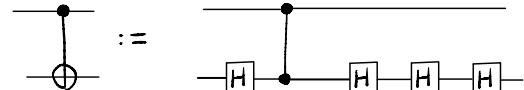
Right-appending both sides of R_{32}^2 by  yields

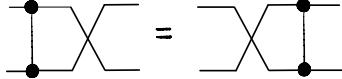
III C₆²

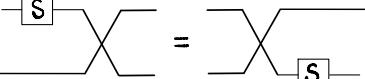
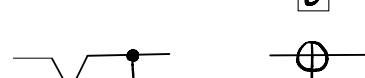
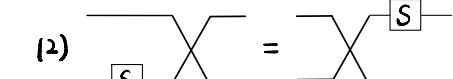
$$\begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \boxed{H} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \bullet \\ | \\ \bullet \end{array} : R_{32}^8$$

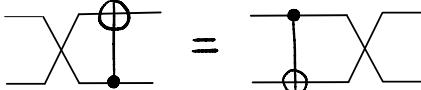
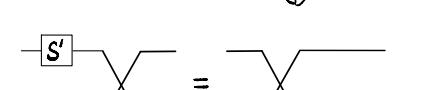
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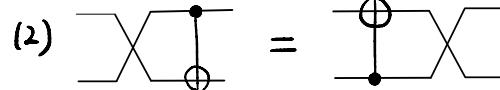
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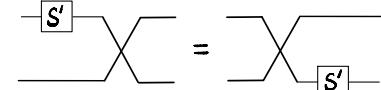
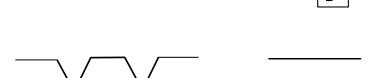
 := 

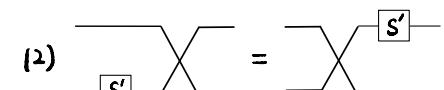
R₁₇:  = 

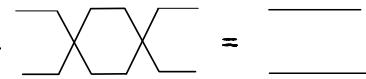
R₁₈: (1)  =  (2)  = 

R₃₁: (1)  = 

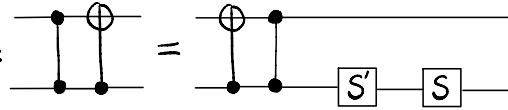
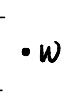
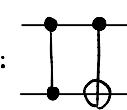
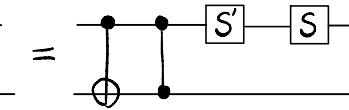
(2)  = 

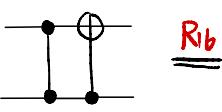
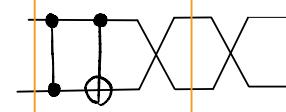
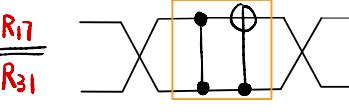
R₂₂: (1)  = 

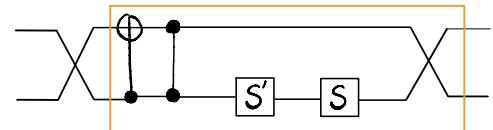
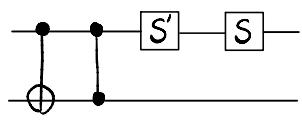
(2)  = 

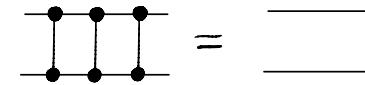
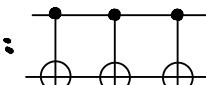
R₁₆:  = 

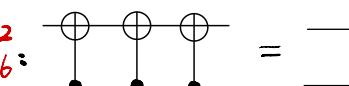
C₂: $H^4 = I$

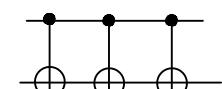
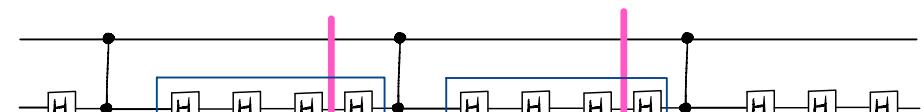
Lem W **R₂₅¹:**  =  $\cdot w$ implies **R₂₅²:**  =  $\cdot w$

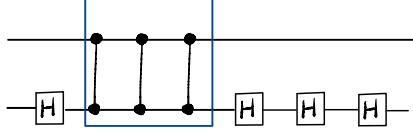
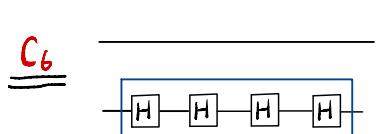
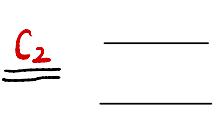
Proof: **R₂₅².RHS** :=  $\stackrel{R_{16}}{=}$  $\stackrel{R_{17} \quad R_{18}}{=} \overline{\overline{R_{22}, R_{31}}}$ 

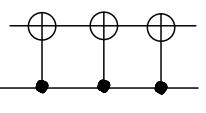
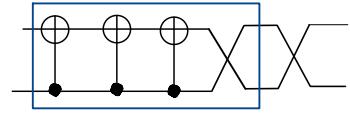
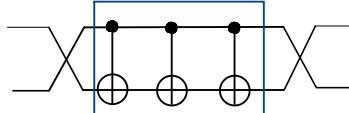
R₂₅¹  $\cdot w$ $\stackrel{R_{17}, R_{18}}{=} \overline{\overline{R_{22}, R_{31}}}$  $\cdot w =: R_{25}^2.LHS$ 

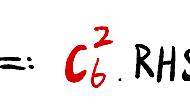
Lem X **C₆**:  =  implies **C₆¹**:  =  &

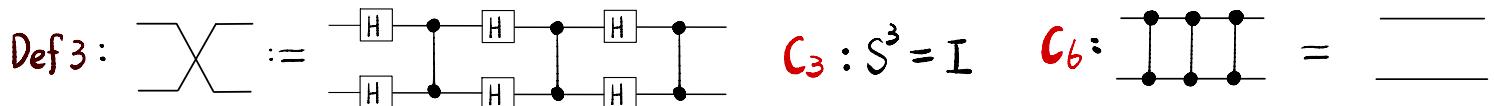
C₆²:  = 

Proof: **C₆¹.LHS** :=  $\stackrel{\text{Def 2}}{=}$ 

C₂  $\stackrel{C_6}{=}$  $\stackrel{C_2}{=}$  $=: C_6^1.RHS$

C₆².LHS :=  $\stackrel{R_{16}}{=}$  $\stackrel{R_{31}}{=}$ 

C₆¹  $\stackrel{R_{16}}{=}$  $=: C_6^2.RHS$ 



Def: $S' := \boxed{H} \boxed{H} \boxed{S} \boxed{H} \boxed{H}$ $S' := H^2 S H^2$ $C_1 : w^3 = I$

$$S'^2 - (H^2 S H^2)(H^2 S H^2) = H^2 S^2 H^2$$

$C_7 : (2)$

$R_{15} : (2)$

$C_2 : H^4 = I$
 $C_5 : S S' = S' S$

$R_{32} :$

$R_{32}^1 :$

$R_{32}^2 :$

$R_{32}^3 :$

$R_{32}^4 :$

$R_{32}^5 :$

$R_{32}^6 :$

$R_{32}^7 :$

$R_{25}^1 :$

$\cdot w$

Lem Y C_8 , R_{25} & R_{32} imply R_{33} :

$\cdot w$

Proof: $R_{33} \text{ RHS} :=$

$\cdot w$

$\stackrel{R_{25}^1}{=} \quad$

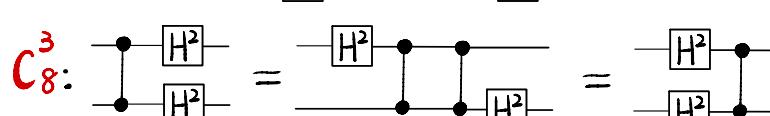
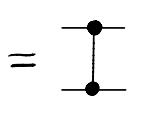
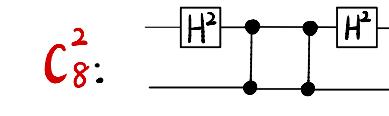
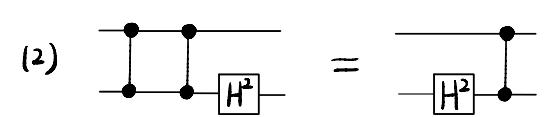
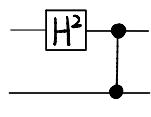
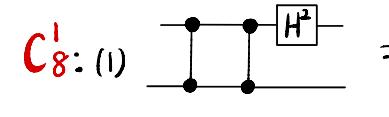
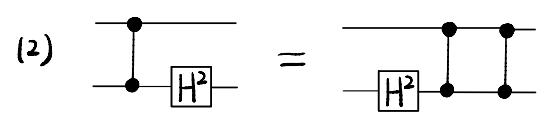
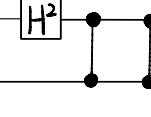
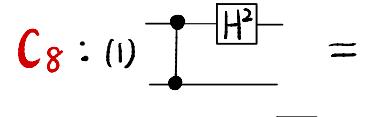
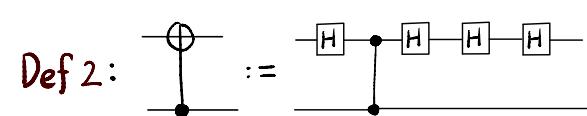
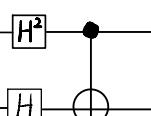
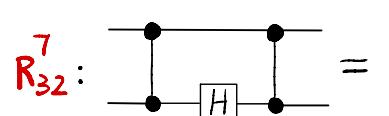
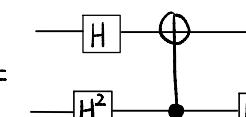
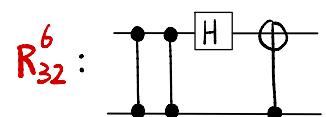
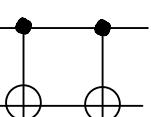
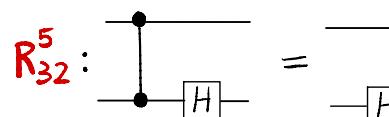
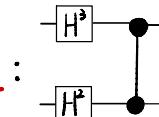
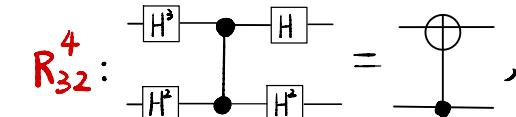
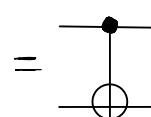
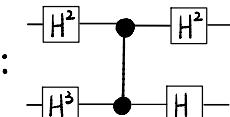
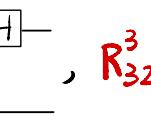
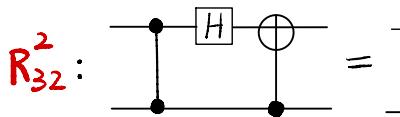
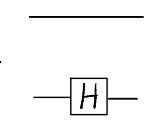
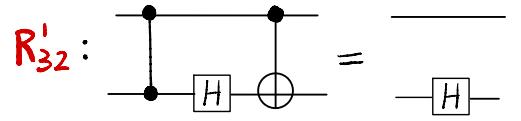
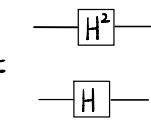
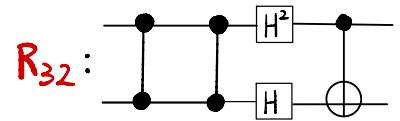
$\cdot w^2$

$\stackrel{C_1}{=} \quad$

$\stackrel{R_{15}}{\underline{\underline{=}}} \quad$

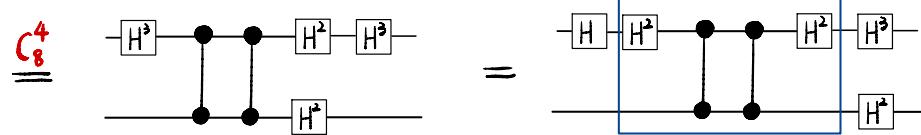
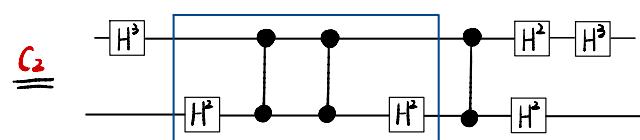
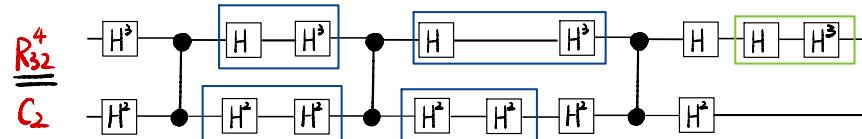
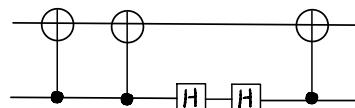
def

$\stackrel{C_2}{=} \quad$

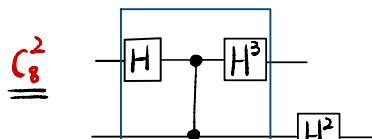
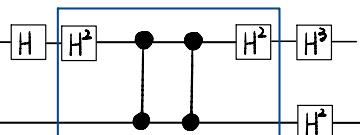


$C_2: H^4 = I$

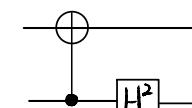
Proof cont. $R_{33}.LHS :=$



$=$

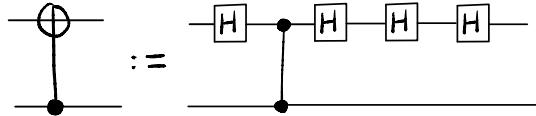
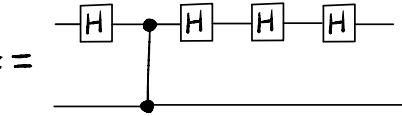


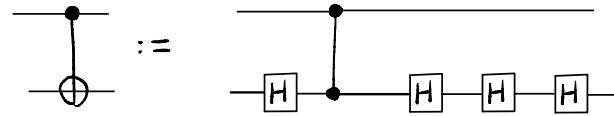
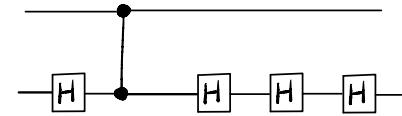
$\underline{\underline{Def 2}}$

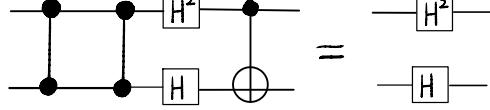
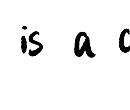


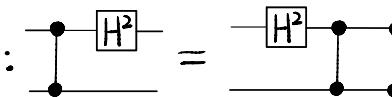
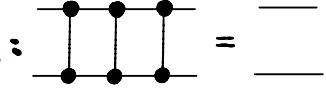
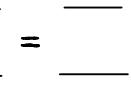
$=: R_{33}.RHS$

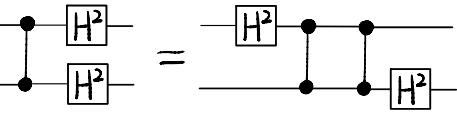
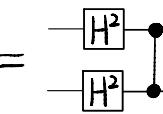
□

Def 2:  := 

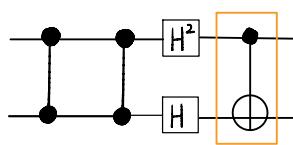
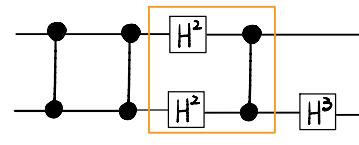
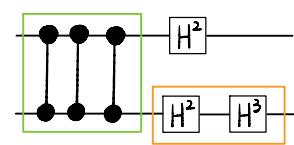
 := 

Lem \nexists R_{32} :  =  is a consequence of

C_8 :  =  , C_2 : $H^4 = I$ & C_6 :  = 

Proof: By Lem K, C_8^3 :  =  = is a consequence of C_8 .

Hence, it is sufficient to show R_{32} is a consequence of C_8^3 .

$R_{32} \cdot \text{LHS} :=$  $\underset{\text{Def 2}}{=}$  $\underset{C_8^3}{=}$ 

$\frac{C_2}{C_6} \underset{\text{Def 2}}{=} \underset{C_8^3}{=} R_{32} \cdot \text{RHS}$

✓