C0: $(-1)^2 = 1$ C1: $w^3 = 1$

C3: $S^3 = I$

R9: $X^3 = I$

C2: $H^4 = I$

---

**Lemma A** $R_1'$: $-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{X}-\boxed{S}- \cdot (-w)$

is a consequence of R1: $-\boxed{H}-\boxed{S}-\boxed{H}- = -\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S^2}-\boxed{X^2}- \cdot (-w^2)$

Proof: Concatenating both sides of $R_1$ by H to the left:

$$-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}- = -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S^2}-\boxed{X^2}- \cdot (-w^2)$$

Concatenating both sides of $R_1$ by X;S to the right:

$$-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{X}-\boxed{S}- = -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S^2}-\boxed{X^2}-\boxed{X}-\boxed{S}- \cdot (-w^2)$$

By C3 & R9, $-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{X}-\boxed{S}- = -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \cdot (-w^2)$

Multiplying both sides of $R_1$ by $-w$:

$$-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{X}-\boxed{S}- \cdot (-w) = -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \cdot (-w^2)(-w)$$

By C0 & $C_1$, $-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{X}-\boxed{S}- \cdot (-w) = -\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \equiv R_1'$. ◪

**Lemma B** $C_5'$: $-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- = -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$ is a consequence of $C_5$: $-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- = -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$

Proof: Concatenating both sides of $C_5$ by $H^2$ to the left:

$$-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- = -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$$

By $C_2$, $-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}- = -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}- \equiv C_5'$.

◪

$C2$: $H^4 = I$

$C3$: $S^3 = I$

---

**Lemma C** $C_4'$: $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}- \;=\; -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}- \cdot (-\omega)$

is a consequence of $C_4$: $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -\omega I$

Proof: Concatenating both sides of $C_4$ by $-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-$ to the right.

$$C_4.\mathrm{LHS} = -\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}-}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-$$

$$\overset{C_2}{=} -\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{-\boxed{S}-\boxed{S}-\boxed{S}-}-\boxed{H}-\boxed{H}-\boxed{H}-$$

$$\overset{C_3}{=} -\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}-}$$

$$\overset{C_2}{=} -\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-$$

$$C_4.\mathrm{RHS} = -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}- \cdot (-\omega)$$

$\Big\} \equiv C_4'$

$\square$

$C_0$: $(-1)^2 = 1$ $C_1$: $w^3 = 1$ $C_2$: $H^4 = I$ $C_3$: $S^3 = I$

$C_4$: $(HS^2)^3 = -wI$

$$-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -wI$$

$C_4'$: $$-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}- = -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}- \cdot(-w)$$

---

**Lem D** By $C_0$, $C_1$ & $C_3$, $C_4^2$: $-\boxed{S}-\boxed{H^\dagger}-\boxed{S}- = -\boxed{H}-\boxed{S^2}-\boxed{H}- \cdot(-w^2)$

is a consequence of $C_4$: $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -wI$

Proof: By Lem C, $C_4'$: $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}- = -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}- \cdot(-w)$

is a consequence of $C_4$: $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -wI$

By $C_3$, concatenating both sides of $C_4'$ by $-\boxed{S}-$ to the right yields

$C_4'$: $$-\boxed{S}-\boxed{S}-\boxed{H}- = -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}- \cdot(-w)$$

By $C_3$, concatenating both sides of $C_4'$ by $-\boxed{S}-$ to the left yields

$C_4'$: $$-\boxed{H}- = -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}- \cdot(-w)$$

By $C_2$ & $C_3$, concatenating both sides of $C_4'$ by $-\boxed{S}-\boxed{S}-\boxed{H}-$ to the right yields

$$-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{-\boxed{S}-\boxed{S}-\boxed{S}-}-\boxed{H}-}$$

$$= -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}- \cdot(-w) \quad (1)$$

By $C_0$ & $C_1$, multiplying both sides of (1) by $(-w^2)$ yields

$$-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \cdot(-w^2) = -\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}- \cdot(-w)\cdot(-w^2)$$

It follows that $C_4^2$: $-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \cdot(-w^2) = -\boxed{S}-\boxed{H^\dagger}-\boxed{S}-$

$\blacksquare$

$C_0$: $(-1)^2 = 1$ $\quad$ $C1$: $w^3 = 1$ $\quad$ $C2$: $H^4 = I$ $\quad$ $C3$: $S^3 = I$

$C_4$ : $(HS^2)^3 = -wI$

$$-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -wI$$

$C_4^1$ : $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}- \;=\; -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}- \cdot (-w)$

$C_4^2$ : $-\boxed{S}-\boxed{H^\dagger}-\boxed{S}- \;=\; -\boxed{H}-\boxed{S^2}-\boxed{H}- \cdot (-w^2)$

Def : $-\boxed{S'}- = -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$ $\quad$ $-\boxed{S'}-\boxed{S'}- = -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{H}-$

$S' := H^2 S H^2$ $\quad$ $S'^2 = (H^2SH^2)(H^2SH^2) = H^2S^2H^2$

---

**Lem E** By $C_0$, $C_1$, $C_2$ & $C_3$, $C_4^3$ : $-\boxed{S'}-\boxed{H^\dagger}-\boxed{S'}- \;=\; -\boxed{H}-\boxed{S'}-\boxed{S'}-\boxed{H}- \cdot (-w^2)$

is a consequence of $C_4$ : $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -wI$

Proof : $C_4^3$. LHS $= -\boxed{S'}-\boxed{H^\dagger}-\boxed{S'}-$

$$\overset{\text{def}}{=\!=} -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$$

$$\overset{C_2}{=\!=} -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$$

$$\overset{C_4^2}{=\!=} -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S^2}-\boxed{H}-\boxed{H}-\boxed{H}- \cdot (-w^2)$$

$$\overset{\text{def}}{=\!=} -\boxed{H}-\boxed{S'}-\boxed{S'}-\boxed{H}- \cdot (-w^2) = C_4^3 . \text{RHS}$$

□

**Lem F** By $C_0$, $C_1$ & $C_2$, $C_4^4$ : $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \cdot (-w^2) \;=\; -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-$

is a consequence of $C_4$ : $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- = -wI$

Proof : By Lem C, $C_4^1$ : $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}- \;=\; -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{H}- \cdot (-w)$

By $C_2$, concatenating both sides of the equation by $-\boxed{H}-$ yields:

$$-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \;=\; -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}- \cdot (-w)$$

By $C_0$ & $C_1$, multiplying both sides of the equation by $(-w^2)$ yields:

$$-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \cdot (-w^2) \;=\; -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}- \;\equiv\; C_4^4$$

□

C2: $H^4 = I$ C3: $S^3 = I$

Def: $-\boxed{S'}- \;=\; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$

$S' := H^2 S H^2$

---

**Lem G** By $C_2$ & $C_3$, $S'^3 = I$.

Proof: $S'^3 = H^2 S \boxed{H^2 H^2} S \boxed{H^2 H^2} S H^2 \overset{C_2}{=\!=} H^2 S^3 H^2 \overset{C_3}{=\!=} H^2 H^2 \overset{C_2}{=\!=} I$ $\blacksquare$