

$$R_{23}^1: \quad \begin{array}{c} S \\ \oplus \\ \bullet \end{array} = \begin{array}{c} \oplus \\ \bullet \end{array} \cdot w^2$$

$$\text{Def 2: } \begin{array}{c} \oplus \\ \bullet \end{array} := \begin{array}{c} H \\ H \\ H \\ H \end{array}$$

$$R_{24}: \quad \begin{array}{c} \oplus \\ \bullet \end{array} = \begin{array}{c} \oplus \\ \bullet \end{array} \cdot S$$

$$\text{Def 1: } \begin{array}{c} S' \end{array} := \begin{array}{c} H \\ H \\ S \\ H \\ H \end{array}$$

$$C_8^6: \quad \begin{array}{c} H^2 \\ \oplus \\ \bullet \end{array} = \begin{array}{c} \oplus \\ \bullet \end{array} \cdot \begin{array}{c} H^2 \\ \oplus \\ \bullet \end{array}$$

$$C_8^1: \quad \begin{array}{c} H^2 \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \bullet \end{array} \cdot \begin{array}{c} H^2 \\ \bullet \end{array}$$

$$R_{15}: \quad \begin{array}{c} \bullet \\ \bullet \end{array} = \begin{array}{c} S' \end{array}$$

Proof cont.

$$R_{35}: \quad \begin{array}{c} H \\ \bullet \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} S^2 \\ S \\ H \\ \bullet \\ S^2 \\ S \\ H^2 \\ S^2 \\ H \\ S \\ S^2 \\ S \\ H^3 \\ S \\ H^2 \\ S \\ H^3 \\ S \\ H^2 \\ S \\ H \end{array} \cdot (-1)$$

$$R_{35}, \text{ RHS} := \begin{array}{c} S^2 \\ S \\ H \\ \bullet \\ S^2 \\ S \\ H^3 \\ \bullet \\ S^2 \\ H^3 \\ S \\ H^2 \\ S \\ H^3 \\ S \\ H^2 \\ S \\ H \end{array} \cdot (-1)$$

$$\text{Def 1} \quad \begin{array}{c} S^2 \\ S \\ H \\ \bullet \\ H^2 \\ S^2 \\ H^2 \\ H \\ S \\ H^2 \\ S^2 \\ H \\ S \\ S^2 \\ S \\ H^3 \\ S \\ H^2 \\ S \\ H \end{array} \cdot (-1)$$

$$C_2 \quad \begin{array}{c} S^2 \\ S \\ H \\ \bullet \\ H^3 \\ H^3 \\ S^2 \\ H^3 \\ S \\ H^2 \\ S^2 \\ H \\ S \\ S^2 \\ S \\ H^3 \\ S \\ H^2 \\ S \\ H \end{array} \cdot (-1)$$

$$\text{Def 2} \quad \begin{array}{c} S^2 \\ S \\ \oplus \\ \bullet \\ H^3 \\ S^2 \\ H^3 \\ S^2 \\ H^3 \\ S \\ H^2 \\ S^2 \\ H \\ S \\ R_{24} \end{array} \cdot (-1)$$

$$R_{23}^1 \quad \begin{array}{c} S^2 \\ \oplus \\ \bullet \\ S' \\ S^2 \\ H^3 \\ S^2 \\ H^3 \\ S \\ H^2 \\ S^2 \\ H \\ S \end{array} \cdot (-1) \cdot w^2$$

$$\text{Def 1} \quad \begin{array}{c} H^2 \\ S^2 \\ H^2 \\ \oplus \\ \bullet \\ S \\ H^3 \\ S^2 \\ H^3 \\ S \\ H^2 \\ S^2 \\ H \\ S \\ C_2 \end{array} \cdot (-w^2)$$

$$C_8^1 \quad \begin{array}{c} H^2 \\ S^2 \\ \oplus \\ \bullet \\ H^2 \\ S \\ H^3 \\ S^2 \\ H^3 \\ S \\ H^2 \\ S^2 \\ H \\ S \\ C_8^6 \end{array} \cdot (-w^2)$$

$$R_{23}^1, R_{23}^1 \quad \begin{array}{c} H^2 \\ S \\ \oplus \\ \bullet \\ S' \\ S^2 \\ H^3 \\ S^2 \\ H^3 \\ S \\ H^2 \\ S^2 \\ H \\ S \\ C_1, R_5 \end{array} \cdot (-w^2) \cdot w^2$$

$$\cdot (-w^2) \cdot w^2 \cdot w^2$$

$$R_{23}^1 \quad \begin{array}{c} H^2 \\ S \\ \oplus \\ \bullet \\ S' \\ S^2 \\ H^3 \\ S^2 \\ H^3 \\ S \\ H^2 \\ S^2 \\ H \\ S \end{array} \cdot (-1) \quad || C_1$$

$$C_7: \quad \text{Diagram} = \text{Diagram}$$

$$R_{15}: \quad \text{Diagram} = \text{Diagram}$$

$$C_6: \quad \text{Diagram} = \text{Diagram}$$

$$\text{Def 1: } S' := H \quad H \quad S \quad H \quad H$$

$$C_1: w^3 = I \quad C_2: H^4 = I \quad C_3: S^3 = I \quad C_5: SS' = S'S$$

$$R_{23}^1: \quad \text{Diagram} = \text{Diagram} \cdot w^2$$

$$R_{25}^2: \quad \text{Diagram} \cdot w^2 = \text{Diagram}$$

$$C_8^5: \quad \text{Diagram} = \text{Diagram}$$

$$\text{Def 2: } \text{Diagram} := \text{Diagram}$$

Proof cont.

$$\begin{array}{c} \text{Diagram} \\ \cdot (-1) \end{array}$$

$$\begin{array}{c} C_7 \\ \equiv \\ R_{15} \end{array} \quad \begin{array}{c} \text{Diagram} \\ \cdot (-1) \end{array}$$

$$\begin{array}{c} \text{Def 1} \\ \equiv \\ C_2, C_3, C_6 \end{array} \quad \begin{array}{c} \text{Diagram} \\ \cdot (-1) \end{array}$$

$$\begin{array}{c} R_{23}^1 \\ \equiv \\ \text{Diagram} \end{array} \quad \begin{array}{c} \cdot (-1) \cdot w^2 \end{array}$$

$$\begin{array}{c} C_7 \\ \equiv \\ R_{15} \end{array} \quad \begin{array}{c} \text{Diagram} \\ \cdot (-1) \cdot w^2 \end{array}$$

$$\begin{array}{c} R_{23}^1 \\ \equiv \\ \text{Diagram} \end{array} \quad \begin{array}{c} \cdot (-1) \cdot w^2 \cdot w^2 \end{array}$$

$$\begin{array}{c} R_{25}^2 \\ \equiv \\ \text{Diagram} \end{array} \quad \begin{array}{c} \cdot (-1) \cdot w^2 \cdot w^2 \cdot w^2 \end{array}$$

$$\begin{array}{c} \text{Def 1, } C_5 \\ \equiv \\ C_1, C_2, C_3, C_6 \end{array} \quad \begin{array}{c} \text{Diagram} \\ \cdot (-1) \end{array}$$

$$\begin{array}{c} C_8^5 \\ \equiv \\ \text{Diagram} \end{array} \quad \begin{array}{c} \cdot (-1) \end{array}$$

$$\begin{array}{c} \text{Def 2} \\ \equiv \\ \text{Diagram} \end{array} \quad \begin{array}{c} \cdot (-1) \end{array}$$

$$\begin{aligned}
 C_4^4 : & \quad S \quad S \quad H \quad S \quad S \quad H \quad \cdot (-w^2) = H \quad H \quad H \quad S \quad \quad C_0: (-1)^2 = 1 \\
 C_1: w^3 = 1 & \quad C_2: H^4 = I \quad C_3: S^3 = I \quad C_5: SS' = S'S \quad \text{Def 1: } S' := H \quad H \quad S \quad H \quad H \\
 C_4^1: & \quad S \quad S \quad H \quad S \quad S \quad = H \quad H \quad H \quad S \quad H \quad H \quad H \quad \cdot (-w) \\
 C_6: & \quad \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = S^2 H S^2 H \cdot (-w^2) = H^3 S
 \end{aligned}$$

Proof cont.

$$\begin{aligned}
 \text{Hence, } R_{35}: & \quad \begin{array}{c} H \\ | \\ \bullet \\ | \\ \bullet \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} H \\ | \\ H^3 \\ | \\ \bullet \\ | \\ H^2 \\ | \\ S^2 \\ | \\ H^2 \\ | \\ S \\ | \\ H^3 \\ | \\ S^2 \\ | \\ H \\ | \\ S \end{array} \cdot (-1) \\
 & \quad S \quad S' \quad H^3 \quad S \quad H^2 \quad S \quad H^3 \quad S \quad H^2 \quad S \quad H \quad S \\
 C_2 \parallel C_6 & \\
 R_{35}: & \quad \begin{array}{c} H^3 \\ | \\ H^2 \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} S^2 \\ | \\ H^2 \\ | \\ S \\ | \\ H^3 \\ | \\ S^2 \\ | \\ H^3 \\ | \\ S \\ | \\ H^2 \\ | \\ S^2 \\ | \\ H \\ | \\ S \end{array} \cdot (-1) \\
 & \quad S \quad S' \quad H^3 \quad S \quad H^2 \quad S \quad H^3 \quad S \quad H^2 \quad S \quad H \\
 C_2 \parallel & \\
 R_{35}: & \quad \begin{array}{c} H \\ | \\ S^2 \\ | \\ H^2 \\ | \\ S \\ | \\ H^3 \\ | \\ S^2 \\ | \\ H^3 \\ | \\ S \\ | \\ H^2 \\ | \\ S^2 \\ | \\ H \\ | \\ S \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} \text{top} \\ \uparrow \\ H \\ | \\ S^2 \\ | \\ H^2 \\ | \\ S \\ | \\ H^3 \\ | \\ S^2 \\ | \\ H^3 \\ | \\ S \\ | \\ H^2 \\ | \\ S^2 \\ | \\ H \\ | \\ S \end{array} \cdot (-1) \\
 & \quad S \quad S' \quad H^3 \quad S \quad H^2 \quad S \quad H^3 \quad S \quad H^2 \quad S \quad H \quad \downarrow \text{bottom}
 \end{aligned}$$

To show the above, we show that top \otimes bottom = $I \otimes I$.

$$\begin{aligned}
 \text{top} := HS^2H^2SH^3S^2H^3SH^2S^2HS \cdot (-1) & \stackrel{C_2}{=} HS^2H^2SH^3S^2H^3S \boxed{H^2S^2H^2H^3S} \cdot (-1) \\
 & \stackrel{\text{Def 1}}{=} HS^2H^2SH^3S^2H^3SS'^2 \boxed{H^3S} \cdot (-1) \stackrel{C_4^4}{=} HS^2H^2SH^3S^2H^3 \boxed{SS'^2S^2} HS^2H \cdot (-w^2) \cdot (-1) \\
 \stackrel{C_0, C_3}{\underline{C_5}} & HS^2H^2SH^3S^2H^3 \boxed{S'^2} HS^2H \cdot w^2 \stackrel{\text{Def 1}}{=} HS^2H^2SH^3S^2H^3 \boxed{H^2H^2} S^2H \cdot w^2 \\
 & \stackrel{C_2}{=} HS^2H^2SH^3 \boxed{S^2} HS^2H^3S^2H \cdot w^2 \stackrel{C_4^1}{=} HS^2H^2SH^3 \boxed{H^3} S \boxed{H^3H^3} S^2H \cdot w^2 \cdot (-w) \\
 & \stackrel{C_2}{=} HS^2 \boxed{H^2S} \boxed{H^2} S H^2S^2H \cdot (-1) \stackrel{\text{Def 1}}{=} HS^2 \boxed{S'S} H^2S^2H \cdot (-1) \stackrel{C_5}{=} H \boxed{S^3} S' H^2S^2H \cdot (-1) \\
 & \stackrel{C_3}{=} H \boxed{S'} H^2S^2H \cdot (-1) \stackrel{\text{Def 1}}{=} H H^2S \boxed{H^2H^2} S^2H \cdot (-1) \stackrel{C_2}{=} H H^2 \boxed{SS^2} H \cdot (-1) \stackrel{C_3}{=} H^4 \cdot (-1) \\
 & \stackrel{C_2}{=} -I.
 \end{aligned}$$

$$\begin{aligned}
 C_4^4 : & \quad \boxed{S} \quad \boxed{S} \quad \boxed{H} \quad \boxed{S} \quad \boxed{S} \quad \boxed{H} \quad \cdot (-w^2) = \boxed{H} \quad \boxed{H} \quad \boxed{H} \quad \boxed{S} \quad S^2 H S^2 H \cdot (-w^2) = H^3 S \\
 C_1 : w^3 = I & \quad C_2 : H^4 = I \quad C_3 : S^3 = I \quad C_5 : SS' = S'S \quad \text{Def 1: } \boxed{S'} := \boxed{H} \quad \boxed{H} \quad \boxed{S} \quad \boxed{H} \quad \boxed{H} \\
 C_4^1 : & \quad \boxed{S} \quad \boxed{S} \quad \boxed{H} \quad \boxed{S} \quad \boxed{S} = \boxed{H} \quad \boxed{H} \quad \boxed{H} \quad \boxed{S} \quad \boxed{H} \quad \boxed{H} \quad \cdot (-w) \quad C_0 : (-I)^2 = I \\
 & \quad S^2 H S^2 \cdot (-w^2) = H^3 S H^3
 \end{aligned}$$

Proof cont.

$$\begin{aligned}
 R_{35}: & \quad \underline{\hspace{2cm}} \quad \underline{\hspace{2cm}} \quad \text{WTS} \quad \begin{array}{c} \boxed{H} \quad \boxed{S^2} \quad \boxed{H^2} \quad \boxed{S} \quad \boxed{H^3} \quad \boxed{S^2} \quad \boxed{H^3} \quad \boxed{S} \quad \boxed{H^2} \quad \boxed{S^2} \quad \boxed{H} \quad \boxed{S} \\ \text{top} \end{array} \cdot (-1) \\
 & \quad \underline{\hspace{2cm}} \quad \underline{\hspace{2cm}} \quad \begin{array}{c} \boxed{S} \quad \boxed{S'} \quad \boxed{H^3} \quad \boxed{S} \quad \boxed{H^2} \quad \boxed{S} \quad \boxed{H^3} \quad \boxed{S} \quad \boxed{H^2} \quad \boxed{S} \quad \boxed{H} \\ \text{bottom} \end{array}
 \end{aligned}$$

$$\text{bottom} := SS' H^3 S H^2 S \boxed{H^3 S} H^2 S H \stackrel{C_4^4}{=} SS' H^3 S H^2 \boxed{SS^2} H S^2 H H^2 S H \cdot (-w^2)$$

$$\stackrel{C_3}{=} SS' \boxed{H^3 S H^2 H} S^2 H H^2 S H \cdot (-w^2) \stackrel{C_4^1}{=} \boxed{SS' S^2 H} \boxed{S^2 S^2} H H^2 S H \cdot (-w^2) \cdot (-w^2)$$

$$\frac{C_0, C_1}{C_3, C_5} \boxed{S'} H S H^3 S H \cdot w \stackrel{\text{Def 1}}{=} H^2 S \boxed{H^2 H S H^3} S H \cdot w$$

$$\stackrel{C_4^1}{=} H^2 S \boxed{(S^2 H} \boxed{S^2) S} H \boxed{\cdot w \cdot (-w^2)} \stackrel{\frac{C_0, C_1}{C_3}}{=} \boxed{H^2 H H} \cdot (-1) \stackrel{C_2}{=} -I.$$

$$\text{Hence, top} \otimes \text{bottom} := (-I) \otimes (-I) = (-1)^2 I \otimes I \stackrel{C_0}{=} I \otimes I.$$

This completes the proof.



$$\begin{array}{l}
 R_{15}: \quad \text{Diagram} = \text{Diagram} \\
 C_1: w^3 = I \quad C_2: H^4 = I \quad C_3: S^3 = I \\
 C_7: (1) \quad \text{Diagram} = \text{Diagram} \quad (2) \quad \text{Diagram} = \text{Diagram} \\
 \text{Def 2:} \quad \text{Diagram} := \text{Diagram} \quad R_{23}^1: \quad \text{Diagram} = \text{Diagram} \cdot w^2
 \end{array}$$

Lem \exists Def 1-2, C1-8, R5, R11, R13, R14, R15, R23, R24, R25 imply

$$\begin{array}{l}
 R_{14}: \quad \text{Diagram} = \text{Diagram} \\
 \equiv R_{14}^1: \quad \text{Diagram} = \text{Diagram}
 \end{array}$$

$$\text{Proof: } R_{14} \cdot \text{RHS} := \text{Diagram} \xrightarrow{\substack{C_7 \\ R_{15}}} \text{Diagram}$$

$$\text{Hence, } R_{14}: \quad \text{Diagram} \xrightarrow{\text{WTS}} \text{Diagram}$$

C2 ||

$$R_{14}: \quad \text{Diagram} \xrightarrow{\text{WTS}} \text{Diagram}$$

C2 || Def 2

$$R_{14}: \quad \text{Diagram} \xrightarrow{\text{WTS}} \text{Diagram}$$

$$\text{Then } R_{14} \cdot \text{RHS} := \text{Diagram}$$

$$\xrightarrow{R_{23}^1} \text{Diagram} \cdot w^2$$

$$\begin{array}{ll}
R_{23}^1: \quad \text{Diagram} = \text{Diagram} \cdot w^2 & C_7: \quad \text{Diagram} = \text{Diagram} \\
R_{15}: \quad \text{Diagram} = \text{Diagram} & C_8^6: \quad \text{Diagram} = \text{Diagram} \\
C_8: \quad \text{Diagram} = \text{Diagram} & R_{25}^1: \quad \text{Diagram} = \text{Diagram} \cdot w \\
\text{Def 1: } [S'] := [H] [H] [S] [H] [H] & R_{24}^1: \quad \text{Diagram} = \text{Diagram} \\
& C_6: \quad \text{Diagram} = \text{Diagram}
\end{array}$$

Proof cont. R_{14} : $\text{Diagram} \underset{\text{WTS}}{=} \text{Diagram}$

$$R_{14} \cdot \text{RHS} := \text{Diagram} \cdot w^2$$

$$\underline{R_{23}^1} \quad \text{Diagram} \cdot w^2 \cdot w^2$$

$$\frac{C_1, G_7}{R_{15}} \quad \text{Diagram} \cdot w$$

$$\frac{C_8^6, C_8}{\text{Def 1, } G_2, G_3} \quad \text{Diagram} \cdot w$$

$$\frac{R_{23}^1}{C_1} \quad \text{Diagram} \cdot w \cdot w^2 = 1$$

$$\frac{R_{23}^1}{R_{15}, R_{24}^1} \quad \text{Diagram} \cdot w^2$$

$$\frac{R_{25}^1}{R_{15}, G_7} \quad \text{Diagram} \cdot w^2 \cdot w$$

$$\frac{C_1, C_2, C_3}{C_5, C_6, C_7} \quad \text{Diagram}$$

$$C_8 : (1) \quad \text{Diagram} = \text{Diagram} \quad C_2 : H^4 = I$$

$$R_5 : \boxed{X} = \boxed{H} \boxed{S} \boxed{H} \boxed{H} \boxed{S} \boxed{S} \boxed{H}$$

$$R_{11} : \boxed{Z^2} = \boxed{H} \boxed{H} \boxed{S} \boxed{H} \boxed{H} \boxed{S} \boxed{S} = \boxed{S'} \boxed{S} \boxed{S}$$

Proof cont. $R_{14} :$

$$\text{Diagram} \xrightarrow{\text{WTS}} \text{Diagram} = \boxed{S' S S}$$

$$R_{14} . \text{RHS} := \text{Diagram}$$

$$\xrightarrow[\substack{\text{Def2} \\ C_2}]{} \text{Diagram}$$

$$\xrightarrow{C_8} \text{Diagram}$$

$$\xrightarrow{C_2} \text{Diagram} = R_{14} . \text{LHS.}$$

Hence, $R_{14} :$

$$\text{Diagram} = \text{Diagram} \quad R_5 \parallel R_{11}$$

$$R_{14}^1 : \text{Diagram} = \text{Diagram}$$

□