

$$R_{19}: \quad (1) \quad \begin{array}{c} \text{H} \\ \text{---} \end{array} \quad \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagdown \\ \diagup \end{array} \quad (2) \quad \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \text{H} \\ \text{---} \end{array}$$

$$C_2: H^4 = I$$

Lem P Def 2, Def 3, C₂, C₈, R₁₉ & R₃₂ imply

$$\text{Def 4: } \text{X} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{ccccccc} \bullet & \bullet & & \oplus & \bullet & \bullet & \bullet \\ | & | & & | & | & | & | \\ \text{---} & \text{---} & & \text{---} & \text{---} & \text{---} & \text{---} \\ | & & & | & & & | \\ \oplus & & & \bullet & & & H^+ \end{array}$$

$$\text{Def 6: } \text{[Diagram of two crossed lines]} = \text{[Diagram of two lines with H^2 box and various symbols]} \quad \checkmark$$

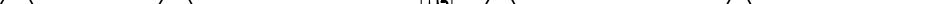
$$\text{Def 5: } \text{[Diagram of two crossed lines]} = \text{[Diagram of five horizontal lines with various symbols and a box labeled H^*]} \quad \checkmark$$

$$\text{Def 7: } \text{X} = \text{H} \circ \text{C}$$

$$\text{Def 8: } \text{Diagram} = \text{Diagram}$$

$$\text{Def 10: } \text{X} = \text{H} \otimes \text{I} \otimes \text{I} \otimes \text{I} \otimes \text{I} \otimes \text{H}$$

$$\text{Def 11: } \text{X gate} = \begin{array}{c} \text{H} \\ \text{---} \\ | \quad | \\ \text{---} \end{array}$$

Proof cont. R_{1g}: (2)  By C₂,

$$\text{Diagram 5: } \text{X} = \text{H}^3 \text{ } \boxed{\text{X}} \text{ } \text{H} = \text{H}^3 \text{ } \text{I} \text{ } \text{H}^3 = \text{H}^3$$

Hence Def 8 :  = 

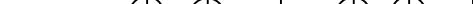
$$\text{CNOT} = \begin{array}{c} \text{H} \\ \times \end{array} = \begin{array}{c} \text{H} \\ \text{H}^{\dagger} \end{array} \quad \underline{\text{Def 7}} \quad \begin{array}{c} \text{H} \\ \text{H}^{\dagger} \end{array} \quad \underline{\text{G}} \quad \begin{array}{c} \text{H} \\ \text{H}^{\dagger} \end{array}$$

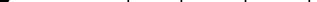
Hence Def 10:  = 

$$R_{19} : (1) \quad \text{Diagram showing } H^3 \text{ and } G_2 \text{ connected in series.} = \text{Diagram showing } H^3 \text{ and } G_2 \text{ connected in parallel.} \equiv \text{Diagram showing } H^3 \text{ and } G_2 \text{ connected in parallel.} = \text{Diagram showing } H^3 \text{ and } G_2 \text{ connected in parallel.}$$

$$\text{Then } \quad \text{Def 4} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}$$

Hence Def 9: $\text{X} = \boxed{\mathbb{H}^3}$

Alternatively,  =  Def 6  C₂

Hence Def 11:  = 

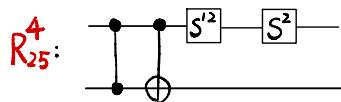
$$\begin{aligned}
R_{23}^4: \quad & \text{Diagram} = \text{Diagram} \cdot w^2 \quad R_{15}: \quad \text{Diagram} = \text{Diagram} \\
R_{24}^3: \quad & \text{Diagram} = \text{Diagram} \quad R_{24}^2: \quad \text{Diagram} = \text{Diagram} \\
R_{10}: \quad & \text{Diagram} = \text{Diagram} \quad C_1: w^3 = I \quad C_2: H^4 = I \quad C_3: S^3 = I \\
C_5: \quad & SH^2SH^2 = H^2SH^2S \quad SS' = s's \quad \text{Def 1: } \text{Diagram} := \text{Diagram} \text{ } H \text{ } S \text{ } H \text{ } H
\end{aligned}$$

Lem Q Def 1-3, $C_1 - C_8$, R_{10} , R_{15} , R_{23} , R_{24} & R_{25} imply

$$\begin{aligned}
R_{27}: \quad & \text{Diagram} = \text{Diagram} \cdot w \\
R_{27}^1: \quad & \text{Diagram} = \text{Diagram} \cdot w \\
R_{27}^2: \quad & \text{Diagram} = \text{Diagram} \cdot w
\end{aligned}$$

$$\text{Proof: } R_{27} \text{ RHS} := \text{Diagram} \cdot w$$

$$\begin{aligned}
& \underline{\underline{R_{23}^4}} \quad \underline{\underline{C_1}} \quad \text{Diagram} \cdot w \\
& \underline{\underline{R_{24}^3}} \quad \text{Diagram} \cdot w \\
& \underline{\underline{R_{23}^4}} \quad \text{Diagram} \cdot w^2 \\
& \underline{\underline{R_{24}^3}} \quad G, G_7 \quad \text{Diagram} \cdot w^2 \\
& \underline{\underline{R_{23}^4}} \quad C_1, R_{10} \quad \text{Diagram} \cdot w \\
& \underline{\underline{R_{23}^4}} \quad R_{15} \quad \text{Diagram} \cdot w \\
& \underline{\underline{R_{23}^4}} \quad R_2, R_3 \quad R_5 \quad \text{Diagram} \cdot w \\
& \underline{\underline{R_{23}^4}} \quad C_1, C_7 \quad \text{Diagram} \cdot w
\end{aligned}$$



$$\cdot w^2 = \text{---}$$

$R_{15}:$

$C_7:$

$$= \text{---}$$

$C_1: w^3 = I$

$C_2: H^4 = I$

$C_3: S^3 = I$

$C_5: SH^2SH^2 = H^2SH^2S$ $SS' = S'S$ $\text{Def 1: } [S'] := [H][H][S][H][H]$

$C_6:$

$\text{Def 7: } \text{---} = \text{---}$

$C_6^1:$

$C_6^2:$

$C_8^4:$

$$= \text{---}$$

$R_{25}^2:$

$$\cdot w^2 = \text{---}$$

Lem Q $R_{27}:$

$R_{27}^1:$

$R_{27}^2:$

Proof cont.

$\underline{\underline{C_7}}$ $\underline{\underline{R_{15}}}:$

$\underline{\underline{R_{25}^4}}$

$\underline{\underline{C_2, C_3, C_5}}$ $\underline{\underline{C_6, G_1, R_{15}}}:$

$\underline{\underline{\text{Def 7}}}$

$\underline{\underline{C_6^1}}$

$\underline{\underline{C_6^2}}$

$\underline{\underline{C_8^4}}$

$\underline{\underline{R_{25}^2}}$

$$C_5: SH^2SH^2 = H^2SH^2S \quad SS' = S'S \quad \text{Def 1: } [S'] := [H][H][S][H][H]$$

$$C_6: \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = C_1: w^3 = I \quad C_2: H^4 = I \quad C_3: S^3 = I$$

$$\text{Def 2: } \begin{array}{c} \oplus \\ \bullet \end{array} := \begin{array}{c} [H] \\ \bullet \\ [H] \\ [H] \\ [H] \end{array} \quad \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} := \begin{array}{c} [H] \\ \bullet \\ [H] \\ [H] \\ [H] \end{array}$$

$$C_8: \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = \begin{array}{c} [H^2] \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \quad R_{25}^2: \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = \begin{array}{c} \oplus \\ \bullet \\ [S^2] \\ [S^2] \\ \bullet \end{array} \cdot w^2 = \begin{array}{c} \oplus \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}$$

$$R_{24}: \begin{array}{c} \bullet \\ \oplus \\ [S] \end{array} = \begin{array}{c} \oplus \\ \bullet \\ [S] \end{array} \quad R_{24}' : \begin{array}{c} \bullet \\ \oplus \\ [S] \end{array} = \begin{array}{c} \oplus \\ \bullet \\ [S'] \end{array}$$

$$\text{Lem Q } R_{27}: \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = \begin{array}{c} [H] \\ \bullet \\ [H] \\ \bullet \\ \bullet \end{array} \times \begin{array}{c} \oplus \\ \bullet \\ [S] \\ \bullet \\ [S] \end{array} \cdot w$$

$$R_{27}': \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = \begin{array}{c} [H] \\ \bullet \\ [H] \\ \bullet \\ \bullet \end{array} \times \begin{array}{c} \oplus \\ \bullet \\ [S] \\ \bullet \\ [S] \end{array} \cdot w$$

$$R_{27}^2: \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = \begin{array}{c} [H] \\ \bullet \\ [H] \\ \bullet \\ \bullet \end{array} \times \begin{array}{c} \oplus \\ \bullet \\ [S] \\ \bullet \\ [S] \end{array} \cdot w$$

Proof cont.

$$\begin{array}{c} [H] \\ \bullet \\ [H] \\ \bullet \\ \bullet \end{array} \times \begin{array}{c} \oplus \\ \bullet \\ [S^2] \\ [S^2] \\ [S^2] \end{array} \cdot w^2 \cdot w^2$$

$$R_{25}^2: \begin{array}{c} [H] \\ \bullet \\ [H] \\ \bullet \\ \bullet \end{array} \times \begin{array}{c} \oplus \\ \bullet \\ [S^2] \\ [S^2] \\ [S^2] \end{array} \cdot w^6$$

$$\begin{array}{c} C_1 \\ \hline R_{24}, R_{24}' \end{array} \begin{array}{c} [H] \\ \bullet \\ [H] \\ \bullet \\ \bullet \end{array} \times \begin{array}{c} \oplus \\ \bullet \\ [S^2] \\ [S^2] \\ [S^2] \end{array}$$

$$\begin{array}{c} C_2, C_3 \\ \hline C_5, C_6 \end{array} \begin{array}{c} [H] \\ \bullet \\ [H] \end{array} \xrightarrow{\text{Def 2}} \begin{array}{c} [H^2] \\ \bullet \\ [H^3] \end{array}$$

$$C_8': \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \xrightarrow{C_2} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = R_{27}. \text{LHS.}$$

$$R_{27}^1. \text{LHS.} := \begin{array}{c} [H] \\ \bullet \\ [H] \end{array} \xrightarrow{R_{27}} \begin{array}{c} [H] \\ \bullet \\ [H] \\ \bullet \\ \bullet \end{array} \times \begin{array}{c} \oplus \\ \bullet \\ [S] \\ \bullet \\ [S] \end{array} \cdot w$$

$$\xrightarrow{\text{Def 2}} \begin{array}{c} [H] \\ \bullet \\ [H] \\ \bullet \\ \bullet \end{array} \times \begin{array}{c} \oplus \\ \bullet \\ [S] \\ \bullet \\ [S] \end{array} \cdot w = R_{27}^1. \text{RHS.}$$

$$C_5: SH^2SH^2 = H^2SH^2S \quad SS' = S'S$$

$$\text{Def 1: } \boxed{S'} = \boxed{H} \boxed{H} \boxed{S} \boxed{H} \boxed{H} \quad \boxed{S'} \boxed{S'} = \boxed{H} \boxed{H} \boxed{S} \boxed{S} \boxed{H} \boxed{H}$$

$$C_6: \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = \quad C_1: w^3 = I \quad C_2: H^4 = I \quad C_3: S^3 = I \quad R10: \boxed{Z} = \boxed{S'} \boxed{S'} \boxed{S}$$

$$C_8^4: \begin{array}{c} \bullet \\ \boxed{H^2} \\ \bullet \\ \boxed{H^2} \end{array} = \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \boxed{H^2} \\ \bullet \\ \boxed{H^2} \end{array}$$

$$\text{Lem Q } R_{27}: \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \oplus \boxed{S} \dots \cdot w$$

$$R_{27}^1: \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \oplus \boxed{S} \dots \cdot w$$

$$R_{27}^2: \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \oplus \boxed{S} \dots \cdot w$$

$$\text{Proof cont. } R_{27}^2 \cdot \text{RHS} := \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \oplus \boxed{S} \dots \cdot w \stackrel{C_2}{=} \quad$$

$$\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \oplus \boxed{S} \dots \cdot w$$

(1)

$$\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \oplus \boxed{S} \dots \cdot w$$

Composition in diagrammatic order:

$$H^3 S^2 \boxed{Z} \stackrel{R10}{=} H^3 \boxed{S^2 S'^2} S \stackrel{C_5}{=} H^3 \boxed{S^2} S'^2 \stackrel{C_3}{=} H^3 S'^2 \stackrel{\text{Def 1}}{=} \boxed{H^3 H^2 S^2 H^2} \stackrel{C_2}{=} HS^2 H^2 \quad (1)$$

$$\text{Hence } R_{27}^1: \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \oplus \boxed{S} \dots \cdot w$$

$$\stackrel{(1)}{=} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \oplus \boxed{S} \dots \cdot w$$

$$\text{Thus, } R_{27}^2 \cdot \text{RHS} = \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \oplus \boxed{S} \dots \cdot w$$

$$\stackrel{R_{27}^1}{=} \boxed{\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}} \stackrel{C_8^4}{=} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = R_{27}^2 \cdot \text{LHS}$$

□

$R_{23}^1:$ = $\cdot w^2$ $C_6:$ = $C_5:$ $S H^2 S H^2 = H^2 S H^2 S$
 $R_{24}^1:$ = $R_{25}^2:$ $\cdot w^2 =$
Def 1: $S' = H H S H H$ $C_1: w^3 = I$ $C_2: H^4 = I$ $C_3: S^3 = I$
 $R_{15}:$ =
 $R_{24}:$ =

Lem R Def 1-3, C1-7, R10, R15, R23, R24 & R25 imply

$R_{28}:$ = $\cdot w$
 $R_{28}^1:$ = $\cdot w$
 $R_{28}^2:$ = $\cdot w$

Proof: $R_{28} \cdot \text{RHS} :=$ $\cdot w$

\equiv $\cdot w \cdot w^2$

$\frac{C_1, R_{23}^1}{R_{24}^1}$ $\cdot w^2$

$\frac{C_2, C_3}{R_{23}^1}$ $\cdot w^2 \cdot w^2$

$\frac{C_1, C_7}{R_{15}, R_{24}^1}$ $\cdot w$

\equiv $\cdot w \cdot w^2$

$\frac{C_1, C_7}{R_{15}}$ $\cdot w^3$

$$C_6: \quad \text{Diagram} = \quad C_5: \quad S H^2 S H^2 S = H^2 S H^2 S \quad C_1: w^3 = I \quad C_2: H^4 = I \quad C_3: S^3 = I$$

$$R_{24}^1: \quad \text{Diagram} = \quad R_{24}: \quad \text{Diagram} = \quad R_{25}^2: \quad \text{Diagram} = \quad \cdot w^2 = \quad \text{Diagram}$$

$$R_{15}: \quad \text{Diagram} = \quad C_7: (1) \quad \text{Diagram} = \quad (2) \quad \text{Diagram} = \quad \text{Diagram}$$

$$\text{Def 2: } \quad \text{Diagram} := \quad \text{Diagram} := \quad \text{Diagram} := \quad \text{Diagram}$$

Lem R

$$R_{28}: \quad \text{Diagram} = \quad \text{Diagram} \cdot w$$

$$R_{28}^1: \quad \text{Diagram} = \quad \text{Diagram} \cdot w$$

$$R_{28}^2: \quad \text{Diagram} = \quad \text{Diagram} \cdot w$$

Proof cont.

$$\text{Diagram}$$

$$\frac{C_2, C_3}{R_{25}^2} \quad \text{Diagram} \cdot w^2$$

$$\frac{C_6, C_7}{R_{15}} \quad \text{Diagram} \cdot w^2$$

$$\frac{R_{25}^2}{\text{Diagram}} \quad \cdot w^2 \cdot w^2$$

$$\frac{C_1, C_2, C_3}{C_5, R_{25}^2} \quad \text{Diagram} \cdot w \cdot w^2$$

$$\frac{C_1, C_2, C_3, C_5}{R_{24}^1, R_{24}} \quad \text{Diagram} \quad \text{Def 2} \quad \text{Diagram}$$

$$\frac{C_2}{\text{Diagram}} =: R_{28} \cdot \text{LHS}$$

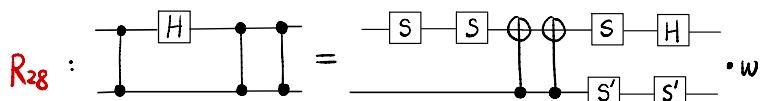
R10: $\boxed{z} = \boxed{s'} \boxed{s'} \boxed{s}$

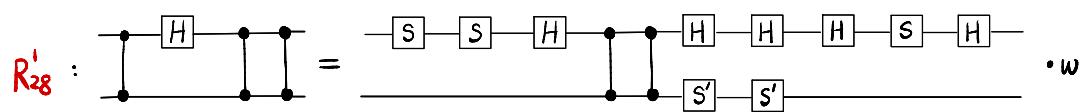
C2: $H^4 = I$ **C3:** $S^3 = I$ **C5:** $SH^2SH^2 = H^2SH^2S$ **SS' = S'S**

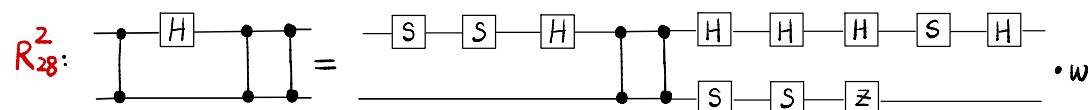
Def 1: $\boxed{s'} = \boxed{H} \boxed{H} \boxed{S} \boxed{H} \boxed{H}$

Def 2: 

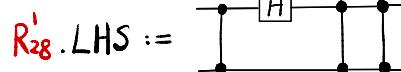
Lem R

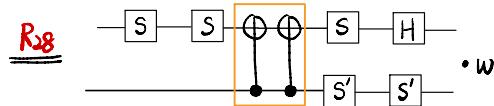
R₂₈:  • w

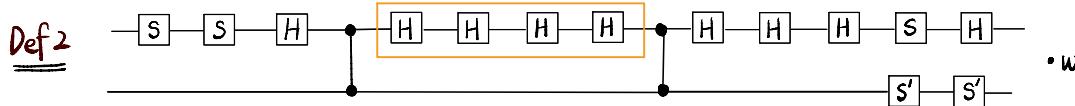
R₂₈¹:  • w

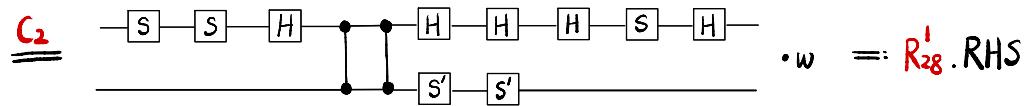
R₂₈²:  • w

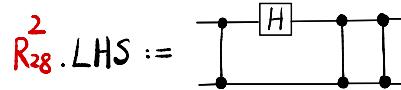
Proof cont.

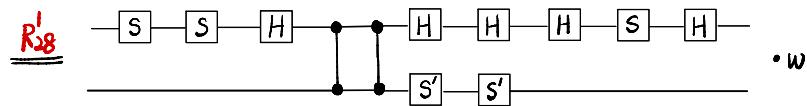
R₂₈¹.LHS := 

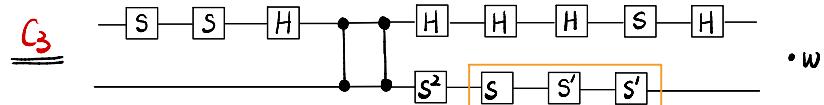
R₂₈  • w

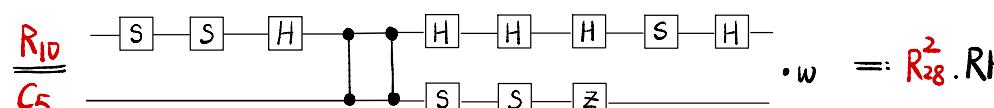
Def 2  • w

C₂  • w =: **R₂₈¹.RHS**

R₂₈².LHS := 

R₂₈¹  • w

C₃  • w

R₁₀  • w =: **R₂₈².RHS**

□