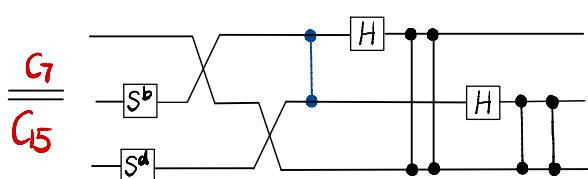
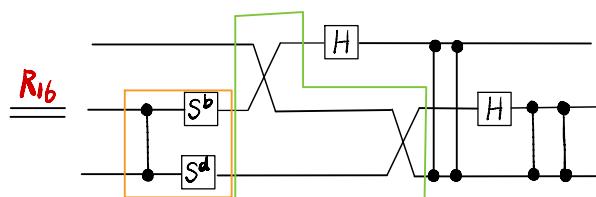
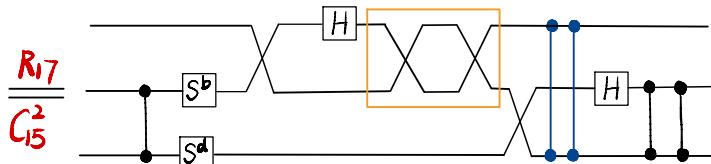
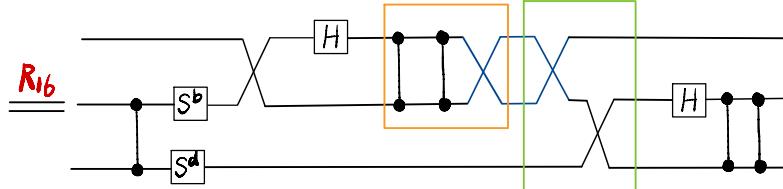
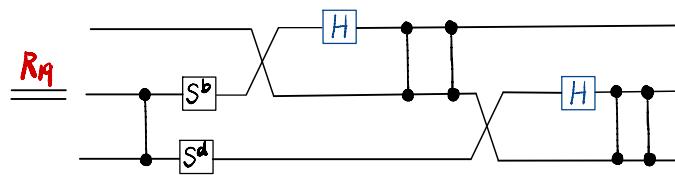
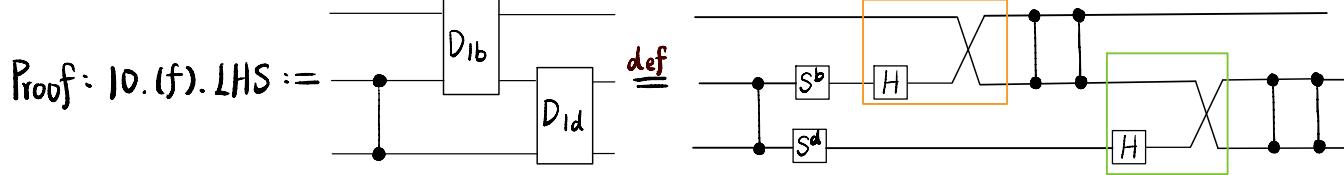
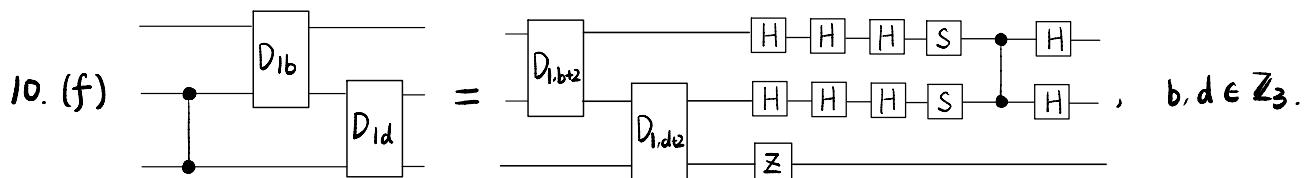
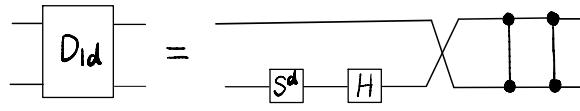
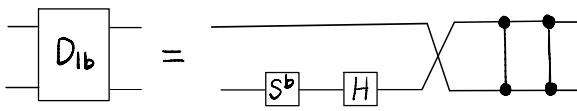


Lem 6 Def 1-2, Def 5, Def 7, C<sub>2-3</sub>, C<sub>6-7</sub>, R<sub>10</sub>, R<sub>16</sub>, R<sub>17</sub>, R<sub>18</sub>, R<sub>19</sub>, R<sub>43</sub>, C<sub>13</sub> & C<sub>15</sub> imply





$$R_{19}: \text{[Circuit diagram: } H \text{ followed by a CNOT gate between the second and third wires.]} = \text{[Circuit diagram: } H \text{ followed by a CNOT gate between the second and third wires.]}$$

$$R_{16}: \text{[Circuit diagram: } H \text{ followed by a CNOT gate between the second and third wires, followed by another CNOT gate between the first and fourth wires.]} = \text{[Circuit diagram: ]}$$

$$R_{17}: \text{[Circuit diagram: } H \text{ followed by a CNOT gate between the second and third wires, followed by another CNOT gate between the first and fourth wires, followed by a CNOT gate between the second and fourth wires.]} = \text{[Circuit diagram: ]}$$

$$C_{15}^2: \text{[Circuit diagram: two CNOT gates between the second and third wires, followed by a CNOT gate between the first and fourth wires, followed by another CNOT gate between the second and third wires, followed by a CNOT gate between the first and fourth wires.]} = \text{[Circuit diagram: two CNOT gates between the second and third wires, followed by a CNOT gate between the first and fourth wires, followed by another CNOT gate between the second and third wires, followed by a CNOT gate between the first and fourth wires.]}$$

$$\text{Def 1: } [S'] := [H] [H] [S] [H] [H] \quad R_{10}: [Z] = [S'] [S'] [S]$$

Lem 6

$$10. (f) \quad D_{1b} \text{ } D_{1d} = D_{1,b+2} \text{ } D_{1,d+2} \text{ } [H] [H] [H] [S] \cdot [H] \text{, } b, d \in \mathbb{Z}_3.$$

Proof cont:

$$10. (f). \text{LHS} = \text{[Circuit diagram: } S^b \text{ followed by a CNOT gate between the second and third wires, then } S^d \text{ followed by a CNOT gate between the second and third wires, then a CNOT gate between the first and fourth wires, then another CNOT gate between the second and third wires, then a CNOT gate between the first and fourth wires.]}$$

$$10. (f). \text{RHS} := D_{1,b+2} \text{ } D_{1,d+2} \text{ } [Z]$$

$$\underline{\underline{\text{def}}} \quad \text{[Circuit diagram: } S^b [S^2] [H] \text{ (orange box), then a CNOT gate between the second and third wires, then } S^d [S^2] [H] \text{ (green box), then a CNOT gate between the second and third wires, then a CNOT gate between the first and fourth wires, then another CNOT gate between the second and third wires, then a CNOT gate between the first and fourth wires, then a CNOT gate between the second and fourth wires, then } [Z] \text{ (blue box).]}$$

$$\underline{\underline{\text{Def1}}} \quad \text{[Circuit diagram: } S^b [S^2] \text{ followed by a CNOT gate between the second and third wires, then } S^d [S^2] \text{ followed by a CNOT gate between the second and third wires, then a CNOT gate between the first and fourth wires, then another CNOT gate between the second and third wires, then a CNOT gate between the first and fourth wires, then a CNOT gate between the second and fourth wires, then } [S^2] [S] \text{.]}$$

$$\underline{\underline{R_{16}}} \quad \text{[Circuit diagram: } S^b [S^2] \text{ followed by a CNOT gate between the second and third wires, then } S^d [S^2] \text{ followed by a CNOT gate between the second and third wires, then a CNOT gate between the first and fourth wires, then another CNOT gate between the second and third wires, then a CNOT gate between the first and fourth wires, then a CNOT gate between the second and fourth wires, then a CNOT gate between the first and fourth wires, then a CNOT gate between the second and fourth wires, then } [S^2] [S] \text{.]}$$

$$\underline{\underline{R_{17}}} \quad \text{[Circuit diagram: } S^b [S^2] \text{ followed by a CNOT gate between the second and third wires, then a CNOT gate between the first and fourth wires, then another CNOT gate between the second and third wires, then a CNOT gate between the first and fourth wires, then a CNOT gate between the second and fourth wires, then a CNOT gate between the first and fourth wires, then a CNOT gate between the second and fourth wires, then a CNOT gate between the first and fourth wires, then a CNOT gate between the second and fourth wires, then } [S^2] [S] \text{.]}$$

$$R_{16} : \text{Diagram} = \text{Diagram} \quad R_{17} : \text{Diagram} = \text{Diagram} \quad R_{18} : \text{Diagram} = \text{Diagram}$$

$$C_{15}^2 : \text{Diagram} = \text{Diagram} \quad \text{Def 7} : \text{Diagram} := \text{Diagram} = \text{Diagram}$$

$$\text{Def 2} : \text{Diagram} := \text{Diagram}$$

$$C_2 : H^4 = I \quad C_3 : S^3 = I$$

Lem 6

$$10. (f) \quad \text{Diagram} = \text{Diagram}, \quad b, d \in \mathbb{Z}_3.$$

Proof cont:

$$10. (f). LHS = \text{Diagram}$$

$$10. (f). RHS = \text{Diagram}$$

$$\frac{R_{16}}{R_{18}} \quad \text{Diagram}$$

$$\frac{C_2, \text{Def 2}}{\text{Def 7}} \quad \text{Diagram}$$

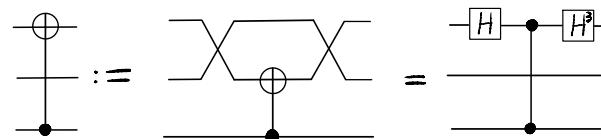
$$\text{Hence} \quad \text{Diagram} \underset{\text{WTS}}{=} \text{Diagram}$$

$$C_3 \parallel R_{16}$$

$$\text{Diagram} \underset{\text{WTS}}{=} \text{Diagram}$$

$$C_2 : H^4 = I$$

Def 7 :



$$R_{43} : \quad \begin{array}{c} \text{---} \\ | \oplus \oplus | \\ | \cdot \cdot | \\ | \cdot \cdot | \end{array} \quad = \quad \begin{array}{c} S^2 \quad \oplus \oplus \quad S \\ | \cdot \cdot | \quad | \cdot \cdot | \quad | \cdot \cdot | \\ | \cdot \cdot | \quad | \cdot \cdot | \quad | \cdot \cdot | \\ | \cdot \cdot | \quad | \cdot \cdot | \quad | \cdot \cdot | \end{array}$$

Lem 6

$$10. (f) \quad \begin{array}{c} \text{---} \\ | \cdot | \\ | D_{1b} | \\ | \cdot | \\ | D_{1d} | \end{array} = \begin{array}{c} D_{1,b+2} \\ | \cdot | \\ | D_{1,d+2} | \\ | \cdot | \\ | Z | \end{array} \quad H \quad H \quad H \quad S \quad | \cdot | \quad H \quad H \quad H \quad S \quad | \cdot | \quad H \quad , \quad b, d \in \mathbb{Z}_3.$$

Proof cont :

$$\begin{array}{c} \text{---} \\ | \cdot | \\ | H | \quad | \cdot | \\ | \cdot | \quad | H | \quad | \cdot | \\ | \cdot | \quad | \cdot | \quad | \cdot | \end{array} \quad \stackrel{\text{WTS}}{=} \quad \begin{array}{c} S^2 \quad \oplus \oplus \quad S \\ | \cdot | \quad | \cdot | \quad | \cdot | \\ | \cdot | \quad | \cdot | \quad | \cdot | \\ | \cdot | \quad | \cdot | \quad | \cdot | \end{array}$$

$C_2 \parallel$

WTS

$$\begin{array}{c} S^2 \quad \oplus \oplus \quad S \\ | \cdot | \quad | \cdot | \quad | \cdot | \\ | \cdot | \quad | \cdot | \quad | \cdot | \\ | \cdot | \quad | \cdot | \quad | \cdot | \end{array}$$

$\parallel \parallel \text{Def 7}$

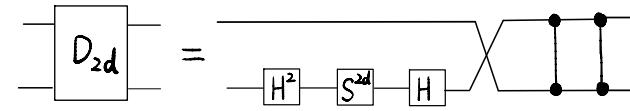
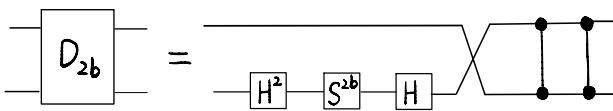
$$\begin{array}{c} \text{---} \\ | \oplus \oplus | \\ | \cdot | \\ | \oplus \oplus | \end{array}$$

WTS

$$\begin{array}{c} S^2 \quad \oplus \oplus \quad S \\ | \cdot | \quad | \cdot | \quad | \cdot | \\ | \cdot | \quad | \cdot | \quad | \cdot | \\ | \cdot | \quad | \cdot | \quad | \cdot | \end{array}$$

By  $R_{43}$ , this completes the proof.





$$R_{19}: \quad \text{Diagram} = \text{Diagram} \quad [H]$$

$$R_{16}: \quad \text{Diagram} = \text{Diagram}$$

$$R_{17}: \quad \text{Diagram} = \text{Diagram}$$

$$C_{15}^2: \quad \text{Diagram} = \text{Diagram}$$

$$C_{15}: \quad \text{Diagram} = \text{Diagram}$$

$$C_7: (1) \quad \text{Diagram} = \text{Diagram} \quad (2) \quad \text{Diagram} = \text{Diagram}$$

$$C_8^3: \quad \text{Diagram} = \text{Diagram}$$

Lem7 Def 1-2, Def 5, Def 7, C2-3, C6-8, R10, R16, R17, R18, R19, R43, C13 & C15 imply

$$10.(g) \quad \text{Diagram} = \text{Diagram} \quad , \quad b, d \in \mathbb{Z}_3.$$

$$\text{Proof: } 10.(g). \text{LHS} := \text{Diagram} \stackrel{\text{def}}{=} \text{Diagram}$$

$$\frac{C_7, C_8^3}{R_{19}} \quad \text{Diagram}$$

$$\frac{C_{15}}{R_{17}} \quad \text{Diagram}$$

$$\frac{C_{15}}{R_{19}} \quad \text{Diagram}$$

$$10.(g).\text{RHS} := \text{Diagram}$$

$$\stackrel{\text{def}}{=} \text{Diagram}$$

$$2(b+1) = 2b+2 \\ 2(d+1) = 2d+2$$

$$R_{19}: \quad \text{Diagram} = \text{Diagram}$$

$$R_{18}: \quad \text{Diagram} = \text{Diagram}$$

$$R_{17}: \quad \text{Diagram} = \text{Diagram}$$

$$C_{15}^2: \quad \text{Diagram} = \text{Diagram}$$

$$R_{16}: \quad \text{Diagram} = \text{Diagram}$$

$$C_2: H^4 = I \quad C_3: S^3 = I$$

Lem 7

$$10. (g) \quad \text{Diagram} = \text{Diagram}, \quad b, d \in \mathbb{Z}_3.$$

Proof cont:

$$10. (g). LHS = \text{Diagram}$$

$$10. (g). RHS = \text{Diagram}$$

$$\frac{C_{15}}{R_{17}} \quad \text{Diagram}$$

$$\frac{R_{18}}{R_{19}} \quad \text{Diagram}$$

$$\text{Hence} \quad \text{Diagram}$$

$$\frac{\text{WTS}}{\text{WTS}} \quad \text{Diagram} \quad \frac{C_2 C_3}{R_{16}}$$

$$\text{WTS} \quad \text{Diagram}$$

$$\text{WTS} \quad \text{Diagram}$$

$$\text{Def 7 : } \begin{array}{c} \oplus \\ \parallel \\ \parallel \end{array} := \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \parallel \quad \parallel \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \parallel \quad \parallel \end{array} = \begin{array}{c} [H] \quad \bullet \quad [H^{\dagger}] \\ \parallel \quad \bullet \quad \parallel \end{array}$$

$$\text{Def 2: } \begin{array}{c} \oplus \\ \parallel \end{array} := \begin{array}{c} \boxed{H} \quad \bullet \quad \boxed{H^3} \\ \parallel \quad \parallel \end{array}$$

$$C_2 : H^4 = I$$

$$C_3 : S^3 = I$$

**Def 1:**  $\boxed{s'} := \boxed{H} \boxed{H} \boxed{S} \boxed{H} \boxed{H}$       **R10:**  $\boxed{z} = \boxed{s'} \boxed{s'} \boxed{s}$

$$RIO : \quad \boxed{\Sigma} = \boxed{S'} \boxed{S'} \boxed{S}$$

$$R_{43} : \quad \begin{array}{c} \text{Circuit Diagram} \\ \text{Left: } R_{43} \text{ (4 inputs, 3 outputs)} \\ \text{Right: } S^2 \text{ (2 inputs, 2 outputs)} \end{array} = \begin{array}{c} \text{Circuit Diagram} \\ \text{Left: } S^2 \text{ (2 inputs, 2 outputs)} \\ \text{Right: } S^{12} \text{ (2 inputs, 2 outputs)} \end{array}$$

## Lem 7

$$10.(g) \quad \text{Diagram showing the equivalence of two circuit representations. The left side shows a sequence of gates: a black dot, a box labeled } D_{2b}, \text{ another black dot, and a box labeled } D_{2d}. \text{ An equals sign follows. The right side shows a sequence of gates: a box labeled } D_{2,b+1}, \text{ a black dot, a box labeled } D_{2,d+1}, \text{ a black dot, and a sequence of four boxes labeled H, H, H, S followed by a vertical line with a dot, then a box labeled H. Below the right side, there is a box labeled } Z.$$

Proof cont :

A quantum circuit diagram consisting of four horizontal lines representing qubits. Two Hadamard gates ( $H$ ) are placed on the top and middle lines. Three CNOT gates connect the top line to the middle line, the middle line to the bottom line, and the bottom line to the top line. The labels 'WTS' and '====' are written in pink at the right end of the circuit.

A quantum circuit diagram illustrating a sequence of operations on three qubits. The circuit is divided into two main horizontal sections by a vertical dashed line. Each section contains the following sequence of operations from left to right:  $S^2$ ,  $H$ , CNOT (indicated by a black dot), CNOT (indicated by a black dot),  $H^3$ ,  $S$ , and  $H$ . The top section is enclosed in an orange box, and the bottom section is enclosed in a green box. A blue box encloses the third qubit, which also contains a  $Z$  gate positioned between the two sections.

## R10 ||| Def 1 , Def 2, Def 7

$C_2$  ||

### III Def 2, Def 7

**WTS**

By  $R_{43}$ , this completes the proof.

1