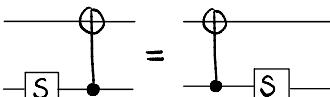


$$C_1: w^3 = I$$

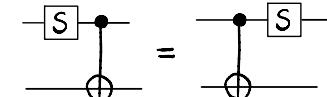
$$C_2: H^4 = I$$

$$C_3: S^3 = I$$

$$R_{24}:$$



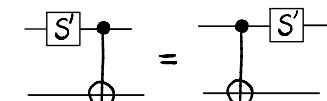
$$R_{24}^2:$$



$$C_6: \quad \text{Diagram showing two wires with three control dots on the top wire and three target dots on the bottom wire.} \quad = \quad \text{Diagram showing two wires with three control dots on the top wire and three target dots on the bottom wire.}$$

$$R_{24}^1:$$

$$R_{24}^3:$$



$$C_{13}: \quad \text{Diagram showing two wires with two control dots on the top wire and two target dots on the bottom wire.} \quad = \quad \text{Diagram showing two wires with two control dots on the top wire and two target dots on the bottom wire.}$$

$$C_{13}^4:$$

$$C_{13}^9:$$

$$C_7: \quad \text{Diagram showing two wires with one control dot on the top wire and one target dot on the bottom wire.} \quad = \quad \text{Diagram showing two wires with one control dot on the top wire and one target dot on the bottom wire.}$$

$$R_{15}: \quad \text{Diagram showing two wires with one control dot on the top wire and one target dot on the bottom wire.} \quad = \quad \text{Diagram showing two wires with one control dot on the top wire and one target dot on the bottom wire.}$$

Lem T

$$R_{54}: \quad \text{Diagram showing two wires. The top wire has four control dots and the bottom wire has four target dots. A Hadamard gate (H) is placed on the bottom wire between the first and second qubits.} \quad = \quad \text{Diagram showing two wires. The top wire has four control dots and the bottom wire has four target dots. A Hadamard gate (H) is placed on the bottom wire between the first and second qubits. Between the second and third qubits, there is a sequence of gates: S', S^2, S^2, S'.$$

Proof cont.

$$R_{54}: \quad \text{Diagram showing two wires. The top wire has four control dots and the bottom wire has four target dots. A sequence of gates S^2, S^2, S^2, S^2 is placed on the bottom wire between the first and fourth qubits.} \quad \underset{\text{WTS}}{=} \quad \text{Diagram showing two wires. The top wire has four control dots and the bottom wire has four target dots. A sequence of gates S^2, S^2 is placed on the bottom wire between the first and fourth qubits.}$$

$$\text{Then } R_{54}. \text{LHS} := \text{Diagram showing two wires. The top wire has four control dots and the bottom wire has four target dots. A sequence of gates S^2, S^2, S^2, S^2 is highlighted with an orange box on the bottom wire between the first and fourth qubits.}$$

$$\frac{\text{C}_7, \text{R}_{15}, \text{R}_4}{R_{24}^1, R_{24}^2, R_{24}^3} \quad \text{Diagram showing two wires. The top wire has four control dots and the bottom wire has four target dots. A sequence of gates S^2, S^2, S^2, S^2 is highlighted with an orange box on the bottom wire between the first and fourth qubits. A sequence of gates S', S^2, S^2, S' is highlighted with a blue box on the bottom wire between the second and fifth qubits.}$$

$$\frac{\text{C}_6}{\text{C}_6^2} \quad \text{Diagram showing two wires. The top wire has four control dots and the bottom wire has four target dots. A sequence of gates S^2, S^2, S^2, S^2 is highlighted with an orange box on the bottom wire between the first and fourth qubits. A sequence of gates S', S^2, S^2, S' is highlighted with a green box on the top wire between the first and second qubits.} \quad \underset{\text{C}_6^9}{=} \quad \text{Diagram showing two wires. The top wire has four control dots and the bottom wire has four target dots. A sequence of gates S^2, S^2, S^2, S^2 is highlighted with an orange box on the bottom wire between the first and fourth qubits. A sequence of gates S', S^2, S^2, S' is highlighted with a green box on the top wire between the first and second qubits.} \quad =: \quad R_{54}. \text{RHS}$$

□

$$C_1: w^3 = I \quad C_2: H^4 = I \quad C_3: S^3 = I \quad C_5: SS' = S'S$$

$$R_{24}^2: \begin{array}{c} S \\ \square \end{array} \otimes \begin{array}{c} S \\ \square \end{array} = \begin{array}{c} S \\ \square \end{array} \otimes \begin{array}{c} S \\ \square \end{array}$$

$$\text{Def 1: } \begin{array}{c} S' \\ \square \end{array} := \begin{array}{ccccc} H & & S & & H \end{array}$$

$$\text{Def 7: } \begin{array}{c} \oplus \\ \square \end{array} := \begin{array}{c} \times \quad \times \\ \times \quad \times \end{array}$$

$$\text{Def 2: } \begin{array}{c} \oplus \\ \square \end{array} := \begin{array}{c} H \\ \square \end{array} \cdot \begin{array}{c} H^3 \\ \square \end{array}$$

$$\text{Def 5: } \begin{array}{c} \cdot \\ \square \end{array} := \begin{array}{c} \times \quad \times \\ \times \quad \times \end{array}$$

$$R_{23-25}^1: \begin{array}{c} \cdot \\ \square \end{array} = \begin{array}{c} \cdot \\ \square \end{array} \cdot w$$

$$\text{Def 4: } \begin{array}{c} \cdot \\ \square \end{array} := \begin{array}{c} H \\ \square \end{array} \cdot \begin{array}{c} H^3 \\ \square \end{array}$$

Lem U Def 1-2, Def 4-5, Def 7, $C_1, C_2, C_3, C_5, C_6, C_7, C_8, C_{13}, C_{16}, R_{15}, R_{23}, R_{24}, R_{25}$ imply

$$R_{58}: \begin{array}{c} \oplus \\ \square \end{array} = \begin{array}{c} H^2 \\ \square \end{array} \cdot \begin{array}{c} \oplus \\ \square \end{array} = \begin{array}{c} H \\ \square \end{array} \cdot \begin{array}{c} H^2 \\ \square \end{array} \cdot \begin{array}{c} S^2 \\ \square \end{array} \cdot \begin{array}{c} S'^2 \\ \square \end{array} \cdot \begin{array}{c} S \\ \square \end{array} \cdot w$$

$$\text{Proof: } R_{58} \cdot \text{RHS} := \begin{array}{c} \oplus \\ \square \end{array} = \begin{array}{c} H^2 \\ \square \end{array} \cdot \begin{array}{c} \oplus \\ \square \end{array} = \begin{array}{c} H \\ \square \end{array} \cdot \begin{array}{c} H^2 \\ \square \end{array} \cdot \begin{array}{c} S^2 \\ \square \end{array} \cdot \begin{array}{c} S^2 \\ \square \end{array} \cdot \begin{array}{c} \oplus \\ \square \end{array} \cdot \begin{array}{c} S \\ \square \end{array} \cdot w$$

$$\frac{R_{24}^2, C_5}{R_{23-25}^1} \begin{array}{c} \oplus \\ \square \end{array} = \begin{array}{c} H^2 \\ \square \end{array} \cdot \begin{array}{c} \oplus \\ \square \end{array} = \begin{array}{c} H \\ \square \end{array} \cdot \begin{array}{c} H^2 \\ \square \end{array} \cdot \begin{array}{c} S^2 \\ \square \end{array} \cdot \begin{array}{c} S^2 \\ \square \end{array} \cdot \begin{array}{c} S \\ \square \end{array} \cdot w \cdot w$$

$$\underline{\underline{C_3}} \begin{array}{c} \oplus \\ \square \end{array} = \begin{array}{c} H^2 \\ \square \end{array} \cdot \begin{array}{c} \oplus \\ \square \end{array} = \begin{array}{c} H \\ \square \end{array} \cdot \begin{array}{c} H^2 \\ \square \end{array} \cdot \begin{array}{c} S^2 \\ \square \end{array} \cdot \begin{array}{c} S^2 \\ \square \end{array} \cdot w^2$$

$$\text{Hence } R_{58}: \begin{array}{c} \oplus \\ \square \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} H^2 \\ \square \end{array} \cdot \begin{array}{c} \oplus \\ \square \end{array} = \begin{array}{c} H \\ \square \end{array} \cdot \begin{array}{c} H^2 \\ \square \end{array} \cdot \begin{array}{c} S^2 \\ \square \end{array} \cdot \begin{array}{c} S^2 \\ \square \end{array} \cdot w^2$$

$\underline{\underline{C_2}} \parallel$

$$R_{58}: \begin{array}{c} \oplus \\ \square \end{array} = \begin{array}{c} H^2 \\ \square \end{array} \cdot \begin{array}{c} \oplus \\ \square \end{array} = \begin{array}{c} H^2 \\ \square \end{array} \cdot \begin{array}{c} H^2 \\ \square \end{array} \cdot \begin{array}{c} H^2 \\ \square \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} H^2 \\ \square \end{array} \cdot \begin{array}{c} \oplus \\ \square \end{array} = \begin{array}{c} H^2 \\ \square \end{array} \cdot \begin{array}{c} H^2 \\ \square \end{array} \cdot \begin{array}{c} S^2 \\ \square \end{array} \cdot \begin{array}{c} S^2 \\ \square \end{array} \cdot w^2$$

$\underline{\underline{C_2, \text{Def 1}}} \parallel \text{Def 4}$

$$R_{58}: \begin{array}{c} \oplus \\ \square \end{array} = \begin{array}{c} H^2 \\ \square \end{array} \cdot \begin{array}{c} \oplus \\ \square \end{array} = \begin{array}{c} H^2 \\ \square \end{array} \cdot \begin{array}{c} H^2 \\ \square \end{array} \cdot \begin{array}{c} S^2 \\ \square \end{array} \cdot \begin{array}{c} H^2 \\ \square \end{array} \cdot \begin{array}{c} S^2 \\ \square \end{array} \cdot \begin{array}{c} H^2 \\ \square \end{array} \cdot w^2$$

$$C_1: w^3 = 1$$

$$C_1: w^3 = I \quad C_2: H^4 = I$$

$$C_1: w^3=I \quad C_2: H^4=I \quad C_3: S^3=I \quad C_5: SS'=S'S$$

$$C_1: w^3 = I \quad C_2: H^4 = I \quad C_3: S^3 = I \quad C_5: SS' = I$$

$$C_1: w^3 = I \quad C_2: H^4 = I \quad C_3: S^3 = I \quad C_4: SS' = S'S \quad C_5: C^3 = C$$

$$\begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \quad \boxed{H^2} = \quad \begin{array}{c} \text{---} \\ \boxed{H^2} \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \end{array}$$

$$C_8^{1*} = \begin{array}{c} \text{Diagram showing } C_8^{1*} \text{ as a } H^2 \text{ box connected to two vertical lines.} \\ \text{Diagram showing } C_8^{1*} \text{ as a } H^2 \text{ box connected to two vertical lines.} \end{array}$$

$$C_{8^*}^6 : \begin{array}{c} \text{---} \\ \oplus \\ \text{---} \\ | \quad | \\ \boxed{H^2} \quad \bullet \end{array} = \begin{array}{c} \text{---} \\ \oplus \\ \text{---} \\ | \quad | \\ \bullet \quad \bullet \quad \boxed{H^2} \end{array}$$

$$C_{16}^{22} : \quad \begin{array}{c} \text{Diagram of } C_{16}^{22} \text{ showing a 4x4 grid with black dots at } (1,1), (1,3), (2,1), (2,3), (3,1), (3,3), (4,1), (4,3) \text{ and circles at } (1,2), (2,2), (3,2), (4,2). \\ = \end{array}$$

$$C_8^{14} : \begin{array}{c} \bullet \\ \text{---} \\ \text{---} \end{array} \boxed{H^2} = \begin{array}{c} \bullet \\ \text{---} \\ \text{---} \end{array} \boxed{H^2}$$

$$C_{13}^4 : \begin{array}{c} \text{Diagram A} \\ \text{Diagram B} \end{array} = \begin{array}{c} \text{Diagram C} \\ \text{Diagram D} \end{array}$$

Lem U

$$R_{58} : \begin{array}{c} \text{Diagram showing } R_{58} \text{ as a sequence of gates: } H^2, S^2, S^{12}, S \\ \text{Diagram showing } R_{58} \text{ as a sequence of gates: } H, S^2, \oplus, S \end{array} = w$$

Proof cont.

R₅₈:

R₅₈:

WTS

$$||| C_8^{1*}, C_8^{6*}$$

R₅₈: **WTS**

$$C_8^3 \quad || \quad C_8^{14}$$

The diagram illustrates the decomposition of the R_{58} gate into two parts: **WTS** and a sequence of gates.

WTS: The first part is a sequence of three H^2 gates (indicated by orange boxes) applied to qubits 5 and 8. This is followed by a double horizontal line representing a swap operation between qubits 5 and 8.

Sequence: The second part consists of a sequence of four gates: H^2 , S^2 , H^2 , and S^2 , followed by another H^2 gate (indicated by an orange box). This sequence is preceded by a double horizontal line.

The entire circuit is labeled with a multiplier $\cdot w^2$.

Def I || C₂

The diagram shows two rows of nodes connected by horizontal lines. The top row has 8 nodes, and the bottom row has 10 nodes. A vertical dashed line connects the 5th node from the left in both rows. The first four nodes in the top row are solid black circles. The next three nodes are open circles with a plus sign. The last two nodes are solid black circles. The first five nodes in the bottom row are solid black circles. The next two nodes are open circles with a plus sign. The last three nodes are solid black circles. An orange box highlights the first four nodes in the top row. To the right of the boxes, there is a horizontal line with the label "WTS" above it. Below the bottom row, there is a sequence of operations: a blue square labeled "S'^{12}" followed by a white square labeled "S^2" followed by a dot followed by "w^2".

III C₁₆²²

The diagram illustrates the WTS (Weighted Tree Search) algorithm for the problem R58. On the left, a search tree is shown with nodes labeled R58:, WTS, and S². The right side shows the resulting search tree after pruning, with nodes labeled S² and w².

$$\begin{array}{ll}
R_{25}^2: & \text{Diagram showing } R_{25}^2 \cdot w^2 = \text{Diagram with } S'^2 \text{ and } S^2 \\
& \text{Diagram with } S'^2 \text{ and } S^2 \quad \text{Diagram with } S \quad \text{Diagram with } S' \\
R_{24}: & \text{Diagram with } S \quad = \quad \text{Diagram with } S \\
R_{24}^1: & \text{Diagram with } S \quad = \quad \text{Diagram with } S' \\
C_{16}^{23}: & \text{Diagram with } C_{16}^{23} = \text{Diagram with } C_{16}^{23} \\
C_{16}^{24}: & \text{Diagram with } C_{16}^{24} = \text{Diagram with } C_{16}^{24} \\
C_{13}^4: & \text{Diagram with } C_{13}^4 = \text{Diagram with } C_{13}^4 \\
C_6^1: & \text{Diagram with } C_6^1 = \text{Diagram with } C_6^1 \\
C_6^{2*}: & \text{Diagram with } C_6^{2*} = \text{Diagram with } C_6^{2*} \\
C_{13}^3: & \text{Diagram with } C_{13}^3 = \text{Diagram with } C_{13}^3
\end{array}$$

Lem U

$$R_{58}: \text{Diagram with } H^2 \text{ and } H \quad = \quad \text{Diagram with } H^2, S^2, S'^2, S \cdot w$$

Proof cont.

$$\begin{array}{ll}
R_{58}: & \text{Diagram with } C_{13}^4 \text{ (orange box)} \text{ and } C_{16}^{23}, C_{13} \text{ (green box)} \quad \text{WTS} \quad \text{Diagram with } S'^2, S^2 \cdot w^2 \\
& C_{13}^4 \parallel C_{16}^{23}, C_{13} \\
R_{58}: & \text{Diagram with } C_{16}^{24} \text{ (orange box)} \text{ and } C_6^{2*} \text{ (green box)} \quad \text{WTS} \quad \text{Diagram with } S'^2, S^2 \cdot w^2 \\
& C_{16}^{24} \parallel C_6^{2*} \\
R_{58}: & \text{Diagram with } C_6^1 \text{ (orange box)} \text{ and } C_6^{2*} \text{ (green box)} \quad \text{WTS} \quad \text{Diagram with } S'^2, S^2 \cdot w^2 \\
& C_6^1 \parallel C_6^{2*} \\
R_{58}: & \text{Diagram with } C_{13}^3 \quad \text{WTS} \quad \text{Diagram with } S'^2, S^2 \cdot w^2
\end{array}$$

Then $R_{58} \cdot \text{LHS} :=$

$$\begin{array}{ccc}
\text{Diagram with } C_{13}^3 & \text{Diagram with } R_{25}^2 & \text{Diagram with } C_{13}^3 \\
& \text{Diagram with } S'^2, S^2 \cdot w^2 & \text{Diagram with } S'^2, S^2 \cdot w^2 \\
\frac{R_{24}}{R_{24}^1} & \text{Diagram with } C_6^2 & \text{Diagram with } C_6^2 \\
& \text{Diagram with } S'^2, S^2 \cdot w^2 & \text{Diagram with } S'^2, S^2 \cdot w^2 =: R_{58} \cdot \text{RHS}
\end{array}$$

$$C_1: w^3 = I \quad C_2: H^4 = I \quad C_3: S^3 = I$$

$$\text{Def 1: } [S'] := [H] [H] [S] [H] [H]$$

$$C_8^{14}: \begin{array}{c} \bullet \\ \oplus \\ \otimes \end{array} \begin{array}{c} H^2 \\ H^2 \end{array} = \begin{array}{c} H^2 \\ H^2 \end{array} \begin{array}{c} \bullet \\ \oplus \end{array}$$

$$C_8^3: \begin{array}{c} \bullet \\ \oplus \end{array} \begin{array}{c} H^2 \\ H^2 \end{array} = \begin{array}{c} H^2 \\ H^2 \end{array} \begin{array}{c} \bullet \\ \oplus \end{array}$$

$$\text{Def 7: } \begin{array}{c} \oplus \\ \otimes \end{array} := \begin{array}{c} X \\ X \end{array}$$

$$\text{Def 5: } \begin{array}{c} \bullet \\ \otimes \end{array} := \begin{array}{c} \bullet \\ X \\ X \end{array}$$

$$\text{Def 2: } \begin{array}{c} \oplus \\ \otimes \end{array} := \begin{array}{c} H \\ H^2 \end{array}$$

$$R_{23-25}^1: \begin{array}{c} \bullet \\ \otimes \\ S^2 \end{array} = \begin{array}{c} \bullet \\ \otimes \\ S^2 \end{array} \cdot w$$

$$\text{Def 4: } \begin{array}{c} \bullet \\ \otimes \\ H \end{array} := \begin{array}{c} H \\ H^2 \end{array}$$

Lem V Def1-2, Def4-5, Def7, $C_1, C_2, C_3, C_5, C_6, C_7, C_8, C_{13}, C_{16}, R_{15}, R_{23}, R_{25}$ imply

$$R_{55}: \begin{array}{c} \bullet \\ \otimes \\ H^2 \end{array} = \begin{array}{c} \bullet \\ \otimes \\ H \end{array} \cdot w \cdot \begin{array}{c} \bullet \\ \otimes \\ H^2 \\ S^2 \\ H \\ H \\ S' \end{array}$$

$$\text{Proof: } R_{55} \cdot \text{RHS} := \begin{array}{c} \bullet \\ \otimes \\ H \end{array} \cdot w \cdot \begin{array}{c} \bullet \\ \otimes \\ H^2 \\ S^2 \\ H \\ H \\ S' \end{array}$$

$$\frac{\text{Def 1}}{C_2} \begin{array}{c} \bullet \\ \otimes \\ H \end{array} \cdot w \cdot \begin{array}{c} \bullet \\ \otimes \\ H^2 \\ S^2 \\ H \\ H^2 \\ S^2 \\ H^2 \\ S \\ H^2 \end{array} \cdot w$$

$$\frac{\text{Def 4}}{C_2} \begin{array}{c} \bullet \\ \otimes \\ H \end{array} \cdot w \cdot \begin{array}{c} \bullet \\ \otimes \\ H^2 \\ S^2 \\ H^2 \\ S^2 \\ H^2 \\ S \\ H^2 \end{array} \cdot w$$

$$\frac{R_{23-25}^1}{C_2} \begin{array}{c} \bullet \\ \otimes \\ H \end{array} \cdot w \cdot w \cdot \begin{array}{c} \bullet \\ \otimes \\ H^2 \\ H^2 \\ H^2 \\ S^2 \\ H^2 \\ S^2 \\ H^2 \\ S^2 \\ H^2 \end{array} \cdot w \cdot w$$

$$\frac{C_8^3, C_8^{14}}{C_3} \begin{array}{c} \bullet \\ \otimes \\ H \end{array} \cdot w^2 \cdot \begin{array}{c} \bullet \\ \otimes \\ H^2 \\ S^2 \\ H^2 \\ S^2 \\ H^2 \\ H^2 \end{array} \cdot w^2$$

$$\frac{\text{Def 1}}{C_2} \begin{array}{c} \bullet \\ \otimes \\ H \end{array} \cdot w^2 \cdot \begin{array}{c} \bullet \\ \otimes \\ S'^2 \\ S^2 \\ H^2 \end{array}$$

$$C_1: w^3 = I \quad C_2: H^4 = I \quad C_3: S^3 = I$$

$$C_8^{14} : \begin{array}{c} \bullet \\ \text{---} \\ | \quad | \\ \oplus \quad \otimes \end{array} \begin{array}{c} H^2 \\ \boxed{H^2} \\ H^2 \end{array} = \begin{array}{c} \text{---} \\ \bullet \\ | \quad | \\ \otimes \quad \oplus \end{array} \begin{array}{c} H^2 \\ \boxed{H^2} \\ H^2 \end{array}$$

$$C_8^3: \quad \begin{array}{c} \text{---} \bullet \\ | \\ \text{---} \bullet \\ | \\ \text{---} \bullet \end{array} \boxed{H^2} = \begin{array}{c} \text{---} \bullet \\ | \\ \text{---} \bullet \\ | \\ \text{---} \bullet \end{array} \boxed{H^2}$$

$$C_8^{1*} = \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array}$$

$$C_8^{6*} : \begin{array}{c} \text{---} \\ \oplus \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \oplus \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \oplus \\ \text{---} \\ \boxed{H^2} \end{array}$$

$$C_8^7: \quad \begin{array}{c} \bullet \\ \text{---} \\ \odot \end{array} = \quad \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \odot \end{array}$$

$$\text{Def 4: } \begin{array}{c} \text{Diagram A} \\ \text{Diagram B} \end{array} := \begin{array}{c} \text{Diagram C} \\ \text{Diagram D} \end{array}$$

$$C_{16}^{19} : \quad \begin{array}{c} \text{Diagram of } C_{16}^{19} \\ \text{Diagram of } C_{16}^{19} \end{array} = \quad \begin{array}{c} \text{Diagram of } C_{16}^{19} \\ \text{Diagram of } C_{16}^{19} \end{array}$$

Lem V

$$R_{55} : \quad \begin{array}{c} \text{Diagram showing } R_{55} \text{ as a sequence of operations: } H^2, H, H^2, S^2, H, S' \\ \text{Diagram showing } R_{55} \text{ as a sequence of operations: } H, H^2, S^2, H^2, S^2, H, S' \end{array} = w$$

Proof cont.

$$R_{55} \cdot RHS = \text{Diagram} \cdot w^2$$

Hence R_{55} :

The diagram illustrates the decomposition of the R_{55} gate into a sequence of operations. On the left, the R_{55} gate is shown as a box labeled "WTS". To its right is an equals sign. Following the equals sign is a quantum circuit consisting of two horizontal lines representing qubits. The top line starts with a control circle, followed by a CNOT gate (represented by a circle with a dot), then a H^2 gate (represented by a blue square). The bottom line starts with a H gate (represented by a blue square), followed by a H^3 gate (represented by a blue square), then a H^2 gate (represented by a blue square). A green bracket groups the first three operations on each line. To the right of this group is another green bracket enclosing the entire sequence of operations. This is followed by another equals sign. To the right of the second equals sign is a quantum circuit consisting of two horizontal lines. The top line has four control circles and two CNOT gates. The bottom line has a H^2 gate, followed by a sequence of four control circles and two CNOT gates, then a S'^2 gate (represented by a blue square), a S^2 gate (represented by a blue square), and a H^2 gate (represented by a blue square). A green bracket groups the first seven operations on each line. To the right of this group is a multiplier $\cdot w^2$.

Def 4 ||| $C_8^{1*}, C_8^{6*}, C_8^7$

R_{55} : WTS

The diagram illustrates a neural network layer. It consists of two horizontal rows of nodes. The top row contains 10 nodes, each represented by a black dot. The bottom row contains 6 nodes, also represented by black dots. Vertical lines connect corresponding nodes between the two rows. Above the top row, there are three small circles containing plus signs (+), representing bias terms. To the right of the bottom row, there are two boxes labeled S'^2 and S^2 , which likely represent activation functions. A label $\cdot w^2$ is positioned at the far right end of the bottom row.

$$C_1: w^3 = I \quad C_2: H^4 = I \quad C_3: S^3 = I \quad C_6^*: \quad = \quad$$

$$C_{16}^{21}: \quad = \quad$$

$$C_6^{2*}: \quad = \quad C_{13}^9: \quad = \quad C_{13}^3: \quad = \quad$$

$$R_{25}'': \quad = \quad [S^2] [S^2] \cdot w^2$$

Lem V

$$R_{55}: \quad = \quad$$

Proof cont.

$$R_{55}: \quad \boxed{\text{WTS}} \quad \boxed{C_6^* \parallel C_{16}^{21}}$$

$$R_{55}: \quad \boxed{\text{WTS}} \quad [S'^2] [S^2] \cdot w^2$$

$$R_{55}: \quad \boxed{\text{WTS}} \quad \boxed{C_6^{7'} \parallel}$$

$$C_6, C_6^{2*} \parallel C_{13}^9, C_{13}^3$$

$$R_{55}: \quad \boxed{\text{WTS}} \quad [S'^2] [S^2] \cdot w^2$$

$$\text{Then } R_{55}.LHS := \frac{\boxed{C_6^2}}{\boxed{R_{25}''}} = [S'^2] [S^2] \cdot w^2 =: R_{55}.RHS$$

$$\begin{aligned}
C_1: w^3 &= I & C_2: H^4 &= I & C_3: S^3 &= I & C_5: SS' &= S'S \\
\text{Def 1: } S' &:= \boxed{H} \quad \boxed{H} \quad \boxed{S} \quad \boxed{H} \quad \boxed{H} & C_8: &= \text{Diagram} & C_8: &= \text{Diagram} \\
&& C_8: &= \text{Diagram} & \text{Def 2: } &:= \text{Diagram} \\
\text{Def 7: } &:= \text{Diagram} & \text{Def 5: } &:= \text{Diagram} & \text{Def 4: } &:= \text{Diagram} \\
\text{R}'_{23}: & \text{Diagram} = \text{Diagram} \cdot w^2
\end{aligned}$$

Lem W Def1-2, Def4-5, Def7, $C_1, C_2, C_3, C_5, C_6, C_7, C_8, C_{13}, C_{16}, R_{15}, R_{23}, R_{25}$ imply

$$R_{56}: \text{Diagram} = \text{Diagram}$$

$$\text{Proof: } R_{56} \cdot \text{RHS} := \text{Diagram}$$

$$C_5: \text{Diagram} = \text{Diagram}$$

$$\text{Def 1: } \text{Diagram} = \text{Diagram}$$

$$\frac{\text{Def 4}}{C_2}: \text{Diagram} = \text{Diagram}$$

$$\frac{R'_{23}}{C_3}: \text{Diagram} = \text{Diagram} \cdot w^2$$

$$\frac{C_8, C_8}{C_3}: \text{Diagram} = \text{Diagram} \cdot w^2$$