

$$R_{23}^{12}: \quad \text{Circuit diagram} = \text{Circuit diagram} \cdot w^2$$

$$R_{23}^{15}: \quad \text{Circuit diagram} = \text{Circuit diagram} \cdot w^2$$

Lem D1

$$R_{59}: \quad \text{Circuit diagram} = \text{Circuit diagram} \cdot w$$

Proof cont.

$$R_{59}: \quad \text{Circuit diagram} \xrightarrow{\text{WTS}} \text{Circuit diagram} \cdot (-w^2)$$

$$R_{59} \cdot \text{RHS} = \text{Circuit diagram} \cdot (-1)$$

$$= \text{Circuit diagram} \cdot (-1)$$

$$R_{23}^{15}: \quad \text{Circuit diagram} \cdot (-1) \cdot w^2$$

$$= \text{Circuit diagram} \cdot (-1) \cdot w^2$$

$$R_{23}^{15}: \quad \text{Circuit diagram} \cdot (-1) \cdot w^2 \cdot w^2$$

$$= \text{Circuit diagram} \cdot (-w)$$

$$R_{23}^{12}: \quad \text{Diagram} = \text{Diagram} \cdot w^2 \quad R_{23}^{15}: \quad \text{Diagram} = \text{Diagram} \cdot w^2 = \text{Diagram} \cdot w^2$$

Lem D1

$$R_{59}: \quad \text{Diagram} = \text{Diagram} \cdot w \quad \text{Diagram} = \text{Diagram} \cdot w$$

Proof cont.

$$R_{59}: \quad \text{Diagram} \stackrel{\text{WTS}}{=} \text{Diagram} \cdot (-w)$$

$$R_{59} \cdot \text{RHS} = \text{Diagram} \cdot (-w) \quad \boxed{\text{Diagram}}_P$$

$$P = \text{Diagram} = \text{Diagram}$$

$$= \text{Diagram} = \text{Diagram}$$

$$\underline{R_{23}^{15}}: \quad \text{Diagram} \cdot w^2$$

$$= \text{Diagram} \cdot w^2 \stackrel{\text{Defl}}{=} S H^2 S' H S'^2 S = S \boxed{H^2 H^2} S \boxed{H^2 H H^2} S^2 H^2 S \\ = S^2 H S^2 H^2 S = H^3 S H^3 H^2 S \cdot (-w) = H^3 S H S \cdot (-w)$$

$$R_{59} \cdot \text{RHS} = \text{Diagram} \cdot (-w) \cdot w$$

$$= \text{Diagram} \cdot (-1) \cdot (-w)$$

$$C_4^1 : \quad \boxed{S} - \boxed{S} - \boxed{H} - \boxed{S} - \boxed{S} = \quad \boxed{H} - \boxed{H} - \boxed{H} - \boxed{S} - \boxed{H} - \boxed{H} - \boxed{H} - \boxed{H} \cdot (-\omega)$$

Lem D1

$$R_{59} : \text{Diagram} = \text{Diagram} \cdot w$$

Proof cont.

The diagram illustrates the decomposition of the R_{59} gate into a sequence of quantum operations. On the left, the R_{59} gate is shown as a rotation around the fifth qubit axis. To its right, the label "WTS" is written above a bracket indicating the first part of the decomposition. The decomposition consists of two main parts: a sequence of operations labeled "WTS" and a final X^2 gate. The "WTS" sequence is composed of several gates: Hadamard (H), controlled-NOT ($CNOT$), and single-qubit operations S^2 , S' , and S'' . The final part of the decomposition is a X^2 gate, which is equivalent to two consecutive X gates.

The diagram illustrates a quantum circuit with four horizontal lines representing qubits. The circuit consists of the following sequence of operations:

- Region A:** A red box containing two CNOT gates with control on the top line and target on the bottom line.
- Region B:** A green box containing a sequence of operations: H, H^2 , S^2 , H, S, S' on the top line, and H, H^2 , S^2 , H, S, S' on the bottom line.
- Region A:** A red box containing two CNOT gates with control on the top line and target on the bottom line.
- Region C:** A blue box containing a sequence of operations: H, S' , H^3 , S on the top line, and H^3 , S, H, S on the bottom line.
- WTS:** A purple box labeled "WTS" positioned between Region B and Region A.
- Measurement:** A meter symbol at the end of the top line.

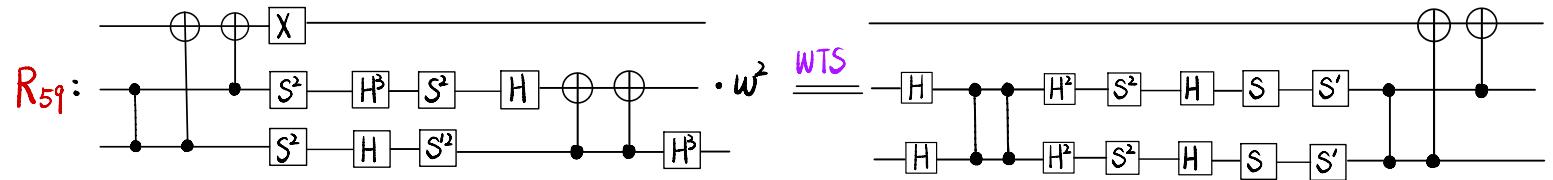
$$A \stackrel{WTS}{=} BAC$$

composition in diagrammatic order

$$AC^{-1} \stackrel{WTS}{=} BA$$

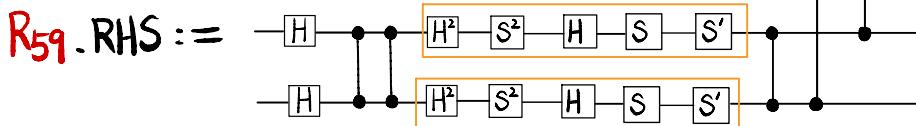
$R_{59}:$

WTS



$$\begin{array}{ccccccccc} X & & & & & & & & \\ \cancel{Z} & H^2 & X^2 & S^2 & H & S & H^2 & X^2 & Z^2 \\ & \cancel{Z^2} & & \cancel{Z^2} & & & \cancel{Z^2} & & \\ & & & & X & & & & \\ & & & & \cancel{XZ} & H^2 & X^2 & S & H^2 \\ & & & & & \cancel{XZ} & \cancel{Z^2} & & \\ & & & & & & & w^2 X^2 & \\ & & & & & & & & w^2 XZ \end{array}$$

$$H^2 - S^2 - H - S - S' = H^3 - S - H - X - Z^2 \cdot (-w) \quad (*)$$



$$(*) = \begin{array}{c} H \\ H \end{array} \cdot (-w)$$

$$= \begin{array}{c} H \\ H \end{array} \cdot (-w)$$

$$R10: \boxed{Z} = \boxed{S'} \boxed{S'} \boxed{S}$$

$$C_4^1: \boxed{S} \boxed{S} \boxed{H} \boxed{S} \boxed{S} = \boxed{H} \boxed{H} \boxed{H} \boxed{S} \boxed{H} \boxed{H} \boxed{H} \cdot (-w)$$

Hence R_{59} :

$\cdot w^2$ WTS

$\cdot (-w)$

R_{59} :

$\cdot (-w^2)$

$\cdot (-w^2)$

$R_{59}, RHS :=$

$\cdot (-w^2)$

$\cdot (-w^2)$

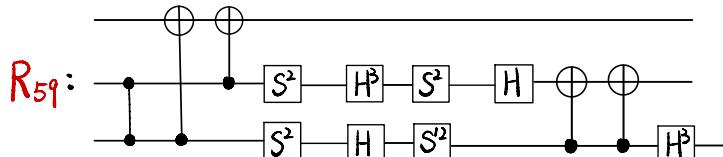
$\cdot (-w^2)$

$\cdot (-w^2)$

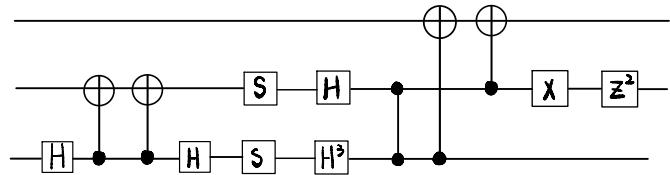
$$H^2 S^2 H S^2 \stackrel{C_4}{=} H^2 H^3 S H^3 \cdot (-w) \stackrel{C_2}{=} H S H^3 \cdot (-w)$$

$\cdot (-w^2)$

$$R_{II} : \underline{\underline{z^2}} = \underline{\underline{s' s s}}$$

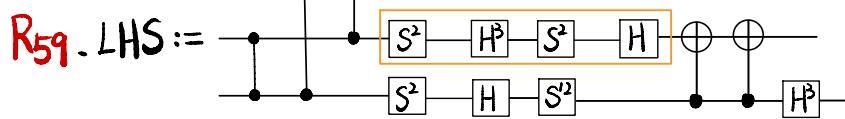


WTS



$$\begin{matrix} X \\ z \end{matrix} \underline{\underline{S^2}} \begin{matrix} Xz \\ z \end{matrix} \underline{\underline{H^3}} \begin{matrix} w^2 x z^2 \\ x \end{matrix} \underline{\underline{S^2}} \begin{matrix} w^2 x \\ xz \end{matrix} \underline{\underline{H}} \begin{matrix} w^2 z \\ w^2 x^2 z \end{matrix}$$

$$\underline{\underline{S^2}} \underline{\underline{H^3}} \underline{\underline{S^2}} \underline{\underline{H}} = \underline{\underline{H}} \underline{\underline{S}} \underline{\underline{X}} \underline{\underline{z^2}} \cdot (-w^2) \quad (**)$$



$$= \underline{\underline{H}} \underline{\underline{S}} \underline{\underline{X}} \underline{\underline{z^2}} \cdot (-w^2)$$

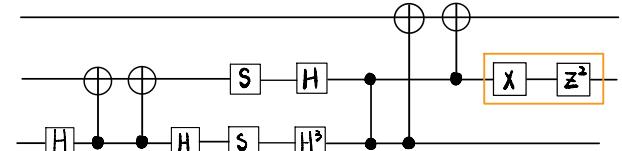
$$= \underline{\underline{H}} \underline{\underline{S}} \underline{\underline{X}} \underline{\underline{z^2}} \cdot (-w^2)$$

$$\begin{matrix} z^2 & z^2 & z^2 \\ I & z & z^2 \end{matrix}$$

$$= \underline{\underline{H}} \underline{\underline{S}} \underline{\underline{X}} \underline{\underline{z^2}} \cdot (-w^2)$$

$R_{59}:$

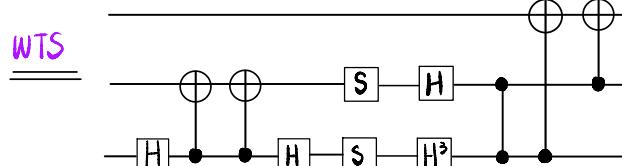
$$\underline{\underline{H}} \underline{\underline{S}} \underline{\underline{X}} \underline{\underline{z^2}} \cdot (-w^2) \quad \text{WTS}$$



|||

$R_{59}:$

$$\underline{\underline{H}} \underline{\underline{S}} \underline{\underline{S^2}} \underline{\underline{z^2}} \cdot (-w^2)$$

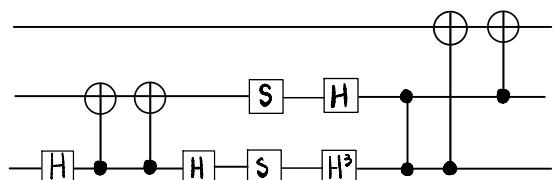


$$S'^2 z^2 \stackrel{R_{II}}{=} S'^2 S' S^2 = S^2$$

|||

$R_{59}:$

$$\underline{\underline{H}} \underline{\underline{S}} \underline{\underline{S^2}} \underline{\underline{H^3}} \cdot (-w^2) \quad \text{WTS}$$



$$C_4^1 : \text{---} S \text{---} S \text{---} H \text{---} S \text{---} S = \text{---} H \text{---} H \text{---} H \text{---} S \text{---} H \text{---} H \text{---} H \cdot (-w)$$

$$R_{23}^1 : \text{---} S \text{---} \oplus \text{---} = \text{---} \oplus \text{---} \bullet S \text{---} \bullet S' \text{---} \cdot w^2$$

$$R_{59}^1 : \text{---} \oplus \text{---} \bullet H \text{---} S \text{---} \bullet \oplus \text{---} \bullet \oplus \text{---} \cdot (-w^2) \xrightarrow{\text{WTS}} \text{---} \bullet \oplus \text{---} \bullet S \text{---} H \text{---} \bullet \oplus \text{---} \bullet H \text{---} S \text{---} H^3 \text{---} \bullet \oplus \text{---}$$

$$R_{59}^1 : \text{---} \oplus \text{---} \bullet H \text{---} S \text{---} \bullet \oplus \text{---} \bullet \oplus \text{---} \cdot (-w^2) \xrightarrow{\text{WTS}} \text{---} \bullet \oplus \text{---} \bullet S \text{---} H \text{---} \bullet \oplus \text{---} \bullet H \text{---} S \text{---} H^3 \text{---} \bullet \oplus \text{---}$$

$$R_{59}^1 : \text{---} \oplus \text{---} \bullet H \text{---} S \text{---} \bullet \oplus \text{---} \bullet \oplus \text{---} \bullet H^3 \text{---} S \text{---} \bullet H^3 \text{---} \bullet H^3 \text{---} \bullet \oplus \text{---} \xrightarrow{\text{WTS}} \text{---} \bullet \oplus \text{---} \bullet S \text{---} H \text{---} \bullet \oplus \text{---} \bullet H \text{---} S \text{---} H^3 \text{---} \bullet \oplus \text{---}$$

$$R_{59}^1 : \text{---} \oplus \text{---} \bullet H \text{---} S \text{---} \bullet \oplus \text{---} \bullet H^3 \text{---} S \text{---} \bullet H \text{---} \bullet \oplus \text{---} \bullet H^2 \text{---} H^3 \text{---} \bullet \oplus \text{---} \xrightarrow{\text{WTS}} \text{---} \bullet \oplus \text{---} \bullet S \text{---} H \text{---} \bullet \oplus \text{---} \bullet H \text{---} S \text{---} H^3 \text{---} \bullet \oplus \text{---}$$

$$R_{59}^1 : \text{---} \bullet \oplus \text{---} \bullet H \text{---} S \text{---} \bullet \oplus \text{---} \bullet H^3 \text{---} S \text{---} \bullet H \text{---} \bullet H \text{---} \bullet \oplus \text{---} \xrightarrow{\text{WTS}} \text{---} \bullet \oplus \text{---} \bullet S \text{---} H \text{---} \bullet \oplus \text{---} \bullet H \text{---} S \text{---} H^3 \text{---} \bullet \oplus \text{---}$$

$$R_{59}^1 : \text{---} \bullet \oplus \text{---} \bullet H \text{---} S \text{---} \bullet \oplus \text{---} \bullet H^3 \text{---} S \text{---} \bullet H \text{---} \bullet H \text{---} \bullet \oplus \text{---} \xrightarrow{\text{WTS}} \text{---} \bullet \oplus \text{---} \bullet S \text{---} H \text{---} \bullet \oplus \text{---} \bullet H \text{---} S \text{---} H^3 \text{---} \bullet \oplus \text{---}$$

Then $R_{59}^1 \cdot \text{RHS} :=$

$$\begin{array}{l}
 R_{23}^1: \quad \text{Circuit diagram} = \text{Circuit diagram} \cdot w^2 \quad R_{23}^4: \quad \text{Circuit diagram} = \text{Circuit diagram} \cdot w^2 \\
 C_{16}^1: \quad \text{Circuit diagram} = \text{Circuit diagram} \\
 C_{13}^2: \quad \text{Circuit diagram} = \text{Circuit diagram}
 \end{array}$$

$$\begin{aligned}
 R_{59} \cdot \text{RHS} &= \text{Circuit diagram} \\
 &= \text{Circuit diagram} \cdot w^2 \\
 &= \text{Circuit diagram} \cdot w^2 \cdot w^2 \\
 &= \text{Circuit diagram} \cdot w \\
 &= \text{Circuit diagram} \cdot w
 \end{aligned}$$

$$R_{23}^{q'}: \quad \text{Diagram} = \text{Diagram} \cdot w^2$$

$$R_{23}^4: \quad \text{Diagram} = \text{Diagram} \cdot w^2$$

$$R_{59}: \quad \text{Diagram} \xrightarrow{\text{WTS}} \text{Diagram}$$

$$R_{59} \cdot \text{RHS} = \text{Diagram} \quad Q = \boxed{\text{Diagram}}$$

$$Q = \text{Diagram} = \text{Diagram} = \text{Diagram}$$

$$= \text{Diagram} \cdot w^2 = \text{Diagram} \cdot w^2$$

$$= \text{Diagram} \cdot w^2$$

$$\text{Then } R_{59} \cdot \text{RHS} = \text{Diagram} \cdot w \cdot w^2$$

$$= \text{Diagram} = \text{Diagram}$$

$$= \text{Diagram} = \text{Diagram}$$

$$= \text{Diagram} = \text{Diagram} \cdot w^2$$