

$$R_{31}: (1) \quad \text{Diagram} = \text{Diagram}$$

$$(2) \quad \text{Diagram} = \text{Diagram}$$

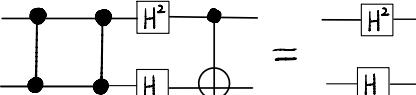
$$R_{16}: \quad \text{Diagram} = \text{Diagram}$$

$$R_{17}: \quad \text{Diagram} = \text{Diagram}$$

$$R_{19}: (1) \quad \text{Diagram} = \text{Diagram}$$

$$(2) \quad \text{Diagram} = \text{Diagram}$$

$$C_2: H^4 = I$$

Lem N R_{32} :  =  ✓

$$R_{32}^1: \quad \text{Diagram} = \text{Diagram} \quad \checkmark$$

$$R_{32}^2: \quad \text{Diagram} = \text{Diagram} \quad \checkmark$$

$$R_{32}^3: \quad \text{Diagram} = \text{Diagram} \quad \checkmark$$

$$R_{32}^4: \quad \text{Diagram} = \text{Diagram} \quad \checkmark$$

$$R_{32}^5: \quad \text{Diagram} = \text{Diagram} \quad \checkmark$$

$$R_{32}^6: \quad \text{Diagram} = \text{Diagram} \quad \checkmark$$

$$R_{32}^7: \quad \text{Diagram} = \text{Diagram} \quad \checkmark$$

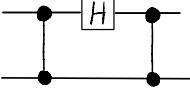
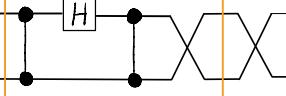
$$R_{32}^8: \quad \text{Diagram} = \text{Diagram} \quad \checkmark$$

$$R_{32}^9: \quad \text{Diagram} = \text{Diagram} \quad \checkmark$$

$$R_{32}^{10}: \quad \text{Diagram} = \text{Diagram} \quad \checkmark$$

$$R_{32}^{11}: \quad \text{Diagram} = \text{Diagram}$$

$$R_{32}^{12}: \quad \text{Diagram} = \text{Diagram}$$

Proof cont. $R_{32}^8 . LHS :=$  $\underset{R_{16}}{=} \quad$  $\underset{R_{17}/R_{19}}{=} \quad$  $\underset{R_{32}^7}{=} \quad$

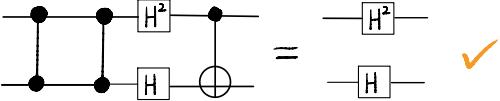
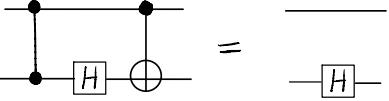
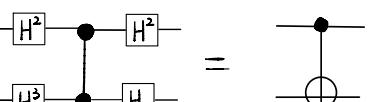
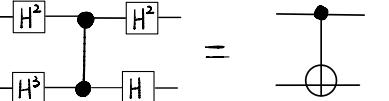
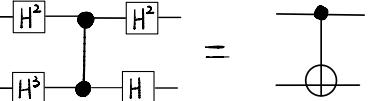
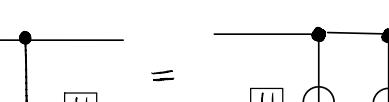
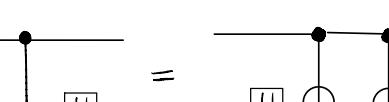
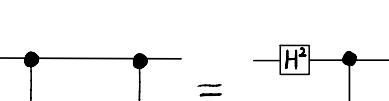
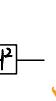
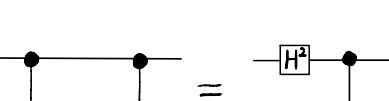
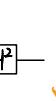
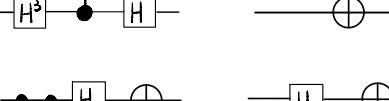
$$\text{Diagram} \underset{R_{31}}{=} \text{Diagram} \quad \underset{R_{16}}{=} \text{Diagram} \quad \underset{R_{16}}{=} \text{Diagram} =: R_{32}^8 . RHS$$

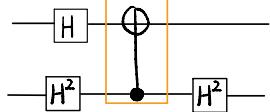
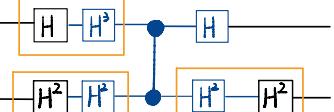
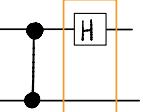
$$R_{32}^3: \quad \text{Diagram} = \text{Diagram} \quad \underset{C_2}{=} \quad \text{Diagram} = \text{Diagram} \quad : R_{32}^9$$

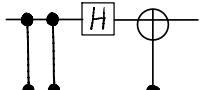
$$R_{32}^4: \quad \text{Diagram} = \text{Diagram} \quad \underset{C_2}{=} \quad \text{Diagram} = \text{Diagram} \quad : R_{32}^9$$

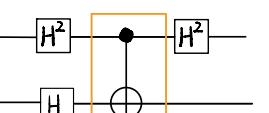
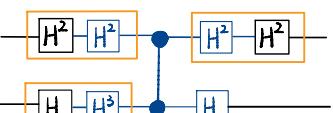
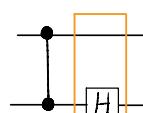
$$\text{Diagram} = \text{Diagram} : R_{32}^{10}$$

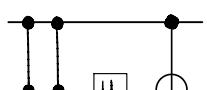
C2: $H^4 = I$

Lem N R_{32} :		$=$		✓
R'_{32} :		$=$		✓
R''_{32} :		$=$		✓
R^3_{32} :		$=$		✓
R^4_{32} :		$=$		✓
R^5_{32} :		$=$		✓
R^6_{32} :		$=$		✓
R^7_{32} :		$=$		✓
R^8_{32} :		$=$		✓
R^9_{32} :		$=$		✓
R^{10}_{32} :		$=$		✓
R^{11}_{32} :		$=$		✓
R^{12}_{32} :		$=$		✓

Proof cont. $R_{32}^{11} \cdot \text{RHS} :=$  $\underset{\text{R32}^4}{\equiv}$  $\underset{\text{C}_2}{\equiv}$ 

$\underset{\text{R32}^2}{\equiv}$  $=: R_{32}^{11} \cdot \text{LHS}$

$R_{32}^{12} \cdot \text{RHS} :=$  $\underset{\text{R32}^3}{\equiv}$  $\underset{\text{C}_2}{\equiv}$ 

$\underset{\text{R32}^1}{\equiv}$  $=: R_{32}^{12} \cdot \text{LHS}$

□

$$C_8 : (1) \quad \begin{array}{c} \text{---} \\ | \\ \bullet \end{array} \quad \boxed{\mathbb{H}^2} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \bullet \end{array} \quad \boxed{\mathbb{H}^2} \quad \begin{array}{c} \text{---} \\ | \\ \bullet \end{array}$$

$$(2) \quad \begin{array}{c} \bullet \\ \hline \bullet \\ \hline \end{array} \quad \boxed{H^2} \quad = \quad \begin{array}{c} \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \end{array}$$

$$C_2: H^4 = I$$

$$C_8^1 : (i) \quad \begin{array}{c} \text{---} \\ | \end{array} \boxed{H^2} \text{---} \bullet \text{---} \bullet \text{---} \text{---} = \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \boxed{H^2} \text{---}$$

$$(2) \quad \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \text{---} \\ | \\ \boxed{H^2} \end{array}$$

$$\text{Def 2: } \begin{array}{c} \textcircled{\text{+}} \\ \text{---} \\ \text{---} \end{array} := \begin{array}{ccccccc} \boxed{\text{H}} & \text{---} & \bullet & \boxed{\text{H}} & \text{---} & \boxed{\text{H}} & \text{---} & \boxed{\text{H}} \end{array}$$

$$\begin{array}{c} \bullet \\ \parallel \\ \circ \end{array} := \begin{array}{c} \bullet \\ \parallel \\ \boxed{\text{H}} \quad \bullet \quad \boxed{\text{H}} \quad \boxed{\text{H}} \quad \boxed{\text{H}} \end{array}$$

Lem 0 Def 2, C₂ & C₈ imply

$$C_8^5: \text{ (1)} \quad \begin{array}{c} \oplus \\ \text{---} \\ \bullet \end{array} \quad \boxed{\mathbb{H}^2} = \quad \boxed{\mathbb{H}^2} \quad \begin{array}{c} \oplus \\ \text{---} \\ \bullet \end{array} \quad \begin{array}{c} \oplus \\ \text{---} \\ \bullet \end{array} \quad \checkmark \quad \text{ (2)} \quad \begin{array}{c} \oplus \\ \text{---} \\ \bullet \end{array} \quad \boxed{\mathbb{H}^2} = \quad \begin{array}{c} \oplus \\ \text{---} \\ \bullet \end{array} \quad \begin{array}{c} \oplus \\ \text{---} \\ \bullet \end{array} \quad \checkmark$$

$$C_8^b: \quad (1) \quad \begin{array}{c} \text{H}^2 \\ \square \end{array} \oplus = \quad \begin{array}{c} \oplus \\ \text{H}^2 \\ \square \end{array} \quad \checkmark \quad (2) \quad \begin{array}{c} \oplus \\ \text{H}^2 \\ \square \end{array} \quad = \quad \begin{array}{c} \oplus \\ \text{H}^2 \\ \square \end{array} \quad \checkmark$$

$$C_8^7: \quad (1) \quad \text{Diagram} = \quad \text{Diagram} \quad \checkmark \quad (2) \quad \text{Diagram} = \quad \text{Diagram}$$

$$C_8^8: \quad (1) \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \quad (2) \quad \begin{array}{c} \boxed{H^2} \\ \text{---} \\ | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \boxed{H^2} \end{array}$$

$$C_8: \quad \begin{array}{c} \text{H}^2 \\ \text{H}^2 \end{array} \oplus \quad = \quad \begin{array}{c} \oplus \\ \ominus \end{array} \quad = \quad \begin{array}{c} \oplus \\ \ominus \end{array} \quad \begin{array}{c} \text{H}^2 \\ \text{H}^2 \end{array}$$

$$C_8^{13} : \begin{array}{c} \oplus \\ \square \\ \bullet \\ \square \\ \bullet \\ \square \end{array} = \begin{array}{c} \square \\ \oplus \\ \bullet \\ \square \\ \bullet \\ \square \end{array}$$

$$C_{8:}^{10} = \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array}$$

$$C_8^{14} : \begin{array}{c} \bullet \\ \text{---} \\ \textcircled{\small 1} \end{array} \begin{array}{c} H^2 \\ \boxed{H^2} \\ H^2 \end{array} = \begin{array}{c} \bullet \\ \text{---} \\ \textcircled{\small 1} \end{array} \begin{array}{c} H^2 \\ \boxed{H^2} \\ H^2 \end{array}$$

$$C_8^{II}: \quad \text{Diagram showing } H^2 \text{ blocks connected by a CNOT gate} = \text{Diagram showing two CNOT gates} = \text{Diagram showing } H^2 \text{ blocks connected by a CNOT gate}$$

$$C_8: \quad \begin{array}{c} \text{---} \\ | \end{array} \bullet \begin{array}{c} \text{---} \\ | \end{array} = \begin{array}{c} \bullet \text{---} \\ | \end{array} = \begin{array}{c} \bullet \text{---} \\ | \end{array} \bullet \begin{array}{c} \text{---} \\ | \end{array}$$

$$\begin{aligned}
 \text{Proof: } C_8^5. \text{LHS.}(1) &:= \text{Diagram 1} \stackrel{\text{Def 2}}{=} \text{Diagram 2} = \text{Diagram 3} \stackrel{\text{C}_8}{=} \text{Diagram 4} \\
 &= \text{Diagram 5} \stackrel{\text{Def 2}}{=} \text{Diagram 6} =: C_8^5. \text{RHS.}(1).
 \end{aligned}$$

$$C_8^5 \cdot LHS_{(2)} := \text{Diagram A} \xrightarrow{\text{Def 2}} \text{Diagram B} \xrightarrow{C_8} \text{Diagram C} \xrightarrow{\text{Def 2}} \text{Diagram D} =: C_8^5 \cdot RHS_{(2)}$$

$$C_8^6 \cdot LHS_{(1)} := \text{Diagram 1} \xrightarrow{\text{Def 2}} \text{Diagram 2} \xrightarrow{C_8^1} \text{Diagram 3} \xrightarrow{\text{Def 2}} \text{Diagram 4} =: C_8^6 \cdot RHS_{(1)}$$

$$C_8^6 \cdot LHS_{(2)} := \text{Diagram} \quad \underline{\text{Def 2}} \quad \text{Diagram} \quad \underline{\underline{C_8^1}} \quad \text{Diagram} \quad \underline{\text{Def 2}} \quad \text{Diagram} =: C_8^6 \cdot RHS_{(2)}$$

$$C_8^7 \cdot LHS.(1) := \text{Diagram 1} \xrightarrow{\text{Def 2}} \text{Diagram 2} \xrightarrow{\text{Def 2}} \text{Diagram 3} =: C_8^7 \cdot RHS.(1).$$

$$C_8 : (1) \quad \begin{array}{c} \text{---} \\ | \\ \bullet \end{array} \quad \boxed{\mathbb{H}^2} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \bullet \end{array} \quad \boxed{\mathbb{H}^2} \quad \begin{array}{c} \text{---} \\ | \\ \bullet \end{array}$$

$$(2) \quad = \quad \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array}$$

$$C_2: H^4 = I$$

$$C_8^1 : (1) \quad \begin{array}{c} \text{---} \\ | \end{array} \boxed{\mathbb{H}^2} \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \boxed{\mathbb{H}^2} \text{---} = \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \boxed{\mathbb{H}^2} \text{---}$$

$$(2) \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \boxed{H^2} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \boxed{H^2}$$

$$\text{Def 2: } \begin{array}{c} \textcircled{\text{+}} \\ \text{---} \\ \bullet \end{array} := \begin{array}{c} \boxed{\text{H}} \\ \text{---} \\ \bullet \quad \boxed{\text{H}} \quad \boxed{\text{H}} \quad \boxed{\text{H}} \quad \boxed{\text{H}} \end{array}$$

$$\vdash := \begin{array}{c} \bullet \\ \hline \end{array} \quad \begin{array}{c} \bullet \\ \hline \end{array} \vdash \begin{array}{c} \bullet \\ \hline \end{array} \quad \vdash \begin{array}{c} \bullet \\ \hline \end{array}$$

Lem 0 Def 2, C₂ & C₈ imply

$$C_8^5: \text{ (1)} \quad \begin{array}{c} \oplus \\ \text{---} \\ \bullet \end{array} \quad \boxed{H^2} = \quad \boxed{H^2} \quad \begin{array}{c} \oplus \\ \text{---} \\ \bullet \\ \bullet \end{array} \quad \checkmark \quad \text{ (2)} \quad \begin{array}{c} \oplus \\ \text{---} \\ \bullet \\ \boxed{H^2} \end{array} = \quad \begin{array}{c} \oplus \\ \text{---} \\ \bullet \\ \bullet \end{array} \quad \checkmark$$

$$C_8^b: \quad (1) \quad \begin{array}{c} \text{H}^2 \\ \square \end{array} \otimes = \quad \begin{array}{c} \square \otimes \text{H}^2 \\ \square \end{array} \quad \checkmark \quad (2) \quad \begin{array}{c} \square \otimes \\ \text{H}^2 \end{array} = \quad \begin{array}{c} \square \otimes \\ \text{H}^2 \end{array}$$

$$C_8: \quad (1) \quad \text{Diagram} = \text{Diagram} \quad \checkmark \quad (2) \quad \text{Diagram} = \text{Diagram} \quad \checkmark$$

$$C_8: \quad (1) \quad \text{Diagram} = \quad (2) \quad \text{Diagram} \quad \checkmark \quad (3) \quad \text{Diagram} = \quad (4) \quad \text{Diagram} \quad \checkmark$$

$$C_8^9: \quad \text{Diagram showing } H^2 \otimes H^2 = H^2 \otimes H^2 = H^2 \otimes H^2 \quad \checkmark \quad C_8^{13}: \quad \text{Diagram showing } H^2 \otimes H^2 = H^2 \otimes H^2$$

$$C_{8:}^{10} = \text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3}$$

$$C_{8:}^{13} = \begin{array}{c} \text{H}^2 \\ \text{H}^2 \end{array} \oplus \begin{array}{c} \text{H}^2 \\ \text{H}^2 \end{array}$$

$$C_8^{\prime \prime}: \quad \begin{array}{c} \text{---} \\ | \end{array} \boxed{H^2} \oplus \begin{array}{c} \text{---} \\ | \end{array} \boxed{H^2} = \quad \begin{array}{c} \text{---} \\ | \end{array} \oplus \begin{array}{c} \text{---} \\ | \end{array} = \quad \begin{array}{c} \text{---} \\ | \end{array} \oplus \begin{array}{c} \text{---} \\ | \end{array} \boxed{H^2}$$

$$C_8: \quad \begin{array}{c} \text{---} \\ | \end{array} \bullet \begin{array}{c} \text{---} \\ | \end{array} = \begin{array}{c} \bullet \text{---} \\ | \end{array} = \begin{array}{c} \boxed{H^2} \text{---} \\ | \end{array} \bullet \begin{array}{c} \text{---} \\ | \end{array} \boxed{H^2}$$

Proof cont.

$$C_8^7 \cdot LHS_{(2)} := \text{Diagram 1} \xrightarrow{\text{Def 2}} \text{Diagram 2} \xrightarrow{\underline{C_8}} \text{Diagram 3} \xrightarrow{\text{Def 2} / C_2} \text{Diagram 4} =: C_8^7 \cdot RHS_{(2)}.$$

$$C_8^8 \cdot LHS.(1) := \text{Diagram 1} \stackrel{\text{Def 2}}{\longrightarrow} \text{Diagram 2} \stackrel{C_8^1}{\longrightarrow} \text{Diagram 3} \stackrel{\text{Def 2}}{\longrightarrow} \text{Diagram 4} = C_8^8 \cdot RHS.(1).$$

$$C_8^8 \cdot LHS.(2) := \text{Diagram} \stackrel{\text{Def 2}}{\longrightarrow} \text{Diagram} \stackrel{C_8^1}{\longrightarrow} \text{Diagram} \stackrel{\text{Def 2}}{\longrightarrow} \text{Diagram} =: C_8^8 \cdot RHS.(2).$$

$$C_8^5 : \text{Diagram} = \text{Diagram} \equiv \text{Diagram} = \text{Diagram} \equiv \text{Diagram} = \dots C_8^9$$

$$C_8^5 : \text{Diagram} = \text{Diagram} \equiv \text{Diagram} \equiv C_2 : \text{Diagram} = \text{Diagram} \vdots C_8^9$$

$$C_8 : (1) \quad \begin{array}{c} \text{---} \\ | \\ \bullet \end{array} \quad \boxed{\mathbb{H}^2} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \bullet \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \bullet \end{array}$$

$$(2) \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array}$$

$$C_2: H^4 = I$$

$$C_8^1 : (1) \quad \begin{array}{c} \text{---} \\ | \end{array} \boxed{H^2} \begin{array}{c} \text{---} \\ | \end{array} \bullet \begin{array}{c} \text{---} \\ | \end{array} = \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \text{---} \\ | | | | \end{array} \boxed{H^2}$$

$$(2) \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \boxed{H^2} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \boxed{H^2} \end{array}$$

$$\text{Def 2: } \begin{array}{c} \oplus \\ \text{---} \\ \bullet \end{array} := \begin{array}{ccccccc} \boxed{\text{H}} & \bullet & \boxed{\text{H}} & \boxed{\text{H}} & \boxed{\text{H}} & \boxed{\text{H}} \end{array}$$

$$\begin{array}{c} \bullet \\ \parallel \\ \circ \end{array} \quad := \quad \begin{array}{c} \bullet \\ \parallel \\ \text{---} \\ \boxed{\text{H}} \quad \bullet \quad \boxed{\text{H}} \quad \boxed{\text{H}} \quad \boxed{\text{H}} \end{array}$$

Lem 0 Def 2, C₂ & C₈ imply

$$C_8^5: \text{(1)} \quad \begin{array}{c} \oplus \\ \text{---} \\ \bullet \end{array} \quad \boxed{\mathbb{H}^2} = \quad \begin{array}{c} \mathbb{H}^2 \\ \oplus \\ \bullet \end{array} \quad \checkmark \quad \text{(2)} \quad \begin{array}{c} \oplus \\ \text{---} \\ \bullet \end{array} \quad \boxed{\mathbb{H}^2} = \quad \begin{array}{c} \oplus \\ \text{---} \\ \bullet \end{array} \quad \boxed{\mathbb{H}^2} \quad \checkmark$$

$$C_8^b: \quad (1) \quad \begin{array}{c} \text{H}^2 \\ \square \end{array} \oplus = \quad \begin{array}{c} \oplus \\ \square \end{array} \quad \checkmark \quad (2) \quad \begin{array}{c} \oplus \\ \square \end{array} \quad = \quad \begin{array}{c} \oplus \\ \square \end{array} \quad \checkmark$$

$$C_8: \quad (1) \quad \text{Diagram} = \text{Diagram} \quad \checkmark \quad (2) \quad \text{Diagram} = \text{Diagram} \quad \checkmark$$

$$C_8: \quad (1) \quad \text{Diagram} = \quad (2) \quad \text{Diagram} \quad \checkmark \quad (3) \quad \text{Diagram} = \quad (4) \quad \text{Diagram} \quad \checkmark$$

$$C_8^9: \quad \text{Diagram showing } H^2 \otimes H^2 = H^2 \otimes H^2 = \text{Diagram showing } H^2 \otimes H^2 \quad \checkmark \quad C_8^{13}: \quad \text{Diagram showing } H^2 \otimes H^2 = H^2 \otimes H^2$$

$$C_8^{10}: \quad \begin{array}{c} \text{---} \\ | \quad | \quad | \\ \text{H}^2 \quad \oplus \quad \oplus \\ | \quad | \quad | \\ \text{H}^2 \end{array} = \quad \begin{array}{c} \text{---} \\ | \\ \text{H}^2 \end{array} = \quad \begin{array}{c} \text{---} \\ | \quad | \quad | \\ \boxed{\text{H}^2} \quad \text{H}^2 \quad \boxed{\text{H}^2} \\ | \quad | \quad | \\ \oplus \quad \oplus \quad \oplus \\ | \quad | \quad | \\ \text{H}^2 \end{array} \quad \checkmark \quad C_8^{14}: \quad \begin{array}{c} \text{---} \\ | \\ \text{H}^2 \end{array} = \quad \begin{array}{c} \text{---} \\ | \quad | \\ \text{H}^2 \quad \text{H}^2 \\ | \quad | \\ \text{H}^2 \end{array}$$

$$C_8'': \quad \text{Diagram} = \quad \text{Diagram} = \quad \text{Diagram} \quad \checkmark$$

$$C_8: \quad \begin{array}{c} \text{---} \\ | \end{array} \bullet \begin{array}{c} \text{---} \\ | \end{array} = \begin{array}{c} \bullet \text{---} \\ | \end{array} = \begin{array}{c} \boxed{H^2} \bullet \text{---} \\ | \end{array} \boxed{H^2} \quad \checkmark$$

Proof cont.

$$C_8^7: \quad \text{Diagram} = \text{Diagram} \equiv \text{Diagram} = \text{Diagram} \equiv C_2 \quad \text{Diagram} = \text{Diagram} : C_8^{10}$$

$$C_8^7 : \text{Diagram} = \text{Diagram} \equiv \text{Diagram} = \text{Diagram} \stackrel{C_2}{=} \text{Diagram} = \text{Diagram} : C_8^{10}$$

$$C_8^6: \text{Diagram} = \text{Diagram} \equiv \text{Diagram} = \text{Diagram} \stackrel{C_2}{\equiv} \text{Diagram} = \text{Diagram} : C_8^{11}$$

$$C_8^6: \text{Diagram} = \text{Diagram} \equiv \text{Diagram} = \text{Diagram} \equiv \text{Diagram} = \text{Diagram} : C_8^1$$

$$C_8^8: \text{Diagram} = \text{Diagram} \equiv \text{Diagram} = \text{Diagram} \stackrel{G_2}{=} \text{Diagram} = \text{Diagram} : C_8^1$$

$$C_8^8 : \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \equiv \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} = \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \stackrel{C_2}{\equiv} \begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} = \begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array} : C_8^{12}$$

$$C_8 : (1) \quad \begin{array}{c} \text{---} \\ | \\ \bullet \end{array} \quad \boxed{\mathbb{H}^2} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \bullet \end{array} \quad \boxed{\mathbb{H}^2} \quad \begin{array}{c} \text{---} \\ | \\ \bullet \end{array}$$

$$(2) \quad \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ | \end{array} \quad \boxed{H^2} \quad = \quad \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \end{array}$$

$$C_2: H^4 = I$$

$$C_8^1 : (1) \quad \begin{array}{c} \text{---} \\ | \end{array} \boxed{\mathbb{H}^2} \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \boxed{\mathbb{H}^2} \text{---} = \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \boxed{\mathbb{H}^2} \text{---}$$

$$(2) \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \boxed{H^2} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \boxed{H^2}$$

$$\text{Def 2: } \begin{array}{c} \oplus \\ \text{---} \\ | \\ \bullet \end{array} := \begin{array}{cccc} \boxed{\text{H}} & \bullet & \boxed{\text{H}} & \boxed{\text{H}} & \boxed{\text{H}} \end{array}$$

$$\begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \textcircled{\text{P}} \\ | \\ \text{---} \end{array} := \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \\ | \\ \text{H} \\ | \\ \text{---} \end{array}$$

Lem 0 Def 2, C₂ & C₈ imply

$$C_8^5: \quad (1) \quad \text{Diagram} = \quad \text{Diagram} \quad \checkmark \quad (2) \quad \text{Diagram} = \quad \text{Diagram}$$

$$C_8^b: \text{(1)} \quad \begin{array}{c} \text{H}^2 \\ \square \end{array} \oplus = \quad \begin{array}{c} \oplus \\ \square \end{array} \oplus \quad \checkmark \quad \text{(2)} \quad \begin{array}{c} \oplus \\ \square \end{array} = \quad \begin{array}{c} \oplus \\ \square \end{array} \quad \begin{array}{c} \text{H}^2 \\ \square \end{array}$$

$$C_8^7: \text{(1)} \quad \begin{array}{c} \bullet \\ \text{---} \\ \oplus \end{array} \quad \boxed{H^2} = \begin{array}{c} \bullet \\ \text{---} \\ \boxed{H^2} \\ \oplus \\ \oplus \end{array} \quad \checkmark \quad \text{(2)} \quad \begin{array}{c} \bullet \\ \text{---} \\ \oplus \end{array} \quad \boxed{H^2} = \begin{array}{c} \bullet \\ \text{---} \\ \boxed{H^2} \\ \oplus \\ \oplus \end{array}$$

$$C_8: \quad (1) \quad \text{Diagram} = \text{Diagram} \quad \checkmark \quad (2) \quad \text{Diagram} = \text{Diagram}$$

$$C_8^9: \quad \begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{H}^2} \quad \begin{array}{c} \oplus \\ | \end{array} \quad \begin{array}{c} \oplus \\ | \end{array} \quad \begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{H}^2} = \quad \begin{array}{c} \oplus \\ | \end{array} = \quad \begin{array}{c} \text{---} \\ | \end{array} \quad \begin{array}{c} \oplus \\ | \end{array} \quad \begin{array}{c} \oplus \\ | \end{array} \quad \begin{array}{c} \text{---} \\ | \end{array} \quad \boxed{\text{H}^2} \quad \boxed{\text{H}^2} \quad \checkmark \quad C_8^{13}: \quad \begin{array}{c} \oplus \\ | \end{array} \quad \boxed{\text{H}^2} = \quad \begin{array}{c} \text{---} \\ | \end{array} \quad \boxed{\text{H}^2} \quad \boxed{\text{H}^2}$$

$$C_{8:}^{10} = \text{Diagram A} = \text{Diagram B} = \text{Diagram C} \quad \checkmark \quad C_{8:}^{14} = \text{Diagram D}$$

$$C_8^{II}: \quad \begin{array}{c} \text{---} \\ | \end{array} \boxed{\mathbb{H}^2} \oplus \boxed{\mathbb{H}^2} \text{---} = \quad \begin{array}{c} \text{---} \\ | \end{array} \oplus \text{---} = \quad \begin{array}{c} \text{---} \\ | \end{array} \oplus \begin{array}{c} \text{---} \\ | \end{array} \boxed{\mathbb{H}^2} \text{---}$$

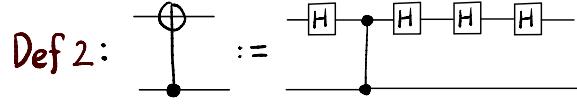
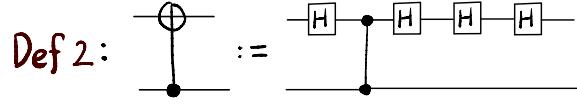
$$C_8^{12}: \quad \begin{array}{c} \text{---} \\ | \end{array} \bullet = \begin{array}{c} \text{---} \\ | \end{array} \bullet \bullet = \begin{array}{c} \boxed{H^2} \\ \text{---} \end{array} \bullet \begin{array}{c} \boxed{H^2} \\ \text{---} \end{array}$$

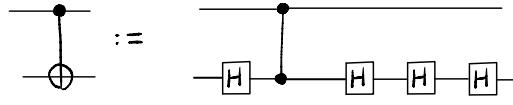
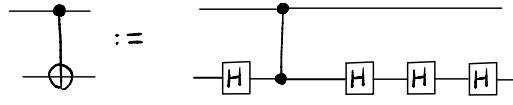
Proof cont.

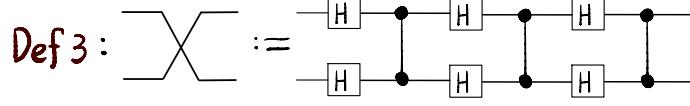
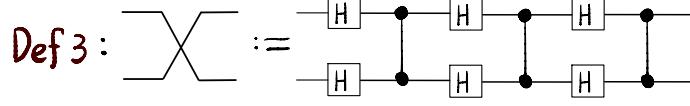
$$C_8^B \cdot LHS := \text{Diagram} \stackrel{C_8^5}{=} \text{Diagram} \stackrel{C_8^6}{=} \text{Diagram} =: C_8^B \cdot RHS$$

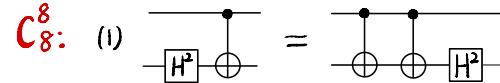
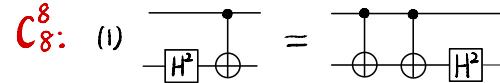
$$C_8^{14} \cdot LHS := \text{Diagram A} \underset{\text{C}_8^7}{=} \text{Diagram B} \underset{\text{C}_8^8}{=} \text{Diagram C} = C_8^{14} \cdot RHS$$

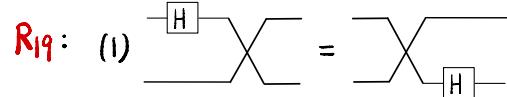
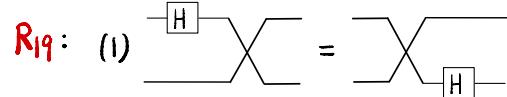
11

Def 2:  := 

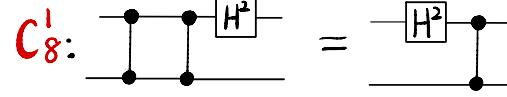
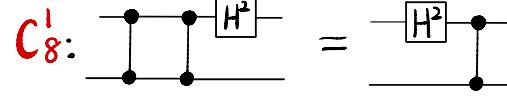
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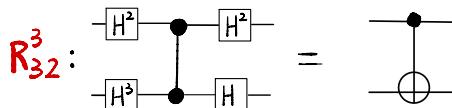
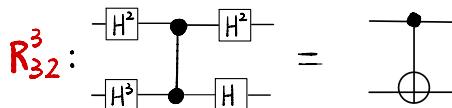
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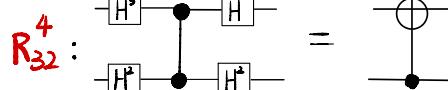
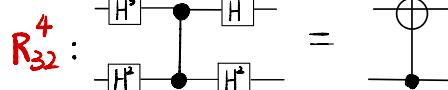
C₈⁸: (1)  = 

R₁₉: (1)  = 

(2)  = 

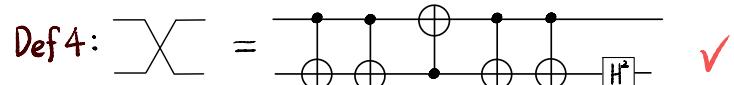
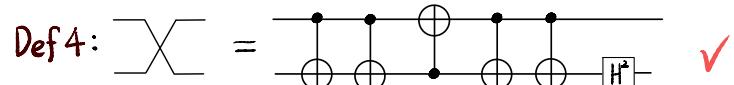
C₈¹:  = 

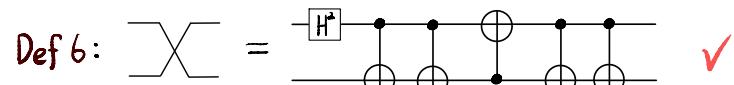
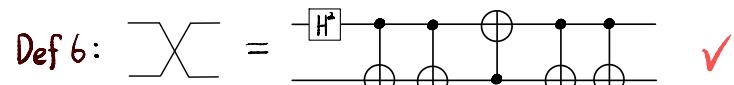
R₃₂³:  = 

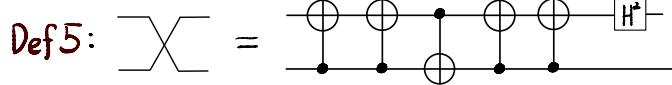
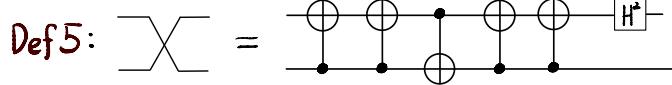
R₃₂⁴:  = 

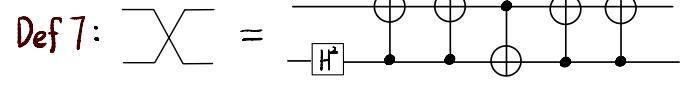
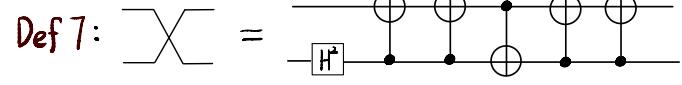
C₂: $H^4 = I$

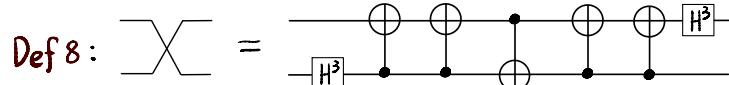
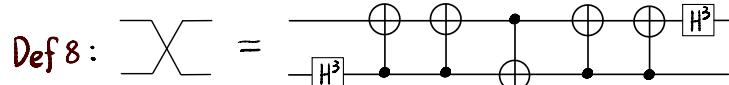
Lem P Def 2, Def 3, C₂, C₈, R₁₉ & R₃₂ imply

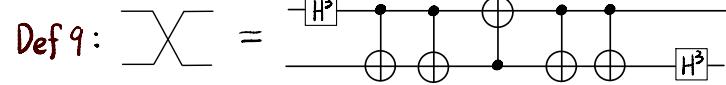
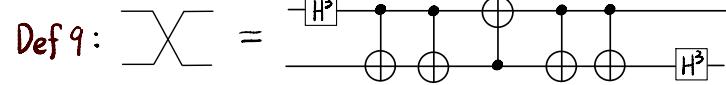
Def 4:  =  ✓

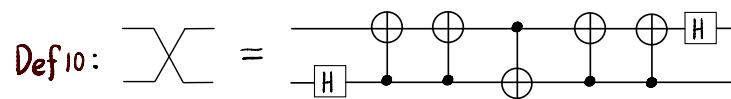
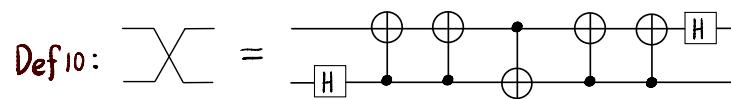
Def 6:  =  ✓

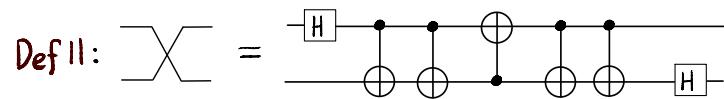
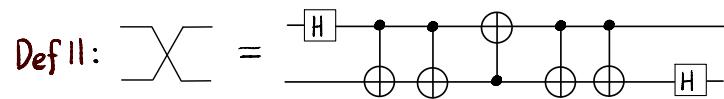
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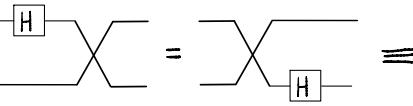
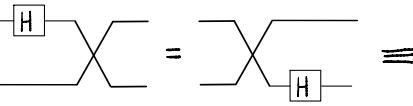
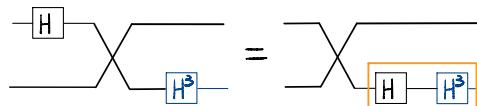
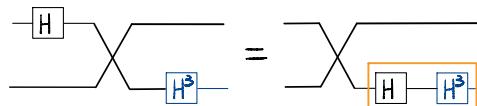
Def 7:  = 

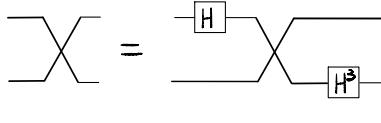
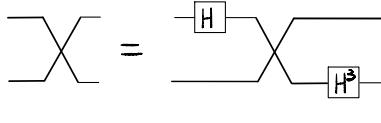
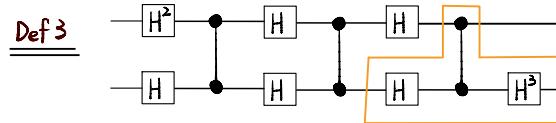
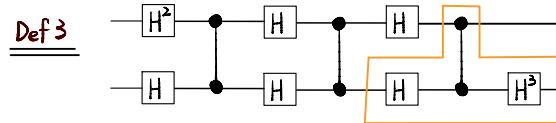
Def 8:  = 

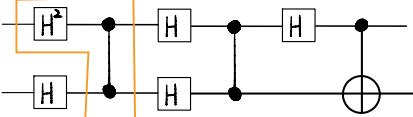
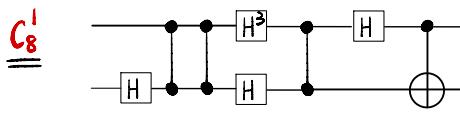
Def 9:  = 

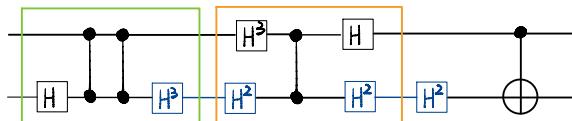
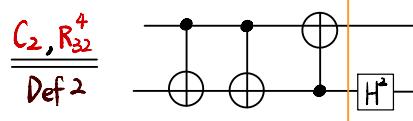
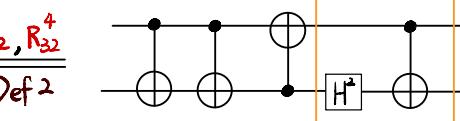
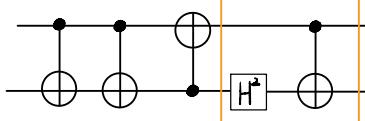
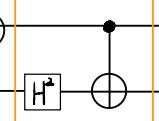
Def 10:  = 

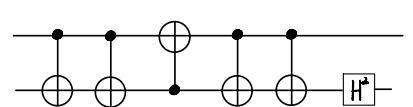
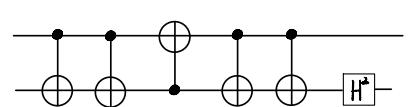
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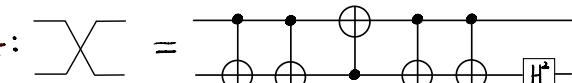
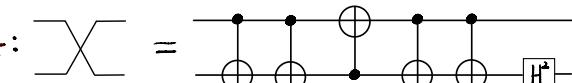
Proof: R₁₉: (1)  =  ≡  = 

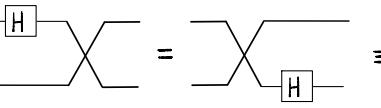
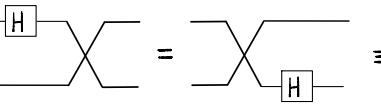
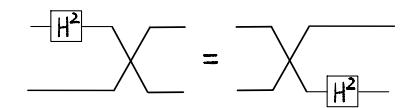
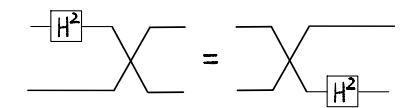
By C₂,  =  Def 3  = 

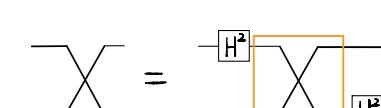
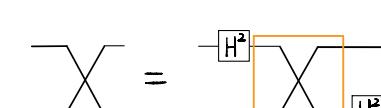
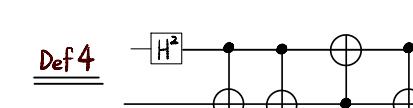
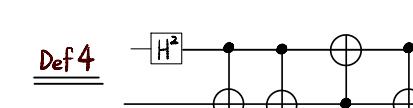
Def 2  =  C₈¹

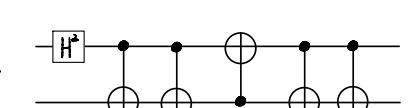
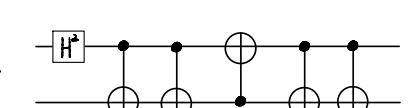
C₂  =  C₂, R₃₂⁴  =  Def 2 

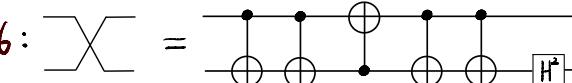
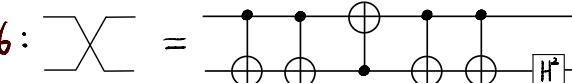
C₈⁸  = 

Hence Def 4:  = 

R₁₉: (1)  =  ≡  = 

By C₂,  =  Def 4  = 

C₂  = 

Hence Def 6:  = 

$$\text{Def 2: } \begin{array}{c} \textcircled{\text{S}} \\ \text{---} \\ \bullet \end{array} := \begin{array}{ccccccc} \text{H} & \bullet & \text{H} & \text{H} & \text{H} & \text{H} & \text{H} \end{array}$$

$$\begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \quad := \quad \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \\ \boxed{\mathbf{H}} \\ | \\ \bullet \\ | \\ \text{---} \\ \boxed{\mathbf{H}} \\ | \\ \text{---} \\ \boxed{\mathbf{H}} \\ | \\ \text{---} \\ \boxed{\mathbf{H}} \end{array}$$

$$\text{Def 3 : } \begin{array}{c} \diagup \\ \diagdown \end{array} := \begin{array}{ccccc} \boxed{\mathbf{H}} & \bullet & \boxed{\mathbf{H}} & \bullet & \boxed{\mathbf{H}} \\ \downarrow & & \downarrow & & \downarrow \\ \boxed{\mathbf{H}} & \bullet & \boxed{\mathbf{H}} & \bullet & \boxed{\mathbf{H}} \end{array}$$

$$C_8^b: \text{ (i)} \quad \begin{array}{c} \text{H}^2 \\ \boxed{\text{H}^2} \end{array} \quad = \quad \begin{array}{c} \text{H}^2 \\ \boxed{\text{H}^2} \end{array}$$

$$R_{1q} : \{2\} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad \boxed{H}$$

$$C_8^1 : (2) \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \quad H^2 \quad = \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \quad H^2$$

$$R_{32}^3 : \quad \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} = \quad \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array}$$

$$R_{32}^4 : \quad \begin{array}{c} \text{---} \boxed{H^3} \text{---} \bullet \text{---} \boxed{H} \text{---} \\ | \qquad \qquad \qquad | \\ \text{---} \boxed{H^3} \text{---} \bullet \text{---} \boxed{H^3} \text{---} \end{array} = \quad \begin{array}{c} \text{---} \bullet \text{---} \text{---} \\ | \\ \text{---} \bullet \text{---} \text{---} \end{array}$$

$$C_2: H^4 = I$$

Lem P Def 2, Def 3, C₂, C₈, R₁₉ & R₃₂ imply

$$\text{Def 4: } \text{X} = \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} = \begin{array}{ccccccccc} \bullet & \bullet & & \oplus & \bullet & \bullet & \bullet & \bullet & \text{H}^2 \end{array}$$

$$\text{Def 6: } \text{X} = \begin{array}{c} \text{H} \\ \text{CNOT} \end{array}$$

$$\text{Def 5: } \text{X} = \text{H}^\dagger$$

Def 7:  = 

$$\text{Def 8: } \text{X} = \text{H}^3 \otimes \text{H}^3$$

$$\text{Def 9: } \text{X} = \begin{array}{c} \text{H}^3 \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{H}^3 \\ \text{---} \\ \text{---} \end{array}$$

$$\text{Def 10: } \text{X} = \text{H} \otimes \text{I} \otimes \text{I} \otimes \text{I} \otimes \text{H}$$

$$\text{Def 11: } \text{X} = \begin{array}{c} \text{H} \\ \text{---} \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ \bullet \quad \bullet \quad \bullet \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ \oplus \quad \oplus \quad \oplus \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ \bullet \quad \bullet \quad \bullet \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ \text{H} \end{array}$$

Proof cont. R₁₉: (2)

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \end{array} \equiv \begin{array}{c} \text{Diagram 4} \\ \text{Diagram 5} \end{array} = \begin{array}{c} \text{Diagram 6} \\ \text{Diagram 7} \end{array}$$

$$\text{By } C_2, \quad \text{Diagram} = \text{Diagram} \quad \underline{\text{Def 3}} \quad \text{Diagram} \quad \underline{\text{Def 2}}$$

The diagram illustrates two quantum circuit decompositions:

- Circuit C₈:** A sequence of operations on four qubits. The first qubit starts with a Hadamard (H) gate, followed by a CNOT gate with control on the second qubit. The second qubit starts with a H² gate, followed by a CNOT gate with control on the first qubit. The third qubit starts with a H gate, followed by a CNOT gate with control on the fourth qubit. The fourth qubit starts with a CNOT gate with control on the third qubit, followed by a H gate.
- Circuit C₂:** An equivalent circuit where the first two qubits are swapped. The first qubit starts with a H gate, followed by a CNOT gate with control on the second qubit. The second qubit starts with a H³ gate, followed by a CNOT gate with control on the first qubit. The third qubit starts with a H gate, followed by a CNOT gate with control on the fourth qubit. The fourth qubit starts with a CNOT gate with control on the third qubit, followed by a H gate.
- Decomposition:** Both circuits are shown to be equivalent to a sequence of basic gates: H, CNOT, H², H³, and H gates.

Def 2  C8^b

Hence Def 5:  = 

$$R_{19} : (1_2) \quad \text{Diagram showing } R_{19} \text{ as a crossing relation between two strands, with labels } H^2 \text{ indicating the crossing points.} \quad \equiv \quad \text{Diagram showing } R_{19} \text{ as a crossing relation between two strands, with labels } H^2 \text{ indicating the crossing points.} \quad = \quad \text{Diagram showing } R_{19} \text{ as a crossing relation between two strands, with labels } H^2 \text{ indicating the crossing points.} \quad . \quad \text{By } C_2, \quad \text{Diagram showing } R_{19} \text{ as a crossing relation between two strands, with labels } H^2 \text{ indicating the crossing points.} \quad =$$