

$$B_{00} = \text{X}$$

Def 3: $\text{X} := \begin{array}{c} \text{H} \\ \text{H} \end{array} \quad C_6 : \quad \begin{array}{c} \text{H} \\ \text{H} \end{array} = \quad \quad \quad$

R_{19} : (1) $\begin{array}{c} \text{H} \\ \text{H} \end{array} \text{X} = \text{X} \begin{array}{c} \text{H} \\ \text{H} \end{array}$ (2) $\begin{array}{c} \text{H} \\ \text{H} \end{array} \text{X} = \text{X} \begin{array}{c} \text{H} \\ \text{H} \end{array}$

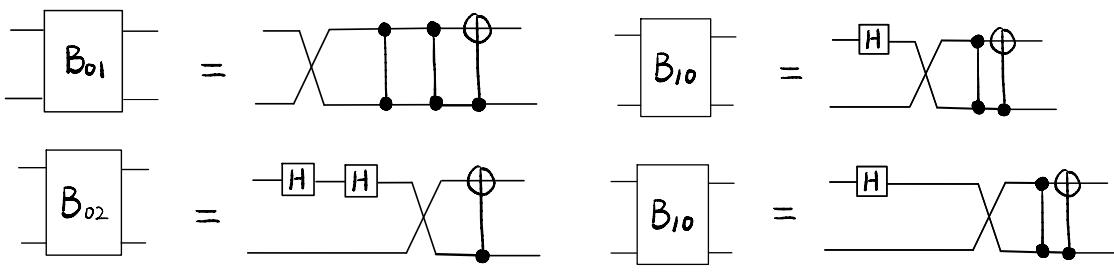
Lem 15 By Def 3, C_6 & R_{19} ,

5. (1) $\begin{array}{c} \text{H} \\ \text{H} \end{array} B_{00} = \begin{array}{c} B_{00} \\ \text{H} \end{array}$

Proof: 5.(1).LHS := $\begin{array}{c} \text{H} \\ \text{H} \end{array} B_{00}$ def $\begin{array}{c} \text{H} \\ \text{H} \end{array} \text{X} = \text{X} \begin{array}{c} \text{H} \\ \text{H} \end{array}$ R_{19}

C_6 $\begin{array}{c} \text{H} \\ \text{H} \end{array} \text{X} = \text{X} \begin{array}{c} \text{H} \\ \text{H} \end{array}$

def $B_{00} = \begin{array}{c} \text{H} \\ \text{H} \end{array}$ $= 5.(1).\text{RHS}$



$R_{25}:$

$$\begin{array}{ccc}
 R_{25} & = & \text{Circuit Diagram} \\
 & & \text{with } H, S, S, Z, Z \text{ gates} \\
 & & \cdot w
 \end{array}$$

Lem 16 By R_{25} ,

$$5.(2) \quad \begin{array}{ccc}
 H & \boxed{B_{01}} & = \boxed{B_{10}} \cdot w \\
 \text{Circuit} & \text{Circuit} & \text{with } S, S, Z, Z \text{ gates}
 \end{array}$$

Proof: $5.(2).LHS :=$

$$\begin{array}{ccc}
 H & \boxed{B_{01}} & \stackrel{\text{def}}{=} \text{Circuit Diagram} \\
 & & \text{with } H, CNOT, \oplus \text{ gates} \\
 & & \text{boxed}
 \end{array}$$

$$\stackrel{R_{25}}{=} \text{Circuit Diagram} \cdot w$$

$$\stackrel{\text{def}}{=} \boxed{B_{10}} \cdot w =: 5.(2).RHS$$

□

Lem 17 By R_{25} ,

$$5.(4) \quad \begin{array}{ccc}
 H & \boxed{B_{10}} & = \boxed{B_{02}} \cdot w \\
 \text{Circuit} & \text{Circuit} & \text{with } S, S, Z, Z \text{ gates}
 \end{array}$$

Proof: $5.(4).LHS :=$

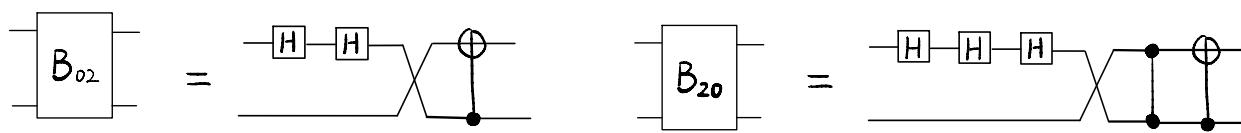
$$\begin{array}{ccc}
 H & \boxed{B_{10}} & \stackrel{\text{def}}{=} \text{Circuit Diagram} \\
 & & \text{with } H, H, CNOT, \oplus \text{ gates} \\
 & & \text{boxed}
 \end{array}$$

$$\stackrel{R_{25}}{=} \text{Circuit Diagram} \cdot w$$

$$\stackrel{\text{def}}{=} \boxed{B_{02}} \cdot w =: 5.(4).RHS$$

□

2



$$R_{25}: \quad = \quad \begin{array}{c} \text{Diagram showing two nodes connected by a vertical line, with a circle containing a plus sign above them.} \\ \text{Diagram showing two nodes connected by a vertical line, with a circle containing a plus sign above them.} \end{array} \cdot w \quad R3: SZ = ZS \quad C1: w^3 = 1$$

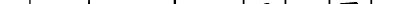
$$C_7: (1) \quad \text{Diagram} = \text{Diagram} \quad (2) \quad \text{Diagram} = \text{Diagram}$$

$$R_B : (1) \quad \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \quad \boxed{z} \quad = \quad \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \quad (2) \quad \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \quad \boxed{z} \quad = \quad \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array}$$

$$C_6 : \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \quad = \quad \underline{\hspace{2cm}} \qquad C_3 : S^3 = I \qquad R12 : Z^3 = I$$

Lem 18 By $C_1, C_3, C_6, C_7, R_3, R_{12}, R_{13}$ & R_{25} ,

$$5.(3) \quad \text{Diagram showing } B_{02} \text{ as a } B_{20} \text{ series circuit with } S \text{ and } Z \text{ components.}$$

Proof: 5.(13). RHS := B_{20}  $\bullet w^3$

R₂₅

• w^2 • w

Quantum circuit diagram for C_7, R_{13} with C_1 . The circuit consists of the following sequence of operations:

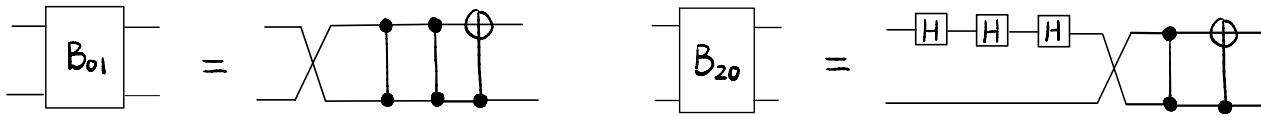
- Initial state: $|0\rangle\langle 0|$
- H gate on the first wire
- H gate on the second wire
- H gate on the third wire
- $CNOT$ gate from the first wire to the second wire
- $CNOT$ gate from the second wire to the third wire
- Two qubits are initialized to $|1\rangle$ and $|0\rangle$ respectively (blue box)
- S gate on the fourth wire
- S gate on the fifth wire
- Z gate on the sixth wire
- Z gate on the seventh wire
- S gate on the eighth wire
- Z gate on the ninth wire

C_6, R_3  C_3, R_{12}

def 

=: 5.(3). LHS

6



R_{25} : = • w R3: SZ = ZS C1: w^3 = I

$C_7: (1)$ = • w (2) = • w

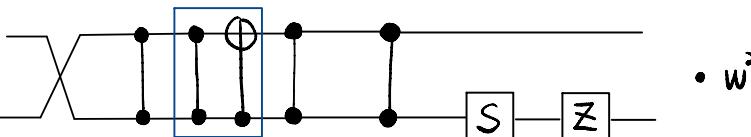
$R_B: (1)$ = • w (2) = • w

$C_6:$ = • w C3: S^3 = I R12: Z^3 = I C2: H^4 = I

Lem 19 By $C_1, C_2, C_3, C_6, C_7, R_3, R_{12}, R_{13}$ & R_{25} ,

5.(7) = • w²

Proof: 5.(7).RHS := • w²

def  • w²

R_{25} • w² • w

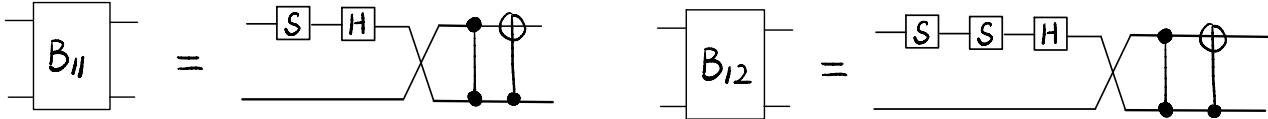
C_7, R_{13} • w²

R_3 • w²

C_3, R_{12} • w²

C_2 • w²

def =: 5.(7).LHS



Def 3: $\begin{array}{c} \diagup \quad \diagdown \\ \text{Control} \end{array} := \begin{array}{c} H \quad \bullet \quad H \quad \bullet \quad H \\ \text{Control} \quad \text{Control} \quad \text{Control} \end{array}$

RI: $[H \quad S \quad H] = [S \quad S \quad H \quad S^2 \quad X^2] \cdot (-w^2)$

R₁₈: (1) $\begin{array}{c} S \\ \diagup \quad \diagdown \\ \text{Control} \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ S \quad \text{Control} \end{array}$ (2) $\begin{array}{c} \diagup \quad \diagdown \\ S \quad \text{Control} \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ S \end{array}$

R₁₉: (1) $\begin{array}{c} H \\ \diagup \quad \diagdown \\ \text{Control} \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \text{Control} \quad H \end{array}$ (2) $\begin{array}{c} \diagup \quad \diagdown \\ H \quad \text{Control} \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ H \end{array}$

C₇: (1) $\begin{array}{c} \bullet \quad S \\ \text{Control} \quad \bullet \end{array} = \begin{array}{c} S \\ \bullet \quad \bullet \end{array}$ (2) $\begin{array}{c} \bullet \quad S \\ \text{Control} \quad \bullet \end{array} = \begin{array}{c} S \\ \bullet \quad \bullet \end{array}$

R₁₄: (1) $\begin{array}{c} \bullet \quad X \\ \text{Control} \quad \bullet \end{array} = \begin{array}{c} X \\ \bullet \quad \bullet \end{array}$ (2) $\begin{array}{c} \bullet \quad X \\ \text{Control} \quad \bullet \end{array} = \begin{array}{c} \bar{z} \quad \bar{z} \\ \bullet \quad X \end{array}$
(3) $\begin{array}{c} X \\ \bullet \quad \bullet \end{array} = \begin{array}{c} X \\ \bullet \quad \bar{z} \end{array}$ (4) $\begin{array}{c} X \\ \bullet \quad \bullet \end{array} = \begin{array}{c} \bar{z} \\ \bullet \quad X \end{array}$

Lem 20 By Def 3, R₁, R₁₂, R₁₄, R₁₈, R₁₉, R₂₄, R₂₆ & C₇,

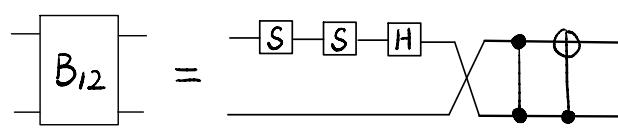
5.(5) $\begin{array}{c} H \\ \diagup \quad \diagdown \\ \text{Control} \end{array} B_{11} = \begin{array}{c} B_{12} \quad X \quad X \quad \bar{z} \quad \bar{z} \\ \diagup \quad \diagdown \quad \text{Control} \quad \text{Control} \end{array} \cdot (-w^2)$

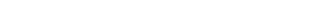
Proof: 5.(5).LHS := $\begin{array}{c} H \\ \diagup \quad \diagdown \\ \text{Control} \end{array} B_{11}$ def $\boxed{\begin{array}{c} H \quad S \quad H \\ \diagup \quad \diagdown \\ \text{Control} \end{array}}$

$\overline{\overline{R_{18}}} \quad \overline{\overline{R_{19}}}$ $\begin{array}{c} \diagup \quad \diagdown \\ \text{Control} \quad \boxed{H \quad S \quad H} \quad \bullet \quad \oplus \end{array}$

$\overline{\overline{R_1}} \quad \begin{array}{c} \diagup \quad \diagdown \\ \text{Control} \quad \boxed{S \quad S \quad H \quad S^2 \quad X^2} \quad \bullet \quad \oplus \end{array} \cdot (-w^2)$

$\overline{\overline{C_7}} \quad \overline{\overline{R_{14}}} \quad \begin{array}{c} \diagup \quad \diagdown \\ \text{Control} \quad \boxed{S \quad S \quad H \quad S^2 \quad X^2} \quad \boxed{\bar{z}^2} \quad \oplus \end{array} \cdot (-w^2)$



R18: (1)  = 

(2)  = 

R12: $Z^3 = I$

R19: (1)  = 

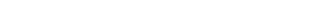
(2)  = 

R26: (1)  = 

(2)  = 

R24:  = 

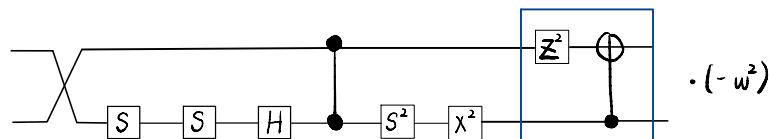
(3)  = 

(4)  = 

(5)  = 

(6)  = 

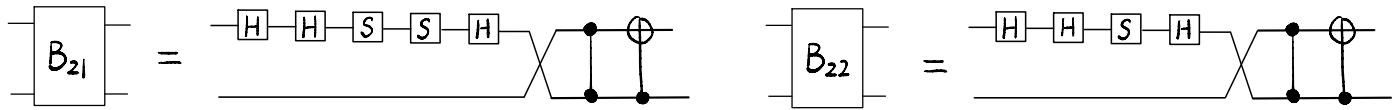
$$5.(5) \quad \begin{array}{c} \text{---} \\ | \end{array} \boxed{\text{H}} \begin{array}{c} \text{---} \\ | \end{array} \boxed{B_{11}} \begin{array}{c} \text{---} \\ | \end{array} = \begin{array}{c} \text{---} \\ | \end{array} \boxed{B_{12}} \begin{array}{c} \text{---} \\ | \end{array} \begin{array}{c} X \\ | \end{array} \begin{array}{c} X \\ | \end{array} \begin{array}{c} Z \\ | \end{array} \begin{array}{c} Z \\ | \end{array} \begin{array}{c} \text{---} \\ | \end{array} \bullet (-w^2)$$

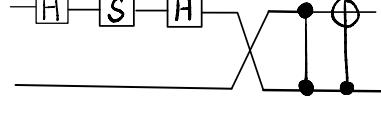
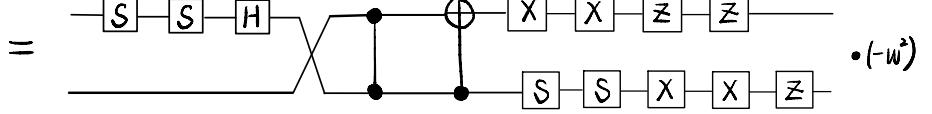


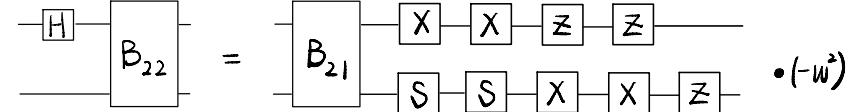
The diagram shows a ladder network with nodes labeled S, H, S², X², Z, and Z². The circuit includes a dependent current source controlled by node H, a dependent voltage source controlled by node S², and two dependent voltage sources controlled by node Z. The output voltage is measured across nodes Z and Z².

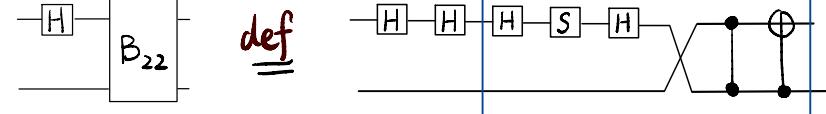
The circuit diagram shows a bridge network with resistors S , S , H , S^2 , X^2 , and Z . A vertical line connects the top node of the S - S branch to the top node of the S^2 - X^2 branch. The bottom node of the S - S branch is connected to the bottom node of the S^2 - X^2 branch. The leftmost node of the S - S branch is connected to the leftmost node of the S^2 - X^2 branch. The rightmost node of the S - S branch is connected to the rightmost node of the S^2 - X^2 branch. The top node of the S^2 - X^2 branch is connected to the top node of the Z branch. The bottom node of the S^2 - X^2 branch is connected to the bottom node of the Z branch. The leftmost node of the S^2 - X^2 branch is connected to the leftmost node of the Z branch. The rightmost node of the S^2 - X^2 branch is connected to the rightmost node of the Z branch. A vertical line also connects the top node of the H branch to the top node of the X^2 branch. The bottom node of the H branch is connected to the bottom node of the X^2 branch. The leftmost node of the H branch is connected to the leftmost node of the X^2 branch. The rightmost node of the H branch is connected to the rightmost node of the X^2 branch. A vertical line connects the top node of the X^2 branch to the top node of the Z branch. The bottom node of the X^2 branch is connected to the bottom node of the Z branch. The leftmost node of the X^2 branch is connected to the leftmost node of the Z branch. The rightmost node of the X^2 branch is connected to the rightmost node of the Z branch. An inductor L is connected between the top node of the S^2 branch and the top node of the X^2 branch.

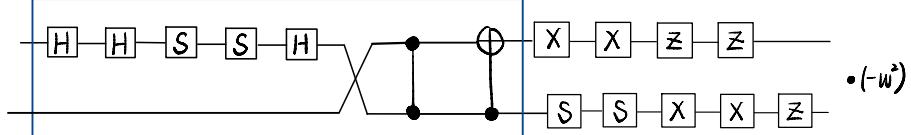
$$\underline{\text{def}} \quad B_{12} = [X \ X \ Z \ \bar{Z}] \quad \bullet (-w^*) =: 5.(15) \text{. RHS}$$

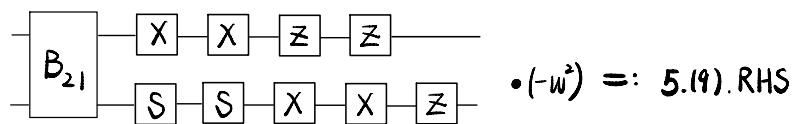


Lem 20  = 

Lem 21 5.(9) 

Proof: 5.(9). LHS := 

Lem 20 

def 

□

$$C_0: (-1)^2 = 1 \quad C_1: w^3 = 1 \quad C_3: S^3 = I \quad R_9: X^3 = I \quad R_{12}: Z^3 = I$$

$$R_3: SZ = ZS \quad R_8: \text{[Z] [X]} = \text{[X] [Z]} \cdot w^2$$

$$R_6: \text{[S] [X]} = \text{[X] [S] [Z]} \cdot w^2$$

$$R'_6: \text{[S] [X] [Z] [X]} \cdot w = \text{[X] [S]}$$

Lem 20

Lem S Lem 20 implies

Proof: By C_0 & C_1 , multiply both sides of (1) by $(-w)$, followed by appending both sides of (1) by H to the left.

By C_3, R_9 & R_{12} , append both sides of (2) by $ZX \otimes Z^2XS$ to the right. Composition in diagrammatic order.

(3). LHS =

R_8

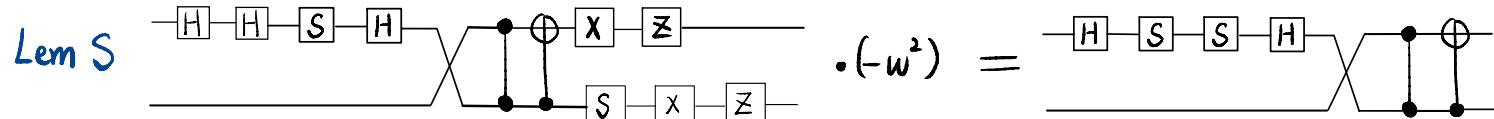
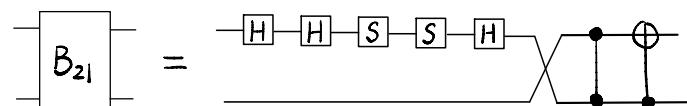
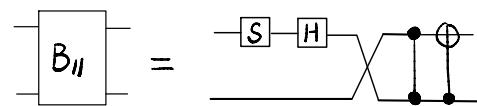
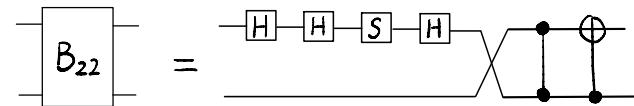
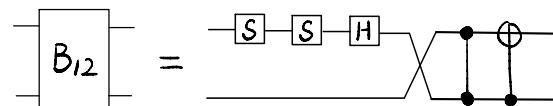
R_3

C_1

R'_6

R_{12}

We can reduce
this process to
the completeness
of single-qutrit
Clifford relations.



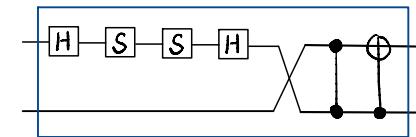
C₂: $H^4 = I$

Lem 22 By Lem S & C₂,

5.(6)

(8)

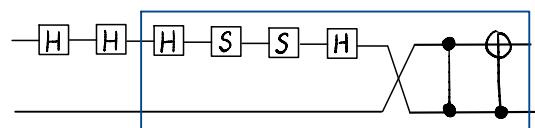
Proof: 5.(6). LHS :=



Lem S

def =: 5.(6). RHS

5.(8). LHS :=



Lem S

C₂

def =: 5.(8). RHS

□