

$$C_2: H^4 = I$$

Def 3:

$R_{19}: (1)$

(2)

Lem 4 By Def 3, C₂ & R₁₉, 8.(1)

(2)

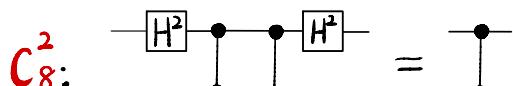
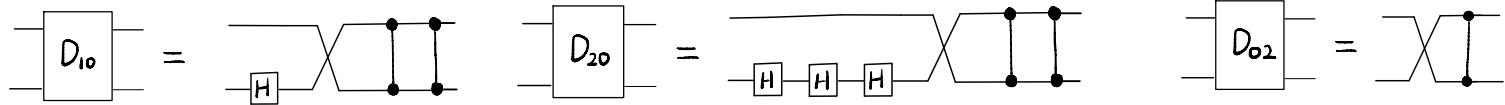
(7)

Proof: 8.(1).LHS = := $\stackrel{R_{19}}{=} \boxed{\text{Diagram for } D_{00}}$ $\stackrel{\text{def}}{=} \boxed{\text{Diagram for } D_{00}}$

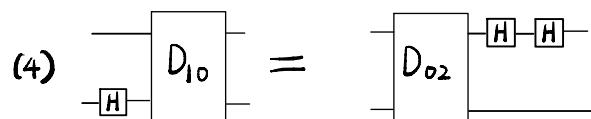
8.(2).LHS = := $\stackrel{\text{def}}{=} \boxed{\text{Diagram for } D_{10}}$

8.(7).LHS = := $\stackrel{C_2}{=} \boxed{\text{Diagram for } D_{20}}$

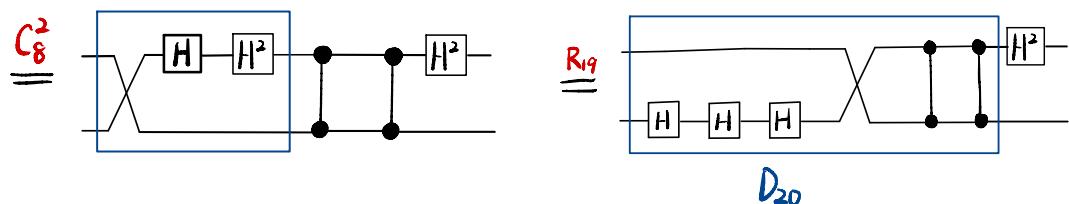
$\stackrel{\text{def}}{=} \boxed{\text{Diagram for } D_{01}}$



Lem 5 By Def 3, C_8 & R_{19} , 8.(3) $\begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} \quad D_{02} = \begin{array}{c} \text{---} \\ | \quad | \quad | \\ \text{---} \end{array}$



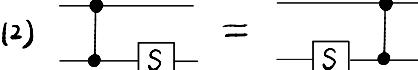
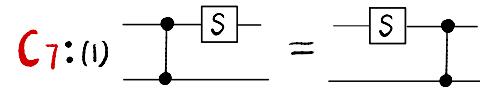
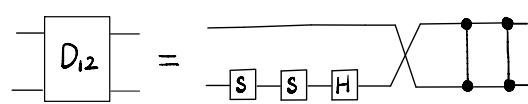
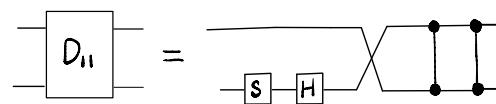
Proof: 8.(3). LHS = $\begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} \quad D_{02} := \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} \quad H = \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array}$



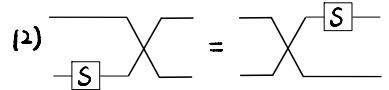
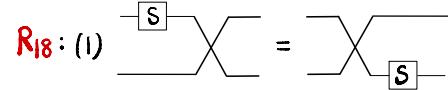
$\stackrel{\text{def}}{=} \begin{array}{c} \text{---} \\ | \quad | \quad | \\ \text{---} \end{array} \quad D_{20} = 8.(3). \text{ RHS}$

8.(4). LHS = $\begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} \quad D_{10} := \begin{array}{c} \text{---} \\ | \quad | \quad | \\ \text{---} \end{array}$ $\stackrel{R_{19}}{=} \begin{array}{c} \text{---} \\ | \quad | \quad | \\ \text{---} \end{array} \quad H = \begin{array}{c} \text{---} \\ | \quad | \quad | \\ \text{---} \end{array}$

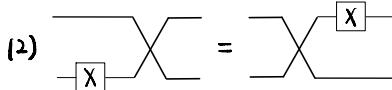
$\stackrel{C_8}{=} \begin{array}{c} \text{---} \\ | \quad | \quad | \\ \text{---} \end{array} \quad H = \begin{array}{c} \text{---} \\ | \quad | \quad | \\ \text{---} \end{array}$ $\stackrel{\text{def}}{=} \begin{array}{c} \text{---} \\ | \quad | \quad | \\ \text{---} \end{array} \quad D_{02} = 8.(4). \text{ RHS}$



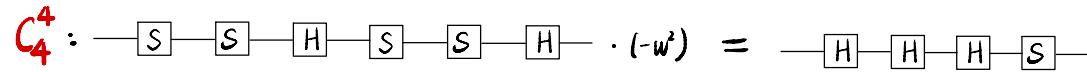
$$R_{12} : Z^3 = I \quad C_3 : S^3 = I$$



$$R_4^2 : -\boxed{H} \rightarrow \boxed{Z} \rightarrow \boxed{Z} \rightarrow \boxed{H^+} = -\boxed{X^2}$$



$$C_5: SH^2SH^2 = H^2SH^2S \quad ss' = s's$$



$$\text{RII: } \boxed{Z^2} = \boxed{H} \boxed{H} \boxed{S} \boxed{H} \boxed{H} \boxed{S} \boxed{S} = \boxed{s'} \boxed{S} \boxed{S}$$

Def : $S' = HSH^2$

$$R_B : (1) \quad \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} = \quad \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \quad (2) \quad \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} = \quad \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array}$$

$$\text{R14: } (1) \quad \begin{array}{c} \bullet \\ \text{---} \end{array} \quad \boxed{X} \quad = \quad \begin{array}{c} \text{---} \\ \boxed{X} \\ \bullet \end{array} \quad (2) \quad \begin{array}{c} \bullet \\ \text{---} \end{array} \quad \boxed{Z} \quad = \quad \begin{array}{c} \text{---} \\ \boxed{Z} \\ \bullet \end{array}$$

$$(3) \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \boxed{X} \text{---} \bullet \text{---} \begin{array}{c} \bullet \\ | \\ \text{---} \end{array} = \begin{array}{c} \bullet \\ | \\ \text{---} \end{array} \text{---} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \boxed{X} \text{---} \begin{array}{c} \bullet \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \boxed{Z} \text{---} \begin{array}{c} \bullet \\ | \\ \text{---} \end{array}$$

$$(2) \quad \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \boxed{Z} \\ \text{---} \\ \boxed{Z} \\ \text{---} \\ | \end{array}$$

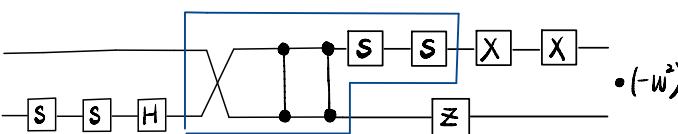
$$(3) \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \boxed{X} \text{---} \bullet \text{---} \begin{array}{c} \bullet \\ | \\ \text{---} \end{array} = \begin{array}{c} \bullet \\ | \\ \text{---} \end{array} \text{---} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \boxed{X} \text{---} \begin{array}{c} \bullet \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \boxed{Z} \text{---} \begin{array}{c} \bullet \\ | \\ \text{---} \end{array}$$

$$(4) \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad = \quad \begin{array}{c} \bullet \\ \text{---} \\ | \\ \text{---} \\ \bullet \\ \text{---} \end{array} \quad \boxed{\Sigma}$$

Lem 6 By $C_3, C_4, C_5, C_7, R_4, R_{11}, R_{12}, R_{13}, R_{14}, R_{18}$ & R_{20} ,

$$8. (5) \quad D_{11} = D_{12} \cdot (-w^2) \quad (9) \quad D_{22} = D_{21} \cdot (-w^2)$$

Proof: 8.(5). RHS =



R_{13}, R_{14}

• $(-w^2)$

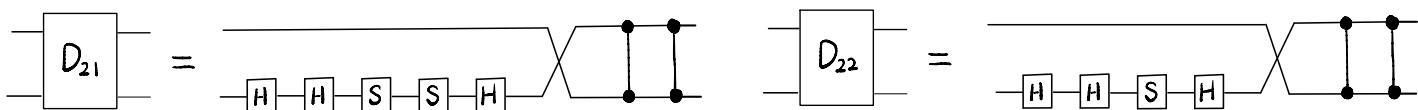
The diagram illustrates the timing sequence of a 4-bit counter connected to a 3-to-2 decoder. The counter's outputs are labeled S₃, S₂, H, S₁, S₀, X₁, and X₀. The decoder's outputs are shown as two vertical columns of four dots each, representing the binary values of the outputs Y₁ and Y₀.

Composition in diagrammatic order

Note that $S^2HS^2X^2 \cdot (-w) \stackrel{R_4}{=} S^2HS^2H^+z^2H^+ \cdot (-w) \stackrel{C_4}{=} H^+S^2\boxed{z^2}H^+ \stackrel{S^2HS^2X^2 \cdot (-w)}{\longrightarrow} HS^2H$.

$$\text{R}_{11} = H^+ \boxed{S S'} S^2 H^+ \stackrel{C_5}{=} H^+ S S' S^3 H^+ \stackrel{C_3}{=} H^+ S' H^+ \stackrel{\text{def}}{=} H^+ H^2 S H^2 H^+ = H S H$$

$$\text{Hence (1)} = \text{Diagram} = 8 \cdot (5) \cdot \text{LHS}$$



$$(9) \quad \begin{array}{c} \text{---} \\ | \end{array} \boxed{D_{22}} \begin{array}{c} \text{---} \\ | \end{array} = \begin{array}{c} \text{---} \\ | \end{array} \boxed{D_{21}} \begin{array}{c} \text{---} \\ | \end{array} \boxed{S} \begin{array}{c} \text{---} \\ | \end{array} \boxed{S} \begin{array}{c} \text{---} \\ | \end{array} \boxed{X} \begin{array}{c} \text{---} \\ | \end{array} \boxed{X} \begin{array}{c} \text{---} \\ | \end{array} \bullet (-w^2)$$

Proof cont. Recall that

$$\text{Diagram showing the equivalence of two circuit representations:} \\ \text{Left: } \text{H} - \text{S} - \text{H} \text{ followed by a controlled NOT gate with control on the second wire and target on the fourth wire.} \\ \text{Right: } \text{S} - \text{S} - \text{H} \text{ followed by a sequence of gates: S, S, X, X, and a measurement gate } \bar{z}. \\ \text{Equation: } \text{Left} = \text{Right} \cdot (-w^2) \quad (2)$$

$$8.(19).RHS := \boxed{D_{21}} - \boxed{s} - \boxed{s} - \boxed{x} - \boxed{x} - \boxed{z} \bullet (-w^2)$$

The diagram illustrates a logic circuit. On the left, two inputs are labeled H . These feed into two AND gates, each with two inputs. The outputs of these AND gates are labeled S . These S outputs feed into another AND gate, which has two inputs. This final output is labeled \bar{z} . A NOT gate, labeled X , also has an input from the same point as the second AND gate's output. The output of the NOT gate is labeled \bar{x} . Finally, the outputs of the first AND gate (H), the second AND gate (S), and the NOT gate (\bar{x}) all feed into an OR gate, whose output is labeled z .

(2)

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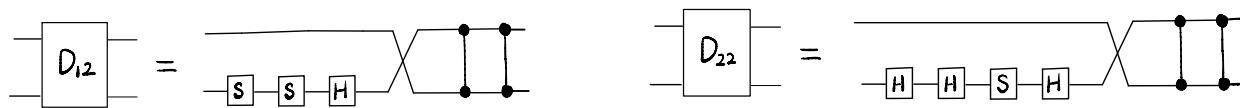
    graph LR
        In(( )) --> H1[H]
        H1 --> H2[H]
        H2 --> S[S]
        S --> H3[H]
        H3 --> CNOT[CNOT]
        CNOT --> Out(( ))
    
```

$$=: \boxed{D_{22}} = 8 \cdot (9) \cdot LHS.$$

II

Lem 6

$$\text{---} \boxed{\mathbf{S}} \text{---} \boxed{\mathbf{S}} \text{---} \boxed{\mathbf{H}} \text{---} \boxed{\mathbf{S}} \text{---} \boxed{\mathbf{S}} \text{---} \boxed{\mathbf{X}} \text{---} \boxed{\mathbf{X}} \text{---} \bullet (-w^*) = \text{---} \boxed{\mathbf{H}} \text{---} \boxed{\mathbf{S}} \text{---} \boxed{\mathbf{H}} \text{---}$$



Def 3:

$C_7 : (1)$ (2)

$R_{12} : z^3 = I$

$R_{18} : (1)$ (2)

$R_{20} : (1)$ (2)

$R_{13} : (1)$ (2)

$R_{14} : (1)$ (2)

(3) (4)

$R_{21} : (1)$ (2)

Lem 7 By Def 3, $C_2, C_3, C_4, C_5, C_7, R_4, R_{10}, R_{11}, R_{12}, R_{13}, R_{14}, R_{18}, R_{20}$ & R_{21} ,

8. (6) (8)

Proof: 8.6.RHS = :=

R_{13}, R_{21}
 C_7, R_{18}

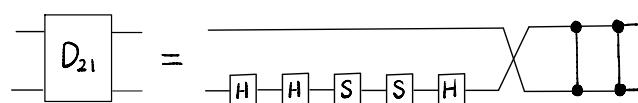
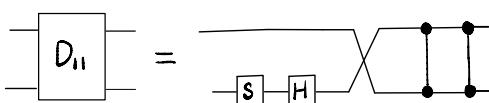
R_{14}
 R_{12}, R_{13}

R_{20}

8.6.LHS =

• It is sufficient to show

$$H^2 S H Z^2 S X \cdot (-w) = H S^2 H$$



$$C_2: H^4 = I \quad C_5: SH^2SH^2 = H^2SH^2S \quad SS' = S'S \quad C_3: S^3 = I \quad R_4: \boxed{H \quad Z \quad H^+} = \boxed{X}$$

$$R_{10}: \boxed{Z} = \boxed{H \quad H \quad S \quad S \quad H \quad H \quad S} = \boxed{S' \quad S' \quad S}$$

$$R_{11}: \boxed{Z^2} = \boxed{H \quad H \quad S \quad H \quad H \quad S \quad S} = \boxed{S' \quad S \quad S}$$

$$C_4: \boxed{S \quad S \quad H \quad S \quad S} = \boxed{H \quad H \quad H \quad S \quad H \quad H \quad H} \cdot (-w)$$

$$\text{Def: } \boxed{S'} = \boxed{H \quad H \quad S \quad H \quad H} \quad S' := H^2SH^2$$

Composition in diagrammatic order →

Note that $H^2SHZ^2SX \cdot (-w) \stackrel{R_4}{=} H^2SH\boxed{Z^2}SH\boxed{Z}H^+ \cdot (-w) \stackrel{\substack{R_{10} \\ R_{11}}}{=} H^2SHS'S^2SHS'S'H^+ \cdot (-w) \stackrel{C_3}{=} H^2SHS'H^2S^2SH^2S'H^+ \cdot (-w)$

$$C_2 = H^2SH^+S^2H^2 \stackrel{\text{def}}{=} H^2SHH^2S^2HH^2S^2H^2SH^+ \stackrel{C_4}{=} H^2SHH^2S^2SH^2H^2SH^+ \stackrel{C_3}{=} H^2H^2S^2H^2H^+ = HS^2H.$$

Hence (1) = $\boxed{H \quad S \quad S \quad H} \times \boxed{Z \quad Z \quad S \quad X} =: \boxed{D_{12}} = 8.(16).LHS.$

$$8.(8) \quad \boxed{D_{21}} = \boxed{D_{11}} \boxed{Z \quad Z \quad S \quad X} \cdot (-w)$$

Recall $\boxed{H \quad H \quad S \quad H} \times \boxed{Z \quad Z \quad S \quad X} \cdot (-w) = \boxed{H \quad S \quad S \quad H} \times \boxed{Z \quad Z \quad S \quad X} \quad (2)$

$$8.(8).RHS = \boxed{D_{11}} \boxed{Z \quad Z \quad S \quad X} \cdot (-w) := \boxed{S \quad H} \times \boxed{Z \quad Z \quad S \quad X} \cdot (-w)$$

$$\stackrel{C_2}{=} \boxed{H \quad H \quad H \quad H \quad S \quad H} \times \boxed{Z \quad Z \quad S \quad X} \cdot (-w)$$

$$\stackrel{(2)}{=} \boxed{H \quad H \quad H \quad S \quad S \quad H} \times \boxed{Z \quad Z \quad S \quad X} =: \boxed{D_{21}} = 8.(8).LHS$$

□

Lem 7 $\boxed{H \quad H \quad S \quad H \quad Z \quad Z \quad S \quad X} \cdot (-w) = \boxed{H \quad S \quad S \quad H}$