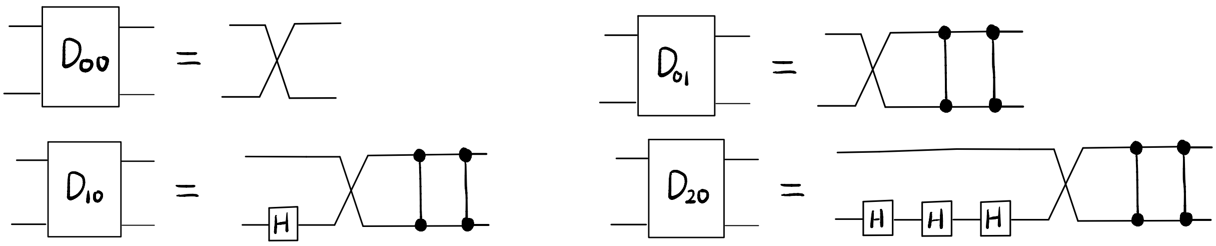
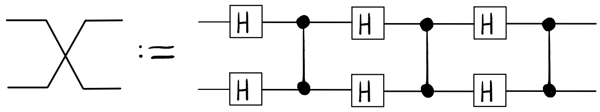
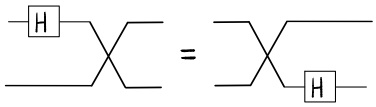
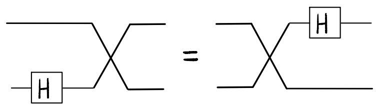
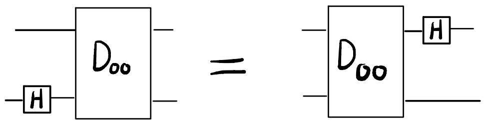
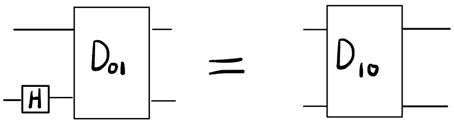


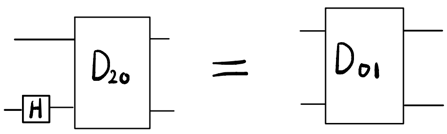
$C2: H^4 = I$ Def 3: 

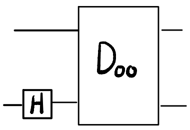
R_{19}: (1) 

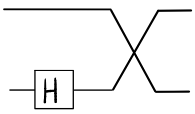
(2) 

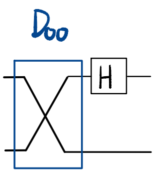
Lem 4 By Def 3, C_2 & R_{19}, 8.(1) 

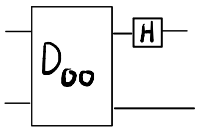
(2) 

(7) 

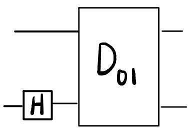
Proof: 8.(1).LHS = 

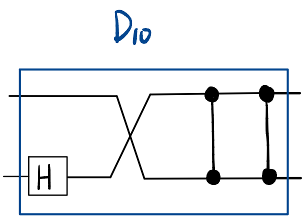
:= 

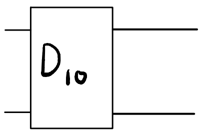
$\overset{R_{19}}{=}$ 

$\overset{\text{def}}{=}$ 

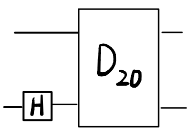
= 8.(1).RHS .

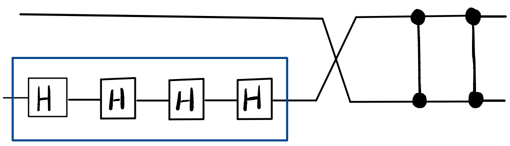
8.(2).LHS = 

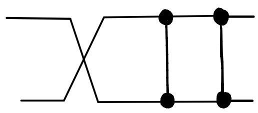
:= 

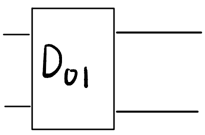
$\overset{\text{def}}{=}$ 

= 8.(2).RHS .

8.(7).LHS = 

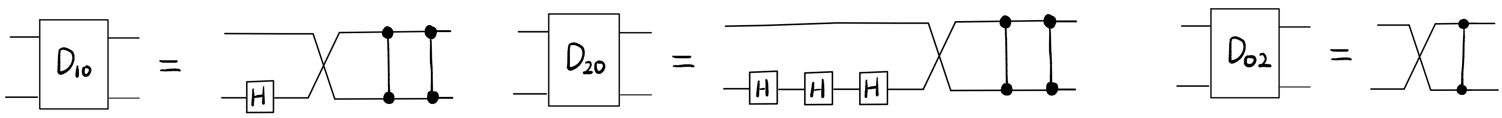
:= 

$\overset{C_2}{=}$ 

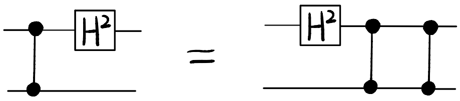
$\overset{\text{def}}{=}$ 

= 8.(7).RHS .

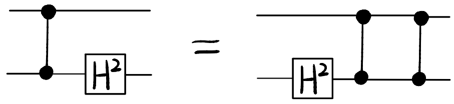
∎



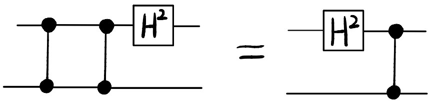
C_8 : (1)



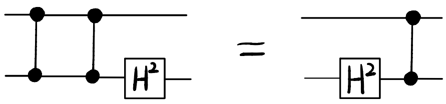
(2)



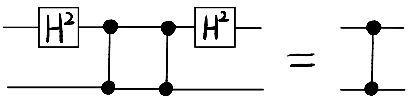
C_8^1: (1)



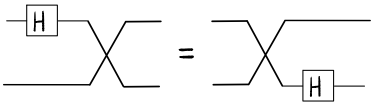
(2)



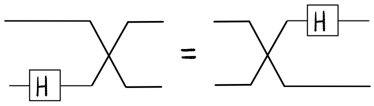
C_8^2:



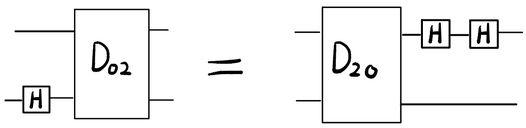
R_{19} : (1)



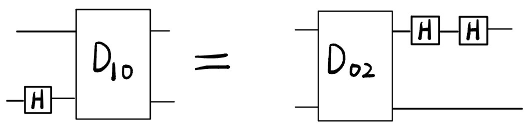
(2)



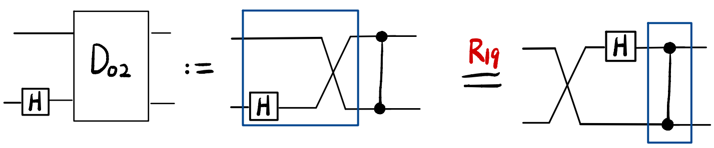
Lem 5 By Def 3, C_8 & R_{19} , 8. (3)



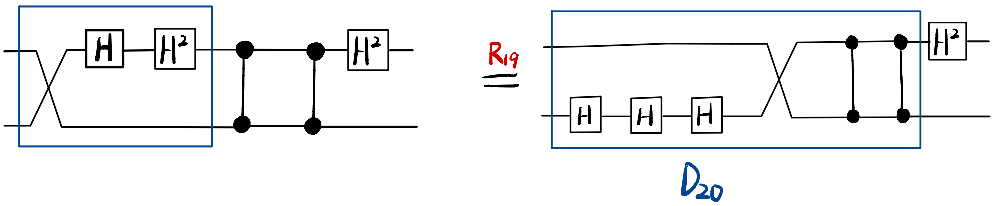
(4)



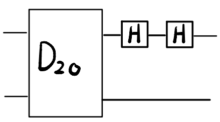
Proof: 8.(3). LHS =



$\overset{C_8^2}{=}$

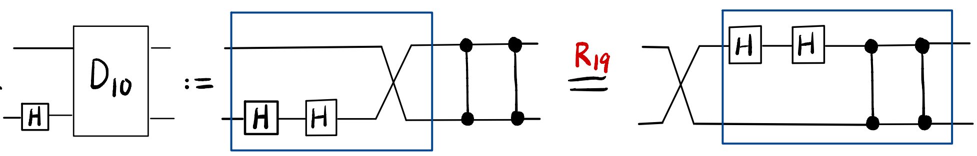


$\overset{def}{=}$

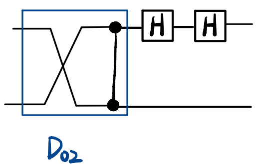


= 8.(3). RHS.

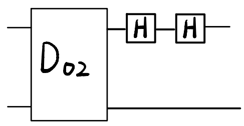
8.(4). LHS =



$\overset{C_8}{=}$

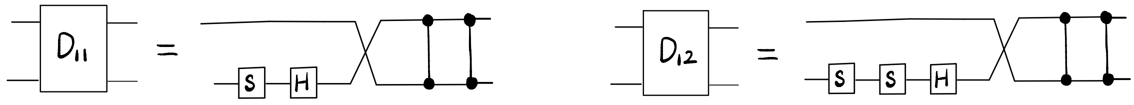


$\overset{def}{=}$

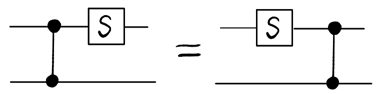


= 8.(4). RHS.

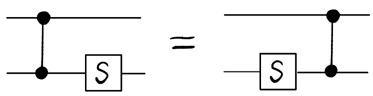
▨



C_7: (1)

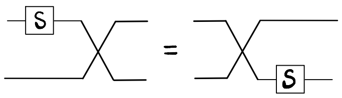


(2)

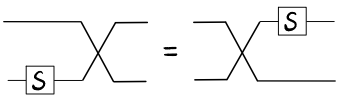


R_{12}: $Z^3 = I$ $\quad$ C_3: $S^3 = I$

R_{18}: (1)

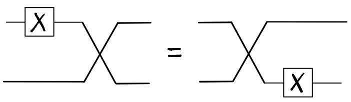


(2)

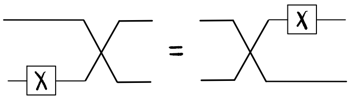


R_4^2: $-\boxed{H}-\boxed{Z}-\boxed{Z}-\boxed{H^\dagger}- \; = \; -\boxed{X^2}-$

R_{20}: (1)



(2)



C_5: $SH^2SH^2 = H^2SH^2S$ $\quad$ $SS' = S'S$

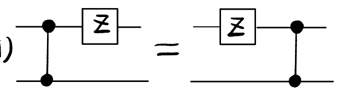
C_4^4: $-\boxed{S}-\boxed{S}-\boxed{H}-\boxed{S}-\boxed{S}-\boxed{H}- \cdot(-\omega^2) \; = \; -\boxed{H}-\boxed{H}-\boxed{H}-\boxed{S}-$

R_{11}: $-\boxed{Z^2}- \; = \; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-\boxed{S}-\boxed{S}- \; = \; -\boxed{S'}-\boxed{S}-\boxed{S}-$

Def: $-\boxed{S'}- \; = \; -\boxed{H}-\boxed{H}-\boxed{S}-\boxed{H}-\boxed{H}-$

$S' := H^2SH^2$

R_{13}: (1)



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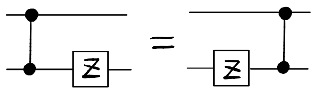
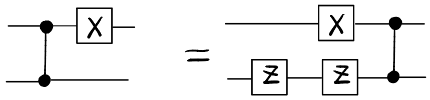
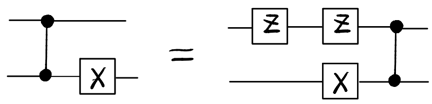


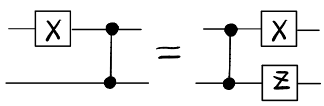
R_{14}: (1)



(2)



(3)



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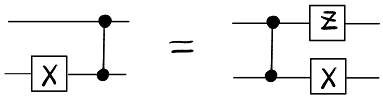
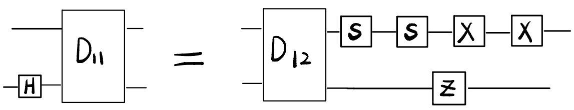


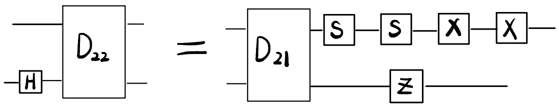
Lem 6 By $C_3, C_4, C_5, C_7, R_4, R_{11}, R_{12}, R_{13}, R_{14}, R_{18}$ & R_{20},

8. (5)



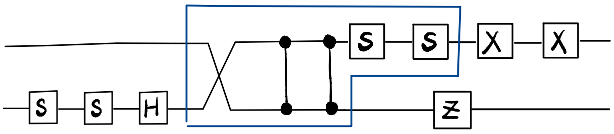
$\cdot(-\omega^2)$

(9)



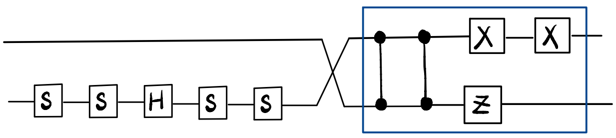
$\cdot(-\omega^2)$

Proof: 8.(5).RHS =



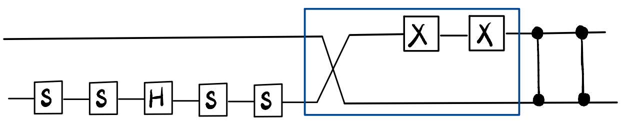
$\cdot(-\omega^2)$

$\overset{C_7}{\underset{R_{18}}{=}}$



$\cdot(-\omega^2)$

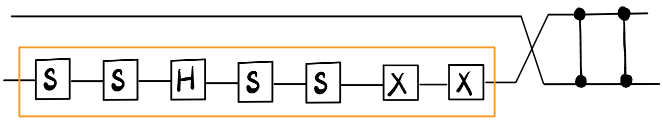
$\overset{R_{13}, R_{14}}{\underset{R_{12}}{=}}$



$\cdot(-\omega^2)$

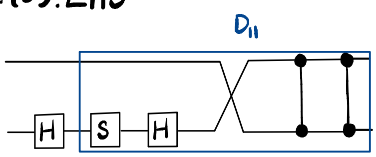
Composition in diagrammatic order

$\overset{R_{20}}{=}$



$\cdot(-\omega^2)$ (1)

8.(5).LHS =

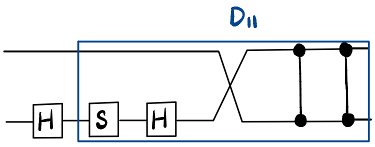


Alternatively, we can say it is sufficient to show $S^2HS^2X^2\cdot(-\omega^2) = HSH$. Since single-qutrit Clifford relations are complete, $\exists$ a sequence of rewrites to send $S^2HS^2X^2\cdot(-\omega^2)$ to HSH.

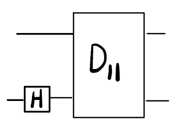
Note that $S^2HS^2X^2\cdot(-\omega^2) \overset{R_4^2}{=} \boxed{S^2HS^2H}Z^2H^\dagger\cdot(-\omega^2) \overset{C_4^4}{=} H^\dagger S\boxed{Z^2}H^\dagger$

$\overset{R_{11}}{=} H^\dagger\boxed{SS'}S^2H^\dagger \overset{C_5}{=} H^\dagger S'S^3H^\dagger \overset{C_3}{=} H^\dagger S'H^\dagger \overset{def}{=} H^\dagger H^2SH^2H^\dagger = HSH$

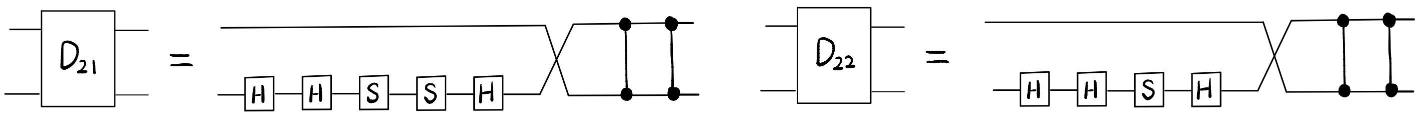
Hence (1) =

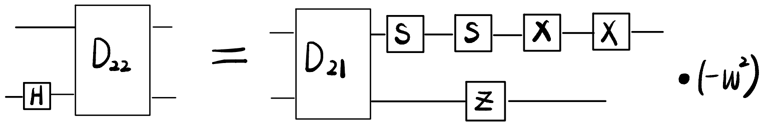


=:

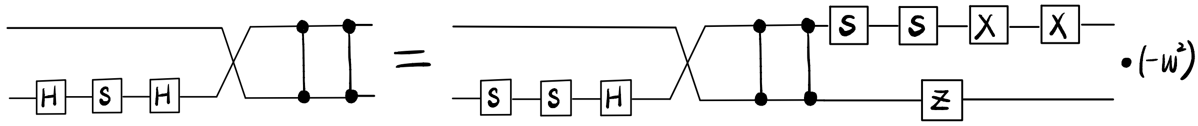


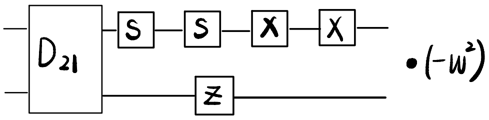
= 8.(5). LHS.

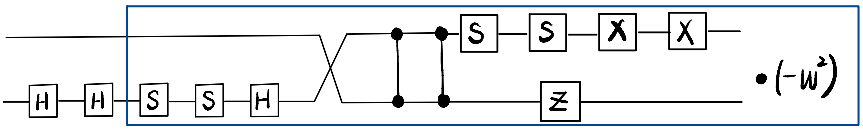


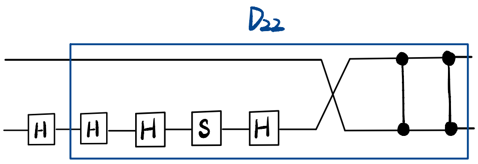
(9) 

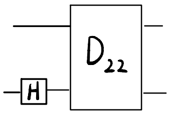
Proof cont. Recall that

 (2)

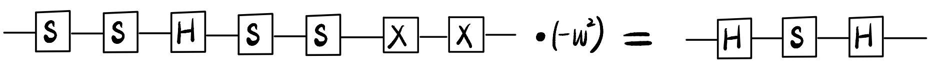
8.(9).RHS := 

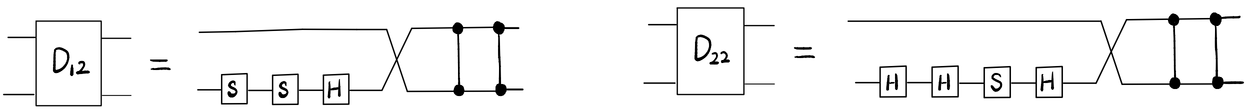
def 

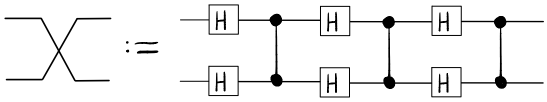
(2) 

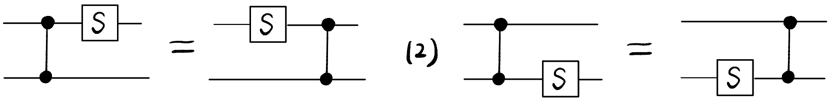
=:  = 8. (9). LHS.

∎

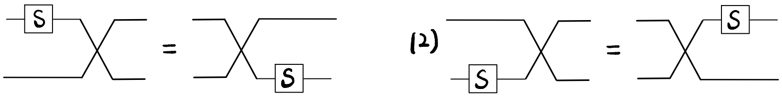
Lem 6 

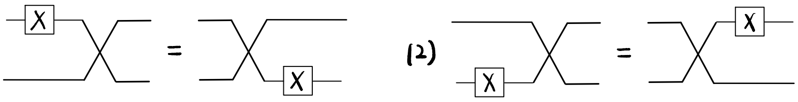


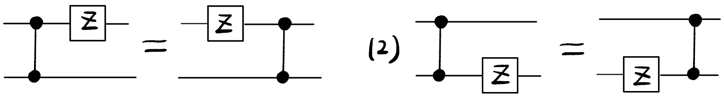
Def 3 : 

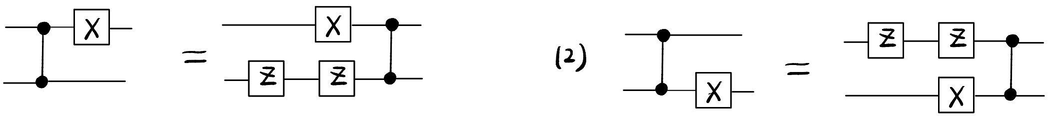
C7: (1) 

R12 : $Z^3 = I$

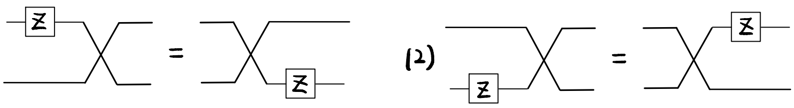
R18 : (1) 

R20 : (1) 

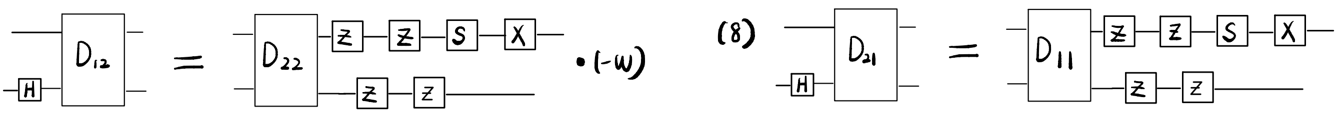
R13 : (1) 

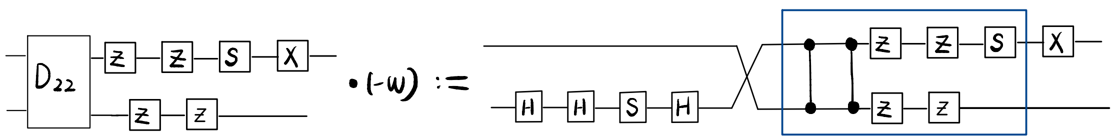
R14 : (1) 

(3) 

R21 : (1) 

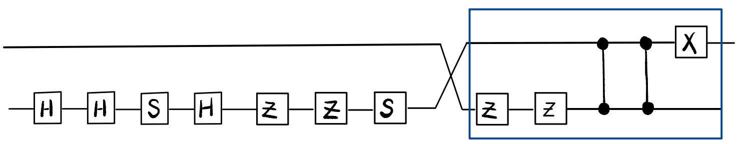
Lem 7 By Def 3, C2, C3, C4, C5, C7, R4, R10, R11, R12, R13, R14, R18, R20 & R21,

8. (6) 

Proof: 8.(6).RHS = 

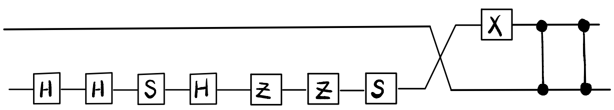
$\overline{\overline{R_{13}, R_{21}}}$

C7 , R18

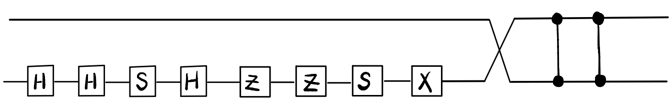


$\overline{\overline{R_{14}}}$

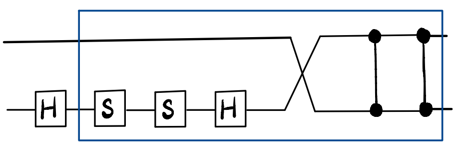
R12, R13



$\overline{\overline{R_{20}}}$

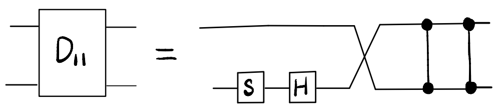


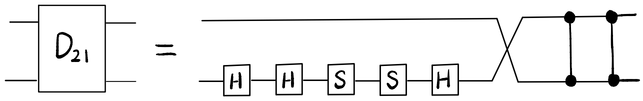
$\cdot(-w)$ (1)

8.(6).LHS = 

. It is sufficient to show

$$H^2 S H Z^2 S X \cdot (-w) = H S^2 H$$





$C2:\ H^4 = I$ $\quad C5:\ SH^2SH^2 = H^2SH^2S$ $\quad SS' = S'S$ $\quad C3:\ S^3 = I$ $\quad R4':\ HZH^\dagger = X$

R10: $Z = HHSSHHS = S'S'S$

R11: $Z^2 = HHSHHSS = S'SS$

$C4'$: $SSHSS = HHHSHHH \cdot (-w)$

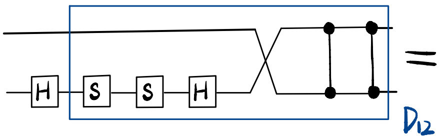
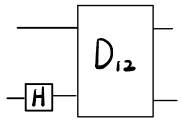
Def: $S' = HHSHH$ $\quad S' := H^2SH^2$

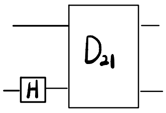
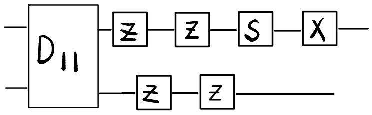
Composition in diagrammatic order →

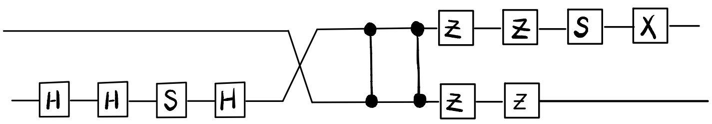
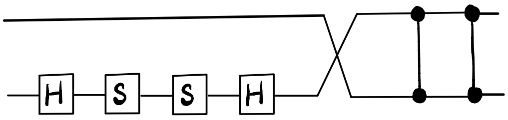
Note that $H^2SHZ^2S\boxed{X}\cdot(-w) \overset{R4'}{=} H^2SH\boxed{Z^2}SH\boxed{Z}H^\dagger\cdot(-w) \overset{R10}{\underset{R11}{=}} H^2SHS'\boxed{S^2S}HS'S'SH^\dagger\cdot(-w) \overset{C3}{=}$

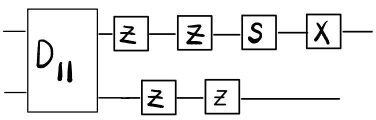
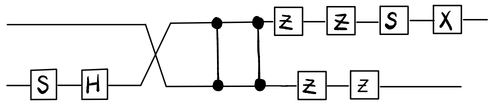
$H^2SH\boxed{S'}H\boxed{S'S'}SH^\dagger\cdot(-w) \overset{def}{=} H^2SHH^2S\boxed{H^2HH^2}S^2H^2SH^\dagger\cdot(-w) \overset{C2}{=} H^2SHH^2S\boxed{H}S^2H^2SH^\dagger\cdot(-w)$

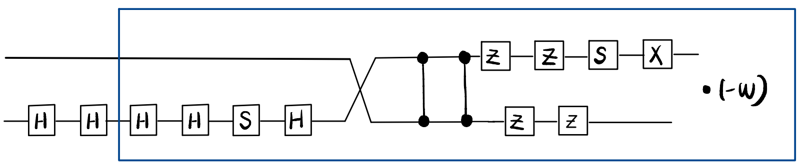
$\overset{C2}{=} H^2S\boxed{H^\dagger SH^\dagger}\boxed{H^2S^2H^2}SH^\dagger\cdot(-w) \overset{C4'}{\underset{def}{=}} H^2\boxed{SS^2}H\boxed{S^2S'S'}SH^\dagger \overset{C3}{\underset{C5}{=}} H^2H\boxed{S'S'}\boxed{S^3}H^\dagger \overset{C3}{\underset{def}{=}} H^\dagger H^2S^2H^2H^\dagger = HS^2H.$

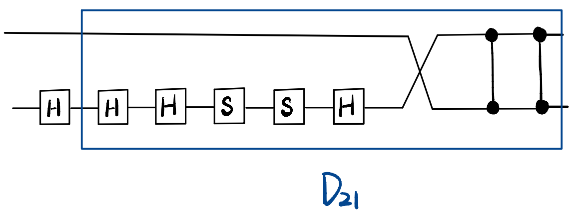
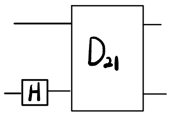
Hence (1) =  =:  = 8.(6).LHS.

8.(8)  =  $\cdot(-w)$

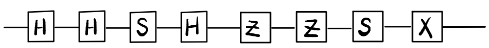
Recall  $\cdot(-w)$ =  (2)

8.(8).RHS =  $\cdot(-w)$:=  $\cdot(-w)$

$\overset{C2}{=}$ 

$\overset{(2)}{=}$  =:  = 8.(8).LHS

∎

Lem 7  $\cdot(-w)$ =