

$$\begin{array}{l}
 R_{23}^{q'}: \quad \text{Diagram} = \text{Diagram} \cdot w^2 \\
 R_{25}^7: \quad \text{Diagram} = \text{Diagram} \cdot w
 \end{array}
 \quad
 \begin{array}{l}
 R_{23}^4: \quad \text{Diagram} = \text{Diagram} \cdot w^2 \\
 C_{16}^{36}: \quad \text{Diagram} = \text{Diagram}
 \end{array}$$

$$R_{59}: \quad \text{Diagram} \xrightarrow{\text{WTS}} \text{Diagram}$$

$$R_{59} \cdot \text{RHS} = \text{Diagram} \cdot w^2$$

$$= \text{Diagram} \cdot w^2$$

$$= \text{Diagram} \cdot w^2$$

$$= \text{Diagram} \cdot w^2 \cdot w^2$$

$$= \text{Diagram} \cdot w \cdot w^2$$

$$= \text{Diagram}$$

$$= \text{Diagram}$$

$$R_{25}^{7'} = \dots \otimes \text{S}' \otimes \text{S} \otimes \dots$$

$$R_{59}: \quad \begin{array}{c} \text{WTS} \\ \text{H} \quad \text{S} \\ \text{H}^3 \quad \text{S} \quad \text{H} \quad \text{H} \end{array} = \dots$$

$$\begin{array}{c} \text{H} \quad \text{S} \quad \text{H} \quad \text{S}' \quad \text{H}^2 \quad \text{S} \quad \text{H} \\ \text{H}^3 \quad \text{S} \quad \text{H} \quad \text{H}^2 \quad \text{S} \quad \text{H}^3 \quad \text{S}' \\ \text{H}^2 \quad \text{S} \quad \text{H}^3 \quad \text{S}' \end{array} = \dots$$

III

$$R_{59}: \quad \begin{array}{c} \text{WTS} \\ \text{H} \quad \text{S} \\ \text{H}^3 \quad \text{S} \quad \text{H} \quad \text{H} \end{array} = \dots$$

$$\begin{array}{c} \text{H}^2 \quad \text{H}^3 \quad \text{H}^2 \quad \text{S} \quad \text{H} \quad \text{S}' \quad \text{H}^2 \quad \text{S} \quad \text{H}^3 \quad \text{S}' \\ \text{H}^2 \quad \text{S} \quad \text{H}^3 \quad \text{S}' \end{array} = \dots$$

III

$$R_{59}: \quad \begin{array}{c} \text{WTS} \\ \text{H} \quad \text{S} \\ \text{H}^3 \quad \text{S} \quad \text{H} \quad \text{H} \end{array} = \dots$$

III → We can stop here by citing two-qutrit Clifford completeness.

$$R_{59}: \quad \begin{array}{c} \text{WTS} \\ \text{H} \quad \text{S} \\ \text{H}^3 \quad \text{S} \quad \text{H} \quad \text{H} \end{array} = \dots$$

$$\text{Hence } R_{59}: \quad \begin{array}{c} \text{WTS} \\ \text{H} \quad \text{S} \\ \text{H}^3 \quad \text{S} \quad \text{H} \quad \text{H} \end{array} = \dots$$

$$R_{59, \text{RHS}} := \begin{array}{c} \text{H} \quad \text{S} \quad \text{H} \quad \text{S}' \quad \text{H}^2 \quad \text{S} \quad \text{H} \\ \text{H}^3 \quad \text{S} \quad \text{H} \quad \text{H}^2 \quad \text{S} \quad \text{H}^3 \quad \text{S}' \end{array}$$

$$= \begin{array}{c} \text{H} \quad \text{S} \quad \text{H} \quad \text{S}' \quad \text{H}^2 \quad \text{S} \quad \text{H} \\ \text{H}^3 \quad \text{S} \quad \text{H} \quad \text{H}^2 \quad \text{S} \quad \text{H}^3 \quad \text{S}' \end{array} \cdot w^2$$

$$= \begin{array}{c} \text{H} \quad \text{S} \quad \text{H} \quad \text{S}' \quad \text{H}^2 \quad \text{S} \quad \text{H} \\ \text{H}^3 \quad \text{S} \quad \text{H} \quad \text{H}^2 \quad \text{S} \quad \text{H}^3 \quad \text{S}' \end{array} \cdot w^2$$

$$R_{23}^4: \quad \begin{array}{c} \text{Diagram of } R_{23}^4 \end{array} = \quad \begin{array}{c} \text{Diagram of } R_{23}^4 \end{array} \cdot w^2$$

R₅₉:

WTS

≡

|||

• w²

$R_{59}:$  \equiv  $\cdot w^2$

$$R_{59} : \begin{array}{c} \text{H} \quad S \quad H \\ \text{H}^3 \quad S \quad H \end{array} \xrightarrow{\text{WTS}} \begin{array}{c} \text{H} \quad S \quad H \quad S^2 \quad S' \\ \text{H}^2 \quad S \quad H \quad S \quad H^2 \end{array} \cdot w^2$$

$R_{59}:$  WTS  $\cdot w^2$

$$R_{59} : \text{Circuit Diagram} \xrightarrow{\text{WTS}} \text{Simpler Circuit Diagram}$$

$$R_{59} : \begin{array}{c} \text{Circuit Diagram} \\ \text{WTS} \end{array} \equiv \begin{array}{c} \text{Circuit Diagram} \\ \text{WTS} \end{array}$$

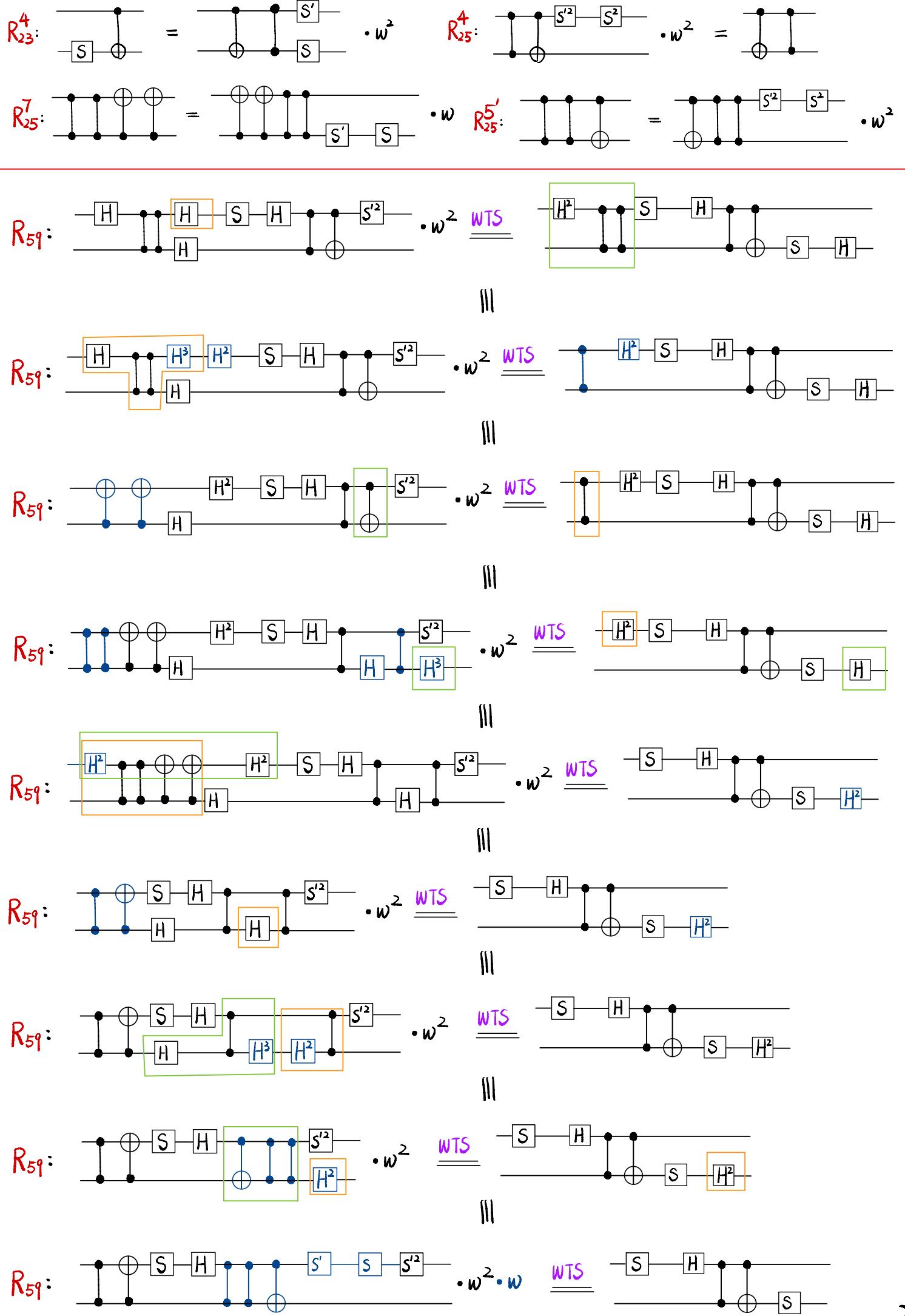
$R_{5^q}:$  $\cdot w^2$ **WTS**

|||



$$R_{59}: \quad \text{Circuit Diagram} \cdot w^2 = \text{WTS} \quad \text{Circuit Diagram}$$

$$R_{5^q}: \quad \text{Circuit Diagram} \cdot w^2 = \text{Simplified Circuit Diagram}$$



$$R_{25}^1: \quad \text{Diagram} = \text{Diagram} \cdot w \quad R_{23}^8: \quad \text{Diagram} = \text{Diagram} \cdot w$$

$$R_{25}^{5'}: \quad \text{Diagram} = \text{Diagram} \cdot w^2$$

$$R_{59}: \quad \text{Diagram} \cdot w^2 \cdot w \xrightarrow{\text{WTS}} \text{Diagram} \quad \text{III}$$

$$R_{59}: \quad \text{Diagram} \xrightarrow{\text{WTS}} \text{Diagram} \quad \text{III}$$

$$R_{59}: \quad \text{Diagram} \cdot w \xrightarrow{\text{WTS}} \text{Diagram} \quad \text{III}$$

$$R_{59}: \quad \text{Diagram} \cdot w \xrightarrow{\text{WTS}} \text{Diagram} \quad \text{III}$$

$$R_{59}: \quad \text{Diagram} \cdot w \cdot w \xrightarrow{\text{WTS}} \text{Diagram} \quad \text{III}$$

$$R_{59}: \quad \text{Diagram} \cdot w^2 \xrightarrow{\text{WTS}} \text{Diagram} \quad \text{III}$$

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$$R_{59}: \quad \text{Diagram} \cdot w^2 \cdot w^2 \xrightarrow{\text{WTS}} \text{Diagram}$$

$$R_{23}^4: \quad \text{Diagram} = \text{Diagram} \cdot w^2$$

$$R_{59}: \quad \text{Diagram} \xrightarrow{\bullet w^2 \cdot w^2 \text{ WTS}} \text{Diagram}$$

$$R_{59}: \quad \text{Diagram} \xrightarrow{\bullet w \text{ WTS}} \text{Diagram}$$

$$R_{59, LHS} := \text{Diagram} \xrightarrow{\bullet w} R_{23}^4 \quad \text{Diagram} \xrightarrow{\bullet w \cdot w^2}$$

$$= \text{Diagram} =: R_{59, RHS} \quad \square$$

$$R_{59} : \begin{array}{c} \text{Diagram of } R_{59} \text{ showing two horizontal wires with } H \text{ gates and vertical wires connecting them.} \end{array} = \begin{array}{c} \text{Diagram showing the decomposition of } R_{59} \text{ into a sequence of } S^2 \text{ and } H \text{ gates. The } S^2 \text{ gates are highlighted in orange boxes, and the } H \text{ gates are highlighted in green boxes. The result is multiplied by } w. \end{array}$$

$$C_3: S^3 = I \quad ||| \quad HS^2 H^2 SH \boxed{S^2} \xrightarrow{R_1} \boxed{HS^2 H^2 SH} \ x^2 \xrightarrow{R_7} X^2 x^2$$

$$\text{R10: } \text{---} \boxed{Z} \text{---} = \text{---} \boxed{S'} \text{---} \boxed{S'} \text{---} \boxed{S} \text{---}$$

$$R_{59} : \text{Circuit Diagram} = \begin{array}{c} \text{Circuit Diagram} \\ \text{with } S^2, H, Z^2 \end{array} \cdot w$$

$$C_7:(1) \quad \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \quad \boxed{S} \quad = \quad \boxed{S} \quad \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \quad (2) \quad \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \quad \boxed{S} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array}$$

Lem D2 Def 0-2, Def 4-5, Def 7, C₀₋₈, C₁₃, C₁₆, R₅, R₇, R₈, R₉, R₁₀, R₁₁, R₁₂, R₁₅, R₂₃, R₂₄, R₂₅ imply

$$R_{60} : \text{Circuit Diagram} = \begin{array}{c} \text{Circuit Diagram} \\ \text{Circuit Diagram} \end{array}$$

Proof: $R_{60} \cdot LHS :=$

The diagram illustrates two equivalent quantum circuit representations for the R_{59} gate, labeled R_{59} and $w \cdot w$. The top row shows a circuit with three horizontal lines. The top line starts with a \oplus gate, followed by two X^2 gates and two Z^2 gates. The middle line starts with two S^2 gates in an orange box, followed by an H gate, and then a sequence of gates: $S^2 H S'^2 H Z S^2 H$, which is highlighted in green. The bottom line starts with two S^2 gates in an orange box, followed by an H gate, and then a sequence of gates: $S^2 H S'^2 H Z S^2 H$, which is also highlighted in green. Below the circuit, the equation $ZS^2 \xrightarrow{R_{10}} S'^2 SS^2 \xrightarrow{C_3} S'^2$ is shown, indicating the simplification of the circuit.

$$R5 : \boxed{x} = \boxed{H} \boxed{S} \boxed{H} \boxed{H} \boxed{S} \boxed{S} \boxed{H} \quad R12 : z^3 = I \quad R9 : x^3 = I$$

$$R10: \quad \boxed{Z} = \boxed{S'} \boxed{S'} \boxed{S} \quad C_1: w^3=I \quad C_2: H^4=I \quad C_3: S^3=I \quad C_5: SS'=S'S$$

$$R8: \boxed{z} - \boxed{x} = \boxed{x} - \boxed{z} \cdot w^2 \quad \text{Def 1: } \boxed{s'} := \boxed{H} - \boxed{H} - \boxed{s} - \boxed{H} - \boxed{H}$$

Lem D2

$R_{60}:$

Proof cont.

$$R_{60} \cdot LHS = \text{Diagram} \cdot w^2$$

The diagram shows two parallel horizontal lines representing neurons. The top line has nodes labeled S, H, S², H, S², H, S², H, S², H, S², H, Z. The bottom line has nodes labeled S, H, S², H, S², H, S², H, S², H, Z. Vertical connections between corresponding nodes on the two lines are indicated by black dots. Horizontal connections between adjacent nodes on each line are shown as solid lines. A yellow box highlights the last four nodes (S², H, S², Z) of the top line.

$$x^2 z^2 x^2 z^2 \stackrel{R_8}{=} x^2 z x^2 z z^2 \cdot w^2 \cdot w^2 \stackrel{R_{12}}{=} x^2 z x^2 \cdot w \stackrel{R_8}{=} x^2 x^2 z \cdot w^2 \cdot w^2 \stackrel{R_9}{=} x z \cdot w^2$$

Quantum circuit diagram showing two parallel chains of operations. The top chain starts with S and H gates, followed by a sequence of controlled operations involving $CNOT$ s and Z gates. The bottom chain is identical. The circuit concludes with measurement symbols at the end of each chain.

$$XZ \xrightarrow[R_{10}]{R_5} HSH^2S^2HS^1S$$

R_5, R_{10}
 $C_i, \text{Def1}$

• w

Hence,  • w WTS

$$C_1: w^3 = I \quad C_2: H^4 = I \quad C_3: S^3 = I \quad C_5: SS' = S'S$$

$$C_6 : \quad \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} \quad = \quad \underline{\hspace{2cm}}$$

$$\text{Def 2: } :=$$

$$\text{Def 4: } := \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}$$

$$C_6^2: \quad \begin{array}{c} \text{---} \\ | \\ \bullet \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \bullet \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \bullet \end{array} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \bullet \end{array}$$

Def 7:  := 

$$\text{Def 5: } := \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array}$$

Lem D2

$$R_{60} : \begin{array}{c} \text{Diagram showing two horizontal lines with vertical connections between them. The top line has a box labeled } H \text{ at its center. The bottom line has a box labeled } H \text{ at its center.} \\ \text{Diagram showing two horizontal lines with vertical connections between them. The top line has boxes labeled } S \text{ and } H \text{ from left to right. The bottom line has boxes labeled } S \text{ and } H \text{ from left to right.} \end{array} = \begin{array}{c} \text{Diagram showing two horizontal lines with vertical connections between them. The top line has boxes labeled } S, H, H^2, S^2, H, S^2, H, S' \text{ from left to right. The bottom line has boxes labeled } S, H, H^2, S^2, H, S^2, H, S' \text{ from left to right.} \end{array}$$

Proof cont.

$$R_{b0} \cdot LHS = \text{Diagram} \cdot w \quad \underline{\text{WTS}}$$

$C_2, C_3, C_5 \equiv C_6^*, C_6^{2*}$, Def 1, Def 2, Def 4-5, Def 7

$$R_{60} \cdot LHS = \begin{array}{ccccccccccccccccccccc} S^2 & H & \bullet & H & S'^2 & H & S^2 & H & \bullet & H & S'^2 & H & S & S' \\ \downarrow & \downarrow \\ S^2 & H & \bullet & H & S'^2 & H & S'^2 & H & \bullet & H & S'^2 & H & S & S' \end{array} \cdot w \quad \text{WTS}$$

$$\begin{array}{ccccccc} \text{---} & \boxed{H^2} & \boxed{S^2} & \text{---} & \text{H} & \bullet & \bullet & \text{---} & \boxed{S^2} & \text{---} & \text{H} \\ \text{---} & \boxed{H^3} & \boxed{S^2} & \text{---} & \text{H} & \bullet & \bullet & \text{---} & \boxed{S^2} & \text{---} & \text{H} \end{array} = R_{b0.} \text{ RHS.}$$

Hence, our problem is reduced to two-qutrit Clifford completeness:

$$R'_{60} \cdot LHS = \begin{array}{ccccccccccccccccc} S^2 & H & \bullet & H & S'^2 & H & S^2 & H & \bullet & H & S'^2 & H & S & S' \\ \hline S^2 & H & \bullet & H & S'^2 & H & S'^2 & H & \bullet & H & S'^2 & H & S & S' \end{array} \cdot w \quad \underline{\text{WTS}}$$

$$= R'_{b0} \cdot \text{RHS}$$

This completes the proof.