

C₁: $w^3 = I$

C₂: $H^4 = I$

C₃: $S^3 = I$

C₇:

R₁₅:

Def 6:

C₆¹:

C₈¹:

Lem Y

R₄:

$\cdot (-w^2)$

Proof cont.

$\cdot w^2$

Def 4

$\cdot w^2$

R₂₃⁴

$\cdot w^2 \cdot w^2$

R₂₃⁴

$\cdot w^2 \cdot w^2 \cdot w^2$

C₁, C₇ R₁₅ $S^2 H^2 S H \xrightarrow{\text{Def 1}} H^2 S^2 H^2 H^2 S H \xrightarrow{\frac{C_2}{C_3}} H^3$

C₂, C₃ Def 1

Def 6

C₆¹

C₂, C₈¹

$$C2: H^4 = I$$

$$C_8 : \quad \text{Diagram} = \quad \text{Diagram}$$

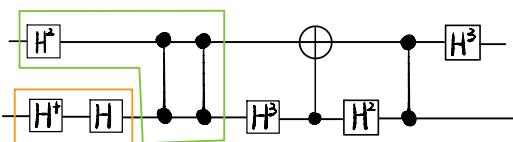
$$R_{32}^4 : \quad \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} \quad \boxed{H^3} \quad \bullet \quad \boxed{H} \quad \text{---} \quad = \quad \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} \quad \bigcirc \quad \text{---}$$

$$\text{Def 2: } \bigoplus := \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \bullet \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \bullet \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \bullet \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}$$

Lem Y

$$R_{24} : \begin{array}{c} \text{Circuit Diagram} \\ \text{with } H \text{ and } \oplus \text{ gates} \end{array} = \begin{array}{c} \text{Circuit Diagram} \\ \text{with } S^2, H, \oplus, S, X^2 \text{ gates} \\ \text{and } S^2, H, S, H, H, S^2, H, S, Z^2 \text{ gates} \end{array} \bullet (-w^2)$$

Proof cont.



A quantum circuit diagram showing two horizontal wires. The top wire has a blue box labeled H^2 , followed by a control dot, and then a blue circle with a plus sign. The bottom wire has a blue box labeled H^3 , followed by a control dot, and then a blue box labeled H^1 . A vertical blue line connects the two wires between the first and second boxes. An orange box highlights the region between the second box on the top wire and the third box on the bottom wire.

R_{32}^4

A quantum circuit diagram consisting of three horizontal wires. The top wire has a red label C_2 with a double underline. A CNOT gate is applied between the first and second wires. The bottom wire has a single H gate. Between the second and third wires, there are two parallel H gates: one on the top wire and one on the bottom wire. A vertical orange box encloses the second and third wires.

$$\underline{\text{Def 2}} \quad \begin{array}{c} \text{Circuit Diagram} \\ \text{with } H \text{ and } \oplus \text{ gates} \end{array} =: R_{34}. \text{LHS.}$$



$$R_{37}: \quad \text{Diagram} = S \oplus S^2 \cdot w^2$$

$$C_7: \quad \text{Diagram} = S$$

$$C_1: w^3 = I$$

$$C_2: H^4 = I$$

$$C_3: S^3 = I$$

$$C_5: SS' = S'S$$

$$R_{11}: \quad \text{Diagram} = S' S S$$

$$R_{14}: \quad \text{Diagram} = X \cdot z$$

$$\text{Def 1: } S' := H H S H H$$

$$R_{13}: \quad \text{Diagram} = z$$

$$R_5: \quad \text{Diagram} = H S H H S S H$$

$$R_{24}: \quad \text{Diagram} = S$$

$$C_8^5: \quad \text{Diagram} = H^2$$

Lem \exists Def 1-2, Def 4, C0-8, R5, R11, R13, R14, R15, R23, R24, R25, R32 & R37 imply

$$R_{35}: \quad \text{Diagram} = S H \oplus S H^2 S H S^2 H S^2 z^2 \cdot w$$

$$\text{Proof: } R_{35}.RHS := S H \oplus S H^2 S H S^2 H S^2 z^2 \cdot w$$

$$\begin{array}{c} C_7, R_8 \\ \hline R_{14} \end{array} \quad \text{Diagram} = S H \oplus S H^2 S H S^2 H S^2 z^2 \cdot w$$

$$z^2 H S^2 z^2 \stackrel{R_{11}}{=} (S' S^2) S^2 H S^2 (S' S^2) \stackrel{C_5}{=} S' S H S S'$$

$$\begin{array}{c} C_3, C_5 \\ \hline R_5, R_{11} \end{array} \quad \text{Diagram} = S H \oplus S H^2 S H S^2 H S^2 H S \cdot w$$

$$\begin{array}{c} R_{37} \\ \hline \end{array} \quad \text{Diagram} = S H S \oplus S^2 H S H S^2 H S^2 H S \cdot w \cdot w^2$$

$$\begin{array}{c} C_1, C_3, C_5 \\ \hline C_7, R_4 \end{array} \quad \text{Diagram} = S H S \oplus S H^2 S H S^2 H S^2 H S \cdot w \cdot w^2$$

$$\begin{array}{c} \text{Def 1} \\ \hline \end{array} \quad \text{Diagram} = S H S \oplus S H^2 S H S^2 H S^2 H S$$

Def 4:

$$:=$$

$$R_{23}^4:$$

$$= \cdot w^2$$

R₂₄³:

$$=$$

$$R_{25}^3:$$

$$= \cdot w$$

C₁: $w^3 = I$ **C₂:** $H^4 = I$ **C₃:** $S^3 = I$ **C₅:** $SS' = S'S$ **C₈⁷:**

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

R₁₅:

$$=$$

C₇:

$$=$$

C₈:

$$=$$

R₂₃¹:

$$=$$

R₂₄:

$$=$$

Proof cont.

$$H^2 = H^6 = H^3 H^3$$

C₂

Def 4

R₂₃⁴

$$\cdot w^2$$

R₂₃⁴

$$\cdot w^2 \cdot w^2 = \cdot w$$

C₁

R₂₅³

$$\cdot w \cdot w$$

C₇, R₅

C₂, C₃, C₅

$$\cdot w^2$$

C₈

C₈⁷

$$\cdot w^2$$

R₂₃¹

R₂₄

$$\cdot w^2 \cdot w^2$$

$$C_7 : \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} [S] = \begin{array}{c} \bullet \\ [S] \\ \bullet \end{array}$$

$$\text{Def 4: } \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} := \begin{array}{c} \bullet \\ \oplus \end{array}$$

$$\begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} [H] [H] [H] [H]$$

$$R_{24}^3: \begin{array}{c} \bullet \\ [S'] \\ \bullet \end{array} \oplus = \begin{array}{c} \bullet \\ \oplus \\ [S'] \end{array}$$

$$C_8^4: \begin{array}{c} \bullet \\ [H^2] \\ \bullet \end{array} [H^2] = \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} = \begin{array}{c} \bullet \\ [H^2] \\ \bullet \end{array} [H^2]$$

$$R_{24}^2: \begin{array}{c} \bullet \\ [S] \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \oplus \\ [S] \end{array}$$

$$R_{15}: \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} [S'] = \begin{array}{c} \bullet \\ [S'] \\ \bullet \end{array}$$

$$\text{Def 1: } [S'] := [H] [H] [S] [H] [H] \quad C_1: w^3 = 1 \quad C_2: H^4 = I \quad C_3: S^3 = I \quad C_5: SS' = S'S$$

$$C_4^1: \begin{array}{c} \bullet \\ [S] [S] [H] [S] [S] \end{array} = \begin{array}{c} \bullet \\ [H] [H] [H] [S] [H] [H] \end{array} \cdot (-w) \quad C_0: (-1)^2 = 1$$

$$R_{23}^4: \begin{array}{c} \bullet \\ [S] \oplus \end{array} = \begin{array}{c} \bullet \\ \oplus \\ [S'] \end{array} \cdot w^2 \quad R_{25}^4: \begin{array}{c} \bullet \\ [S^2] [S^2] \end{array} \cdot w^2 = \begin{array}{c} \bullet \\ \oplus \end{array}$$

Proof cont.

$$\begin{array}{c} \bullet \\ [S] \end{array} [H] [S^2] [H] [S] \dots \cdot w$$

$$\begin{array}{c} \bullet \\ [S] \end{array} [H] [S'] [H^2] [H^2] \dots \cdot w$$

$$\begin{array}{c} \bullet \\ [S] \end{array} [H] [S'] [H^2] [H^2] \dots \cdot w$$

$$\begin{array}{c} \bullet \\ [S] \end{array} [H] [S'] [H^3] [H^3] \dots \cdot w$$

$$\begin{array}{c} \bullet \\ [S] \end{array} [H] [S'] [H^3] [H^3] \dots \cdot w$$

$$\begin{array}{c} \bullet \\ [S] \end{array} [H] [S'] [H^3] [H^3] [S^2] [H^2] [S^2] [H] [S] \quad H^3 S H^3 = S^2 H S^2 \cdot (-w^2)$$

$$\begin{array}{c} \bullet \\ [S] \end{array} [H] [S^2] [H] [S^2] \dots \cdot w \cdot (-w^2)$$

$$\begin{array}{c} \bullet \\ [S] \end{array} [H] [S^2] [H] [S^2] \dots \cdot (-1) \cdot w^2$$

$$\begin{array}{c} \bullet \\ [S^2] [S^2] [S'] \end{array} [S] [H] [S^2] [H] [S] \dots \cdot (-1) \cdot w^2 \cdot w^2$$

$$\begin{array}{c} \bullet \\ [S] \end{array} [H] [S^2] [H] [S^2] \dots \cdot (-w)$$

$$R_{32}^4 : \begin{array}{c} H^3 \\ \text{---} \\ | \quad | \\ \text{---} \\ | \quad | \\ H^3 \end{array} = \begin{array}{c} \oplus \\ \text{---} \\ | \quad | \\ \text{---} \end{array} \quad R_{23}^1 : \begin{array}{c} S \\ \oplus \\ \text{---} \\ | \quad | \\ \text{---} \end{array} = \begin{array}{c} \oplus \\ \text{---} \\ | \quad | \\ \text{---} \\ S' \end{array} \cdot w^2 \quad C_0 : (-1)^2 = 1$$

$$\text{Def 1} : \begin{array}{c} S' \\ \text{---} \\ | \quad | \\ \text{---} \\ | \quad | \\ H \end{array} := \begin{array}{c} H \quad H \quad S \quad H \quad H \\ \text{---} \end{array} \quad C_1 : w^3 = 1 \quad C_2 : H^4 = I \quad C_3 : S^3 = I \quad C_5 : SS' = S'S$$

$$R_{23}^4 : \begin{array}{c} \oplus \\ \text{---} \\ | \quad | \\ \text{---} \\ | \quad | \\ S \end{array} = \begin{array}{c} \oplus \\ \text{---} \\ | \quad | \\ \text{---} \\ | \quad | \\ S' \end{array} \cdot w^2 \quad R_{25}^4 : \begin{array}{c} S^2 \quad S^2 \\ \text{---} \\ | \quad | \\ \text{---} \\ | \quad | \\ \oplus \end{array} \cdot w^2 = \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \\ | \quad | \\ \oplus \end{array}$$

$$C_6 : \begin{array}{c} \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \\ | \quad | \\ \text{---} \end{array} \quad R_{24}^3 : \begin{array}{c} S' \\ \text{---} \\ | \quad | \\ \text{---} \\ | \quad | \\ \oplus \end{array} = \begin{array}{c} S \\ \text{---} \\ | \quad | \\ \text{---} \\ | \quad | \\ \oplus \end{array} \quad R_{24}^2 : \begin{array}{c} S \\ \text{---} \\ | \quad | \\ \text{---} \\ | \quad | \\ \oplus \end{array} = \begin{array}{c} S \\ \text{---} \\ | \quad | \\ \text{---} \\ | \quad | \\ \oplus \end{array}$$

Proof cont.

$$\cdot (-w)$$

$$R_{23}^4 : \begin{array}{c} \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \end{array} \cdot (-w) \cdot w^2$$

$$C_1 \quad C_3 : \begin{array}{c} \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \end{array} \cdot (-1)$$

$$R_{25}^4 : \begin{array}{c} \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \end{array} \cdot (-1) \cdot w^2$$

$$C_5 : \begin{array}{c} \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \end{array} \cdot (-w^2)$$

$$R_{25}^4 : \begin{array}{c} \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \end{array} \cdot (-w^2) \cdot w^2$$

$$R_{24} : \begin{array}{c} \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \end{array} \cdot (-w)$$

$$R_{35} : \begin{array}{c} H \quad \oplus \\ \text{---} \\ | \quad | \\ \text{---} \\ | \quad | \\ H \end{array} \quad \text{WTS} \quad \underline{\underline{\underline{\quad}}}$$

$$\begin{array}{c} \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \\ | \quad | \quad | \quad | \\ \text{---} \end{array} \cdot (-w)$$

$\parallel C_6$

$$R_{35} : \begin{array}{c} H \quad \oplus \\ \text{---} \\ | \quad | \\ \text{---} \\ | \quad | \\ H \end{array} \quad \text{WTS} \quad \underline{\underline{\underline{\quad}}} \cdot (-w)$$

$$\text{Def 4: } \begin{array}{c} \bullet \\ \parallel \\ \oplus \end{array} := \begin{array}{c} \bullet \\ \parallel \\ \boxed{\text{H}} \quad \bullet \quad \boxed{\text{H}} \quad \boxed{\text{H}} \quad \boxed{\text{H}} \end{array}$$

$$C2: H^4 = I \quad R_{25}^1: \quad \begin{array}{c} \bullet \\ \oplus \\ \bullet \end{array} = \quad \begin{array}{c} \oplus \\ \bullet \\ \bullet \end{array} \quad [S'] \quad [S]$$

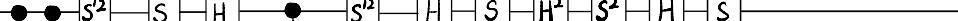
$$\text{Def 2: } \begin{array}{c} \textcircled{\text{+}} \\ \text{---} \\ \text{---} \end{array} := \begin{array}{ccccccc} \boxed{\text{H}} & \text{---} & \bullet & \boxed{\text{H}} & \text{---} & \boxed{\text{H}} & \text{---} & \boxed{\text{H}} \end{array}$$

$$C_8^1: \quad \begin{array}{c} \text{---} \\ | \\ \boxed{H^2} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \boxed{H^2} \\ | \\ \text{---} \end{array}$$

Proof cont.

$$R_{35} : \begin{array}{c} \text{Circuit Diagram} \\ \text{WTS} \end{array} = \begin{array}{c} \text{Circuit Diagram} \\ \text{Def 4} \parallel C_2 \end{array} \cdot (-w)$$

$$\begin{array}{c}
 \text{R}_{35}: \quad \begin{array}{c} \text{H} \\ \text{H} \end{array} \xrightarrow{\text{WTS}} \begin{array}{c} \text{S}^2 \text{ S} \text{ H} \text{ S}^2 \text{ H} \text{ S} \text{ H}^2 \text{ S}^2 \text{ H} \text{ S} \\ \text{H}^3 \text{ H} \text{ S}^2 \text{ H}^3 \text{ S} \text{ H}^2 \text{ S} \text{ H}^3 \text{ S} \text{ H}^2 \text{ S}' \text{ H} \text{ S}' \text{ S} \end{array} \cdot (-w) \\
 \parallel C_2
 \end{array}$$

$R_{35}:$  $\xrightarrow{\text{WTS}}$ 

$$R_{35} : \begin{array}{c} \text{---} | H \text{---} \bullet \text{---} \bullet \text{---} \oplus \\ \text{---} \end{array} \xrightarrow{\text{WTS}} \begin{array}{ccccccccc} \bullet & \bullet & S^2 & S & H & \bullet & S^2 & H & S \\ \bullet & \bullet & S^2 & H^3 & S & H^2 & S & H^3 & S \\ \bullet & \bullet & S^2 & H^3 & S & H^2 & S & H^3 & S \\ \bullet & \bullet & S^2 & H^3 & S & H^2 & S & H^3 & S' \\ \bullet & \bullet & S^2 & H^3 & S & H^2 & S & H & S' \end{array} \cdot (-w)$$

$$R_{35}.LHS := \text{Diagram} \stackrel{R'_{25}}{=} \text{Diagram} \cdot w \stackrel{\text{Def 4}}{=} \text{Diagram} \cdot w$$

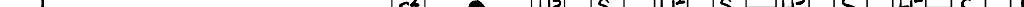
C_8

\equiv

$\cdot w$

Hence, R_{35} :

$$C_1, C_2 \parallel C_3, C_6$$

R₃₅:  **WTS**