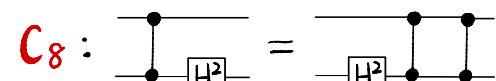
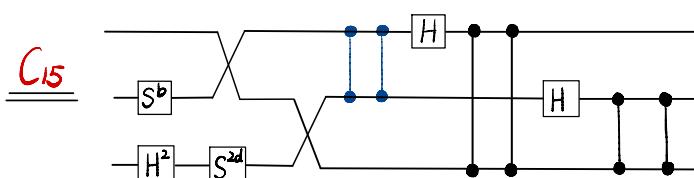
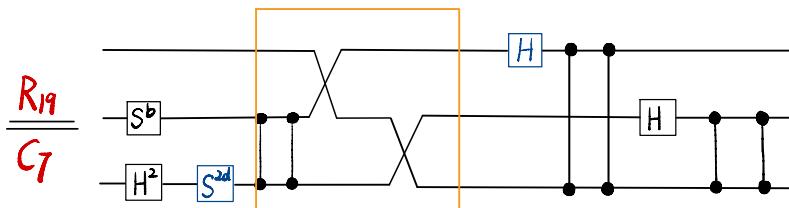
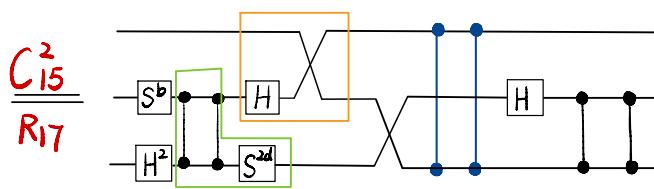
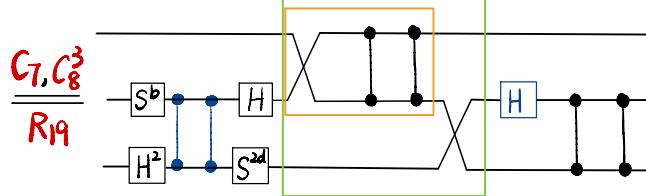
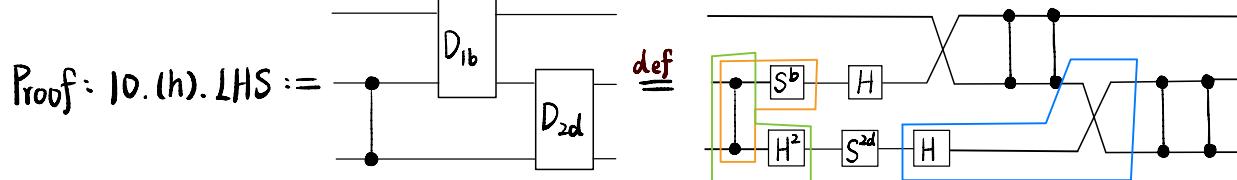
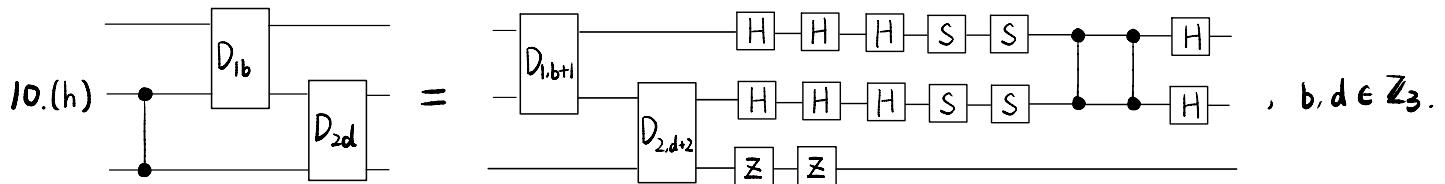


$$(2) \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}$$



Lem 8 Def 1-2, Def 5, Def 7, C<sub>2-3</sub>, C<sub>6-8</sub>, R<sub>11</sub>, R<sub>16</sub>, R<sub>17</sub>, R<sub>18</sub>, R<sub>19</sub>, R<sub>43</sub>, C<sub>13</sub> & C<sub>15</sub> imply



$$D_{lb} = \begin{array}{c} \text{S}^b \\ \text{H} \end{array}$$

$$C_{15}^2 : \quad \text{Diagram showing two horizontal lines with vertical connections forming a loop-like structure} \quad = \quad \text{Diagram showing a single horizontal line with a vertical connection at one end.}$$

$$C_2 : H^4 = I$$

$$D_{2d} = \begin{array}{c} \text{Circuit Diagram} \end{array}$$

$$R_{1q} : \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \boxed{\text{H}} \quad \times \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \times \begin{array}{c} \text{---} \\ | \\ \boxed{\text{H}} \end{array}$$

$$R_{18} : \begin{array}{c} \text{---} \\ \text{---} \\ | \quad | \\ S \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ | \quad | \\ S \end{array}$$

$$R_{17}: \quad \text{Diagram} = \text{Diagram}$$

$$R11 : \underline{\underline{Z}}^2 = \underline{\underline{S}}' \underline{\underline{S}} \underline{\underline{S}}$$

$$R_{16} : \text{Diagram} = \text{Diagram}$$

$$\text{Lem 8} \quad 10.(h) \quad \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \quad D_{1b} \quad \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \quad D_{2d} \quad = \quad \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \quad D_{1,b+1} \quad \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \quad H \quad H \quad H \quad S \quad S \quad \bullet \quad \bullet \quad H \quad , \quad b, d \in \mathbb{Z}_3. \\ \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \quad D_{2,d+2} \quad \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \quad H \quad H \quad H \quad S \quad S \quad \bullet \quad \bullet \quad H \\ \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \quad Z \quad Z \quad \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array}$$

Proof cont :

$$10.(h).LHS = \begin{array}{c} S^b \\ H^2 \end{array} \quad \begin{array}{c} S^{2d} \end{array}$$

**10.(h).RHS :=**

def

$$2(d+2) = 2d + 1$$

The diagram illustrates a quantum circuit with two main horizontal lines representing qubits. The top line starts with a sequence of gates:  $S^\dagger$ ,  $S$ , and  $H$ . A control dot is placed above the  $S$  gate. This line then branches into two paths. The left path contains a sequence of gates:  $H^2$ ,  $S^{2d}$ ,  $S$ , and  $H$ . A control dot is placed above the  $S^{2d}$  gate. The right path contains a sequence of gates:  $H^3$ ,  $S^2$ ,  $H^3$ ,  $S^2$ , and  $H$ . A control dot is placed above the first  $H^3$  gate. The bottom line also branches into two paths. The left path contains a sequence of gates:  $H^2$ ,  $S^{2d}$ ,  $S$ , and  $H$ . A control dot is placed above the  $S^{2d}$  gate. The right path contains a sequence of gates:  $Z^2$  and  $H$ . A control dot is placed above the  $Z^2$  gate. Various colored boxes highlight specific regions: an orange box highlights the initial sequence of gates on the top line; a green box highlights the sequence from  $S^{2d}$  to  $H$  on the bottom line; and a blue box highlights the  $Z^2$  gate on the bottom line.

$$\text{Def 7 : } \begin{array}{c} \text{Diagram} \\ \oplus \end{array} := \begin{array}{c} \text{Diagram} \\ \times \end{array} = \begin{array}{c} \text{Diagram} \\ H \cdot H^3 \end{array} \quad \text{Def 2 : } \begin{array}{c} \text{Diagram} \\ \oplus \end{array} := \begin{array}{c} \text{Diagram} \\ H \cdot H^3 \end{array} \quad \text{C2 : } H^4 = I$$

$$R_{44} : \begin{array}{c} \text{Diagram} \\ \oplus \end{array} = \begin{array}{c} \text{Diagram} \\ S \end{array} \oplus \begin{array}{c} \text{Diagram} \\ S^2 \end{array} = \begin{array}{c} \text{Diagram} \\ S \end{array} \oplus \begin{array}{c} \text{Diagram} \\ S^2 \end{array} \oplus \begin{array}{c} \text{Diagram} \\ S' \end{array} \oplus \begin{array}{c} \text{Diagram} \\ S^2 \end{array}$$

Lem 8

$$10.(h) \begin{array}{c} \text{Diagram} \\ D_{1b} \end{array} \begin{array}{c} \text{Diagram} \\ D_{2d} \end{array} = \begin{array}{c} \text{Diagram} \\ D_{1,b+1} \end{array} \begin{array}{c} \text{Diagram} \\ D_{2,d+2} \end{array} \begin{array}{c} \text{Diagram} \\ H \end{array} \begin{array}{c} \text{Diagram} \\ H \end{array} \begin{array}{c} \text{Diagram} \\ H \end{array} \begin{array}{c} \text{Diagram} \\ S \end{array} \begin{array}{c} \text{Diagram} \\ S \end{array} \begin{array}{c} \text{Diagram} \\ H \end{array} \quad , \quad b, d \in \mathbb{Z}_3.$$

Proof cont : Hence

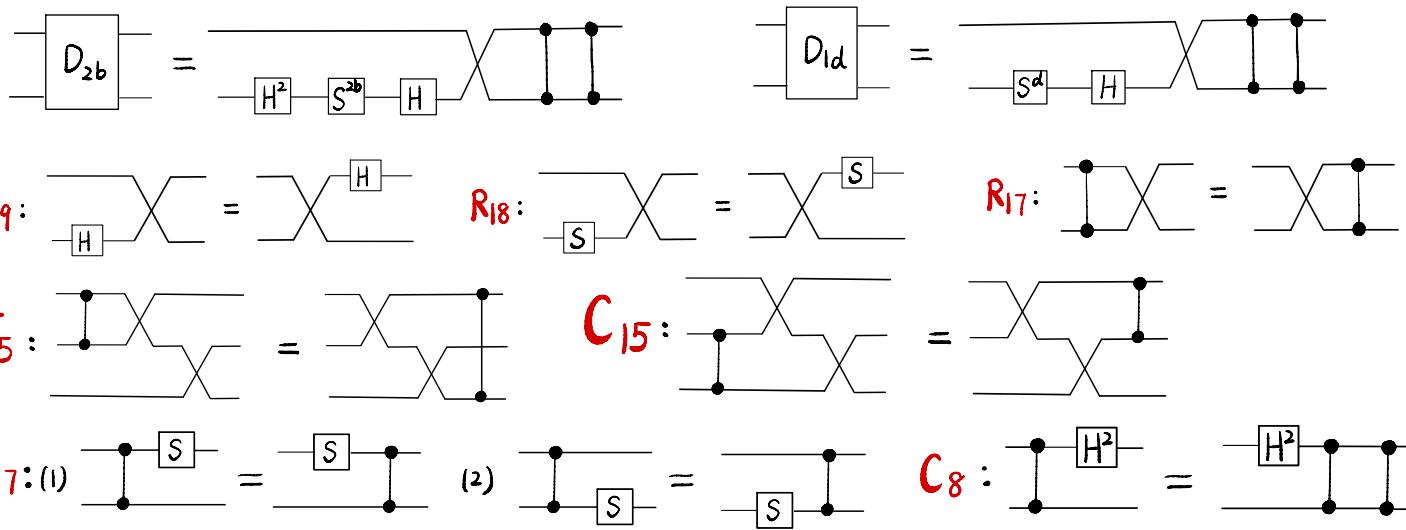
$$\begin{array}{c} \text{Diagram} \\ S^b \end{array} \begin{array}{c} \text{Diagram} \\ H^2 \end{array} \begin{array}{c} \text{Diagram} \\ S^{2d} \end{array} \xrightarrow{\text{WTS}} \begin{array}{c} \text{Diagram} \\ S^b \end{array} \begin{array}{c} \text{Diagram} \\ H \end{array} \begin{array}{c} \text{Diagram} \\ S \end{array} \begin{array}{c} \text{Diagram} \\ H^3 \end{array} \begin{array}{c} \text{Diagram} \\ S^2 \end{array} \begin{array}{c} \text{Diagram} \\ H \end{array} \quad C_2, C_3 \parallel R_{1b}$$

$$\begin{array}{c} \text{Diagram} \\ H \end{array} \begin{array}{c} \text{Diagram} \\ H \end{array} \xrightarrow{\text{WTS}} \begin{array}{c} \text{Diagram} \\ S \end{array} \begin{array}{c} \text{Diagram} \\ H \end{array} \begin{array}{c} \text{Diagram} \\ H^3 \end{array} \begin{array}{c} \text{Diagram} \\ S^2 \end{array} \begin{array}{c} \text{Diagram} \\ H \end{array} \quad C_2 \parallel$$

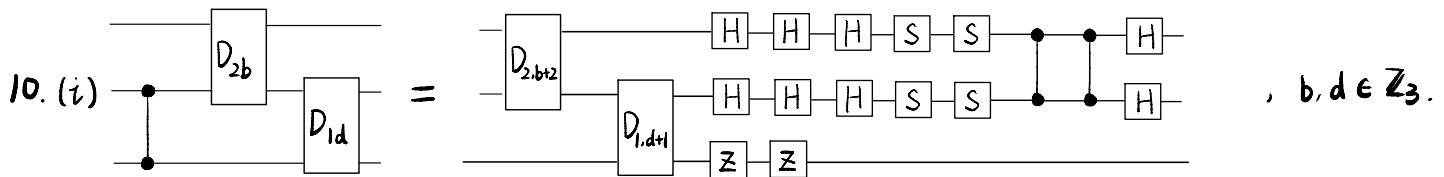
$$\begin{array}{c} \text{Diagram} \\ H \end{array} \begin{array}{c} \text{Diagram} \\ H^3 \end{array} \begin{array}{c} \text{Diagram} \\ H \end{array} \begin{array}{c} \text{Diagram} \\ H^3 \end{array} \xrightarrow{\text{WTS}} \begin{array}{c} \text{Diagram} \\ S \end{array} \begin{array}{c} \text{Diagram} \\ H \end{array} \begin{array}{c} \text{Diagram} \\ H^3 \end{array} \begin{array}{c} \text{Diagram} \\ S^2 \end{array} \begin{array}{c} \text{Diagram} \\ H \end{array} \quad C_2 \parallel \text{Def 2, Def 7}$$

$$\begin{array}{c} \text{Diagram} \\ \oplus \end{array} \xrightarrow{\text{WTS}} \begin{array}{c} \text{Diagram} \\ S \end{array} \oplus \begin{array}{c} \text{Diagram} \\ S^2 \end{array} \oplus \begin{array}{c} \text{Diagram} \\ S' \end{array} \oplus \begin{array}{c} \text{Diagram} \\ S^2 \end{array}$$

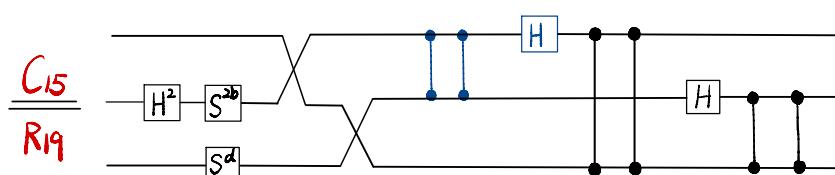
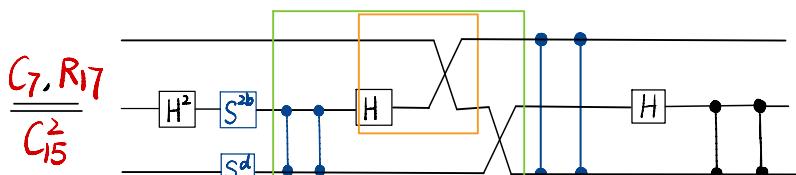
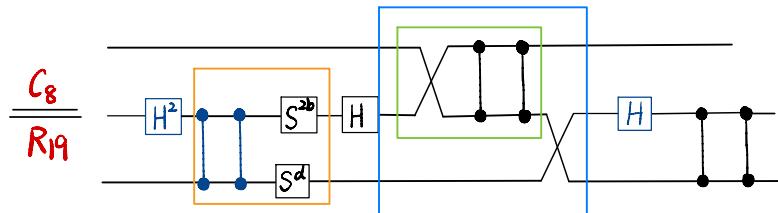
By  $R_{44}$ , this completes the proof.



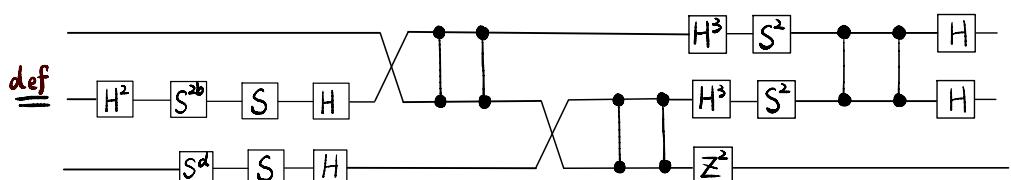
Lem 9 Def 1-2, Def 5, Def 7, C<sub>2-3</sub>, C<sub>6-8</sub>, R<sub>11</sub>, R<sub>16</sub>, R<sub>17</sub>, R<sub>18</sub>, R<sub>19</sub>, R<sub>43</sub>, C<sub>13</sub> & C<sub>15</sub> imply



Proof: 10.(i). LHS :=   $\stackrel{\text{def}}{=}$  



$$10.(i).RHS := \text{Diagram } D_{1,d+1} \text{ connected to } H^3 \text{ and } S^2 \text{ via } V_{2,b+2} \text{ and } \Xi^2.$$



$$R_{19}: \quad \text{Diagram} = \text{Diagram}$$

$$R_{18}: \quad \text{Diagram} = \text{Diagram}$$

$$R_{17}: \quad \text{Diagram} = \text{Diagram}$$

$$C_{15}^2: \quad \text{Diagram} = \text{Diagram}$$

$$R_{11}: \quad \bar{z}^2 = \text{Diagram}$$

$$R_{16}: \quad \text{Diagram} = \text{Diagram}$$

$$\text{Def 1: } S' := H H S H H$$

Lem 9

$$10. (i) \quad \text{Diagram} = \text{Diagram}, \quad b, d \in \mathbb{Z}_3.$$

Proof cont:

$$10. (i). \text{LHS} = \text{Diagram}$$

$$10. (i). \text{RHS} = \text{Diagram}$$

$$\begin{array}{c} R_{17} \\ C_{15}^2 \end{array} \quad \text{Diagram}$$

$$\begin{array}{c} R_{18}, R_{19} \\ R_{11}, \text{Def 1} \end{array} \quad \text{Diagram}$$

Hence

$$\text{Diagram} \xrightarrow{\text{WTS}} \text{Diagram}$$

$$\text{Diagram} \xrightarrow{\text{C}_2, \text{C}_3} \text{Diagram}$$

$$\text{Diagram} \xrightarrow{\text{WTS}} \text{Diagram}$$

$$\text{Def 7 : } \begin{array}{c} \oplus \\ \parallel \end{array} := \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} [H] \bullet [H^3] \\ \parallel \end{array} \quad \text{Def 2 : } \begin{array}{c} \oplus \\ \parallel \end{array} := \begin{array}{c} [H] \bullet [H^3] \\ \parallel \end{array} \quad \text{C2 : } H^4 = I$$

$$R_{44} : \begin{array}{c} \bullet \bullet \oplus \oplus \\ \bullet \bullet \oplus \oplus \\ \bullet \bullet \bullet \bullet \end{array} = \begin{array}{c} S \oplus S^2 \\ S \oplus S^2 \\ S' S^2 \end{array}$$

Lem 9

$$10. (i) \begin{array}{c} D_{2b} \\ \parallel \\ D_{1d} \end{array} = \begin{array}{c} D_{2,b+2} \\ \parallel \\ D_{1,d+1} \\ \parallel \\ Z \quad Z \end{array} \begin{array}{c} H \quad H \quad H \quad S \quad S \\ H \quad H \quad H \quad S \quad S \\ H \end{array}, \quad b, d \in \mathbb{Z}_3.$$

Proof cont :

$$\begin{array}{c} \bullet \bullet H \bullet \bullet \\ \bullet \bullet \parallel H \bullet \bullet \\ \bullet \bullet \bullet \bullet \end{array} \stackrel{\text{WTS}}{=} \begin{array}{c} S \quad H \quad H^3 \quad S^2 \quad H \\ S \quad H \quad H^3 \quad S^2 \quad H \\ S' \quad S^2 \end{array}$$

Reason analogously as in Lem 8 , by R<sub>44</sub>, we complete the proof. □