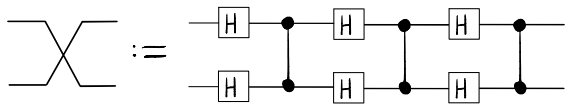
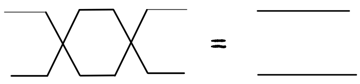
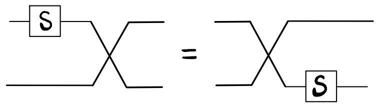
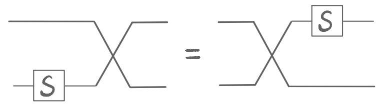
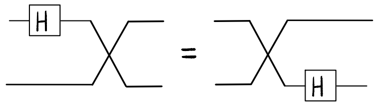
Two – Qutrit Derived Relations : T_3

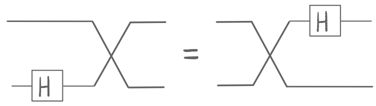
Def 3 : 

R_{16} : 

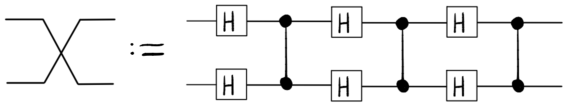
R_{18} : (1) 

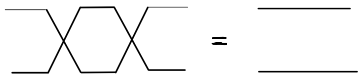
(2) 

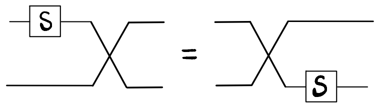
R_{19} : (1) 

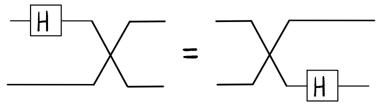
(2) 

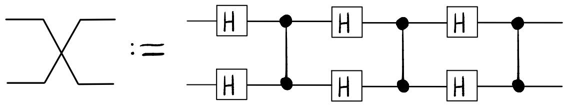
Two – Qutrit Derived Relations : T_2

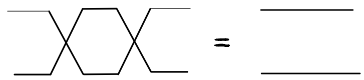
Def 3 : 

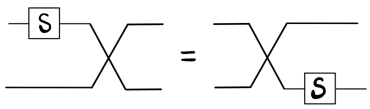
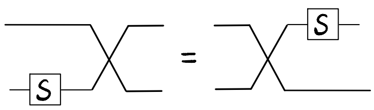
R_{16} : 

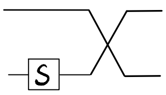
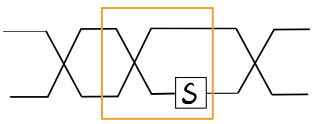
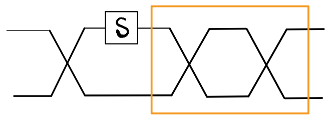
R_{18} : 

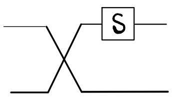
R_{19} : 

Def 3 : 

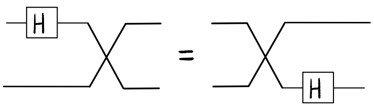
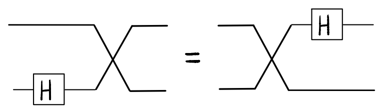
R_{16} : 

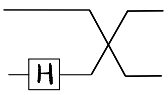
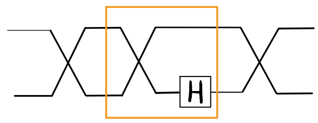
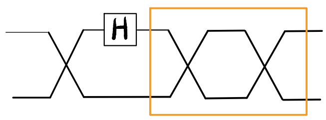
Lem A By Def 3 & R_{16}, R_{18} : (1)  implies R_{18} : (2) 

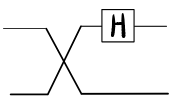
Proof : R_{18} : (2) . LHS :=  $\overset{R_{16}}{=\!=}$  $\overset{R_{18}:(1)}{=\!=}$ 

$\overset{R_{16}}{=\!=}$  =: R_{18} : (2) . RHS

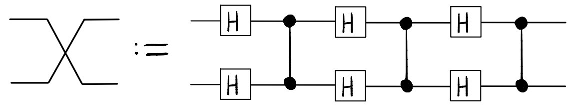
$\square$

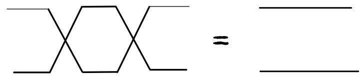
Lem B By Def 3 & R_{16}, R_{19} : (1)  implies R_{19} : (2) 

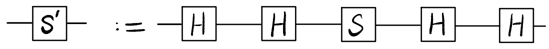
Proof : R_{19} : (2) . LHS :=  $\overset{R_{16}}{=\!=}$  $\overset{R_{19}:(1)}{=\!=}$ 

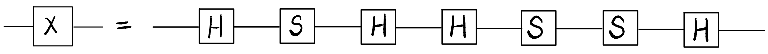
$\overset{R_{16}}{=\!=}$  =: R_{19} : (2) . RHS

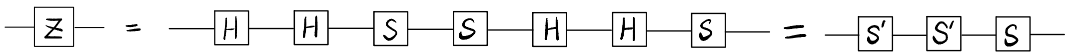
$\square$

Def 3 : 

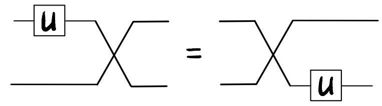
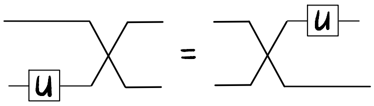
R_{16} : 

Def 1 : 

R5 : 

R10 : 

Prop 1 Let $\mathcal{C}_1 := \langle H, S, -\omega \rangle$. $\forall U \in \mathcal{C}_1$

 (1) and  (2)

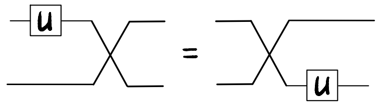
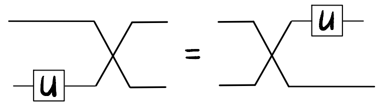
Proof: $\forall U \in \mathcal{C}_1$, $\exists\, t \in \mathbb{Z}_6$ and $m \in \mathbb{N}$ s.t. $U = (-\omega)^t H^{a_0} S^{b_0} H^{a_1} S^{b_1} \cdots H^{a_m} S^{b_m}$, composition in diagrammatic order, $a_i, b_i \in \mathbb{Z}_2$, $0 \le i \le m$. We proceed by induction on m.

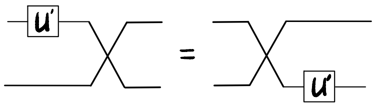
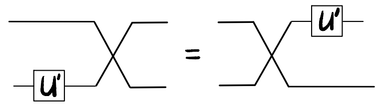
Base Case: ① $m=0$, $a_0 = b_0 = 0$. $U = (-\omega)^t$ is a scalar. (1) & (2) hold trivially.

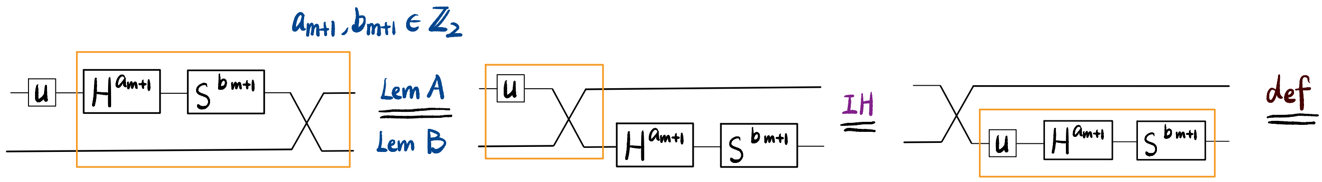
② $m=0$, $a_0 = 1$ and $b_0 = 0$. $U = (-\omega)^t H$. By Lem A, (1) & (2) hold.

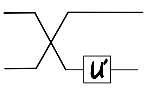
③ $m=0$, $a_0 = 0$ and $b_0 = 1$. $U = (-\omega)^t S$. By Lem B, (1) & (2) hold.

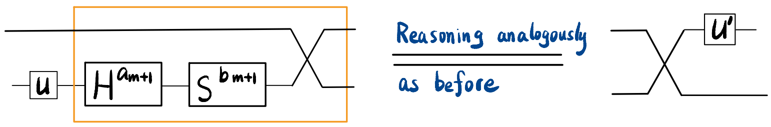
Induction Hypothesis: (1) & (2) hold for $m \geqslant 0$. That is, when $U = (-\omega)^t H^{a_0} S^{b_0} H^{a_1} S^{b_1} \cdots H^{a_m} S^{b_m}$

 (3) and  (4)

Induction Step: When $U' = U\, H^{a_{m+1}} S^{b_{m+1}}$, WTS  (5) &  (6).

(5). LHS := 

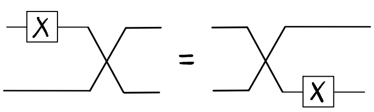
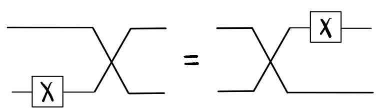
 =: (5). RHS

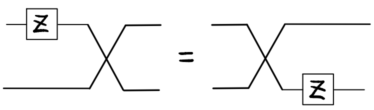
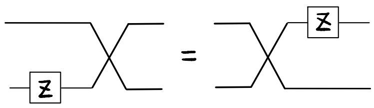
(6). LHS := 

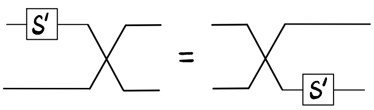
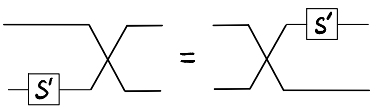
=: (6). RHS

This completes the proof. ☒

Cor 1

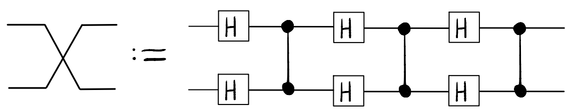
R_{20}: (1)  (2) 

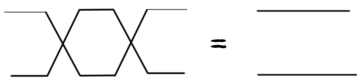
R_{21}: (1)  (2) 

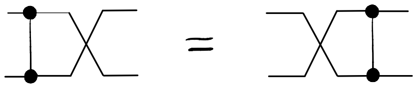
R_{22}: (1)  (2) 

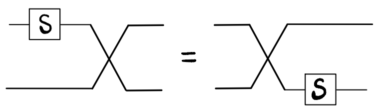
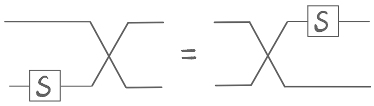
Proof: By Def 1, R_5, R_{10} & Prop 1. ☒

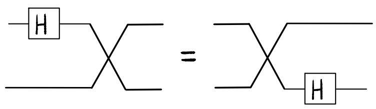
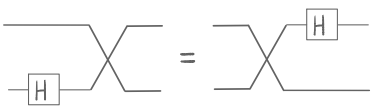
Two - Qutrit Derived Relations : T_3

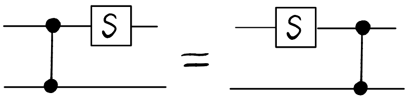
Def 3 :


R_{16} :


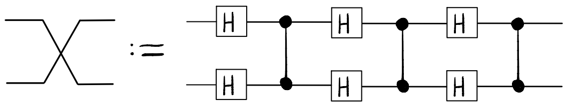
R_{17} :


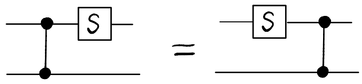
R_{18} : (1)

(2)


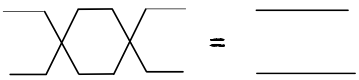
R_{19} : (1)

(2)


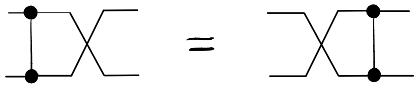
C_7 : (1)

(2)

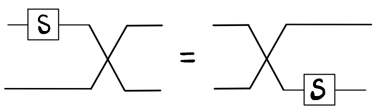

Two - Qutrit Derived Relations : T_2

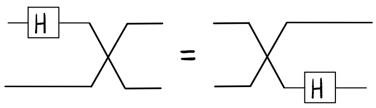
Def 3 :


C_7 :


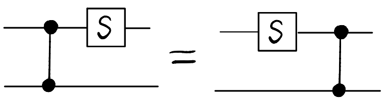
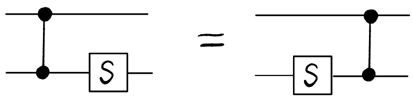
R_{16} :


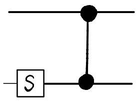
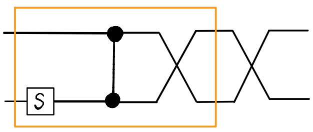
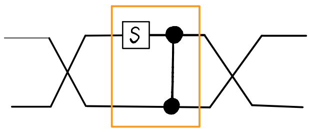
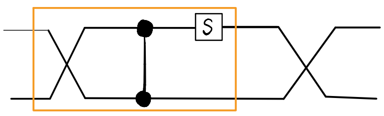
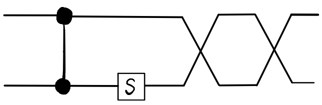
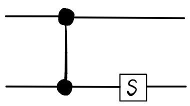
R_{17} :


R_{18} :


R_{19} :


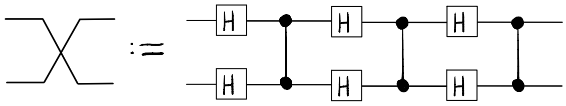
Lem C By Def 3, R_{16}, R_{17} & R_{18},

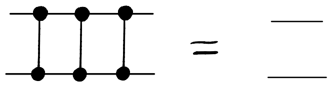
C_7 : (1)

implies (2)


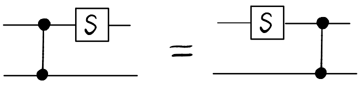
Proof: C_7.(2). RHS :=

$\overset{R_{16}}{=}$

$\overset{R_{17}}{\underset{R_{18}}{=}}$

$\overset{C_7.(1)}{=}$

$\overset{R_{17}}{\underset{R_{18}}{=}}$

$\overset{R_{16}}{=}$

=: C_7.(2). LHS

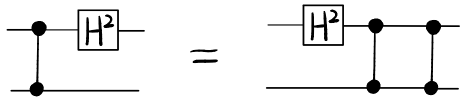
∎

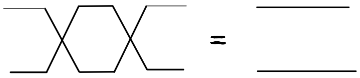
Two - Qutrit T_2

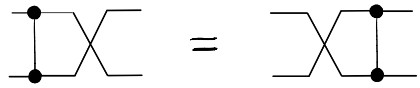
Def 3 : 

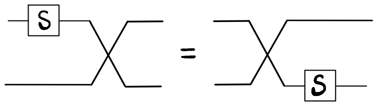
C_6 :  ✓

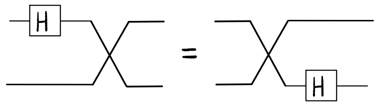
C_7 :  ✓

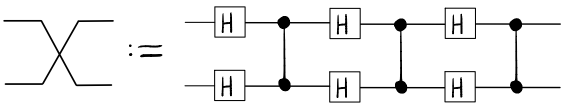
C_8 :  ✓

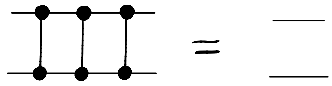
R_{16} :  ✓

R_{17} :  ✓

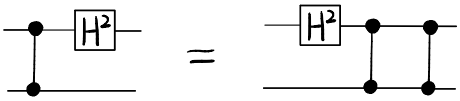
R_{18} : 

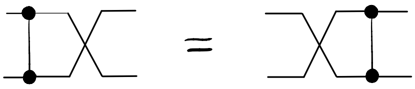
R_{19} :  ✓

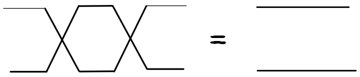
Def 3 : 

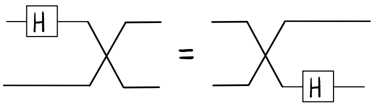
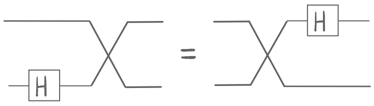
C_6 : 

$C2$: $H^4 = I$

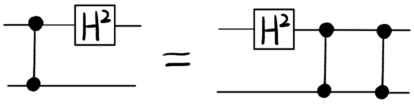
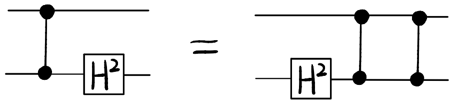
C_8 : 

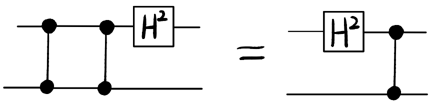
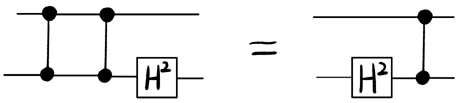
R_{17} : 

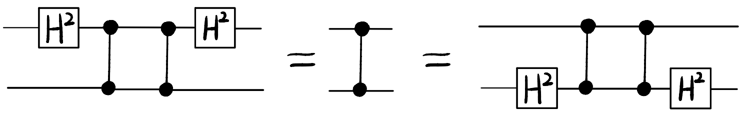
R_{16} : 

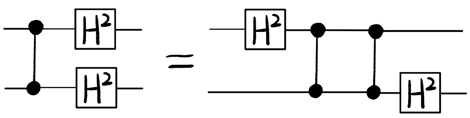
R_{19} : (1)  (2) 

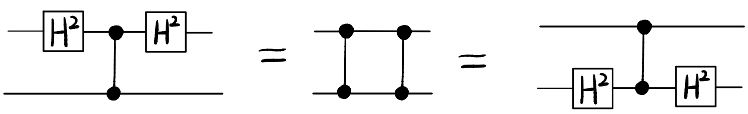
Lem D By Def 3, R_{16}, R_{17}, R_{19}, C_6, C_2 & C_8,

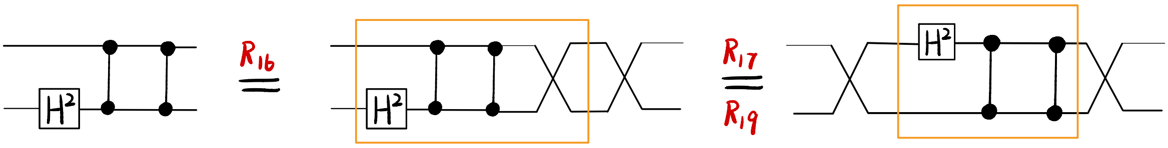
C_8 : (1)  implies (2) 

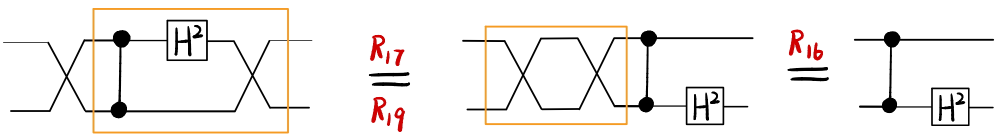
C_8^1 : (1)  (2) 

C_8^2 : 

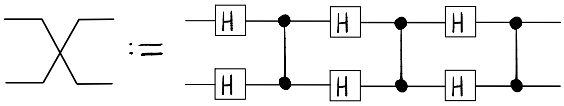
C_8^3 : 

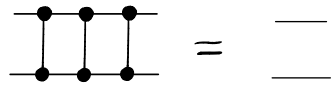
C_8^4 : 

Proof: C_8 .(2). RHS := 

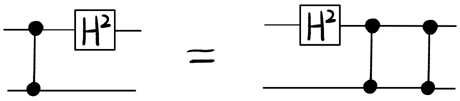
$\underline{\underline{C_8\ .(1)}}$ 

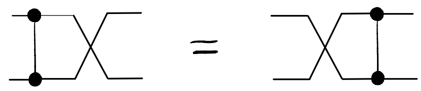
$=:$ C_8 .(2). LHS

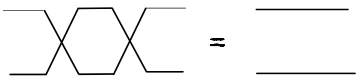
Def 3 : 

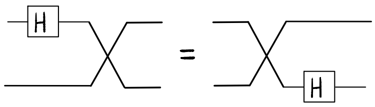
C_6 : 

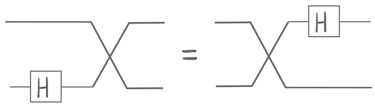
$C2$: $H^4 = I$

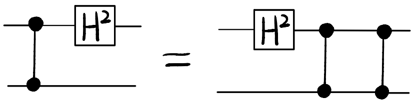
C_8 : 

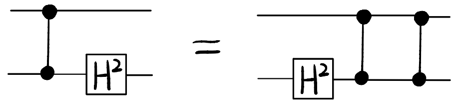
R_{17} : 

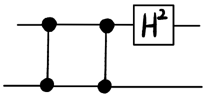
R_{16} : 

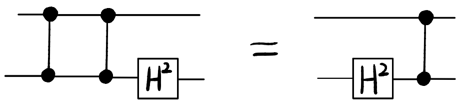
R_{19} : (1) 

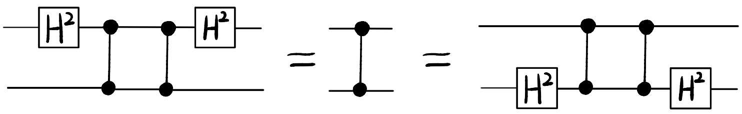
(2) 

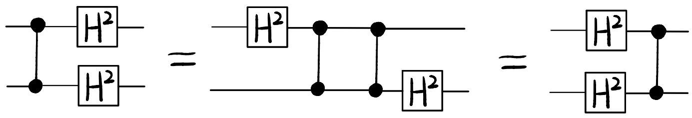
C_8 : (1) 

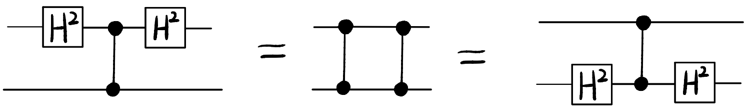
implies (2) 

C_8^1 : (1) 

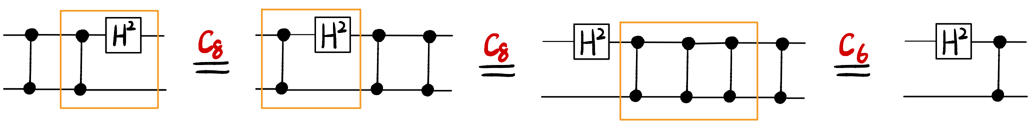
(2) 

C_8^2 : 

C_8^3 : 

C_8^4 : 

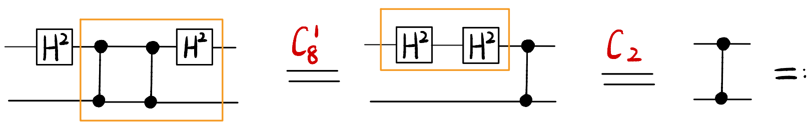
Proof cont :

C_8^1.(1). LHS := 

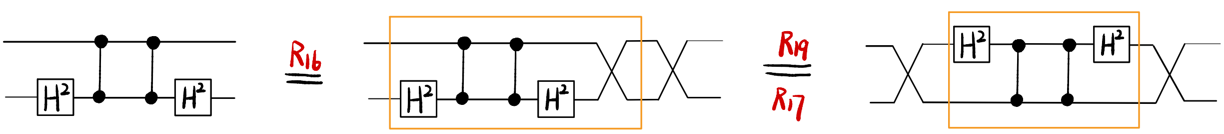
=: C_8^1.(1).RHS .

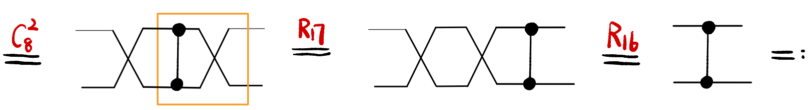
C_8^1.(2). LHS := 

=: C_8^1.(2).RHS .

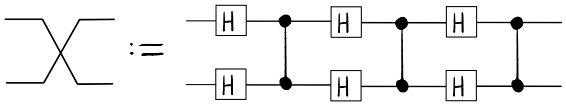
C_8^2. LHS := 

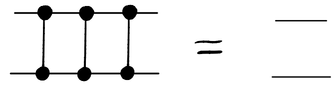
=: C_8^2. MID

C_8^2. RHS := 

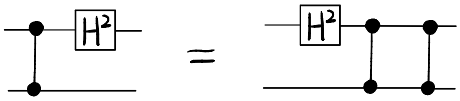


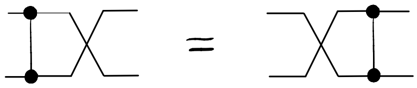
=: C_8^2. MID

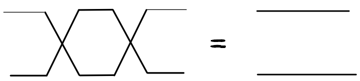
Def 3 : 

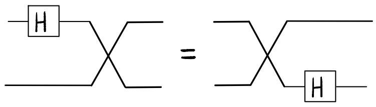
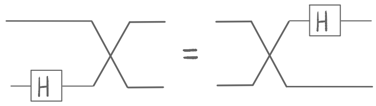
C_6: 

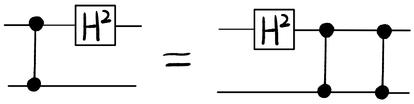
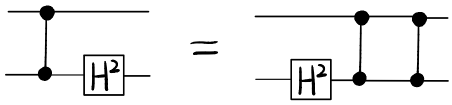
C_2: $H^4 = I$

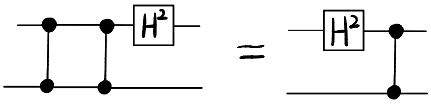
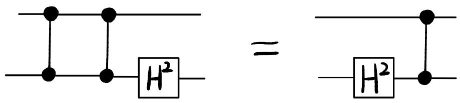
C_8: 

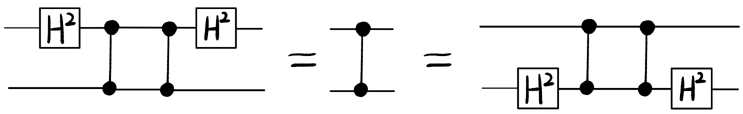
R_{17}: 

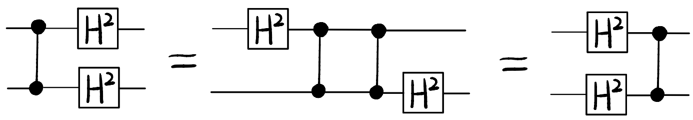
R_{16}: 

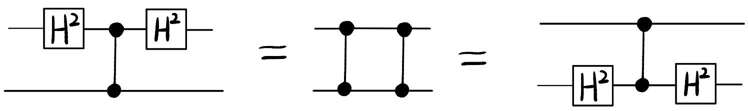
R_{19}: (1)  (2) 

C_8 : (1)  implies (2) 

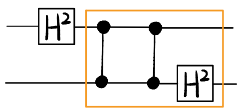
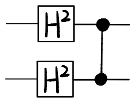
C_8^1: (1)  (2) 

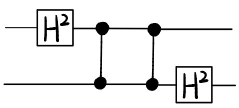
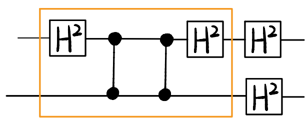
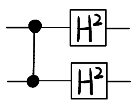
C_8^2: 

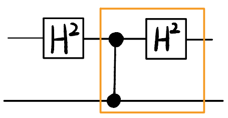
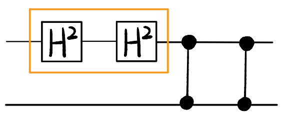
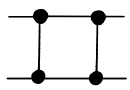
C_8^3: 

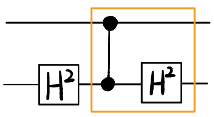
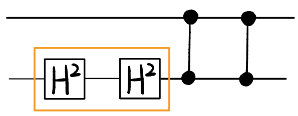
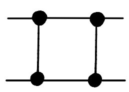
C_8^4: 

Proof cont :

C_8^3 . MID :=  $\overset{C_8^1}{=}$  $=:$ C_8^3 . RHS.

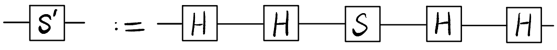
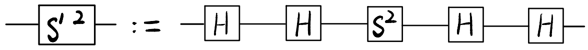
C_8^3 . MID :=  $\overset{C_2}{=}$  $\overset{C_8^2}{=}$  $=:$ C_8^3 . LHS.

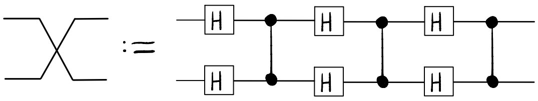
C_8^4 . LHS :=  $\overset{C_8}{=}$  $\overset{C_2}{=}$  $=:$ C_8^4 . MID

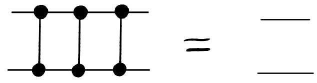
C_8^4 . RHS :=  $\overset{C_8}{=}$  $\overset{C_2}{=}$  $=:$ C_8^4 . MID

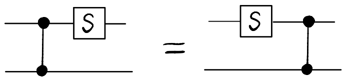
∎

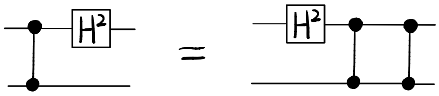
Two - Qutrit Derived Relations : T_2

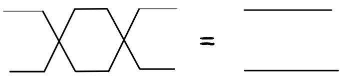
Def 1 :  

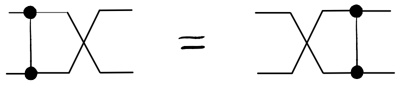
Def 3 : 

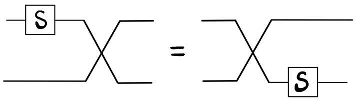
C_6 : 

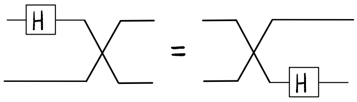
C_7 : 

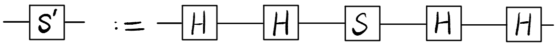
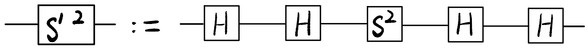
C_8 : 

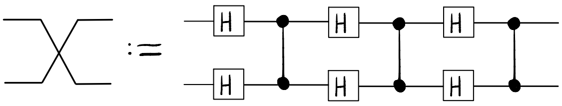
R_{16} : 

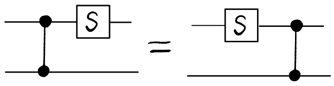
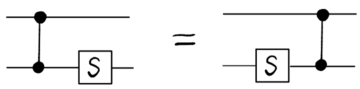
R_{17} : 

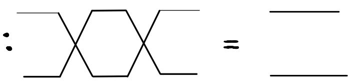
R_{18} : 

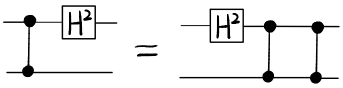
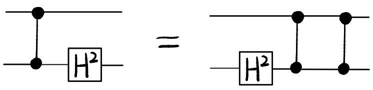
R_{19} : 

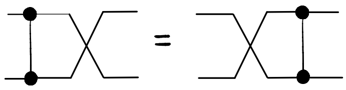
Def 1 :  

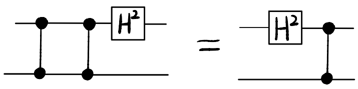
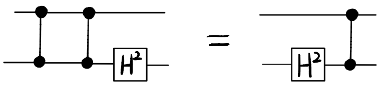
Def 3 : 

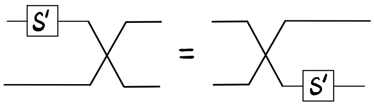
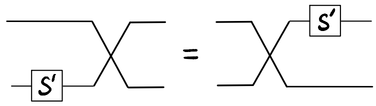
C_7 : (1)  (2) 

R_{16} : 

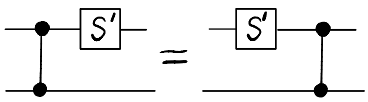
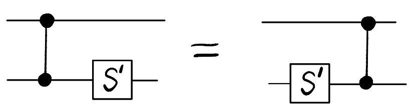
C_8 : (1)  (2) 

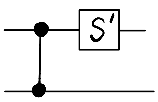
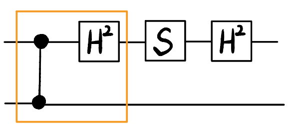
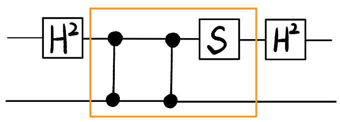
R_{17} : 

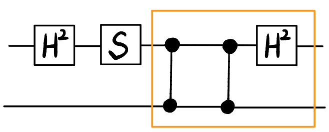
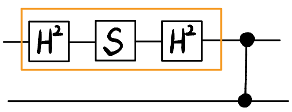
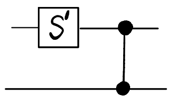
C_8' : (1)  (2) 

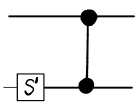
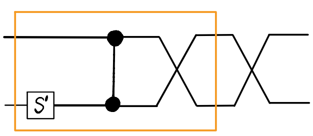
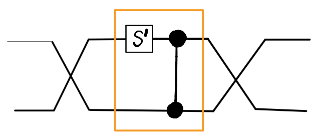
R_{22} : (1)  (2) 

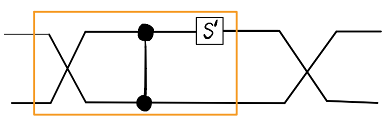
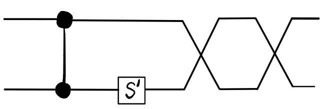
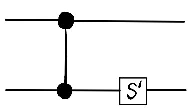
Lem E By Def 1, Def 3, C_7, C_8, R_{16}, R_{17} & R_{22},

R_{15} : (1)  & (2) 

Proof : R_{15}.(1). LHS :=  $\overset{\text{Def 1}}{=}$  $\overset{C_8}{=}$  $\overset{C_7}{=}$

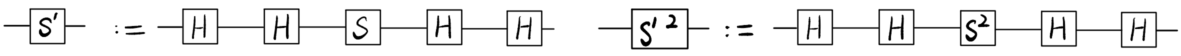
 $\overset{C_8'}{=}$  $\overset{\text{Def 1}}{=}$  =: R_{15}.(1). RHS

R_{15} .(2). RHS :=  $\overset{R_{16}}{=}$  $\overset{R_{17}}{\underset{R_{22}}{=}}$  $\overset{R_{15}.(1)}{=}$

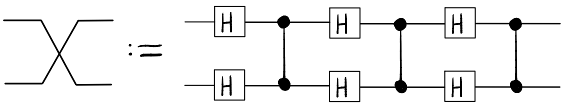
 $\overset{R_{17}}{\underset{R_{22}}{=}}$  $\overset{R_{16}}{=}$  =: R_{15} .(2). LHS ∎

Two - Qutrit Derived Relations : T_2

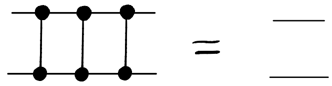
Def 1 :



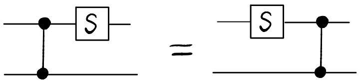
Def 3 :



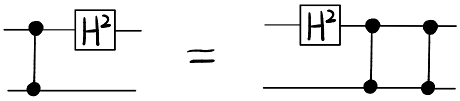
C_6 :



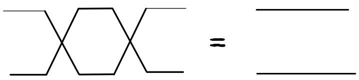
C_7 :



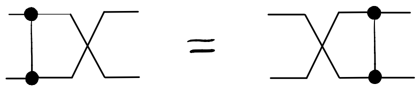
C_8 :



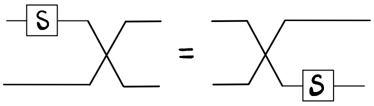
R_{16} :



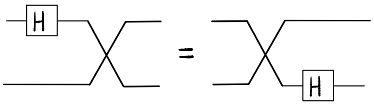
R_{17} :



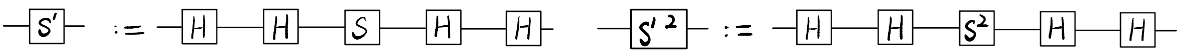
R_{18} :



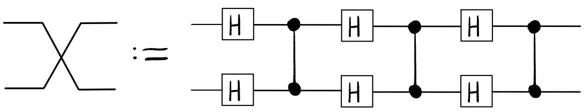
R_{19} :



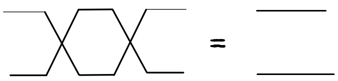
Def 1 :



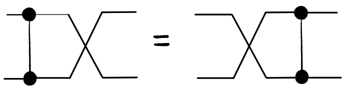
Def 3 :



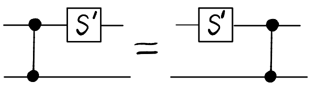
R_{16} :



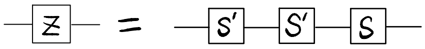
R_{17} :



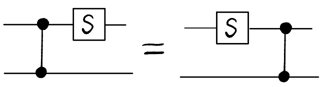
R_{15} : (1)



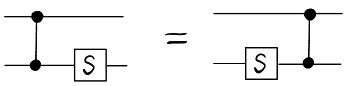
R10 :



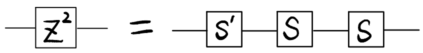
C_7 : (1)



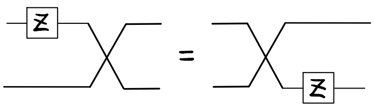
(2)



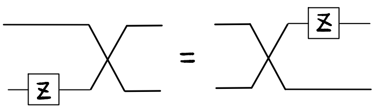
R11 :



R_{21} : (1)

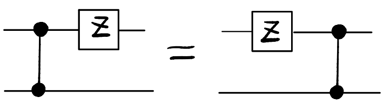


(2)

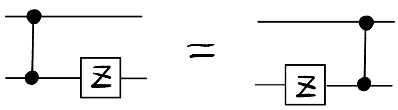


Lem F By Def 1, Def 3, C_7, R_{10}, R_{11}, R_{15}, R_{16}, R_{17} & R_{21},

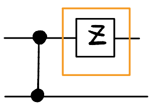
R_{13} : (1)



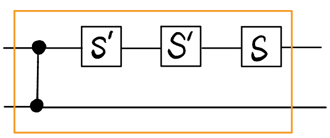
& (2)



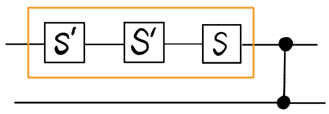
Proof: R_{13} . (1). LHS :=



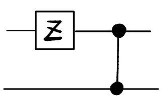
$\overset{R_{10}}{=}$



$\underset{R_{15}}{\overset{C_7}{=}}$

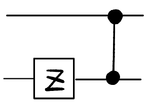


$\overset{R_{10}}{=}$

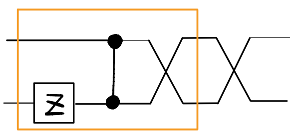


=: R_{13} . (1). RHS

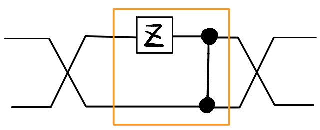
R_{13} .(2). RHS :=



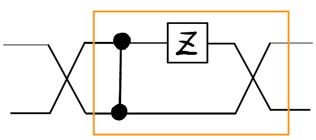
$\overset{R_{16}}{=}$



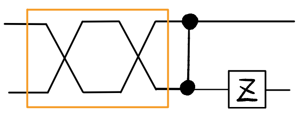
$\underset{R_{21}}{\overset{R_{17}}{=}}$



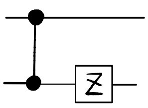
$\overset{R_{13}.(1)}{=}$



$\underset{R_{21}}{\overset{R_{17}}{=}}$



$\overset{R_{16}}{=}$



=: R_{13} .(2). LHS

∎