MT3121 Algebra Problem Sheet 1 : Saral 30 618 428

Question 1

a) YES, $a * b$ takes two integers and returns another.

b) YES, $a * b$ takes two non - negative integers and returns another non - negative integer.

c) NO, the logarithm of a positive real can be negative. Thus $a * b$ could be negative.

d) NO, $a * b$ could give an irrational number.

e) YES, $a * b$ returns another element of the set.

f) NO, $a * b$ could give a complex number.

g) YES, $a * b$ returns a real number for all x and y.

h) NO, $a * b$ is not defined when $x = 1$ and $y = -1$.

Question 2 f

First we need to show that $(a, b) * (c, d)$
actually gives something in S for all $(a, b) \in S$ and $(c, d) \in S$.

$(a, b) * (c, d) = (ac - bd, ad + bc)$

$ac - bd$ is defined for all $a, b, c, d \in R$
$ad + bc$ is defined for all $a, b, c, d \in R$

Also $ac - bd$ and $ad + bc$ will never be 0 at the same time

Thus $(ac - bd, ad + bc)$ is in S

Now is there an identity?

For this to be possible we need something in S, lets call it (e, f) such that :

$$(a, b) * (e, f) = (ae - bf, af + be) = (a, b)$$
$$(e, f) * (a, b) = (ea - fb, fa + eb) = (a, b)$$

The only way this is possible is if :

$(a, b) = (ae - bf, af + be)$

Therefore
$a = ae - bf$ (1)
$b = af + be$ (2)

These equations are satisfied if $e = 1$ and $f = 0$.

Thus the identity is $(1, 0)$

Now that we have found the identity, is there an inverse?

For an inverse we would need something in S, lets call this (g, h) for each (a, b) such that :

$(a, b) * (g, h) = (1, 0)$ and $(g, h) * (a, b) = (1, 0)$

This is possible if :

$ag - bh = 1 \quad (3)$

$ah + bg = 0 \quad (4)$

From making g the subject of (4) and (5) we find that :

$$g = \frac{1 + bh}{a} \quad (5)$$

$$g = \frac{-ah}{b} \quad (6)$$

For an inverse we would need (5) and (6) to be equal thus :

$$\frac{1 + bh}{a} = \frac{-ah}{a} \quad (7)$$

Making h the subject of (7) we find that :

$$h = \frac{-b}{a^2 + b^2} \quad (8)$$

Therefore by substituting (8) into (6) we find that :

$$g = \frac{a}{a^2 + b^2}$$

Thus our inverse is $\left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$

Now is the binary operation associative ?

After doing loads of arithmetic (I did it on paper) you get to a point where you find that for the binary operation to be associative

You would need that for (a, b), (c, d), $(e, f) \in S$:

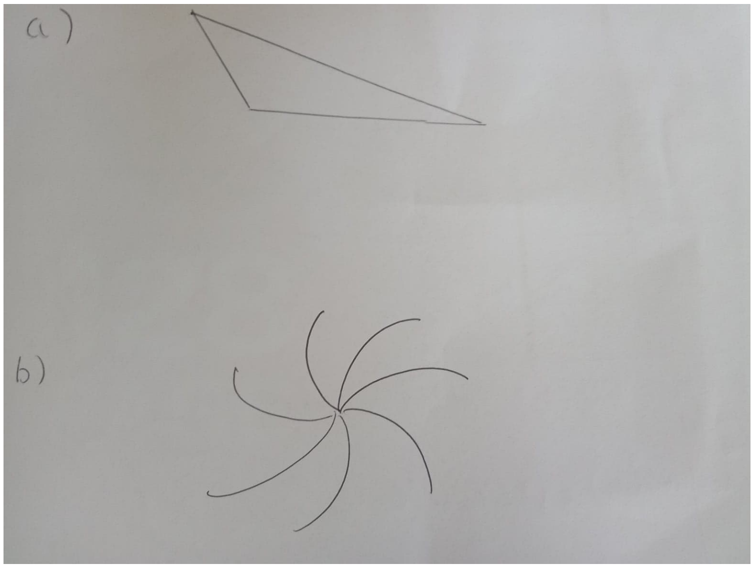
$(ace - bde - adf - bcf, acf - bdf + ade + bce) =$
$(ace - bdf - bcf - bde, acf + ade + bce - bdf) \quad (9)$

Since LHS is not the same as RHS of (9) the binary operation is not associative.

Question 5

a) A scalene triangle

b) A 7 sided flower like figure with equal line lengths.



c) A regular 500 sided polygon

d) No. If there are two reflectional symmetries this means that there is also rotational symmetry.

Question 8

a) $e = a * a^{-1}$ (Definition of inverse)
$a * a^{-1} = a^{-1} * (a^{-1})^{-1}$ (Taking the inverse of each term)

Thus we have that :

$a * a^{-1} = a^{-1} * (a^{-1})^{-1}$

Beform the binary operation with a on both sides to give :

$a * a^{-1} * a = a^{-1} * (a^{-1})^{-1} * a$

Thus :

$a * e = e * (a^{-1})^{-1}$ (since $a * a^{-1} = e$)

Therefore :

$a = (a^{-1})^{-1}$

b) $a * b = e$
$a * b * b^{-1} = e * b^{-1}$ (doing binary operation with b^{-1} on both sides)

$a * e = e * b^{-1}$ (since $b * b^{-1} = e$)

$a = e * b^{-1}$ (since $a * e = e$)

$b * a = b * e * b^{-1}$ (doing the binary operation with b on both sides)

$b * a = b * b^{-1} \left(e * b^{-1} = b^{-1}\right)$

$b * a = e \quad \left(\text{since } b * b^{-1} = e\right)$

Quesiton 9

a)

□	f_0	f_1	f_2	f_3	s_0	s_1	s_2	s_3
f_0	f_0	f_1	f_2	f_3	s_0	s_1	s_2	s_3
f_1	f_1	f_0	f_3	f_2	s_1	s_0	s_3	s_2
f_2	f_2	f_3	f_0	f_1	s_2	s_3	s_0	s_1
f_3	f_3	f_2	f_1	f_0	s_3	s_2	s_1	s_0
s_0	s_0	s_1	s_3	s_2	f_0	f_1	f_3	f_2
s_1	s_1	s_0	s_2	s_3	f_1	f_0	f_2	f_3
s_2	s_2	s_3	s_1	s_0	f_2	f_3	f_1	f_0
s_3	s_3	s_2	s_0	s_1	f_3	f_2	f_0	f_1

For G to be a group we need that first need an identity.

The identity from the cayley table is f_0.

f_0 is present in each row which means we can ' reverse ' every move in some way. Thus there is an inverse.

Now we have to show that the binary operation is associative. This can be observed from the cayley table since any three operation done in succession give the same result regardless of order.

b) To be isomorphic to the symmetry group of a square, the orders of the symmetries of a square and the transformations for the coins would need to give the same list of values.

The order for each transformation in G is : [1, 2, 2, 2, 2, 2, 4, 4]

And for a square is [1, 2 , 2 , 2, 2, 2 , 4 , 4]

These lists are the same, therefore G is isomorphic to the symmetry group of a square.