$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \dfrac{\partial^2 u}{\partial x^2}(t,x) = 0 & -1<x<1 \ \& \ 0<t<T \\ u(0,x) = u_0(x) & -1<x<1 \\ u(t,-1) = u(t,1) = 0. & 0\leq t<T \end{cases}$$

1) Montrer par une estimation d'énergie que la solution est unique.

2) Montrer que si $u_0 \geq 0 \Rightarrow u \geq 0$

Indications :

- On pose $v(t,x) = e^{-t}u(t,x)$ quelle équation vérifie v ?
- Soit $\delta>0$, soit $\boxed{t_0 \text{ le premier}}$ temps / $\exists\, x_0 \,/\, v(t_0,x_0) = -\delta$

Montrer que l'on ne peut pas avoir $\left(\dfrac{\partial v}{\partial t} - \dfrac{\partial^2 v}{\partial x^2} + v\right)(t_0,x_0) = 0$

1) Spz v_1 & v_2 solutions de (E)

$w = v_1 - v_2$

$\dfrac{\partial w}{\partial t} - \dfrac{\partial^2 w}{\partial x^2} = 0$ & $w(0,x) = 0$

$$\frac{\partial w}{\partial t}(t,x)\, w(t,x) - \frac{\partial^2 w}{\partial x^2}(t,x)\, w(t,x) = 0$$

$$\int_{-1}^{1} \frac{\partial w}{\partial t}(t,x)\, w(t,x) - \int_{-1}^{1} \frac{\partial^2 w}{\partial x^2}(t,x)\, w(t,x) = 0$$

$$= \frac{1}{2}\frac{d}{dt}\int_{-1}^{1} w^2(t,x)\,dx + \int_{-1}^{1}\left(\frac{\partial w}{\partial x}\right)^2(t,x)\,dx = 0$$

(pas de termes aux bords car $w=0$ sur le bord)

donc $\dfrac{d}{dt}\displaystyle\int_{-1}^{1} w^2(t,x)\,dx \leq 0$

$$\int_{-1}^{1} w^2(t,x)\,dx \leq \int_{-1}^{1} \underbrace{w^2(0,x)}_{=0}\,dx = 0 \Rightarrow w = 0$$

d'où l'unicité

2) $\dfrac{\partial v}{\partial t} = -e^{-t} u(t,x) + e^{-t} \dfrac{\partial u(t,x)}{\partial t}$

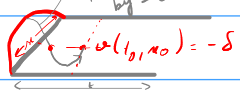
$\dfrac{\partial v}{\partial x} = e^{-t} \dfrac{\partial u}{\partial x}(t,x) \Rightarrow \dfrac{\partial^2 v}{\partial x^2} = e^{-t} \dfrac{\partial^2 u(t,x)}{\partial x^2}$.

$$\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial^2 x} = -e^{-t}u(t,x) + e^{-t}\frac{\partial u}{\partial t}(t,x) - e^{-t}\frac{\partial^2 u}{\partial x^2}(t,x)$$
$$= -v + e^{-t}\underbrace{\left[\frac{\partial u}{\partial t}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x)\right]}_{=0}$$

$\dfrac{\partial v}{\partial t}(t,x) + \dfrac{\partial^2 v}{\partial x^2}(t,x) + v = 0$ Montrer que u reste positif est équivalent à montrer que v reste positif

Signe de $\dfrac{\partial v}{\partial t}(t_0, x_0)$?? $\dfrac{\partial v}{\partial t}(t_0, x_0) < 0$

if $\frac{\partial v}{\partial t} \geq 0$ on t_0 means we already passed by $-\delta$ but we defined t_0 as the first t where we pass by $-\delta$



Signe de $\dfrac{\partial^2 v}{\partial x^2}(t_0, x_0)$?? $\dfrac{\partial^2 v}{\partial x^2}(t_0, x_0) \geq 0$

$$\underbrace{0 > \left(\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2}\right)(t_0, x_0)}_{} = \underbrace{-v(t_0, x_0) > 0}_{}$$

contradiction

3) $u(t,x) = \displaystyle\sum_{n=0}^{+\infty} \boxed{a_n} e^{-n^2\pi^2 t} \sin(n\pi x)$ → coefficients de Fourier de la donnée initiale

Hypothèse : $\displaystyle\sum_{n=0}^{\infty} n^2 |a_n| < +\infty$

$|u(t,x)| \leq \displaystyle\sum_{n=0}^{+\infty} |a_n| < +\infty$

La fonction u est régulière et solution de l'équation de la chaleur

$\left|\dfrac{\partial u}{\partial t}(t,x)\right| \leq \displaystyle\sum_{n=0}^{+\infty} n^2\pi^2 |a_n| < +\infty$

$\left|\dfrac{\partial^2 u}{\partial x^2}(t,x)\right| \leq \displaystyle\sum_{n=0}^{+\infty} n^2\pi^2 |a_n| < +\infty$

$t_0 > 0$, on va regarder la dérivée k-ième de u

$$\left| \sum_{n=0}^{+\infty} a_n (-1)^k (n^2\pi^2)^k e^{-n^2\pi^2 t} \sin(n\pi x) \right| \leftarrow \text{dérivée } \%\, x.$$

$$\leq \sum_{n=0}^{+\infty} |a_n| (n^2)^k (\pi^2)^k e^{-n^2\pi^2 t}$$

$$\leq \sum_{n=0}^{+\infty} |a_n| \frac{n^{2k}\pi^{2k}}{t_0^{k/2}} t_0^{k/2} e^{-n^2\pi^2 t_0}$$

$$\leq \sum_{n=0}^{+\infty} |a_n| \frac{n^{2k}\pi^{2k} t_0^{k/2}}{t_0^{k/2}} e^{-n^2\pi^2 t_0/2} e^{-n^2\pi^2 t_0/2}$$

$$\leq \frac{1}{t_0^{k/2}} \sum_{n=0}^{+\infty} |a_n| (n\pi t_0)^k e^{-n^2\pi^2 t_0/2} e^{-n^2\pi^2 t_0/2}$$

formellement $\left| \frac{\partial^k u}{\partial x^k}(t_0, x) \right| \leq \frac{1}{t_0^{k/2}} \sum_{n=0}^{+\infty} |a_n| e^{-n^2\pi^2 t_0} \boxed{(n\pi t_0^{1/2})^k e^{-n^2\pi^2 t_0}}$

$\leq C$

$x \longrightarrow x^k e^{-x^2}$