Gradients des coordonnées barycentriques en 2D.

$$\lambda_1(x) = \frac{(x - x_2^\tau) \times (x_3^\tau - x_2^\tau) \cdot \hat{e}_3}{(x_1^\tau - x_2^\tau) \times (x_3^\tau - x_2^\tau) \cdot \hat{e}_3} = \frac{\det(x - x_2^\tau \,;\, x_3^\tau - x_2^\tau)}{\det(x_1^\tau - x_2^\tau \,;\, x_3^\tau - x_2^\tau)}$$

$$\lambda_2(x) = \frac{(x - x_3^\tau) \times (x_1^\tau - x_3^\tau) \cdot \hat{e}_3}{(x_2^\tau - x_3^\tau) \times (x_1^\tau - x_3^\tau) \cdot \hat{e}_3} = \frac{\det(x - x_3^\tau \,;\, x_1^\tau - x_3^\tau)}{\det(x_2^\tau - x_3^\tau \,;\, x_1^\tau - x_3^\tau)}$$

$$\lambda_3(x) = \frac{(x - x_3^\tau) \times (x_2^\tau - x_1^\tau) \cdot \hat{e}_3}{(x_3^\tau - x_1^\tau) \times (x_2^\tau - x_1^\tau) \cdot \hat{e}_3} = \frac{\det(x - x_3^\tau \,;\, x_2^\tau - x_1^\tau)}{\det(x_3^\tau - x_1^\tau \,;\, x_2^\tau - x_1^\tau)}$$

Sur un triangle K, les coordonnées barycentriques sont affines :

$$\lambda_i(x,y) = a_i + b_i x + c_i y$$

$$\Rightarrow \quad \nabla \lambda_i = \begin{pmatrix} b_i \\ c_i \end{pmatrix}.$$

pour l'elt e ayant les sommets $X_1 = (x_1, y_1)$

$X_2 = (x_2, y_2)$

$X_3 = (x_3, y_3)$

on a aussi :

$$\lambda_i(x_j) = \delta_{ij}$$

pour trouver les coefficients on résoud le système

$$\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} = \begin{pmatrix} \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \end{pmatrix}.$$

N_p : nb de points geom du maillage
N_v : nb de DOF global
N_{ve} : nb de DOF par elt
N_e : nb d'elts

Soit A la matrice globale.

$$A_{ij} = a(\varphi_j, \varphi_i) = \sum_e a^e(\varphi_j, \varphi_i)$$
$$= \sum_e A^e_{ij}$$

On peut aussi le faire avec $\hat{\varphi}_i$ qui sont les fcts de base sur l'élt de référence

Soit une matrice locale $\hat{A}^e$:

$$\hat{A}^e_{k,k'} = a^e(\hat{\varphi}_{k'}, \hat{\varphi}_k).$$
$$= \begin{pmatrix} a^e(\hat{\varphi}_1, \hat{\varphi}_1) & a^e(\hat{\varphi}_2, \hat{\varphi}_1) \\ a^e(\hat{\varphi}_1, \hat{\varphi}_2) & a^e(\hat{\varphi}_2, \hat{\varphi}_2) \end{pmatrix}$$

On a : $A^e_{D(k,e), D(k',e)} = \hat{A}^e_{k,k'}$

global DOF index associated with local DOF k on e.

$D(k,e)$ tells you that:
this local basis fct "k" lives at that global index "for the elt"

$\theta(k,e)$ points you only to the vertices from local to global

Ex 0) $-u''(x) + u(x) = x$ over Ω

→ derive the weak formulation :
- multiply by v & integrate

$$\int_\Omega -u''(x)\,v(x)\,dx + \int_\Omega u(x)\,v(x)\,dx = \int_\Omega x\,v(x)\,dx$$

- integrate by parts the first term

$u''v = uv - \int u'v'$

$$\int_\Omega -u''(x)\,v(x)\,dx = -[u'v]_{\partial\Omega} + \int_\Omega u'(x)\,v'(x)\,dx$$

- replace

$$\int_\Omega u'(x)\,v'(x)\,dx + \int_\Omega u(x)\,v(x)\,dx = \int_\Omega x\,v(x)\,dx + [u'v]_{\partial\Omega}$$

case 1, homogeneous DC $u=0$ over $\partial\Omega$

$\Rightarrow v = 0$ over $\partial\Omega$ $(H_0^1(\Omega))$

$\Rightarrow [u'v]_{\partial\Omega} = u'(L)v(L) - u'(0)\,v(0) = 0$

case 2 : Neuman B.C.

$-u'(0) = g_0 \quad u'(L) = g_L$

$\Rightarrow [u'v]_{\partial\Omega} = g_L\,v(L) + g_0\,v(0)$.

→ Discretization : introduce V_h, u_h, v_h

Choose F.E space $V_h \subset V$ & $\dim(V_h) < +\infty \Rightarrow$

Trouver $u_h \in V_h \,/\, a(u_h, v_h) = l(v_h) \quad \forall v_h \in V_h$

so we have N_e elts $/ \ \Omega_h = \bigcup^{N_e} e$

$$a(u_h, v_h) = \sum_{e=1}^{N_e} \int_e u'(x)\,v'(x) + u(x)\,v(x)\,dx = \sum_{e=1}^{N_e} \int_e A(u_h, v_h)\,dx$$

$$l(v_h) = \sum_{e=1}^{N_e} \left[\int_e L(x, v_h)\,dx + \int_{\partial e \cap \partial\Omega} L_N(x, v_h)\,dS \right.$$

Dans l'exo 2.1.1 ; on veut la matrice d'un elt qui ne touche pas les bords ⇒ on peut se débarasser du term aux bords de Ω_h

$$\sum_{e=1}^{\vec{N_e}} \int_e A(u_h, v_h)\,dx = \sum_{e=1}^{N_e} \int_e \mathcal{L}(x, v(x))\,dx$$

→ u_h ; $v_h \in V_h$ ayant comme base canonique $\{\varphi_i\}_{i=1,\dots,N_V}$ / $\varphi_i(x_j) = \delta_{ij}$

$$\Rightarrow u_h = \sum_{i=1}^{N_V} U_i\, \varphi_i(x)$$

on somme sur tous les elts
$$= \sum_{e=1}^{N_e} \sum_{p=1}^{N_{ve}} u_{G(p,e)}\, \hat{\varphi}_p(\xi^e(x))$$

on passe le sommets de l'elt de ref à l'elt phy

on somme sur les DOF — mapping des DOF du local au global — base sur l'elt de référence

→ On se restraint sur un elt ⇒ problème local

Trouver $u_h^e(x) = \sum_{p=1}^{N_{ve}} u_{G(p,e)}\, \hat{\varphi}_p(\xi^e(x))$

→ on prend v_h ; une fct de base (la formulation faible est vrai $\forall v_h \in V_h$) ; $\hat{\varphi}_q$

→ problème local :

$\varphi_p^e(x) = \hat{\varphi}_p(\xi^e(x))$

Find $u_h^e(x) = \sum_{p=1}^{N_{ve}} u_{G(p,e)}\, \hat{\varphi}_p(\xi^e(x))$ /

$$a^e(u_h^e, \hat{\varphi}_q) = \ell^e(\hat{\varphi}_q) \qquad q = 1, \dots, N_{ve}$$

$$\int_e \left[\left[\sum_{p=1}^{N_{ve}} u_{G(p,e)} \frac{d}{dx}\hat{\varphi}_p(\xi^e(x))\right] \frac{d}{dx}\hat{\varphi}_q(\xi^e(x)) + \sum_{p=1}^{N_{ve}} u_{G(p,e)}\hat{\varphi}_p(\xi^e(x))\,\hat{\varphi}_q(\xi^e(x))\right] dx = \int_e x\, \hat{\varphi}_q(\xi^e(x))\,dx$$

$$\sum_{p=1}^{N_{ve}} \underbrace{\int_e \left(\frac{d}{dx}\hat{\varphi}_p(\xi^e(x))\frac{d}{dx}\hat{\varphi}_q(\xi^e(x)) + \hat{\varphi}_p(\xi^e(x))\hat{\varphi}_q(\xi^e(x))\right) dx}_{A_{pq}^e} = \underbrace{\int_e x\, \hat{\varphi}_q(\xi^e(x))\,dx}_{F_p^e}$$

$$\sum_{p=1}^{N_{ve}} A^e_{pq}\, u_{G(p,e)} = F^e_q \quad q = 1, \dots, N_{ve}$$

→ elt de référence

$x = x^e(\xi) \Rightarrow dx = J_e\, d\xi$

$[2,3] \qquad [0,1]$

• changement de variable : $e \longrightarrow \hat{e}$ (elt phy → elt ref)

$$A^e_{pq} = \int_{\hat{e}} \left[\frac{1}{J_e} \hat{\varphi}'_p(\xi)\, \hat{\varphi}'_q(\xi) + J_e\, \hat{\varphi}_p(\xi)\, \hat{\varphi}_q(\xi) \right] d\xi$$

$\xi = \xi^e(x) = \dfrac{x - x^e_1}{x^e_2 - x^e_1}$

en générale, en 1D ; $x = x^e(\xi) = x^e_1 + (x^e_2 - x^e_1)\xi$

elt physique de l'exo $[2,3]$ (x^e_1, x^e_2) elt ref $[0,1]$ (ξ^e_1, ξ^e_2) $\quad \hat{\varphi}_1(\xi) = 1 - \xi$

$\hat{\varphi}_2(\xi) = \xi$

$\xi^e_1 = x^e_1 - 2 \qquad \xi^e_2 = x^e_2 - 2 \quad \Rightarrow \xi = x - 2$

$\Rightarrow x = \xi + 2. \qquad \varphi^e_i(x) = \hat{\varphi}_i(\xi^e(x))$

$dx = 1 . d\xi$

$$\Rightarrow A^e_{pq} = \int_0^1 1.\hat{\varphi}'_p(\xi)\, \hat{\varphi}'_q(\xi) + 1.\hat{\varphi}_q(\xi)\, \hat{\varphi}_p(\xi)\, d\xi$$

$$A^e_{11} = \int_0^1 1.\, \hat{\varphi}'_1(\xi)\, \hat{\varphi}'_1(\xi) + 1.\, \hat{\varphi}_1(\xi)\, \hat{\varphi}_1(\xi)\, d\xi.$$

$$= \int_0^1 \hat{\varphi}'_1(\xi)\, \hat{\varphi}'_1(\xi)\, d\xi + \int_0^1 \hat{\varphi}_1(\xi)\, \hat{\varphi}_1(\xi)\, d\xi$$

$$= \int_0^1 (-1)(-1)\, d\xi + \int_0^1 (1-\xi)^2\, d\xi.$$

$$= [\xi]_0^1 + \left[\frac{-(1-\xi)^3}{3} \right]_0^1 = 1 + \frac{1}{3}\left[-\frac{0}{3} + 1 \right] = 1 + \frac{1}{3} = \frac{4}{3}$$

$$A^e_{12} = \int_0^1 (-1)(1)\, d\xi + \int_0^1 (1-\xi)\,\xi\, d\xi$$

$$= -1 + \int_0^1 \xi - \xi^2\, d\xi$$

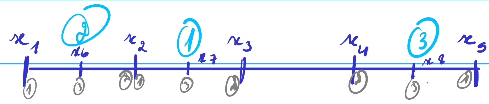
$$= -1 + \left[\frac{\xi^2}{2} - \frac{\xi^3}{3} \right]_0^1 = -1 + \left[\frac{1}{2} - \frac{1}{3} \right] = -\frac{5}{6}$$

A est sym, le calcul du rhs.

2024 ($\hat{m}$ réponses pour 2025)

1D

Q1–Q2:



Q3: $(1)\int_{\Omega_h} \varphi_1(x)\,\varphi_m(x)\,dx = 0$ & $\int_{\Omega_h} \varphi_4(x)\,\varphi_m(x)\,dx = 0\ (2)$

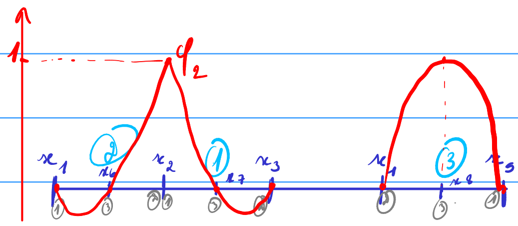
Les fonctions de base sont des fonctions à support compacte $\quad \text{supp}(\varphi_i) = \bigcup\{K \mid K \text{ elt} / x_i \in K\}$.

$$\Rightarrow \varphi_i(x)\,\varphi_j(x) \neq 0 \iff \text{supp}(\varphi_i) \cap \text{supp}(\varphi_j) \neq \emptyset$$

$$\Rightarrow \begin{cases} \int_{\Omega_h} \varphi_1(x)\,\varphi_m(x)\,dx = 0 \iff m = \{3,4,5,7,8\} \\ \& \\ \int_{\Omega_h} \varphi_4(x)\,\varphi_m(x)\,dx = 0 \iff m = \{1,2,3,6,7\} \end{cases}$$

$\iff$ final answer $(1) = 0$ & $(2) = 0$ for $m = \{3,7\}$

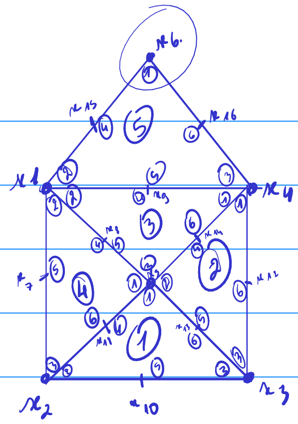
Q4



pour les elts P2; les fonctions de bases sont des polynômes de degrée 2; et $\varphi_i(x_j) = \delta_{ij}$

$$\Rightarrow \varphi_2(x_2) = 1 \ \& \ \varphi_2(x_j) = 0 \quad \forall j = \{1,3,4,5,6,7,8\}.$$
$$\& \ \varphi_8(x_8) = 1 \ \& \ \varphi_8(x_j) = 0 \quad \forall j = \{1,2,3,4,5,6,7\}.$$

(2D) $Q_1 - Q_2$



Q3) Les fonctions de base sont des fonctions à support compacte

$$\mathrm{supp}(\varphi_i) = \bigcup \{K \mid K \text{ elt} \,/\, x_i \in K\}.$$

$$\Rightarrow \quad \varphi_i(x)\, \varphi_j(x) \neq 0 \iff \mathrm{supp}(\varphi_i) \cap \mathrm{supp}(\varphi_j) \neq \emptyset$$

$$\Rightarrow \begin{cases} \displaystyle\int_{\Omega_h} \varphi_6(x)\, \varphi_m(x)\, dx = 0 \iff m = \{2, \cancel{3}, \cancel{5}, 7, 8, 10, 11, \cancel{12}, \cancel{13}, \cancel{14} \\ \& \\ \displaystyle\int_{\Omega_h} \varphi_{12}(x)\, \varphi_m(x)\, dx = 0 \iff m = \{\cancel{1}, 2, \cancel{6}, 7, 8, \cancel{9}, 10, 11, \cancel{15}, \cancel{16}\} \end{cases}$$

$$m = \{2, 7, 8, 10, 11\}$$

Q4) $e = 6 \rightarrow$ sommets : x_6 ; x_1 ; x_4

$x_6 = (1, 3) \qquad x_1 = (0, 2) \qquad x_4 = (2, 2).$

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & 2 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} a_1 = -2 \\ b_1 = 0 \\ c_1 = 1 \end{matrix}$$

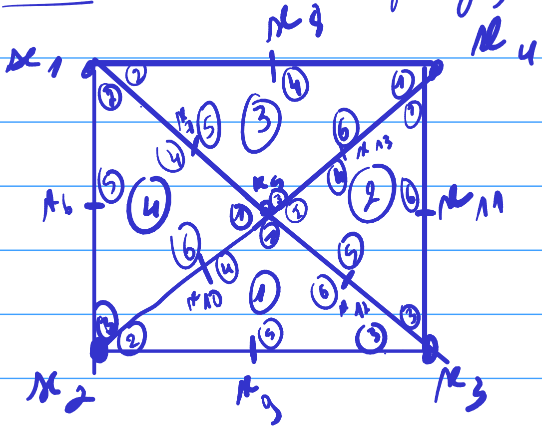
$$a_2 = 2 \quad b_2 = -0.5 \quad c_2 = -0.5$$

$$a_3 = 1 \quad b_3 = 0.5 \quad c_3 = -0.5$$

$$\nabla \lambda_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \nabla \lambda_2 = \begin{pmatrix} -0.5 \\ -0.5 \end{pmatrix} \quad \nabla \lambda_3 = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix}$$

EXO : (Dans le poly)

Q1 - Q2

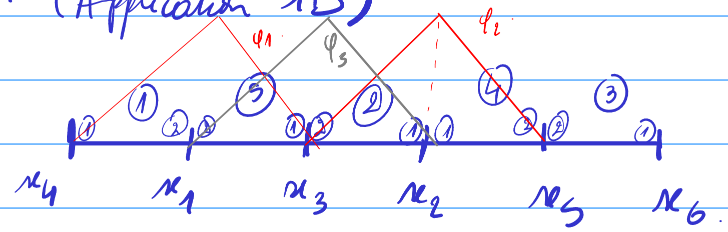


Q3 $\hat{M}$ explication

$$\int_{\Omega_h} \varphi_{11} \varphi_m \, dx = 0 \iff m = \{1, 2, 6, 7, 8, 9, 10\}$$

ExO. (poly). (Application 1D)

Q1)



$N_v = 6$ $\quad N_{ve}$ = nb de degree de liberté par elt = 2

Q2) $A_{ij} = a(\hat{\varphi}_j, \hat{\varphi}_i) = \sum_e a^e(\hat{\varphi}_j, \hat{\varphi}_i)$

Q3) A^e est la restriction de A sur l'elt e

$\Rightarrow A^e$ a m̂ taille que A ; avec juste les contributions de l'elt e

$\Rightarrow A = \sum_e A^e$.

- $A^e_{D(k,e), D(k',e)} = \hat{A}^e_{k,k'}$?? $\quad k = 1, 2. \quad k' = 1, 2. \quad D(1,1) = 4 \quad D(2,1) = 1$

$\hat{A}^1_{1,1} = A^1_{4,4} \qquad \hat{A}^1_{2,1} = A^1_{1,4}$

$\hat{A}^1_{1,2} = A^1_{4,1} \qquad \hat{A}^1_{2,2} = A^1_{1,1}$