Chapter 11

Parametric Equations and Polar Coordinates

11.1 Curves Defined by Parametric Equations

Suppose that x and y are both given as functions of a third variable t (called a **parameter** by the equations)

$$x = f(t) \quad y = g(t)$$

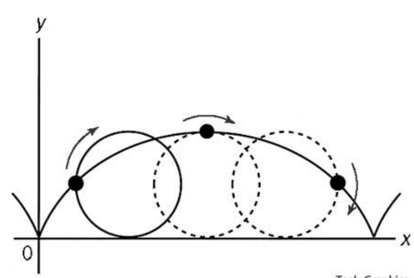
(called **parametric equations**). Each value of t determines a point (x,y). As t changes, $(x, y) = (f(t), g(t))$ changes and traces out a curve C, which is called a **parametric curve**. The direction of the arrows on curve C show the change in the position of the equation as t increases.

We can also restrict t to a finite interval. In general, the curve with parametric equations

$$x = f(t) \quad y = g(t) \quad a \leq t \leq b$$

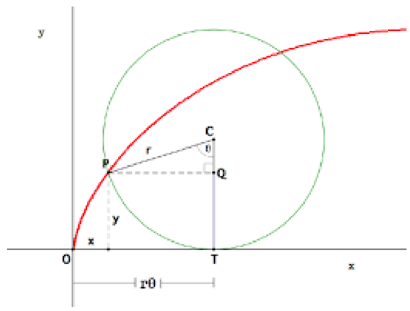
has **initial point** $(f(a), g(a))$ and **terminal point** $(f(b), g(b))$.

The Cycloid



Example 11.1.1. A circle with radius r rolls along the x-axis. The curve traced out by a point P on the circumference of the circle is called a **cycloid**. Find parametric eqations for the cycloid.

Solution. We will use the angle of rotation θ as the parameter ($\theta = 0$ when P is at the origin).



Suppose the circle has rotated θ radians. Using the figure, the distance it has rolled from the origin is

$$|OT| = arc\ PT = r\theta$$

because P starts at the origin. Therefore, the center of the circle is $C(r\theta, r)$. Let the coordinates of P be (x, y). Then from the figure,

$$x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$$
$$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$$

Definition 11.1.1. Paremetric equations of the cycloid are

$$x = r(\theta - \sin\theta) \quad y = r(1 - \cos\theta)$$

11.2 Calculus with Parametric Curves

We will mainly solve problems involving tangents, area, arc length, and surface area.

Tangents

In the previous section, we saw that some curves defined by parametric equations $x = f(t)$ and $y = g(t)$ can also be expressed, by eliminating the parameter, in the form $y = F(x)$. If we substitute $x = f(t)$ and $y = g(t)$ in the equation $y = F(x)$, we get

$$g(t) = F(f(t))$$

If g, f, and F are differentiable, the Chain Rule gives

$$g'(t) = F'(f(t))f'(t) = F'(x)f'(t)$$

If $f'(t) \neq 0$, we can solve for $F'(x)$:

Definition 11.2.1. The slope of the tangent to the parametric curve $y = F(x)$ is $F'(x)$.

$$F'(x) = \frac{g'(t)}{f'(t)}$$

This enables us to find tangents to parametric curves without having to eliminate the parameter. We can rewrite the previous equation in an easily remembered form.

Definition 11.2.2. We can use this to find tangents to parametric curves without having to eliminate the parameter.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \text{ if } \frac{dx}{dt} \neq 0$$

The curve has a

- horizontal tangent when $\frac{dy}{dt} = 0$ (provided that $\frac{dx}{dt} \neq 0$)
- vertical tangent when $\frac{dx}{dt} = 0$ (provided that $\frac{dy}{dt} \neq 0$)

This is useful when sketching parametric curves.

Definition 11.2.3. We can also find $\frac{d^2y}{dx^2}$ by replacing y with $\frac{dy}{dx}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Proof. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ considering $y(t)$ and $g(t)$.

1.

$$\text{Chain rule: } \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \implies \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad (\implies \text{ means "implies"})$$

2.

$$\begin{aligned}
\text{Chain rule: } \frac{d}{dt}\left(\frac{dy}{dx}\right) &= \left(\frac{d}{dx}\frac{dy}{dx}\right)\frac{dx}{dt} = \frac{d^2y}{dx^2}\frac{dx}{dt} \\
\text{Substitute: } \frac{d}{dt}\left(\frac{dy}{dx}\right) &= \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) \\
\text{Quotient rule: } &= \frac{\frac{d^2y}{dt^2}\frac{d^x}{dt} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2}
\end{aligned}$$

Set equation from line 1 and line 3 equal and divide both sides by $\frac{dx}{dt}$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{\frac{d^2y}{dt^2}\frac{d^x}{dt} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2\left(\frac{dx}{dt}\right)} \\
&= \frac{\frac{d^2y}{dt^2}\frac{d^x}{dt} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}
\end{aligned}$$

Example 11.2.1. A curve C is defined by the parametric equations $x = t^2,\ y = t^3 - 3t$.

1. Show that C has two tangents at the point (3,0) and find their equations.
2. Find the points on C where the tangent is horizontal or vertical.
3. Determine where the curve is concave upward or downward.

Solution. A curve C is defined by the parametric equations $x = t^2,\ y = t^3 - 3t$.

1. Rewrite $y = t^3 - 3t = t(t^2 - 3) = 0$ when $t = 0$ or $t = \pm\sqrt{3}$. This indicates that C intersects itself at (3.0).

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right) \\
t = \pm\sqrt{3} &\to dy/dx = \pm 6/(2\sqrt{3})
\end{aligned}$$

so the equations of the tangents at (3,0) are

$$y = \sqrt{3}(x-3) \quad \textit{and} \quad y = -\sqrt{3}(x-3)$$

2. C has a horizontal tangent when $dy/dx = 0$. In other words, when $dy/dt = 0$ and $dx/dt \neq 0$. $dy/dt = 3t^2 - 3 = 0$ when $t^2 = 1$ so $t = \pm 1$. This means there are horizontal tangents on C at (1,-2) and (1,2). C has a vertical tangent when $dx/dt = 2t = 0$, so $t = 0$. This means C has a vertical tangent at (0,0).

3. To determine concavity we calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{3}{2}\left(1 + \frac{1}{t^2}\right)}{2t} = \frac{3(t^2+1)}{4t^3}$$

   The curve is concave upward when $t > 0$ and concave downward when $t < 0$.

Area

We already know that area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)dx$. We can apply this to parametric equations using the Substitution Rule for Definite Integrals.

Definition 11.2.4. If the curve C is given by parametric equations $x = f(t)$ and $y = g(t)$ and t increases from α to β,

$$A = \int_a^b y dx = \int_\alpha^\beta g(t) f'(t) dt$$

(Switch α to β if the point on C at β is more left than α.

Example 11.2.2. Find the area under one arch of the cycloid $x = r(\theta - \sin\theta),\ y = r(1 - \cos\theta)$.

Solution. One arch of the cycloid is given by $0 \leq \theta \leq 2\pi$. Using the Substitution Rule with $y = r(1 - \cos\theta)$ and $dx = r(1 - \cos\theta)d\theta$, we have

$$\begin{aligned}
A = \int_0^{2\pi} y dx = A &= \int_0^{2\pi} r(1-\cos\theta) r(1-\cos\theta) d\theta \\
&= r^2 \int_0^{2\pi} (1-\cos\theta)^2 d\theta = r^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta) d\theta \\
&= r^2 \int_0^{2\pi} \left[1 - 2\cos\theta + \frac{1}{2}(1 + \cos 2\theta)\right] d\theta \\
&= r^2 \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} \\
&= r^2 \left(\frac{3}{2} \cdot 2\pi\right) = 3\pi r^2
\end{aligned}$$

Arc Length

We already know how to find length L of a curve C given in the form $y = F(x)$, $a \leq x \leq b$.

Definition 11.2.5. If F' is continuous, then

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If C can describe the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, where $dx/dt = f'(t) > 0$. Using the substitution rule, we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt} dt$$

Since $dx/dt > 0$, we have

Theorem 11.1. If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

This is consistent with the general formula $L = \int ds$ and $(ds^2) = (dx^2) + (dy^2)$.

Proof. Prove the length formula of a parametric curve

$$\begin{aligned}
\overrightarrow{ds} &= \overrightarrow{i}\, dx + \overrightarrow{j}\, dy \\
ds^2 &= \overrightarrow{ds} \cdot \overrightarrow{ds} = \left(\overrightarrow{i}\, dx + \overrightarrow{j}\, dy\right) \cdot \left(\overrightarrow{i}\, dx + \overrightarrow{j}\, dy\right) = dx^2 + dy^2 \\
ds &= \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
L &= \int_\alpha^\beta ds = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt
\end{aligned}$$

Example 11.2.3. Find the length of the unit circle as (x,y) moves both once and twice around the circle.

Solution. For one traversal around the unit circle,

$$x = \cos t \quad y = \sin t \quad 0 \leq t \leq 2\pi$$

so $dx/dt = -\sin t$ and $dy/dt = \cos t$

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt \\
&= \int_0^{2\pi} dt = 2\pi
\end{aligned}$$

For two traversals around the unit circle,

$$x = \sin 2t \quad y = \cos 2t \quad 0 \le t \le 2\pi$$

so $dx/dt = 2\cos 2t$ and $dy/dt = -2\sin 2t$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^{2\pi} \sqrt{4\cos^2 2t + 4\sin^2 2t}\ dt \quad = \int_0^{2\pi} 2\ dt = 4\pi$$

Surface Area

We can also adapt the surface area formula to a parametric curve.

Definition 11.2.6. If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$, is rotated about the **x-axis**, where f',g' are continuous and $g(t) \ge 0$, the surface area is

$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

If the curve C is rotated about the **y-axis**, the surface area is

$$S = \int_\alpha^\beta 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The generic formulas $S = \int 2\pi y\ ds$ for rotation about the x-axis and $S = \int 2\pi x\ ds$ for rotation about the y-axis are still valid, but for parametric curves we use

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Example 11.2.4. Show that the surface area of a sphere of radius r is $4\pi r^2$

Solution. The sphere is obtained by rotating the semicircle

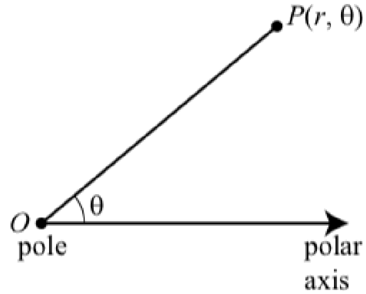
$$x = r\cos t \quad y = r\sin t \quad 0 \le t \le \pi$$

about the x-axis.

$$\begin{aligned}
S &= \int_0^\pi 2\pi r \sin t \sqrt{(-r\sin t)^2 + (r\cos t)^2}\ dt \\
&= 2\pi \int_0^\pi r\sin t \sqrt{r^2(sin^2 t + \cos^2 t)}\ dt \\
&= 2\pi \int_0^\pi r\sin t \cdot r\ dt = 2\pi r^2 \int_0^\pi \sin t\ dt \\
&= 2\pi r^2(-\cos t)]_0^\pi = 4\pi r^2
\end{aligned}$$

11.3 Polar Coordinates

In addition to Cartisian coordinates, we can also use a **polar coordinate system**.



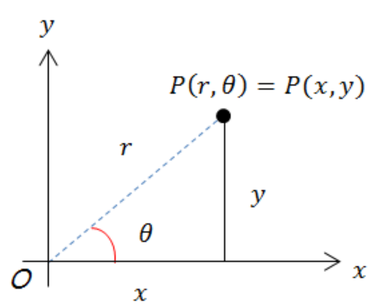
Point P is represented by the ordered pair (r, θ), where r is the distance to the point from the center and θ is the angle from the polar axis to the point.

The points (r, θ) and $(-r, \theta)$ are on the same line and have the same distance $|r|$ from the center but are on opposite sides of the center. Additionally, $(-r, \theta)$ and $(r, \theta + \pi)$ are also on the same line.

This means a complete counterclockwise rotation is given by an angle 2π, so (r, θ) is also represented by

$$(r, \theta + 2n\pi) \text{ and } (-r, \theta + (2n+1)\pi)$$

Relationship Between Cartesian and Polar Coordinates



$$\cos\theta = \tfrac{x}{r} \quad \sin\theta = \tfrac{y}{r}$$

$$x = r\cos\theta \quad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \quad \tan\theta = \tfrac{y}{x}$$

Example 11.3.1. Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

Solution.

$$r = 2,\ \theta = \pi/3$$

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1$$

$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$$

So the point is $(1, \sqrt{3})$ in Cartesian coordinates.

Example 11.3.2. Represent the Cartesian coordinates $(1, -1)$ in polar coordinates.

Solution.

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\tan\theta = \frac{y}{x} = -1$$

Since the point (1,-1) lies in the fourth quadrant, we can choose $\theta = -pi/4$ or $\theta = 7pi/4$. So the possible answers are either $(\sqrt{2}, -\pi/4$ or $(\sqrt{2}, 7\pi/4$.

Polar Curves

The **graph of a polar equation** $r = f(\theta)$, or $F(r, \theta) = 0$, consists of all of the points where (r, θ) satisfies the equation.

Tangents to Polar Curves

To find a tangent line to a polar curce $r = f(\theta)$, we regard θ as a parameter and write the parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \quad y = r\sin\theta = f(\theta)\sin\theta$$

So

Definition 11.3.1.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dy}{d\theta} sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

- horizontal tangent when $\frac{dy}{d\theta} = 0$ (provided that $\frac{dx}{d\theta} \neq 0$)
- vertical tangent when $\frac{dx}{d\theta} = 0$ (provided that $\frac{dy}{d\theta} \neq 0$)

NOTE tangent lines at the pole have r=0 and the slope of the tangent simplifies to

$$\frac{dy}{dx} = \tan\theta \text{ if } \frac{dr}{d\theta} \neq 0$$

Example 11.3.3. For the cardiod $r = 1 + \sin\theta$, find the slope of the tangent line when r=3

Solution.

$$\begin{aligned} r &= 1 + \sin\theta \\ \frac{dy}{dx} &= \frac{\frac{dy}{d\theta} sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta} \\ &= \frac{\cos\theta(1+2\sin\theta)}{1 - 2\sin^2\theta - \sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-\sin\theta)} \end{aligned}$$

The slope of the tangent where $\theta = \pi/3$ is

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{\theta=\pi/3} &= \frac{\cos(\pi/3)(1+2\sin(\pi/3))}{(1+\sin(\pi/3))(1-\sin(\pi/3))} \\ &= \frac{\frac{1}{2}(1+\sqrt{3})}{(1+\sqrt{3}/2)(1-\sqrt{3})} = \frac{1+\sqrt{3}}{(2+\sqrt{3})(1-\sqrt{3})} \\ &= \frac{1+\sqrt{3}}{-1-\sqrt{3}} = -1 \end{aligned}$$

NOTE Instead of memorizing the equation, we can instead use the same method we used to derive it.

$$\begin{aligned} x &= r\cos\theta = (1+\sin\theta)\cos\theta = \cos\theta + \frac{1}{2}\sin 2\theta \\ y &= r\sin\theta = (1+\sin\theta)\sin\theta = \sin\theta + \sin^2\theta \\ \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} = \frac{cos\theta + 2\sin\theta\cos\theta}{-\sin\theta + \cos 2\theta} = \frac{\cos\theta + \sin 2\theta}{-\sin\theta + \cos 2\theta} \end{aligned}$$

This is equivalent to the previous equation.

11.4 Areas and Lengths in Polar Coordinates

Area

We can determine the formula for the area of a region whose boundary is given by a polar equation by taking the limit of a Riemann Sum starting with the formula for the area of a sector of a circle $A = \frac{1}{2}r^2\theta$.

Definition 11.4.1. The formula for the area A of the polar region $\mathcal{R}$ is

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 \, d\theta = \int_a^b \frac{1}{2}r^2 d\theta$$

with the understanding that $r = f(\theta)$.

Example 11.4.1. Find the area enclosed by one loop of the four-leaved rose $r = 2\cos 2\theta$.

Solution. The right loop rotates from $\theta = -\pi/4$ to $\theta = \pi/4$.

$$\begin{aligned} A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta \ d\theta \\ &= \int_0^{\pi/4} \cos^2 2\theta \ d\theta = \int_0^{\pi/4} \frac{1}{2}(1+\cos 4\theta) \ d\theta \\ &= \frac{1}{2}[\theta + \frac{1}{4}\sin 4\theta] = \pi/8 \end{aligned}$$

We can also adapt the formula to find the area of a region bounded by two polar curves.

Definition 11.4.2. Let $\mathcal{R}$ be a region that is bounded by curves with polar equations $r = f(\theta)$, $r = g(\theta)$, $\theta = a$, and $\theta = b$, where $f(\theta) \geq g(\theta) \geq 0$ and $0 < b - a \leq 2\pi$. The area A of $\mathcal{R}$ is found by subtracting the area inside $r = g(\theta)$ from the area inside $r = f(\theta)$, so

$$\begin{aligned} A &= \int_a^b \frac{1}{2}[f(\theta)]^2 \ d\theta - \int_a^b \frac{1}{2}[g(\theta)]^2 \ d\theta \\ &= \int_a^b \frac{1}{2}([f(\theta)]^2 - [g(\theta)]^2) \ d\theta \end{aligned}$$

Arc Length

To find the length of a polar curve $r = f(\theta)$, $a \leq \theta \leq b$, we regard θ as a parameter and write the parametric equations of the curve as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Using the projecut Rule and differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

so, using $\cos^2\theta + \sin^2\theta = 1$, we have

$$\begin{aligned} \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\ &+ \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2 \end{aligned}$$

Assuming that f' is continuous, we can use the theorem from 11.2 about the arc length of a curve defined by parametric equations to write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Definition 11.4.3. The length of a curve with polar equation $r = f(\theta),\ a \leq \theta \leq b$, is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example 11.4.2. Find the arc length of the cardiod $r = 1 + \sin\theta$.

Solution. The full length of the cardiod is given by the parameter interval $0 \leq \theta \leq 2\pi$.

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta = 8 \ (\textit{by rationalizing the integrand by } \sqrt{2 - 2\sin\theta}) \end{aligned}$$

11.5 Conic Sections

11.6 Conic Sections in Polar Coordinates