

# Multivariable Calculus and Linear Algebra

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**Part I**

**Multivariable Calculus**

## Chapter 11

# Parametric Equations and Polar Coordinates

### 11.1 Curves Defined by Parametric Equations

Suppose that  $x$  and  $y$  are both given as functions of a third variable  $t$  (called a **parameter** by the equations)

$$x = f(t) \quad y = g(t)$$

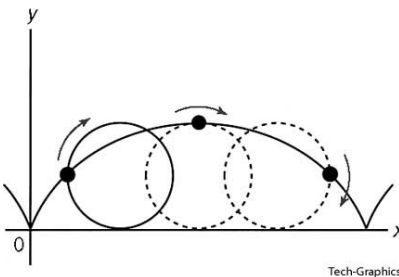
(called **parametric equations**). Each value of  $t$  determines a point  $(x,y)$ . As  $t$  changes,  $(x,y) = (f(t), g(t))$  changes and traces out a curve  $C$ , which is called a **parametric curve**. The direction of the arrows on curve  $C$  show the change in the position of the equation as  $t$  increases.

We can also restrict  $t$  to a finite interval. In general, the curve with parametric equations

$$x = f(t) \quad y = g(t) \quad a \leq t \leq b$$

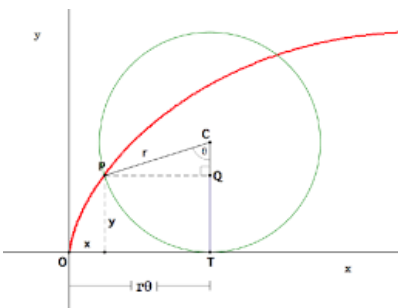
has **initial point**  $(f(a), g(a))$  and **terminal point**  $(f(b), g(b))$ .

#### The Cycloid



**Example 11.1.1.** A circle with radius  $r$  rolls along the  $x$ -axis. The curve traced out by a point  $P$  on the circumference of the circle is called a **cycloid**. Find parametric equations for the cycloid.

*Solution.* We will use the angle of rotation  $\theta$  as the parameter ( $\theta = 0$  when  $P$  is at the origin).



Suppose the circle has rotated  $\theta$  radians. Using the figure, the distance it has rolled from the origin is

$$|OT| = \text{arc } PT = r\theta$$

because  $P$  starts at the origin. Therefore, the center of the circle is  $C(r\theta, r)$ . Let the coordinates of  $P$  be  $(x, y)$ . Then from the figure,

$$x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$$

$$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$$

**Definition 11.1.1.** Parametric equations of the cycloid are

$$x = r(\theta - \sin\theta) \quad y = r(1 - \cos\theta)$$

## 11.2 Calculus with Parametric Curves

We will mainly solve problems involving tangents, area, arc length, and surface area.

### Tangents

In the previous section, we saw that some curves defined by parametric equations  $x = f(t)$  and  $y = g(t)$  can also be expressed, by eliminating the parameter, in the form  $y = F(x)$ . If we substitute  $x = f(t)$  and  $y = g(t)$  in the equation  $y = F(x)$ , we get

$$g(t) = F(f(t))$$

If  $g$ ,  $f$ , and  $F$  are differentiable, the Chain Rule gives

$$g'(t) = F'(f(t))f'(t) = F'(x)f'(t)$$

If  $f'(t) \neq 0$ , we can solve for  $F'(x)$ :



**Definition 11.2.1.** The slope of the tangent to the parametric curve  $y = F(x)$  is  $F'(x)$ .

$$F'(x) = \frac{g'(t)}{f'(t)}$$

This enables us to find tangents to parametric curves without having to eliminate the parameter. We can rewrite the previous equation in an easily remembered form.

**Definition 11.2.2.** We can use this to find tangents to parametric curves without having to eliminate the parameter.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0$$

The curve has a

- horizontal tangent when  $\frac{dy}{dt} = 0$  (provided that  $\frac{dx}{dt} \neq 0$ )
- vertical tangent when  $\frac{dx}{dt} = 0$  (provided that  $\frac{dy}{dt} \neq 0$ )

This is useful when sketching parametric curves.

**Definition 11.2.3.** We can also find  $\frac{d^2y}{dx^2}$  by replacing  $y$  with  $\frac{dy}{dx}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left( \frac{dy}{dx} \right)}{\frac{dx}{dt}}$$

*Proof.* Find  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  considering  $y(t)$  and  $g(t)$ .

1.

$$\text{Chain rule: } \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} \implies \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad (\implies \text{ means "implies" })$$

2.

$$\text{Chain rule: } \frac{d}{dt} \left( \frac{dy}{dx} \right) = \left( \frac{d}{dx} \frac{dy}{dx} \right) \frac{dx}{dt} = \frac{d^2y}{dx^2} \frac{dx}{dt}$$

$$\text{Substitute: } \frac{d}{dt} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right)$$

$$\text{Quotient rule: } = \frac{\frac{d^2y}{dt^2} \frac{dx}{dt} - \frac{dy}{dt} \frac{d^2x}{dt^2}}{\left( \frac{dx}{dt} \right)^2}$$

Set equation from line 1 and line 3 equal and divide both sides by  $\frac{dx}{dt}$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{\frac{d^2y}{dt^2} \frac{dx}{dt} - \frac{dy}{dt} \frac{d^2x}{dt^2}}{\left( \frac{dx}{dt} \right)^2 \left( \frac{dx}{dt} \right)} \\ &= \frac{\frac{d^2y}{dt^2} \frac{dx}{dt} - \frac{dy}{dt} \frac{d^2x}{dt^2}}{\left( \frac{dx}{dt} \right)^3} \end{aligned}$$

**Example 11.2.1.** A curve C is defined by the parametric equations  $x = t^2$ ,  $y = t^3 - 3t$ .

1. Show that  $C$  has two tangents at the point  $(3,0)$  and find their equations.
2. Find the points on  $C$  where the tangent is horizontal or vertical.
3. Determine where the curve is concave upward or downward.

*Solution.* A curve C is defined by the parametric equations  $x = t^2$ ,  $y = t^3 - 3t$ .

1. Rewrite  $y = t^3 - 3t = t(t^2 - 3) = 0$  when  $t = 0$  or  $t = \pm\sqrt{3}$ . This indicates that C intersects itself at  $(3,0)$ .

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t} = \frac{3}{2} \left( t - \frac{1}{t} \right)$$

$$t = \pm\sqrt{3} \rightarrow dy/dx = \pm 6/(2\sqrt{3})$$

so the equations of the tangents at  $(3,0)$  are

$$y = \sqrt{3}(x - 3) \quad \text{and} \quad y = -\sqrt{3}(x - 3)$$

2. C has a horizontal tangent when  $dy/dx = 0$ . In other words, when  $dy/dt = 0$  and  $dx/dt \neq 0$ .  $dy/dt = 3t^2 - 3 = 0$  when  $t^2 = 1$  so  $t = \pm 1$ . This means there are horizontal tangents on C at (1,-2) and (-1,2). C has a vertical tangent when  $dx/dt = 2t = 0$ , so  $t = 0$ . This means C has a vertical tangent at (0,0).
3. To determine concavity we calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{3}{2}\left(1 + \frac{1}{t^2}\right)}{2t} = \frac{3(t^2 + 1)}{4t^3}$$

The curve is concave upward when  $t > 0$  and concave downward when  $t < 0$ .

## Area

We already know that area under a curve  $y = F(x)$  from  $a$  to  $b$  is  $A = \int_a^b F(x)dx$ . We can apply this to parametric equations using the Substitution Rule for Definite Integrals.

**Definition 11.2.4.** If the curve  $C$  is given by parametric equations  $x = f(t)$  and  $y = g(t)$  and  $t$  increases from  $\alpha$  to  $\beta$ ,

$$A = \int_a^b ydx = \int_\alpha^\beta g(t)f'(t)dt$$

(Switch  $\alpha$  to  $\beta$  if the point on  $C$  at  $\beta$  is more left than  $\alpha$ .)

**Example 11.2.2.** Find the area under one arch of the cycloid  $x = r(\theta - \sin \theta)$ ,  $y = r(1 - \cos \theta)$ .

*Solution.* One arch of the cycloid is given by  $0 \leq \theta \leq 2\pi$ . Using the Substitution Rule with  $y = r(1 - \cos \theta)$  and  $dx = r(1 - \cos \theta)d\theta$ , we have

$$\begin{aligned} A &= \int_0^{2\pi} ydx = \int_0^{2\pi} r(1 - \cos \theta)r(1 - \cos \theta)d\theta \\ &= r^2 \int_0^{2\pi} (1 - \cos \theta)^2 d\theta = r^2 \int_0^{2\pi} (1 - 2\cos \theta + \cos^2 \theta) d\theta \\ &= r^2 \int_0^{2\pi} \left[ 1 - 2\cos \theta + \frac{1}{2}(1 + \cos 2\theta) \right] d\theta \\ &= r^2 \left[ \frac{3}{2}\theta - 2\sin \theta + \frac{1}{4}\sin 2\theta \right]_0^{2\pi} \\ &= r^2 \left( \frac{3}{2} \cdot 2\pi \right) = 3\pi r^2 \end{aligned}$$

## Arc Length

We already know how to find length  $L$  of a curve  $C$  given in the form  $y = F(x)$ ,  $a \leq x \leq b$ .

**Definition 11.2.5.** If  $F'$  is continuous, then

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

If  $C$  can describe the parametric equations  $x = f(t)$  and  $y = g(t)$ ,  $\alpha \leq t \leq \beta$ , where  $dx/dt = f'(t) > 0$ . Using the substitution rule, we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2} \frac{dx}{dt} dt$$

Since  $dx/dt > 0$ , we have

**Theorem 11.2.1.** If a curve  $C$  is described by the parametric equations  $x = f(t)$ ,  $y = g(t)$ ,  $\alpha \leq t \leq \beta$ , where  $f'$  and  $g'$  are continuous on  $[\alpha, \beta]$  and  $C$  is traversed exactly once as  $t$  increases from  $\alpha$  to  $\beta$ , then the length of  $C$  is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

This is consistent with the general formula  $L = \int ds$  and  $(ds^2) = (dx^2) + (dy^2)$ .

*Proof.* Prove the length formula of a parametric curve

$$\vec{ds} = \vec{i} dx + \vec{j} dy$$

$$ds^2 = \vec{ds} \cdot \vec{ds} = \left(\vec{i} dx + \vec{j} dy\right) \cdot \left(\vec{i} dx + \vec{j} dy\right) = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$L = \int_\alpha^\beta ds = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

**Example 11.2.3.** Find the length of the unit circle as  $(x, y)$  moves both once and twice around the circle.

*Solution.* For one traversal around the unit circle,

$$x = \cos t \quad y = \sin t \quad 0 \leq t \leq 2\pi$$

so  $dx/dt = -\sin t$  and  $dy/dt = \cos t$

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t} dt \\ &= \int_0^{2\pi} dt = 2\pi \end{aligned}$$

For two traversals around the unit circle,

$$x = \sin 2t \quad y = \cos 2t \quad 0 \leq t \leq 2\pi$$

so  $dx/dt = 2 \cos 2t$  and  $dy/dt = -2 \sin 2t$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^{2\pi} \sqrt{4 \cos^2 2t + 4 \sin^2 2t} dt = \int_0^{2\pi} 2 dt = 4\pi$$

## Surface Area

We can also adapt the surface area formula to a parametric curve.

**Definition 11.2.6.** If a curve  $C$  is described by the parametric equations  $x = f(t)$ ,  $y = g(t)$ ,  $\alpha \leq t \leq \beta$ , is rotated about the **x-axis**, where  $f', g'$  are continuous and  $g(t) \geq 0$ , the surface area is

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

If the curve  $C$  is rotated about the **y-axis**, the surface area is

$$S = \int_{\alpha}^{\beta} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The generic formulas  $S = \int 2\pi y ds$  for rotation about the x-axis and  $S = \int 2\pi x ds$  for rotation about the y-axis are still valid, but for parametric curves we use

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

**Example 11.2.4.** Show that the surface area of a sphere of radius  $r$  is  $4\pi r^2$

*Solution.* The sphere is obtained by rotating the semicircle

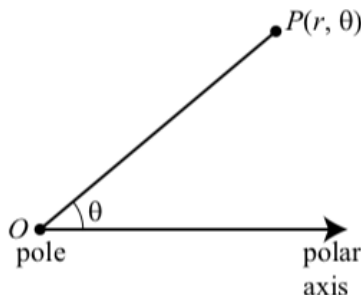
$$x = r \cos t \quad y = r \sin t \quad 0 \leq t \leq \pi$$

about the x-axis.

$$\begin{aligned} S &= \int_0^{\pi} 2\pi r \sin t \sqrt{(-r \sin t)^2 + (r \cos t)^2} dt \\ &= 2\pi \int_0^{\pi} r \sin t \sqrt{r^2(\sin^2 t + \cos^2 t)} dt \\ &= 2\pi \int_0^{\pi} r \sin t \cdot r dt = 2\pi r^2 \int_0^{\pi} \sin t dt \\ &= 2\pi r^2 (-\cos t) \Big|_0^{\pi} = 4\pi r^2 \end{aligned}$$

## 11.3 Polar Coordinates

In addition to Cartesian coordinates, we can also use a **polar coordinate system**.



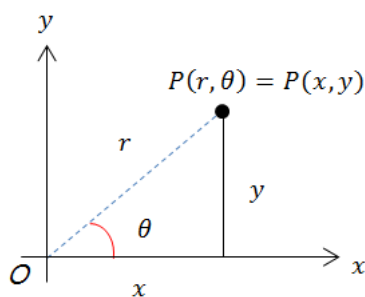
Point  $P$  is represented by the ordered pair  $(r, \theta)$ , where  $r$  is the distance to the point from the center and  $\theta$  is the angle from the polar axis to the point.

The points  $(r, \theta)$  and  $(-r, \theta)$  are on the same line and have the same distance  $|r|$  from the center but are on opposite sides of the center. Additionally,  $(-r, \theta)$  and  $(r, \theta + \pi)$  are also on the same line.

This means a complete counterclockwise rotation is given by an angle  $2\pi$ , so  $(r, \theta)$  is also represented by

$$(r, \theta + 2n\pi) \text{ and } (-r, \theta + (2n + 1)\pi)$$

### Relationship Between Cartesian and Polar Coordinates



$$\cos \theta = \frac{x}{r} \quad \sin \theta = \frac{y}{r}$$

$$x = r \cos \theta \quad y = r \sin \theta$$

$$r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x}$$

**Example 11.3.1.** Convert the point  $(2, \pi/3)$  from polar to Cartesian coordinates.

*Solution.*

$$r = 2, \theta = \pi/3$$

$$x = r \cos \theta = 2 \cos \frac{\pi}{3} = 2 \cdot \frac{1}{2} = 1$$

$$y = r \sin \theta = 2 \sin \frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

So the point is  $(1, \sqrt{3})$  in Cartesian coordinates.

**Example 11.3.2.** Represent the Cartesian coordinates  $(1, -1)$  in polar coordinates.

*Solution.*

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\tan \theta = \frac{y}{x} = -1$$

Since the point  $(1, -1)$  lies in the fourth quadrant, we can choose  $\theta = -\pi/4$  or  $\theta = 7\pi/4$ . So the possible answers are either  $(\sqrt{2}, -\pi/4)$  or  $(\sqrt{2}, 7\pi/4)$ .

## Polar Curves

The **graph of a polar equation**  $r = f(\theta)$ , or  $F(r, \theta) = 0$ , consists of all of the points where  $(r, \theta)$  satisfies the equation.

## Tangents to Polar Curves

To find a tangent line to a polar curve  $r = f(\theta)$ , we regard  $\theta$  as a parameter and write the parametric equations as

$$x = r \cos \theta = f(\theta) \cos \theta \quad y = r \sin \theta = f(\theta) \sin \theta$$

So

**Definition 11.3.1.**

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dy}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}$$

- horizontal tangent when  $\frac{dy}{d\theta} = 0$  (provided that  $\frac{dx}{d\theta} \neq 0$ )
- vertical tangent when  $\frac{dx}{d\theta} = 0$  (provided that  $\frac{dy}{d\theta} \neq 0$ )

NOTE tangent lines at the pole have  $r=0$  and the slope of the tangent simplifies to

$$\frac{dy}{dx} = \tan \theta \text{ if } \frac{dr}{d\theta} \neq 0$$

**Example 11.3.3.** For the cardioid  $r = 1 + \sin \theta$ , find the slope of the tangent line when  $r=3$

*Solution.*

$$\begin{aligned} r &= 1 + \sin \theta \\ \frac{dy}{dx} &= \frac{\frac{dy}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} = \frac{\cos \theta \sin \theta + (1 + \sin \theta) \cos \theta}{\cos \theta \cos \theta - (1 + \sin \theta) \sin \theta} \\ &= \frac{\cos \theta (1 + 2 \sin \theta)}{1 - 2 \sin^2 \theta - \sin \theta} = \frac{\cos \theta (1 + 2 \sin \theta)}{(1 + \sin \theta)(1 - \sin \theta)} \end{aligned}$$

The slope of the tangent where  $\theta = \pi/3$  is

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{\theta=\pi/3} &= \frac{\cos(\pi/3)(1 + 2 \sin(\pi/3))}{(1 + \sin(\pi/3))(1 - \sin(\pi/3))} \\ &= \frac{\frac{1}{2}(1 + \sqrt{3})}{(1 + \sqrt{3}/2)(1 - \sqrt{3})} = \frac{1 + \sqrt{3}}{(2 + \sqrt{3})(1 - \sqrt{3})} \\ &= \frac{1 + \sqrt{3}}{-1 - \sqrt{3}} = -1 \end{aligned}$$

NOTE Instead of memorizing the equation, we can instead use the same method we used to derive it.

$$\begin{aligned} x &= r \cos \theta = (1 + \sin \theta) \cos \theta = \cos \theta + \frac{1}{2} \sin 2\theta \\ y &= r \sin \theta = (1 + \sin \theta) \sin \theta = \sin \theta + \sin^2 \theta \\ \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} = \frac{\cos \theta + 2 \sin \theta \cos \theta}{-\sin \theta + \cos 2\theta} = \frac{\cos \theta + \sin 2\theta}{-\sin \theta + \cos 2\theta} \end{aligned}$$

This is equivalent to the previous equation.

## 11.4 Areas and Lengths in Polar Coordinates

### Area

We can determine the formula for the area of a region whose boundary is given by a polar equation by taking the limit of a Riemann Sum starting with the formula for the area of a sector of a circle  $A = \frac{1}{2}r^2\theta$ .

**Definition 11.4.1.** The formula for the area  $A$  of the polar region  $\mathcal{R}$  is

$$A = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta = \int_a^b \frac{1}{2} r^2 d\theta$$

with the understanding that  $r = f(\theta)$ .



**Example 11.4.1.** Find the area enclosed by one loop of the four-leaved rose  $r = 2 \cos 2\theta$ .

*Solution.* The right loop rotates from  $\theta = -\pi/4$  to  $\theta = \pi/4$ .

$$\begin{aligned} A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta d\theta \\ &= \int_0^{\pi/4} \cos^2 2\theta d\theta = \int_0^{\pi/4} \frac{1}{2} (1 + \cos 4\theta) d\theta \\ &= \frac{1}{2} \left[ \theta + \frac{1}{4} \sin 4\theta \right] = \pi/8 \end{aligned}$$

We can also adapt the formula to find the area of a region bounded by two polar curves.

**Definition 11.4.2.** Let  $\mathcal{R}$  be a region that is bounded by curves with polar equations  $r = f(\theta)$ ,  $r = g(\theta)$ ,  $\theta = a$ , and  $\theta = b$ , where  $f(\theta) \geq g(\theta) \geq 0$  and  $0 < b - a \leq 2\pi$ . The area  $A$  of  $\mathcal{R}$  is found by subtracting the area inside  $r = g(\theta)$  from the area inside  $r = f(\theta)$ , so

$$\begin{aligned} A &= \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta - \int_a^b \frac{1}{2} [g(\theta)]^2 d\theta \\ &= \int_a^b \frac{1}{2} ([f(\theta)]^2 - [g(\theta)]^2) d\theta \end{aligned}$$

## Arc Length

To find the length of a polar curve  $r = f(\theta)$ ,  $a \leq \theta \leq b$ , we regard  $\theta$  as a parameter and write the parametric equations of the curve as

$$x = r \cos \theta = f(\theta) \cos \theta \quad y = r \sin \theta = f(\theta) \sin \theta$$

Using the project Rule and differentiating with respect to  $\theta$ , we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta \quad \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta$$

so, using  $\cos^2 \theta + \sin^2 \theta = 1$ , we have

$$\begin{aligned} \left( \frac{dx}{d\theta} \right)^2 + \left( \frac{dy}{d\theta} \right)^2 &= \left( \frac{dr}{d\theta} \right)^2 \cos^2 \theta - 2r \frac{dr}{d\theta} \cos \theta \sin \theta + r^2 \sin^2 \theta \\ &\quad + \left( \frac{dr}{d\theta} \right)^2 \sin^2 \theta + 2r \frac{dr}{d\theta} \sin \theta \cos \theta + r^2 \cos^2 \theta \\ &= \left( \frac{dr}{d\theta} \right)^2 + r^2 \end{aligned}$$

Assuming that  $f'$  is continuous, we can use the theorem from 11.2 about the arc length of a curve defined by parametric equations to write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

**Definition 11.4.3.** The length of a curve with polar equation  $r = f(\theta)$ ,  $a \leq \theta \leq b$ , is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

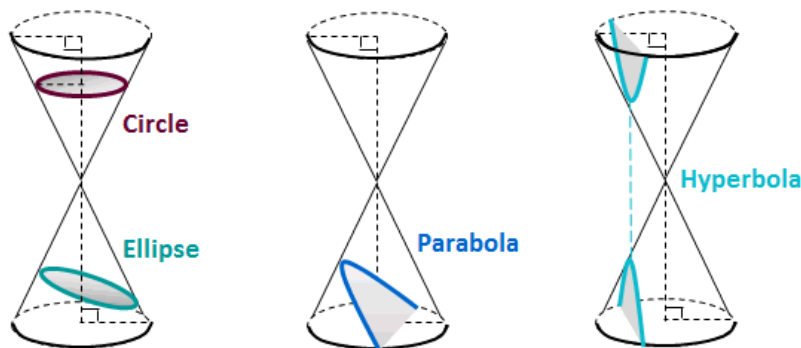
**Example 11.4.2.** Find the arc length of the cardioid  $r = 1 + \sin \theta$ .

*Solution.* The full length of the cardioid is given by the parameter interval  $0 \leq \theta \leq 2\pi$ .

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_0^{2\pi} \sqrt{(1 + \sin \theta)^2 + \cos^2 \theta} d\theta \\ &= \int_0^{2\pi} \sqrt{2 + 2 \sin \theta} d\theta = 8 \text{ (by rationalizing the integrand by } \sqrt{2 - 2 \sin \theta} \text{)} \end{aligned}$$

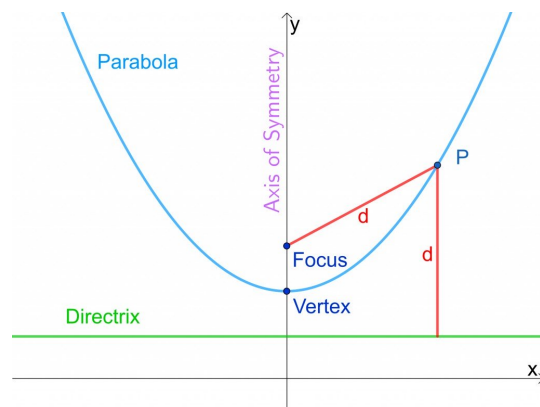
## 11.5 Conic Sections

Parabolas, ellipses, and hyperbolas are called **conic sections**, or **conics**, because they result from intersecting a cone with a plane.



### Parabolas

A **parabola** is the set of points in a plane that are equidistant from a fixed point  $F$  (called the **focus**) and a fixed line (called the **directrix**). The halfway point between the focus and directrix is on the parabola and is called the **vertex**. The line through the focus and the vertex and perpendicular to the directrix is the **axis** of the parabola.



As seen in the figure, the focus is always inside the region of the parabola and the directrix is the same distance away on the opposite side.

**Definition 11.5.1.** An equation of the parabola with focus  $(0, p)$  and directrix  $y = -p$  is

$$x^2 = 4py$$

. If we set  $a = \frac{1}{4p}$ , then the standard equation of a parabola is  $y = ax^2$ . This opens upward if  $p > 0$  and downward if  $p < 0$ , and is symmetric with respect to the y-axis.

**Definition 11.5.2.** If we switch  $x$  and  $y$ , we get

$$y^2 = 4px$$

(reflection about the diagonal line  $y=x$ ). This parabola opens to the right if  $p > 0$  and to the left if  $p < 0$ .

**Definition 11.5.3.** The vertex form of a parabola is

$$y = a(x - h)^2 + k$$

where  $(h, k)$  is the vertex of the parabola and  $x = h$  is the axis of symmetry. We can also switch  $x$  and  $y$  to get the vertex form of the rotated parabola.

**Example 11.5.1.** Find the focus and directrix of the parabola  $y^2 + 10x = 0$ .

*Solution.* We rewrite the equation as  $y^2 = -10x$ . We know  $y^2 = 4px$ , so  $4px = -10x$  and  $p = -\frac{5}{2}$ . Thus, the focus is  $(p, 0) = (-\frac{5}{2}, 0)$  and the directrix is  $x = \frac{5}{2}$ .

## Ellipses

An **ellipse** is the set of points in a plane surrounding two fixed focal points  $F_1$  and  $F_2$  such that the sum of the two distances to the focal points is a constant. Imagine tracing a line along the furthest path of a string stretched across two different points.

**Definition 11.5.4.** The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a \geq b \geq 0$$

has foci  $(\pm c, 0)$ , where  $c^2 = a^2 - b^2$ , and vertices  $(\pm a, 0)$  (**lies on x-axis**).

The **vertices** are on the **major axis**, where  $a$  is the distance to the center of the ellipse from each vertex. This distance is greater than the distance from a **co-vertex** to the center of the ellipse,  $b$ . The co-vertices lie on the **minor axis**. Because the sum of the two distances from a point on the ellipse to the foci is a constant, the distance from a co-vertex to a focal point is also  $a$ . If the foci coincide, then  $c = 0$ , so  $a = b$  and the ellipse becomes a circle with radius  $r = a = b$ .

**Definition 11.5.5.** The ellipse

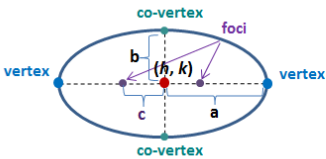
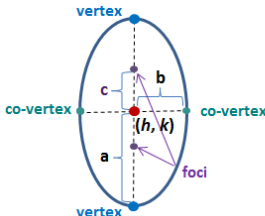
$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad a \geq b \geq 0$$

has foci  $(0, \pm c)$ , where  $c^2 = a^2 - b^2$ , and vertices  $(0, \pm a)$  (**lies on y-axis**).

**Definition 11.5.6.** The general form of a horizontal ellipse is

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

where  $(h, k)$  is the center of the ellipse. The same transformation can be done to the standard form of a vertical ellipse.

Horizontal Ellipse	Vertical Ellipse
At $(0, 0)$ : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	At $(0, 0)$ : $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$
General: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ $a^2 - b^2 = c^2$	General: $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$ $a^2 - b^2 = c^2$
Center: $(h, k)$ Foci: $(h \pm c, k)$	Center: $(h, k)$ Foci: $(h, k \pm c)$
Vertices: $(h \pm a, k)$ Co-Vertices: $(h, k \pm b)$	Vertices: $(h, k \pm a)$ Co-Vertices: $(h \pm b, k)$
	

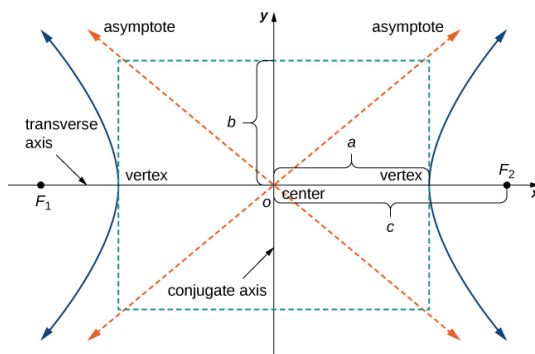
**Example 11.5.2.** Find an equation of the ellipse with foci  $(0, \pm 2)$  and vertices  $(0, \pm 3)$ .

*Solution.* This equation represents a vertical ellipse because the foci and vertices lie on the  $y$ -axis. The distance from a focal point to the center is  $c = 2$  and the distance from a vertex to the center is  $a = 3$ . Then we obtain  $b^2 = a^2 + c^2 = 9 - 4 = 5$ , so the equation of the ellipse is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = \frac{x^2}{5} + \frac{y^2}{9} = 1$$

## Hyperbolas

An **ellipse** is the set of points in a plane surrounding two fixed focal points  $F_1$  and  $F_2$  such that the difference of the two distances to the focal points is a constant. The **transverse axis** is the axis of a hyperbola that passes through the two foci. The **conjugate axis** is perpendicular to the transverse axis and passes through the center of the hyperbola.



**Definition 11.5.7.** The hyperbola along a horizontal transverse axis

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has foci  $(\pm c, 0)$ , where  $c^2 = a^2 + b^2$ , vertices  $(\pm a, 0)$  (**lies on x-axis**), and asymptotes  $y = \pm \frac{b}{a}x$ .

**Definition 11.5.8.** The hyperbola along a vertical transverse axis

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

has foci  $(0, \pm c)$ , where  $c^2 = a^2 + b^2$ , vertices  $(0, \pm a)$  (**lies on y-axis**), and asymptotes  $y = \pm \frac{a}{b}x$ .

**Definition 11.5.9.** The general form of a hyperbola along a horizontal transverse axis is

$$\frac{(x - h)^2}{b^2} - \frac{(y - k)^2}{a^2} = 1$$

where  $(h, k)$  is the center of the ellipse. The same transformation can be done to the standard form a hyperbola along a vertical transverse axis.

**Example 11.5.3.** Find the foci and asymptotes of the hyperbola  $9x^2 - 16y^2 = 144$ .

*Solution.* If we divide both sides of the equation by 144, it becomes

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

which is a hyperbola along a horizontal transverse axis. Therefore, we get  $a = 4$  and  $b = 3$ . Since  $c^2 = a^2 + b^2 = 16 + 9 = 25$ ,  $c = 5$ . The foci are  $(\pm 5, 0)$ , and the asymptotes are  $y = \pm \frac{3}{4}x$ .

**Example 11.5.4.** Find the foci and asymptotes of the hyperbola  $ax^2 - by^2 = r$  in terms of  $a$ ,  $b$ ,  $r$ .

*Solution.* We first put the equation in standard form to get

$$\frac{x^2}{(\sqrt{\frac{r}{a}})^2} - \frac{y^2}{(\sqrt{\frac{r}{b}})^2} = 1$$

which is a hyperbola along a horizontal transverse axis. The foci are at

$$(\pm c, 0) = \left( \pm \sqrt{\frac{r}{a} + \frac{r}{b}}, 0 \right) = \left( \pm \sqrt{\frac{r(a+b)}{ab}}, 0 \right)$$

the vertices are at  $(\pm \sqrt{\frac{r}{a}}, 0)$ , and the asymptotes are at  $y = \pm \left( \sqrt{\frac{a}{b}} \right) x$ .

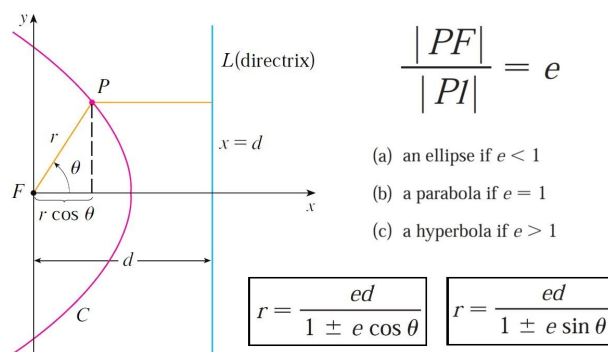
## 11.6 Conic Sections in Polar Coordinates

**Theorem 11.6.1.** Let  $F$  be a fixed point (called the **focus**) and  $l$  be a fixed line (called the **directrix**). Let  $e$  be a fixed positive number (called the **eccentricity**). The set of all points  $P$  in the plane such that

$$\frac{|PF|}{|Pl|} = e \quad (\text{the ratio of the distance from } F \text{ to the distance from } l \text{ is the constant } e)$$

is a conic section. The conic is

1. an ellipse if  $e < 1$
2. a parabola if  $e = 1$
3. a hyperbola if  $e > 1$

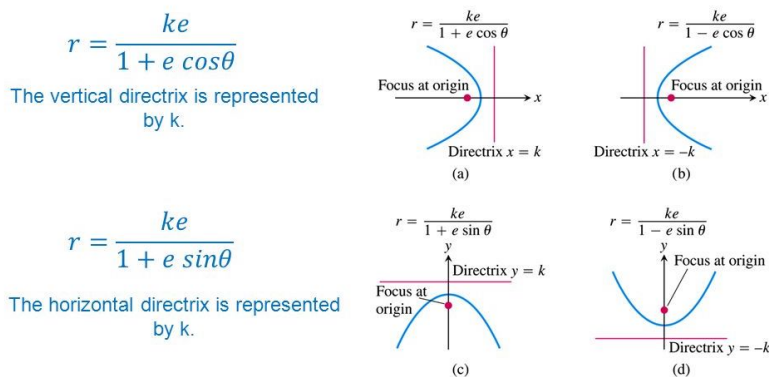


**Theorem 11.6.2.** A polar equation of the form

$$r = \frac{ed}{1 \pm e \cos \theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e \sin \theta}$$

represents a conic section with eccentricity  $e$  and distance  $d$  from the center to the directrix, with the focus at the origin. The conic is an ellipse if  $e < 1$ , a parabola if  $e = 1$ , or a hyperbola if  $e > 1$ .

### Polar Equation for a Conic with Eccentricity $e$



To use these polar equations, a focus is located at the origin.

Use “ $\cos \theta$ ” when the conic section opens rightward or leftward, and use “ $\sin \theta$ ” when the conic section opens upward or downward. Use “+” if the conic section opens leftward or downward, and use “−” if the conic section opens rightward or upward.

**Example 11.6.1.** Find a polar equation for a parabola that has its focus at the origin and whose directrix is the line  $y = -6$ .

*Solution.* The eccentricity  $e = 1$  because the conic section is a parabola, and the distance from the center to the directrix is  $d = 6$ . The directrix is on the  $y$ -axis and is underneath the center, so the parabola opens upward. Therefore, we use the “ $\sin \theta$ ” equation and use “ $-$ ” in the denominator. The polar equation of the parabola is

$$r = \frac{6}{1 - \sin \theta}$$

**Example 11.6.2.** A conic is given by the polar equation

$$r = \frac{10}{3 - 2 \cos \theta}$$

Find the eccentricity, identify the conic, and locate the directrix.

*Solution.* Divide the numerator and denominator by 3 to get

$$r = \frac{\frac{10}{3}}{1 - \frac{2}{3} \cos \theta}$$

This represents an ellipses with eccentricity  $e = \frac{2}{3}$ . Since  $ed = \frac{10}{3}$ ,

$$d = \frac{\frac{10}{3}}{e} = \frac{\frac{10}{3}}{\frac{2}{3}} = 5$$

so the directrix has Cartesian equation  $x = -5$ . When  $\theta = 0$ ,  $r = 10$ ; when  $\theta = \pi$ ,  $r = 2$ , so the vertices have polar coordinates  $(10, 0)$ , and  $(2, \pi)$ .





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**Part II**

**Linear Algebra**

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