12.7 Strategy for Testing Series

The main strategy for testing series is to classify the series according to its *form*.

1. If the series is of the form $\sum 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leq 1$.
2. If the series has form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric series, which converges is $|r| < 1$ and diverges if $|r| \geq 1$. You may need to manipulate the equation to bring the series into this form.
3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series (the values of p should be chosen by keeping only the highest powers of n in the numerator and denominator). The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, we can apply the Comparison Tests to $\sum |a_n|$ and test for absolute convergence.
4. If it is obvious that $\lim_{n\to\infty} \neq 0$, then use the Test for Divergence.
5. If the series is of the form $\sum(-1)^{n-1}b_n$ or $\sum(-1)^n b_n$, then the Alternating Series Test is an obvious possibility.
6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus, the Ratio Test should not be used for such series.
7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.
8. If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

Example 12.7.1. These examples show demonstrate how to identify which test should be used.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

Since $a_n \to \frac{1}{2} \neq 0$ as $n \to \infty$, we should use the Test for Divergence.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

Since a_n is an algebraic function of n, we should compare the given series with a p-series. The comparison series for the Limit Comparison Test is b_n, where

$$b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{n^{3/2}}{3n^3} = \frac{1}{3n^{3/2}}$$

(c) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

Since the integral $\int_1^\infty xe^{-x^2}\, dx$ is easily evaluated, we use the Integral Test. The Ratio Test also works.

(d) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{n^4+1}$

Since the series is alternating, we use the Alternating Series Test.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{2^k}{k!}$

Since the series is involves $k!$, we use the Ratio Test.

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

Since the series is closely related to the geometric series $\sum 1/3^n$, we use the Comparison Test.