

## 12.1 Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

The number  $a_1$  is the *first term*,  $a_2$  is the *second term*, and in general  $a_n$  is the *nth term*. We will deal with infinite sequences exclusively so each term  $a_n$  will have a successor  $a_{n+1}$ .

Notice that for every positive integer  $n$  there is a corresponding number  $a_n$  so a sequence can be defined as a function whose domain is the set of positive integers. We usually write  $a_n$  instead of the function notation  $f(n)$ .

NOTATION The sequence  $a_1, a_2, a_3, \dots$  is also denoted by

$$a_n \quad \text{or} \quad a_{n=1}^{\infty}$$

**Example 12.1.1.** Some sequences can be defined by giving a formula for the  $n$ th term. In this example, we will describe a sequence in 3 ways: the previous notation, defining a formula, and writing out the terms of the sequence. Note that  $n$  doesn't have to start at 1.

1. 
$$\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty} \quad a_n = \frac{n}{n+1} \quad \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots \right\}$$
2. 
$$\left\{ \frac{(-1)^n(n+1)}{3^n} \right\} \quad a_n = \frac{(-1)^n(n+1)}{3^n} \quad \left\{ -\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \dots, \frac{(-1)^n(n+1)}{3^n}, \dots \right\}$$
3. 
$$\{\sqrt{n-3}\}_{n=3}^{\infty} \quad a_n = \sqrt{n-3}, \quad n \geq 3 \quad \{0, 1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n-3}, \dots\}$$
4. 
$$\left\{ \cos \frac{n\pi}{6} \right\}_{n=0}^{\infty} \quad a_n = \cos \frac{n\pi}{6}, \quad n \geq 0 \quad \left\{ 1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \dots, \cos \frac{n\pi}{6}, \dots \right\}$$

**Example 12.1.2.** Find a formula for the general term  $a_n$  of the sequence

$$\left\{ \frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \dots \right\}$$

*Solution.* We are given the first five terms. The numerator of the fractions start at 3 and increase by 1, so the  $n$ th term will have numerator  $n+1$ . The denominators are the powers of 5, so  $a_n$  has denominator  $5^n$ . The signs of the terms alternate between positive and negative, so we need to multiply by a power of  $-1$ . The factor  $(-1)^n$  means we start with a negative term, so here we use  $(-1)^{n-1}$  or  $(-1)^{n+1}$  because we start with a positive term. Therefore,

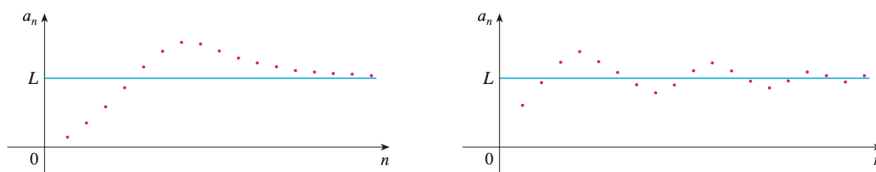
$$a_n = (-1)^{n-1} \frac{n+1}{5^n}$$

**Definition 12.1.1.** A sequence  $\{a_n\}$  has the **limit**  $L$  and we write

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \text{ as } n \rightarrow \infty$$

if we can make the terms  $a_n$  as close to  $L$  as we like by taking  $n$  sufficiently large. If  $\lim_{n \rightarrow \infty} a_n$  exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

The figure below graphs examples of two sequences that have the limit  $L$ .



A more precise version of the previous definition is

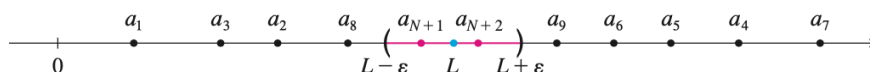
**Definition 12.1.2.** A sequence  $\{a_n\}$  has the **limit**  $L$  and we write

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \text{ as } n \rightarrow \infty$$

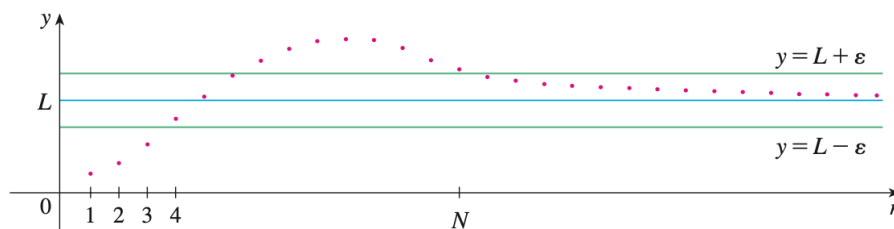
if for every  $\varepsilon > 0$  there is a corresponding integer  $N$  such that

$$|a_n - L| < \varepsilon \quad \text{whenever } n > N$$

No matter how small an interval  $(L - \varepsilon, L + \varepsilon)$  is chosen, there exists an  $N$  such that all terms of the sequence from  $a_{N+1}$  onward must lie in that interval.



The points on the graph of  $a_n$  must lie between the horizontal lines  $y = L + \varepsilon$  and  $y = L - \varepsilon$  if  $n > N$ . This picture must be valid no matter how small  $\varepsilon$  is chosen, but usually a smaller  $\varepsilon$  requires a larger  $N$ .



The only difference between  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{x \rightarrow \infty} f(x) = L$  is that  $n$  is required to be an integer.

**Theorem 12.1.1.** If  $\lim_{x \rightarrow \infty} f(x) = L$  and  $f(n) = a_n$  when  $n$  is an integer, then  $\lim_{n \rightarrow \infty} a_n = L$ .

Since we know that  $\lim_{x \rightarrow \infty} (1/x^r) = 0$  when  $r > 0$ , we can use the previous theorem to get

**Definition 12.1.3.**

$$\lim_{n \rightarrow \infty} \frac{1}{n^r} = 0 \quad \text{if } r > 0$$

If  $a_n$  grows as  $n$  grows, we use the notation  $\lim_{n \rightarrow \infty} a_n = \infty$ . We say that  $a_n$  diverges to  $\infty$ .

**Definition 12.1.4.**  $\lim_{n \rightarrow \infty} a_n = \infty$  means that for every positive number  $M$  there is an integer  $N$  such that

$$a_n > M \quad \text{whenever } n > N$$

**Definition 12.1.5 (Limit Laws for Sequences)** (similar to original Limit Laws)). If  $\{a_n\}$  and  $\{b_n\}$  are convergent sequences and  $c$  is a constant, then

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n$$

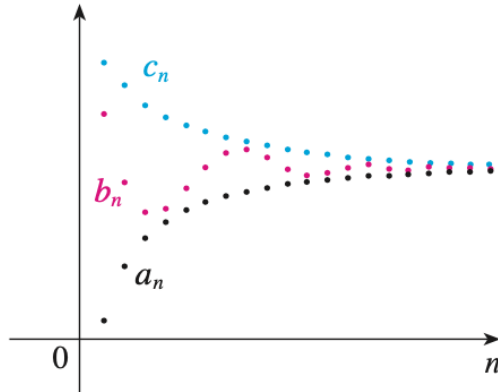
$$\lim_{n \rightarrow \infty} ca_n = c \lim_{n \rightarrow \infty} a_n \qquad \lim_{n \rightarrow \infty} c = c$$

$$\lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} \quad \text{if } b_n \neq 0$$

$$\lim_{n \rightarrow \infty} (a_n^p) = \left[ \lim_{n \rightarrow \infty} a_n \right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

**Theorem 12.1.2 (The Squeeze Theorem for Sequences)** (same as original)). If  $a_n \leq b_n \leq c_n$  for  $n \geq n_0$  and  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$ , then  $\lim_{n \rightarrow \infty} b_n = L$ .



**Theorem 12.1.3.** If  $\lim_{n \rightarrow \infty} |a_n| = 0$ , then  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Example 12.1.3.** Evaluate  $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n}$  if it exists.

*Solution.*

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \quad \text{so} \quad \lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$$

**Example 12.1.4.** Find  $\lim_{n \rightarrow \infty} \frac{n}{n+1}$ .

*Solution.* Divide the numerator and denominator by the highest power of  $n$  and then use the Limit Laws.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n}{n+1} &= \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = \frac{\lim_{n \rightarrow \infty} 1}{\lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{1}{n}} \\ &= \frac{1}{1+0} = 1 \end{aligned}$$

**Example 12.1.5.** Calculate  $\lim_{n \rightarrow \infty} \frac{\ln n}{n}$ .

*Solution.* Both the numerator and denominator approach infinity as  $n \rightarrow \infty$ . We can't apply l'Hospital's Rule directly because it applies to functions, not sequences. However, we can apply it to the related function.

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0 \quad \text{so} \quad \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$$

**Example 12.1.6.** Determine whether the sequence  $a_n = (-1)^n$  is convergent or divergent.

*Solution.* If we write out the terms of the sequence, we get  $\{-1, 1, -1, 1, -1, 1, -1, \dots\}$ . Since the terms oscillate between 1 and -1,  $a_n$  does not approach any number. Thus,  $\lim_{n \rightarrow \infty} (-1)^n$  does not exist so the sequence  $\{(-1)^n\}$  is divergent.

**Example 12.1.7.** Discuss the convergence of the sequence  $a_n = n!/n^n$ .

*Solution.* Both the numerator and denominator approach infinity as  $n \rightarrow \infty$ , but we have no corresponding functions to use l'Hospital's Rule because  $x!$  is not defined when  $x$  is not an integer. If we write the general formula for the sequence, we get

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot n}{n \cdot n \cdot n \cdot \dots \cdot n} = \frac{1}{n} \left( \frac{2 \cdot 3 \cdot \dots \cdot n}{n \cdot n \cdot n \cdot \dots \cdot n} \right)$$

The expression in the parenthesis is at most 1 because the numerator is less than (or equal) to the denominator, so

$$0 < a_n \leq \frac{1}{n}$$

We can use the squeeze theorem because both  $0$  and  $1/n \rightarrow 0$  as  $n \rightarrow \infty$ , so  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ .

**Example 12.1.8.** Determine if the sequences below converge. If they do, find the limits as  $n \rightarrow \infty$ .

1.  $\frac{\sin n}{n}$
2.  $ne^{-n}$

*Solution.* 1.  $\frac{\sin n}{n}$  converges to 0 by the squeeze theorem

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}, \quad \lim_{n \rightarrow \infty} \frac{1}{n} = 0, \text{ so}$$

$$0 \leq \frac{\sin n}{n} \leq 0 \implies \lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$$

2.  $ne^{-n} = \frac{n}{e^n}$ . The denominator  $e^n$  converges faster than the numerator  $n$  does. Use l'Hospital's Rule to get

$$\lim_{x \rightarrow \infty} \frac{x}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0 \text{ so } \lim_{n \rightarrow \infty} ne^{-n} = 0$$

**Example 12.1.9.** Show that if  $\lim_{n \rightarrow \infty} a_{2n} = L$  and  $\lim_{n \rightarrow \infty} a_{2n+1} = L$ ,  $\{a_n\}$  is convergent and  $\lim_{n \rightarrow \infty} a_n = L$ .

*Solution.* The solution uses the symbols  $\exists$  ("exists") and  $\implies$  ("implies").

$$\begin{aligned} \text{Since } \lim_{n \rightarrow \infty} a_{2n} = L, \quad & \exists N_1 \implies |a_{2n} - L| < \varepsilon \text{ for } n > N_1 \\ \text{Since } \lim_{n \rightarrow \infty} a_{2n+1} = L, \quad & \exists N_2 \implies |a_{2n+1} - L| < \varepsilon \text{ for } n > N_2 \end{aligned}$$

Let  $N = \max\{2N_1, 2N_2 + 1\}$  and let  $n > N$ .

$$\begin{aligned} \text{If } n \text{ is even,} \quad & n = 2m, m > N_1, |a_n - L| = |a_{2m} - L| < \varepsilon \\ \text{If } n \text{ is odd,} \quad & n = 2m + 1, m > N_2, |a_n - L| = |a_{2m+1} - L| < \varepsilon \end{aligned}$$

Therefore,  $\{a_n\}$  is convergent and  $\lim_{n \rightarrow \infty} a_n = L$ .

**Definition 12.1.6.** The sequence  $\{r^n\}$  is convergent if  $-1 < r \leq 1$  and divergent for all other values of  $r$ .

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

**Definition 12.1.7.** A sequence  $\{a_n\}$  is **increasing** if  $a_n < a_{n+1}$  for all  $n \geq 1$  ( $a_1 < a_2 < a_3 < \dots$ ). It is **decreasing** if  $a_n > a_{n+1}$  for all  $n \geq 1$ . It is **monotonic** if it is either increasing or decreasing.

**Example 12.1.10.** The sequence  $\left\{\frac{3}{n+5}\right\}$  is decreasing because

$$\frac{3}{n+5} < \frac{3}{(n+1)+5} = \frac{3}{n+6}$$

for all  $n \geq 1$  (the right side is smaller because it has a larger denominator).

**Example 12.1.11.** Show that the sequence  $a_n = \frac{n}{n^2+1}$  is decreasing.

*Solution (1).* We must show that  $a_{n+1} < a_n$ .

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$

We can simplify this by cross-multiplying.  $\iff$  means "if and only if".

$$\begin{aligned} \frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} &\iff (n+1)(n^2+1) < n[(n+1)^2+1] \\ &\iff n^3 + n^2 + n + 1 < n^3 + 2n^2 + 2n \\ &\iff 1 < n^2 + n \end{aligned}$$

Since  $n \geq 1$ , we know that the inequality  $n^2 + n > 1$  is true. Therefore,  $a_{n+1} < a_n$  so  $\{a_n\}$  is decreasing.

*Solution (2).* Consider the function  $f(x) = \frac{x}{x^2+1}$ :

$$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0 \quad \text{whenever } x^2 > 1$$

This,  $f$  is decreasing on  $(1, \infty)$  so  $f(n) > f(n+1)$ . Therefore,  $\{a_n\}$  is decreasing.

**Theorem 12.1.4** (Monotonic Sequence Theorem). Every bounded, monotonic sequence is convergent.