Chapter 11

Parametric Equations and Polar Coordinates

11.1 Curves Defined by Parametric Equations

Suppose that x and y are both given as functions of a third variable t (called a **parameter** by the equations)

$$x = f(t) \quad y = g(t)$$

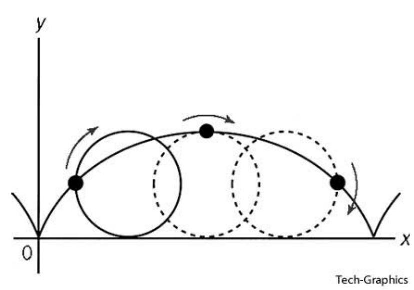
(called **parametric equations**). Each value of t determines a point (x,y). As t changes, $(x, y) = (f(t), g(t))$ changes and traces out a curve C, which is called a **parametric curve**. The direction of the arrows on curve C show the change in the position of the equation as t increases.

We can also restrict t to a finite interval. In general, the curve with parametric equations

$$x = f(t) \quad y = g(t) \quad a \leq t \leq b$$

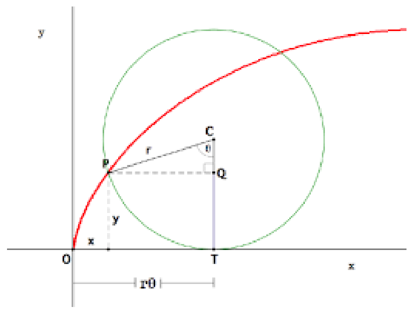
has **initial point** $(f(a), g(a))$ and **terminal point** $(f(b), g(b))$.

The Cycloid



Example 11.1.1. A circle with radius r rolls along the x-axis. The curve traced out by a point P on the circumference of the circle is called a **cycloid**. Find parametric eqations for the cycloid.

Solution. We will use the angle of rotation θ as the parameter ($\theta = 0$ when P is at the origin).



Suppose the circle has rotated θ radians. Using the figure, the distance it has rolled from the origin is

$$|OT| = arc\ PT = r\theta$$

because P starts at the origin. Therefore, the center of the circle is $C(r\theta, r)$. Let the coordinates of P be (x, y). Then from the figure,

$$x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$$
$$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$$

Definition 11.1.1. Paremetric equations of the cycloid are

$$x = r(\theta - \sin\theta) \quad y = r(1 - \cos\theta)$$

11.2 Calculus with Parametric Curves

We will mainly solve problems involving tangents, area, arc length, and surface area.

Tangents

In the previous section, we saw that some curves defined by parametric equations $x = f(t)$ and $y = g(t)$ can also be expressed, by eliminating the parameter, in the form $y = F(x)$. If we substitute $x = f(t)$ and $y = g(t)$ in the equation $y = F(x)$, we get

$$g(t) = F(f(t))$$

If g, f, and F are differentiable, the Chain Rule gives

$$g'(t) = F'(f(t))f'(t) = F'(x)f'(t)$$

If $f'(t) \neq 0$, we can solve for $F'(x)$:

Definition 11.2.1. The slope of the tangent to the parametric curve $y = F(x)$ is $F'(x)$.

$$F'(x) = \frac{g'(t)}{f'(t)}$$

This enables us to find tangents to parametric curves without having to eliminate the parameter. We can rewrite the previous equation in an easily remembered form.

Definition 11.2.2. We can use this to find tangents to parametric curves without having to eliminate the parameter.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \text{ if } \frac{dx}{dt} \neq 0$$

The curve has a

- horizontal tangent when $\frac{dy}{dt} = 0$ (provided that $\frac{dx}{dt} \neq 0$)
- vertical tangent when $\frac{dx}{dt} = 0$ (provided that $\frac{dy}{dt} \neq 0$)

This is useful when sketching parametric curves.

Definition 11.2.3. We can also find $\frac{d^2y}{dx^2}$ by replacing y with $\frac{dy}{dx}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Proof. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ considering $y(t)$ and $g(t)$.

1.

$$\text{Chain rule: } \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \implies \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad (\implies \text{ means "implies"})$$

2.

$$\begin{aligned}
\text{Chain rule: } \frac{d}{dt}\left(\frac{dy}{dx}\right) &= \left(\frac{d}{dx}\frac{dy}{dx}\right)\frac{dx}{dt} = \frac{d^2y}{dx^2}\frac{dx}{dt} \\
\text{Substitute: } \frac{d}{dt}\left(\frac{dy}{dx}\right) &= \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) \\
\text{Quotient rule: } &= \frac{\frac{d^2y}{dt^2}\frac{d^x}{dt} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2}
\end{aligned}$$

Set equation from line 1 and line 3 equal and divide both sides by $\frac{dx}{dt}$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{\frac{d^2y}{dt^2}\frac{d^x}{dt} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2\left(\frac{dx}{dt}\right)} \\
&= \frac{\frac{d^2y}{dt^2}\frac{d^x}{dt} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}
\end{aligned}$$

Example 11.2.1. A curve C is defined by the parametric equations $x = t^2,\ y = t^3 - 3t$.

1. Show that C has two tangents at the point (3,0) and find their equations.
2. Find the points on C where the tangent is horizontal or vertical.
3. Determine where the curve is concave upward or downward.

Solution. A curve C is defined by the parametric equations $x = t^2,\ y = t^3 - 3t$.

1. Rewrite $y = t^3 - 3t = t(t^2 - 3) = 0$ when $t = 0$ or $t = \pm\sqrt{3}$. This indicates that C intersects itself at (3.0).

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$$
$$t = \pm\sqrt{3} \to dy/dx = \pm 6/(2\sqrt{3})$$

so the equations of the tangents at (3,0) are

$$y = \sqrt{3}(x - 3) \quad \textit{and} \quad y = -\sqrt{3}(x - 3)$$

2. C has a horizontal tangent when $dy/dx = 0$. In other words, when $dy/dt = 0$ and $dx/dt \neq 0$. $dy/dt = 3t^2 - 3 = 0$ when $t^2 = 1$ so $t = \pm 1$. This means there are horizontal tangents on C at (1,-2) and (1,2). C has a vertical tangent when $dx/dt = 2t = 0$, so $t = 0$. This means C has a vertical tangent at (0,0).

3. To determine concavity we calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{3}{2}\left(1+\frac{1}{t^2}\right)}{2t} = \frac{3(t^2+1)}{4t^3}$$

   The curve is concave upward when $t > 0$ and concave downward when $t < 0$.

Area

We already know that area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)dx$. We can apply this to parametric equations using the Substitution Rule for Definite Integrals.

Definition 11.2.4. If the curve C is given by parametric equations $x = f(t)$ and $y = g(t)$ and t increases from α to β,

$$A = \int_a^b y dx = \int_\alpha^\beta g(t)f'(t)dt$$

(Switch α to β if the point on C at β is more left than α.

Example 11.2.2. Find the area under one arch of the cycloid $x = r(\theta - \sin\theta),\ y = r(1-\cos\theta)$.

Solution. One arch of the cycloid is given by $0 \leq \theta \leq 2\pi$. Using the Substitution Rule with $y = r(1-\cos\theta)$ and $dx = r(1-\cos\theta)d\theta$, we have

$$\begin{aligned}
A = \int_0^{2\pi} y dx = A &= \int_0^{2\pi} r(1-\cos\theta)r(1-\cos\theta)d\theta \\
&= r^2\int_0^{2\pi}(1-\cos\theta)^2 d\theta = r^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)d\theta \\
&= r^2\int_0^{2\pi}\left[1-2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right]d\theta \\
&= r^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} \\
&= r^2\left(\frac{3}{2}\cdot 2\pi\right) = 3\pi r^2
\end{aligned}$$

Arc Length

We already know how to find length L of a curve C given in the form $y = F(x)$, $a \leq x \leq b$.

Definition 11.2.5. If F' is continuous, then

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If C can describe the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, where $dx/dt = f'(t) > 0$. Using the substitution rule, we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt} dt$$

Since $dx/dt > 0$, we have

Theorem 11.2.1. If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

This is consistent with the general formula $L = \int ds$ and $(ds^2) = (dx^2) + (dy^2)$.

Proof. Prove the length formula of a parametric curve

$$\begin{aligned}
\overrightarrow{ds} &= \overrightarrow{i}\, dx + \overrightarrow{j}\, dy \\
ds^2 &= \overrightarrow{ds} \cdot \overrightarrow{ds} = \left(\overrightarrow{i}\, dx + \overrightarrow{j}\, dy\right) \cdot \left(\overrightarrow{i}\, dx + \overrightarrow{j}\, dy\right) = dx^2 + dy^2 \\
ds &= \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
L &= \int_\alpha^\beta ds = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt
\end{aligned}$$

Example 11.2.3. Find the length of the unit circle as (x,y) moves both once and twice around the circle.

Solution. For one traversal around the unit circle,

$$x = \cos t \quad y = \sin t \quad 0 \leq t \leq 2\pi$$

so $dx/dt = -\sin t$ and $dy/dt = \cos t$

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt \\
&= \int_0^{2\pi} dt = 2\pi
\end{aligned}$$

For two traversals around the unit circle,

$$x = \sin 2t \quad y = \cos 2t \quad 0 \leq t \leq 2\pi$$

so $dx/dt = 2\cos 2t$ and $dy/dt = -2\sin 2t$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^{2\pi} \sqrt{4\cos^2 2t + 4\sin^2 2t}\ dt \quad = \int_0^{2\pi} 2\ dt = 4\pi$$

Surface Area

We can also adapt the surface area formula to a parametric curve.

Definition 11.2.6. If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, is rotated about the **x-axis**, where f',g' are continuous and $g(t) \geq 0$, the surface area is

$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

If the curve C is rotated about the **y-axis**, the surface area is

$$S = \int_\alpha^\beta 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The generic formulas $S = \int 2\pi y\ ds$ for rotation about the x-axis and $S = \int 2\pi x\ ds$ for rotation about the y-axis are still valid, but for parametric curves we use

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Example 11.2.4. Show that the surface area of a sphere of radius r is $4\pi r^2$

Solution. The sphere is obtained by rotating the semicircle

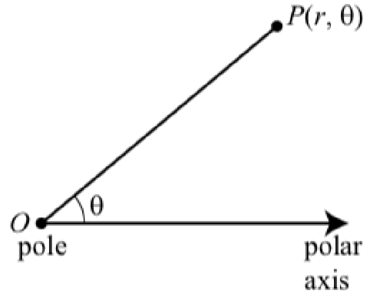
$$x = r\cos t \quad y = r\sin t \quad 0 \leq t \leq \pi$$

about the x-axis.

$$\begin{aligned} S &= \int_0^\pi 2\pi r \sin t \sqrt{(-r\sin t)^2 + (r\cos t)^2}\ dt \\ &= 2\pi \int_0^\pi r\sin t \sqrt{r^2(sin^2 t + \cos^2 t)}\ dt \\ &= 2\pi \int_0^\pi r\sin t \cdot r\ dt = 2\pi r^2 \int_0^\pi \sin t\ dt \\ &= 2\pi r^2(-\cos t)]_0^\pi = 4\pi r^2 \end{aligned}$$

11.3 Polar Coordinates

In addition to Cartisian coordinates, we can also use a **polar coordinate system**.



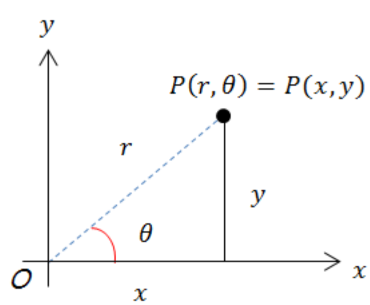
Point P is represented by the ordered pair (r, θ), where r is the distance to the point from the center and θ is the angle from the polar axis to the point.

The points (r, θ) and $(-r, \theta)$ are on the same line and have the same distance $|r|$ from the center but are on opposite sides of the center. Additionally, $(-r, \theta)$ and $(r, \theta + \pi)$ are also on the same line.

This means a complete counterclockwise rotation is given by an angle 2π, so (r, θ) is also represented by

$$(r, \theta + 2n\pi) \text{ and } (-r, \theta + (2n+1)\pi)$$

Relationship Between Cartesian and Polar Coordinates



$$\cos\theta = \tfrac{x}{r} \quad \sin\theta = \tfrac{y}{r}$$

$$x = r\cos\theta \quad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \quad \tan\theta = \tfrac{y}{x}$$

Example 11.3.1. Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

Solution.

$$
\begin{aligned}
&r = 2,\ \theta = \pi/3 \\
&\qquad x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1 \\
&\qquad y = r\sin\theta = 2\sin\frac{\pi}{3} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}
\end{aligned}
$$

So the point is $(1, \sqrt{3})$ in Cartesian coordinates.

Example 11.3.2. Represent the Cartesian coordinates $(1, -1)$ in polar coordinates.

Solution.

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2} \\
\tan\theta &= \frac{y}{x} = -1
\end{aligned}
$$

Since the point (1,-1) lies in the fourth quadrant, we can choose $\theta = -pi/4$ or $\theta = 7pi/4$. So the possible answers are either $(\sqrt{2}, -\pi/4$ or $(\sqrt{2}, 7\pi/4$.

Polar Curves

The **graph of a polar equation** $r = f(\theta)$, or $F(r, \theta) = 0$, consists of all of the points where (r, θ) satisfies the equation.

Tangents to Polar Curves

To find a tangent line to a polar curce $r = f(\theta)$, we regard θ as a parameter and write the parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \quad y = r\sin\theta = f(\theta)\sin\theta$$

So

Definition 11.3.1.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dy}{d\theta} sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

- horizontal tangent when $\frac{dy}{d\theta} = 0$ (provided that $\frac{dx}{d\theta} \neq 0$)
- vertical tangent when $\frac{dx}{d\theta} = 0$ (provided that $\frac{dy}{d\theta} \neq 0$)

NOTE tangent lines at the pole have r=0 and the slope of the tangent simplifies to

$$\frac{dy}{dx} = \tan\theta \text{ if } \frac{dr}{d\theta} \neq 0$$

Example 11.3.3. For the cardiod $r = 1 + \sin\theta$, find the slope of the tangent line when r=3

Solution.

$$\begin{aligned} r &= 1 + \sin\theta \\ \frac{dy}{dx} &= \frac{\frac{dy}{d\theta} sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta} \\ &= \frac{\cos\theta(1+2\sin\theta)}{1 - 2\sin^2\theta - \sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-\sin\theta)} \end{aligned}$$

The slope of the tangent where $\theta = \pi/3$ is

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{\theta=\pi/3} &= \frac{\cos(\pi/3)(1+2\sin(\pi/3))}{(1+\sin(\pi/3))(1-\sin(\pi/3))} \\ &= \frac{\frac{1}{2}(1+\sqrt{3})}{(1+\sqrt{3}/2)(1-\sqrt{3})} = \frac{1+\sqrt{3}}{(2+\sqrt{3})(1-\sqrt{3})} \\ &= \frac{1+\sqrt{3}}{-1-\sqrt{3}} = -1 \end{aligned}$$

NOTE Instead of memorizing the equation, we can instead use the same method we used to derive it.

$$\begin{aligned} x &= r\cos\theta = (1+\sin\theta)\cos\theta = \cos\theta + \frac{1}{2}\sin 2\theta \\ y &= r\sin\theta = (1+\sin\theta)\sin\theta = \sin\theta + \sin^2\theta \\ \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} = \frac{cos\theta + 2\sin\theta\cos\theta}{-\sin\theta + \cos 2\theta} = \frac{\cos\theta + \sin 2\theta}{-\sin\theta + \cos 2\theta} \end{aligned}$$

This is equivalent to the previous equation.

11.4 Areas and Lengths in Polar Coordinates

Area

We can determine the formula for the area of a region whose boundary is given by a polar equation by taking the limit of a Riemann Sum starting with the formula for the area of a sector of a circle $A = \frac{1}{2}r^2\theta$.

Definition 11.4.1. The formula for the area A of the polar region $\mathcal{R}$ is

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2\ d\theta = \int_a^b \frac{1}{2}r^2 d\theta$$

with the understanding that $r = f(\theta)$.

Example 11.4.1. Find the area enclosed by one loop of the four-leaved rose $r = 2\cos 2\theta$.

Solution. The right loop rotates from $\theta = -\pi/4$ to $\theta = \pi/4$.

$$\begin{aligned}
A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta \; d\theta \\
&= \int_0^{\pi/4} \cos^2 2\theta \; d\theta = \int_0^{\pi/4} \frac{1}{2}(1 + \cos 4\theta) \; d\theta \\
&= \frac{1}{2}[\theta + \frac{1}{4}\sin 4\theta] = \pi/8
\end{aligned}$$

We can also adapt the formula to find the area of a region bounded by two polar curves.

Definition 11.4.2. Let $\mathcal{R}$ be a region that is bounded by curves with polar equations $r = f(\theta)$, $r = g(\theta)$, $\theta = a$, and $\theta = b$, where $f(\theta) \geq g(\theta) \geq 0$ and $0 < b - a \leq 2\pi$. The area A of $\mathcal{R}$ is found by subtracting the area inside $r = g(\theta)$ from the area inside $r = f(\theta)$, so

$$\begin{aligned}
A &= \int_a^b \frac{1}{2}[f(\theta)]^2 \; d\theta - \int_a^b \frac{1}{2}[g(\theta)]^2 \; d\theta \\
&= \int_a^b \frac{1}{2}([f(\theta)]^2 - [g(\theta)]^2) \; d\theta
\end{aligned}$$

Arc Length

To find the length of a polar curve $r = f(\theta)$, $a \leq \theta \leq b$, we regard θ as a parameter and write the parametric equations of the curve as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Using the projecut Rule and differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

so, using $\cos^2\theta + \sin^2\theta = 1$, we have

$$\begin{aligned}
\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\
&+ \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta \\
&= \left(\frac{dr}{d\theta}\right)^2 + r^2
\end{aligned}$$

Assuming that f' is continuous, we can use the theorem from 11.2 about the arc length of a curve defined by parametric equations to write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Definition 11.4.3. The length of a curve with polar equation $r = f(\theta)$, $a \le \theta \le b$, is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

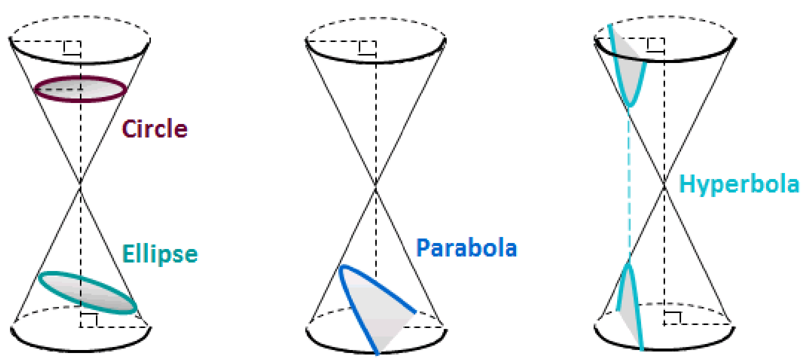
Example 11.4.2. Find the arc length of the cardiod $r = 1 + \sin\theta$.

Solution. The full length of the cardiod is given by the parameter interval $0 \le \theta \le 2\pi$.

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta$$
$$= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta = 8 \ (\textit{by rationalizing the integrand by } \sqrt{2 - 2\sin\theta})$$

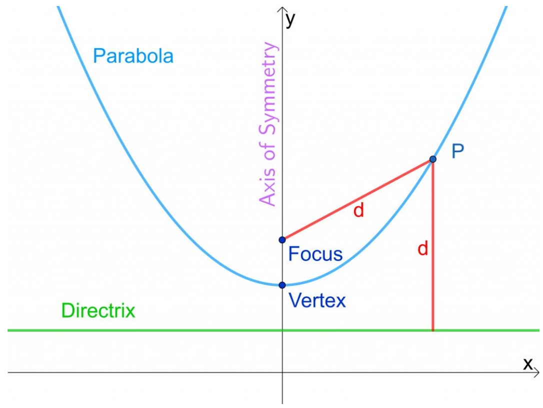
11.5 Conic Sections

Parabolas, ellipses, and hyperbolas are called **conic sections**, or **conics**, because they result from intersecting a cone with a plane.



Parabolas

A **parabola** is the set of points in a plane that are equidistant from a fixed point F (called the **focus**) and a fixed line (called the **directrix**). The halfway point between the focus and directrix is on the parabola and is called the **vertex**. The line through the focus and the vertex and perpendicular to the directrix is the **axis** of the parabola.



As seen in the figure, the focus is always inside the region of the parabola and the directrix is the same distance away on the opposite side.

Definition 11.5.1. An equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is

$$x^2 = 4py$$

. If we set $a = \frac{1}{4p}$, then the standard equation of a parabola is $y = ax^2$. This opens upward if $p > 0$ and downard if $p < 0$, and is symmetric with respect to the y-axis.

Definition 11.5.2. If we switch x and y, we get

$$y^2 = 4px$$

(reflection about the diagonal line y=x). This parabola opens to the right if $p > 0$ and to the left if $p < 0$.

Definition 11.5.3. The vertex form of a parabola is

$$y = a(x - h)^2 + k$$

where (h, k) is the vertex of the parabola and $x = h$ is the axis of symmetry. We can also switch x and y to get the vertex form of the rotated parabola.

Example 11.5.1. Find the focus and directrix of the parabola $y^2 + 10x = 0$.

Solution. We rewrite the equation as $y^2 = -10x$. We know $y^2 = 4px$, so $4px = -10x$ and $p = -\frac{5}{2}$. Thus, the focus is $(p, 0) = -\frac{5}{2}, 0)$ and the directrix is $x = \frac{5}{2}$.

Ellipses

An **ellipse** is the set of points in a plane surrounding two fixed focal points F_1 and F_2 such that the <u>sum</u> of the two distances to the focal points is a constant. Imagine tracing a line along the furthest path of a string stretched across two different points.

Definition 11.5.4. The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a \geq b \geq 0$$

has foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$ **(lies on x-axis)**.

The **vertices** are on the **major axis**, where a is the distance to the center of the ellipse from each vertex. This distance is greater than the distance from a **co-vertex** to the center of the ellipse, b. The co-vertices lie on the **minor axis**. Because the sum of the two distances from a point on the ellipse to the foci is a constant, the distance from a co-vertex to a focal point is also a. If the foci coincide, then $c = 0$, so $a = b$ and the ellipse becomes a circle with radius $r = a = b$.

Definition 11.5.5. The ellipse

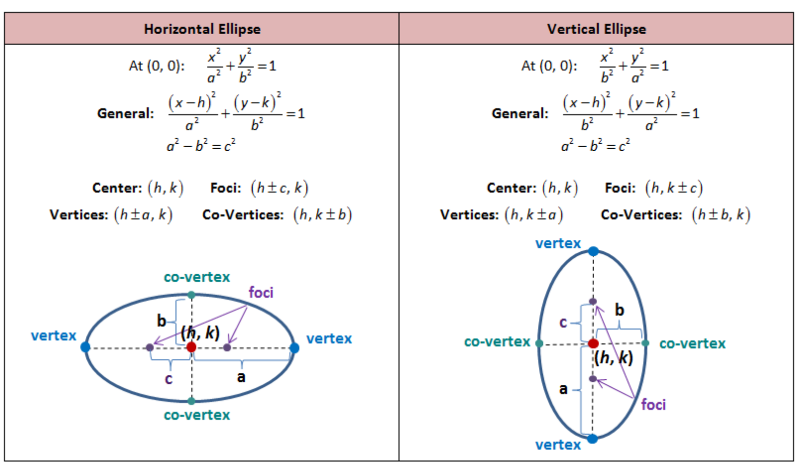
$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad a \geq b \geq 0$$

has foci $(0, \pm c)$, where $c^2 = a^2 - b^2$, and vertices $(0, \pm a)$ **(lies on y-axis)**.

Definition 11.5.6. The general form of a horizontal ellipse is

$$\frac{(x-h)^2}{b^2} + \frac{(y-h)^2}{a^2} = 1$$

where (h, k) is the center of the ellipse. The same transformation can be done to the standard form of a vertical ellipse.



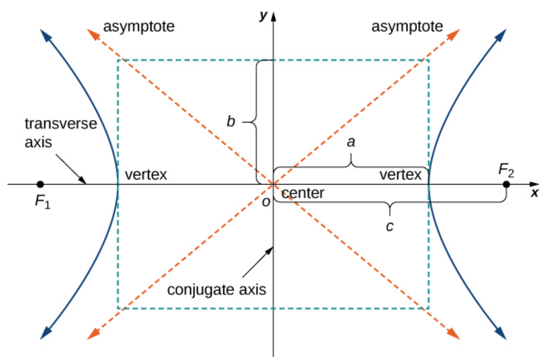
Example 11.5.2. Find an equation of the ellipse with foci $(0, \pm 2)$ and vertices $(0, \pm 3)$.

Solution. This equation represents a vertical ellipse because the foci and vertices lie on the y-axis. The distance from a focal point to the center is $c = 2$ and the distance from a vertex to the center is $a = 3$. Then we obtain $b^2 = a^2 + c^2 = 9 - 4 = 5$, so the equation of the ellipse is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = \frac{x^2}{5} + \frac{y^2}{9} = 1$$

Hyperbolas

An **ellipse** is the set of points in a plane surrounding two fixed focal points F_1 and F_2 such that the difference of the two distances to the focal points is a constant. The **transverse axis** is the axis of a hyperbola that passes through the two foci. The **conjugate axis** is perpendicular to the transverse axis and passes through the center of the hyperbola.



Definition 11.5.7. The hyperbola along a horizontal transverse axis

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has foci $(\pm c, 0)$, where $c^2 = a^2 + b^2$, vertices $(\pm a, 0)$ **(lies on x-axis)**, and asympotes $y = \pm \frac{b}{a} x$.

Definition 11.5.8. The hyperbola along a vertical transverse axis

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

has foci $(0, \pm c)$, where $c^2 = a^2 + b^2$, vertices $(0, \pm a)$ **(lies on y-axis)**, and asympotes $y = \pm \frac{a}{b} x$.

Definition 11.5.9. The general form of a hyperbola along a horizontal transverse axis is

$$\frac{(x-h)^2}{b^2} - \frac{(y-h)^2}{a^2} = 1$$

where (h, k) is the center of the ellipse. The same transformation can be done to the standard form a hyperbola along a vertical transverse axis.

Example 11.5.3. Find the foci and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$.

Solution. If we divide both sides of the equation by 144, it becomes

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

which is a hyperbola along a horizontal transverse axis. Therefore, we get $a = 4$ and $b = 3$. Since $c^2 = a^2 + b^2 = 16 + 9 = 25$, $c = 5$. The foci are $(\pm 5, 0)$, and the asymptotes are $y = \pm\frac{3}{4}x$.

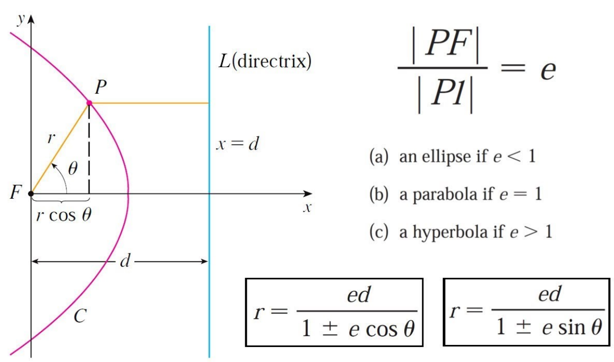
11.6 Conic Sections in Polar Coordinates

Theorem 11.6.1. Let F be a fixed point (called the **focus**) and l be a fixed line (called the **directrix**). Let e be a fixed positive number (called the **eccentricity**). The set of all points P in the plane such that

$$\frac{|PF|}{|Pl|} = e \quad \text{(the ratio of the distance from } F \text{ to the distance from } l \text{ is the constant e)}$$

is a conic section. The conic is

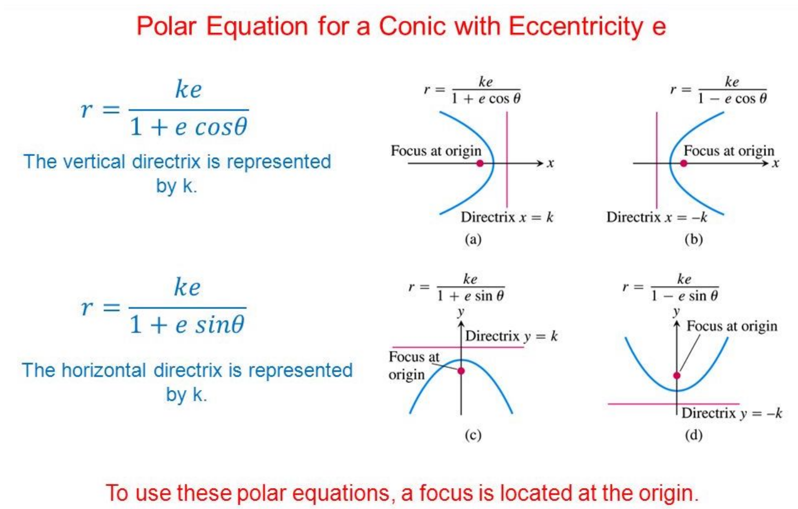
1. an ellipse if $e < 1$
2. a parabola if $e = 1$
3. a hyperbola if $e > 1$



Theorem 11.6.2. A polar equation of the form

$$r = \frac{ed}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic section with eccentricity e and distance d from the center to the directrix, with the focus at the origin. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola is $e > 1$.



Use "$\cos\theta$" when the conic section opens rightward or leftward, and use "$\sin\theta$" when the conic section opens upward or downward. Use "+" if the conic section opens leftward or downward, and use "−" if the conic section opens righttward or upward.

Example 11.6.1. Find a polar equation for a parabola that has its focus at the origin and whose directrix is the line $y = -6$.

Solution. The eccentricity $e = 1$ because the conic section is a parabola, and the distance from the center to the directrix is $d = 6$. The directrix is on the y-axis and is underneath the center, so the parabola opens upward. Therefore, we use the "$\sin\theta$" equation and use "−" in the denominator. The polar equation of the parabola is

$$r = \frac{6}{1 - \sin\theta}$$

.

Example 11.6.2. A conic is given by the polar equation

$$r = \frac{10}{3 - 2\cos\theta}$$

Find the eccentricity, identify the conic, and locate the directrix.

Solution. Divide the numerator and denominator by 3 to get

$$r = \frac{\frac{10}{3}}{1 - \frac{2}{3}\cos\theta}$$

This represents an ellipses with eccentricity $e = \frac{2}{3}$. Since $ed = \frac{10}{3}$,

$$d = \frac{\frac{10}{3}}{e} = \frac{\frac{10}{3}}{\frac{2}{3}} = 5$$

so the directrix has Cartesian equation $x = -5$. When $\theta = 0,\ r = 10$; when $\theta = \pi,\ r = 2$, so the vertices have polar coordinates $(10, 0)$, and $(2, \pi)$.