12.5 Alternating Series

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$
$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$$

We see from these examples that the nth term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n \qquad \text{or} a_n = (-1)^n b_n$$

where b_n is a positive number. (In fact, $b_n = |a_n|$).

Definition 12.5.1 (**The Alternating Series Test**)**.** If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad (b_n > 0)$$

satisfies

$$\text{(i)} \;\; b_{n+1} \leq b_n \quad \text{for all } n$$
$$\text{(ii)} \;\; \lim_{n\to\infty} b_n = 0$$

then the series is convergent.

"The alternating series converges if its terms decrease toward 0 in absolute value".

Proof. We consider the even and odd partial sums separately.

We first consider *even* partial sums:

$$s_2 = b_1 - b_2 \geq 0 \qquad \text{since } b_2 \leq b_1$$
$$s_4 = s_2 + (b_3 - b_4) \geq s_2 \qquad \text{since } b_4 \leq b_3$$

In general $\quad s_{2n} = s_{2n-2} + (b_{2n-1} - b_{2n}) \geq s_{2n-2} \qquad \text{since } b_{2n} \leq b_{2n-1}$

Thus $\quad 0 \leq s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_{2n} \leq \cdots$

But we can also write

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}$$

Every term in brackets is positive, so $s_{2n} \leq b_1$ for all n. Therefore, the sequence $\{s_{2n}\}$ of even partial sums is increasing and bounded above. It is therefore convergent by the Monotonic Sequence Theorem. Let's call this limit s:

$$\lim_{n\to\infty} s_{2n} = s$$

Next we compute the limit of the *odd* partial sums:

$$\begin{aligned}
\lim_{n\to\infty} s_{2n+1} &= \lim_{n\to\infty} (s_{2n} + b_{2n+1}) \\
&= \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} \\
&= s + 0 \\
&= s
\end{aligned}$$

Since both partial sums converge to s, we have $\lim_{n\to\infty} s_{2n} = s$ and so the series is convergent.

Example 12.5.1. The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies

(i) $b_{n+1} \leq b_n$ because $\dfrac{1}{n+1} < \dfrac{1}{n}$

(ii) $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

so the series is convergent by the Alternating Series Test

Example 12.5.2. The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is alternating but

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \lim_{n\to\infty} \frac{3}{4 - \frac{1}{n}} = \frac{3}{4} \neq 0$$

Instead, we look at the nth term of the series:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n 3n}{4n-1}$$

The limit does not exist, so the series diverges by the Test for Divergence.

Estimating Sums

Theorem 12.5.1 (Alternating Series Estimation Theorem). If $s = \sum(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies

$$\text{(i) } 0 \leq b_{n+1} \leq b_n \quad \text{and} \quad \lim_{n\to\infty} b_n = 0$$

then
$$|R_n| = |s - s_n| \leq b_{n+1}$$

Proof. We know from the proof of the Alternating Series Test that s lies between any two consecutive partial sums s_n and s_{n+1}. It follows that

$$|s - s_n| \leq |s_{n+1} - s_n| = b_{n+1}$$

"The size of the error is smaller than b_{n+1}, which is the absolute value of the first neglected term" (valid only for alternating series that satisfy the Alternating Series Estimation, not other theorems.

Example 12.5.3. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places (By definition, $0! = 1$).

Solution. We first observe that the series is convergent by the Alternating Series Test because

$$\text{(i)} \quad \frac{1}{(n+1)!} = \frac{1}{n!(n+1)} \leq \frac{1}{n!}$$

$$\text{(ii)} \quad 0 < \frac{1}{n!} < \frac{1}{n} \to 0 \quad \text{so } \frac{1}{n!} \to 0 \text{ as } n \to \infty$$

To get a feel for how many terms we need to use in our approximation, let's write out the first few terms of the series:

$$\begin{aligned} s &= \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots \\ &= 1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120} + \tfrac{1}{720} - \tfrac{1}{5040} + \cdots \end{aligned}$$

Notice that
$$b_7 = \tfrac{1}{5040} < \tfrac{1}{5000} = 0.0002$$

and
$$s_n = 1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120} + \tfrac{1}{720} \approx 0.369056$$

By the Alternating Series Estimation Theorem, we know that

$$|s - s_6| \leq b_7 < 0.0002$$

This error of less than 0.0002 does not affect the third decimal place, so we have

$$s \approx 0.368$$

correct to three decimal places.