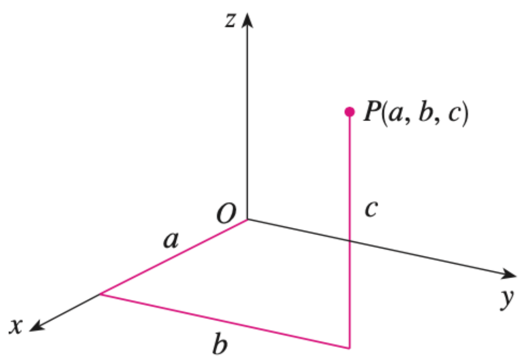
13.1 Three-Dimensional Coordinate Systems

An ordered pair $(a,\ b)$ of real numbers is used to represent a point in a plane, which is two-dimensional. To locate a point in space, which is three-dimensional, we use an ordered triple $(a,\ b,\ c)$ of real numbers.



To represent points in space we draw three perpendicular lines, called the **coordinate axes** and labeled the x-axis, y-axis, and z-axis, through a fixed point O (the origin). The three coordinate axes determine the three **coordinate planes**: the xy-plane contains the x- and y-axes; the yz-plane contains the y- and z-axes; the xz-plane contains the x- and z-axes. The three coordinate planes divided space into eight parts called **octants**. The **first octant** is the side we typically see and represents the positive axes.

If P is any point in space, let a be the x-coordinate, let b be the y-coordinate, and let c be the z-coordinate. We represent point P by the ordered triple $(a,\ b,\ c)$. If we drop a perpendicular from P to the xy-plane, we get a point Q with coordinates $(a,\ b,\ 0)$ called the **projection** of P on the xy-plane. Similarly, $R(0,\ b,\ c)$ is the projection of P on the yz-plane and $S(a,\ 0,\ c)$ is the projection of P on the xz-plane.

The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) | x, y, z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$. This is called a **three-dimensional rectangular coordinate system**.

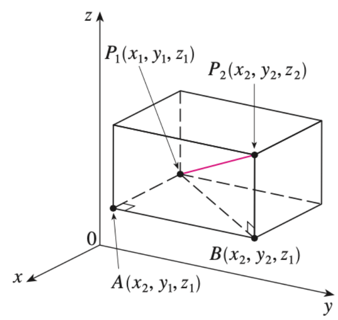
In two-dimensional analytic geomemtry, the graph of an equation involving x and y is a **curve** in $\mathbb{R}^2$. In three-dimensional analytic geomemtry, an equation in x, y, and z is a **surface** in $\mathbb{R}^3$.

The formula for distance between two points in a plane is easily extended to a formula for three dimensions.

Definition 13.1.1 (**Distance Formula in Three Dimensions**)**.** The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| - \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Proof. Construct a rectangular box where P_1 and P_2 are opposite vertices and the sides of the box is parallel to the coordinate planes.



If $A(x_2, y_1, z_1)$ and $B(x_2, y_2, z_1)$ are the vertices of the box indicated in the figure, then

$$|P_1A| = |x_2 - x_1| \qquad |AB| = |y_2 - y_1| \qquad |BP_2| = |z_2 - z_1|$$

Because triangles P_1BP_2 and P_1AB are both right triangles, two applications of the Pythagorean Theorem give

$$\begin{aligned} |P_1P_2|^2 &= |P_1B|^2 + |BP_2|^2 \\ |P_1B|^2 &= |P_1A|^2 + |AB|^2 \end{aligned}$$

Combine these equations through substitution to get

$$\begin{aligned} |P_1P_2|^2 &= |P_1A|^2 + |AB|^2 + |BP_2|^2 \\ &= |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \end{aligned}$$

Example 13.1.1. The distance from point $P(2, -1, 7)$ to the point $Q(1, -3, 5)$ is

$$|PQ| = \sqrt{(1-2)^2 + (-3-1)^2 + (5-7)^2} = \sqrt{1+4+4} = 3$$

Just as the two-dimensional distance formula can be used to define the equation of a circle, the three-dimensional distance formula can be used to define the equaition of a sphere.

Definition 13.1.2 (Equation of a Sphere). An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

In particular, if the center is the origin O, then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$

Proof. By definition, a sphere is the set of all points $P(x, y, z)$ whos distance from center $C(h, k, l)$ is radius r. Thus, P is on the sphere if and only if $|PC| = r$. Squaring both sides, we have $|PC|^2 = r^2$, or

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

Example 13.1.2. Show that $x^2+y^2+z^2+4x-6y+2z+6=0$ is the equation of a sphere, and find its center and radius.

Solution. We can rewrite the given equation in the form of an equation of a sphere by completing the square:

$$(x^2+4x+4)+(y^2-6y+9)(z^2+2z+1) = -6+4+9+1$$
$$(x+2)^2+(y-3)^2+(z+1)^2 = 8$$

Comparing this equation with the standard form, we see that it is the equation of a sphere with center $(-2, 3, -1)$ and radius $\sqrt{8} = 2\sqrt{2}$.

Example 13.1.3. What region in $\mathbb{R}^3$ is represented by $1 \leq x^2 + y^2 + z^2 \leq 4,\ z \leq 0$?

Solution. Rewrite the inequality as $1 \leq \sqrt{x^2+y^2+z^2} \leq 2$, which represents the points whose distance from the origin is at least 1 and at most 2. Since $z \leq 0$, the points lie on or below the xy-plane. The inequalities represent the lower hemisphere between the radii 1 and 2.