12.8 Power Series

Definition 12.8.1. A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the **coefficients** of the series.

For each fixed x, the series is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for other values of x. The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

whose domain is the set of all x for which the series converges. Notice that f resembles a polynomial. The only difference is that f has infinitely many terms.

For instance, if we take $c_n = 1$ for all n, the power series becomes the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$.

Definition 12.8.2. A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

is called a **power series in** $(x-a)$ or a **power series centered at** a or a **power series about** a.

Notice that when $x = a$ all of the terms are 0 for $n \geq 1$, so the power series always converges when $x = a$.

Example 12.8.1. For what values of x is the series $\displaystyle\sum_{n=0}^{\infty} n! x^n$ convergent?

Solution. We use the Ratio Test. If we let a_n denote the nth term of the series, then $a_n = n!x^n$. If $x \neq 0$, we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| = \lim_{n\to\infty} (n+1)|x| = \infty$$

By the Ratio Test, the series diverges when $x \neq 0$, so it converges only when $x = 0$.

Example 12.8.2. For what values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

Solution. Let $a_n = (x-3)^n/n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right|$$
$$= \frac{1}{1+\frac{1}{n}}|x-3| \to |x-3| \quad \text{as } n \to \infty$$

By the Ratio Test, the given series is absolutely convergent, and therefore convergent, when $|x-3| < 1$ and divergent when $|x-3| > 1$. We rewrite the inequality as

$$|x-3| < 1 \quad \Longleftrightarrow \quad -1 < x-3 < 1 \quad \Longleftrightarrow \quad 2 < x < 4$$

Now we know the series converges when $2 < x < 4$ and diverges when $x < 2$ or $x > 4$.

The Ratio Test gives no information when $|x-3| = 1$, so we must consider $x = 2$ and $x = 4$ separately.

(a) If we put $x = 4$ in the series, it becomes the harmonic series $\sum 1/n$, which is divergent.

(b) If $x = 2$, the series is $\sum(-1)^n/n$, which converges by the Alternating Series Test.

We can summarize our results by concluding that the power series converges for $2 \leq x < 4$.

Theorem 12.8.1. For a given power series $\sum_{n=0}^{\infty}$ there are only three possibilities:

1. The series converges only when $x = a$.
2. The series converges for all x.
3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

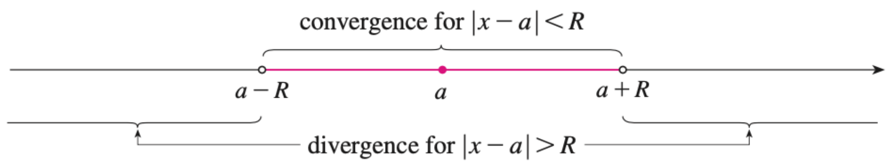
The proof of this theorem is at the end of this chapter because this theorem is more relevant than the proof.

The number R in case 3 is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case 1 and $R = \infty$ in case (ii).

The **interval of convergence** of a power series is the interval that consists of just a single point a. in case 2, the interval is $(-\infty, \infty)$. In case 3, note that the inequality $|x-a| < R$ can be rewritten as $a - R < x < a + R$. When x is an *endpoint* of the interval ($x = a \pm R$), anything can happen—the series might converge at one or both endpoints, or it might diverge at both endpoints.

Thus, in case 3 there are four possibilities for the interval of convergence:

$$(a-R, a+R) \qquad (a-R, a+] \qquad [a-R, a+R) \qquad [a-R, a+R]$$



We summarize the radius and interval for convergence for each of the examples in this section.

	Series	Radius of Convergence	Interval of Convergence
Geometric Series	$\sum_{n=0}^{\infty} x^n$	$R=1$	$(-1,1)$
Example 12.8.1	$\sum_{n=0}^{\infty} n!x^n$	$R=0$	$\{0\}$
Example 12.8.2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R=1$	$[2,4)$

In general, the Ratio Test (or sometimes the Root Test) should always be used to determine the radius of convergence R. The Ratio and Root Tests always fail when x is an endpoint of the interval of convergence, so the endpoints must be checked with some other test.

Example 12.8.3. Find the radius and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

Solution. Let $a_n = (-3)^n x^n / \sqrt{n+1}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n}\right| = \left|-3x\sqrt{\frac{n+1}{n+2}}\right|$$
$$= 3\sqrt{\frac{1+(1/n)}{1+(2/n)}}|x| \to 3|x| \quad \text{as } n \to \infty$$

By the Ratio Test, the given series converges if $3|x| < 1$ and diverges if $3|x| > 1$. Thus, it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$, meaning that the radius of convergences is $R = \frac{1}{3}$.

We know the series converges in the interval $(-\frac{1}{3}, \frac{1}{3})$, but we must now test for convergence at the endpoints of this interval.

(a) If $x = -\frac{1}{3}$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$ which diverges (using the Integral Test or simply observing that it is a p-series with $p = \frac{1}{2} < 1$).

(b) If $x = \frac{1}{3}$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ which converges by the Alternating Series Test. Therefore, the given power series converges when $-\frac{1}{3} < x \le \frac{1}{3}$, so the interval of convergence is $(-\frac{1}{3}, \frac{1}{3}]$.

Proof

To prove the theorem that is found earlier in this section, we first need to prove 2 theorems.

Theorem 12.8.2.

1. If a power series $\sum c_n x^n$ converges when $x = b$ (where $b \neq 0$), then it converges whenever $|x| < |b|$.
2. If a power series $\sum c_n x^n$ diverges when $x = d$ (where $d \neq 0$), then it converges whenever $|x| > |d|$.

Proof.

1. Suppose that $\sum c_n x^n$ converges. Then, we know $\lim_{n\to\infty} c_n b^n = 0$. According to the definition of a limit of a sequence with $\varepsilon = 1$, there is a positive integer N such that $|c_n b^n| < 1$ whenever $n \ge N$,

$$|c_n x^n| = \left|\frac{c_n b^n x^n}{b^n}\right| = |c_n b^n| \left|\frac{x}{b}\right|^n < \left|\frac{x}{b}\right|^n$$

If $|x| < |b|$, then $|x/b| < 1$, so $\sum |x/b|^n$ is a convergent geometric series. Therefore, by the Comparison Test, the series $\sum_{n=N}^{\infty} |c_n b^n|$ is convergent. Thus, the series $\sum c_n b^n$ is absolutely convergent and therefore convergent.

2. Suppose that $\sum c_n d^n$ diverges. If x is any number such that $|x| > |d|$, then $\sum c_n x^n$ cannot converge because, by part 1, the convergence of $\sum c_n x^n$ would imply the convergence of $\sum c_n d^n$. Therefore $\sum c_n x^n$ diverges whenever $|x| > |d|$.

Theorem 12.8.3. For a power series $\sum c_n x^n$ there are only three possibilities

1. The series converges only when $x = 0$.

2. The series converges for all x.

3. There is a positive number R such that the series converges if $|x| < R$ and diverges if $|x| > R$.

Proof. We use the preceding theorem to prove this theorem. The symbol $\in$ means "is an element of" or "in".

Suppose that neither case 1 nor case 2 is true. Then there are nonzero numbers b and d such that $\sum c_n x^n$ converges for $x = b$ and diverges for $x = d$. Therefore, the set $S = \{x | \sum c_n x^n \text{ converges}\}$ is not empty. By the preceding theorem, the series diverges if $|x| > |d|$, so $|x| \geq |d|$ for all $x \in S$. This says that $|d|$ is an upper bound for the set S. Thus, by the Completeness Axiom (see Section 12.1), S has a least upper bound R. If $|x| > R$, then $x \notin S$, so $\sum c_n x^n$ diverges. If $|x| < R$, then $|x|$ is not an upper bound for S and so there exists $b \in S$ such that $b > |x|$. Since $b \in S$, $\sum c_n b^n$ converges, so by the preceding theorem $\sum c_n x^n$ converges.

Now we are ready for the proof of the main theorem found earlier in the section. We use the preceding theorem to prove it.

Proof. If we can make the change of variable $u = x - a$, then the power series becomes $\sum c_n u^n$ and we can apply the preceding theorem to this series. In case 3 we have convergence for $|u| < R$ and divergence for $|u| > R$. Thus, we have convergence for $|x - a| < R$ and divergence for $|x - a| > R$.