12.11 The Binomial Series

The **Binomial Theorem** states that if a and b are any real numbers and k is a positive integer, then

$$\begin{aligned}(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{2!}a^{k-2}b^2 + \frac{k(k-1)(k-2)}{3!}a^{k-3}b^3 \\ + \cdots + \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}a^{k-n}b^n \\ + \cdots + kab^{k-1} + b^k\end{aligned}$$

The traditional notation for the binomial coefficients is

$$\binom{k}{0} = 1 \qquad \binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} \qquad n = 1, 2, \ldots, k$$

which enables us to write the Binomial Theorem in the abbreviated form

$$(a+b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n}b^n$$

Definition 12.11.1 (The Binomial Series). If k is any real number and $|x| < 1$, then

$$\begin{aligned}(1+x)^k &= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \\ &= \sum_{n=0}^{\infty} \binom{k}{n} x^n\end{aligned}$$

where $\binom{k}{n} = \dfrac{k(k-1)\cdots(k-n+1)}{n!}$ $(n \geq 1)$ and $\binom{k}{0} = 1$

Proof. Newton extended the Binomial Theorem to the case in which k is no longer a positive integer. In particular, if we put $a = 1$ and $b = k$, we get

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

To find this series we compute the Maclaurin series of $(1+x)^k$ in the usual way.

$$\begin{aligned}
f(x) &= (1+x)^k & f(0) &= 1 \\
f'(x) &= k(1+x)^{k-1} & f'(0) &= k \\
f''(x) &= k(k-1)(1+x)^{k-2} & f''(0) &= k(k-1) \\
f'''(x) &= k(k-1)(k-2)(1+x)^{k-3} & f'''(0) &= k(k-1)(k-2) \\
&\vdots & &\vdots \\
f^{(n)}(x) &= k(k-1)\cdots(k-n+1)(1+x)^{k-n} & f^{(n)}(0) &= k(k-1)\cdots(k-n+1)
\end{aligned}$$

Therefore the Maclaurin series of $f(x) = (1+x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

Now we use the Ratio Test to test the binomial series for convergence. If the nth term is a_n, then

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k-1)\cdots(k-n+1)x^n}\right| \\
&= \frac{|k-n|}{x+1}|x| = \frac{\left|1-\dfrac{k}{n}\right|}{1+\dfrac{1}{n}}|x| \to |x| \quad \text{as } n \to \infty
\end{aligned}$$

The binomial series converges if $|x| < 1$ and diverges if $|x| > 1$ by the Ratio Test.

Example 12.11.1. Expand $\dfrac{1}{(1+x)^2}$ as a power series.

Solution. Use the binomial series with $k = 2$. The binomial coefficient is

$$\begin{aligned}
\binom{-2}{n} &= \frac{(-2)(-3)(-4)\cdots(-2-n+1)}{n!} \\
&= \frac{(-1)^n 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n(n+1)}{n!} = (-1)^n(n+1)
\end{aligned}$$

and so, when $|x| < 1$,

$$\begin{aligned}
\frac{1}{(1+x)^2} &= (1+x)^{-2} = \sum_{n=0}^{\infty} \binom{-2}{n} x^n \\
&= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n = 1 - 2x + 3x^2 - 4x^3 + \cdots
\end{aligned}$$