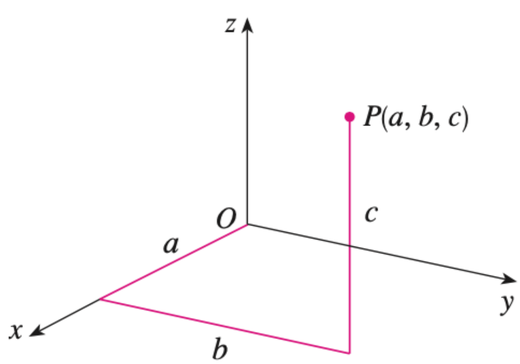
Chapter 13

Vectors and the Geometry of Space

13.1 Three-Dimensional Coordinate Systems

An ordered pair $(a,\ b)$ of real numbers is used to represent a point in a plane, which is two-dimensional. To locate a point in space, which is three-dimensional, we use an ordered triple $(a,\ b,\ c)$ of real numbers.



To represent points in space we draw three perpendicular lines, called the **coordinate axes** and labeled the x-axis, y-axis, and z-axis, through a fixed point O (the origin). The three coordinate axes determine the three **coordinate planes**: the xy-plane contains the x- and y-axes; the yz-plane contains the y- and z-axes; the xz-plane contains the x- and z-axes. The three coordinate planes divided space into eight parts called **octants**. The **first octant** is the side we typically see and represents the positive axes.

If P is any point in space, let a be the x-coordinate, let b be the y-coordinate, and let c be the z-coordinate. We represent point P by the ordered triple

$(a,\ b,\ c)$. If we drop a perpendicular from P to the xy-plane, we get a point Q with coordinates $(a,\ b,\ 0)$ called the **projection** of P on the xy-plane. Similarly, $R(0,\ b,\ c)$ is the projection of P on the yz-plane and $S(a,\ 0,\ c)$ is the projection of P on the xz-plane.

The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) | x, y, z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$. This is called a **three-dimensional rectangular coordinate system**.

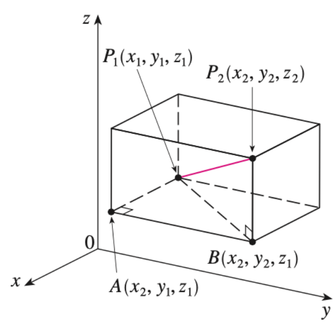
In two-dimensional analytic geomemtry, the graph of an equation involving x and y is a **curve** in $\mathbb{R}^2$. In three-dimensional analytic geomemtry, an equation in x, y, and z is a **surface** in $\mathbb{R}^3$.

The formula for distance between two points in a plane is easily extended to a formula for three dimensions.

Definition 13.1.1 (**Distance Formula in Three Dimensions**)**.** The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| - \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Proof. Construct a rectangular box where P_1 and P_2 are opposite vertices and the sides of the box is parallel to the coordinate planes.



If $A(x_2, y_1, z_1)$ and $B(x_2, y_2, z_1)$ are the vertices of the box indicated in the figure, then

$$|P_1A| = |x_2 - x_1| \qquad |AB| = |y_2 - y_1| \qquad |BP_2| = |z_2 - z_1|$$

Because triangles P_1BP_2 and P_1AB are both right triangles, two applications of the Pythagorean Theorem give

$$|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2$$
$$|P_1B|^2 = |P_1A|^2 + |AB|^2$$

Combine these equations through substitution to get

$$\begin{aligned}|P_1P_2|^2 &= |P_1A|^2 + |AB|^2 + |BP_2|^2\\ &= |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2\\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\end{aligned}$$

Example 13.1.1. The distance from point $P(2, -1, 7)$ to the point $Q(1, -3, 5)$ is

$$|PQ| = \sqrt{(1-2)^2 + (-3-1)^2 + (5-7)^2} = \sqrt{1+4+4} = 3$$

Just as the two-dimensional distance formula can be used to define the equation of a circle, the three-dimensional distance formula can be used to define the equaition of a sphere.

Definition 13.1.2 (Equation of a Sphere). An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

In particular, if the center is the origin O, then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$

Proof. By definition, a sphere is the set of all points $P(x, y, z)$ whos distance from center $C(h, k, l)$ is radius r. Thus, P is on the sphere if and only if $|PC| = r$. Squaring both sides, we have $|PC|^2 = r^2$, or

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

Example 13.1.2. Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ is the equation of a sphere, and find its center and radius.

Solution. We can rewrite the given equation in the form of an equation of a sphere by completing the square:

$$\begin{gathered}(x^2 + 4x + 4) + (y^2 - 6y + 9)(z^2 + 2z + 1) = -6 + 4 + 9 + 1\\ (x+2)^2 + (y-3)^2 + (z+1)^2 = 8\end{gathered}$$

Comparing this equation with the standard form, we see that it is the equation of a sphere with center $(-2, 3, -1)$ and radius $\sqrt{8} = 2\sqrt{2}$.

Example 13.1.3. What region in $\mathbb{R}^3$ is represented by $1 \leq x^2 + y^2 + z^2 \leq 4,\ z \leq 0$?

Solution. Rewrite the inequality as $1 \leq \sqrt{x^2 + y^2 + z^2} \leq 2$, which represents the points whose distance from the origin is at least 1 and at most 2. Since $z \leq 0$, the points lie on or below the xy-plane. The inequalities represent the lower hemisphere between the radii 1 and 2.

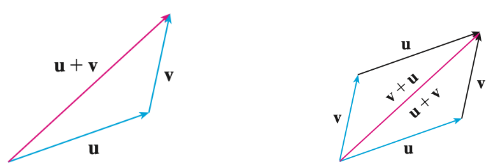
13.2 Vectors

A **vector** indicates a quantity that has both magnitude and direction and is often represented by an arrow. The length of the arrow represents its magnitude and the direction represents the vector's direction. A vector is generally typed in boldface ($\mathbf{v}$) and written with an arrow above the letter ($\vec{v}$).

Suppose a particle moves from A to B, so its **displacement vector** $\mathbf{v}$ is $\vec{AB}$. The vector has **initial point** A and **terminal point** B and the vector is indicated by $\mathbf{v} = \vec{AB}$. Suppose another vector $\mathbf{u}$ has the same length and direction as $\mathbf{v}$ even though it is in a different position. We can say that $\mathbf{u}$ and $\mathbf{v}$ are **equivalent** (or **equal**) and we write $\mathbf{u} = \mathbf{v}$. The **zero vector**, denoted by $\mathbf{0}$, has length 0. It is the only vector with no specific direction.

Definition 13.2.1 (**Definition of Vector Addition**)**.** If $\mathbf{u}$ and $\mathbf{v}$ are vectors positioned so the initial point of $\mathbf{v}$ is at the terminal point of $\mathbf{u}$, then the **sum** $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$.

The definition of vector addition is sometimes called the **Triangle Law**. We can also use what we know about vectors to visualize the **Parallelogram Law**.



Definition 13.2.2 (**Definition of Scalar Multiplication**)**.** If c is a scalar and $\mathbf{v}$ is a vector, the the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

For instance, $2\mathbf{v}$ is the same as $\mathbf{v} + \mathbf{v}$, which has the same direction as $\mathbf{v}$ but is twice as long.

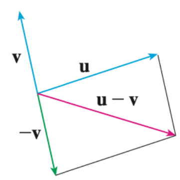
Two nonzero vectors are **parallel** if they are scaler multiples of one another. In particular, the vector $-\mathbf{v} = (-1)\mathbf{v}$ has the same length as $\mathbf{v}$ but points in the opposite direction and is called the **negative** of $\mathbf{v}$.

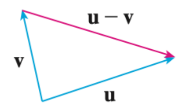
The **difference** $\mathbf{u} - \mathbf{v}$ is

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

We can construct $\mathbf{u} - \mathbf{v}$ in two ways:

1. Draw the negative of $\mathbf{v}$, $-\mathbf{v}$, and add it to $\mathbf{u}$ by the Parallelogram Law.
2. $\mathbf{v} + (\mathbf{u} - \mathbf{v} = \mathbf{u}$, which also equals $\mathbf{u}$, so we could construct $\mathbf{u} - \mathbf{v}$ by the Triangle Law.





Components

If we place the initial point of a vector **a** at the origin of a rectangular coordinate system, then the terminal point of **a** has coordinates of the form (a_1, a_2) or (a_1, a_2, a_3) depending on the dimensions of the coordinate system. These coordinates are called the **components** of **a**.

$$\mathbf{a} = \langle a_1, a_2 \rangle \qquad \text{or} \qquad \mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

We use the notation $\langle a_1, a_2 \rangle$ that refers to a vector so we don't confuse it with the ordered pair (a_1, a_2) that refers to a point in the plane.

Definition 13.2.3. Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector **a** with representation $\vec{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Example 13.2.1. Find the vector represented by the directed line segment with initial point $A(2, -3, 4)$ and $B(-2, 1, 1)$.

Solution. The vector corresponding to $\vec{AB}$ is

$$\mathbf{a} = \langle -2 - 2,\ 1 - (-3),\ 1 - 4 \rangle = \langle -4,\ 4,\ -3 \rangle$$

The **magnitude** or **length** of the vector **v** is the length of any of its representations and is denoted by the symbol $|v|$ or $\|v\|$. By using the distance formula to compute the length of a segment OP, we obtain the following formulas.

Definition 13.2.4.
The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Definition 13.2.5. If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \langle a_1 + b_1, a_2 + b_2 \rangle \\ \mathbf{a} - \mathbf{b} &= \langle a_1 - b_1, a_2 - b_2 \rangle \\ c\mathbf{a} &= \langle ca_1, ca_2 \rangle \end{aligned}$$

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3\rangle + \langle b_1, b_2, b_3\rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$$
$$\langle a_1, a_2, a_3\rangle - \langle b_1, b_2, b_3\rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3\rangle$$
$$c\langle a_1, a_2, a_3\rangle = \langle ca_1, ca_2, ca_3\rangle$$

We denote V_2 as the set of all two-dimensional vectors and V_3 as the set of all three-dimensional vectors. More generally, we consider V_n the set of all n-dimensional vectors. An n-dimensional vector is an ordered n-tuple:

$$\mathbf{a} = \langle a_1, a_2, \ldots, a_n\rangle$$

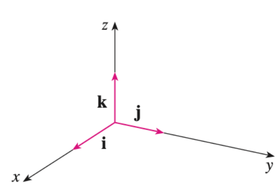
Definition 13.2.6 (**Properties of Vectors**). If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

Any vector in V_3 can be expressed in terms of the **standard basis vectors** $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. Such vectors are typically written with a hat. Let

$$\hat{\mathbf{i}} = \langle 1, 0, 0\rangle \qquad \hat{\mathbf{j}} = \langle 0, 1, 0\rangle \qquad \hat{\mathbf{k}} = \langle 0, 0, 1\rangle$$

Then $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are vectors that have length 1 and point in the direction of the positive x-, y-, and z-axes. Similarly, in two dimensions we define $\hat{\mathbf{i}} = \langle 1, 0\rangle$ and $\hat{\mathbf{j}} = \langle 0, 1\rangle$.



Proof. We prove that these any vectors in V_3 can be in terms of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$.

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then we write

$$\begin{aligned}\mathbf{a} &= \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\ &= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle \\ \mathbf{a} &= a_1 \hat{\mathbf{i}} + a_2 \hat{\mathbf{j}} + a_3 \hat{\mathbf{k}}\end{aligned}$$

Similarly, in two dimensions, we write

$$\mathbf{a} = \langle a_1, a_2 \rangle = a_1 \hat{\mathbf{i}} + a_2 \hat{\mathbf{j}}$$

Example 13.2.2. For instance,

$$\langle 1, -2, 6 \rangle = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$$

A **unit vector** is a vector whose length is 1. For instance $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are all unit vectors.

Definition 13.2.7. In general, if $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as $\mathbf{a}$ is

$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

Proof. Let $c = 1/|\mathbf{a}|$. Then $\mathbf{u} = c\mathbf{a}$ and c is a positive scalar, so $\mathbf{u}$ has the same direction as $\mathbf{a}$. Also

$$|\mathbf{u}| = |c\mathbf{a}| = |c||\mathbf{a}| = \frac{1}{|\mathbf{a}|}\mathbf{a} = 1$$

Example 13.2.3. Find the unit vector in the same direction of the vector $2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$.

Solution. The given vector has length

$$|2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

We divide the vector by its length to find the unit vector with the same direction:

$$\tfrac{1}{3}(2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}}) = \tfrac{2}{3}\hat{\mathbf{i}} - \tfrac{1}{3}\hat{\mathbf{j}} - \tfrac{2}{3}\hat{\mathbf{k}}$$

13.3 The Dot Product

13.4 The Cross Product

13.5 Equations of Lines and Planes

13.6 Cylinders and Quadric Surfaces

13.7 Cylindrical and Spherical Coordinates