12.1 Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_n,\ \ldots$$

The number a_1 is the *first term*, a_2 is the *second term*, and in general a_n is the *nth term.* We will deal with infinite sequences exclusively so each term a_n will have a successor $a_n + 1$.

Notice that for every positive integer n there is a corresponding number a_n so a sequence can be defined as a function whose domain is the set of positive integers. We usually write a_n instead of the function notation $f(n)$.

NOTATION The sequence $a_1,\ a_2,\ a_3,\ \ldots$ is also denoted by

$$a_n \quad \text{or} \quad a_{n}{}_{n=1}^{\infty}$$

Example 12.1.1. Some sequences can be defined by giving a formula for the nth term. In this example, we will describe a sequence in 3 ways: the previous notation, defining a formula, and writing out the terms of the sequence. Note that n doesn't have to start at 1.

1. $$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} \qquad a_n = \frac{n}{n+1} \qquad \left\{\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5}, \ldots,\ \frac{n}{n+1}, \ldots\right\}$$

2. $$\left\{\frac{(-1)^n(n+1)}{3^n}\right\} \qquad a_n = \frac{(-1)^n(n+1)}{3^n} \qquad \left\{-\frac{2}{3},\ \frac{3}{9},\ -\frac{4}{27},\ \frac{5}{81}, \ldots,\ \frac{(-1)^n(n+1)}{3^n}, \ldots\right\}$$

3. $$\{\sqrt{n-3}\}_{n=3}^{\infty} \qquad a_n = \sqrt{n-3},\ n \geq 3 \qquad \{0,\ 1,\ \sqrt{2},\ \sqrt{3}, \ldots,\ \sqrt{n-3}, \ldots\}$$

4. $$\left\{\cos\frac{n\pi}{6}\right\}_{n=0}^{\infty} \qquad a_n = \cos\frac{n\pi}{6},\ n \geq 0 \qquad \left\{1,\ \frac{\sqrt{3}}{2},\ \frac{1}{2},\ 0, \ldots,\ \cos\frac{n\pi}{6}, \ldots\right\}$$

Example 12.1.2. Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5},\ -\frac{4}{25},\ \frac{5}{125},\ -\frac{6}{625},\ \frac{7}{3125}, \ldots\right\}$$

Solution. We are given the first five terms. The numerator of the fractions start at 3 and increase by 1, so the nth term will have numerator $n + 1$. The denominators are the powers of 5, so a_n has denominator 5^n. The signs of the terms alternate between positive and negative, so we need to multiply by a power of 1. The factor $(-1)^n$ means we start with a negative term, so here we use $(-1)^{n-1}$ or $(-1)^{n+1}$ because we start with a positive term. Therefore,

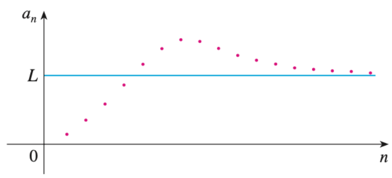
$$a_n = (-1)^{n-1}\frac{n+2}{5^n}$$

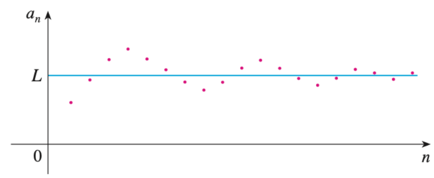
Definition 12.1.1. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

The figure below graphs examples of two sequences that have the limit L.





A more precise version of the previous definition is

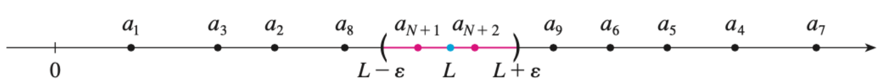
Definition 12.1.2. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

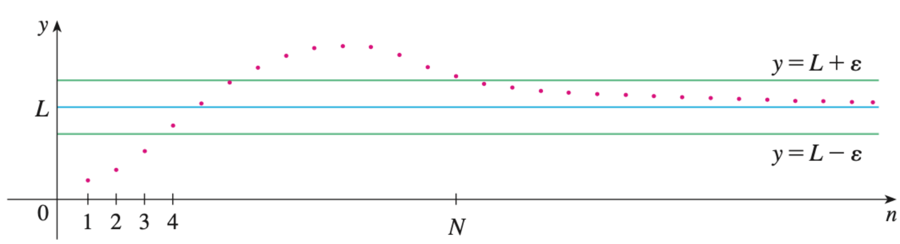
if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$|a_n - L| < \varepsilon \quad \text{whenever } n > N$$

No matter how small an interval $(L - \varepsilon\ L + \varepsilon)$ is chosen, there exists an N such that all terms of the sequence from a_{N+1} onward must lie in that interval.



The points on the graph of a_n must lie between the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ if $n > N$. This picture must be valid no matter how small ε is chosen, but usually a smaller ε requires a larger N.



The only difference between $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$ is that n is required to be an integer.

Theorem 12.1.1. If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} = a_n = L$.

Since we know that $\lim_{x\to\infty}(1/x^r) = 0$ when $r > 0$, we can use the previous theorem to get

Definition 12.1.3.

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \quad \text{if } r > 0$$

If a_n grows as n grows, we use the notation $\lim_{n\to\infty} a_n = \infty$. We say that a_n diverges to ∞.

Definition 12.1.4. $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$a_n > M \quad \text{whenever } n > N$$

Definition 12.1.5 (Limit Laws for Sequences (similar to original Limit Laws)**).** If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

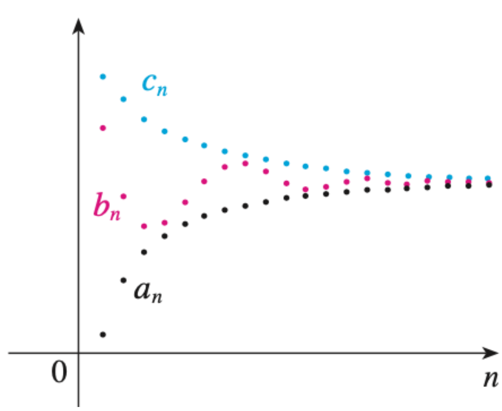
$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad\qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \tfrac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } b_n \neq 0$$

$$\lim_{n\to\infty}(a_n^p) = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

Theorem 12.1.2 (The Squeeze Theorem for Sequences (same as original)**).** If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.



Theorem 12.1.3. If $\lim\limits_{n\to\infty} |a_n| = 0, then \lim\limits_{n\to\infty} a_n = 0.$

Example 12.1.3. Evaluate $\lim\limits_{n\to\infty} \frac{(-1)^n}{n}$ if it exists.

Solution.
$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{so} \quad \lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

Example 12.1.4. Find $\lim\limits_{n\to\infty} \frac{n}{n+1}$.

Solution. Divide the numerator and denominator by the highest power of n and then use the Limit Laws.

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \frac{1}{n}}$$
$$= \frac{1}{1+0} = 1$$

Example 12.1.5. Calculate $\lim\limits_{n\to\infty} \frac{\ln n}{n}$.

Solution. Both the numerator and denominator approach infinity as $n \to \infty$. We can't apply l'Hospital;s Rule directly because it applies to functions, not sequences. However, we can apply it to the related function.

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0 \quad \text{so} \quad \lim_{n\to\infty} \frac{\ln n}{n} = 0$$

Example 12.1.6. Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

Solution. If we write out the terms of the sequence, we get $\{-1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1, \ldots\}$. Since the terms oscillate between 1 and -1, a_n does not approach any number. Thus, $\lim_{n\to\infty}(-1)^n$ does not exist so the sequence $\{(-1)^n\}$ is divergent.

Example 12.1.7. Discuss the convergence of the the sequence $a_n = n!/n^n$.

Solution. Both the numerator and denominator approach infinity as $n \to \infty$, but we have no corresponding functions to use l'Hospital's Rule because $x!$ is not defined when x is not an integer. if we write the general formula for the sequence, we get

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}{n \cdot n \cdot n \cdot \ldots \cdot n} = \frac{1}{n}\left(\frac{2 \cdot 3 \cdot \ldots \cdot n}{n \cdot n \cdot n \cdot \ldots \cdot n}\right)$$

The expression in the parenthesis is at most 1 because the numerator is less than (or equal) to the denominator, so

$$0 < a_n \leq \frac{1}{n}$$

We can use the squeeze theorem because both 0 and $1/n \to 0$ as $n \to \infty$, so $a_n \to \infty$ as $n \to \infty$.

Example 12.1.8. Determine if the sequences below converge. If they do, find the limits as $n \to \infty$.

1. $\frac{\sin n}{n}$
2. ne^{-n}

Solution. 1. $\frac{\sin n}{n}$ converges to 0 by the squeeze theorem

$$-\frac{1}{n} \le \frac{\sin n}{n} \le -\frac{1}{n} \ , \ \lim_{n\to\infty} \frac{1}{n} = 0 \ , \text{so}$$
$$0 \le \frac{\sin n}{n} \le 0 \implies \lim_{n\to\infty} \frac{\sin n}{n} = 0$$

2. $ne^{-n} = \frac{n}{e^n}$. The denominator e^n converges faster than the numerator n does. Use l'Hospital's Rule to get

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0 \text{ so } \lim_{n\to\infty} ne^{-n} = 0$$

Example 12.1.9. Show that if $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$, $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.

Solution. The solution uses the symbols $\exists$ ("exists") and $\implies$ ("implies").

$$\begin{array}{ll} \text{Since } \lim_{n\to\infty} a_{2n} = L, & \exists\ N_1 \implies |a_{2n} - L| < \varepsilon \text{ for } n > N_1 \\ \text{Since } \lim_{n\to\infty} a_{2n+1} = L, & \exists\ N_2 \implies |a_{2n+1} - L| < \varepsilon \text{ for } n > N_2 \end{array}$$

Let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$.

$$\begin{array}{ll} \text{If } n \text{ is even,} & n = 2m,\ m > N_1,\ |a_n - L| = |a_{2m} - L| < \varepsilon \\ \text{If } n \text{ is odd,} & n = 2m+1,\ m > N_2,\ |a_n - L| = |a_{2m+1} - L| < \varepsilon \end{array}$$

Therefore, $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.

Definition 12.1.6. The sequence $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 0 & \text{if } r = 1 \end{cases}$$

Definition 12.1.7. A sequence $\{a_n\}$ is **increasing** if $a_n < a_{n+1}$ for all $n \ge 1$ ($a_1 < a_2 < a_3 < \cdots$). It is **decreasing** is $a_n < a_{n+1}$ for all $n \ge 1$. It is **monotonic** if the is either increasing or decreasing.

Example 12.1.10. The sequence $\left\{\frac{3}{n+5}\right\}$ is decreasing because

$$\frac{3}{n+5} < \frac{3}{(n+1)+5} = \frac{3}{n+6}$$

for all $n \geq 1$ (the right side is smaller because it has a larger denominator).

Example 12.1.11. Show that the sequence $a_n = \frac{n}{n^2+1}$ is decreasing.

Solution (1). We must show that $a_{n+1} < a_n$.

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$

We can simplify this by cross-multiplying. $\Longleftrightarrow$ means "if and only if".

$$\begin{aligned}\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} &\Longleftrightarrow (n+1)(n^2+1) < n[(n+1)^2+1] \\ &\Longleftrightarrow n^3+n^2+n+1 < n^3+2n^2+2n \\ &\Longleftrightarrow 1 < n^2+n\end{aligned}$$

Since $n \geq 1$, we know that the inequality $n^2+n > 1$ is true. Therefore, $a_{n+1} < a_n$ so $\{a_n\}$ is decreasing.

Solution (2). Consider the function $f(x) = \frac{x}{x^2+1}$:

$$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0 \quad \text{whenever } x^2 > 1$$

This, f is decreasing on $(1, \infty)$ so $f(n) > f(n+1)$. Therefore, $\{a_n\}$ is decreasing.

Theorem 12.1.4 (Monotonic Sequence Theorem)**.** Every bounded, monotonic sequence is convergent.

The proof is based on the **Completeness Axiom** for the set $\mathbb{R}$ of real numbers, which says that if S is a nonempty set of real numbers that has an upper bound M ($x \leq M$ for all x in S), then S has a **least upper bound** b (This means that b is an upper bound for S, but if M is any other upper bound, then $b \leq M$). The Completeness Axiom is an expression of the fact that there is no gap or hole in the real number line.

Proof. Suppose $\{a_n\}$ is an increasing sequence. Since $\{a_n\}$ is bounded, the set S=$\{a_n | n \geq 1\}$ has an upper bound. By the Completeness Axiom it has a least upper bound L. Given $\varepsilon > 0$, $L - \varepsilon$ is *not* an upper bound for S (since L) is the *least* upper bound). Therefore

$$a_N > L - \varepsilon \qquad \text{for some integer } N$$

But the sequence is increasing so $a_n \geq a_N$ for every $n > N$. Thus, if $n > N$ we have

$$\begin{aligned} a_n &> L - \varepsilon \\ 0 \leq L - a_n &< \varepsilon \end{aligned}$$

since $a_n \leq L$. Thus

$$|L - a_n| < \varepsilon \qquad \text{whenever } n > N$$

so $\lim_{n\to\infty} a_n = L$.

A similar proof (using the greatest lower bound) works if $\{a_n\}$ is decreasing.