Contents

Part I

Multivariable Calculus

Chapter 11

Parametric Equations and Polar Coordinates

11.1 Curves Defined by Parametric Equations

Suppose that x and y are both given as functions of a third variable t (called a **parameter** by the equations)

$$x = f(t) \quad y = g(t)$$

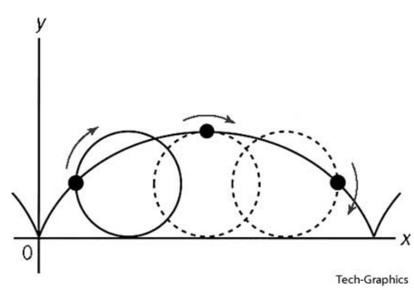
(called **parametric equations**). Each value of t determines a point (x,y). As t changes, $(x, y) = (f(t), g(t))$ changes and traces out a curve C, which is called a **parametric curve**. The direction of the arrows on curve C show the change in the position of the equation as t increases.

We can also restrict t to a finite interval. In general, the curve with parametric equations

$$x = f(t) \quad y = g(t) \quad a \leq t \leq b$$

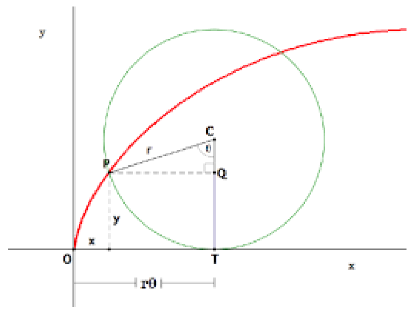
has **initial point** $(f(a), g(a))$ and **terminal point** $(f(b), g(b))$.

The Cycloid



Example 11.1.1. A circle with radius r rolls along the x-axis. The curve traced out by a point P on the circumference of the circle is called a **cycloid**. Find parametric eqations for the cycloid.

Solution. We will use the angle of rotation θ as the parameter ($\theta = 0$ when P is at the origin).



Suppose the circle has rotated θ radians. Using the figure, the distance it has rolled from the origin is

$$|OT| = arc\ PT = r\theta$$

because P starts at the origin. Therefore, the center of the circle is $C(r\theta, r)$. Let the coordinates of P be (x, y). Then from the figure,

$$x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$$
$$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$$

Definition 11.1.1. Paremetric equations of the cycloid are

$$x = r(\theta - \sin\theta) \quad y = r(1 - \cos\theta)$$

11.2 Calculus with Parametric Curves

We will mainly solve problems involving tangents, area, arc length, and surface area.

Tangents

In the previous section, we saw that some curves defined by parametric equations $x = f(t)$ and $y = g(t)$ can also be expressed, by eliminating the parameter, in the form $y = F(x)$. If we substitute $x = f(t)$ and $y = g(t)$ in the equation $y = F(x)$, we get

$$g(t) = F(f(t))$$

If g, f, and F are differentiable, the Chain Rule gives

$$g'(t) = F'(f(t))f'(t) = F'(x)f'(t)$$

If $f'(t) \neq 0$, we can solve for $F'(x)$:

Definition 11.2.1. The slope of the tangent to the parametric curve $y = F(x)$ is $F'(x)$.

$$F'(x) = \frac{g'(t)}{f'(t)}$$

This enables us to find tangents to parametric curves without having to eliminate the parameter. We can rewrite the previous equation in an easily remembered form.

Definition 11.2.2. We can use this to find tangents to parametric curves without having to eliminate the parameter.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0$$

The curve has a

- horizontal tangent when $\frac{dy}{dt} = 0$ (provided that $\frac{dx}{dt} \neq 0$)
- vertical tangent when $\frac{dx}{dt} = 0$ (provided that $\frac{dy}{dt} \neq 0$)

This is useful when sketching parametric curves.

Definition 11.2.3. We can also find $\frac{d^2y}{dx^2}$ by replacing y with $\frac{dy}{dx}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Proof. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ considering $y(t)$ and $g(t)$.

1.

$$\text{Chain rule: } \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \implies \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad (\implies \text{ means "implies"})$$

2.

$$\begin{aligned}
\text{Chain rule: } \frac{d}{dt}\left(\frac{dy}{dx}\right) &= \left(\frac{d}{dx}\frac{dy}{dx}\right)\frac{dx}{dt} = \frac{d^2y}{dx^2}\frac{dx}{dt} \\
\text{Substitute: } \frac{d}{dt}\left(\frac{dy}{dx}\right) &= \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) \\
\text{Quotient rule: } &= \frac{\frac{d^2y}{dt^2}\frac{d^x}{dt} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2}
\end{aligned}$$

Set equation from line 1 and line 3 equal and divide both sides by $\frac{dx}{dt}$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{\frac{d^2y}{dt^2}\frac{d^x}{dt} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2\left(\frac{dx}{dt}\right)} \\
&= \frac{\frac{d^2y}{dt^2}\frac{d^x}{dt} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}
\end{aligned}$$

Example 11.2.1. A curve C is defined by the parametric equations $x = t^2,\ y = t^3 - 3t$.

1. Show that C has two tangents at the point (3,0) and find their equations.
2. Find the points on C where the tangent is horizontal or vertical.
3. Determine where the curve is concave upward or downward.

Solution. A curve C is defined by the parametric equations $x = t^2,\ y = t^3 - 3t$.

1. Rewrite $y = t^3 - 3t = t(t^2 - 3) = 0$ when $t = 0$ or $t = \pm\sqrt{3}$. This indicates that C intersects itself at (3.0).

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$$
$$t = \pm\sqrt{3} \to dy/dx = \pm 6/(2\sqrt{3})$$

so the equations of the tangents at (3,0) are

$$y = \sqrt{3}(x - 3) \quad \textit{and} \quad y = -\sqrt{3}(x - 3)$$

2. C has a horizontal tangent when $dy/dx = 0$. In other words, when $dy/dt = 0$ and $dx/dt \neq 0$. $dy/dt = 3t^2 - 3 = 0$ when $t^2 = 1$ so $t = \pm 1$. This means there are horizontal tangents on C at (1,-2) and (1,2). C has a vertical tangent when $dx/dt = 2t = 0$, so $t = 0$. This means C has a vertical tangent at (0,0).

3. To determine concavity we calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{3}{2}\left(1+\frac{1}{t^2}\right)}{2t} = \frac{3(t^2+1)}{4t^3}$$

The curve is concave upward when $t > 0$ and concave downward when $t < 0$.

Area

We already know that area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)dx$. We can apply this to parametric equations using the Substitution Rule for Definite Integrals.

Definition 11.2.4. If the curve C is given by parametric equations $x = f(t)$ and $y = g(t)$ and t increases from α to β,

$$A = \int_a^b y dx = \int_\alpha^\beta g(t)f'(t)dt$$

(Switch α to β if the point on C at β is more left than α.

Example 11.2.2. Find the area under one arch of the cycloid $x = r(\theta - \sin\theta),\ y = r(1-\cos\theta)$.

Solution. One arch of the cycloid is given by $0 \leq \theta \leq 2\pi$. Using the Substitution Rule with $y = r(1-\cos\theta)$ and $dx = r(1-\cos\theta)d\theta$, we have

$$\begin{aligned}
A &= \int_0^{2\pi} y dx = A = \int_0^{2\pi} r(1-\cos\theta)r(1-\cos\theta)d\theta \\
&= r^2\int_0^{2\pi}(1-\cos\theta)^2 d\theta = r^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)d\theta \\
&= r^2\int_0^{2\pi}\left[1-2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right]d\theta \\
&= r^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} \\
&= r^2\left(\frac{3}{2}\cdot 2\pi\right) = 3\pi r^2
\end{aligned}$$

Arc Length

We already know how to find length L of a curve C given in the form $y = F(x)$, $a \leq x \leq b$.

Definition 11.2.5. If F' is continuous, then

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If C can describe the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, where $dx/dt = f'(t) > 0$. Using the substitution rule, we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt} dt$$

Since $dx/dt > 0$, we have

Theorem 11.2.1. If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

This is consistent with the general formula $L = \int ds$ and $(ds^2) = (dx^2) + (dy^2)$.

Proof. Prove the length formula of a parametric curve

$$\begin{aligned}
\overrightarrow{ds} &= \overrightarrow{i}\, dx + \overrightarrow{j}\, dy \\
ds^2 &= \overrightarrow{ds} \cdot \overrightarrow{ds} = \left(\overrightarrow{i}\, dx + \overrightarrow{j}\, dy\right) \cdot \left(\overrightarrow{i}\, dx + \overrightarrow{j}\, dy\right) = dx^2 + dy^2 \\
ds &= \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
L &= \int_\alpha^\beta ds = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt
\end{aligned}$$

Example 11.2.3. Find the length of the unit circle as (x,y) moves both once and twice around the circle.

Solution. For one traversal around the unit circle,

$$x = \cos t \quad y = \sin t \quad 0 \leq t \leq 2\pi$$

so $dx/dt = -\sin t$ and $dy/dt = \cos t$

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt \\
&= \int_0^{2\pi} dt = 2\pi
\end{aligned}$$

For two traversals around the unit circle,

$$x = \sin 2t \quad y = \cos 2t \quad 0 \le t \le 2\pi$$

so $dx/dt = 2\cos 2t$ and $dy/dt = -2\sin 2t$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^{2\pi} \sqrt{4\cos^2 2t + 4\sin^2 2t}\ dt \quad = \int_0^{2\pi} 2\ dt = 4\pi$$

Surface Area

We can also adapt the surface area formula to a parametric curve.

Definition 11.2.6. If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$, is rotated about the **x-axis**, where f',g' are continuous and $g(t) \ge 0$, the surface area is

$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

If the curve C is rotated about the **y-axis**, the surface area is

$$S = \int_\alpha^\beta 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The generic formulas $S = \int 2\pi y\ ds$ for rotation about the x-axis and $S = \int 2\pi x\ ds$ for rotation about the y-axis are still valid, but for parametric curves we use

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Example 11.2.4. Show that the surface area of a sphere of radius r is $4\pi r^2$

Solution. The sphere is obtained by rotating the semicircle

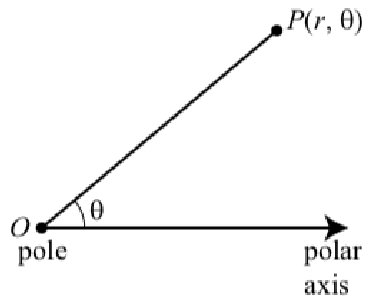
$$x = r\cos t \quad y = r\sin t \quad 0 \le t \le \pi$$

about the x-axis.

$$\begin{aligned} S &= \int_0^\pi 2\pi r \sin t \sqrt{(-r\sin t)^2 + (r\cos t)^2}\ dt \\ &= 2\pi \int_0^\pi r\sin t \sqrt{r^2(sin^2 t + \cos^2 t)}\ dt \\ &= 2\pi \int_0^\pi r\sin t \cdot r\ dt = 2\pi r^2 \int_0^\pi \sin t\ dt \\ &= 2\pi r^2(-\cos t)]_0^\pi = 4\pi r^2 \end{aligned}$$

11.3 Polar Coordinates

In addition to Cartisian coordinates, we can also use a **polar coordinate system**.



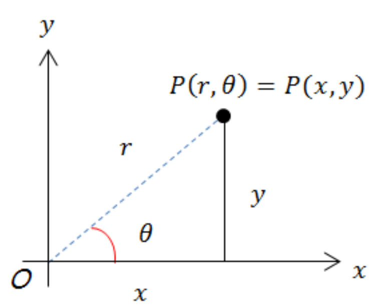
Point P is represented by the ordered pair (r, θ), where r is the distance to the point from the center and θ is the angle from the polar axis to the point.

The points (r, θ) and $(-r, \theta)$ are on the same line and have the same distance $|r|$ from the center but are on opposite sides of the center. Additionally, $(-r, \theta)$ and $(r, \theta + \pi)$ are also on the same line.

This means a complete counterclockwise rotation is given by an angle 2π, so (r, θ) is also represented by

$$(r, \theta + 2n\pi) \text{ and } (-r, \theta + (2n+1)\pi)$$

Relationship Between Cartesian and Polar Coordinates



$$\cos\theta = \tfrac{x}{r} \quad \sin\theta = \tfrac{y}{r}$$

$$x = r\cos\theta \quad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \quad \tan\theta = \tfrac{y}{x}$$

Example 11.3.1. Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

Solution.

$$r = 2,\ \theta = \pi/3$$
$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1$$
$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$$

So the point is $(1, \sqrt{3})$ in Cartesian coordinates.

Example 11.3.2. Represent the Cartesian coordinates $(1, -1)$ in polar coordinates.

Solution.

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$
$$\tan\theta = \frac{y}{x} = -1$$

Since the point (1,-1) lies in the fourth quadrant, we can choose $\theta = -pi/4$ or $\theta = 7pi/4$. So the possible answers are either $(\sqrt{2}, -\pi/4$ or $(\sqrt{2}, 7\pi/4$.

Polar Curves

The **graph of a polar equation** $r = f(\theta)$, or $F(r, \theta) = 0$, consists of all of the points where (r, θ) satisfies the equation.

Tangents to Polar Curves

To find a tangent line to a polar curce $r = f(\theta)$, we regard θ as a parameter and write the parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \quad y = r\sin\theta = f(\theta)\sin\theta$$

So

Definition 11.3.1.
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dy}{d\theta} sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

- horizontal tangent when $\frac{dy}{d\theta} = 0$ (provided that $\frac{dx}{d\theta} \neq 0$)
- vertical tangent when $\frac{dx}{d\theta} = 0$ (provided that $\frac{dy}{d\theta} \neq 0$)

NOTE tangent lines at the pole have r=0 and the slope of the tangent simplifies to

$$\frac{dy}{dx} = \tan\theta \text{ if } \frac{dr}{d\theta} \neq 0$$

Example 11.3.3. For the cardiod $r = 1 + \sin\theta$, find the slope of the tangent line when r=3

Solution.

$$\begin{aligned} r &= 1 + \sin\theta \\ \frac{dy}{dx} &= \frac{\frac{dy}{d\theta} sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta} \\ &= \frac{\cos\theta(1+2\sin\theta)}{1-2\sin^2\theta - \sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-\sin\theta)} \end{aligned}$$

The slope of the tangent where $\theta = \pi/3$ is

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{\theta=\pi/3} &= \frac{\cos(\pi/3)(1+2\sin(\pi/3))}{(1+\sin(\pi/3))(1-\sin(\pi/3))} \\ &= \frac{\frac{1}{2}(1+\sqrt{3})}{(1+\sqrt{3}/2)(1-\sqrt{3})} = \frac{1+\sqrt{3}}{(2+\sqrt{3})(1-\sqrt{3})} \\ &= \frac{1+\sqrt{3}}{-1-\sqrt{3}} = -1 \end{aligned}$$

Note Instead of memorizing the equation, we can instead use the same method we used to derive it.

$$\begin{aligned} x &= r\cos\theta = (1+\sin\theta)\cos\theta = \cos\theta + \frac{1}{2}\sin 2\theta \\ y &= r\sin\theta = (1+\sin\theta)\sin\theta = \sin\theta + \sin^2\theta \\ \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} = \frac{cos\theta + 2\sin\theta\cos\theta}{-\sin\theta + \cos 2\theta} = \frac{\cos\theta + \sin 2\theta}{-\sin\theta + \cos 2\theta} \end{aligned}$$

This is equivalent to the previous equation.

11.4 Areas and Lengths in Polar Coordinates

Area

We can determine the formula for the area of a region whose boundary is given by a polar equation by taking the limit of a Riemann Sum starting with the formula for the area of a sector of a circle $A = \frac{1}{2}r^2\theta$.

Definition 11.4.1. The formula for the area A of the polar region $\mathcal{R}$ is

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 \, d\theta = \int_a^b \frac{1}{2}r^2 d\theta$$

with the understanding that $r = f(\theta)$.

Example 11.4.1. Find the area enclosed by one loop of the four-leaved rose $r = 2\cos 2\theta$.

Solution. The right loop rotates from $\theta = -\pi/4$ to $\theta = \pi/4$.

$$\begin{aligned} A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta \ d\theta \\ &= \int_0^{\pi/4} \cos^2 2\theta \ d\theta = \int_0^{\pi/4} \frac{1}{2}(1 + \cos 4\theta) \ d\theta \\ &= \frac{1}{2}[\theta + \frac{1}{4}\sin 4\theta] = \pi/8 \end{aligned}$$

We can also adapt the formula to find the area of a region bounded by two polar curves.

Definition 11.4.2. Let $\mathcal{R}$ be a region that is bounded by curves with polar equations $r = f(\theta)$, $r = g(\theta)$, $\theta = a$, and $\theta = b$, where $f(\theta) \geq g(\theta) \geq 0$ and $0 < b - a \leq 2\pi$. The area A of $\mathcal{R}$ is found by subtracting the area inside $r = g(\theta)$ from the area inside $r = f(\theta)$, so

$$\begin{aligned} A &= \int_a^b \frac{1}{2}[f(\theta)]^2 \ d\theta - \int_a^b \frac{1}{2}[g(\theta)]^2 \ d\theta \\ &= \int_a^b \frac{1}{2}([f(\theta)]^2 - [g(\theta)]^2) \ d\theta \end{aligned}$$

Arc Length

To find the length of a polar curve $r = f(\theta)$, $a \leq \theta \leq b$, we regard θ as a parameter and write the parametric equations of the curve as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Using the projecut Rule and differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

so, using $\cos^2\theta + \sin^2\theta = 1$, we have

$$\begin{aligned} \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\ &+ \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2 \end{aligned}$$

Assuming that f' is continuous, we can use the theorem from 11.2 about the arc length of a curve defined by parametric equations to write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Definition 11.4.3. The length of a curve with polar equation $r = f(\theta)$, $a \leq \theta \leq b$, is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

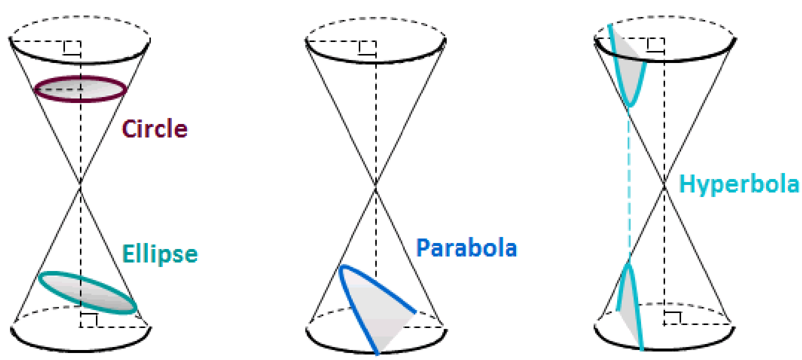
Example 11.4.2. Find the arc length of the cardiod $r = 1 + \sin\theta$.

Solution. The full length of the cardiod is given by the parameter interval $0 \leq \theta \leq 2\pi$.

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta = 8 \ (\textit{by rationalizing the integrand by } \sqrt{2 - 2\sin\theta}) \end{aligned}$$

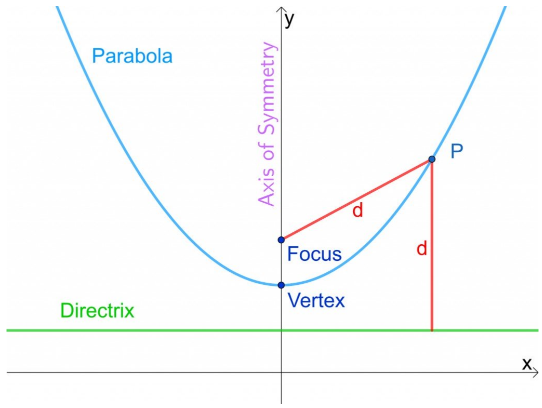
11.5 Conic Sections

Parabolas, ellipses, and hyperbolas are called **conic sections**, or **conics**, because they result from intersecting a cone with a plane.



Parabolas

A **parabola** is the set of points in a plane that are equidistant from a fixed point F (called the **focus**) and a fixed line (called the **directrix**). The halfway point between the focus and directrix is on the parabola and is called the **vertex**. The line through the focus and the vertex and perpendicular to the directrix is the **axis** of the parabola.



As seen in the figure, the focus is always inside the region of the parabola and the directrix is the same distance away on the opposite side.

Definition 11.5.1. An equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is

$$x^2 = 4py$$

. If we set $a = \frac{1}{4p}$, then the standard equation of a parabola is $y = ax^2$. This opens upward if $p > 0$ and downard if $p < 0$, and is symmetric with respect to the y-axis.

Definition 11.5.2. If we switch x and y, we get

$$y^2 = 4px$$

(reflection about the diagonal line y=x). This parabola opens to the right if $p > 0$ and to the left if $p < 0$.

Definition 11.5.3. The vertex form of a parabola is

$$y = a(x - h)^2 + k$$

where (h, k) is the vertex of the parabola and $x = h$ is the axis of symmetry. We can also switch x and y to get the vertex form of the rotated parabola.

Example 11.5.1. Find the focus and directrix of the parabola $y^2 + 10x = 0$.

Solution. We rewrite the equation as $y^2 = -10x$. We know $y^2 = 4px$, so $4px = -10x$ and $p = -\frac{5}{2}$. Thus, the focus is $(p, 0) = -\frac{5}{2}, 0)$ and the directrix is $x = \frac{5}{2}$.

Ellipses

An **ellipse** is the set of points in a plane surrounding two fixed focal points F_1 and F_2 such that the <u>sum</u> of the two distances to the focal points is a constant. Imagine tracing a line along the furthest path of a string stretched across two different points.

Definition 11.5.4. The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a \geq b \geq 0$$

has foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$ **(lies on x-axis)**.

The **vertices** are on the **major axis**, where a is the distance to the center of the ellipse from each vertex. This distance is greater than the distance from a **co-vertex** to the center of the ellipse, b. The co-vertices lie on the **minor axis**. Because the sum of the two distances from a point on the ellipse to the foci is a constant, the distance from a co-vertex to a focal point is also a. If the foci coincide, then $c = 0$, so $a = b$ and the ellipse becomes a circle with radius $r = a = b$.

Definition 11.5.5. The ellipse

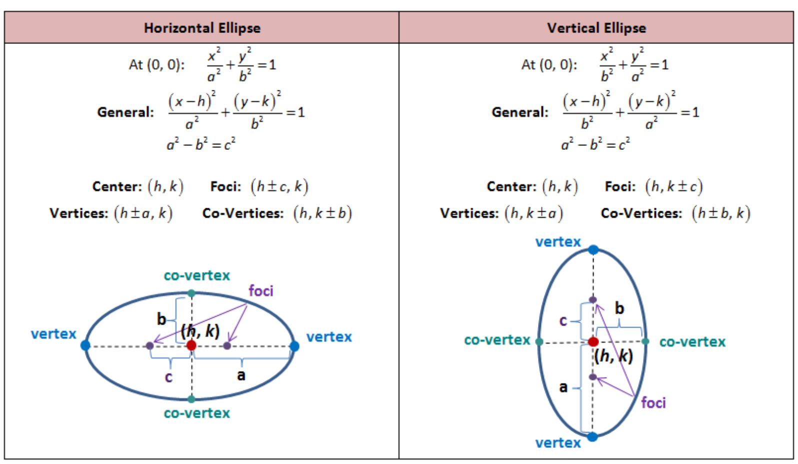
$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad a \geq b \geq 0$$

has foci $(0, \pm c)$, where $c^2 = a^2 - b^2$, and vertices $(0, \pm a)$ **(lies on y-axis)**.

Definition 11.5.6. The general form of a horizontal ellipse is

$$\frac{(x-h)^2}{b^2} + \frac{(y-h)^2}{a^2} = 1$$

where (h, k) is the center of the ellipse. The same transformation can be done to the standard form of a vertical ellipse.



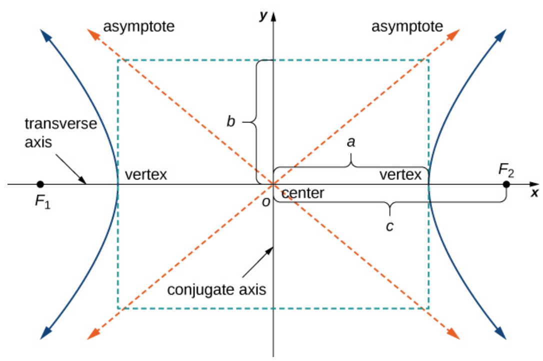
Example 11.5.2. Find an equation of the ellipse with foci $(0, \pm 2)$ and vertices $(0, \pm 3)$.

Solution. This equation represents a vertical ellipse because the foci and vertices lie on the y-axis. The distance from a focal point to the center is $c = 2$ and the distance from a vertex to the center is $a = 3$. Then we obtain $b^2 = a^2 + c^2 = 9 - 4 = 5$, so the equation of the ellipse is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = \frac{x^2}{5} + \frac{y^2}{9} = 1$$

Hyperbolas

An **ellipse** is the set of points in a plane surrounding two fixed focal points F_1 and F_2 such that the difference of the two distances to the focal points is a constant. The **transverse axis** is the axis of a hyperbola that passes through the two foci. The **conjugate axis** is perpendicular to the transverse axis and passes through the center of the hyperbola.



Definition 11.5.7. The hyperbola along a horizontal transverse axis

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has foci $(\pm c, 0)$, where $c^2 = a^2 + b^2$, vertices $(\pm a, 0)$ **(lies on x-axis)**, and asympotes $y = \pm \frac{b}{a} x$.

Definition 11.5.8. The hyperbola along a vertical transverse axis

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

has foci $(0, \pm c)$, where $c^2 = a^2 + b^2$, vertices $(0, \pm a)$ **(lies on y-axis)**, and asympotes $y = \pm \frac{a}{b} x$.

Definition 11.5.9. The general form of a hyperbola along a horizontal transverse axis is

$$\frac{(x-h)^2}{b^2} - \frac{(y-h)^2}{a^2} = 1$$

where (h, k) is the center of the ellipse. The same transformation can be done to the standard form a hyperbola along a vertical transverse axis.

Example 11.5.3. Find the foci and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$.

Solution. If we divide both sides of the equation by 144, it becomes

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

which is a hyperbola along a horizontal transverse axis. Therefore, we get $a = 4$ and $b = 3$. Since $c^2 = a^2 + b^2 = 16 + 9 = 25$, $c = 5$. The foci are $(\pm 5, 0)$, and the asymptotes are $y = \pm\frac{3}{4}x$.

Example 11.5.4. Find the foci and asymptotes of the hyperbola $ax^2 - by^2 = r$ in terms of $a,\ b,\ r$.

Solution. We first put the equation in standard form to get

$$\frac{x^2}{\left(\sqrt{\frac{r}{a}}\right)^2} - \frac{y^2}{\left(\sqrt{\frac{r}{b}}\right)^2} = 1$$

which is a hyperbola along a horizontal transverse axis. The foci are at

$$(\pm c, 0) = \left(\pm\sqrt{\frac{r}{a} + \frac{r}{b}},\ 0\right) = \left(\pm\sqrt{\frac{r(a+b)}{ab}},\ 0\right)$$

the vertices are at $\left(\pm\sqrt{\frac{r}{a}},\ 0\right)$, and the asymptotes are at $y = \pm\left(\sqrt{\frac{a}{b}}\right)x$.

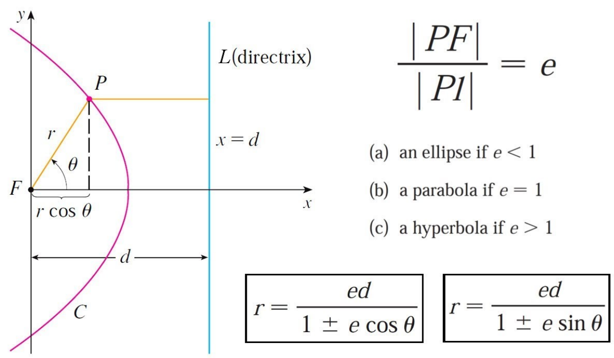
11.6 Conic Sections in Polar Coordinates

Theorem 11.6.1. Let F be a fixed point (called the **focus**) and l be a fixed line (called the **directrix**). Let e be a fixed positive number (called the **eccentricity**). The set of all points P in the plane such that

$$\frac{|PF|}{|Pl|} = e \quad \text{(the ratio of the distance from } F \text{ to the distance from } l \text{ is the constant e)}$$

is a conic section. The conic is

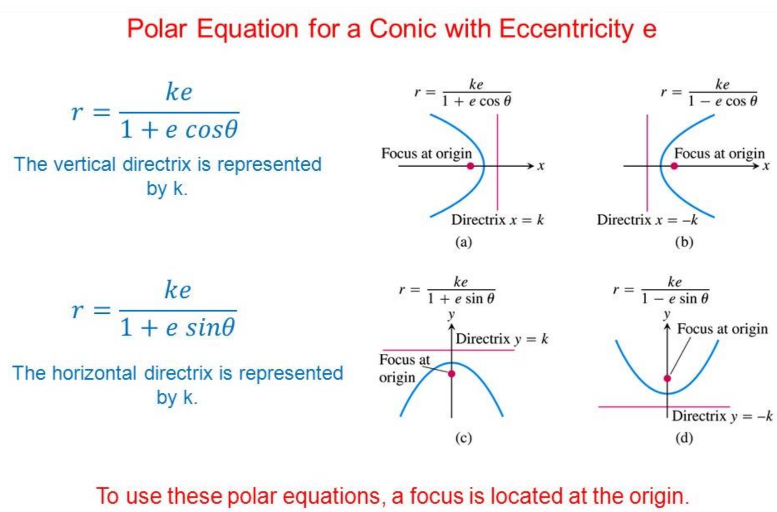
1. an ellipse if $e < 1$
2. a parabola if $e = 1$
3. a hyperbola if $e > 1$



Theorem 11.6.2. A polar equation of the form

$$r = \frac{ed}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic section with eccentricity e and distance d from the center to the directrix, with the focus at the origin. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola is $e > 1$.



Use "$\cos\theta$" when the conic section opens rightward or leftward, and use "$\sin\theta$" when the conic section opens upward or downward. Use "+" if the conic section opens leftward or downward, and use "−" if the conic section opens righttward or upward.

Example 11.6.1. Find a polar equation for a parabola that has its focus at the origin and whose directrix is the line $y = -6$.

Solution. The eccentricity $e = 1$ because the conic section is a parabola, and the distance from the center to the directrix is $d = 6$. The directrix is on the y-axis and is underneath the center, so the parabola opens upward. Therefore, we use the "$\sin\theta$" equation and use "$-$" in the denominator. The polar equation of the parabola is

$$r = \frac{6}{1 - \sin\theta}$$

.

Example 11.6.2. A conic is given by the polar equation

$$r = \frac{10}{3 - 2\cos\theta}$$

Find the eccentricity, identify the conic, and locate the directrix.

Solution. Divide the numerator and denominator by 3 to get

$$r = \frac{\frac{10}{3}}{1 - \frac{2}{3}\cos\theta}$$

This represents an ellipses with eccentricity $e = \frac{2}{3}$. Since $ed = \frac{10}{3}$,

$$d = \frac{\frac{10}{3}}{e} = \frac{\frac{10}{3}}{\frac{2}{3}} = 5$$

so the directrix has Cartesian equation $x = -5$. When $\theta = 0,\ r = 10$; when $\theta = \pi,\ r = 2$, so the vertices have polar coordinates $(10, 0)$, and $(2, \pi)$.

Chapter 12

Infinite Sequences and Series

12.1 Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_n,\ \ldots$$

The number a_1 is the *first term*, a_2 is the *second term*, and in general a_n is the *nth term*. We will deal with infinite sequences exclusively so each term a_n will have a successor $a_n + 1$.

Notice that for every positive integer n there is a corresponding number a_n so a sequence can be defined as a function whose domain is the set of positive integers. We usually write a_n instead of the function notation $f(n)$.

NOTATION The sequence $a_1,\ a_2,\ a_3,\ \ldots$ is also denoted by

$$a_n \quad \text{or} \quad a_n{}_{n=1}^{\infty}$$

Example 12.1.1. Some sequences can be defined by giving a formula for the nth term. In this example, we will describe a sequence in 3 ways: the previous notation, defining a formula, and writing out the terms of the sequence. Note that n doesn't have to start at 1.

1. $$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} \qquad a_n = \frac{n}{n+1} \qquad \left\{\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5}, \ldots,\ \frac{n}{n+1}, \ldots\right\}$$

2. $$\left\{\frac{(-1)^n(n+1)}{3^n}\right\} \qquad a_n = \frac{(-1)^n(n+1)}{3^n} \qquad \left\{-\frac{2}{3},\ \frac{3}{9},\ -\frac{4}{27},\ \frac{5}{81}, \ldots,\ \frac{(-1)^n(n+1)}{3^n}, \ldots\right\}$$

3.

$$\{\sqrt{n-3}\}_{n=3}^{\infty} \quad a_n = \sqrt{n-3},\ n \geq 3 \quad \{0,\ 1,\ \sqrt{2},\ \sqrt{3}, \ldots,\ \sqrt{n-3}, \ldots\}$$

4.

$$\left\{\cos\frac{n\pi}{6}\right\}_{n=0}^{\infty} \quad a_n = \cos\frac{n\pi}{6},\ n \geq 0 \quad \left\{1,\ \frac{\sqrt{3}}{2},\ \frac{1}{2},\ 0, \ldots,\ \cos\frac{n\pi}{6}, \ldots\right\}$$

Example 12.1.2. Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5},\ -\frac{4}{25},\ \frac{5}{125},\ -\frac{6}{625},\ \frac{7}{3125}, \ldots\right\}$$

Solution. We are given the first five terms. The numerator of the fractions start at 3 and increase by 1, so the nth term will have numerator $n + 1$. The demoninators are the powers of 5, so a_n has denominator 5^n. The signs of the terms alternate between positive and negative, so we need to multiply by a power of 1. The factor $(-1)^n$ means we start with a negative term, so here we use $(-1)^{n-1}$ or $(-1)^{n+1}$ because we start with a positive term. Therefore,

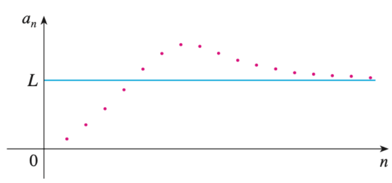
$$a_n = (-1)^{n-1}\frac{n+2}{5^n}$$

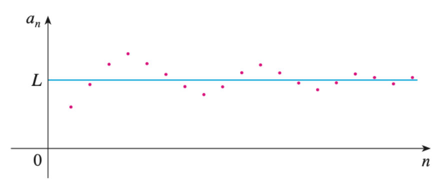
Definition 12.1.1. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as wel like by taking n sufficiently large. If $lim_{n\to\infty}a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

The figure below graphs examples of two sequences that have the limit L.





A more precise version of the previous definition is

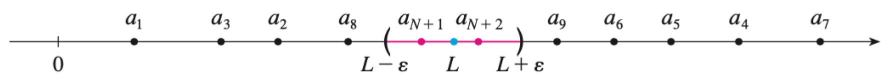
Definition 12.1.2. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

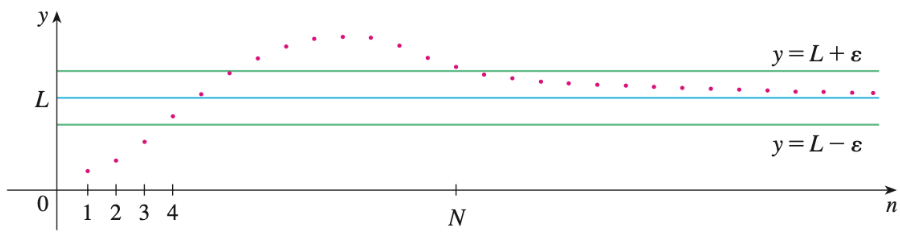
if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$|a_n - L| < \varepsilon \quad \text{whenever } n > N$$

No matter how small an interval $(L-\varepsilon\ L+\varepsilon)$ is chosen, there exists an N such that all terms of the sequence from a_{N+1} onward must lie in that interval.



The points on the graph of a_n must lie between the horizontal lines $y = L+\varepsilon$ and $y = L-\varepsilon$ if $n > N$. This picture must be valid no matter how small ε is chosen, but usually a smaller ε requires a larger N.



The only difference between $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{x\to\infty} f(x) = L$ is that n is required to be an integer.

Theorem 12.1.1. If $\lim\limits_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an interger, then $\lim_{n\to\infty} = a_n = L$.

Since we know that $\lim\limits_{x\to\infty}(1/x^r) = 0$ when $r > 0$, we can use the previous theorem to get

Definition 12.1.3.

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \quad \text{if } r > 0$$

If a_n grows as n grows, we use the notation $\lim_{n\to\infty} a_n = \infty$. We say that a_n diverges to ∞.

Definition 12.1.4. $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$a_n > M \quad \text{whenever } n > N$$

Definition 12.1.5 (Limit Laws for Sequences (similar to original Limit Laws)**).** If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

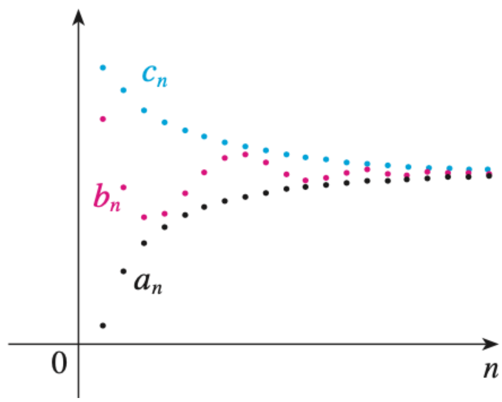
$$\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n \qquad\qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } b_n \neq 0$$

$$\lim_{n\to\infty} (a_n^p) = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

Theorem 12.1.2 (**The Squeeze Theorem for Sequences** (same as original))**.** If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.



Theorem 12.1.3. If $\lim_{n\to\infty} |a_n| = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Example 12.1.3. Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ if it exists.

Solution.

$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{so} \quad \lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

Example 12.1.4. Find $\lim_{n\to\infty} \frac{n}{n+1}$.

Solution. Divide the numberator and denominator by the highest power of n and then use the Limit Laws.

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}}$$
$$= \frac{1}{1+0} = 1$$

Example 12.1.5. Calculate $\lim_{n\to\infty} \frac{\ln n}{n}$.

Solution. Both the numerator and demoninator approach infinity as $n \to \infty$. We can't apply l'Hospital;s Rule directly because it applies to functions, not sequences. However, we can apply it to the related function.

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0 \quad \text{so} \quad \lim_{n\to\infty} \frac{\ln n}{n} = 0$$

Example 12.1.6. Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

Solution. If we write out the terms of the sequence, we get $\{-1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1, \ldots\}$. Since the terms oscillate between 1 and -1, a_n does not approach any number. Thus, $\lim_{n\to\infty}(-1)^n$ does not exist so the sequence $\{(-1)^n\}$ is divergent.

Example 12.1.7. Discuss the convergence of the the sequence $a_n = n!/n^n$.

Solution. Both the numerator and denominator approach infinity as $n \to \infty$, but we have no corresponding functions to use l'Hospital's Rule because $x!$ is not defined when x is not an integer. if we write the general forumula for the sequence, we get

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}{n \cdot n \cdot n \cdot \ldots \cdot n} = \frac{1}{n}\left(\frac{2 \cdot 3 \cdot \ldots \cdot n}{n \cdot n \cdot n \cdot \ldots \cdot n}\right)$$

The expression in the parenthesis is at most 1 because the numerator is less than (or equal) to the denominator, so

$$0 < a_n \leq \frac{1}{n}$$

We can use the squeeze theorem because both 0 and $1/n \to 0$ as $n \to \infty$, so $a_n \to \infty$ as $n \to \infty$.

Example 12.1.8. Determine if the sequences below converge. If they do, find the limits as $n \to \infty$.

1. $\frac{\sin n}{n}$
2. ne^{-n}

Solution. 1. $\frac{\sin n}{n}$ converges to 0 by the squeeze theorem

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq -\frac{1}{n}\ ,\ \lim_{n\to\infty} \frac{1}{n} = 0\ \text{, so}$$
$$0 \leq \frac{\sin n}{n} \leq 0 \implies \lim_{n\to\infty} \frac{\sin n}{n} = 0$$

2. $ne^{-n} = \frac{n}{e^n}$. The denominator e^n converges faster than the numerator n does. Use l'Hospital's Rule to get

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0 \text{ so } \lim_{n\to\infty} ne^{-n} = 0$$

Example 12.1.9. Show that if $\lim\limits_{n\to\infty} a_{2n} = L$ and $\lim\limits_{n\to\infty} a_{2n+1} = L$, $\{a_n\}$ is convergent and $\lim\limits_{n\to\infty} a_n = L$.

Solution. The solution uses the symbols $\exists$ ("exists") and $\implies$ ("implies").

$$\begin{array}{ll} \text{Since } \lim_{n\to\infty} a_{2n} = L, & \exists\, N_1 \implies |a_{2n} - L| < \varepsilon \text{ for } n > N_1 \\ \text{Since } \lim_{n\to\infty} a_{2n+1} = L, & \exists\, N_2 \implies |a_{2n+1} - L| < \varepsilon \text{ for } n > N_2 \end{array}$$

Let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$.

$$\begin{array}{ll} \text{If } n \text{ is even,} & n = 2m,\ m > N_1,\ |a_n - L| = |a_{2m} - L| < \varepsilon \\ \text{If } n \text{ is odd,} & n = 2m+1,\ m > N_2,\ |a_n - L| = |a_{2m+1} - L| < \varepsilon \end{array}$$

Therefore, $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.

Definition 12.1.6. The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 0 & \text{if } r = 1 \end{cases}$$

Definition 12.1.7. A sequence $\{a_n\}$ is **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$ ($a_1 < a_2 < a_3 < \cdots$). It is **decreasing** is $a_n < a_{n+1}$ for all $n \geq 1$. It is **monotonic** if the is either increasing or decreasing.

Example 12.1.10. The sequence $\left\{\frac{3}{n+5}\right\}$ is decreasing because

$$\frac{3}{n+5} < \frac{3}{(n+1)+5} = \frac{3}{n+6}$$

for all $n \geq 1$ (the right side is smaller because it has a larger denominator).

Example 12.1.11. Show that the sequence $a_n = \frac{n}{n^2+1}$ is decreasing.

Solution (1). We must show that $a_{n+1} < a_n$.

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$

We can simplify this by cross-multiplying. $\iff$ means "if and only if".

$$\begin{aligned} \frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} &\iff (n+1)(n^2+1) < n[(n+1)^2+1] \\ &\iff n^3 + n^2 + n + 1 < n^3 + 2n^2 + 2n \\ &\iff 1 < n^2 + n \end{aligned}$$

Since $n \geq 1$, we know that the inequality $n^2 + n > 1$ is true. Therefore, $a_{n+1} < a_n$ so $\{a_n\}$ is decreasing.

Solution (2). Consider the function $f(x) = \frac{x}{x^2+1}$:

$$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0 \quad \text{whenever } x^2 > 1$$

This, f is decreasing on $(1, \infty)$ so $f(n) > f(n+1)$. Therefore, $\{a_n\}$ is decreasing.

Theorem 12.1.4 (Monotonic Sequence Theorem)**.** Every bounded, monotonic sequence is convergent.

12.2 Series

If we try to add the terms of an infinite sequence $a_n{}_{n=1}^{\infty}$ we get the expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a **series**) and is denoted by the symbol

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

We also consider the **partial sums**

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
s_4 &= a_1 + a_2 + a_3 + a_4 \\
s_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{\infty} a_i
\end{aligned}$$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit. If the $\lim_{n\to\infty} s_n = s$ exists (as a finite number), then we call it the sum of the infinite series $\sum a_n$.

Definition 12.2.1. Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is **convergent** and we write

$$s_n = a_1 + a_2 + \cdots + a_n = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number s is the **sum** of the series. Otherwise, the series is **divergent**.

Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{\infty} a_i$$

Definition 12.2.2 (**Geometric Series**)**.** The **geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geq 1$, the geometric series is divergent.

"The sum of a convergent geometric series is $\frac{\text{first term}}{1-\text{common ratio}}$".

Proof.

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

Each term is obtained from the preceding one by multiplying it by the **common ratio** r.

If $r = 1$, then $s_n = a + a + \cdots + a = na \to \pm\infty$. Since $\lim_{n\to\infty} s_n$ doesn't exist, the geometric series diverges in this case. If $r \neq 1$, then

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rs_n &= \phantom{a + {}} ar + ar^2 + \cdots + ar^{n-1} + ar^n \end{aligned}$$

Subtracting these equations, we get

$$s_n - rs_n = a - ar^n$$

Definition 12.2.3 (**Partial Sum of a Geometric Series**)**.**

$$s_n = \frac{a(1-r^n)}{1-r}$$

If $-1 < r < 1$, we know that $r^n \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{a}{1-r} \lim_{n\to\infty} r^n = \frac{a}{1-r}$$

Thus, when $|r| < 1$, the geometric series is convergent and its sum is $a/(1-r)$.

If $r \leq -1$ or $r > 1$, the sequence $\{r^n\}$ is divergent, so $\lim_{n\to\infty} s_n$ does not exist.

Example 12.2.1. Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

Solution. The first time is $a = 5$ and the common ratio is $r = -\frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series is convergent and its sum is

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \frac{5}{1-(-\frac{2}{3})} = \frac{5}{\frac{5}{3}} = 3$$

Example 12.2.2. Write the number $2.3\overline{17} = 2.3171717\ldots$ as a ratio of integers.

Solution.

$$2.3171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

After the first term, we have a geometric series with $a = \frac{17}{10^3}$ and $r = 1/10^2$.

$$\begin{aligned} 2.3\overline{17} &= 2.3 + \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} = 2.3 + \frac{\frac{17}{1000}}{\frac{99}{100}} \\ &= \frac{23}{10} + \frac{17}{990} = \frac{1147}{495} \end{aligned}$$

Example 12.2.3. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

Solution. THis is not a geometric series, so we go back to the definition of a convergent series and compute the partial sums.

$$s_n = \sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \ + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$$

We simplify this expression if we use the **partial fraction decomposition**

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

Thus, we have

$$\begin{aligned} s_n &= \sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1}\right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \qquad \text{so} \\ \lim_{n\to\infty} s_n &= \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1 \end{aligned}$$

Therefore, the given series is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Definition 12.2.4. The **harmonic series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Theorem 12.2.1. If the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

NOTE 1 With any *series* $\sum a_n$, we associate two *sequences*: the sequence $\{s_n\}$ of its partial sums and the sequence $\{a_n\}$ of its terms. If $\sum a_n$ is convergent, then the limit of the sequence $\{s_n\}$ is s (the sum of the series) and the limit of the sequence $\{a_n\}$ is 0.

NOTE 2 The converse is not true in general. If $\displaystyle\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

Proof. Let $s_n = \sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots + a_n$. Then $a_n = s_n - s_{n-1}$. Since $\sum a_n$ is convergent, the sequence $\{s_n\}$ is convergent. Let $\lim_{n\to\infty} s_n = s$. Since $n-1 \to \infty$ as $n \to \infty$, we also have $\lim_{n\to\infty} s_{n-1} = s$. Therefore

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} \\ &= s - s = 0\end{aligned}$$

Definition 12.2.5 (The Test for Divergence). If $\displaystyle\lim_{n\to\infty} a_n$ does not exist or if $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

Example 12.2.4. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

Solution.

$$\lim_{n\to\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2}{5n^2+4} = \sum_{n=1}^{\infty} \frac{1}{5+4/n^2} = \frac{1}{5} \neq 0$$

So the series diverges by the Test for Divergence.

NOTE 3 If we find that $\lim_{n\to\infty} a_n \neq 0$, we know that $\sum a_n$ is divergent. If we find that $\lim_{n\to\infty} a_n = 0$, we know *nothing* about the convergence or divergence about $\sum a_n$.

Theorem 12.2.2. If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$.

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Example 12.2.5. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$.

Solution. The series $\sum 1/2^n$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$, so

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

We found that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

So the given series is convergent and

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) &= 3\sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= 3 \cdot 1 + 1 = 4\end{aligned}$$

NOTE 4 A finite number of terms doesn't affect the convergence or divergence of a series. For instance, suppose that we were able to show that the series $\displaystyle\sum_{n=4}^{\infty} \frac{n}{n^3+1}$ is convergent. Since

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} = \sum_{n=4}^{\infty} \frac{n}{n^3+1}$$

we can conclude that the entire series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ is convergent.

Similarly, if it is known that the series $\displaystyle\sum_{n=N+1}^{\infty} a_n$ converges, then the full series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

is also convergent.

12.3 The Integral Test and Estimates of Sums

Definition 12.3.1 (**The Integral Test**)**.** Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty}$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\ dx$ is convergent. In other words:

(i) If $\displaystyle\int_1^{\infty} f(x)\ dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty}$ is convergent.

(i) If $\displaystyle\int_1^{\infty} f(x)\ dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty}$ is divergent.

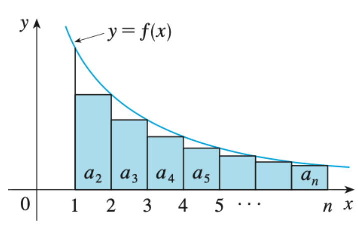
Note When we use the Integral Test, it is not necessary to start the series or the integral at $n = 1$. For instance, in testing the series

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} \qquad \text{we use} \qquad \int_1^{\infty} \frac{1}{(n-3)^2}\ dx$$

Also, it is not necessary that f is always decreasing; it is important that f is *ultimately* decreasing.

Proof. We will prove the convergence and divergence of the Integral Test for the general series $\sum a_n$

(i) **Convergence**



The area of the first shaded rectangle is $f(2) = a_2$. Because there is always space underneath the curve, the sum of the area of the shaded triangles from 1 to n is always less than the area under the curve (since f is decreasing).

$$a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)\ dx$$

If $\int_1^{\infty} f(x)\ dx$ is convergent, then

$$\sum_{i=2}^{n} a_i \leq \int_1^n f(x)\ dx \leq \int_1^{\infty} f(x)\ dx$$

since $f(x) \geq 0$. Therefore,

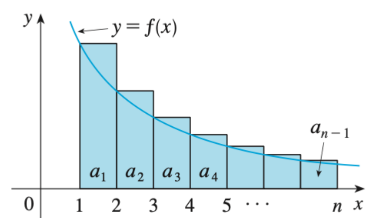
$$s_n = a_1 + \sum_{i=2}^{n} a_i \leq a_1 + \int_1^{\infty} f(x)\ dx = M \quad \text{(random variable)}$$

Since $s_n \leq M$ for all n, the sequence $\{s_n\}$ is bounded above. Also

$$s_{n+1} = s_n + a_{n+1} \geq s_n$$

since $a_{n+1} = f(n+1) \geq 0$. Thus, $\{s_n\}$ is an increasing bounded sequence so it it convergent by the Monotonic Sequence Theorem. This means that $\sum a_n$ is convergent.

(ii) **Divergence**



Because there is always space above the curve, the sum of the area of the shaded triangles from 1 to n is always greater than the area under the curve.

$$\int_1^n f(x)\ dx \leq a_1 + a_2 + \cdots + a_{n-1}$$

If $\int_1^{\infty} f(x)\ dx$ is divergent, then $\int_1^n f(x)\ dx \to \infty$ as $n \to \infty$ because $f(x) \geq 0$. But

$$\int_1^n f(x)\ dx \leq \sum_{i=1}^{n-1} a_i = s_{n-1}$$

so $s_{n-1} \to \infty$. This implies that $s_n \to \infty$ so $\sum a_n$ is diverges.

Example 12.3.1. Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

Solution. The function $f(x) = 1/(x^2+1)$ is continuous, positive, and decreasing on $[1, \infty)$ so we use the Integral Test:

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^2+1}\ dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\ dx = \lim_{t\to\infty} \tan^{-1} x \Big]_1^t \\ &= \lim_{t\to\infty} \left(\tan^{-1} t - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

Thus, $\int_1^{\infty} \frac{1}{x^2+1}\ dx$ is a convergent integral. The series $\sum 1/(n^2+1)$ is convergent by the Integral Test.

Definition 12.3.2. The ***p*-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

For $p = 1$, the series is a harmonic series.

Proof. If $p < 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$. If $p = 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 1$. In either case, $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$, so the p-series diverges by the Test for Divergence.

If $p > 0$, then the function $f(x) = \frac{1}{x^p}$ is clearly continuous, positive, and decreasing on $[1, \infty)$. We know that

$$\int_1^{\infty} \frac{1}{x^p} \text{ converges if } p > 1 \text{ and diverges if } p \leq 1$$

Using the Integral Test, the series $\sum 1/n^p$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

Example 12.3.2.

(a) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$$

is convergent because it is a p-series with $p = 3 > 1$.

(b) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$$

is divergent because it is a p-series with $p = \frac{1}{3} < 1$.

NOTE We should *not* infer that the sum of the series is equal to the value of the integral from the Integral Test. In fact,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{whereas} \quad \int_n^{\infty} \frac{1}{n^2} = 1$$

Therefore, in general,

$$\sum_{n=1}^{\infty} a_n \neq \int_n^{\infty} f(x)\, dx$$

Example 12.3.3. Determine whether the series $\sum -n = 1^{\infty} \frac{\ln n}{n}$ converges or diverges.

Solution. The function $\frac{\ln x}{x}$ is positive and continous for $x > 1$ because the logarithm function is continuous, but it is not obvious whether or not f is decreasing, so we take its derivative:

$$f'(x) = \frac{(1/x)x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

Thus, $f'(x) < 0$ when $\ln x > 1$, which is when $x > e$. We conclude that f is decreasing when $x > e$ so we can apply the Integral Test:

$$\int_1^{\infty} \frac{\ln x}{x}\, dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\, dx = \lim_{t\to\infty} \frac{(\ln x)^2}{2}\Bigg]_1^t$$
$$= \lim_{t\to\infty} \frac{(\ln t)^2}{2} = \infty$$

Since this improper integral is divergent, the series $\sum(\ln n)/n$ is also divergent by the Integral Test.

Estimating the Sum of a Series

We can show if a series $\sum a_n$ is converging. Now we want to find an approximation to the sum s of the series. ANy partial sum s_n is an approximation to s because $\lim_{n\to\infty} s_n = s$, but *how good is that approximation*? To find out, we need to estimate the size of the **remainder**

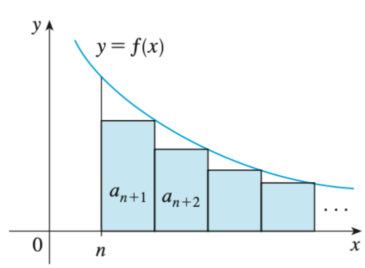
$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

The remainder R_n is the *error* made when s_n, the sum of the first n terms, is used as an approximation of the total sum.

Definition 12.3.3 (**Remainder Estimate for the Integral Test**). Suppose that $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

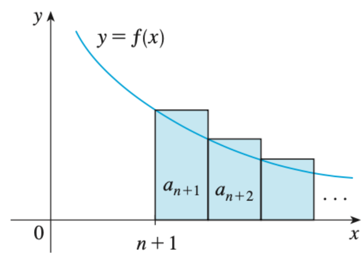
$$\int_{n+1}^{\infty} f(x)\, dx \leq R_n \leq \int_n^{\infty} f(x)\, dx$$

Proof. We use the same concept as the Integral test, assuming that f is decreasing on $[n, \infty)$.



We compare the sum of the area of the rectangles with the area under $y = f(x)$ for $x > n$ to see that

$$R_n = a_{n+1} + a_{n+2} + \cdots \le \int_n^\infty f(x)\ dx$$



Similarly, we see that

$$R_n = a_{n+1} + a_{n+2} + \cdots \ge \int_n^\infty f(x)\ dx$$

Example 12.3.4. (a) Approximate the sum of the series $\sum 1/n^3$ by using the sum of the first 10 terms. Approximate the error involved in the approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

Example 12.3.5.

$$\int_n^\infty \frac{1}{x^3}\ dx = \lim_{t\to\infty} \left[-\frac{1}{2x^2}\right]_n^t = \lim_{t\to\infty} \left(-\frac{1}{2t^2} + \frac{1}{2n^2}\right) = \frac{1}{2n^2}$$

(a)

$$\sum_{n=1}^\infty \frac{1}{n^3} \approx \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{10^3} \approx 1.1975$$

According to the remainder estimate, we have

$$R_{10} \le \int_{10}^\infty \frac{1}{x^3}\ dx = \frac{1}{2(10)^2} = \frac{1}{200}$$

So the size of the error is at most 0.005.

(b) Accuracy to within 0.0005 means that we have to find a value of n such that $R_n \le 0.0005$. Since

$$R_n \le \int_{10}^\infty \frac{1}{x^3}\ dx = \frac{1}{2n^2}$$

We want $$\frac{1}{2n^2} \le 0.0005$$

Solving this inequality, we get

$$n^2 > \frac{1}{0.001} = 1000 \quad \text{or} \quad n > \sqrt{1000} \approx 31.6$$

We need 32 terms to ensure accuracy to within 0.0005.

If we add s_n to each side of the inequality of the Remainder Estimate for the Integral Test, we get

Definition 12.3.4.

$$s_n + \int_{n+1}^{\infty} f(x)\, dx \le s \le s_n + \int_{n}^{\infty} f(x)\, dx$$

because $s_n + R_n = s$. These inequalities give a lower bound and an upper bound for s. They provide a more accurate approximation than the partial sum s_n does.

Example 12.3.6. Use the improved remainder estimate with $n = 10$ to estimate the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$.

Solution.

$$s_{11} + \int_{11}^{\infty} \frac{1}{x^3}\, dx \le s \le s_{10} + \int_{10}^{\infty} \frac{1}{x^3}\, dx$$

We know from the previous example that

$$\int_{n}^{\infty} \frac{1}{x^3}\, dx = \frac{1}{2n^2}$$

so $$s_{11} + \frac{1}{2(11)^2} \le s \le s_{10} + \frac{1}{2(10)^2}$$

Using $s_{10} \approx 1.197532$, we get

$$1.201664 \le s \le 1.202532$$

If we approximate s by the midpoint of this interval, then the error is at most half the length of the interval, so

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2021 \quad \text{with error} < 0.0005$$

We get a much better estimate with this method than the estimate $s \approx s_n$ in the previous example. Also, we only had to use 10 terms to get the error smaller than 0.0005 instead of 32 terms.

12.4 The Comparison Tests

In comparison tests, the idea is to compare a given series with a series that is know to be convergent or divergent.

Definition 12.4.1 (The Comparison Test). Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

In other words,

(i) If we have a series whose terms are *smaller* than those of a known *convergent* series, then our series is also convergent.

(ii) If we have a series whose terms are *larger* than those of a known *divergent* series, then our series is also divergent.

Proof. Let

$$s_n = \sum_{i=1}^{n} a_i \qquad t_n = \sum_{i=1}^{n} b_i \qquad t = \sum_{n=1}^{n} b_n$$

(i) **Convergence**

The sequences $\{s_n\}$ and $\{t_n\}$ are increasing $(s_{n+1} = s_n + a_{n+1} \geq s_n)$ because both series have positive terms. Also $t_n \to t$, so $t_n \leq t$ for all n. This means that $\{s_n\}$ is increasing and bounded above and therefore converges by the Monotonic Sequence Theorem. Thus, $\sum a_n$ converges.

(ii) **Divergence**

If $\sum b_n$ is divergent, then $t \to \infty$ (since $\{t_n\}$ is increasing). BUt $a_i \geq b_i$ so $s_n \geq t_n$. Thus, $s_n \to \infty$. Therefore, $\sum a_n$ diverges.

Example 12.4.1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges or diverges.

Solution. As n gets larger, the dominant term in the denominator is $2n^2$, so we compare the given series with the series $\sum 5/(2n^2)$. Observe that

$$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$$

because the left side has a bigger denominator. We know the

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent because it is a constant times a p-series with $p = 2 > 1$. Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ is convergent by the Comparison Test.

NOTE Although the condition $a_n \leq b_n$ for $a_n \geq b_n$ in the Comparison Test is given for all n, we only need to verify it for $n \geq N$, where N is some fixed integer, because the convergence of a series is not affected by a finite number.

Definition 12.4.2 (The Limit Comparison Test). Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or diverge.

Proof. Let m and M be positive numbers such that $m < c < M$. Because a_n/b_n is close to c for a large n, there is an integer N such that

$$m < \frac{a_n}{b_n} < M \quad \text{when } n > N \quad \text{so}$$

$$mb_n < a_n < Mb_n \quad \text{when } n > N$$

We can conclude the following:

(i) If $\sum b_n$ converges, so does $\sum Mb_n$, so $\sum a_n$ converges by the Comparison Test.

(i) If $\sum b_n$ diverges, so does $\sum Mb_n$, so $\sum a_n$ diverges by the Comparison Test.

Example 12.4.2. Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

Solution. We use the limit comparison test with

$$a_n = \frac{1}{2^n - 1} \qquad b_n = \frac{1}{2^n}$$

and obtain

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1(2^n - 1)}{1/2^n} = \lim_{n \to \infty} \frac{2^n}{2^n - 1} = \lim_{n \to \infty} \frac{1}{1 - 1/2^n} = 1 > 0$$

Since this limit exists and $\sum 1/2^n$ is a convergent geometric series, the given series converges by the Limit Comparison Test.

Estimating Sums

We used the Comparison test to series $\sum a_n$ by comparison with $\sum b_n$. We can also use it to estimate the sum by compparing remaindeds. We continue to consider the remainder R_n and consider T_n for the comparison series $\sum b_n$ as the corresponding remainder.

$$\begin{aligned} R_n &= s - s_n = a_{n+1} + a_{n+2} + \cdots \\ T_n &= t - t_n = b_{n+1} + b_{n+2} + \cdots \end{aligned}$$

Since $a_n \leq b_n$, $R_n \leq T_n$.

Example 12.4.3. Use the sum of the first 100 terms to approximate the sum of the series $\sum 1/(n^3+1)$. Estimate the error involved in this approximation.

Solution. Since

$$\frac{1}{n^3+1} < \frac{1}{n^3}$$

the given series is convergent by the Comparison Test. Using the Remainder Estimate for the Integral Test in section 12.3 we found that

$$T_n \leq \int_n^\infty \frac{1}{x^3}\, dx = \frac{1}{2n^2}$$

Therefore, the remainder R_n for the given series satisfies

$$R_n \leq T_n \leq \frac{1}{2n^2}$$

With $n = 100$ we have

$$R_{100} \leq \frac{1}{2(100)^2} = 0.00005$$

Using a calculator, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^3+1} \approx \sum_{n=1}^{100} \frac{1}{n^3+1} \approx 0.6864538$$

with error less than 0.00005.

12.5 Alternating Series

12.6 Absolute Convergence and the Ratio and Root Tests

12.7 Strategy for Testing Series

12.8 Power Series

12.9 Representation of Functions as Power Series

12.10 Taylor and Maclaurin Series

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Vectors and the Geometry of Space

Chapter 14

Vector Functions

14.1 Vector Functions and Space Curves

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