12.10 Taylor and Maclaurin Series

Theorem 12.10.1. If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Substituting this formula for c_n back into the series, we see that *if* f has a power series expansion at a, then it must be of the following form.

Definition 12.10.1. The **Taylor series of the function** f **at** a (or **about** a or **centered at** a) is of the form

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

Definition 12.10.2. The **Maclaurin series** is a special case of the Taylor series when $a = 0$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

Example 12.10.1. Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence. Also, find the Taylor series at a.

Solution. If $f(x) = e^x$, then $f^n(x) = e^x$, so $f^n(0) = e^x = e^0 = 1$ for all n. Therefore, the Taylor series for f at 0 (Maclaurin series) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

To find the radius of convergence we let $a_n = x^n/n!$, so

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 < 1$$

so the series converges for all x by the Ratio Test and the radius of convergence is $R = \infty$.

The Taylor series at a is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n$$

Theorem 12.10.2. If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth degree polynomial if f at a, R_n is the remainder of the Taylor series, and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Proof. We are determining under what circumstances is a function equal to the sum of its Taylor Series. In other words, if f has derivatives of all orders, when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

As with any convergent series, this means that $f(x)$ is the limit of the sequence of partial sums. In the case of the Taylor series, the partial sums are

$$\begin{aligned} T_n(x) &= \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{n!}(x-a)^i \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

For example, for the polynomial function $f(x) = e^x$, the Taylor polynomials at 0 (or Maclaurin polynomials) with $n = 1,\ 2,$ and 3 are

$$T_1(x) = 1 + x \qquad T_2(x) = 1 + x + \frac{x^2}{2!} \qquad T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n\to\infty} T_n(x)$$

If we let

$$R_n(x) = f(x) - T_n(x) \quad \text{so that} \quad f(x) = T_n(x) + R_n(x)$$

then $R_n(x)$ is the remainder of the Taylor series. If we can somehow show that $\lim_{n\to\infty} R_n(x) = 0$, then it follows that

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty}[f(x) - R_n(x)] = f(x) - \lim_{n\to\infty} R_n(x) = f(x)$$

In trying to show that $\lim_{n\to\infty} R_n(x) = 0$ for a specific function f, we usually use the following fact.

Definition 12.10.3 (Taylor's Inequality). If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \le d$$

It is helpful to use the following fact.

Definition 12.10.4. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for every real number x

Definition 12.10.5.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for every real number } x$$

If we plug in $x = 1$, we get the following expression for the number e as a sum of an infinite series:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{2}{2!} + \frac{3}{3!} + \cdots$$

Proof. If $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$ for all n. If d is a positive number and $|x| \le d$, then $|f^{(n+1)}(x)| = e^x \le e^d$. So Taylor's inequality, with $a = 0$, and $M = e^d$, says that

$$|R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1} \qquad \text{for} |x| \le d$$

Notice that we have the same constant $M = e^d$ for every value of n. But because $\lim_{n\to\infty} \frac{x^n}{n!} = 0$, we have

$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

It follows from the Squeeze Theorem that $\lim_{n\to\infty} |R_n(x)| = 0$ and therefore $\lim_{n\to\infty} R_n(x) = 0$ for all values of x.

Definition 12.10.6.

(i) $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x

(ii) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ for all x

Proof. The strategy is to find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x, and then differentiate it to find the Maclaurin series for $\cos x$.

(i) We arrange our computation in two columns:

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

Since the derivatives repeat in a cycle of four, we can write the Maclaurin series as

$$\begin{aligned} f(0) &+ \frac{f'(0)}{1!} + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!} + \cdots \\ &= x - \frac{x^3}{x!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

Since $f^{(n+1)}(x)$ is $\pm \sin x$ or $\pm \cos x$, we know that $|f^{(n+1)}(x)| <\leq 1$ for all x, so we can take $M = 1$ in Taylor's inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x|^{n+1} = \frac{|x|^{n+1}}{(n+1)!}$$

Since $\lim_{n\to\infty} \frac{x^n}{n!} = 0$, the right side of this inequality approaches 0 as $n \to \infty$, so $|R_n(x)| \to 0$ by the Squeeze Theorem. It follows that $R_n(x) \to 0$ as $n \to \infty$, so $\sin x$ is equal to the sum of its Maclaurin series.

(ii) We could proceed directly as the previous proof but it is easier to differentiate the Maclaurin series for $\sin x$.

$$\begin{aligned} \cos x &= \frac{d}{dx} \sin x = \frac{d}{dx}\left(x - \frac{x^3}{x!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right) \\ &= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \end{aligned}$$

Since the Maclaurin series for $\sin x$ converges for all x, the differentiated series for $\cos x$ also converges for all x, so

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{for all } x$$

Example 12.10.2. Find the Maclaurin series for the function $f(x) = x \cos x$.

Solution. Multiply the series for $\cos x$ by x.

$$x \cos x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}$$

Important Maclaurin series	Interval of Convergence
$\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$(-1, 1)$
$e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3} + \cdots$	$(-\infty, \infty)$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{x!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$	$(-\infty, \infty)$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$	$(-\infty, \infty)$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = x - \dfrac{x^3}{x!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$	$(-1, 1]$

Example 12.10.3.

(a) Evaluate $\int e^{-x^2}\, dx$ as an infinite series.

(b) Evaluate $\int_0^1 e^{-x^2}\, dx$ correct to within an error of 0.001.

Solution.

(a) First we find the Maclaurin series for $e^{-x^2}\, dx$ by replacing x with $-x^2$ in the series for e^x given in the table for Maclaurin series. Thus, for all values of x,

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3} + \cdots$$

Now integrate by term:

$$\begin{aligned}\int e^{-x^2}\, dx &= \int \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3} + \cdots\right) dx \\ &= C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots\end{aligned}$$

The series converges for all x because the original series e^{-x^2} converges for all x.

(b)

$$\int_0^1 e^{-x^2}\, dx = \left[x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 3!} - \cdots\right]_0^1$$
$$= 1 - \tfrac{1}{3} + \tfrac{1}{10} - \tfrac{1}{42} + \tfrac{1}{216} - \cdots$$
$$\approx 1 - \tfrac{1}{3} + \tfrac{1}{10} - \tfrac{1}{42} + \tfrac{1}{216} \approx 0.7475$$

The Alternating Series Estimation THeorem shows that the error involved in this approximation is less than

$$\frac{1}{11 \cdot 5!} = \frac{1}{1320} < 0.001$$

Multiplication and Division of Power Series

If power series are added or subtracted they behave like polynomials. In fact, they can also be multiplied and divided like polynomials. We only find the first few terms because they are the most important ones and the calculations for the later terms become tedious.

Example 12.10.4. Find the first three nonzero terms in the Maclaurin series for (a) $e^x \sin x$ and (b) $\tan x$.

Solution.

(a) Using the Maclaurin series for e^x and $\sin x$ from the table,

$$e^x \sin^x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3} + \cdots\right)\left(x - \frac{x^3}{x!} + \cdots\right)$$

We multiply these expressions like polynomials and get

$$e^x \sin^x = x + x^2 + \tfrac{1}{3}x^3 + \cdots$$

(b) Using the Maclaurin series in the table, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}$$

Use long-division to get

$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots$$