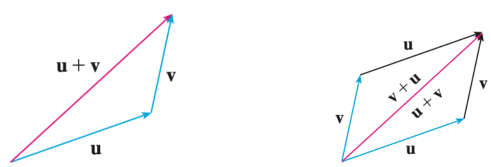
## 13.2 Vectors

A **vector** indicates a quantity that has both magnitude and direction and is often represented by an arrow. The length of the arrow represents its magnitude and the direction represents the vector's direction. A vector is generally typed in boldface ($\mathbf{v}$) and written with an arrow above the letter ($\vec{v}$).

Suppose a particle moves from $A$ to $B$, so its **displacement vector** $\mathbf{v}$ is $\vec{AB}$. The vector has **initial point** $A$ and **terminal point** $B$ and the vector is indicated by $\mathbf{v} = \vec{AB}$. Suppose another vector $\mathbf{u}$ has the same length and direction as $\mathbf{v}$ even though it is in a different position. We can say that $\mathbf{u}$ and $\mathbf{v}$ are **equivalent** (or **equal**) and we write $\mathbf{u} = \mathbf{v}$. The **zero vector**, denoted by $\mathbf{0}$, has length 0. It is the only vector with no specific direction.

**Definition 13.2.1** (**Definition of Vector Addition**)**.** If $\mathbf{u}$ and $\mathbf{v}$ are vectors positioned so the initial point of $\mathbf{v}$ is at the terminal point of $\mathbf{u}$, then the **sum** $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$.

The definition of vector addition is sometimes called the **Triangle Law**. We can also use what we know about vectors to visualize the **Parallelogram Law**.




**Definition 13.2.2** (**Definition of Scalar Multiplication**)**.** If $c$ is a scalar and $\mathbf{v}$ is a vector, the the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

For instance, $2\mathbf{v}$ is the same as $\mathbf{v} + \mathbf{v}$, which has the same direction as $\mathbf{v}$ but is twice as long.

Two nonzero vectors are **parallel** if they are scaler multiples of one another. In particular, the vector $-\mathbf{v} = (-1)\mathbf{v}$ has the same length as $\mathbf{v}$ but points in the opposite direction and is called the **negative** of $\mathbf{v}$.

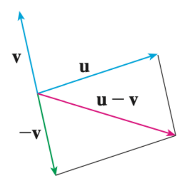
The **difference** $\mathbf{u} - \mathbf{v}$ is

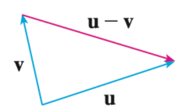
$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

We can construct $\mathbf{u} - \mathbf{v}$ in two ways:

1. Draw the negative of $\mathbf{v}$, $-\mathbf{v}$, and add it to $\mathbf{u}$ by the Parallelogram Law.

2. $\mathbf{v} + (\mathbf{u} - \mathbf{v} = \mathbf{u}$, which also equals $\mathbf{u}$, so we could construct $\mathbf{u} - \mathbf{v}$ by the Triangle Law.




## Components

If we place the initial point of a vector **a** at the origin of a rectangular coordinate system, then the terminal point of **a** has coordinates of the form $(a_1, a_2)$ or $(a_1, a_2, a_3)$ depending on the dimensions of the coordinate system. These coordinates are called the **components** of **a**.

$$\mathbf{a} = \langle a_1, a_2 \rangle \qquad \text{or} \qquad \mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

We use the notation $\langle a_1, a_2 \rangle$ that refers to a vector so we don't confuse it with the ordered pair $(a_1, a_2)$ that refers to a point in the plane.

**Definition 13.2.3.** Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector **a** with representation $\vec{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

**Example 13.2.1.** Find the vector represented by the directed line segment with initial point $A(2, -3, 4)$ and $B(-2, 1, 1)$.

*Solution.* The vector corresponding to $\vec{AB}$ is

$$\mathbf{a} = \langle -2 - 2,\ 1 - (-3),\ 1 - 4 \rangle = \langle -4,\ 4,\ -3 \rangle$$

The **magnitude** or **length** of the vector **v** is the length of any of its representations and is denoted by the symbol $|v|$ or $\|v\|$. By using the distance formula to compute the length of a segment $OP$, we obtain the following formulas.

**Definition 13.2.4.**
The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

**Definition 13.2.5.** If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \langle a_1 + b_1, a_2 + b_2 \rangle \\ \mathbf{a} - \mathbf{b} &= \langle a_1 - b_1, a_2 - b_2 \rangle \\ c\mathbf{a} &= \langle ca_1, ca_2 \rangle \end{aligned}$$

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$
$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$
$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

We denote $V_2$ as the set of all two-dimensional vectors and $V_3$ as the set of all three-dimensional vectors. More generally, we consider $V_n$ the set of all $n$-dimensional vectors. An $n$-dimensional vector is an ordered $n$-tuple:

$$\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$$

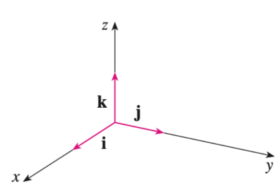
**Definition 13.2.6** (**Properties of Vectors**). If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in $V_n$ and $c$ and $d$ are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

Any vector in $V_3$ can be expressed in terms of the **standard basis vectors** $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. Such vectors are typically written with a hat. Let

$$\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle \qquad \hat{\mathbf{j}} = \langle 0, 1, 0 \rangle \qquad \hat{\mathbf{k}} = \langle 0, 0, 1 \rangle$$

Then $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are vectors that have length 1 and point in the direction of the positive $x$-, $y$-, and $z$-axes. Similarly, in two dimensions we define $\hat{\mathbf{i}} = \langle 1, 0 \rangle$ and $\hat{\mathbf{j}} = \langle 0, 1 \rangle$.



*Proof.* We prove that these any vectors in $V_3$ can be in terms of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$.

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then we write

$$\begin{aligned}\mathbf{a} &= \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\ &= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle \\ \mathbf{a} &= a_1 \hat{\mathbf{i}} + a_2 \hat{\mathbf{j}} + a_3 \hat{\mathbf{k}}\end{aligned}$$

Similarly, in two dimensions, we write

$$\mathbf{a} = \langle a_1, a_2 \rangle = a_1 \hat{\mathbf{i}} + a_2 \hat{\mathbf{j}}$$

**Example 13.2.2.** For instance,

$$\langle 1, -2, 6 \rangle = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$$

A **unit vector** is a vector whose length is 1. For instance $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are all unit vectors.

**Definition 13.2.7.** In general, if $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as $\mathbf{a}$ is

$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

*Proof.* Let $c = 1/|\mathbf{a}|$. Then $\mathbf{u} = c\mathbf{a}$ and $c$ is a positive scalar, so $\mathbf{u}$ has the same direction as $\mathbf{a}$. Also

$$|\mathbf{u}| = |c\mathbf{a}| = |c||\mathbf{a}| = \frac{1}{|\mathbf{a}|}\mathbf{a} = 1$$

**Example 13.2.3.** Find the unit vector in the same direction of the vector $2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$.

*Solution.* The given vector has length

$$|2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

We divide the vector by its length to find the unit vector with the same direction:

$$\tfrac{1}{3}(2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}}) = \tfrac{2}{3}\hat{\mathbf{i}} - \tfrac{1}{3}\hat{\mathbf{j}} - \tfrac{2}{3}\hat{\mathbf{k}}$$