

## 11.2 Calculus with Parametric Curves

We will mainly solve problems involving tangents, area, arc length, and surface area.

### Tangents

In the previous section, we saw that some curves defined by parametric equations  $x = f(t)$  and  $y = g(t)$  can also be expressed, by eliminating the parameter, in the form  $y = F(x)$ . If we substitute  $x = f(t)$  and  $y = g(t)$  in the equation  $y = F(x)$ , we get

$$g(t) = F(f(t))$$

If  $g$ ,  $f$ , and  $F$  are differentiable, the Chain Rule gives

$$g'(t) = F'(f(t))f'(t) = F'(x)f'(t)$$

If  $f'(t) \neq 0$ , we can solve for  $F'(x)$ :

**Definition 11.2.1.** The slope of the tangent to the parametric curve  $y = F(x)$  is  $F'(x)$ .

$$F'(x) = \frac{g'(t)}{f'(t)}$$

This enables us to find tangents to parametric curves without having to eliminate the parameter. We can rewrite the previous equation in an easily remembered form.

**Definition 11.2.2.** We can use this to find tangents to parametric curves without having to eliminate the parameter.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0$$

The curve has a

- horizontal tangent when  $\frac{dy}{dt} = 0$  (provided that  $\frac{dx}{dt} \neq 0$ )
- vertical tangent when  $\frac{dx}{dt} = 0$  (provided that  $\frac{dy}{dt} \neq 0$ )

This is useful when sketching parametric curves.

**Definition 11.2.3.** We can also find  $\frac{d^2y}{dx^2}$  by replacing  $y$  with  $\frac{dy}{dx}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left( \frac{dy}{dx} \right)}{\frac{dx}{dt}}$$

*Proof.* Find  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  considering  $y(t)$  and  $g(t)$ .

1.

$$\text{Chain rule: } \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} \implies \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad (\implies \text{ means "implies" })$$

2.

$$\text{Chain rule: } \frac{d}{dt} \left( \frac{dy}{dx} \right) = \left( \frac{d}{dx} \frac{dy}{dx} \right) \frac{dx}{dt} = \frac{d^2y}{dx^2} \frac{dx}{dt}$$

$$\text{Substitute: } \frac{d}{dt} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right)$$

$$\text{Quotient rule: } = \frac{\frac{d^2y}{dt^2} \frac{dx}{dt} - \frac{dy}{dt} \frac{d^2x}{dt^2}}{\left( \frac{dx}{dt} \right)^2}$$

Set equation from line 1 and line 3 equal and divide both sides by  $\frac{dx}{dt}$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{\frac{d^2y}{dt^2} \frac{dx}{dt} - \frac{dy}{dt} \frac{d^2x}{dt^2}}{\left( \frac{dx}{dt} \right)^2 \left( \frac{dx}{dt} \right)} \\ &= \frac{\frac{d^2y}{dt^2} \frac{dx}{dt} - \frac{dy}{dt} \frac{d^2x}{dt^2}}{\left( \frac{dx}{dt} \right)^3} \end{aligned}$$

**Example 11.2.1.** A curve C is defined by the parametric equations  $x = t^2$ ,  $y = t^3 - 3t$ .

1. Show that  $C$  has two tangents at the point  $(3,0)$  and find their equations.
2. Find the points on  $C$  where the tangent is horizontal or vertical.
3. Determine where the curve is concave upward or downward.

*Solution.* A curve C is defined by the parametric equations  $x = t^2$ ,  $y = t^3 - 3t$ .

1. Rewrite  $y = t^3 - 3t = t(t^2 - 3) = 0$  when  $t = 0$  or  $t = \pm\sqrt{3}$ . This indicates that C intersects itself at  $(3,0)$ .

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t} = \frac{3}{2} \left( t - \frac{1}{t} \right)$$

$$t = \pm\sqrt{3} \rightarrow dy/dx = \pm 6/(2\sqrt{3})$$

so the equations of the tangents at  $(3,0)$  are

$$y = \sqrt{3}(x - 3) \quad \text{and} \quad y = -\sqrt{3}(x - 3)$$

2. C has a horizontal tangent when  $dy/dx = 0$ . In other words, when  $dy/dt = 0$  and  $dx/dt \neq 0$ .  $dy/dt = 3t^2 - 3 = 0$  when  $t^2 = 1$  so  $t = \pm 1$ . This means there are horizontal tangents on C at (1,-2) and (-1,2). C has a vertical tangent when  $dx/dt = 2t = 0$ , so  $t = 0$ . This means C has a vertical tangent at (0,0).
3. To determine concavity we calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{3}{2}\left(1 + \frac{1}{t^2}\right)}{2t} = \frac{3(t^2 + 1)}{4t^3}$$

The curve is concave upward when  $t > 0$  and concave downward when  $t < 0$ .

## Area

We already know that area under a curve  $y = F(x)$  from  $a$  to  $b$  is  $A = \int_a^b F(x)dx$ . We can apply this to parametric equations using the Substitution Rule for Definite Integrals.

**Definition 11.2.4.** If the curve  $C$  is given by parametric equations  $x = f(t)$  and  $y = g(t)$  and  $t$  increases from  $\alpha$  to  $\beta$ ,

$$A = \int_a^b ydx = \int_\alpha^\beta g(t)f'(t)dt$$

(Switch  $\alpha$  to  $\beta$  if the point on  $C$  at  $\beta$  is more left than  $\alpha$ .)

**Example 11.2.2.** Find the area under one arch of the cycloid  $x = r(\theta - \sin \theta)$ ,  $y = r(1 - \cos \theta)$ .

*Solution.* One arch of the cycloid is given by  $0 \leq \theta \leq 2\pi$ . Using the Substitution Rule with  $y = r(1 - \cos \theta)$  and  $dx = r(1 - \cos \theta)d\theta$ , we have

$$\begin{aligned} A &= \int_0^{2\pi} ydx = \int_0^{2\pi} r(1 - \cos \theta)r(1 - \cos \theta)d\theta \\ &= r^2 \int_0^{2\pi} (1 - \cos \theta)^2 d\theta = r^2 \int_0^{2\pi} (1 - 2\cos \theta + \cos^2 \theta) d\theta \\ &= r^2 \int_0^{2\pi} \left[ 1 - 2\cos \theta + \frac{1}{2}(1 + \cos 2\theta) \right] d\theta \\ &= r^2 \left[ \frac{3}{2}\theta - 2\sin \theta + \frac{1}{4}\sin 2\theta \right]_0^{2\pi} \\ &= r^2 \left( \frac{3}{2} \cdot 2\pi \right) = 3\pi r^2 \end{aligned}$$

## Arc Length

We already know how to find length  $L$  of a curve  $C$  given in the form  $y = F(x)$ ,  $a \leq x \leq b$ .

**Definition 11.2.5.** If  $F'$  is continuous, then

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

If  $C$  can describe the parametric equations  $x = f(t)$  and  $y = g(t)$ ,  $\alpha \leq t \leq \beta$ , where  $dx/dt = f'(t) > 0$ . Using the substitution rule, we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2} \frac{dx}{dt} dt$$

Since  $dx/dt > 0$ , we have

**Theorem 11.2.1.** If a curve  $C$  is described by the parametric equations  $x = f(t)$ ,  $y = g(t)$ ,  $\alpha \leq t \leq \beta$ , where  $f'$  and  $g'$  are continuous on  $[\alpha, \beta]$  and  $C$  is traversed exactly once as  $t$  increases from  $\alpha$  to  $\beta$ , then the length of  $C$  is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

This is consistent with the general formula  $L = \int ds$  and  $(ds^2) = (dx^2) + (dy^2)$ .

*Proof.* Prove the length formula of a parametric curve

$$\vec{ds} = \vec{i} dx + \vec{j} dy$$

$$ds^2 = \vec{ds} \cdot \vec{ds} = \left(\vec{i} dx + \vec{j} dy\right) \cdot \left(\vec{i} dx + \vec{j} dy\right) = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$L = \int_\alpha^\beta ds = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

**Example 11.2.3.** Find the length of the unit circle as  $(x, y)$  moves both once and twice around the circle.

*Solution.* For one traversal around the unit circle,

$$x = \cos t \quad y = \sin t \quad 0 \leq t \leq 2\pi$$

so  $dx/dt = -\sin t$  and  $dy/dt = \cos t$

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t} dt \\ &= \int_0^{2\pi} dt = 2\pi \end{aligned}$$

For two traversals around the unit circle,

$$x = \sin 2t \quad y = \cos 2t \quad 0 \leq t \leq 2\pi$$

so  $dx/dt = 2 \cos 2t$  and  $dy/dt = -2 \sin 2t$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^{2\pi} \sqrt{4 \cos^2 2t + 4 \sin^2 2t} dt = \int_0^{2\pi} 2 dt = 4\pi$$

## Surface Area

We can also adapt the surface area formula to a parametric curve.

**Definition 11.2.6.** If a curve  $C$  is described by the parametric equations  $x = f(t)$ ,  $y = g(t)$ ,  $\alpha \leq t \leq \beta$ , is rotated about the **x-axis**, where  $f', g'$  are continuous and  $g(t) \geq 0$ , the surface area is

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

If the curve  $C$  is rotated about the **y-axis**, the surface area is

$$S = \int_{\alpha}^{\beta} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The generic formulas  $S = \int 2\pi y ds$  for rotation about the x-axis and  $S = \int 2\pi x ds$  for rotation about the y-axis are still valid, but for parametric curves we use

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

**Example 11.2.4.** Show that the surface area of a sphere of radius  $r$  is  $4\pi r^2$

*Solution.* The sphere is obtained by rotating the semicircle

$$x = r \cos t \quad y = r \sin t \quad 0 \leq t \leq \pi$$

about the x-axis.

$$\begin{aligned} S &= \int_0^{\pi} 2\pi r \sin t \sqrt{(-r \sin t)^2 + (r \cos t)^2} dt \\ &= 2\pi \int_0^{\pi} r \sin t \sqrt{r^2(\sin^2 t + \cos^2 t)} dt \\ &= 2\pi \int_0^{\pi} r \sin t \cdot r dt = 2\pi r^2 \int_0^{\pi} \sin t dt \\ &= 2\pi r^2 (-\cos t) \Big|_0^{\pi} = 4\pi r^2 \end{aligned}$$