12.6 Absolute Convergence and the Ratio and Root Tests

Given any series $\sum a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute value of the terms of the original series.

Definition 12.6.1. A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Notice that if $\sum a_n$ is a series with positive terms, then $\sum |a_n| = \sum a_n$ and so absolute convergence is the same as convergence in this case.

Example 12.6.1. The series

$$\sum_{n\to\infty}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent because

$$\sum_{n\to\infty}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is a convergent p-series ($p = 2$).

Definition 12.6.2. A series $\sum a_n$ is **conditionally convergent** if it is convergent but not absolutely convergent.

Definition 12.6.3. If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Proof. Observe that the inequality

$$0 \leq a_n + |a_n| \leq 2|a_n|$$

is true because $|a_n|$ is either a_n or $-a_n$. If $\sum a_n$ is absolutely convergent, then $\sum |a_n|$ is convergent, $\sum 2|a_n|$. Therefore, by the comparison test, $\sum(a_n + |a_n|)$ is convergent. Then

$$\sum a_n = \sum(a_n + |a_n|) - \sum |a_n|$$

is the difference of the two convergent series and is therefore convergent.

Definition 12.6.4. The Ratio Test

(i) If $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = L > 1$ or If $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive (no conclusion can be drawn about the convergence or divergence of $\sum a_n$).

Proof.

(i) The idea is to compare the given series with a convergent geometric series. Since $L > 1$, we can choose a number r such that $L < r < 1$. Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \quad \text{and} \quad L < r$$

the ratio $|a_{n+1}/a_n|$ will eventually be less than r; this means that there exists an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| < r \quad \text{whenever } n \geq N$$

or, equivalently,

$$a_{n+1} < r|a_n| \quad \text{whenever } n \geq N$$

Putting n successively equal to $N,\ N+1,\ N+2,\ldots$ in the previous equation, we obtain

$$\begin{aligned}
|a_{N+1}| &< |a_N|r \\
|a_{N+2}| &< |a_{N+1}|r < |a_N|r^2 \\
|a_{N+3}| &< |a_{N+2}|r < |a_N|r^3
\end{aligned}$$

and, in general,

$$|a_{N+k}| < |a_N|r^k \qquad \text{for all } k \geq 1$$

Now the series

$$\sum_{k=1}^{\infty} |a_N|r^k = |a_N|r + |a_N|r^2 + |a_N|r^3 + \cdots$$

is convergent because it is a geometric series with $0 < r < 1$. So the previous inequality , together with the Comparison Test, shows that the series

$$\sum_{n=N+1}^{\infty} |a_n| = \sum_{k=1}^{\infty} |a_{N+k}| = |a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \cdots$$

is also convergent. It follows that the series $\sum_{n=1}^{\infty} |a_n|$ is also convergent. Therefore, $\sum a_n$ is absolutely convergent.

(ii) If $|a_{n+1}/a_n| \to L > 1$ or $|a_{n+1}/a_n| \to \infty$, then the ratio $|a_{n+1}/a_n|$ will eventually be greater than 1. This means that there exists an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \quad \text{whenever } n \geq N$$

This means that $|a_{n+1}/a_n| > |a_n|$ whenever $n \geq N$ and so

$$\lim_{n\to\infty} a_n \neq 0$$

Therefore $\sum a_n$ diverges by the Test for Divergence.

(iii) The Ratio Test gives no information if $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$. For instance,

for the convergent series $\sum 1/n^2$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{1}{(n+1)^2}}{\dfrac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \frac{1}{\left(1+\dfrac{1}{n}\right)^2} \to 1 \quad \text{as } n \to \infty$$

whereas for the divergent series $\sum 1/n$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{1}{n+1}}{\dfrac{1}{n}} = \frac{n}{n+1} = \frac{1}{1+\dfrac{1}{n}} \to 1 \quad \text{as } n \to \infty$$

Therefore, if $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$, the series $\sum a_n$ might converge or diverge. In this case, the Ratio Test fails and we must use some other test.

Example 12.6.2. Test the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{3^n} for absolute convergence.$

Solution. We use the Ratio Test with $a_n = (-1)^n n^3/3^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\dfrac{(-1)^{n+1}(n+1)^3}{3(n+1)}}{\dfrac{(-1)^n n^3}{3^n}}\right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$
$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \to \frac{1}{3} < 1$$

The given series is absolutely convergent by the Ratio Test and therefore convergent.

Definition 12.6.5. The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root Test is inconclusive.

NOTE If $L = 1$ in the Ratio Test, don't try the Root Test because L will also be 1.

Example 12.6.3. Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$

Solution.

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$
$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2+\dfrac{3}{n}}{3+\dfrac{2}{n}} \to \frac{2}{3} < 1$$

The given series converges by the Root Test.