## 12.8 Power Series

**Definition 12.8.1.** A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where $x$ is a variable and the $c_n$'s are constants called the **coefficients** of the series.

For each fixed $x$, the series is a series of constants that we can test for convergence or divergence. A power series may converge for some values of $x$ and diverge for other values of $x$. The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

whose domain is the set of all $x$ for which the series converges. Notice that $f$ resembles a polynomial. The only difference is that $f$ has infinitely many terms.

For instance, if we take $c_n = 1$ for all $n$, the power series becomes the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$.

**Definition 12.8.2.** A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

is called a **power series in** $(x-a)$ or a **power series centered at** $a$ or a **power series about** $a$.

Notice that when $x = a$ all of the terms are 0 for $n \geq 1$, so the power series always converges when $x = a$.

FUN FACT A power series is a series in which each term is a power function. A **trignometric series**

$$\sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is a series whose terms are trignometric functions.

**Example 12.8.1.** For what values of $x$ is the series $\sum_{n=0}^{\infty} n! x^n$ convergent?

*Solution.* We use the Ratio Test. If we let $a_n$ denote the $n$th term of the series, then $a_n = n!x^n$. If $x \neq 0$, we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| = \lim_{n\to\infty} (n+1)|x| = \infty$$

By the Ratio Test, the series diverges when $x \neq 0$, so it converges only when $x = 0$.

**Example 12.8.2.** For what values of $x$ does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

*Solution.* Let $a_n = (x-3)^n/n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right|$$
$$= \frac{1}{1+\dfrac{1}{n}}|x-3| \to |x-3| \quad \text{as } n \to \infty$$

By the Ratio Test, the given series is absolutely convergent, and therefore convergent, when $|x-3| < 1$ and divergent when $|x-3| > 1$. We rewrite the inequality as

$$|x-3| < 1 \quad \Longleftrightarrow \quad -1 < x-3 < 1 \quad \Longleftrightarrow \quad 2 < x < 4$$

Now we know the series converges when $2 < x < 4$ and diverges when $x < 2$ or $x > 4$.

The Ratio Test gives no information when $|x-3| = 1$, so we must consider $x = 2$ and $x = 4$ separately.

(a) If we put $x = 4$ in the series, it becomes the harmonic series $\sum 1/n$, which is divergent.

(b) If $x = 2$, the series is $\sum(-1)^n/n$, which converges by the Alternating Series Test.

We can summarize our results by concluding that the power series converges for $2 \leq x < 4$.

**Theorem 12.8.1.** For a given power series $\displaystyle\sum_{n=0}^{\infty}$ there are only three possibilities:

1. The series converges only when $x = a$.
2. The series converges for all $x$.
3. There is a positive number $R$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

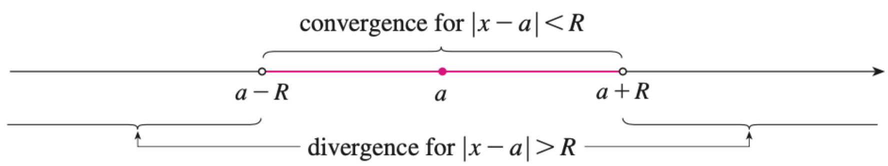
The proof of this theorem is at the end of this chapter because this theorem is more relevant than the proof.

The number $R$ in case 3 is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case 1 and $R = \infty$ in case (ii).

The **interval of convergence** of a power series is the interval that consists of just a single point $a$. in case 2, the interval is $(-\infty, \infty)$. In case 3, note that the inequality $|x - a| < R$ can be rewritten as $a - R < x < a + R$. When $x$ is an *endpoint* of the interval ($x = a \pm R$), anything can happen—the series might converge at one or both endpoints, or it might diverge at both endpoints.

Thus, in case 3 there are four possibilities for the interval of convergence:

$$(a - R, a + R) \qquad (a - R, a+] \qquad [a - R, a + R) \qquad [a - R, a + R]$$




We summarize the radius and interval for convergence for each of the examples in this section.

| | Series | Radius of Convergence | Interval of Convergence |
|---|---|---|---|
| Geometric Series | $\sum_{n=0}^{\infty} x^n$ | $R = 1$ | $(-1, 1)$ |
| Example 12.8.1 | $\sum_{n=0}^{\infty} n!x^n$ | $R = 0$ | $\{0\}$ |
| Example 12.8.2 | $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ | $R = 1$ | $[2, 4)$ |

In general, the Ratio Test (or sometimes the Root Test) should always be used to determine the radius of convergence $R$. The Ratio and Root Tests always fail when $x$ is an endpoint of the interval of convergence, so the endpoints must be checked with some other test.

**Example 12.8.3.** Find the radius and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

*Solution.* Let $a_n = (-3)^n x^n / \sqrt{n+1}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n}\right| = \left|-3x\sqrt{\frac{n+1}{n+2}}\right|$$
$$= 3\sqrt{\frac{1+(1/n)}{1+(2/n)}}|x| \to 3|x| \quad \text{as } n \to \infty$$

By the Ratio Test, the given series converges if $3|x| < 1$ and diverges if $3|x| > 1$. Thus, it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$, meaning that the radius of convergences is $R = \frac{1}{3}$.

We know the series converges in the interval $(-\frac{1}{3}, \frac{1}{3})$, but we must now test for convergence at the endpoints of this interval.

(a) If $x = -\frac{1}{3}$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$ which diverges (using the Integral Test or simply observing that it is a $p$-series with $p = \frac{1}{2} < 1$).

(b) If $x = \frac{1}{3}$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ which converges by the Alternating Series Test. Therefore, the given power series converges when $-\frac{1}{3} < x \leq \frac{1}{3}$, so the interval of convergence is $(-\frac{1}{3}, \frac{1}{3}]$.

## Proof

To prove the theorem that is found earlier in this section, we first need to prove 2 theorems.

**Theorem 12.8.2.**

1. If a power series $\sum c_n x^n$ converges when $x = b$ (where $b \neq 0$), then it converges whenever $|x| < |b|$.
2. If a power series $\sum c_n x^n$ diverges when $x = d$ (where $d \neq 0$), then it converges whenever $|x| > |d|$.

*Proof.*

1. Suppose that $\sum c_n x^n$ converges. Then, we know $\lim_{n\to\infty} c_n b^n = 0$. According to the definition of a limit of a sequence with $\varepsilon = 1$, there is a positive integer $N$ such that $|c_n b^n| < 1$ whenever $n \geq N$,

$$|c_n x^n| = \left|\frac{c_n b^n x^n}{b^n}\right| = |c_n b^n| \left|\frac{x}{b}\right|^n < \left|\frac{x}{b}\right|^n$$

If $|x| < |b|$, then $|x/b| < 1$, so $\sum |x/b|^n$ is a convergent geometric series. Therefore, by the Comparison Test, the series $\sum_{n=N}^{\infty} |c_n b^n|$ is convergent. Thus, the series $\sum c_n b^n$ is absolutely convergent and therefore convergent.

2. Suppose that $\sum c_n d^n$ diverges. If $x$ is any number such that $|x| > |d|$, then $\sum c_n x^n$ cannot converge because, by part 1, the convergence of $\sum c_n x^n$ would imply the convergence of $\sum c_n d^n$. Therefore $\sum c_n x^n$ diverges whenever $|x| > |d|$.

**Theorem 12.8.3.** For a power series $\sum c_n x^n$ there are only three possibilities

1. The series converges only when $x = 0$.
2. The series converges for all $x$.
3. There is a positive number $R$ such that the series converges if $|x| < R$ and diverges if $|x| > R$.

*Proof.* We use the preceding theorem to prove this theorem. The symbol $\in$ means "is an element of" or "in".

Suppose that neither case 1 nor case 2 is true. Then there are nonzero numbers $b$ and $d$ such that $\sum c_n x^n$ converges for $x = b$ and diverges for $x = d$. Therefore, the set $S = \{x | \sum c_n x^n \text{ converges}\}$ is not empty. By the preceding theorem, the series diverges if $|x| > |d|$, so $|x| \geq |d|$ for all $x \in S$. This says that $|d|$ is an upper bound for the set $S$. Thus, by the Completeness Axiom (see Section 12.1), $S$ has a least upper bound $R$. If $|x| > R$, then $x \notin S$, so $\sum c_n x^n$ diverges. If $|x| < R$, then $|x|$ is not an upper bound for $S$ and so there exists $b \in S$ such that $b > |x|$. Since $b \in S$, $\sum c_n b^n$ converges, so by the preceding theorem $\sum c_n x^n$ converges.

Now we are ready for the proof of the main theorem found earlier in the section. We use the preceding theorem to prove it.

*Proof.* If we can make the change of variable $u = x - a$, then the power series becomes $\sum c_n u^n$ and we can apply the preceding theorem to this series. In case 3 we have convergence for $|u| < R$ and divergence for $|u| > R$. Thus, we have convergence for $|x - a| < R$ and divergence for $|x - a| > R$.