Chapter 12

Infinite Sequences and Series

12.1 Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_n,\ \ldots$$

The number a_1 is the *first term*, a_2 is the *second term*, and in general a_n is the *nth term*. We will deal with infinite sequences exclusively so each term a_n will have a successor $a_n + 1$.

Notice that for every positive integer n there is a corresponding number a_n so a sequence can be defined as a function whose domain is the set of positive integers. We usually write a_n instead of the function notation $f(n)$.

NOTATION The sequence $a_1,\ a_2,\ a_3,\ \ldots$ is also denoted by

$$a_n \quad \text{or} \quad a_n{}_{n=1}^{\infty}$$

Example 12.1.1. Some sequences can be defined by giving a formula for the nth term. In this example, we will describe a sequence in 3 ways: the previous notation, defining a formula, and writing out the terms of the sequence. Note that n doesn't have to start at 1.

1. $$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} \qquad a_n = \frac{n}{n+1} \qquad \left\{\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5}, \ldots,\ \frac{n}{n+1}, \ldots\right\}$$

2. $$\left\{\frac{(-1)^n(n+1)}{3^n}\right\} \qquad a_n = \frac{(-1)^n(n+1)}{3^n} \qquad \left\{-\frac{2}{3},\ \frac{3}{9},\ -\frac{4}{27},\ \frac{5}{81}, \ldots,\ \frac{(-1)^n(n+1)}{3^n}, \ldots\right\}$$

3.

$$\{\sqrt{n-3}\}_{n=3}^{\infty} \quad a_n = \sqrt{n-3},\ n \geq 3 \quad \{0,\ 1,\ \sqrt{2},\ \sqrt{3}, \ldots,\ \sqrt{n-3}, \ldots\}$$

4.

$$\left\{\cos\frac{n\pi}{6}\right\}_{n=0}^{\infty} \quad a_n = \cos\frac{n\pi}{6},\ n \geq 0 \quad \left\{1,\ \frac{\sqrt{3}}{2},\ \frac{1}{2},\ 0, \ldots,\ \cos\frac{n\pi}{6}, \ldots\right\}$$

Example 12.1.2. Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots\right\}$$

Solution. We are given the first five terms. The numerator of the fractions start at 3 and increase by 1, so the nth term will have numerator $n+1$. The demoninators are the powers of 5, so a_n has denominator 5^n. The signs of the terms alternate between positive and negative, so we need to multiply by a power of 1. The factor $(-1)^n$ means we start with a negative term, so here we use $(-1)^{n-1}$ or $(-1)^{n+1}$ because we start with a positive term. Therefore,

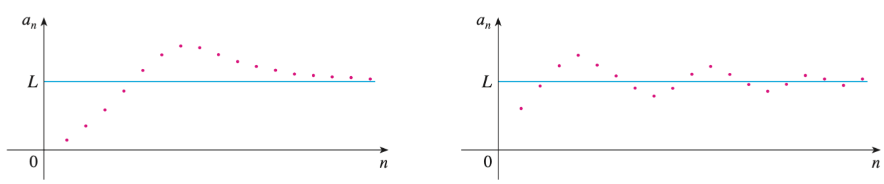
$$a_n = (-1)^{n-1}\frac{n+2}{5^n}$$

Definition 12.1.1. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as wel like by taking n sufficiently large. If $lim_{n\to\infty}a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

The figure below graphs examples of two sequences that have the limit L.



A more precise version of the previous definition is

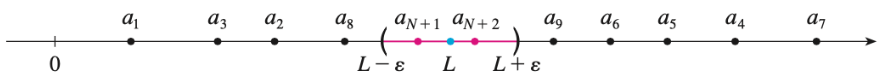
Definition 12.1.2. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

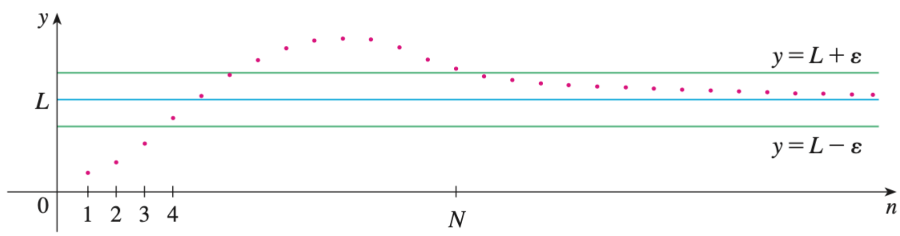
if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$|a_n - L| < \varepsilon \quad \text{whenever } n > N$$

No matter how small an interval $(L-\varepsilon\ L+\varepsilon)$ is chosen, there exists an N such that all terms of the sequence from a_{N+1} onward must lie in that interval.



The points on the graph of a_n must lie between the horizontal lines $y = L+\varepsilon$ and $y = L-\varepsilon$ if $n > N$. This picture must be valid no matter how small ε is chosen, but usually a smaller ε requires a larger N.



The only difference between $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{x\to\infty} f(x) = L$ is that n is required to be an integer.

Theorem 12.1.1. If $\lim\limits_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an interger, then $\lim_{n\to\infty} = a_n = L$.

Since we know that $\lim\limits_{x\to\infty} (1/x^r) = 0$ when $r > 0$, we can use the previous theorem to get

Definition 12.1.3.

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \quad \text{if } r > 0$$

If a_n grows as n grows, we use the notation $\lim_{n\to\infty} a_n = \infty$. We say that a_n diverges to ∞.

Definition 12.1.4. $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$a_n > M \quad \text{whenever } n > N$$

Definition 12.1.5 (Limit Laws for Sequences (similar to original Limit Laws)**).** If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

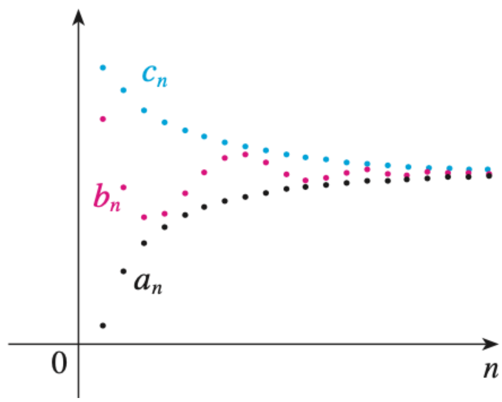
$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad\qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty}(a_nb_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } b_n \neq 0$$

$$\lim_{n\to\infty}(a_n^p) = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

Theorem 12.1.2 (**The Squeeze Theorem for Sequences** (same as original))**.** If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.



Theorem 12.1.3. If $\lim_{n\to\infty} |a_n| = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Example 12.1.3. Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ if it exists.

Solution.
$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{so} \quad \lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

Example 12.1.4. Find $\lim_{n\to\infty} \frac{n}{n+1}$.

Solution. Divide the numberator and denominator by the highest power of n and then use the Limit Laws.

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}}$$
$$= \frac{1}{1+0} = 1$$

Example 12.1.5. Calculate $\lim_{n\to\infty} \frac{\ln n}{n}$.

Solution. Both the numerator and demoninator approach infinity as $n \to \infty$. We can't apply l'Hospital;s Rule directly because it applies to functions, not sequences. However, we can apply it to the related function.

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0 \quad \text{so} \quad \lim_{n\to\infty} \frac{\ln n}{n} = 0$$

Example 12.1.6. Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

Solution. If we write out the terms of the sequence, we get $\{-1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1, \ldots\}$. Since the terms oscillate between 1 and -1, a_n does not approach any number. Thus, $\lim_{n\to\infty}(-1)^n$ does not exist so the sequence $\{(-1)^n\}$ is divergent.

Example 12.1.7. Discuss the convergence of the the sequence $a_n = n!/n^n$.

Solution. Both the numerator and denominator approach infinity as $n \to \infty$, but we have no corresponding functions to use l'Hospital's Rule because $x!$ is not defined when x is not an integer. if we write the general forumula for the sequence, we get

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}{n \cdot n \cdot n \cdot \ldots \cdot n} = \frac{1}{n}\left(\frac{2 \cdot 3 \cdot \ldots \cdot n}{n \cdot n \cdot n \cdot \ldots \cdot n}\right)$$

The expression in the parenthesis is at most 1 because the numerator is less than (or equal) to the denominator, so

$$0 < a_n \leq \frac{1}{n}$$

We can use the squeeze theorem because both 0 and $1/n \to 0$ as $n \to \infty$, so $a_n \to \infty$ as $n \to \infty$.

Example 12.1.8. Determine if the sequences below converge. If they do, find the limits as $n \to \infty$.

1. $\frac{\sin n}{n}$
2. ne^{-n}

Solution. 1. $\frac{\sin n}{n}$ converges to 0 by the squeeze theorem

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq -\frac{1}{n}\ ,\ \lim_{n\to\infty}\frac{1}{n} = 0\ ,\ \text{so}$$
$$0 \leq \frac{\sin n}{n} \leq 0 \implies \lim_{n\to\infty}\frac{\sin n}{n} = 0$$

2. $ne^{-n} = \frac{n}{e^n}$. The denominator e^n converges faster than the numerator n does. Use l'Hospital's Rule to get

$$\lim_{x\to\infty}\frac{x}{e^x} = \lim_{x\to\infty}\frac{1}{e^x} = 0 \text{ so } \lim_{n\to\infty} ne^{-n} = 0$$

Example 12.1.9. Show that if $\lim\limits_{n\to\infty} a_{2n} = L$ and $\lim\limits_{n\to\infty} a_{2n+1} = L$, $\{a_n\}$ is convergent and $\lim\limits_{n\to\infty} a_n = L$.

Solution. The solution uses the symbols $\exists$ ("exists") and $\implies$ ("implies").

$$\begin{array}{ll} \text{Since } \lim_{n\to\infty} a_{2n} = L, & \exists\, N_1 \implies |a_{2n} - L| < \varepsilon \text{ for } n > N_1 \\ \text{Since } \lim_{n\to\infty} a_{2n+1} = L, & \exists\, N_2 \implies |a_{2n+1} - L| < \varepsilon \text{ for } n > N_2 \end{array}$$

Let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$.

$$\begin{array}{ll} \text{If } n \text{ is even,} & n = 2m,\ m > N_1,\ |a_n - L| = |a_{2m} - L| < \varepsilon \\ \text{If } n \text{ is odd,} & n = 2m+1,\ m > N_2,\ |a_n - L| = |a_{2m+1} - L| < \varepsilon \end{array}$$

Therefore, $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.

Definition 12.1.6. The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 0 & \text{if } r = 1 \end{cases}$$

Definition 12.1.7. A sequence $\{a_n\}$ is **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$ ($a_1 < a_2 < a_3 < \cdots$). It is **decreasing** is $a_n < a_{n+1}$ for all $n \geq 1$. It is **monotonic** if the is either increasing or decreasing.

Example 12.1.10. The sequence $\left\{\frac{3}{n+5}\right\}$ is decreasing because

$$\frac{3}{n+5} < \frac{3}{(n+1)+5} = \frac{3}{n+6}$$

for all $n \geq 1$ (the right side is smaller because it has a larger denominator).

Example 12.1.11. Show that the sequence $a_n = \frac{n}{n^2+1}$ is decreasing.

Solution (1). We must show that $a_{n+1} < a_n$.

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$

We can simplify this by cross-multiplying. $\iff$ means "if and only if".

$$\begin{aligned} \frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} &\iff (n+1)(n^2+1) < n[(n+1)^2+1] \\ &\iff n^3 + n^2 + n + 1 < n^3 + 2n^2 + 2n \\ &\iff 1 < n^2 + n \end{aligned}$$

Since $n \geq 1$, we know that the inequality $n^2 + n > 1$ is true. Therefore, $a_{n+1} < a_n$ so $\{a_n\}$ is decreasing.

Solution (2). Consider the function $f(x) = \frac{x}{x^2+1}$:

$$f'(x) = \frac{x^2 + 1 - 2x^2}{(x^2+1)^2} = \frac{1 - x^2}{(x^2+1)^2} < 0 \quad \text{whenever } x^2 > 1$$

This, f is decreasing on $(1, \infty)$ so $f(n) > f(n+1)$. Therefore, $\{a_n\}$ is decreasing.

Theorem 12.1.4 (Monotonic Sequence Theorem)**.** Every bounded, monotonic sequence is convergent.

12.2 Series

If we try to add the terms of an infinite sequence $a_n{}_{n=1}^{\infty}$ we get the expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a **series**) and is denoted by the symbol

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

We also consider the **partial sums**

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
s_4 &= a_1 + a_2 + a_3 + a_4 \\
s_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{\infty} a_i
\end{aligned}$$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit. If the $\lim_{n\to\infty} s_n = s$ exists (as a finite number), then we call it the sum of the infinite series $\sum a_n$.

Definition 12.2.1. Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is **convergent** and we write

$$s_n = a_1 + a_2 + \cdots + a_n = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number s is the **sum** of the series. Otherwise, the series is **divergent**.

Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{\infty} a_i$$

Definition 12.2.2 (**Geometric Series**)**.** The **geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geq 1$, the geometric series is divergent.

"The sum of a convergent geometric series is $\frac{\text{first term}}{1-\text{common ratio}}$".

Proof.

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

Each term is obtained from the preceding one by multiplying it by the **common ratio** r.

If $r = 1$, then $s_n = a + a + \cdots + a = na \to \pm\infty$. Since $\lim_{n\to\infty} s_n$ doesn't exist, the geometric series diverges in this case. If $r \neq 1$, then

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rs_n &= \quad\;\; ar + ar^2 + \cdots + ar^{n-1} + ar^n \end{aligned}$$

Subtracting these equations, we get

$$s_n - rs_n = a - ar^n$$

Definition 12.2.3 (**Partial Sum of a Geometric Series**)**.**

$$s_n = \frac{a(1-r^n)}{1-r}$$

If $-1 < r < 1$, we know that $r^n \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{a}{1-r} \lim_{n\to\infty} r^n = \frac{a}{1-r}$$

Thus, when $|r| < 1$, the geometric series is convergent and its sum is $a/(1-r)$.

If $r \leq -1$ or $r > 1$, the sequence $\{r^n\}$ is divergent, so $\lim_{n\to\infty} s_n$ does not exist.

Example 12.2.1. Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

Solution. The first time is $a = 5$ and the common ratio is $r = -\frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series is convergent and its sum is

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \frac{5}{1 - (-\frac{2}{3})} = \frac{5}{\frac{5}{3}} = 3$$

Example 12.2.2. Write the number $2.3\overline{17} = 2.3171717\ldots$ as a ratio of integers.

Solution.

$$2.3171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

After the first term, we have a geometric series with $a = \frac{17}{10^3}$ and $r = 1/10^2$.

$$\begin{aligned} 2.3\overline{17} &= 2.3 + \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} = 2.3 + \frac{\frac{17}{1000}}{\frac{99}{100}} \\ &= \frac{23}{10} + \frac{17}{990} = \frac{1147}{495} \end{aligned}$$

Example 12.2.3. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

Solution. THis is not a geometric series, so we go back to the definition of a convergent series and compute the partial sums.

$$s_n = \sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \ + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$$

We simplify this expression if we use the **partial fraction decomposition**

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

Thus, we have

$$\begin{aligned} s_n &= \sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1} \right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \qquad \text{so} \\ \lim_{n\to\infty} s_n &= \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1 \end{aligned}$$

Therefore, the given series is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Definition 12.2.4. The **harmonic series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Theorem 12.2.1. If the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

NOTE 1 With any *series* $\sum a_n$, we associate two *sequences*: the sequence $\{s_n\}$ of its partial sums and the sequence $\{a_n\}$ of its terms. If $\sum a_n$ is convergent, then the limit of the sequence $\{s_n\}$ is s (the sum of the series) and the limit of the sequence $\{a_n\}$ is 0.

NOTE 2 The converse is not true in general. If $\displaystyle\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

Proof. Let $s_n = \sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots + a_n$. Then $a_n = s_n - s_{n-1}$. Since $\sum a_n$ is convergent, the sequence $\{s_n\}$ is convergent. Let $\lim_{n\to\infty} s_n = s$. Since $n-1 \to \infty$ as $n \to \infty$, we also have $\lim_{n\to\infty} s_{n-1} = s$. Therefore

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} \\ &= s - s = 0\end{aligned}$$

Definition 12.2.5 (**The Test for Divergence**)**.** If $\displaystyle\lim_{n\to\infty} a_n$ does not exist or if $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

Example 12.2.4. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

Solution.

$$\lim_{n\to\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2}{5n^2+4} = \sum_{n=1}^{\infty} \frac{1}{5+4/n^2} = \frac{1}{5} \neq 0$$

So the series diverges by the Test for Divergence.

NOTE 3 If we find that $\lim_{n\to\infty} a_n \neq 0$, we know that $\sum a_n$ is divergent. If we find that $\lim_{n\to\infty} a_n = 0$, we know *nothing* about the convergence or divergence about $\sum a_n$.

Theorem 12.2.2. If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$.

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Example 12.2.5. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$.

Solution. The series $\sum 1/2^n$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$, so

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

We found that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

So the given series is convergent and

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) &= 3\sum_{n=1}^{\infty}\frac{1}{n(n+1)} + \sum_{n=1}^{\infty}\frac{1}{2^n} \\ &= 3\cdot 1 + 1 = 4\end{aligned}$$

Note 4 A finite number of terms doesn't affect the convergence or divergence of a series. For instance, suppose that we were able to show that the series $\displaystyle\sum_{n=4}^{\infty} \frac{n}{n^3+1}$ is convergent. Since

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} = \sum_{n=4}^{\infty} \frac{n}{n^3+1}$$

we can conclude that the entire series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ is convergent.

Similarly, if it is known that the series $\displaystyle\sum_{n=N+1}^{\infty} a_n$ converges, then the full series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

is also convergent.

12.3 The Integral Test and Estimates of Sums

12.4 The Comparison Tests

12.5 Alternating Series

12.6 Absolute Convergence and the Ratio and Root Tests

12.7 Strategy for Testing Series

12.8 Power Series

12.9 Representation of Functions as Power Series

12.10 Taylor and Maclaurin Series

12.11 The Binomial Series

12.12 Applications of Taylor Polynomials