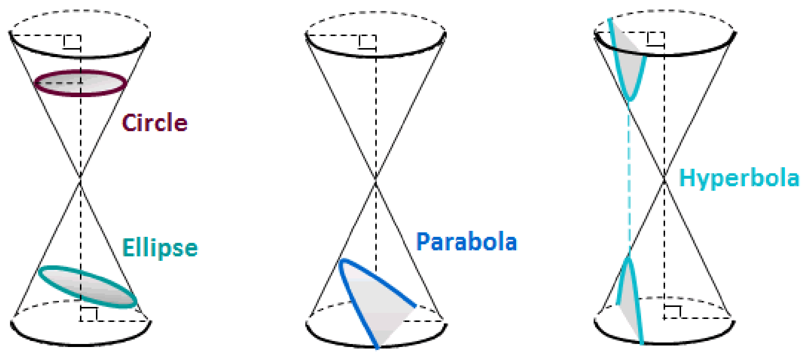
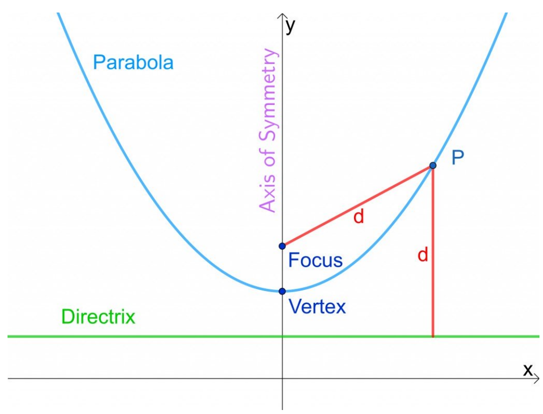
11.5 Conic Sections

Parabolas, ellipses, and hyperbolas are called **conic sections**, or **conics**, because they result from intersecting a cone with a plane.



Parabolas

A **parabola** is the set of points in a plane that are equidistant from a fixed point F (called the **focus**) and a fixed line (called the **directrix**). The halfway point between the focus and directrix is on the parabola and is called the **vertex**. The line through the focus and the vertex and perpendicular to the directrix is the **axis** of the parabola.



As seen in the figure, the focus is always inside the region of the parabola and the directrix is the same distance away on the opposite side.

Definition 11.5.1. An equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is

$$x^2 = 4py$$

. If we set $a = \frac{1}{4p}$, then the standard equation of a parabola is $y = ax^2$. This opens upward if $p > 0$ and downard if $p < 0$, and is symmetric with respect to the y-axis.

Definition 11.5.2. If we switch x and y, we get

$$y^2 = 4px$$

(reflection about the diagonal line y=x). This parabola opens to the right if $p > 0$ and to the left if $p < 0$.

Definition 11.5.3. The vertex form of a parabola is

$$y = a(x-h)^2 + k$$

where (h, k) is the vertex of the parabola and $x = h$ is the axis of symmetry. We can also switch x and y to get the vertex form of the rotated parabola.

Example 11.5.1. Find the focus and directrix of the parabola $y^2 + 10x = 0$.

Solution. We rewrite the equation as $y^2 = -10x$. We know $y^2 = 4px$, so $4px = -10x$ and $p = -\frac{5}{2}$. Thus, the focus is $(p, 0) = -\frac{5}{2}, 0)$ and the directrix is $x = \frac{5}{2}$.

Ellipses

An **ellipse** is the set of points in a plane surrounding two fixed focal points F_1 and F_2 such that the sum of the two distances to the focal points is a constant. Imagine tracing a line along the furthest path of a string stretched across two different points.

Definition 11.5.4. The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a \geq b \geq 0$$

has foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$ **(lies on x-axis)**.

The **vertices** are on the **major axis**, where a is the distance to the center of the ellipse from each vertex. This distance is greater than the distance from a **co-vertex** to the center of the ellipse, b. The co-vertices lie on the **minor axis**. Because the sum of the two distances from a point on the ellipse to the foci is a constant, the distance from a co-vertex to a focal point is also a. If the foci coincide, then $c = 0$, so $a = b$ and the ellipse becomes a circle with radius $r = a = b$.

Definition 11.5.5. The ellipse

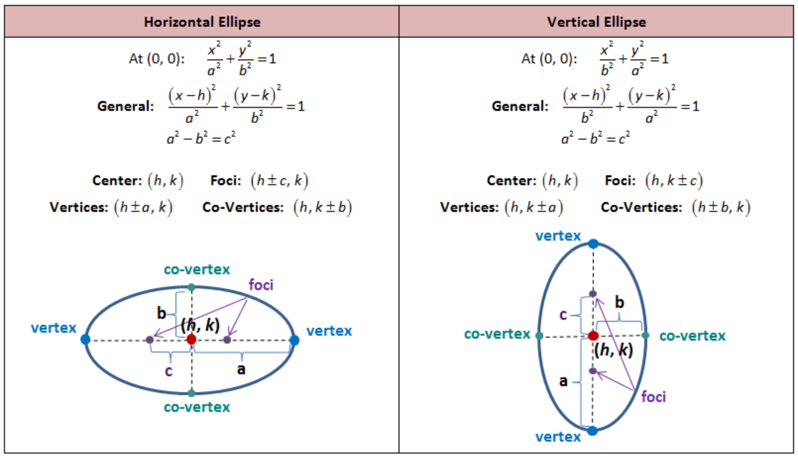
$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad a \geq b \geq 0$$

has foci $(0, \pm c)$, where $c^2 = a^2 - b^2$, and vertices $(0, \pm a)$ **(lies on y-axis)**.

Definition 11.5.6. The general form of a horizontal ellipse is

$$\frac{(x-h)^2}{b^2} + \frac{(y-h)^2}{a^2} = 1$$

where (h, k) is the center of the ellipse. The same transformation can be done to the standard form of a vertical ellipse.



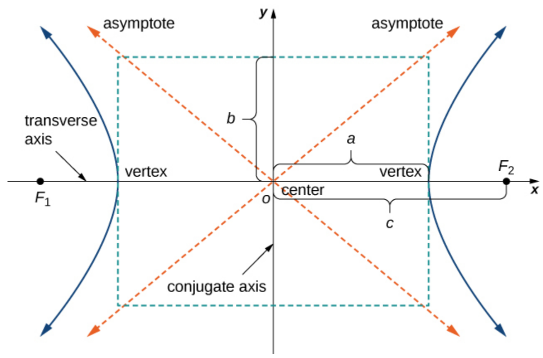
Example 11.5.2. Find an equation of the ellipse with foci $(0, \pm 2)$ and vertices $(0, \pm 3)$.

Solution. This equation represents a vertical ellipse because the foci and vertices lie on the y-axis. The distance from a focal point to the center is $c = 2$ and the distance from a vertex to the center is $a = 3$. Then we obtain $b^2 = a^2 + c^2 = 9 - 4 = 5$, so the equation of the ellipse is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = \frac{x^2}{5} + \frac{y^2}{9} = 1$$

Hyperbolas

An **ellipse** is the set of points in a plane surrounding two fixed focal points F_1 and F_2 such that the <u>difference</u> of the two distances to the focal points is a constant. The **transverse axis** is the axis of a hyperbola that passes through the two foci. The **conjugate axis** is perpendicular to the transverse axis and passes through the center of the hyperbola.



Definition 11.5.7. The hyperbola along a horizontal transverse axis

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has foci $(\pm c, 0)$, where $c^2 = a^2 + b^2$, vertices $(\pm a, 0)$ **(lies on x-axis)**, and asympotes $y = \pm \frac{b}{a}x$.

Definition 11.5.8. The hyperbola along a vertical transverse axis

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

has foci $(0, \pm c)$, where $c^2 = a^2 + b^2$, vertices $(0, \pm a)$ **(lies on y-axis)**, and asympotes $y = \pm \frac{a}{b}x$.

Definition 11.5.9. The general form of a hyperbola along a horizontal transverse axis is

$$\frac{(x-h)^2}{b^2} - \frac{(y-h)^2}{a^2} = 1$$

where (h, k) is the center of the ellipse. The same transformation can be done to the standard form a hyperbola along a vertical transverse axis.

Example 11.5.3. Find the foci and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$.

Solution. If we divide both sides of the equation by 144, it becomes

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

which is a hyperbola along a horizontal transverse axis. Therefore, we get $a = 4$ and $b = 3$. Since $c^2 = a^2 + b^2 = 16 + 9 = 25$, $c = 5$. The foci are $(\pm 5, 0)$, and the asymptotes are $y = \pm \frac{3}{4}x$.

Example 11.5.4. Find the foci and asymptotes of the hyperbola $ax^2 - by^2 = r$ in terms of $a,\ b,\ r$.

Solution. We first put the equation in standard form to get

$$\frac{x^2}{\left(\sqrt{\frac{r}{a}}\right)^2} - \frac{y^2}{\left(\sqrt{\frac{r}{b}}\right)^2} = 1$$

which is a hyperbola along a horizontal transverse axis. The foci are at

$$(\pm c, 0) = \left(\pm\sqrt{\frac{r}{a} + \frac{r}{b}},\ 0\right) = \left(\pm\sqrt{\frac{r(a+b)}{ab}},\ 0\right)$$

the vertices are at $\left(\pm\sqrt{\frac{r}{a}},\ 0\right)$, and the asymptotes are at $y = \pm\left(\sqrt{\frac{a}{b}}\right)x$.