

12.1 Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

The number a_1 is the *first term*, a_2 is the *second term*, and in general a_n is the *nth term*. We will deal with infinite sequences exclusively so each term a_n will have a successor a_{n+1} .

Notice that for every positive integer n there is a corresponding number a_n so a sequence can be defined as a function whose domain is the set of positive integers. We usually write a_n instead of the function notation $f(n)$.

NOTATION The sequence a_1, a_2, a_3, \dots is also denoted by

$$a_n \quad \text{or} \quad a_{n=1}^{\infty}$$

Example 12.1.1. Some sequences can be defined by giving a formula for the n th term. In this example, we will describe a sequence in 3 ways: the previous notation, defining a formula, and writing out the terms of the sequence. Note that n doesn't have to start at 1.

1.
$$\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty} \quad a_n = \frac{n}{n+1} \quad \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots \right\}$$
2.
$$\left\{ \frac{(-1)^n(n+1)}{3^n} \right\} \quad a_n = \frac{(-1)^n(n+1)}{3^n} \quad \left\{ -\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \dots, \frac{(-1)^n(n+1)}{3^n}, \dots \right\}$$
3.
$$\{\sqrt{n-3}\}_{n=3}^{\infty} \quad a_n = \sqrt{n-3}, \quad n \geq 3 \quad \{0, 1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n-3}, \dots\}$$
4.
$$\left\{ \cos \frac{n\pi}{6} \right\}_{n=0}^{\infty} \quad a_n = \cos \frac{n\pi}{6}, \quad n \geq 0 \quad \left\{ 1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \dots, \cos \frac{n\pi}{6}, \dots \right\}$$

Example 12.1.2. Find a formula for the general term a_n of the sequence

$$\left\{ \frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \dots \right\}$$

Solution. We are given the first five terms. The numerator of the fractions start at 3 and increase by 1, so the n th term will have numerator $n+1$. The denominators are the powers of 5, so a_n has denominator 5^n . The signs of the terms alternate between positive and negative, so we need to multiply by a power of -1 . The factor $(-1)^n$ means we start with a negative term, so here we use $(-1)^{n-1}$ or $(-1)^{n+1}$ because we start with a positive term. Therefore,

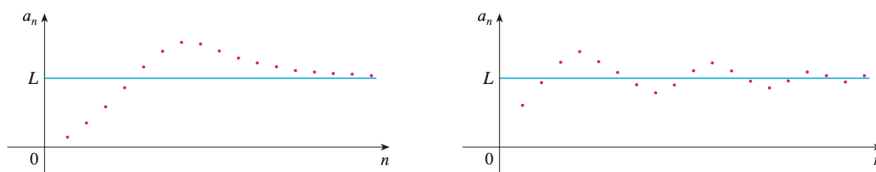
$$a_n = (-1)^{n-1} \frac{n+1}{5^n}$$

Definition 12.1.1. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \text{ as } n \rightarrow \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n \rightarrow \infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

The figure below graphs examples of two sequences that have the limit L .



A more precise version of the previous definition is

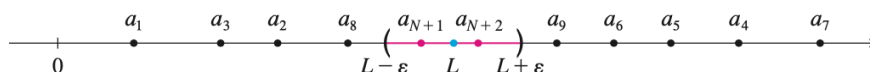
Definition 12.1.2. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \text{ as } n \rightarrow \infty$$

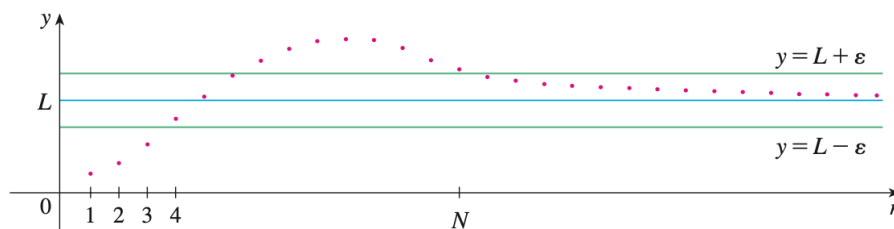
if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$|a_n - L| < \varepsilon \quad \text{whenever } n > N$$

No matter how small an interval $(L - \varepsilon, L + \varepsilon)$ is chosen, there exists an N such that all terms of the sequence from a_{N+1} onward must lie in that interval.



The points on the graph of a_n must lie between the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ if $n > N$. This picture must be valid no matter how small ε is chosen, but usually a smaller ε requires a larger N .



The only difference between $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{x \rightarrow \infty} f(x) = L$ is that n is required to be an integer.

Theorem 12.1.1. If $\lim_{x \rightarrow \infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n \rightarrow \infty} a_n = L$.

Since we know that $\lim_{x \rightarrow \infty} (1/x^r) = 0$ when $r > 0$, we can use the previous theorem to get

Definition 12.1.3.

$$\lim_{n \rightarrow \infty} \frac{1}{n^r} = 0 \quad \text{if } r > 0$$

If a_n grows as n grows, we use the notation $\lim_{n \rightarrow \infty} a_n = \infty$. We say that a_n diverges to ∞ .

Definition 12.1.4. $\lim_{n \rightarrow \infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$a_n > M \quad \text{whenever } n > N$$

Definition 12.1.5 (Limit Laws for Sequences (similar to original Limit Laws)). If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n$$

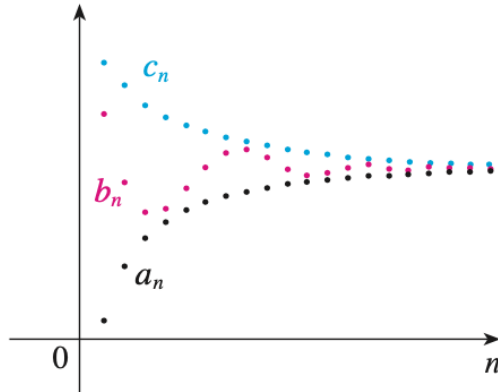
$$\lim_{n \rightarrow \infty} ca_n = c \lim_{n \rightarrow \infty} a_n \qquad \lim_{n \rightarrow \infty} c = c$$

$$\lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} \quad \text{if } b_n \neq 0$$

$$\lim_{n \rightarrow \infty} (a_n^p) = \left[\lim_{n \rightarrow \infty} a_n \right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

Theorem 12.1.2 (The Squeeze Theorem for Sequences (same as original)). If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$.



Theorem 12.1.3. If $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Example 12.1.3. Evaluate $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n}$ if it exists.

Solution.

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \quad \text{so} \quad \lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$$

Example 12.1.4. Find $\lim_{n \rightarrow \infty} \frac{n}{n+1}$.

Solution. Divide the numerator and denominator by the highest power of n and then use the Limit Laws.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n}{n+1} &= \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = \frac{\lim_{n \rightarrow \infty} 1}{\lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{1}{n}} \\ &= \frac{1}{1+0} = 1 \end{aligned}$$

Example 12.1.5. Calculate $\lim_{n \rightarrow \infty} \frac{\ln n}{n}$.

Solution. Both the numerator and denominator approach infinity as $n \rightarrow \infty$. We can't apply l'Hospital's Rule directly because it applies to functions, not sequences. However, we can apply it to the related function.

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0 \quad \text{so} \quad \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$$

Example 12.1.6. Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

Solution. If we write out the terms of the sequence, we get $\{-1, 1, -1, 1, -1, 1, -1, \dots\}$. Since the terms oscillate between 1 and -1, a_n does not approach any number. Thus, $\lim_{n \rightarrow \infty} (-1)^n$ does not exist so the sequence $\{(-1)^n\}$ is divergent.

Example 12.1.7. Discuss the convergence of the sequence $a_n = n!/n^n$.

Solution. Both the numerator and denominator approach infinity as $n \rightarrow \infty$, but we have no corresponding functions to use l'Hospital's Rule because $x!$ is not defined when x is not an integer. If we write the general formula for the sequence, we get

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot n}{n \cdot n \cdot n \cdot \dots \cdot n} = \frac{1}{n} \left(\frac{2 \cdot 3 \cdot \dots \cdot n}{n \cdot n \cdot n \cdot \dots \cdot n} \right)$$

The expression in the parenthesis is at most 1 because the numerator is less than (or equal) to the denominator, so

$$0 < a_n \leq \frac{1}{n}$$

We can use the squeeze theorem because both 0 and $1/n \rightarrow 0$ as $n \rightarrow \infty$, so $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Example 12.1.8. Determine if the sequences below converge. If they do, find the limits as $n \rightarrow \infty$.

1. $\frac{\sin n}{n}$
2. ne^{-n}

Solution. 1. $\frac{\sin n}{n}$ converges to 0 by the squeeze theorem

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}, \quad \lim_{n \rightarrow \infty} \frac{1}{n} = 0, \text{ so}$$

$$0 \leq \frac{\sin n}{n} \leq 0 \implies \lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$$

2. $ne^{-n} = \frac{n}{e^n}$. The denominator e^n converges faster than the numerator n does. Use l'Hospital's Rule to get

$$\lim_{x \rightarrow \infty} \frac{x}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0 \text{ so } \lim_{n \rightarrow \infty} ne^{-n} = 0$$

Example 12.1.9. Show that if $\lim_{n \rightarrow \infty} a_{2n} = L$ and $\lim_{n \rightarrow \infty} a_{2n+1} = L$, $\{a_n\}$ is convergent and $\lim_{n \rightarrow \infty} a_n = L$.

Solution. The solution uses the symbols \exists ("exists") and \implies ("implies").

$$\begin{aligned} \text{Since } \lim_{n \rightarrow \infty} a_{2n} = L, \quad & \exists N_1 \implies |a_{2n} - L| < \varepsilon \text{ for } n > N_1 \\ \text{Since } \lim_{n \rightarrow \infty} a_{2n+1} = L, \quad & \exists N_2 \implies |a_{2n+1} - L| < \varepsilon \text{ for } n > N_2 \end{aligned}$$

Let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$.

$$\begin{aligned} \text{If } n \text{ is even,} \quad & n = 2m, m > N_1, |a_n - L| = |a_{2m} - L| < \varepsilon \\ \text{If } n \text{ is odd,} \quad & n = 2m + 1, m > N_2, |a_n - L| = |a_{2m+1} - L| < \varepsilon \end{aligned}$$

Therefore, $\{a_n\}$ is convergent and $\lim_{n \rightarrow \infty} a_n = L$.

Definition 12.1.6. The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r .

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Definition 12.1.7. A sequence $\{a_n\}$ is **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$ ($a_1 < a_2 < a_3 < \dots$). It is **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. It is **monotonic** if it is either increasing or decreasing.

Example 12.1.10. The sequence $\left\{\frac{3}{n+5}\right\}$ is decreasing because

$$\frac{3}{n+5} < \frac{3}{(n+1)+5} = \frac{3}{n+6}$$

for all $n \geq 1$ (the right side is smaller because it has a larger denominator).

Example 12.1.11. Show that the sequence $a_n = \frac{n}{n^2+1}$ is decreasing.

Solution (1). We must show that $a_{n+1} < a_n$.

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$

We can simplify this by cross-multiplying. \iff means "if and only if".

$$\begin{aligned} \frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} &\iff (n+1)(n^2+1) < n[(n+1)^2+1] \\ &\iff n^3 + n^2 + n + 1 < n^3 + 2n^2 + 2n \\ &\iff 1 < n^2 + n \end{aligned}$$

Since $n \geq 1$, we know that the inequality $n^2 + n > 1$ is true. Therefore, $a_{n+1} < a_n$ so $\{a_n\}$ is decreasing.

Solution (2). Consider the function $f(x) = \frac{x}{x^2+1}$:

$$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0 \quad \text{whenever } x^2 > 1$$

This, f is decreasing on $(1, \infty)$ so $f(n) > f(n+1)$. Therefore, $\{a_n\}$ is decreasing.

Theorem 12.1.4 (Monotonic Sequence Theorem). Every bounded, monotonic sequence is convergent.