Chapter 12

Infinite Sequences and Series

12.1 Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_n,\ \ldots$$

The number a_1 is the *first term*, a_2 is the *second term*, and in general a_n is the *nth term*. We will deal with infinite sequences exclusively so each term a_n will have a successor $a_n + 1$.

Notice that for every positive integer n there is a corresponding number a_n so a sequence can be defined as a function whose domain is the set of positive integers. We usually write a_n instead of the function notation $f(n)$.

NOTATION The sequence $a_1,\ a_2,\ a_3,\ \ldots$ is also denoted by

$$a_n \quad \text{or} \quad a_{n\,n=1}^{\ \infty}$$

Example 12.1.1. Some sequences can be defined by giving a formula for the nth term. In this example, we will describe a sequence in 3 ways: the previous notation, defining a formula, and writing out the terms of the sequence. Note that n doesn't have to start at 1.

1. $$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} \qquad a_n = \frac{n}{n+1} \qquad \left\{\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ldots,\ \frac{n}{n+1},\ldots\right\}$$

2. $$\left\{\frac{(-1)^n(n+1)}{3^n}\right\} \qquad a_n = \frac{(-1)^n(n+1)}{3^n} \qquad \left\{-\frac{2}{3},\ \frac{3}{9},\ -\frac{4}{27},\ \frac{5}{81},\ldots,\ \frac{(-1)^n(n+1)}{3^n},\ldots\right\}$$

3.

$$\{\sqrt{n-3}\}_{n=3}^{\infty} \quad a_n = \sqrt{n-3},\ n \geq 3 \quad \{0,\ 1,\ \sqrt{2},\ \sqrt{3}, \ldots,\ \sqrt{n-3}, \ldots\}$$

4.

$$\left\{\cos\frac{n\pi}{6}\right\}_{n=0}^{\infty} \quad a_n = \cos\frac{n\pi}{6},\ n \geq 0 \quad \left\{1,\ \frac{\sqrt{3}}{2},\ \frac{1}{2},\ 0, \ldots,\ \cos\frac{n\pi}{6}, \ldots\right\}$$

Example 12.1.2. Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5},\ -\frac{4}{25},\ \frac{5}{125},\ -\frac{6}{625},\ \frac{7}{3125}, \ldots\right\}$$

Solution. We are given the first five terms. The numerator of the fractions start at 3 and increase by 1, so the nth term will have numerator $n+1$. The denominators are the powers of 5, so a_n has denominator 5^n. The signs of the terms alternate between positive and negative, so we need to multiply by a power of 1. The factor $(-1)^n$ means we start with a negative term, so here we use $(-1)^{n-1}$ or $(-1)^{n+1}$ because we start with a positive term. Therefore,

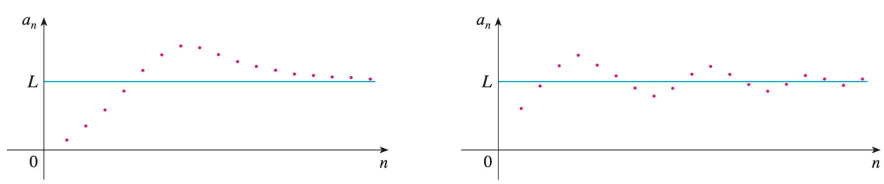
$$a_n = (-1)^{n-1}\frac{n+2}{5^n}$$

Definition 12.1.1. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

The figure below graphs examples of two sequences that have the limit L.



A more precise version of the previous definition is

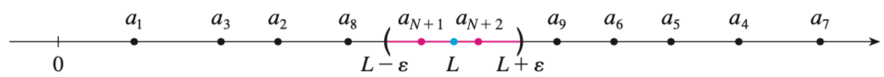
Definition 12.1.2. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

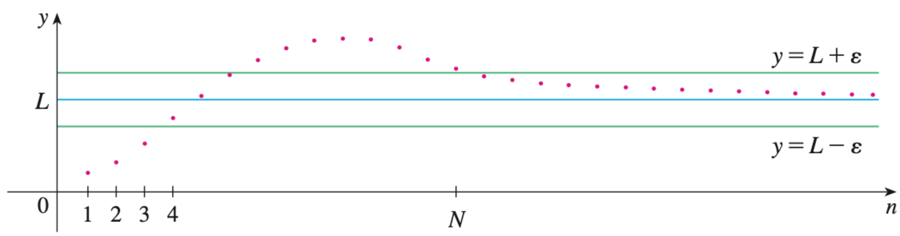
if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$|a_n - L| < \varepsilon \quad \text{whenever } n > N$$

No matter how small an interval $(L - \varepsilon\ L + \varepsilon)$ is chosen, there exists an N such that all terms of the sequence from a_{N+1} onward must lie in that interval.



The points on the graph of a_n must lie between the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ if $n > N$. This picture must be valid no matter how small ε is chosen, but usually a smaller ε requires a larger N.



The only difference between $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{x\to\infty} f(x) = L$ is that n is required to be an integer.

Theorem 12.1.1. If $\lim\limits_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} = a_n = L$.

Since we know that $\lim\limits_{x\to\infty} (1/x^r) = 0$ when $r > 0$, we can use the previous theorem to get

Definition 12.1.3.

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \quad \text{if } r > 0$$

If a_n grows as n grows, we use the notation $\lim_{n\to\infty} a_n = \infty$. We say that a_n diverges to ∞.

Definition 12.1.4. $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$a_n > M \quad \text{whenever } n > N$$

Definition 12.1.5 (Limit Laws for Sequences (similar to original Limit Laws)**).** If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

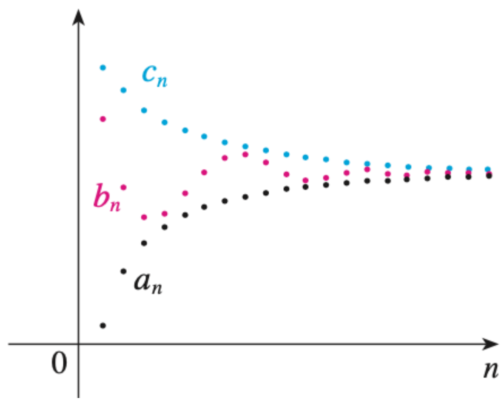
$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad\qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \tfrac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } b_n \neq 0$$

$$\lim_{n\to\infty}(a_n^p) = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

Theorem 12.1.2 (**The Squeeze Theorem for Sequences** (same as original))**.** If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.



Theorem 12.1.3. If $\lim_{n\to\infty} |a_n| = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Example 12.1.3. Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ if it exists.

Solution.

$$\lim_{n\to\infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{so} \quad \lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

Example 12.1.4. Find $\lim_{n\to\infty} \frac{n}{n+1}$.

Solution. Divide the numerator and denominator by the highest power of n and then use the Limit Laws.

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}}$$

$$= \frac{1}{1+0} = 1$$

Example 12.1.5. Calculate $\lim_{n\to\infty} \frac{\ln n}{n}$.

Solution. Both the numerator and denominator approach infinity as $n \to \infty$. We can't apply l'Hospital;s Rule directly because it applies to functions, not sequences. However, we can apply it to the related function.

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0 \quad \text{so} \quad \lim_{n\to\infty} \frac{\ln n}{n} = 0$$

Example 12.1.6. Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

Solution. If we write out the terms of the sequence, we get $\{-1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1, \ldots\}$. Since the terms oscillate between 1 and -1, a_n does not approach any number. Thus, $\lim_{n\to\infty}(-1)^n$ does not exist so the sequence $\{(-1)^n\}$ is divergent.

Example 12.1.7. Discuss the convergence of the the sequence $a_n = n!/n^n$.

Solution. Both the numerator and denominator approach infinity as $n \to \infty$, but we have no corresponding functions to use l'Hospital's Rule because $x!$ is not defined when x is not an integer. if we write the general formula for the sequence, we get

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}{n \cdot n \cdot n \cdot \ldots \cdot n} = \frac{1}{n}\left(\frac{2 \cdot 3 \cdot \ldots \cdot n}{n \cdot n \cdot n \cdot \ldots \cdot n}\right)$$

The expression in the parenthesis is at most 1 because the numerator is less than (or equal) to the denominator, so

$$0 < a_n \leq \frac{1}{n}$$

We can use the squeeze theorem because both 0 and $1/n \to 0$ as $n \to \infty$, so $a_n \to \infty$ as $n \to \infty$.

Example 12.1.8. Determine if the sequences below converge. If they do, find the limits as $n \to \infty$.

1. $\frac{\sin n}{n}$
2. ne^{-n}

Solution.

1. $\frac{\sin n}{n}$ converges to 0 by the squeeze theorem

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq -\frac{1}{n}\ ,\ \lim_{n\to\infty}\frac{1}{n} = 0\ , \text{ so}$$
$$0 \leq \frac{\sin n}{n} \leq 0 \implies \lim_{n\to\infty}\frac{\sin n}{n} = 0$$

2. $ne^{-n} = \frac{n}{e^n}$. The denominator e^n converges faster than the numerator n does. Use l'Hospital's Rule to get

$$\lim_{x\to\infty}\frac{x}{e^x} = \lim_{x\to\infty}\frac{1}{e^x} = 0 \text{ so } \lim_{n\to\infty} ne^{-n} = 0$$

Example 12.1.9. Show that if $\lim\limits_{n\to\infty} a_{2n} = L$ and $\lim\limits_{n\to\infty} a_{2n+1} = L$, $\{a_n\}$ is convergent and $\lim\limits_{n\to\infty} a_n = L$.

Solution. The solution uses the symbols $\exists$ ("exists") and $\implies$ ("implies").

$$\begin{array}{ll} \text{Since } \lim_{n\to\infty} a_{2n} = L, & \exists\ N_1 \implies |a_{2n} - L| < \varepsilon \text{ for } n > N_1 \\ \text{Since } \lim_{n\to\infty} a_{2n+1} = L, & \exists\ N_2 \implies |a_{2n+1} - L| < \varepsilon \text{ for } n > N_2 \end{array}$$

Let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$.

$$\begin{array}{ll} \text{If } n \text{ is even,} & n = 2m,\ m > N_1,\ |a_n - L| = |a_{2m} - L| < \varepsilon \\ \text{If } n \text{ is odd,} & n = 2m+1,\ m > N_2,\ |a_n - L| = |a_{2m+1} - L| < \varepsilon \end{array}$$

Therefore, $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.

Definition 12.1.6. The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 0 & \text{if } r = 1 \end{cases}$$

Definition 12.1.7. A sequence $\{a_n\}$ is **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$ ($a_1 < a_2 < a_3 < \cdots$). It is **decreasing** is $a_n < a_{n+1}$ for all $n \geq 1$. It is **monotonic** if the is either increasing or decreasing.

Example 12.1.10. The sequence $\left\{\frac{3}{n+5}\right\}$ is decreasing because

$$\frac{3}{n+5} < \frac{3}{(n+1)+5} = \frac{3}{n+6}$$

for all $n \geq 1$ (the right side is smaller because it has a larger denominator).

Example 12.1.11. Show that the sequence $a_n = \frac{n}{n^2+1}$ is decreasing.

Solution (1). We must show that $a_{n+1} < a_n$.

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$

We can simplify this by cross-multiplying. $\iff$ means "if and only if".

$$\begin{aligned} \frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} &\iff (n+1)(n^2+1) < n[(n+1)^2+1] \\ &\iff n^3 + n^2 + n + 1 < n^3 + 2n^2 + 2n \\ &\iff 1 < n^2 + n \end{aligned}$$

Since $n \geq 1$, we know that the inequality $n^2 + n > 1$ is true. Therefore, $a_{n+1} < a_n$ so $\{a_n\}$ is decreasing.

Solution (2). Consider the function $f(x) = \frac{x}{x^2+1}$:

$$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0 \quad \text{whenever } x^2 > 1$$

This, f is decreasing on $(1, \infty)$ so $f(n) > f(n+1)$. Therefore, $\{a_n\}$ is decreasing.

Theorem 12.1.4 (Monotonic Sequence Theorem)**.** Every bounded, monotonic sequence is convergent.

The proof is based on the **Completeness Axiom** for the set $\mathbb{R}$ of real numbers, which says that if S is a nonempty set of real numbers that has an upper bound M ($x \leq M$ for all x in S), then S has a **least upper bound** b (This means that b is an upper bound for S, but if M is any other upper bound, then $b \leq M$). The Completeness Axiom is an expression of the fact that there is no gap or hole in the real number line.

Proof. Suppose $\{a_n\}$ is an increasing sequence. Since $\{a_n\}$ is bounded, the set S=$\{a_n | n \geq 1\}$ has an upper bound. By the Completeness Axiom it has a least upper bound L. Given $\varepsilon > 0$, $L - \varepsilon$ is *not* an upper bound for S (since L) is the *least* upper bound). Therefore

$$a_N > L - \varepsilon \qquad \text{for some integer } N$$

But the sequence is increasing so $a_n \geq a_N$ for every $n > N$. Thus, if $n > N$ we have

$$\begin{aligned} a_n &> L - \varepsilon \\ 0 \leq L - a_n &< \varepsilon \end{aligned}$$

since $a_n \leq L$. Thus

$$|L - a_n| < \varepsilon \qquad \text{whenever } n > N$$

so $\lim_{n\to\infty} a_n = L$.

A similar proof (using the greatest lower bound) works if $\{a_n\}$ is decreasing.

12.2 Series

If we try to add the terms of an infinite sequence $a_{n}{}_{n=1}^{\infty}$ we get the expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a **series**) and is denoted by the symbol

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

We also consider the **partial sums**

$$
\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
s_4 &= a_1 + a_2 + a_3 + a_4 \\
s_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{\infty} a_i
\end{aligned}
$$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit. If the $\lim_{n\to\infty} s_n = s$ exists (as a finite number), then we call it the sum of the infinite series $\sum a_n$.

Definition 12.2.1. Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is **convergent** and we write

$$s_n = a_1 + a_2 + \cdots + a_n = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number s is the **sum** of the series. Otherwise, the series is **divergent**.

Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{\infty} a_i$$

Definition 12.2.2 (Geometric Series). The **geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geq 1$, the geometric series is divergent.

"The sum of a convergent geometric series is $\frac{\text{first term}}{1-\text{common ratio}}$".

Proof.

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

Each term is obtained from the preceding one by multiplying it by the **common ratio** r.

If $r = 1$, then $s_n = a + a + \cdots + a = na \to \pm\infty$. Since $\lim_{n\to\infty} s_n$ doesn't exist, the geometric series diverges in this case. If $r \neq 1$, then

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rs_n &= \qquad ar + ar^2 + \cdots + ar^{n-1} + ar^n \end{aligned}$$

Subtracting these equations, we get

$$s_n - rs_n = a - ar^n$$

Definition 12.2.3 (Partial Sum of a Geometric Series).

$$s_n = \frac{a(1 - r^n)}{1 - r}$$

If $-1 < r < 1$, we know that $r^n \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{a}{1 - r} \lim_{n\to\infty} r^n = \frac{a}{1 - r}$$

Thus, when $|r| < 1$, the geometric series is convergent and its sum is $a/(1 - r)$.

If $r \leq -1$ or $r > 1$, the sequence $\{r^n\}$ is divergent, so $\lim_{n\to\infty} s_n$ does not exist.

Example 12.2.1. Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

Solution. The first time is $a = 5$ and the common ratio is $r = -\frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series is convergent and its sum is

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \frac{5}{1 - (-\frac{2}{3})} = \frac{5}{\frac{5}{3}} = 3$$

Example 12.2.2. Write the number $2.3\overline{17} = 2.3171717\ldots$ as a ratio of integers.

Solution.

$$2.3171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

After the first term, we have a geometric series with $a = \frac{17}{10^3}$ and $r = 1/10^2$.

$$\begin{aligned} 2.3\overline{17} &= 2.3 + \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} = 2.3 + \frac{\frac{17}{1000}}{\frac{99}{100}} \\ &= \frac{23}{10} + \frac{17}{990} = \frac{1147}{495} \end{aligned}$$

Example 12.2.3. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

Solution. THis is not a geometric series, so we go back to the definition of a convergent series and compute the partial sums.

$$s_n = \sum_{i=1}^{\infty} \frac{1}{i(i+1)} = +\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$$

We simplify this expression if we use the **partial fraction decomposition**

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

Thus, we have

$$\begin{aligned}
s_n &= \sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1}\right) \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= 1 - \frac{1}{n+1} \qquad \text{so} \\
\lim_{n\to\infty} s_n &= \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1
\end{aligned}$$

Therefore, the given series is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Definition 12.2.4. The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Theorem 12.2.1. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Note 1 With any *series* $\sum a_n$, we associate two *sequences*: the sequence $\{s_n\}$ of its partial sums and the sequence $\{a_n\}$ of its terms. If $\sum a_n$ is convergent, then the limit of the sequence $\{s_n\}$ is s (the sum of the series) and the limit of the sequence $\{a_n\}$ is 0.

Note 2 The converse is not true in general. If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\sum_{n=1}^{\infty} a_n$ is convergent.

Proof. Let $s_n = \sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots + a_n$. Then $a_n = s_n - s_{n-1}$. Since $\sum a_n$ is convergent, the sequence $\{s_n\}$ is convergent. Let $\lim_{n\to\infty} s_n = s$. Since $n - 1 \to \infty$ as $n \to \infty$, we also have $\lim_{n\to\infty} s_{n-1} = s$. Therefore

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1}\\ &= s - s = 0\end{aligned}$$

Definition 12.2.5 (The Test for Divergence). If $\lim\limits_{n\to\infty} a_n$ does not exist or if $\lim\limits_{n\to\infty} a_n \neq 0$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

Example 12.2.4. Show that the series $\sum\limits_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

Solution.

$$\lim_{n\to\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2}{5n^2+4} = \sum_{n=1}^{\infty} \frac{1}{5+4/n^2} = \frac{1}{5} \neq 0$$

So the series diverges by the Test for Divergence.

Note 3 If we find that $\lim_{n\to\infty} a_n \neq 0$, we know that $\sum a_n$ is divergent. If we find that $\lim_{n\to\infty} a_n = 0$, we know *nothing* about the convergence or divergence about $\sum a_n$.

Theorem 12.2.2. If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$.

(i) $\sum\limits_{n=1}^{\infty} ca_n = c\sum\limits_{n=1}^{\infty} a_n$

(ii) $\sum\limits_{n=1}^{\infty} (a_n + b_n) = \sum\limits_{n=1}^{\infty} a_n + \sum\limits_{n=1}^{\infty} b_n$

(iii) $\sum\limits_{n=1}^{\infty} (a_n - b_n) = \sum\limits_{n=1}^{\infty} a_n - \sum\limits_{n=1}^{\infty} b_n$

Example 12.2.5. Find the sum of the series $\sum\limits_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$.

Solution. The series $\sum 1/2^n$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$, so

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

We found that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

So the given series is convergent and

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\frac{1}{2^n}\right) &= 3\sum_{n=1}^{\infty}\frac{1}{n(n+1)}+\sum_{n=1}^{\infty}\frac{1}{2^n}\\ &= 3\cdot 1+1=4\end{aligned}$$

NOTE 4 A finite number of terms doesn't affect the convergence or divergence of a series. For instance, suppose that we were able to show that the series $\sum_{n=4}^{\infty}\frac{n}{n^3+1}$ is convergent. Since

$$\sum_{n=1}^{\infty}\frac{n}{n^3+1}=\frac{1}{2}+\frac{2}{9}+\frac{3}{28}=\sum_{n=4}^{\infty}\frac{n}{n^3+1}$$

we can conclude that the entire series $\sum_{n=1}^{\infty}\frac{n}{n^3+1}$ is convergent.

Similarly, if it is known that the series $\sum_{n=N+1}^{\infty} a_n$ converges, then the full series

$$\sum_{n=1}^{\infty} a_n=\sum_{n=1}^{N} a_n+\sum_{n=N+1}^{\infty} a_n$$

is also convergent.

12.3 The Integral Test and Estimates of Sums

Definition 12.3.1 (The Integral Test). Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty}$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ is convergent. In other words:

(i) If $\displaystyle\int_1^{\infty} f(x)\, dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty}$ is convergent.

(i) If $\displaystyle\int_1^{\infty} f(x)\, dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty}$ is divergent.

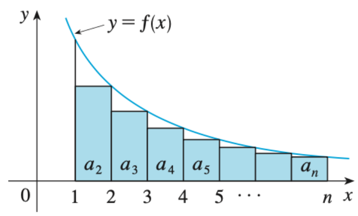
NOTE When we use the Integral Test, it is not necessary to start the series or the integral at $n=1$. For instance, in testing the series

$$\sum_{n=4}^{\infty}\frac{1}{(n-3)^2} \qquad \text{we use} \qquad \int_1^{\infty}\frac{1}{(n-3)^2}\, dx$$

Also, it is not necessary that f is always decreasing; it is important that f is *ultimately* decreasing.

Proof. We will prove the convergence and divergence of the Integral Test for the general series $\sum a_n$

(i) **Convergence**



The area of the first shaded rectangle is $f(2) = a_2$. Because there is always space underneath the curve, the sum of the area of the shaded triangles from 1 to n is always less than the area under the curve (since f is decreasing).

$$a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)\ dx$$

If $\int_1^\infty f(x)\ dx$ is convergent, then

$$\sum_{i=2}^n a_i \leq \int_1^n f(x)\ dx \leq \int_1^\infty f(x)\ dx$$

since $f(x) \geq 0$. Therefore,

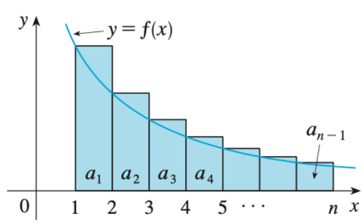
$$s_n = a_1 + \sum_{i=2}^n a_i \leq a_1 + \int_1^\infty f(x)\ dx = M \quad \text{(random variable)}$$

Since $s_n \leq M$ for all n, the sequence $\{s_n\}$ is bounded above. Also

$$s_{n+1} = s_n + a_{n+1} \geq s_n$$

since $a_{n+1} = f(n+1) \geq 0$. Thus, $\{s_n\}$ is an increasing bounded sequence so it it convergent by the Monotonic Sequence Theorem. This means that $\sum a_n$ is convergent.

(ii) **Divergence**



Because there is always space above the curve, the sum of the area of the shaded triangles from 1 to n is always greater than the area under the curve.

$$\int_1^n f(x)\ dx \leq a_1 + a_2 + \cdots + a_{n-1}$$

If $\int_1^\infty f(x)\ dx$ is divergent, then $\int_1^n f(x)\ dx \to \infty$ as $n \to \infty$ because $f(x) \geq 0$. But

$$\int_1^n f(x)\ dx \leq \sum_{i=1}^{n-1} a_i = s_{n-1}$$

so $s_{n-1} \to \infty$. This implies that $s_n \to \infty$ so $\sum a_n$ is diverges.

Example 12.3.1. Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

Solution. The function $f(x) = 1/(x^2+1)$ is continuous, positive, and decreasing on $[1, \infty)$ so we use the Integral Test:

$$\begin{aligned}\int_1^\infty \frac{1}{x^2+1}\ dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\ dx = \lim_{t\to\infty} \tan^{-1} x\Big]_1^t \\ &= \lim_{t\to\infty} \left(\tan^{-1} t - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}\end{aligned}$$

Thus, $\int_1^\infty \frac{1}{x^2+1}\ dx$ is a convergent integral. The series $\sum 1/(n^2+1)$ is convergent by the Integral Test.

Definition 12.3.2. The ***p*-series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

For $p = 1$, the series is a harmonic series.

Proof. If $p < 0$, then $\displaystyle\lim_{n\to\infty} \frac{1}{n^p} = 0$. If $p = 0$, then $\displaystyle\lim_{n\to\infty} \frac{1}{n^p} = 1$. In either case, $\displaystyle\lim_{n\to\infty} \frac{1}{n^p} \neq 0$, so the p-series diverges by the Test for Divergence.

If $p > 0$, then the function $f(x) = \dfrac{1}{x^p}$ is clearly continuous, positive, and decreasing on $[1, \infty)$. We know that

$$\int_1^\infty \frac{1}{x^p} \text{ converges if } p > 1 \text{ and diverges if } p \leq 1$$

Using the Integral Test, the series $\sum 1/n^p$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

Example 12.3.2.

(a) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$$

is convergent because it is a p-series with $p = 3 > 1$.

(b) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$$

is divergent because it is a p-series with $p = \frac{1}{3} < 1$.

Note We should *not* infer that the sum of the series is equal to the value of the integral from the Integral Test. In fact,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{whereas} \quad \int_{n}^{\infty} \frac{1}{n^2} = 1$$

Therefore, in general,

$$\sum_{n=1}^{\infty} a_n \neq \int_{n}^{\infty} f(x)\ dx$$

Example 12.3.3. Determine whether the series $\sum -n = 1^{\infty} \dfrac{\ln n}{n}$ converges or diverges.

Solution. The function $\frac{\ln x}{x}$ is positive and continous for $x > 1$ because the logarithm function is continuous, but it is not obvious whether or not f is decreasing, so we take its derivative:

$$f'(x) = \frac{(1/x)x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

Thus, $f'(x) < 0$ when $\ln x > 1$, which is when $x > e$. We conclude that f is decreasing when $x > e$ so we can apply the Integral Test:

$$\begin{aligned}\int_1^{\infty} \frac{\ln x}{x}\ dx &= \lim_{t\to\infty} \int_1^{t} \frac{\ln x}{x}\ dx = \lim_{t\to\infty} \frac{(\ln x)^2}{2}\Bigg]_1^t \\ &= \lim_{t\to\infty} \frac{(\ln t)^2}{2} = \infty\end{aligned}$$

Since this improper integral is divergent, the series $\sum(\ln n)/n$ is also divergent by the Integral Test.

Estimating the Sum of a Series

We can show if a series $\sum a_n$ is converging. Now we want to find an approximation to the sum s of the series. ANy partial sum s_n is an approximation to s because $\lim_{n\to\infty} s_n = s$, but *how good is that approximation*? To find out, we need to estimate the size of the **remainder**

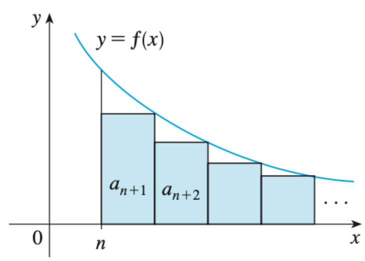
$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

The remainder R_n is the *error* made when s_n, the sum of the first n terms, is used as an approximation of the total sum.

Definition 12.3.3 (**Remainder Estimate for the Integral Test**)**.** Suppose that $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

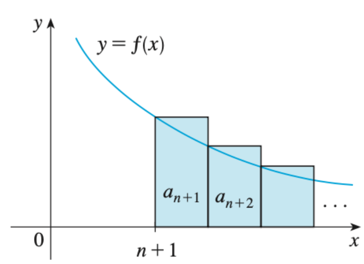
$$\int_{n+1}^{\infty} f(x)\ dx \leq R_n \leq \int_{n}^{\infty} f(x)\ dx$$

Proof. We use the same concept as the Integral test, assuming that f is decreasing on $[n, \infty)$.



We compare the sum of the area of the rectangles with the area under $y = f(x)$ for $x > n$ to see that

$$R_n = a_{n+1} + a_{n+2} + \cdots \leq \int_{n}^{\infty} f(x)\ dx$$



Similarly, we see that

$$R_n = a_{n+1} + a_{n+2} + \cdots \geq \int_{n}^{\infty} f(x)\ dx$$

Example 12.3.4. (a) Approximate the sum of the series $\sum 1/n^3$ by using the sum of the first 10 terms. Approximate the error involved in the approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

Example 12.3.5.

$$\int_n^\infty \frac{1}{x^3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_n^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2}+\frac{1}{2n^2}\right) = \frac{1}{2n^2}$$

(a)

$$\sum_{n=1}^{\infty}\frac{1}{n^3} \approx \frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots+\frac{1}{10^3} \approx 1.1975$$

According to the remainder estimate, we have

$$R_{10} \le \int_{10}^\infty \frac{1}{x^3}\,dx = \frac{1}{2(10)^2} = \frac{1}{200}$$

So the size of the error is at most 0.005.

(b) Accuracy to within 0.0005 means that we have to find a value of n such that $R_n \le 0.0005$. Since

$$R_n \le \int_{10}^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$$

We want

$$\frac{1}{2n^2} \le 0.0005$$

Solving this inequality, we get

$$n^2 > \frac{1}{0.001} = 1000 \quad \text{or} \quad n > \sqrt{1000} \approx 31.6$$

We need 32 terms to ensure accuracy to within 0.0005.

If we add s_n to each side of the inequality of the Remainder Estimate for the Integral Test, we get

Definition 12.3.4.

$$s_n + \int_{n+1}^\infty f(x)\,dx \le s \le s_n + \int_n^\infty f(x)\,dx$$

because $s_n + R_n = s$. These inequalities give a lower bound and an upper bound for s. They provide a more accurate approximation than the partial sum s_n does.

Example 12.3.6. Use the improved remainder estimate with $n = 10$ to estimate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

Solution.

$$s_{11} + \int_{11}^{\infty} \frac{1}{x^3}\, dx \leq s \leq s_{10} + \int_{10}^{\infty} \frac{1}{x^3}\, dx$$

We know from the previous example that

$$\int_{n}^{\infty} \frac{1}{x^3}\, dx = \frac{1}{2n^2}$$

so
$$s_{11} + \frac{1}{2(11)^2} \leq s \leq s_{10} + \frac{1}{2(10)^2}$$

Using $s_{10} \approx 1.197532$, we get

$$1.201664 \leq s \leq 1.202532$$

If we approximate s by the midpoint of this interval, then the error is at most half the length of the interval, so

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2021 \quad \text{with error} < 0.0005$$

We get a much better estimate with this method than the estimate $s \approx s_n$ in the previous example. Also, we only had to use 10 terms to get the error smaller than 0.0005 instead of 32 terms.

12.4 The Comparison Tests

In comparison tests, the idea is to compare a given series with a series that is know to be convergent or divergent.

Definition 12.4.1 (The Comparison Test). Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

In other words,

(i) If we have a series whose terms are *smaller* than those of a known *convergent* series, then our series is also convergent.

(ii) If we have a series whose terms are *larger* than those of a known *divergent* series, then our series is also divergent.

Proof. Let

$$s_n = \sum_{i=1}^{n} a_i \qquad t_n = \sum_{i=1}^{n} b_i \qquad t = \sum_{n=1}^{n} b_n$$

(i) **Convergence**

The sequences $\{s_n\}$ and $\{t_n\}$ are increasing ($s_{n+1} = s_n + a_{n+1} \geq s_n$) because both series have positive terms. Also $t_n \to t$, so $t_n \leq t$ for all n. This means that $\{s_n\}$ is increasing and bounded above and therefore converges by the Monotonic Sequence Theorem. Thus, $\sum a_n$ converges.

(ii) **Divergence**

If $\sum b_n$ is divergent, then $t \to \infty$ (since $\{t_n\}$ is increasing). BUt $a_i \geq b_i$ so $s_n \geq t_n$. Thus, $s_n \to \infty$. Therefore, $\sum a_n$ diverges.

Example 12.4.1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ converges or diverges.

Solution. As n gets larger, the dominant term in the denominator is $2n^2$, so we compare the given series with the series $\sum 5/(2n^2)$. Observe that

$$\frac{5}{2n^2 + 4n + 3} < \frac{5}{2n^2}$$

because the left side has a bigger denominator. We know the

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent because it is a constant times a p-series with $p = 2 > 1$. Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ is convergent by the Comparison Test.

Note Although the condition $a_n \leq b_n$ for $a_n \geq b_n$ in the Comparison Test is given for all n, we only need to verify it for $n \geq N$, where N is some fixed integer, because the convergence of a series is not affected by a finite number.

Definition 12.4.2 (**The Limit Comparison Test**)**.** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or diverge.

Proof. Let m and M be positive numbers such that $m < c < M$. Because a_n/b_n is close to c for a large n, there is an integer N such that

$$\begin{aligned} m < \frac{a_n}{b_n} < M \quad &\text{when } n > N \quad \text{so} \\ mb_n < a_n < Mb_n \quad &\text{when } n > N \end{aligned}$$

We can conclude the following:

(i) If $\sum b_n$ converges, so does $\sum Mb_n$, so $\sum a_n$ converges by the Comparison Test.

(i) If $\sum b_n$ diverges, so does $\sum Mb_n$, so $\sum a_n$ diverges by the Comparison Test.

Example 12.4.2. Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

Solution. We use the limit comparison test with

$$a_n = \frac{1}{2^n - 1} \qquad b_n = \frac{1}{2^n}$$

and obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1(2^n - 1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - 1/2^n} = 1 > 0$$

Since this limit exists and $\sum 1/2^n$ is a convergent geometric series, the given series converges by the Limit Comparison Test.

Estimating Sums

We used the Comparison test to series $\sum a_n$ by comparison with $\sum b_n$. We can also use it to estimate the sum by compparing remaindeds. We continue to consider the remainder R_n and consider T_n for the comparison series $\sum b_n$ as the corresponding remainder.

$$\begin{aligned} R_n &= s - s_n = a_{n+1} + a_{n+2} + \cdots \\ T_n &= t - t_n = b_{n+1} + b_{n+2} + \cdots \end{aligned}$$

Since $a_n \leq b_n$, $R_n \leq T_n$.

Example 12.4.3. Use the sum of the first 100 terms to approximate the sum of the series $\sum 1/(n^3 + 1)$. Estimate the error involved in this approximation.

Solution. Since

$$\frac{1}{n^3 + 1} < \frac{1}{n^3}$$

the given series is convergent by the Comparison Test. Using the Remainder Estimate for the Integral Test in section 12.3 we found that

$$T_n \leq \int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$$

Therefore, the remainder R_n for the given series satisfies

$$R_n \leq T_n \leq \frac{1}{2n^2}$$

With $n = 100$ we have

$$R_{100} \leq \frac{1}{2(100)^2} = 0.00005$$

Using a calculator, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^3+1} \approx \sum_{n=1}^{100} \frac{1}{n^3+1} \approx 0.6864538$$

with error less than 0.00005.

12.5 Alternating Series

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$$

We see from these examples that the nth term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n \qquad \text{or} a_n = (-1)^n b_n$$

where b_n is a positive number. (In fact, $b_n = |a_n|$).

Definition 12.5.1 (**The Alternating Series Test**)**.** If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad (b_n > 0)$$

satisfies

$$\text{(i)}\ \ b_{n+1} \leq b_n \quad \text{for all } n$$

$$\text{(ii)}\ \ \lim_{n \to \infty} b_n = 0$$

then the series is convergent.

"The alternating series converges if its terms decrease toward 0 in absolute value".

Proof. We consider the even and odd partial sums separately.

We first consider *even* partial sums:

$$\begin{aligned} &s_2 = b_1 - b_2 \geq 0 && \text{since } b_2 \leq b_1 \\ &s_4 = s_2 + (b_3 - b_4) \geq s_2 && \text{since } b_4 \leq b_3 \\ \text{In general} \quad &s_{2n} = s_{2n-2} + (b_{2n-1} - b_{2n}) \geq s_{2n-2} && \text{since } b_{2n} \leq b_{2n-1} \end{aligned}$$

Thus $$0 \leq s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_{2n} \leq \cdots$$

But we can also write

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}$$

Every term in brackets is positive, so $s_{2n} \leq b_1$ for all n. Therefore, the sequence $\{s_{2n}\}$ of even partial sums is increasing and bounded above. It is therefore convergent by the Monotonic Sequence Theorem. Let's call this limit s:

$$\lim_{n\to\infty} s_{2n} = s$$

Next we compute the limit of the *odd* partial sums:

$$\begin{aligned} \lim_{n\to\infty} s_{2n+1} &= \lim_{n\to\infty} (s_{2n} + b_{2n+1}) \\ &= \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} \\ &= s + 0 \\ &= s \end{aligned}$$

Since both partial sums converge to s, we have $\lim_{n\to\infty} s_{2n} = s$ and so the series is convergent.

Example 12.5.1. The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies

(i) $b_{n+1} \leq b_n$ because $\dfrac{1}{n+1} < \dfrac{1}{n}$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \dfrac{1}{n} = 0$

so the series is convergent by the Alternating Series Test

Example 12.5.2. The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is alternating but

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \lim_{n\to\infty} \frac{3}{4-\frac{1}{n}} = \frac{3}{4} \neq 0$$

Instead, we look at the nth term of the series:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n 3n}{4n-1}$$

The limit does not exist, so the series diverges by the Test for Divergence.

Estimating Sums

Theorem 12.5.1 (Alternating Series Estimation Theorem). If $s = \sum(-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

$$\text{(i) } 0 \le b_{n+1} \le b_n \quad \text{and} \quad \lim_{n\to\infty} b_n = 0$$

then
$$|R_n| = |s - s_n| \le b_{n+1}$$

Proof. We know from the proof of the Alternating Series Test that s lies between any two consecutive partial sums s_n and s_{n+1}. It follows that

$$|s - s_n| \le |s_{n+1} - s_n| = b_{n+1}$$

"The size of the error is smaller than b_{n+1}, which is the absolute value of the first neglected term" (valid only for alternating series that satisfy the Alternating Series Estimation, not other theorems.

Example 12.5.3. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places (By definition, $0! = 1$).

Solution. We first observe that the series is convergent by the Alternating Series Test because

$$\text{(i)} \quad \frac{1}{(n+1)!} = \frac{1}{n!(n+1)} \le \frac{1}{n!}$$

$$\text{(ii)} \quad 0 < \frac{1}{n!} < \frac{1}{n} \to 0 \quad \text{so } \frac{1}{n!} \to 0 \text{ as } n \to \infty$$

To get a feel for how many terms we need to use in our approximation, let's write out the first few terms of the series:

$$\begin{aligned} s &= \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots \\ &= 1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120} + \tfrac{1}{720} - \tfrac{1}{5040} + \cdots \end{aligned}$$

Notice that $$b_7 = \tfrac{1}{5040} < \tfrac{1}{5000} = 0.0002$$

and $$s_n = 1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120} + \tfrac{1}{720} \approx 0.369056$$

By the Alternating Series Estimation Theorem, we know that

$$|s - s_6| \le b_7 < 0.0002$$

This error of less than 0.0002 does not affect the third decimal place, so we have

$$s \approx 0.368$$

correct to three decimal places.

12.6 Absolute Convergence and the Ratio and Root Tests

Given any series $\sum a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute value of the terms of the original series.

Definition 12.6.1. A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Notice that if $\sum a_n$ is a series with positive terms, then $\sum |a_n| = \sum a_n$ and so absolute convergence is the same as convergence in this case.

Example 12.6.1. The series

$$\sum_{n\to\infty}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent because

$$\sum_{n\to\infty}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is a convergent p-series ($p = 2$).

Definition 12.6.2. A series $\sum a_n$ is **conditionally convergent** if it is convergent but not absolutely convergent.

Definition 12.6.3. If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Proof. Observe that the inequality

$$0 \leq a_n + |a_n| \leq 2|a_n|$$

is true because $|a_n|$ is either a_n or $-a_n$. If $\sum a_n$ is absolutely convergent, then $\sum |a_n|$ is convergent, $\sum 2|a_n|$. Therefore, by the comparison test, $\sum(a_n + |a_n|)$ is convergent. Then

$$\sum a_n = \sum(a_n + |a_n|) - \sum |a_n|$$

is the difference of the two convergent series and is therefore convergent.

Definition 12.6.4. The Ratio Test

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive (no conclusion can be drawn about the convergence or divergence of $\sum a_n$).

Proof.

(i) The idea is to compare the given series with a convergent geometric series. Since $L > 1$, we can choose a number r such that $L < r < 1$. Since

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \quad \text{and} \quad L < r$$

the ratio $|a_{n+1}/a_n|$ will eventually be less than r; this means that there exists an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| < r \quad \text{whenever } n \geq N$$

or, equivalently,

$$a_{n+1} < r|a_n| \quad \text{whenever } n \geq N$$

Putting n successively equal to $N,\ N+1,\ N+2,\ldots$ in the previous equation, we obtain

$$\begin{aligned}
|a_{N+1}| &< |a_N|r \\
|a_{N+2}| &< |a_{N+1}|r < |a_N|r^2 \\
|a_{N+3}| &< |a_{N+2}|r < |a_N|r^3
\end{aligned}$$

and, in general,

$$|a_{N+k}| < |a_N|r^k \qquad \text{for all } k \geq 1$$

Now the series

$$\sum_{k=1}^{\infty} |a_N|r^k = |a_N|r + |a_N|r^2 + |a_N|r^3 + \cdots$$

is convergent because it is a geometric series with $0 < r < 1$. So the previous inequality , together with the Comparison Test, shows that the series

$$\sum_{n=N+1}^{\infty} |a_n| = \sum_{k=1}^{\infty} |a_{N+k}| = |a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \cdots$$

is also convergent. It follows that the series $\sum_{n=1}^{\infty} |a_n|$ is also convergent. Therefore, $\sum a_n$ is absolutely convergent.

(ii) If $|a_{n+1}/a_n| \to L > 1$ or $|a_{n+1}/a_n| \to \infty$, then the ratio $|a_{n+1}/a_n|$ will eventually be greater than 1. This means that there exists an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \quad \text{whenever } n \geq N$$

This means that $|a_{n+1}/a_n| > |a_n|$ whenever $n \geq N$ and so

$$\lim_{n\to\infty} a_n \neq 0$$

Therefore $\sum a_n$ diverges by the Test for Divergence.

(iii) The Ratio Test gives no information if $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$. For instance,

for the convergent series $\sum 1/n^2$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{1}{(n+1)^2}}{\dfrac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \frac{1}{\left(1+\dfrac{1}{n}\right)^2} \to 1 \quad \text{as } n \to \infty$$

whereas for the divergent series $\sum 1/n$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{1}{n+1}}{\dfrac{1}{n}} = \frac{n}{n+1} = \frac{1}{1+\dfrac{1}{n}} \to 1 \quad \text{as } n \to \infty$$

Therefore, if $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$, the series $\sum a_n$ might converge or diverge. In this case, the Ratio Test fails and we must use some other test.

Example 12.6.2. Test the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{3^n} for absolute convergence.$

Solution. We use the Ratio Test with $a_n = (-1)^n n^3/3^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\dfrac{(-1)^{n+1}(n+1)^3}{3(n+1)}}{\dfrac{(-1)^n n^3}{3^n}}\right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$
$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \to \frac{1}{3} < 1$$

The given series is absolutely convergent by the Ratio Test and therefore convergent.

Definition 12.6.5. The Root Test

(i) If $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is is absolutely convergent (and therefore convergent).

(ii) If $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root Test is inconclusive.

NOTE If $L = 1$ in the Ratio Test, don't try the Root Test because L will also be 1.

Example 12.6.3. Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$

Solution.

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2+\dfrac{3}{n}}{3+\dfrac{2}{n}} \to \frac{2}{3} < 1$$

The given series converges by the Root Test.

12.7 Strategy for Testing Series

The main strategy for testing series is to classify the series according to its *form*.

1. If the series is of the form $\sum 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leq 1$.

2. If the series has form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric series, which converges is $|r| < 1$ and diverges if $|r| \geq 1$. You may need to manipulate the equation to bring the series into this form.

3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series (the values of p should be chosen by keeping only the highest powers of n in the numerator and denominator). The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, we can apply the Comparison Tests to $\sum |a_n|$ and test for absolute convergence.

4. If it is obvious that $\lim_{n\to\infty} \neq 0$, then use the Test for Divergence.

5. If the series is of the form $\sum(-1)^{n-1}b_n$ or $\sum(-1)^n b_n$, then the Alternating Series Test is an obvious possibility.

6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus, the Ratio Test should not be used for such series.

7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.

8. If $a_n = f(n)$, where $\int_1^\infty f(x)\, dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

Example 12.7.1. These examples show demonstrate how to identify which test should be used.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

Since $a_n \to \frac{1}{2} \neq 0$ as $n \to \infty$, we should use the Test for Divergence.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

Since a_n is an algebraic function of n, we should compare the given series with a p-series. The comparison series for the Limit Comparison Test is b_n, where

$$b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{n^{3/2}}{3n^3} = \frac{1}{3n^{3/2}}$$

(c) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

Since the integral $\int_1^{\infty} xe^{-x^2}\, dx$ is easily evaluated, we use the Integral Test. The Ratio Test also works.

(d) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$

Since the series is alternating, we use the Alternating Series Test.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{2^k}{k!}$

Since the series is involves $k!$, we use the Ratio Test.

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

Since the series is closely related to the geometric series $\sum 1/3^n$, we use the Comparison Test.

12.8 Power Series

Definition 12.8.1. A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the **coefficients** of the series.

For each fixed x, the series is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for other values of x. The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

whose domain is the set of all x for which the series converges. Notice that f resembles a polynomial. The only difference is that f has infinitely many terms.

For instance, if we take $c_n = 1$ for all n, the power series becomes the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$.

Definition 12.8.2. A series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

is called a **power series in** $(x-a)$ or a **power series centered at** a or a **power series about** a.

Notice that when $x = a$ all of the terms are 0 for $n \geq 1$, so the power series always converges when $x = a$.

FUN FACT A power series is a series in which each term is a power function. A **trignometric series**

$$\sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is a series whose terms are trignometric functions.

Example 12.8.1. For what values of x is the series $\displaystyle\sum_{n=0}^{\infty} n!x^n$ convergent?

Solution. We use the Ratio Test. If we let a_n denote the nth term of the series, then $a_n = n!x^n$. If $x \neq 0$, we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty} (n+1)|x| = \infty$$

By the Ratio Test, the series diverges when $x \neq 0$, so it converges only when $x = 0$.

Example 12.8.2. For what values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

Solution. Let $a_n = (x-3)^n/n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right|$$

$$= \frac{1}{1 + \frac{1}{n}}|x-3| \to |x-3| \quad \text{as } n \to \infty$$

By the Ratio Test, the given series is absolutely convergent, and therefore convergent, when $|x-3|<1$ and divergent when $|x-3|>1$. We rewrite the inequality as

$$|x-3|<1 \iff -1<x-3<1 \iff 2<x<4$$

Now we know the series converges when $2<x<4$ and diverges when $x<2$ or $x>4$.

The Ratio Test gives no information when $|x-3|=1$, so we must consider $x=2$ and $x=4$ separately.

(a) If we put $x=4$ in the series, it becomes the harmonic series $\sum 1/n$, which is divergent.

(b) If $x=2$, the series is $\sum(-1)^n/n$, which converges by the Alternating Series Test.

We can summarize our results by concluding that the power series converges for $2 \leq x < 4$.

Theorem 12.8.1. For a given power series $\displaystyle\sum_{n=0}^{\infty}$ there are only three possibilities:

1. The series converges only when $x=a$.
2. The series converges for all x.
3. There is a positive number R such that the series converges if $|x-a|<R$ and diverges if $|x-a|>R$.

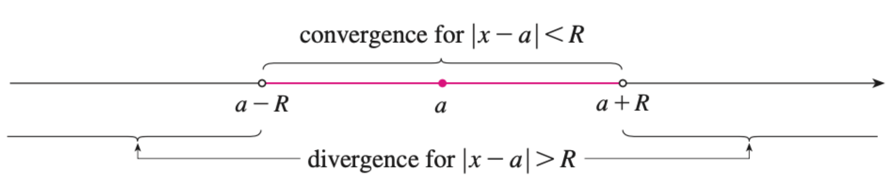
The proof of this theorem is at the end of this chapter because this theorem is more relevant than the proof.

The number R in case 3 is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R=0$ in case 1 and $R=\infty$ in case (ii).

The **interval of convergence** of a power series is the interval that consists of just a single point a. in case 2, the interval is $(-\infty,\infty)$. In case 3, note that the inequality $|x-a|<R$ can be rewritten as $a-R<x<a+R$. When x is an *endpoint* of the interval ($x=a\pm R$), anything can happen—the series might converge at one or both endpoints, or it might diverge at both endpoints.

Thus, in case 3 there are four possibilities for the interval of convergence:

$$(a-R,a+R) \qquad (a-R,a+] \qquad [a-R,a+R) \qquad [a-R,a+R]$$



We summarize the radius and interval for convergence for each of the examples in this section.

	Series	Radius of Convergence	Interval of Convergence
Geometric Series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 12.8.1	$\sum_{n=0}^{\infty} n!x^n$	$R = 0$	$\{0\}$
Example 12.8.2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$

In general, the Ratio Test (or sometimes the Root Test) should always be used to determine the radius of convergence R. The Ratio and Root Tests always fail when x is an endpoint of the interval of convergence, so the endpoints must be checked with some other test.

Example 12.8.3. Find the radius and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

Solution. Let $a_n = (-3)^n x^n / \sqrt{n+1}$.

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n}\right| = \left|-3x\sqrt{\frac{n+1}{n+2}}\right| \\ &= 3\sqrt{\frac{1+(1/n)}{1+(2/n)}}|x| \to 3|x| \quad \text{as } n \to \infty\end{aligned}$$

By the Ratio Test, the given series converges if $3|x| < 1$ and diverges if $3|x| > 1$. Thus, it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$, meaning that the radius of convergences is $R = \frac{1}{3}$.

We know the series converges in the interval $(-\frac{1}{3}, \frac{1}{3})$, but we must now test for convergence at the endpoints of this interval.

(a) If $x = -\frac{1}{3}$, the series becomes $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$ which diverges (using the Integral Test or simply observing that it is a p-series with $p = \frac{1}{2} < 1$).

(b) If $x = \frac{1}{3}$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ which converges by the Alternating Series Test. Therefore, the given power series converges when $-\frac{1}{3} < x \leq \frac{1}{3}$, so the interval of convergence is $(-\frac{1}{3}, \frac{1}{3}]$.

Proof

To prove the theorem that is found earlier in this section, we first need to prove 2 theorems.

Theorem 12.8.2.

1. If a power series $\sum c_n x^n$ converges when $x = b$ (where $b \neq 0$), then it converges whenever $|x| < |b|$.
2. If a power series $\sum c_n x^n$ diverges when $x = d$ (where $d \neq 0$), then it converges whenever $|x| > |d|$.

Proof.

1. Suppose that $\sum c_n x^n$ converges. Then, we know $\lim_{n\to\infty} c_n b^n = 0$. According to the definition of a limit of a sequence with $\varepsilon = 1$, there is a positive integer N such that $|c_n b^n| < 1$ whenever $n \geq N$,

   $$|c_n x^n| = \left|\frac{c_n b^n x^n}{b^n}\right| = |c_n b^n| \left|\frac{x}{b}\right|^n < \left|\frac{x}{b}\right|^n$$

   If $|x| < |b|$, then $|x/b| < 1$, so $\sum |x/b|^n$ is a convergent geometric series. Therefore, by the Comparison Test, the series $\sum_{n=N}^{\infty} |c_n b^n|$ is convergent. Thus, the series $\sum c_n b^n$ is absolutely convergent and therefore convergent.

2. Suppose that $\sum c_n d^n$ diverges. If x is any number such that $|x| > |d|$, then $\sum c_n x^n$ cannot converge because, by part 1, the convergence of $\sum c_n x^n$ would imply the convergence of $\sum c_n d^n$. Therefore $\sum c_n x^n$ diverges whenever $|x| > |d|$.

Theorem 12.8.3. For a power series $\sum c_n x^n$ there are only three possibilities

1. The series converges only when $x = 0$.
2. The series converges for all x.
3. There is a positive number R such that the series converges if $|x| < R$ and diverges if $|x| > R$.

Proof. We use the preceding theorem to prove this theorem. The symbol $\in$ means "is an element of" or "in".

Suppose that neither case 1 nor case 2 is true. Then there are nonzero numbers b and d such that $\sum c_n x^n$ converges for $x = b$ and diverges for $x = d$. Therefore, the set $S = \{x | \sum c_n x^n \text{ converges}\}$ is not empty. By the preceding theorem, the series diverges if $|x| > |d|$, so $|x| \geq |d|$ for all $x \in S$. This says that $|d|$ is an upper bound for the set S. Thus, by the Completeness Axiom (see Section 12.1), S has a least upper bound R. If $|x| > R$, then $x \notin S$, so $\sum c_n x^n$ diverges. If $|x| < R$, then $|x|$ is not an upper bound for S and so there exists $b \in S$ such that $b > |x|$. Since $b \in S$, $\sum c_n b^n$ converges, so by the preceding theorem $\sum c_n x^n$ converges.

Now we are ready for the proof of the main theorem found earlier in the section. We use the preceding theorem to prove it.

Proof. If we can make the change of variable $u = x - a$, then the power series becomes $\sum c_n u^n$ and we can apply the preceding theorem to this series. In case 3 we have convergence for $|u| < R$ and divergence for $|u| > R$. Thus, we have convergence for $|x - a| < R$ and divergence for $|x - a| > R$.

12.9 Representation of Functions as Power Series

In this section we learn how to represent certain types of functions as sums of power series by manipulating geometric series or by differentiating or integrating such a series.

Example 12.9.1. Express $\dfrac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.

Solution. Replace x with $-x^2$.

$$\begin{aligned}\frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n \\ &= \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots\end{aligned}$$

Because this is a geometric series, it converges when $|x^2| < 1$, which is the same as $x^2 < 1$ or $|x| < 1$. Therefore, the interval of convergence is $(-1, 1)$.

Differentiation and Integration of Power Series

The sum of a power series is a function $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ whose domain is the interval of convergence of the series. We would like to be able to differentiate and inte- grate such functions, and the following theorem says that we can do so by differentiating or integrating each individual term in the series, just as we would for a polynomial. This is called **term-by-term differentiation and integration**.

Theorem 12.9.1. If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-2) + c_2(x-2)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R,\ a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$

(i) $\displaystyle\int f(x)\ dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in both equations are R.

Note The equations can be rewritten in the form

(i) $\displaystyle\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty}\frac{d}{dx}[c_n(x-a)^n]$

(ii) $\displaystyle\int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] dx = \sum_{n=0}^{\infty}\int[c_n(x-a)^n]\ dx$

Example 12.9.2. Express $\dfrac{1}{(1-x)^2}$ as a power series by differenting the sum of a geometric series.

Solution. Differentiate both sides of the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=0}^{\infty} nx^{n-1}$$

If we wish, we can replace n by $n+1$ and write the answer as

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n$$

The radius of convergence of the differentiated series is the same as the radius of convergence of the original series, so $R = 1$.

Example 12.9.3. FInd a power series representation for $\ln(1-x)$ and its radius of convergence.

Solution. We notice that, besides from a factor of -1, the derivitive of this function is $1/(1-x)$. So we integrate both sides of the equation:

$$\begin{aligned}
-\ln(1-x) &= \int \frac{1}{1-x}\,dx = \int (1+x+x^2+\cdots)\,dx \\
&= x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + C = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C \\
&= \sum_{n=1}^{\infty} \frac{x^n}{n} + C \qquad |x| < 1
\end{aligned}$$

To determine the value of C, we plug in $x = 0$ into the equation and get $-\ln(1-0) = C$, so $C = 0$ and

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n} \qquad |x| < 1$$

The radius of convergence is the same as the original series: $R = 1$.

Example 12.9.4.

(a) Evaluate $\displaystyle\int \frac{1}{1+x^7}\,dx$ as a power series.

(b) Use part (a) to approximate $\displaystyle\int_0^{0.5} \frac{1}{1+x^7}\,dx$ correct to within 10^{-7}.

Solution.

(a) First express the integrand, $\dfrac{1}{1+x^7}$, as the sum of apower series. We start with the sum of a geometric series and replace it x with $-x^7$.

$$\begin{aligned}
\frac{1}{1+x^7} &= \frac{1}{1-(-x^7)} = \sum_{n=0}^{\infty}(-x^7)^n \\
&= \sum_{n=0}^{\infty}(-1)^n x^{7n} = 1 - x^7 + x^{14} - \cdots
\end{aligned}$$

Now we integrate by term:

$$\begin{aligned}
\int \frac{1}{1+x^7}\,dx &= \int \sum_{n=0}^{\infty}(-1)^n x^{7n}\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{7n+1}}{7n+1} \\
&= C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots
\end{aligned}$$

THe series converges for $|-x^7| < 1$ which is the same as $|x| < 1$.

(b) We apply the Fundamental Theorem of Calculus to the antiderivative from part (a) with $C = 0$:

$$\begin{aligned}\int_0^{0.5} \frac{1}{1+x^7}\,dx &= \left[-\frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots\right]_0^{1/2} \\ &= \frac{1}{2} - \frac{1}{8\cdot 2^8} + \frac{1}{15\cdot 2^{15}} - \frac{1}{22\cdot 2^{22}} + \cdots + \frac{(-1)^n}{(7n+1)2^{7n+1}} + \cdots\end{aligned}$$

This infinite series is the exact value of the definite integral, but since it is an alternating series, we can approximate the sum using the Alternating Series Estimation Theorem. If we stop adding after the term $n = 3$, the error is smaller than the term with $n = 4$:

$$\frac{1}{29\cdot 2^29} \approx 6.4\times 10^{-11}$$

So we have

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx \approx \frac{1}{2} - \frac{1}{8\cdot 2^8} + \frac{1}{15\cdot 2^{15}} - \frac{1}{22\cdot 2^{22}} \approx 0.49951374$$

12.10 Taylor and Maclaurin Series

Theorem 12.10.1. If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Substituting this formula for c_n back into the series, we see that *if* f has a power series expansion at a, then it must be of the following form.

Definition 12.10.1. The **Taylor series of the function** f **at** a (or **about** a or **centered at** a) is of the form

$$\begin{aligned}f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots\end{aligned}$$

Definition 12.10.2. The **Maclaurin series** is a special case of the Taylor series when $a = 0$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

Example 12.10.1. Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence. Also, find the Taylor series at a.

Solution. If $f(x) = e^x$, then $f^n(x) = e^x$, so $f^n(0) = e^x = e^0 = 1$ for all n. Therefore, the Taylor series for f at 0 (Maclaurin series) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

To find the radius of convergence we let $a_n = x^n/n!$, so

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 < 1$$

so the series converges for all x by the Ratio Test and the radius of convergence is $R = \infty$.

The Taylor series at a is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n$$

Theorem 12.10.2. If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth degree polynomial if f at a, R_n is the remainder of the Taylor series, and

$$\lim_{n \to \infty} R_n(x) = 0$$

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

Proof. We are determining under what circumstances is a function equal to the sum of its Taylor Series. In other words, if f has derivatives of all orders, when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

As with any convergent series, this means that $f(x)$ is the limit of the sequence of partial sums. In the case of the Taylor series, the partial sums are

$$\begin{aligned} T_n(x) &= \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{n!}(x-a)^i \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

For example, for the polynomial function $f(x) = e^x$, the Taylor polynomials at 0 (or Maclaurin polynomials) with $n = 1,\ 2,$ and 3 are

$$T_1(x) = 1 + x \qquad T_2(x) = 1 + x + \frac{x^2}{2!} \qquad T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n\to\infty} T_n(x)$$

If we let

$$R_n(x) = f(x) - T_n(x) \quad \text{so that} \quad f(x) = T_n(x) + R_n(x)$$

then $R_n(x)$ is the remainder of the Taylor series. If we can somehow show that $\lim_{n\to\infty} R_n(x) = 0$, then it follows that

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} [f(x) - R_n(x)] = f(x) - \lim_{n\to\infty} R_n(x) = f(x)$$

In trying to show that $\lim_{n\to\infty} R_n(x) = 0$ for a specific function f, we usually use the following fact.

Definition 12.10.3 (Taylor's Inequality). If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leq d$$

It is helpful to use the following fact.

Definition 12.10.4. $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad$ for every real number x

Definition 12.10.5.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for every real number } x$$

If we plug in $x = 1$, we get the following expression for the number e as a sum of an infinite series:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{2}{2!} + \frac{3}{3!} + \cdots$$

Proof. If $f(x) = e^x$, then $f^{(n+1)}(x) = e^x$ for all n. If d is a positive number and $|x| \leq d$, then $|f^{(n+1)}(x)| = e^x \leq e^d$. So Taylor's inequality, with $a = 0$, and $M = e^d$, says that

$$|R_n(x)| \leq \frac{e^d}{(n+1)!}|x|^{n+1} \qquad \text{for} |x| \leq d$$

Notice that we have the same constant $M = e^d$ for every value of n. But because $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!} = 0$, we have

$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

It follows from the Squeeze Theorem that $\lim_{n\to\infty} |R_n(x)| = 0$ and therefore $\lim_{n\to\infty} R_n(x) = 0$ for all values of x.

Definition 12.10.6.

(i) $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x

(ii) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ for all x

Proof. The strategy is to find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x, and then differentiate it to find the Maclaurin series for $\cos x$.

(i) We arrange our computation in two columns:

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

Since the derivatives repeat in a cycle of four, we can write the Maclaurin series as

$$\begin{aligned} f(0) &+ \frac{f'(0)}{1!} + \frac{f''(0)}{2!} + \frac{f'''(0)}{3!} + \cdots \\ &= x - \frac{x^3}{x!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

Since $f^{(n+1)}(x)$ is $\pm \sin x$ or $\pm \cos x$, we know that $|f^{(n+1)}(x)| <\leq 1$ for all x, so we can take $M = 1$ in Taylor's inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x|^{n+1} = \frac{|x|^{n+1}}{(n+1)!}$$

Since $\lim_{n\to\infty} \frac{x^n}{n!} = 0$, the right side of this inequality approaches 0 as $n \to \infty$, so $|R_n(x)| \to 0$ by the Squeeze Theorem. It follows that $R_n(x) \to 0$ as $n \to \infty$, so $\sin x$ is equal to the sum of its Maclaurin series.

(ii) We could proceed directly as the previous proof but it is easier to differentiate the Maclaurin series for $\sin x$.

$$\begin{aligned} \cos x &= \frac{d}{dx} \sin x = \frac{d}{dx}\left(x - \frac{x^3}{x!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\ &= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \end{aligned}$$

Since the Maclaurin series for $\sin x$ converges for all x, the differentiated series for $\cos x$ also converges for all x, so

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{for all } x$$

Example 12.10.2. Find the Maclaurin series for the function $f(x) = x \cos x$.

Solution. Multiply the series for $\cos x$ by x.

$$x \cos x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}$$

Important Maclaurin series	Interval of Convergence
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$(-1, 1)$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3} + \cdots$	$(-\infty, \infty)$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{x!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$(-\infty, \infty)$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$(-\infty, \infty)$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{x!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$(-1, 1]$

Example 12.10.3.

(a) Evaluate $\int e^{-x^2}\, dx$ as an infinite series.

(b) Evaluate $\int_0^1 e^{-x^2}\, dx$ correct to within an error of 0.001.

Solution.

(a) First we find the Maclaurin series for $e^{-x^2}\ dx$ by replacing x with $-x^2$ in the series for e^x given in the table for Maclaurin series. Thus, for all values of x,

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3} + \cdots$$

Now integrate by term:

$$\begin{aligned} \int e^{-x^2}\ dx &= \int \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3} + \cdots\right)\ dx \\ &= C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots \end{aligned}$$

The series converges for all x because the original series e^{-x^2} converges for all x.

(b)

$$\begin{aligned} \int_0^1 e^{-x^2}\ dx &= \left[x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 3!} - \cdots\right]_0^1 \\ &= 1 - \tfrac{1}{3} + \tfrac{1}{10} - \tfrac{1}{42} + \tfrac{1}{216} - \cdots \\ &\approx 1 - \tfrac{1}{3} + \tfrac{1}{10} - \tfrac{1}{42} + \tfrac{1}{216} \approx 0.7475 \end{aligned}$$

The Alternating Series Estimation THeorem shows that the error involved in this approximation is less than

$$\frac{1}{11 \cdot 5!} = \frac{1}{1320} < 0.001$$

Multiplication and Division of Power Series

If power series are added or subtracted they behave like polynomials. In fact, they can also be multiplied and divided like polynomials. We only find the first few terms because they are the most important ones and the calculations for the later terms become tedious.

Example 12.10.4. Find the first three nonzero terms in the Maclaurin series for (a) $e^x \sin x$ and (b) $\tan x$.

Solution.

(a) Using the Maclaurin series for e^x and $\sin x$ from the table,

$$e^x \sin^x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3} + \cdots\right)\left(x - \frac{x^3}{x!} + \cdots\right)$$

We multiply these expressions like polynomials and get

$$e^x \sin^x = x + x^2 + \tfrac{1}{3}x^3 + \cdots$$

(b) Using the Maclaurin series in the table, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}$$

Use long-division to get

$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots$$

12.11 The Binomial Series

The **Binomial Theorem** states that if a and b are any real numbers and k is a positive integer, then

$$\begin{aligned}(a+b)^k = a^k &+ ka^{k-1}b + \frac{k(k-1)}{2!}a^{k-2}b^2 + \frac{k(k-1)(k-2)}{3!}a^{k-3}b^3 \\ &+ \cdots + \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}a^{k-n}b^n \\ &+ \cdots + kab^{k-1} + b^k\end{aligned}$$

The traditional notation for the binomial coefficients is

$$\binom{k}{0} = 1 \qquad \binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} \qquad n = 1, 2, \ldots, k$$

which enables us to write the Binomial Theorem in the abbreviated form

$$(a+b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n}b^n$$

Definition 12.11.1 (The Binomial Series). If k is any real number and $|x| < 1$, then

$$\begin{aligned}(1+x)^k &= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \\ &= \sum_{n=0}^{\infty} \binom{k}{n} x^n\end{aligned}$$

where $\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!} \quad (n \geq 1)$ and $\binom{k}{0} = 1$

Proof. Newton extended the Binomial Theorem to the case in which k is no longer a positive integer. In particular, if we put $a = 1$ and $b = k$, we get

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

To find this series we compute the Maclaurin series of $(1+x)^k$ in the usual way.

$$\begin{aligned}
f(x) &= (1+x)^k & f(0) &= 1 \\
f'(x) &= k(1+x)^{k-1} & f'(0) &= k \\
f''(x) &= k(k-1)(1+x)^{k-2} & f''(0) &= k(k-1) \\
f'''(x) &= k(k-1)(k-2)(1+x)^{k-3} & f'''(0) &= k(k-1)(k-2) \\
&\vdots & &\vdots \\
f^{(n)}(x) &= k(k-1)\cdots(k-n+1)(1+x)^{k-n} & f^{(n)}(0) &= k(k-1)\cdots(k-n+1)
\end{aligned}$$

Therefore the Maclaurin series of $f(x) = (1+x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

Now we use the Ratio Test to test the binomial series for convergence. If the nth term is a_n, then

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k-1)\cdots(k-n+1)x^n}\right| \\
&= \frac{|k-n|}{x+1}|x| = \frac{\left|1 - \frac{k}{n}\right|}{1 + \frac{1}{n}}|x| \to |x| \quad \text{as } n \to \infty
\end{aligned}$$

The binomial series converges if $|x| < 1$ and diverges if $|x| > 1$ by the Ratio Test.

Example 12.11.1. Expand $\dfrac{1}{(1+x)^2}$ as a power series.

Solution. Use the binomial series with $k = 2$. The binomial coefficient is

$$\begin{aligned}
\binom{-2}{n} &= \frac{(-2)(-3)(-4)\cdots(-2-n+1)}{n!} \\
&= \frac{(-1)^n 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n(n+1)}{n!} = (-1)^n(n+1)
\end{aligned}$$

and so, when $|x| < 1$,

$$\begin{aligned}
\frac{1}{(1+x)^2} &= (1+x)^{-2} = \sum_{n=0}^{\infty} \binom{-2}{n} x^n \\
&= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n = 1 - 2x + 3x^2 - 4x^3 + \cdots
\end{aligned}$$

12.12 Applications of Taylor Polynomials

Two applications of Taylor polynomials are to approximate functions and use them in physics.

Approximating Functions by Polynomials

In section 12.10, we introduced the notion for $T_n(x)$ for the nth partial sum of this series. Since f is the sum of its Taylor series, we know that $T_n(x) \to \infty$ as $n \to \infty$ so T_n can be used as an approximation to f: $f(x) \approx T_n(x)$.

When using a Taylor polynomial T_n to approximate a function f, we have to ask the questions: How good an approximation is it? How large should we take n to be in order to achieve a desired accuracy? To answer these questions we need to look at the absolute value of the remainder:

$$|R_n(x)| = |f(x) - T_n(x)|$$

There are three possible methods for estimating the size of the error:

1. If a graphing device is available, we can use it to graph $|R_n(x)|$ and thereby graphing the error.
2. If the series happens to be an alternating series, we can use the Alternating Series Estimation Theorem.
3. In all cases we can use Taylor's Inequality, which says that if $|f^{(n+1)}(x)| \leq M$, then
$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

Example 12.12.1.

(a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.

(b) How accurate is this approximation when $7 \leq x \leq 9$?

Solution.

(a)
$$\begin{aligned} f(x) &= \sqrt[3]{x} = x^{1/3} & f(8) &= 2 \\ f'(x) &= \tfrac{1}{3}x^{-2/3} & f'(8) &= \tfrac{1}{12} \\ f''(x) &= -\tfrac{2}{9}x^{-5/3} & f'(8) &= -\tfrac{1}{144} \\ f'''(x) &= \tfrac{10}{27}x^{-8/3} \end{aligned}$$

Thus, the second-degree Taylor polynomial is

$$\begin{aligned} T_2(x) &= f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f'(8)}{2!}(x-8)^2 \\ &= 2 + \tfrac{1}{12}(x-8) - \tfrac{1}{288}(x-8)^2 \end{aligned}$$

The desired approximation is

$$\sqrt[3]{x} \approx T_2(x) = 2 + \tfrac{1}{12}(x-8) - \tfrac{1}{288}(x-8)^2$$

(b) The Taylor series is not alternating when $x < 8$, so we can't use the Alternating Series Estimation Theorem, but we can use Taylor's Inequality with $n = 2$ and $a = 8$:

$$|R_2(x)| \le \frac{M}{3!}|x-8|^3$$

where $|f'''(x)| \le M$. Because $x \ge 7$, we have $x^{8/3} \ge 7^{8/3}$ and so

$$f'''(x) = \frac{10}{27} \cdot \frac{1}{x^{8/3}} \le \frac{10}{27} \cdot \frac{1}{7^{8/3}} < 0.0021$$

Therefore, we can take $M = 0.0021$. Also $7 \le x \le 9$ so $-1 \le x - 8 \le 1$ and $|x-8| \le 1$. Taylor's inequality gives

$$|R_2(x)| \le \frac{0.0021}{3!} \cdot 1^3 = \frac{0.0021}{6} \le 0.0004$$

Thus, if $7 \le x \le 9$, the approximation in part (a) is accurate to within 0.0004.