11.4 Areas and Lengths in Polar Coordinates

Area

We can determine the formula for the area of a region whose boundary is given by a polar equation by taking the limit of a Riemann Sum starting with the formula for the area of a sector of a circle $A = \frac{1}{2}r^2\theta$.

Definition 11.4.1. The formula for the area A of the polar region $\mathcal{R}$ is

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 \; d\theta = \int_a^b \frac{1}{2}r^2 d\theta$$

with the understanding that $r = f(\theta)$.

Example 11.4.1. Find the area enclosed by one loop of the four-leaved rose $r = 2\cos 2\theta$.

Solution. The right loop rotates from $\theta = -\pi/4$ to $\theta = \pi/4$.

$$\begin{aligned} A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2}r^2 d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta \; d\theta \\ &= \int_0^{\pi/4} \cos^2 2\theta \; d\theta = \int_0^{\pi/4} \frac{1}{2}(1 + \cos 4\theta) \; d\theta \\ &= \frac{1}{2}[\theta + \frac{1}{4}\sin 4\theta] = \pi/8 \end{aligned}$$

We can also adapt the formula to find the area of a region bounded by two polar curves.

Definition 11.4.2. Let $\mathcal{R}$ be a region that is bounded by curves with polar equations $r = f(\theta)$, $r = g(\theta)$, $\theta = a$, and $\theta = b$, where $f(\theta) \geq g(\theta) \geq 0$ and $0 < b-a \leq 2\pi$. The area A of $\mathcal{R}$ is found by subtracting the area inside $r = g(\theta)$ from the area inside $r = f(\theta)$, so

$$\begin{aligned} A &= \int_a^b \frac{1}{2}[f(\theta)]^2 \; d\theta - \int_a^b \frac{1}{2}[g(\theta)]^2 \; d\theta \\ &= \int_a^b \frac{1}{2}([f(\theta)]^2 - [g(\theta)]^2) \; d\theta \end{aligned}$$

Arc Length

To find the length of a polar curve $r = f(\theta)$, $a \leq \theta \leq b$, we regard θ as a parameter and write the parametric equations of the curve as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Using the projecut Rule and differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

so, using $\cos^2\theta + \sin^2\theta = 1$, we have

$$\begin{aligned}
\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\
&+ \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta \\
&= \left(\frac{dr}{d\theta}\right)^2 + r^2
\end{aligned}$$

Assuming that f' is continuous, we can use the theorem from 11.2 about the arc length of a curve defined by parametric equations to write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Definition 11.4.3. The length of a curve with polar equation $r = f(\theta)$, $a \leq \theta \leq b$, is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example 11.4.2. Find the arc length of the cardiod $r = 1 + \sin\theta$.

Solution. The full length of the cardiod is given by the parameter interval $0 \leq \theta \leq 2\pi$.

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta \\
&= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta = 8 \ (\textit{by rationalizing the integrand by } \sqrt{2 - 2\sin\theta})
\end{aligned}$$