# Chapter 11

# Parametric Equations and Polar Coordinates

## 11.1 Curves Defined by Parametric Equations

Suppose that $x$ and $y$ are both given as functions of a third variable $t$ (called a **parameter** by the equations)

$$x = f(t) \quad y = g(t)$$

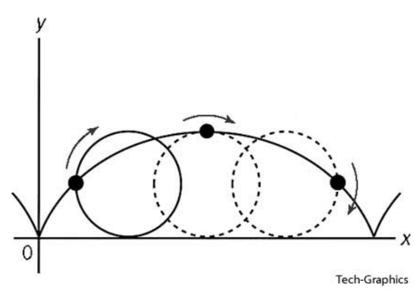
(called **parametric equations**). Each value of t determines a point (x,y). As t changes, $(x, y) = (f(t), g(t))$ changes and traces out a curve $C$, which is called a **parametric curve**. The direction of the arrows on curve $C$ show the change in the position of the equation as $t$ increases.

We can also restrict $t$ to a finite interval. In general, the curve with parametric equations

$$x = f(t) \quad y = g(t) \quad a \leq t \leq b$$

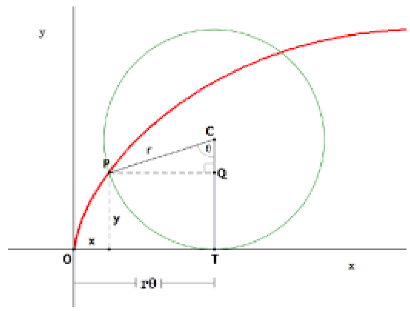
has **initial point** $(f(a), g(a))$ and **terminal point** $(f(b), g(b))$.

### The Cycloid



**Example 11.1.1.** A circle with radius $r$ rolls along the $x$-axis. The curve traced out by a point $P$ on the circumference of the circle is called a **cycloid**. Find parametric eqations for the cycloid.

*Solution.* We will use the angle of rotation $\theta$ as the parameter ($\theta = 0$ when $P$ is at the origin).




Suppose the circle has rotated $\theta$ radians. Using the figure, the distance it has rolled from the origin is

$$|OT| = arc\ PT = r\theta$$

because P starts at the origin. Therefore, the center of the circle is $C(r\theta, r)$. Let the coordinates of $P$ be $(x, y)$. Then from the figure,

$$x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$$
$$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$$

**Definition 11.1.1.** Paremetric equations of the cycloid are

$$x = r(\theta - \sin\theta) \quad y = r(1 - \cos\theta)$$

## 11.2 Calculus with Parametric Curves

We will mainly solve problems involving tangents, area, arc length, and surface area.

### Tangents

In the previous section, we saw that some curves defined by parametric equations $x = f(t)$ and $y = g(t)$ can also be expressed, by eliminating the parameter, in the form $y = F(x)$. If we substitute $x = f(t)$ and $y = g(t)$ in the equation $y = F(x)$, we get

$$g(t) = F(f(t))$$

If g, f, and F are differentiable, the Chain Rule gives

$$g'(t) = F'(f(t))f'(t) = F'(x)f'(t)$$

If $f'(t) \neq 0$, we can solve for $F'(x)$:

**Definition 11.2.1.** The slope of the tangent to the parametric curve $y = F(x)$ is $F'(x)$.

$$F'(x) = \frac{g'(t)}{f'(t)}$$

This enables us to find tangents to parametric curves without having to eliminate the parameter. We can rewrite the previous equation in an easily remembered form.

**Definition 11.2.2.** We can use this to find tangents to parametric curves without having to eliminate the parameter.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0$$

The curve has a

- horizontal tangent when $\frac{dy}{dt} = 0$ (provided that $\frac{dx}{dt} \neq 0$)
- vertical tangent when $\frac{dx}{dt} = 0$ (provided that $\frac{dy}{dt} \neq 0$)

This is useful when sketching parametric curves.

**Definition 11.2.3.** We can also find $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

**Example 11.2.1.** A curve C is defined by the parametric equations $x = t^2,\ y = t^3 - 3t$.

1. Show that $C$ has two tangents at the point (3,0) and find their equations.
2. Find the points on $C$ where the tangent is horizontal or vertical.
3. Determine where the curve is concave upward or downward.

*Solution.* A curve C is defined by the parametric equations $x = t^2,\ y = t^3 - 3t$.

1. Rewrite $y = t^3 - 3t = t(t^2 - 3) = 0$ when $t = 0$ or $t = \pm\sqrt{3}$. This indicates that C intersects itself at (3.0).

   $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$$
   $$t = \pm\sqrt{3} \rightarrow dy/dx = \pm 6/(2\sqrt{3})$$

   so the equations of the tangents at (3,0) are

   $$y = \sqrt{3}(x - 3) \quad and \quad y = -\sqrt{3}(x - 3)$$

2. C has a horizontal tangent when $dy/dx = 0$. In other words, when $dy/dt = 0$ and $dx/dt \neq 0$. $dy/dt = 3t^2 - 3 = 0$ when $t^2 = 1$ so $t = \pm 1$. This means there are horizontal tangents on C at (1,-2) and (1,2). C has a vertical tangent when $dx/dt = 2t = 0$, so $t = 0$. This means C has a vertical tangent at (0,0).

3. To determine concavity we calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{3}{2}\left(1+\frac{1}{t^2}\right)}{2t} = \frac{3(t^2+1)}{4t^3}$$

   The curve is concave upward when $t > 0$ and concave downward when $t < 0$.

## Area

We already know that area under a curve $y = F(x)$ from $a$ to $b$ is $A = \int_a^b F(x)dx$. We can apply this to parametric equations using the Substitution Rule for Definite Integrals.

**Definition 11.2.4.** If the curve $C$ is given by parametric equations $x = f(t)$ and $y = g(t)$ and $t$ increases from $\alpha$ to $\beta$,

$$A = \int_a^b y dx = \int_\alpha^\beta g(t)f'(t)dt$$

(Switch $\alpha$ to $\beta$ if the point on $C$ at $\beta$ is more left than $\alpha$.

**Example 11.2.2.** Find the area under one arch of the cycloid $x = r(\theta - \sin\theta),\ y = r(1-\cos\theta)$.

*Solution.* One arch of the cycloid is given by $0 \leq \theta \leq 2\pi$. Using the Substitution Rule with $y = r(1-\cos\theta)$ and $dx = r(1-\cos\theta)d\theta$, we have

$$\begin{aligned}
A &= \int_0^{2\pi} y dx = A = \int_0^{2\pi} r(1-\cos\theta)r(1-\cos\theta)d\theta \\
&= r^2\int_0^{2\pi}(1-\cos\theta)^2 d\theta = r^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)d\theta \\
&= r^2\int_0^{2\pi}\left[1-2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right]d\theta \\
&= r^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} \\
&= r^2\left(\frac{3}{2}\cdot 2\pi\right) = 3\pi r^2
\end{aligned}$$

### Arc Length

### Surface Area

## 11.3 Polar Coordinates

## 11.4 Areas and Lengths in Polar Coordinates

## 11.5 Conic Sections

## 11.6 Conic Sections in Polar Coordinates