12.2 Series

If we try to add the terms of an infinite sequence $a_n{}_{n=1}^{\infty}$ we get the expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a **series**) and is denoted by the symbol

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

We also consider the **partial sums**

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ s_4 &= a_1 + a_2 + a_3 + a_4 \\ s_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{\infty} a_i \end{aligned}$$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit. If the $\lim_{n\to\infty} s_n = s$ exists (as a finite number), then we call it the sum of the infinite series $\sum a_n$.

Definition 12.2.1. Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is **convergent** and we write

$$s_n = a_1 + a_2 + \cdots + a_n = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number s is the **sum** of the series. Otherwise, the series is **divergent**.

Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{\infty} a_i$$

Definition 12.2.2 (**Geometric Series**)**.** The **geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geq 1$, the geometric series is divergent.

"The sum of a convergent geometric series is $\frac{\text{first term}}{1-\text{common ratio}}$".

Proof.

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

Each term is obtained from the preceding one by multiplying it by the **common ratio** r.

If $r = 1$, then $s_n = a + a + \cdots + a = na \to \pm\infty$. Since $\lim_{n\to\infty}\ s_n$ doesn't exist, the geometric series diverges in this case. If $r \neq 1$, then

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rs_n &= \qquad ar + ar^2 + \cdots + ar^{n-1} + ar^n \end{aligned}$$

Subtracting these equations, we get

$$s_n - rs_n = a - ar^n$$

Definition 12.2.3 (**Partial Sum of a Geometric Series**)**.**

$$s_n = \frac{a(1-r^n)}{1-r}$$

If $-1 < r < 1$, we know that $r^n \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{a}{1-r}\lim_{n\to\infty} r^n = \frac{a}{1-r}$$

Thus, when $|r| < 1$, the geometric series is convergent and its sum is $a/(1-r)$.

If $r \leq -1$ or $r > 1$, the sequence $\{r^n\}$ is divergent, so $\lim_{n\to\infty} s_n$ does not exist.

Example 12.2.1. Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

Solution. The first time is $a = 5$ and the common ratio is $r = -\frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series is convergent and its sum is

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \frac{5}{1 - (-\frac{2}{3})} = \frac{5}{\frac{5}{3}} = 3$$

Example 12.2.2. Write the number $2.3\overline{17} = 2.3171717\ldots$ as a ratio of integers.

Solution.

$$2.3171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

After the first term, we have a geometric series with $a = \frac{17}{10^3}$ and $r = 1/10^2$.

$$\begin{aligned} 2.3\overline{17} &= 2.3 + \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} = 2.3 + \frac{\frac{17}{1000}}{\frac{99}{100}} \\ &= \frac{23}{10} + \frac{17}{990} = \frac{1147}{495} \end{aligned}$$

Example 12.2.3. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

Solution. THis is not a geometric series, so we go back to the definition of a convergent series and compute the partial sums.

$$s_n = \sum_{i=1}^{\infty} \frac{1}{i(i+1)} = +\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$$

We simplify this expression if we use the **partial fraction decomposition**

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

Thus, we have

$$\begin{aligned} s_n &= \sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1}\right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \qquad \text{so} \\ \lim_{n\to\infty} s_n &= \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1 \end{aligned}$$

Therefore, the given series is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Definition 12.2.4. The **harmonic series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Theorem 12.2.1. If the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

NOTE 1 With any *series* $\sum a_n$, we associate two *sequences*: the sequence $\{s_n\}$ of its partial sums and the sequence $\{a_n\}$ of its terms. If $\sum a_n$ is convergent, then the limit of the sequence $\{s_n\}$ is s (the sum of the series) and the limit of the sequence $\{a_n\}$ is 0.

NOTE 2 The converse is not true in general. If $\displaystyle\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

Proof. Let $s_n = \sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots + a_n$. Then $a_n = s_n - s_{n-1}$. Since $\sum a_n$ is convergent, the sequence $\{s_n\}$ is convergent. Let $\lim_{n\to\infty} s_n = s$. Since $n - 1 \to \infty$ as $n \to \infty$, we also have $\lim_{n\to\infty} s_{n-1} = s$. Therefore

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1}\\ &= s - s = 0\end{aligned}$$

Definition 12.2.5 (The Test for Divergence). If $\displaystyle\lim_{n\to\infty} a_n$ does not exist or if $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

Example 12.2.4. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

Solution.

$$\lim_{n\to\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2}{5n^2+4} = \sum_{n=1}^{\infty} \frac{1}{5+4/n^2} = \frac{1}{5} \neq 0$$

So the series diverges by the Test for Divergence.

NOTE 3 If we find that $\lim_{n\to\infty} a_n \neq 0$, we know that $\sum a_n$ is divergent. If we find that $\lim_{n\to\infty} a_n = 0$, we know *nothing* about the convergence or divergence about $\sum a_n$.

Theorem 12.2.2. If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$.

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Example 12.2.5. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$.

Solution. The series $\sum 1/2^n$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$, so

$$\sum_{n=1}^{\infty}\frac{1}{2^n} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

We found that

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$$

So the given series is convergent and

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) &= 3\sum_{n=1}^{\infty}\frac{1}{n(n+1)} + \sum_{n=1}^{\infty}\frac{1}{2^n} \\ &= 3 \cdot 1 + 1 = 4\end{aligned}$$

Note 4 A finite number of terms doesn't affect the convergence or divergence of a series. For instance, suppose that we were able to show that the series $\displaystyle\sum_{n=4}^{\infty}\frac{n}{n^3+1}$ is convergent. Since

$$\sum_{n=1}^{\infty}\frac{n}{n^3+1} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} = \sum_{n=4}^{\infty}\frac{n}{n^3+1}$$

we can conclude that the entire series $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^3+1}$ is convergent.

Similarly, if it is known that the series $\displaystyle\sum_{n=N+1}^{\infty} a_n$ converges, then the full series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

is also convergent.