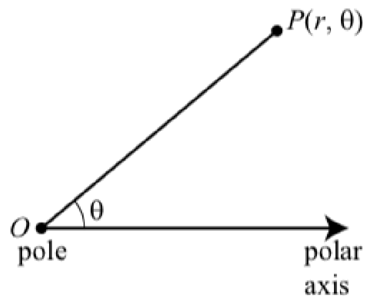
11.3 Polar Coordinates

In addition to Cartisian coordinates, we can also use a **polar coordinate system**.



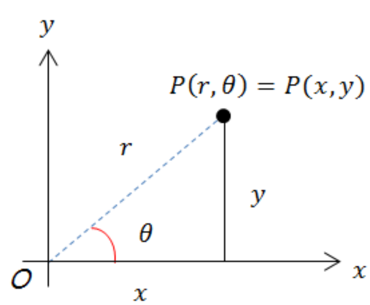
Point P is represented by the ordered pair (r, θ), where r is the distance to the point from the center and θ is the angle from the polar axis to the point.

The points (r, θ) and $(-r, \theta)$ are on the same line and have the same distance $|r|$ from the center but are on opposite sides of the center. Additionally, $(-r, \theta)$ and $(r, \theta + \pi)$ are also on the same line.

This means a complete counterclockwise rotation is given by an angle 2π, so (r, θ) is also represented by

$$(r, \theta + 2n\pi) \text{ and } (-r, \theta + (2n + 1)\pi)$$

Relationship Between Cartesian and Polar Coordinates



$$\cos\theta = \tfrac{x}{r} \quad \sin\theta = \tfrac{y}{r}$$

$$x = r\cos\theta \quad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \quad \tan\theta = \tfrac{y}{x}$$

Example 11.3.1. Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

Solution.

$$r = 2,\ \theta = \pi/3$$
$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1$$
$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$$

So the point is $(1, \sqrt{3})$ in Cartesian coordinates.

Example 11.3.2. Represent the Cartesian coordinates $(1, -1)$ in polar coordinates.

Solution.

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$
$$\tan\theta = \frac{y}{x} = -1$$

Since the point (1,-1) lies in the fourth quadrant, we can choose $\theta = -pi/4$ or $\theta = 7pi/4$. So the possible answers are either $(\sqrt{2}, -\pi/4$ or $(\sqrt{2}, 7\pi/4$.

Polar Curves

The **graph of a polar equation** $r = f(\theta)$, or $F(r, \theta) = 0$, consists of all of the points where (r, θ) satisfies the equation.

Tangents to Polar Curves

To find a tangent line to a polar curce $r = f(\theta)$, we regard θ as a parameter and write the parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \quad y = r\sin\theta = f(\theta)\sin\theta$$

So

Definition 11.3.1.
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dy}{d\theta} sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

- horizontal tangent when $\frac{dy}{d\theta} = 0$ (provided that $\frac{dx}{d\theta} \neq 0$)
- vertical tangent when $\frac{dx}{d\theta} = 0$ (provided that $\frac{dy}{d\theta} \neq 0$)

NOTE tangent lines at the pole have r=0 and the slope of the tangent simplifies to
$$\frac{dy}{dx} = \tan\theta \text{ if } \frac{dr}{d\theta} \neq 0$$

Example 11.3.3. For the cardiod $r = 1 + \sin\theta$, find the slope of the tangent line when r=3

Solution.

$$\begin{aligned}
r &= 1 + \sin\theta \\
\frac{dy}{dx} &= \frac{\frac{dy}{d\theta} sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta} \\
&= \frac{\cos\theta(1+2\sin\theta)}{1-2\sin^2\theta - \sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}
\end{aligned}$$

The slope of the tangent where $\theta = \pi/3$ is

$$\begin{aligned}
\left.\frac{dy}{dx}\right|_{\theta=\pi/3} &= \frac{\cos(\pi/3)(1+2\sin(\pi/3))}{(1+\sin(\pi/3))(1-\sin(\pi/3))} \\
&= \frac{\frac{1}{2}(1+\sqrt{3})}{(1+\sqrt{3}/2)(1-\sqrt{3})} = \frac{1+\sqrt{3}}{(2+\sqrt{3})(1-\sqrt{3})} \\
&= \frac{1+\sqrt{3}}{-1-\sqrt{3}} = -1
\end{aligned}$$

Note Instead of memorizing the equation, we can instead use the same method we used to derive it.

$$\begin{aligned}
x &= r\cos\theta = (1+\sin\theta)\cos\theta = \cos\theta + \frac{1}{2}\sin 2\theta \\
y &= r\sin\theta = (1+\sin\theta)\sin\theta = \sin\theta + \sin^2\theta \\
\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} = \frac{cos\theta + 2\sin\theta\cos\theta}{-\sin\theta + \cos 2\theta} = \frac{\cos\theta + \sin 2\theta}{-\sin\theta + \cos 2\theta}
\end{aligned}$$

This is equivalent to the previous equation.