12.3 The Integral Test and Estimates of Sums

Definition 12.3.1 (**The Integral Test**)**.** Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty}$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ is convergent. In other words:

(i) If $\displaystyle\int_1^{\infty} f(x)\, dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty}$ is convergent.

(i) If $\displaystyle\int_1^{\infty} f(x)\, dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty}$ is divergent.

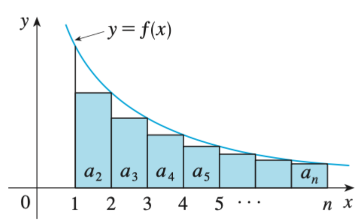
Note When we use the Integral Test, it is not necessary to start the series or the integral at $n = 1$. For instance, in testing the series

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} \qquad \text{we use} \qquad \int_1^{\infty} \frac{1}{(n-3)^2}\, dx$$

Also, it is not necessary that f is always decreasing; it is important that f is *ultimately* decreasing.

Proof. We will prove the convergence and divergence of the Integral Test for the general series $\sum a_n$

(i) **Convergence**



The area of the first shaded rectangle is $f(2) = a_2$. Because there is always space underneath the curve, the sum of the area of the shaded triangles from 1 to n is always less than the area under the curve (since f is decreasing).

$$a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)\, dx$$

If $\int_1^{\infty} f(x)\, dx$ is convergent, then

$$\sum_{i=2}^{n} a_i \leq \int_1^n f(x)\, dx \leq \int_1^{\infty} f(x)\, dx$$

since $f(x) \geq 0$. Therefore,

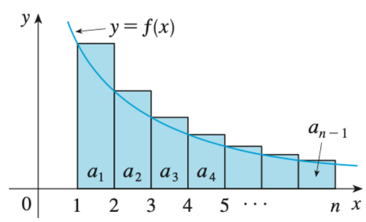
$$s_n = a_1 + \sum_{i=2}^{n} a_i \leq a_1 + \int_1^{\infty} f(x)\ dx = M \quad \text{(random variable)}$$

Since $s_n \leq M$ for all n, the sequence $\{s_n\}$ is bounded above. Also

$$s_{n+1} = s_n + a_{n+1} \geq s_n$$

since $a_{n+1} = f(n+1) \geq 0$. Thus, $\{s_n\}$ is an increasing bounded sequence so it it convergent by the Monotonic Sequence Theorem. This means that $\sum a_n$ is convergent.

(ii) **Divergence**



Because there is always space above the curve, the sum of the area of the shaded triangles from 1 to n is always greater than the area under the curve.

$$\int_1^n f(x)\ dx \leq a_1 + a_2 + \cdots + a_{n-1}$$

If $\int_1^\infty f(x)\ dx$ is divergent, then $\int_1^n f(x)\ dx \to \infty$ as $n \to \infty$ because $f(x) \geq 0$. But

$$\int_1^n f(x)\ dx \leq \sum_{i=1}^{n-1} a_i = s_{n-1}$$

so $s_{n-1} \to \infty$. This implies that $s_n \to \infty$ so $\sum a_n$ is diverges.

Example 12.3.1. Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

Solution. The function $f(x) = 1/(x^2+1)$ is continuous, positive, and decreasing on $[1, \infty)$ so we use the Integral Test:

$$\int_1^\infty \frac{1}{x^2+1}\ dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\ dx = \lim_{t\to\infty} \tan^{-1} x \Big]_1^t$$
$$= \lim_{t\to\infty} \left(\tan^{-1} t - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Thus, $\int_1^\infty \frac{1}{x^2+1}\ dx$ is a convergent integral. The series $\sum 1/(n^2+1)$ is convergent by the Integral Test.

Definition 12.3.2. The ***p*-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

For $p = 1$, the series is a harmonic series.

Proof. If $p < 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$. If $p = 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 1$. In either case, $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$, so the p-series diverges by the Test for Divergence.

If $p > 0$, then the function $f(x) = \frac{1}{x^p}$ is clearly continuous, positive, and decreasing on $[1, \infty)$. We know that

$$\int_1^{\infty} \frac{1}{x^p} \text{ converges if } p > 1 \text{ and diverges if } p \leq 1$$

Using the Integral Test, the series $\sum 1/n^p$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

Example 12.3.2.

(a) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$$

is convergent because it is a p-series with $p = 3 > 1$.

(b) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$$

is divergent because it is a p-series with $p = \frac{1}{3} < 1$.

NOTE We should *not* infer that the sum of the series is equal to the value of the integral from the Integral Test. In fact,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{whereas} \quad \int_n^{\infty} \frac{1}{n^2} = 1$$

Therefore, in general,

$$\sum_{n=1}^{\infty} a_n \neq \int_n^{\infty} f(x)\ dx$$

Example 12.3.3. Determine whether the series $\sum -n = 1^{\infty} \frac{\ln n}{n}$ converges or diverges.

Solution. The function $\frac{\ln x}{x}$ is positive and continous for $x > 1$ because the logarithm function is continuous, but it is not obvious whether or not f is decreasing, so we take its derivative:

$$f'(x) = \frac{(1/x)x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

Thus, $f'(x) < 0$ when $\ln x > 1$, which is when $x > e$. We conclude that f is decreasing when $x > e$ so we can apply the Integral Test:

$$\int_1^\infty \frac{\ln x}{x}\, dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\, dx = \lim_{t\to\infty} \frac{(\ln x)^2}{2}\Bigg]_1^t$$
$$= \lim_{t\to\infty} \frac{(\ln t)^2}{2} = \infty$$

Since this improper integral is divergent, the series $\sum(\ln n)/n$ is also divergent by the Integral Test.

Estimating the Sum of a Series

We can show if a series $\sum a_n$ is converging. Now we want to find an approximation to the sum s of the series. ANy partial sum s_n is an approximation to s because $\lim_{n\to\infty} s_n = s$, but *how good is that approximation*? To find out, we need to estimate the size of the **remainder**

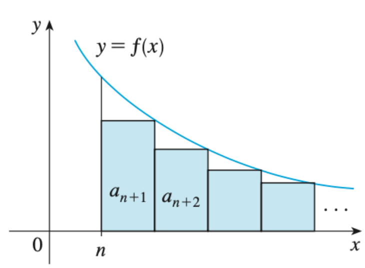
$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

The remainder R_n is the *error* made when s_n, the sum of the first n terms, is used as an approximation of the total sum.

Definition 12.3.3 (**Remainder Estimate for the Integral Test**). Suppose that $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

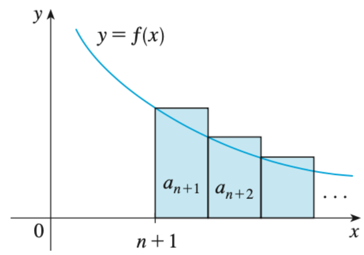
$$\int_{n+1}^\infty f(x)\, dx \leq R_n \leq \int_n^\infty f(x)\, dx$$

Proof. We use the same concept as the Integral test, assuming that f is decreasing on $[n, \infty)$.



We compare the sum of the area of the rectangles with the area under $y = f(x)$ for $x > n$ to see that

$$R_n = a_{n+1} + a_{n+2} + \cdots \leq \int_n^\infty f(x)\ dx$$



Similarly, we see that

$$R_n = a_{n+1} + a_{n+2} + \cdots \geq \int_n^\infty f(x)\ dx$$

Example 12.3.4. (a) Approximate the sum of the series $\sum 1/n^3$ by using the sum of the first 10 terms. Approximate the error involved in the approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

Example 12.3.5.

$$\int_n^\infty \frac{1}{x^3}\ dx = \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_n^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2} + \frac{1}{2n^2}\right) = \frac{1}{2n^2}$$

(a)

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{10^3} \approx 1.1975$$

According to the remainder estimate, we have

$$R_{10} \leq \int_{10}^\infty \frac{1}{x^3}\ dx = \frac{1}{2(10)^2} = \frac{1}{200}$$

So the size of the error is at most 0.005.

(b) Accuracy to within 0.0005 means that we have to find a value of n such that $R_n \leq 0.0005$. Since

$$R_n \leq \int_{10}^\infty \frac{1}{x^3}\ dx = \frac{1}{2n^2}$$

We want $$\frac{1}{2n^2} \leq 0.0005$$

Solving this inequality, we get

$$n^2 > \frac{1}{0.001} = 1000 \quad \text{or} \quad n > \sqrt{1000} \approx 31.6$$

We need 32 terms to ensure accuracy to within 0.0005.

If we add s_n to each side of the inequality of the Remainder Estimate for the Integral Test, we get

Definition 12.3.4.

$$s_n + \int_{n+1}^{\infty} f(x)\ dx \leq s \leq s_n + \int_{n}^{\infty} f(x)\ dx$$

because $s_n + R_n = s$. These inequalities give a lower bound and an upper bound for s. They provide a more accurate approximation than the partial sum s_n does.

Example 12.3.6. Use the improved remainder estimate with $n = 10$ to estimate the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$.

Solution.

$$s_{11} + \int_{11}^{\infty} \frac{1}{x^3}\ dx \leq s \leq s_{10} + \int_{10}^{\infty} \frac{1}{x^3}\ dx$$

We know from the previous example that

$$\int_{n}^{\infty} \frac{1}{x^3}\ dx = \frac{1}{2n^2}$$

so $$s_{11} + \frac{1}{2(11)^2} \leq s \leq s_{10} + \frac{1}{2(10)^2}$$

Using $s_{10} \approx 1.197532$, we get

$$1.201664 \leq s \leq 1.202532$$

If we approximate s by the midpoint of this interval, then the error is at most half the length of the interval, so

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2021 \quad \text{with error} < 0.0005$$

We get a much better estimate with this method than the estimate $s \approx s_n$ in the previous example. Also, we only had to use 10 terms to get the error smaller than 0.0005 instead of 32 terms.