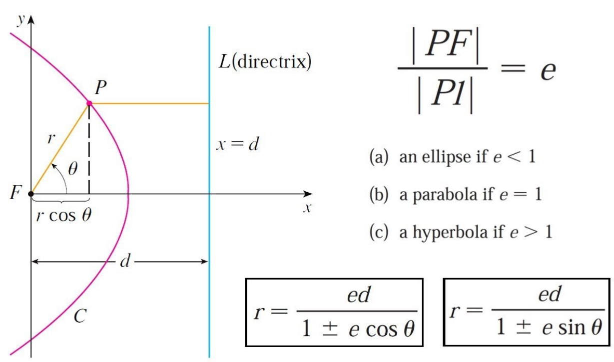
11.6 Conic Sections in Polar Coordinates

Theorem 11.6.1. Let F be a fixed point (called the **focus**) and l be a fixed line (called the **directrix**). Let e be a fixed positive number (called the **eccentricity**). The set of all points P in the plane such that

$$\frac{|PF|}{|Pl|} = e \quad \text{(the ratio of the distance from } F \text{ to the distance from } l \text{ is the constant e)}$$

is a conic section. The conic is

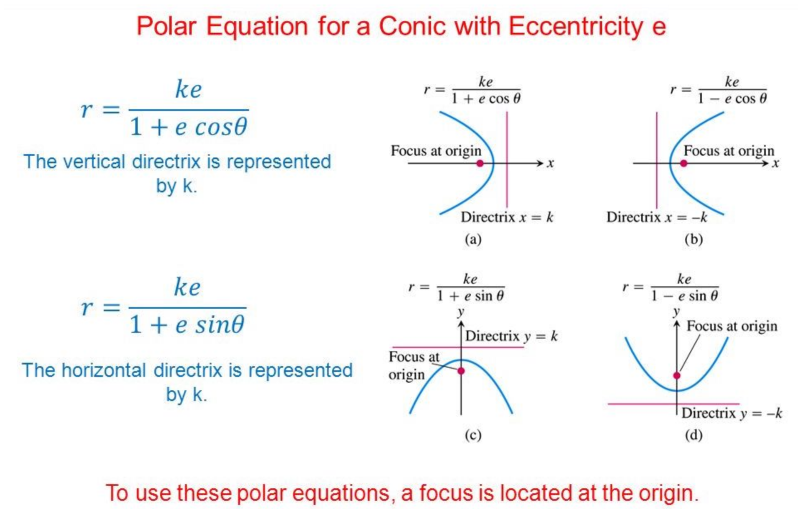
1. an ellipse if $e < 1$
2. a parabola if $e = 1$
3. a hyperbola if $e > 1$



Theorem 11.6.2. A polar equation of the form

$$r = \frac{ed}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic section with eccentricity e and distance d from the center to the directrix, with the focus at the origin. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola is $e > 1$.



Use "$\cos\theta$" when the conic section opens rightward or leftward, and use "$\sin\theta$" when the conic section opens upward or downward. Use "+" if the conic section opens leftward or downward, and use "−" if the conic section opens righttward or upward.

Example 11.6.1. Find a polar equation for a parabola that has its focus at the origin and whose directrix is the line $y = -6$.

Solution. The eccentricity $e = 1$ because the conic section is a parabola, and the distance from the center to the directrix is $d = 6$. The directrix is on the y-axis and is underneath the center, so the parabola opens upward. Therefore, we use the "$\sin\theta$" equation and use "−" in the denominator. The polar equation of the parabola is

$$r = \frac{6}{1 - \sin\theta}$$

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Example 11.6.2. A conic is given by the polar equation

$$r = \frac{10}{3 - 2\cos\theta}$$

Find the eccentricity, identify the conic, and locate the directrix.

Solution. Divide the numerator and denominator by 3 to get

$$r = \frac{\frac{10}{3}}{1 - \frac{2}{3}\cos\theta}$$

This represents an ellipses with eccentricity $e = \frac{2}{3}$. Since $ed = \frac{10}{3}$,

$$d = \frac{\frac{10}{3}}{e} = \frac{\frac{10}{3}}{\frac{2}{3}} = 5$$

so the directrix has Cartesian equation $x = -5$. When $\theta = 0$, $r = 10$; when $\theta = \pi$, $r = 2$, so the vertices have polar coordinates $(10, 0)$, and $(2, \pi)$.