

## Chapter 12

# Infinite Sequences and Series

### 12.1 Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

The number  $a_1$  is the *first term*,  $a_2$  is the *second term*, and in general  $a_n$  is the *nth term*. We will deal with infinite sequences exclusively so each term  $a_n$  will have a successor  $a_n + 1$ .

Notice that for every positive integer  $n$  there is a corresponding number  $a_n$  so a sequence can be defined as a function whose domain is the set of positive integers. We usually write  $a_n$  instead of the function notation  $f(n)$ .

NOTATION The sequence  $a_1, a_2, a_3, \dots$  is also denoted by

$$a_n \quad \text{or} \quad a_{n=1}^{\infty}$$

**Example 12.1.1.** Some sequences can be defined by giving a formula for the  $n$ th term. In this example, we will describe a sequence in 3 ways: the previous notation, defining a formula, and writing out the terms of the sequence. Note that  $n$  doesn't have to start at 1.

1.

$$\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty} \quad a_n = \frac{n}{n+1} \quad \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots \right\}$$

2.

$$\left\{ \frac{(-1)^n(n+1)}{3^n} \right\} \quad a_n = \frac{(-1)^n(n+1)}{3^n} \quad \left\{ -\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \dots, \frac{(-1)^n(n+1)}{3^n}, \dots \right\}$$

3.

$$\{\sqrt{n-3}\}_{n=3}^{\infty} \quad a_n = \sqrt{n-3}, \quad n \geq 3 \quad \{0, 1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n-3}, \dots\}$$

4.

$$\left\{\cos \frac{n\pi}{6}\right\}_{n=0}^{\infty} \quad a_n = \cos \frac{n\pi}{6}, \quad n \geq 0 \quad \left\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \dots, \cos \frac{n\pi}{6}, \dots\right\}$$

**Example 12.1.2.** Find a formula for the general term  $a_n$  of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \dots\right\}$$

*Solution.* We are given the first five terms. The numerator of the fractions start at 3 and increase by 1, so the  $n$ th term will have numerator  $n+1$ . The denominators are the powers of 5, so  $a_n$  has denominator  $5^n$ . The signs of the terms alternate between positive and negative, so we need to multiply by a power of  $-1$ . The factor  $(-1)^n$  means we start with a negative term, so here we use  $(-1)^{n-1}$  or  $(-1)^{n+1}$  because we start with a positive term. Therefore,

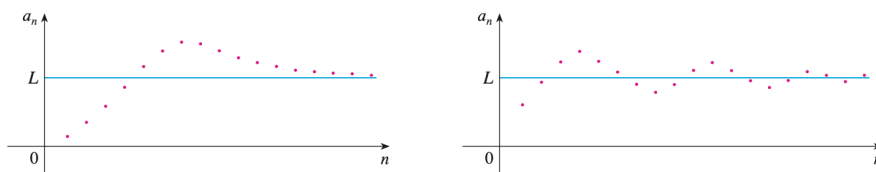
$$a_n = (-1)^{n-1} \frac{n+2}{5^n}$$

**Definition 12.1.1.** A sequence  $\{a_n\}$  has the **limit**  $L$  and we write

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \text{ as } n \rightarrow \infty$$

if we can make the terms  $a_n$  as close to  $L$  as we like by taking  $n$  sufficiently large. If  $\lim_{n \rightarrow \infty} a_n$  exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

The figure below graphs examples of two sequences that have the limit  $L$ .



A more precise version of the previous definition is

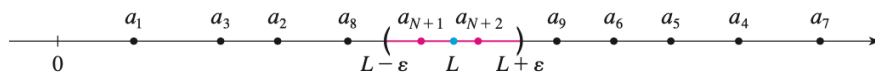
**Definition 12.1.2.** A sequence  $\{a_n\}$  has the **limit**  $L$  and we write

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \text{ as } n \rightarrow \infty$$

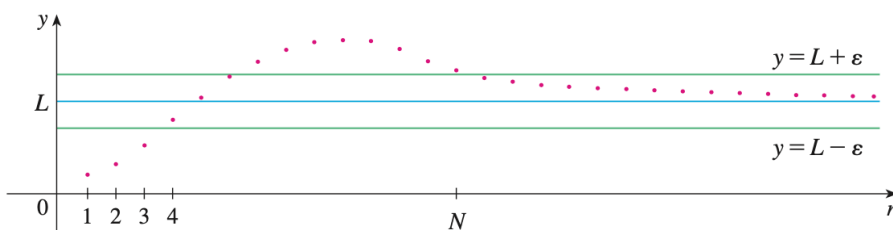
if for every  $\varepsilon > 0$  there is a corresponding integer  $N$  such that

$$|a_n - L| < \varepsilon \quad \text{whenever } n > N$$

No matter how small an interval  $(L - \varepsilon, L + \varepsilon)$  is chosen, there exists an  $N$  such that all terms of the sequence from  $a_{N+1}$  onward must lie in that interval.



The points on the graph of  $a_n$  must lie between the horizontal lines  $y = L + \varepsilon$  and  $y = L - \varepsilon$  if  $n > N$ . This picture must be valid no matter how small  $\varepsilon$  is chosen, but usually a smaller  $\varepsilon$  requires a larger  $N$ .



The only difference between  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{x \rightarrow \infty} f(x) = L$  is that  $n$  is required to be an integer.

**Theorem 12.1.1.** If  $\lim_{x \rightarrow \infty} f(x) = L$  and  $f(n) = a_n$  when  $n$  is an integer, then  $\lim_{n \rightarrow \infty} a_n = L$ .

Since we know that  $\lim_{x \rightarrow \infty} (1/x^r) = 0$  when  $r > 0$ , we can use the previous theorem to get

**Definition 12.1.3.**

$$\lim_{n \rightarrow \infty} \frac{1}{n^r} = 0 \quad \text{if } r > 0$$

If  $a_n$  grows as  $n$  grows, we use the notation  $\lim_{n \rightarrow \infty} a_n = \infty$ . We say that  $a_n$  diverges to  $\infty$ .

**Definition 12.1.4.**  $\lim_{n \rightarrow \infty} a_n = \infty$  means that for every positive number  $M$  there is an integer  $N$  such that

$$a_n > M \quad \text{whenever } n > N$$

**Definition 12.1.5 (Limit Laws for Sequences** (similar to original Limit Laws)). If  $\{a_n\}$  and  $\{b_n\}$  are convergent sequences and  $c$  is a constant, then

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} ca_n = c \lim_{n \rightarrow \infty} a_n$$

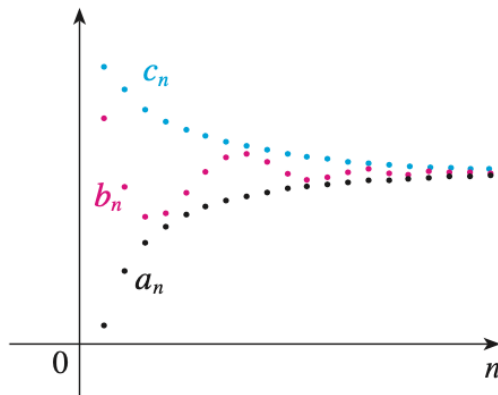
$$\lim_{n \rightarrow \infty} c = c$$

$$\lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} \quad \text{if } b_n \neq 0$$

$$\lim_{n \rightarrow \infty} (a_n^p) = \left[ \lim_{n \rightarrow \infty} a_n \right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

**Theorem 12.1.2 (The Squeeze Theorem for Sequences** (same as original)). If  $a_n \leq b_n \leq c_n$  for  $n \geq n_0$  and  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$ , then  $\lim_{n \rightarrow \infty} b_n = L$ .



**Theorem 12.1.3.** If  $\lim_{n \rightarrow \infty} |a_n| = 0$ , then  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Example 12.1.3.** Evaluate  $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n}$  if it exists.

*Solution.*

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \quad \text{so} \quad \lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$$

**Example 12.1.4.** Find  $\lim_{n \rightarrow \infty} \frac{n}{n+1}$ .

*Solution.* Divide the numerator and denominator by the highest power of  $n$  and then use the Limit Laws.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n}{n+1} &= \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = \frac{\lim_{n \rightarrow \infty} 1}{\lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{1}{n}} \\ &= \frac{1}{1 + 0} = 1 \end{aligned}$$

**Example 12.1.5.** Calculate  $\lim_{n \rightarrow \infty} \frac{\ln n}{n}$ .

*Solution.* Both the numerator and denominator approach infinity as  $n \rightarrow \infty$ . We can't apply l'Hospital's Rule directly because it applies to functions, not sequences. However, we can apply it to the related function.

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0 \quad \text{so} \quad \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$$

**Example 12.1.6.** Determine whether the sequence  $a_n = (-1)^n$  is convergent or divergent.

*Solution.* If we write out the terms of the sequence, we get  $\{-1, 1, -1, 1, -1, 1, -1, \dots\}$ . Since the terms oscillate between 1 and -1,  $a_n$  does not approach any number. Thus,  $\lim_{n \rightarrow \infty} (-1)^n$  does not exist so the sequence  $\{(-1)^n\}$  is divergent.

**Example 12.1.7.** Discuss the convergence of the the sequence  $a_n = n!/n^n$ .

*Solution.* Both the numerator and denominator approach infinity as  $n \rightarrow \infty$ , but we have no corresponding functions to use l'Hospital's Rule because  $x!$  is not defined when  $x$  is not an integer. if we write the general formula for the sequence, we get

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot n}{n \cdot n \cdot n \cdot \dots \cdot n} = \frac{1}{n} \left( \frac{2 \cdot 3 \cdot \dots \cdot n}{n \cdot n \cdot n \cdot \dots \cdot n} \right)$$

The expression in the parenthesis is at most 1 because the numerator is less than (or equal) to the denominator, so

$$0 < a_n \leq \frac{1}{n}$$

We can use the squeeze theorem because both  $0$  and  $1/n \rightarrow 0$  as  $n \rightarrow \infty$ , so  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ .

**Example 12.1.8.** Determine if the sequences below converge. If they do, find the limits as  $n \rightarrow \infty$ .

1.  $\frac{\sin n}{n}$
2.  $ne^{-n}$

*Solution.* 1.  $\frac{\sin n}{n}$  converges to 0 by the squeeze theorem

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}, \quad \lim_{n \rightarrow \infty} \frac{1}{n} = 0, \text{ so}$$

$$0 \leq \frac{\sin n}{n} \leq 0 \implies \lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$$

2.  $ne^{-n} = \frac{n}{e^n}$ . The denominator  $e^n$  converges faster than the numerator  $n$  does. Use l'Hospital's Rule to get

$$\lim_{x \rightarrow \infty} \frac{x}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0 \text{ so } \lim_{n \rightarrow \infty} ne^{-n} = 0$$

**Example 12.1.9.** Show that if  $\lim_{n \rightarrow \infty} a_{2n} = L$  and  $\lim_{n \rightarrow \infty} a_{2n+1} = L$ ,  $\{a_n\}$  is convergent and  $\lim_{n \rightarrow \infty} a_n = L$ .

*Solution.* The solution uses the symbols  $\exists$  ("exists") and  $\implies$  ("implies").

$$\begin{aligned} \text{Since } \lim_{n \rightarrow \infty} a_{2n} &= L, & \exists N_1 &\implies |a_{2n} - L| < \varepsilon \text{ for } n > N_1 \\ \text{Since } \lim_{n \rightarrow \infty} a_{2n+1} &= L, & \exists N_2 &\implies |a_{2n+1} - L| < \varepsilon \text{ for } n > N_2 \end{aligned}$$

Let  $N = \max\{2N_1, 2N_2 + 1\}$  and let  $n > N$ .

$$\begin{aligned} \text{If } n \text{ is even,} & & n = 2m, m > N_1, & |a_n - L| = |a_{2m} - L| < \varepsilon \\ \text{If } n \text{ is odd,} & & n = 2m + 1, m > N_2, & |a_n - L| = |a_{2m+1} - L| < \varepsilon \end{aligned}$$

Therefore,  $\{a_n\}$  is convergent and  $\lim_{n \rightarrow \infty} a_n = L$ .

**Definition 12.1.6.** The sequence  $\{r^n\}$  is convergent if  $-1 < r \leq 1$  and divergent for all other values of  $r$ .

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

**Definition 12.1.7.** A sequence  $\{a_n\}$  is **increasing** if  $a_n < a_{n+1}$  for all  $n \geq 1$  ( $a_1 < a_2 < a_3 < \dots$ ). It is **decreasing** if  $a_n > a_{n+1}$  for all  $n \geq 1$ . It is **monotonic** if it is either increasing or decreasing.

**Example 12.1.10.** The sequence  $\left\{\frac{3}{n+5}\right\}$  is decreasing because

$$\frac{3}{n+5} < \frac{3}{(n+1)+5} = \frac{3}{n+6}$$

for all  $n \geq 1$  (the right side is smaller because it has a larger denominator).

**Example 12.1.11.** Show that the sequence  $a_n = \frac{n}{n^2+1}$  is decreasing.

*Solution (1).* We must show that  $a_{n+1} < a_n$ .

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$

We can simplify this by cross-multiplying.  $\iff$  means "if and only if".

$$\begin{aligned} \frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} &\iff (n+1)(n^2+1) < n[(n+1)^2+1] \\ &\iff n^3 + n^2 + n + 1 < n^3 + 2n^2 + 2n \\ &\iff 1 < n^2 + n \end{aligned}$$

Since  $n \geq 1$ , we know that the inequality  $n^2 + n > 1$  is true. Therefore,  $a_{n+1} < a_n$  so  $\{a_n\}$  is decreasing.

*Solution (2).* Consider the function  $f(x) = \frac{x}{x^2+1}$ :

$$f'(x) = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} < 0 \quad \text{whenever } x^2 > 1$$

This,  $f$  is decreasing on  $(1, \infty)$  so  $f(n) > f(n+1)$ . Therefore,  $\{a_n\}$  is decreasing.

**Theorem 12.1.4** (Monotonic Sequence Theorem). Every bounded, monotonic sequence is convergent.

## 12.2 Series

If we try to add the terms of an infinite sequence  $a_{n=1}^\infty$  we get the expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a **series**) and is denoted by the symbol

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

We also consider the **partial sums**

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^n a_i$$

These partial sums form a new sequence  $\{s_n\}$ , which may or may not have a limit. If the  $\lim_{n \rightarrow \infty} s_n = s$  exists (as a finite number), then we call it the sum of the infinite series  $\sum a_n$ .

**Definition 12.2.1.** Given a series  $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ , let  $s_n$  denote its  $n$ th partial sum:

$$s_n = \sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence  $\{s_n\}$  is convergent and  $\lim_{n \rightarrow \infty} s_n = s$  exists as a real number, then the series  $\sum a_n$  is **convergent** and we write

$$s_n = a_1 + a_2 + \cdots + a_n = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number  $s$  is the **sum** of the series. Otherwise, the series is **divergent**.

Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} a_i$$

**Definition 12.2.2 (Geometric Series).** The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if  $|r| < 1$  and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r| < 1$$

If  $|r| \geq 1$ , the geometric series is divergent.

“The sum of a convergent geometric series is  $\frac{\text{first term}}{1 - \text{common ratio}}$ ”.

*Proof.*

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

Each term is obtained from the preceding one by multiplying it by the **common ratio**  $r$ .

If  $r = 1$ , then  $s_n = a + a + \cdots + a = na \rightarrow \pm\infty$ . Since  $\lim_{n \rightarrow \infty} s_n$  doesn't exist, the geometric series diverges in this case. If  $r \neq 1$ , then

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rs_n &= ar + ar^2 + \cdots + ar^{n-1} + ar^n \end{aligned}$$

Subtracting these equations, we get

$$s_n - rs_n = a - ar^n$$

**Definition 12.2.3 (Partial Sum of a Geometric Series).**

$$s_n = \frac{a(1-r^n)}{1-r}$$

If  $-1 < r < 1$ , we know that  $r^n \rightarrow 0$  as  $n \rightarrow \infty$ , so

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{a}{1-r} \lim_{n \rightarrow \infty} r^n = \frac{a}{1-r}$$

Thus, when  $|r| < 1$ , the geometric series is convergent and its sum is  $a/(1-r)$ .

If  $r \leq -1$  or  $r > 1$ , the sequence  $\{r^n\}$  is divergent, so  $\lim_{n \rightarrow \infty} s_n$  does not exist.



**Example 12.2.1.** Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

*Solution.* The first term is  $a = 5$  and the common ratio is  $r = -\frac{2}{3}$ . Since  $|r| = \frac{2}{3} < 1$ , the series is convergent and its sum is

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \frac{5}{1 - (-\frac{2}{3})} = \frac{5}{\frac{1}{3}} = 3$$

**Example 12.2.2.** Write the number  $2.3\overline{17} = 2.3171717\ldots$  as a ratio of integers.

*Solution.*

$$2.3171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

After the first term, we have a geometric series with  $a = \frac{17}{10^3}$  and  $r = 1/10^2$ .

$$\begin{aligned} 2.3\overline{17} &= 2.3 + \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} = 2.3 + \frac{\frac{17}{1000}}{\frac{99}{100}} \\ &= \frac{23}{10} + \frac{17}{990} = \frac{1147}{495} \end{aligned}$$

**Example 12.2.3.** Show that the series  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$  is convergent and find its sum.

*Solution.* This is not a geometric series, so we go back to the definition of a convergent series and compute the partial sums.

$$s_n = \sum_{i=1}^n \frac{1}{i(i+1)} = \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$$

We simplify this expression if we use the **partial fraction decomposition**

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

Thus, we have

$$\begin{aligned} s_n &= \sum_{i=1}^n \frac{1}{i(i+1)} = \sum_{i=1}^n \left( \frac{1}{i} - \frac{1}{i+1} \right) \\ &= \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \cdots + \left( \frac{1}{n} - \frac{1}{n+1} \right) \\ &= 1 - \frac{1}{n+1} \quad \text{so} \\ \lim_{n \rightarrow \infty} s_n &= \lim_{n \rightarrow \infty} \left( 1 - \frac{1}{n+1} \right) = 1 - 0 = 1 \end{aligned}$$

Therefore, the given series is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

**Definition 12.2.4.** The **harmonic series**  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent.

**Theorem 12.2.1.** If the series  $\sum_{n=1}^{\infty} a_n$  is convergent, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

**NOTE 1** With any *series*  $\sum a_n$ , we associate two *sequences*: the sequence  $\{s_n\}$  of its partial sums and the sequence  $\{a_n\}$  of its terms. If  $\sum a_n$  is convergent, then the limit of the sequence  $\{s_n\}$  is  $s$  (the sum of the series) and the limit of the sequence  $\{a_n\}$  is 0.

**NOTE 2** The converse is not true in general. If  $\lim_{n \rightarrow \infty} a_n = 0$ , we cannot conclude

that  $\sum_{n=1}^{\infty} a_n$  is convergent.

*Proof.* Let  $s_n = \sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots + a_n$ . Then  $a_n = s_n - s_{n-1}$ . Since  $\sum a_n$  is convergent, the sequence  $\{s_n\}$  is convergent. Let  $\lim_{n \rightarrow \infty} s_n = s$ . Since  $n-1 \rightarrow \infty$  as  $n \rightarrow \infty$ , we also have  $\lim_{n \rightarrow \infty} s_{n-1} = s$ . Therefore

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} (s_n - s_{n-1}) = \lim_{n \rightarrow \infty} s_n - \lim_{n \rightarrow \infty} s_{n-1} \\ &= s - s = 0 \end{aligned}$$

**Definition 12.2.5 (The Test for Divergence).** If  $\lim_{n \rightarrow \infty} a_n$  does not exist or

if  $\lim_{n \rightarrow \infty} a_n \neq 0$ , then the series  $\sum_{n=1}^{\infty} a_n$  is divergent.

**Example 12.2.4.** Show that the series  $\sum_{n=1}^{\infty} \frac{n^2}{5n^2 + 4}$  diverges.

*Solution.*

$$\lim_{n \rightarrow \infty} a_n = \sum_{n=1}^{\infty} \frac{n^2}{5n^2 + 4} = \sum_{n=1}^{\infty} \frac{1}{5 + 4/n^2} = \frac{1}{5} \neq 0$$

So the series diverges by the Test for Divergence.

**NOTE 3** If we find that  $\lim_{n \rightarrow \infty} a_n \neq 0$ , we know that  $\sum a_n$  is divergent. If we find that  $\lim_{n \rightarrow \infty} a_n = 0$ , we know *nothing* about the convergence or divergence about  $\sum a_n$ .

**Theorem 12.2.2.** If  $\sum a_n$  and  $\sum b_n$  are convergent series, then so are the series  $\sum ca_n$  (where  $c$  is a constant),  $\sum(a_n + b_n)$ , and  $\sum(a_n - b_n)$ .

$$(i) \sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$$

$$(ii) \sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$$(iii) \sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

**Example 12.2.5.** Find the sum of the series  $\sum_{n=1}^{\infty} \left( \frac{3}{n(n+1)} + \frac{1}{2^n} \right)$ .

*Solution.* The series  $\sum 1/2^n$  is a geometric series with  $a = \frac{1}{2}$  and  $r = \frac{1}{2}$ , so

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

We found that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

So the given series is convergent and

$$\begin{aligned} \sum_{n=1}^{\infty} \left( \frac{3}{n(n+1)} + \frac{1}{2^n} \right) &= 3 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= 3 \cdot 1 + 1 = 4 \end{aligned}$$

NOTE 4 A finite number of terms doesn't affect the convergence or divergence of a series. For instance, suppose that we were able to show that the series

$\sum_{n=4}^{\infty} \frac{n}{n^3+1}$  is convergent. Since

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} + \sum_{n=4}^{\infty} \frac{n}{n^3+1}$$

we can conclude that the entire series  $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$  is convergent.

Similarly, if it is known that the series  $\sum_{n=N+1}^{\infty} a_n$  converges, then the full series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^N a_n + \sum_{n=N+1}^{\infty} a_n$$

is also convergent.

## 12.3 The Integral Test and Estimates of Sums

**Definition 12.3.1 (The Integral Test).** Suppose  $f$  is a continuous, positive, decreasing function on  $[1, \infty)$  and let  $a_n = f(n)$ . Then the series  $\sum_{n=1}^{\infty}$  is convergent if and only if the improper integral  $\int_1^{\infty} f(x) dx$  is convergent. In other words:

- (i) If  $\int_1^{\infty} f(x) dx$  is convergent, then  $\sum_{n=1}^{\infty}$  is convergent.
- (i) If  $\int_1^{\infty} f(x) dx$  is divergent, then  $\sum_{n=1}^{\infty}$  is divergent.

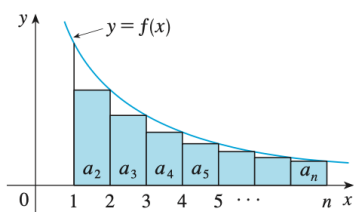
**NOTE** When we use the Integral Test, it is not necessary to start the series or the integral at  $n = 1$ . For instance, in testing the series

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} \quad \text{we use} \quad \int_1^{\infty} \frac{1}{(n-3)^2} dx$$

Also, it is not necessary that  $f$  is always decreasing; it is important that  $f$  is *ultimately* decreasing.

*Proof.* We will prove the convergence and divergence of the Integral Test for the general series  $\sum a_n$

### (i) Convergence



The area of the first shaded rectangle is  $f(2) = a_2$ . Because there is always space underneath the curve, the sum of the area of the shaded rectangles from 1 to  $n$  is always less than the area under the curve (since  $f$  is decreasing).

$$a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x) dx$$

If  $\int_1^{\infty} f(x) dx$  is convergent, then

$$\sum_{i=2}^n a_i \leq \int_1^n f(x) dx \leq \int_1^{\infty} f(x) dx$$

since  $f(x) \geq 0$ . Therefore,

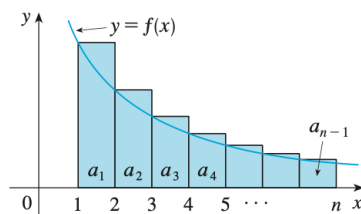
$$s_n = a_1 + \sum_{i=2}^n a_i \leq a_1 + \int_1^\infty f(x) dx = M \quad (\text{random variable})$$

Since  $s_n \leq M$  for all  $n$ , the sequence  $\{s_n\}$  is bounded above. Also

$$s_{n+1} = s_n + a_{n+1} \geq s_n$$

since  $a_{n+1} = f(n+1) \geq 0$ . Thus,  $\{s_n\}$  is an increasing bounded sequence so it is convergent by the Monotonic Sequence Theorem. This means that  $\sum a_n$  is convergent.

(ii) **Divergence**



Because there is always space above the curve, the sum of the area of the shaded triangles from 1 to  $n$  is always greater than the area under the curve.

$$\int_1^n f(x) dx \leq a_1 + a_2 + \cdots + a_{n-1}$$

If  $\int_1^\infty f(x) dx$  is divergent, then  $\int_1^n f(x) dx \rightarrow \infty$  as  $n \rightarrow \infty$  because  $f(x) \geq 0$ . But

$$\int_1^n f(x) dx \leq \sum_{i=1}^{n-1} a_i = s_{n-1}$$

so  $s_{n-1} \rightarrow \infty$ . This implies that  $s_n \rightarrow \infty$  so  $\sum a_n$  is divergent.

**Example 12.3.1.** Test the series  $\sum_{n=1}^\infty \frac{1}{n^2+1}$  for convergence or divergence.

*Solution.* The function  $f(x) = 1/(x^2+1)$  is continuous, positive, and decreasing on  $[1, \infty)$  so we use the Integral Test:

$$\begin{aligned} \int_1^\infty \frac{1}{x^2+1} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2+1} dx = \lim_{t \rightarrow \infty} \left[ \tan^{-1} x \right]_1^t \\ &= \lim_{t \rightarrow \infty} \left( \tan^{-1} t - \frac{\pi}{4} \right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

Thus,  $\int_1^\infty \frac{1}{x^2+1} dx$  is a convergent integral. The series  $\sum 1/(n^2+1)$  is convergent by the Integral Test.

**Definition 12.3.2.** The  **$p$ -series**  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  is convergent if  $p > 1$  and divergent if  $p \leq 1$ .

For  $p = 1$ , the series is a harmonic series.

*Proof.* If  $p < 0$ , then  $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$ . If  $p = 0$ , then  $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 1$ . In either case,  $\lim_{n \rightarrow \infty} \frac{1}{n^p} \neq 0$ , so the  $p$ -series diverges by the Test for Divergence.

If  $p > 0$ , then the function  $f(x) = \frac{1}{x^p}$  is clearly continuous, positive, and decreasing on  $[1, \infty)$ . We know that

$$\int_1^{\infty} \frac{1}{x^p} \text{ converges if } p > 1 \text{ and diverges if } p \leq 1$$

Using the Integral Test, the series  $\sum 1/n^p$  converges if  $p > 1$  and diverges if  $0 < p \leq 1$ .

**Example 12.3.2.**

(a) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$$

is convergent because it is a  $p$ -series with  $p = 3 > 1$ .

(b) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$$

is divergent because it is a  $p$ -series with  $p = \frac{1}{3} < 1$ .

NOTE We should *not* infer that the sum of the series is equal to the value of the integral from the Integral Test. In fact,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{whereas} \quad \int_1^{\infty} \frac{1}{n^2} = 1$$

Therefore, in general,

$$\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x) \, dx$$

**Example 12.3.3.** Determine whether the series  $\sum -n = 1^{\infty} \frac{\ln n}{n}$  converges or diverges.

*Solution.* The function  $\frac{\ln x}{x}$  is positive and continuous for  $x > 1$  because the logarithm function is continuous, but it is not obvious whether or not  $f$  is decreasing, so we take its derivative:

$$f'(x) = \frac{(1/x)x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

Thus,  $f'(x) < 0$  when  $\ln x > 1$ , which is when  $x > e$ . We conclude that  $f$  is decreasing when  $x > e$  so we can apply the Integral Test:

$$\begin{aligned} \int_1^\infty \frac{\ln x}{x} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{\ln x}{x} dx = \lim_{t \rightarrow \infty} \left. \frac{(\ln x)^2}{2} \right|_1^t \\ &= \lim_{t \rightarrow \infty} \frac{(\ln t)^2}{2} = \infty \end{aligned}$$

Since this improper integral is divergent, the series  $\sum (\ln n)/n$  is also divergent by the Integral Test.

### Estimating the Sum of a Series

We can show if a series  $\sum a_n$  is converging. Now we want to find an approximation to the sum  $s$  of the series. Any partial sum  $s_n$  is an approximation to  $s$  because  $\lim_{n \rightarrow \infty} s_n = s$ , but *how good is that approximation?* To find out, we need to estimate the size of the **remainder**

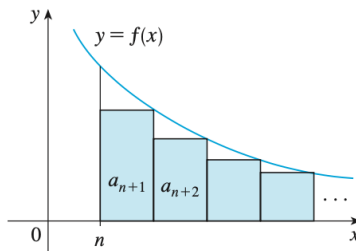
$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

The remainder  $R_n$  is the *error* made when  $s_n$ , the sum of the first  $n$  terms, is used as an approximation of the total sum.

**Definition 12.3.3 (Remainder Estimate for the Integral Test).** Suppose that  $f(k) = a_k$ , where  $f$  is a continuous, positive, decreasing function for  $x \geq n$  and  $\sum a_n$  is convergent. If  $R_n = s - s_n$ , then

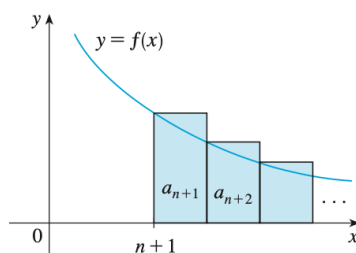
$$\int_{n+1}^\infty f(x) dx \leq R_n \leq \int_n^\infty f(x) dx$$

*Proof.* We use the same concept as the Integral test, assuming that  $f$  is decreasing on  $[n, \infty)$ .



We compare the sum of the area of the rectangles with the area under  $y = f(x)$  for  $x > n$  to see that

$$R_n = a_{n+1} + a_{n+2} + \cdots \leq \int_n^\infty f(x) \, dx$$



Similarly, we see that

$$R_n = a_{n+1} + a_{n+2} + \cdots \geq \int_n^\infty f(x) \, dx$$

**Example 12.3.4.** (a) Approximate the sum of the series  $\sum 1/n^3$  by using the sum of the first 10 terms. Approximate the error involved in the approximation.

- (b) How many terms are required to ensure that the sum is accurate to within 0.0005?

**Example 12.3.5.**

$$\int_n^\infty \frac{1}{x^3} \, dx = \lim_{t \rightarrow \infty} \left[ -\frac{1}{2x^2} \right]_n^t = \lim_{t \rightarrow \infty} \left( -\frac{1}{2t^2} + \frac{1}{2n^2} \right) = \frac{1}{2n^2}$$

(a)

$$\sum_{n=1}^\infty \frac{1}{n^3} \approx \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{10^3} \approx 1.1975$$

According to the remainder estimate, we have

$$R_{10} \leq \int_{10}^\infty \frac{1}{x^3} \, dx = \frac{1}{2(10)^2} = \frac{1}{200}$$

So the size of the error is at most 0.005.

- (b) Accuracy to within 0.0005 means that we have to find a value of  $n$  such that  $R_n \leq 0.0005$ . Since

$$R_n \leq \int_n^\infty \frac{1}{x^3} \, dx = \frac{1}{2n^2}$$



We want  $\frac{1}{2n^2} \leq 0.0005$

Solving this inequality, we get

$$n^2 > \frac{1}{0.001} = 1000 \quad \text{or} \quad n > \sqrt{1000} \approx 31.6$$

We need 32 terms to ensure accuracy to within 0.0005.

If we add  $s_n$  to each side of the inequality of the Remainder Estimate for the Integral Test, we get

**Definition 12.3.4.**

$$s_n + \int_{n+1}^{\infty} f(x) \, dx \leq s \leq s_n + \int_n^{\infty} f(x) \, dx$$

because  $s_n + R_n = s$ . These inequalities give a lower bound and an upper bound for  $s$ . They provide a more accurate approximation than the partial sum  $s_n$  does.

**Example 12.3.6.** Use the improved remainder estimate with  $n = 10$  to estimate the sum of the series  $\sum_{n=1}^{\infty} \frac{1}{n^3}$ .

*Solution.*

$$s_{11} + \int_{11}^{\infty} \frac{1}{x^3} \, dx \leq s \leq s_{10} + \int_{10}^{\infty} \frac{1}{x^3} \, dx$$

We know from the previous example that

$$\int_n^{\infty} \frac{1}{x^3} \, dx = \frac{1}{2n^2}$$

so 
$$s_{11} + \frac{1}{2(11)^2} \leq s \leq s_{10} + \frac{1}{2(10)^2}$$

Using  $s_{10} \approx 1.197532$ , we get

$$1.201664 \leq s \leq 1.202532$$

If we approximate  $s$  by the midpoint of this interval, then the error is at most half the length of the interval, so

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2021 \quad \text{with error} < 0.0005$$

We get a much better estimate with this method than the estimate  $s \approx s_n$  in the previous example. Also, we only had to use 10 terms to get the error smaller than 0.0005 instead of 32 terms.

**12.4 The Comparison Tests**

**12.5 Alternating Series**

**12.6 Absolute Convergence and the Ratio and Root Tests**

**12.7 Strategy for Testing Series**

**12.8 Power Series**

**12.9 Representation of Functions as Power Series**

**12.10 Taylor and Maclaurin Series**

**12.11 The Binomial Series**

**12.12 Applications of Taylor Polynomials**