Chapter 12

Infinite Sequences and Series

12.1 Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_n,\ \ldots$$

The number a_1 is the *first term*, a_2 is the *second term*, and in general a_n is the *nth term*. We will deal with infinite sequences exclusively so each term a_n will have a successor $a_n + 1$.

Notice that for every positive integer n there is a corresponding number a_n so a sequence can be defined as a function whose domain is the set of positive integers. We usually write a_n instead of the function notation $f(n)$.

NOTATION   The sequence $a_1,\ a_2,\ a_3,\ \ldots$ is also denoted by

$$a_n \quad \text{or} \quad {a_n}_{n=1}^{\infty}$$

Example 12.1.1. Some sequences can be defined by giving a formula for the nth term. In this example, we will describe a sequence in 3 ways: the previous notation, defining a formula, and writing out the terms of the sequence. Note that n doesn't have to start at 1.

1. $$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} \qquad a_n = \frac{n}{n+1} \qquad \left\{\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5}, \ldots,\ \frac{n}{n+1}, \ldots\right\}$$

2. $$\left\{\frac{(-1)^n(n+1)}{3^n}\right\} \qquad a_n = \frac{(-1)^n(n+1)}{3^n} \qquad \left\{-\frac{2}{3},\ \frac{3}{9},\ -\frac{4}{27},\ \frac{5}{81}, \ldots,\ \frac{(-1)^n(n+1)}{3^n}, \ldots\right\}$$

3.

$$\{\sqrt{n-3}\}_{n=3}^{\infty} \quad a_n = \sqrt{n-3},\ n \geq 3 \quad \{0,\ 1,\ \sqrt{2},\ \sqrt{3}, \ldots,\ \sqrt{n-3}, \ldots\}$$

4.

$$\left\{\cos\frac{n\pi}{6}\right\}_{n=0}^{\infty} \quad a_n = \cos\frac{n\pi}{6},\ n \geq 0 \quad \left\{1,\ \frac{\sqrt{3}}{2},\ \frac{1}{2},\ 0, \ldots,\ \cos\frac{n\pi}{6}, \ldots\right\}$$

Example 12.1.2. Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5},\ -\frac{4}{25},\ \frac{5}{125},\ -\frac{6}{625},\ \frac{7}{3125}, \ldots\right\}$$

Solution. We are given the first five terms. The numerator of the fractions start at 3 and increase by 1, so the nth term will have numerator $n+1$. The demoninators are the powers of 5, so a_n has denominator 5^n. The signs of the terms alternate between positive and negative, so we need to multiply by a power of 1. The factor $(-1)^n$ means we start with a negative term, so here we use $(-1)^{n-1}$ or $(-1)^{n+1}$ because we start with a positive term. Therefore,

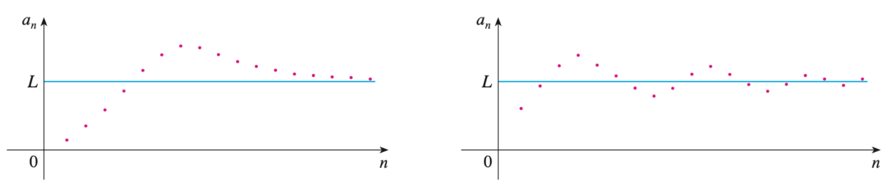
$$a_n = (-1)^{n-1}\frac{n+2}{5^n}$$

Definition 12.1.1. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as wel like by taking n sufficiently large. If $lim_{n\to\infty}a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

The figure below graphs examples of two sequences that have the limit L.



A more precise version of the previous definition is

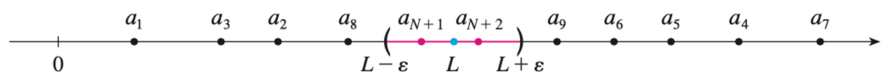
Definition 12.1.2. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

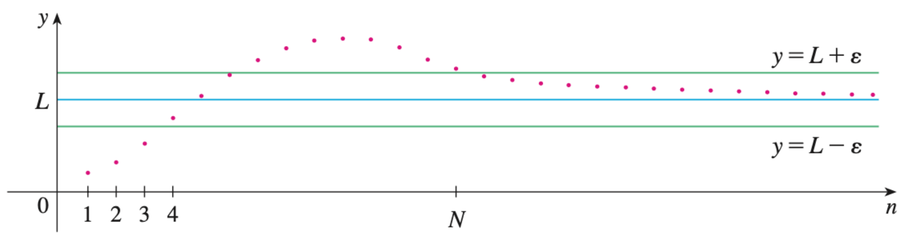
if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$|a_n - L| < \varepsilon \quad \text{whenever } n > N$$

No matter how small an interval $(L - \varepsilon\ L + \varepsilon)$ is chosen, there exists an N such that all terms of the sequence from a_{N+1} onward must lie in that interval.



The points on the graph of a_n must lie between the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ if $n > N$. This picture must be valid no matter how small ε is chosen, but usually a smaller ε requires a larger N.



The only difference between $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{x\to\infty} f(x) = L$ is that n is required to be an integer.

Theorem 12.1.1. If $\lim\limits_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an interger, then $\lim_{n\to\infty} = a_n = L$.

Since we know that $\lim\limits_{x\to\infty} (1/x^r) = 0$ when $r > 0$, we can use the previous theorem to get

Definition 12.1.3.
$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \quad \text{if } r > 0$$

If a_n grows as n grows, we use the notation $\lim_{n\to\infty} a_n = \infty$. We say that a_n diverges to ∞.

Definition 12.1.4. $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$a_n > M \quad \text{whenever } n > N$$

Definition 12.1.5 (**Limit Laws for Sequences** (similar to original Limit Laws))**.** If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

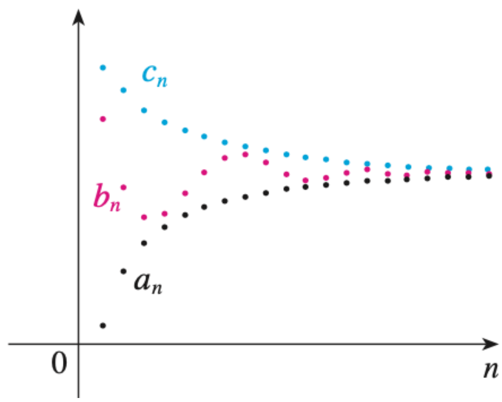
$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad\qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \tfrac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } b_n \neq 0$$

$$\lim_{n\to\infty}(a_n^p) = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

Theorem 12.1.2 (**The Squeeze Theorem for Sequences** (same as original))**.** If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.



Theorem 12.1.3. If $\lim_{n\to\infty} |a_n| = 0, then \lim_{n\to\infty} a_n = 0$.

Example 12.1.3. Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ if it exists.

Solution.

$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{so} \quad \lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

Example 12.1.4. Find $\lim_{n\to\infty} \frac{n}{n+1}$.

Solution. Divide the numberator and denominator by the highest power of n and then use the Limit Laws.

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}}$$
$$= \frac{1}{1+0} = 1$$

Example 12.1.5. Calculate $\lim_{n\to\infty} \frac{\ln n}{n}$.

Solution. Both the numerator and demoninator approach infinity as $n \to \infty$. We can't apply l'Hospital;s Rule directly because it applies to functions, not sequences. However, we can apply it to the related function.

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0 \quad \text{so} \quad \lim_{n\to\infty} \frac{\ln n}{n} = 0$$

Example 12.1.6. Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

Solution. If we write out the terms of the sequence, we get $\{-1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1, \ldots\}$. Since the terms oscillate between 1 and -1, a_n does not approach any number. Thus, $\lim_{n\to\infty}(-1)^n$ does not exist so the sequence $\{(-1)^n\}$ is divergent.

Example 12.1.7. Discuss the convergence of the the sequence $a_n = n!/n^n$.

Solution. Both the numerator and denominator approach infinity as $n \to \infty$, but we have no corresponding functions to use l'Hospital's Rule because $x!$ is not defined when x is not an integer. if we write the general forumula for the sequence, we get

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}{n \cdot n \cdot n \cdot \ldots \cdot n} = \frac{1}{n}\left(\frac{2 \cdot 3 \cdot \ldots \cdot n}{n \cdot n \cdot n \cdot \ldots \cdot n}\right)$$

The expression in the parenthesis is at most 1 because the numerator is less than (or equal) to the denominator, so

$$0 < a_n \leq \frac{1}{n}$$

We can use the squeeze theorem because both 0 and $1/n \to 0$ as $n \to \infty$, so $a_n \to \infty$ as $n \to \infty$.

Example 12.1.8. Determine if the sequences below converge. If they do, find the limits as $n \to \infty$.

1. $\frac{\sin n}{n}$
2. ne^{-n}

Solution.

1. $\frac{\sin n}{n}$ converges to 0 by the squeeze theorem

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq -\frac{1}{n}\ ,\ \lim_{n\to\infty}\frac{1}{n} = 0\ ,\ \text{so}$$
$$0 \leq \frac{\sin n}{n} \leq 0 \implies \lim_{n\to\infty}\frac{\sin n}{n} = 0$$

2. $ne^{-n} = \frac{n}{e^n}$. The denominator e^n converges faster than the numerator n does. Use l'Hospital's Rule to get

$$\lim_{x\to\infty}\frac{x}{e^x} = \lim_{x\to\infty}\frac{1}{e^x} = 0 \text{ so } \lim_{n\to\infty} ne^{-n} = 0$$

Example 12.1.9. Show that if $\lim\limits_{n\to\infty} a_{2n} = L$ and $\lim\limits_{n\to\infty} a_{2n+1} = L$, $\{a_n\}$ is convergent and $\lim\limits_{n\to\infty} a_n = L$.

Solution. The solution uses the symbols $\exists$ ("exists") and $\implies$ ("implies").

$$
\begin{array}{ll}
\text{Since } \lim_{n\to\infty} a_{2n} = L, & \exists\, N_1 \implies |a_{2n} - L| < \varepsilon \text{ for } n > N_1 \\
\text{Since } \lim_{n\to\infty} a_{2n+1} = L, & \exists\, N_2 \implies |a_{2n+1} - L| < \varepsilon \text{ for } n > N_2
\end{array}
$$

Let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$.

$$
\begin{array}{ll}
\text{If } n \text{ is even,} & n = 2m,\ m > N_1,\ |a_n - L| = |a_{2m} - L| < \varepsilon \\
\text{If } n \text{ is odd,} & n = 2m+1,\ m > N_2,\ |a_n - L| = |a_{2m+1} - L| < \varepsilon
\end{array}
$$

Therefore, $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.

Definition 12.1.6. The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$
\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 0 & \text{if } r = 1 \end{cases}
$$

Definition 12.1.7. A sequence $\{a_n\}$ is **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$ ($a_1 < a_2 < a_3 < \cdots$). It is **decreasing** is $a_n < a_{n+1}$ for all $n \geq 1$. It is **monotonic** if the is either increasing or decreasing.

Example 12.1.10. The sequence $\left\{\frac{3}{n+5}\right\}$ is decreasing because

$$
\frac{3}{n+5} < \frac{3}{(n+1)+5} = \frac{3}{n+6}
$$

for all $n \geq 1$ (the right side is smaller because it has a larger denominator).

Example 12.1.11. Show that the sequence $a_n = \frac{n}{n^2+1}$ is decreasing.

Solution (1). We must show that $a_{n+1} < a_n$.

$$
\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}
$$

We can simplify this by cross-multiplying. $\iff$ means "if and only if".

$$
\begin{aligned}
\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} &\iff (n+1)(n^2+1) < n[(n+1)^2+1] \\
&\iff n^3 + n^2 + n + 1 < n^3 + 2n^2 + 2n \\
&\iff 1 < n^2 + n
\end{aligned}
$$

Since $n \geq 1$, we know that the inequality $n^2 + n > 1$ is true. Therefore, $a_{n+1} < a_n$ so $\{a_n\}$ is decreasing.

Solution (2). Consider the function $f(x) = \frac{x}{x^2+1}$:

$$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0 \quad \text{whenever } x^2 > 1$$

This, f is decreasing on $(1, \infty)$ so $f(n) > f(n+1)$. Therefore, $\{a_n\}$ is decreasing.

Theorem 12.1.4 (Monotonic Sequence Theorem)**.** Every bounded, monotonic sequence is convergent.

12.2 Series

If we try to add the terms of an infinite sequence $a_{n}{}_{n=1}^{\infty}$ we get the expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a **series**) and is denoted by the symbol

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

We also consider the **partial sums**

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
s_4 &= a_1 + a_2 + a_3 + a_4 \\
s_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{\infty} a_i
\end{aligned}$$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit. If the $\lim_{n\to\infty} s_n = s$ exists (as a finite number), then we call it the sum of the infinite series $\sum a_n$.

Definition 12.2.1. Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is **convergent** and we write

$$s_n = a_1 + a_2 + \cdots + a_n = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number s is the **sum** of the series. Otherwise, the series is **divergent**.

Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{\infty} a_i$$

Definition 12.2.2 (**Geometric Series**)**.** The **geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geq 1$, the geometric series is divergent.

"The sum of a convergent geometric series is $\frac{\text{first term}}{1-\text{common ratio}}$".

Proof.

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

Each term is obtained from the preceding one by multiplying it by the **common ratio** r.

If $r = 1$, then $s_n = a + a + \cdots + a = na \to \pm\infty$. Since $\lim_{n\to\infty} s_n$ doesn't exist, the geometric series diverges in this case. If $r \neq 1$, then

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rs_n &= \quad\;\; ar + ar^2 + \cdots + ar^{n-1} + ar^n \end{aligned}$$

Subtracting these equations, we get

$$s_n - rs_n = a - ar^n$$

Definition 12.2.3 (**Partial Sum of a Geometric Series**)**.**

$$s_n = \frac{a(1-r^n)}{1-r}$$

If $-1 < r < 1$, we know that $r^n \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{a}{1-r} \lim_{n\to\infty} r^n = \frac{a}{1-r}$$

Thus, when $|r| < 1$, the geometric series is convergent and its sum is $a/(1-r)$.

If $r \leq -1$ or $r > 1$, the sequence $\{r^n\}$ is divergent, so $\lim_{n\to\infty} s_n$ does not exist.

Example 12.2.1. Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

Solution. The first time is $a = 5$ and the common ratio is $r = -\frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series is convergent and its sum is

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \frac{5}{1 - (-\frac{2}{3})} = \frac{5}{\frac{5}{3}} = 3$$

Example 12.2.2. Write the number $2.3\overline{17} = 2.3171717\ldots$ as a ratio of integers.

Solution.

$$2.3171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

After the first term, we have a geometric series with $a = \frac{17}{10^3}$ and $r = 1/10^2$.

$$\begin{aligned} 2.3\overline{17} &= 2.3 + \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} = 2.3 + \frac{\frac{17}{1000}}{\frac{99}{100}} \\ &= \frac{23}{10} + \frac{17}{990} = \frac{1147}{495} \end{aligned}$$

Example 12.2.3. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

Solution. THis is not a geometric series, so we go back to the definition of a convergent series and compute the partial sums.

$$s_n = \sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \ + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$$

We simplify this expression if we use the **partial fraction decomposition**

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

Thus, we have

$$\begin{aligned} s_n &= \sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1} \right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \qquad \text{so} \\ \lim_{n\to\infty} s_n &= \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1 \end{aligned}$$

Therefore, the given series is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Definition 12.2.4. The **harmonic series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Theorem 12.2.1. If the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

NOTE 1 With any *series* $\sum a_n$, we associate two *sequences*: the sequence $\{s_n\}$ of its partial sums and the sequence $\{a_n\}$ of its terms. If $\sum a_n$ is convergent, then the limit of the sequence $\{s_n\}$ is s (the sum of the series) and the limit of the sequence $\{a_n\}$ is 0.

NOTE 2 The converse is not true in general. If $\displaystyle\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

Proof. Let $s_n = \sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots + a_n$. Then $a_n = s_n - s_{n-1}$. Since $\sum a_n$ is convergent, the sequence $\{s_n\}$ is convergent. Let $\lim_{n\to\infty} s_n = s$. Since $n-1 \to \infty$ as $n \to \infty$, we also have $\lim_{n\to\infty} s_{n-1} = s$. Therefore

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} \\ &= s - s = 0\end{aligned}$$

Definition 12.2.5 (The Test for Divergence). If $\displaystyle\lim_{n\to\infty} a_n$ does not exist or if $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

Example 12.2.4. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

Solution.

$$\lim_{n\to\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2}{5n^2+4} = \sum_{n=1}^{\infty} \frac{1}{5+4/n^2} = \frac{1}{5} \neq 0$$

So the series diverges by the Test for Divergence.

NOTE 3 If we find that $\lim_{n\to\infty} a_n \neq 0$, we know that $\sum a_n$ is divergent. If we find that $\lim_{n\to\infty} a_n = 0$, we know *nothing* about the convergence or divergence about $\sum a_n$.

Theorem 12.2.2. If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$.

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Example 12.2.5. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$.

Solution. The series $\sum 1/2^n$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$, so

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

We found that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

So the given series is convergent and

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) &= 3\sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= 3 \cdot 1 + 1 = 4\end{aligned}$$

Note 4 A finite number of terms doesn't affect the convergence or divergence of a series. For instance, suppose that we were able to show that the series $\displaystyle\sum_{n=4}^{\infty} \frac{n}{n^3+1}$ is convergent. Since

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} = \sum_{n=4}^{\infty} \frac{n}{n^3+1}$$

we can conclude that the entire series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ is convergent.

Similarly, if it is known that the series $\displaystyle\sum_{n=N+1}^{\infty} a_n$ converges, then the full series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

is also convergent.

12.3 The Integral Test and Estimates of Sums

Definition 12.3.1 (**The Integral Test**)**.** Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty}$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\ dx$ is convergent. In other words:

(i) If $\displaystyle\int_1^{\infty} f(x)\ dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty}$ is convergent.

(i) If $\displaystyle\int_1^{\infty} f(x)\ dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty}$ is divergent.

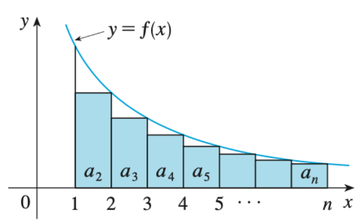
NOTE When we use the Integral Test, it is not necessary to start the series or the integral at $n = 1$. For instance, in testing the series

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} \qquad \text{we use} \qquad \int_1^{\infty} \frac{1}{(n-3)^2}\ dx$$

Also, it is not necessary that f is always decreasing; it is important that f is *ultimately* decreasing.

Proof. We will prove the convergence and divergence of the Integral Test for the general series $\sum a_n$

(i) **Convergence**



The area of the first shaded rectangle is $f(2) = a_2$. Because there is always space underneath the curve, the sum of the area of the shaded triangles from 1 to n is always less than the area under the curve (since f is decreasing).

$$a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)\ dx$$

If $\int_1^{\infty} f(x)\ dx$ is convergent, then

$$\sum_{i=2}^{n} a_i \leq \int_1^n f(x)\ dx \leq \int_1^{\infty} f(x)\ dx$$

since $f(x) \geq 0$. Therefore,

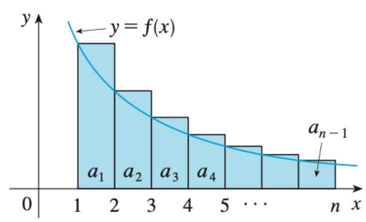
$$s_n = a_1 + \sum_{i=2}^{n} a_i \leq a_1 + \int_1^{\infty} f(x)\, dx = M \quad \text{(random variable)}$$

Since $s_n \leq M$ for all n, the sequence $\{s_n\}$ is bounded above. Also

$$s_{n+1} = s_n + a_{n+1} \geq s_n$$

since $a_{n+1} = f(n+1) \geq 0$. Thus, $\{s_n\}$ is an increasing bounded sequence so it it convergent by the Monotonic Sequence Theorem. This means that $\sum a_n$ is convergent.

(ii) **Divergence**



Because there is always space above the curve, the sum of the area of the shaded triangles from 1 to n is always greater than the area under the curve.

$$\int_1^{n} f(x)\, dx \leq a_1 + a_2 + \cdots + a_{n-1}$$

If $\int_1^{\infty} f(x)\, dx$ is divergent, then $\int_1^{n} f(x)\, dx \to \infty$ as $n \to \infty$ because $f(x) \geq 0$. But

$$\int_1^{n} f(x)\, dx \leq \sum_{i=1}^{n-1} a_i = s_{n-1}$$

so $s_{n-1} \to \infty$. This implies that $s_n \to \infty$ so $\sum a_n$ is diverges.

Example 12.3.1. Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

Solution. The function $f(x) = 1/(x^2+1)$ is continuous, positive, and decreasing on $[1, \infty)$ so we use the Integral Test:

$$\int_1^{\infty} \frac{1}{x^2+1}\, dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1}\, dx = \lim_{t\to\infty} \tan^{-1} x \Big]_1^t$$
$$= \lim_{t\to\infty} \left(\tan^{-1} t - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Thus, $\int_1^{\infty} \frac{1}{x^2+1}\, dx$ is a convergent integral. The series $\sum 1/(n^2+1)$ is convergent by the Integral Test.

Definition 12.3.2. The ***p*-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

For $p = 1$, the series is a harmonic series.

Proof. If $p < 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$. If $p = 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 1$. In either case, $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$, so the p-series diverges by the Test for Divergence.

If $p > 0$, then the function $f(x) = \frac{1}{x^p}$ is clearly continuous, positive, and decreasing on $[1, \infty)$. We know that

$$\int_1^{\infty} \frac{1}{x^p} \text{ converges if } p > 1 \text{ and diverges if } p \leq 1$$

Using the Integral Test, the series $\sum 1/n^p$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

Example 12.3.2.

(a) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$$

is convergent because it is a p-series with $p = 3 > 1$.

(b) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$$

is divergent because it is a p-series with $p = \frac{1}{3} < 1$.

NOTE We should *not* infer that the sum of the series is equal to the value of the integral from the Integral Test. In fact,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{whereas} \quad \int_n^{\infty} \frac{1}{n^2} = 1$$

Therefore, in general,

$$\sum_{n=1}^{\infty} a_n \neq \int_n^{\infty} f(x)\, dx$$

Example 12.3.3. Determine whether the series $\sum -n = 1^{\infty} \frac{\ln n}{n}$ converges or diverges.

Solution. The function $\frac{\ln x}{x}$ is positive and continous for $x > 1$ because the logarithm function is continuous, but it is not obvious whether or not f is decreasing, so we take its derivative:

$$f'(x) = \frac{(1/x)x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

Thus, $f'(x) < 0$ when $\ln x > 1$, which is when $x > e$. We conclude that f is decreasing when $x > e$ so we can apply the Integral Test:

$$\int_1^\infty \frac{\ln x}{x}\ dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\ dx = \lim_{t\to\infty} \frac{(\ln x)^2}{2}\bigg]_1^t$$
$$= \lim_{t\to\infty} \frac{(\ln t)^2}{2} = \infty$$

Since this improper integral is divergent, the series $\sum(\ln n)/n$ is also divergent by the Integral Test.

Estimating the Sum of a Series

We can show if a series $\sum a_n$ is converging. Now we want to find an approximation to the sum s of the series. ANy partial sum s_n is an approximation to s because $\lim_{n\to\infty} s_n = s$, but *how good is that approximation*? To find out, we need to estimate the size of the **remainder**

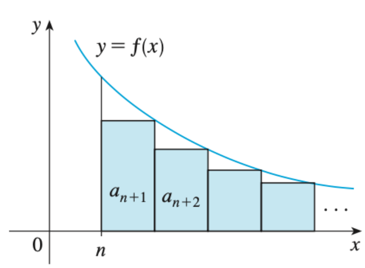
$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

The remainder R_n is the *error* made when s_n, the sum of the first n terms, is used as an approximation of the total sum.

Definition 12.3.3 (**Remainder Estimate for the Integral Test**)**.** Suppose that $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

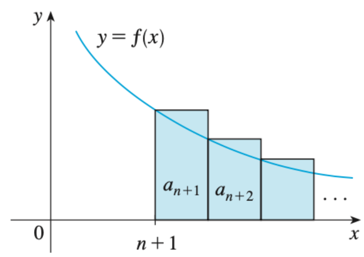
$$\int_{n+1}^\infty f(x)\ dx \leq R_n \leq \int_n^\infty f(x)\ dx$$

Proof. We use the same concept as the Integral test, assuming that f is decreasing on $[n, \infty)$.



We compare the sum of the area of the rectangles with the area under $y = f(x)$ for $x > n$ to see that

$$R_n = a_{n+1} + a_{n+2} + \cdots \le \int_n^\infty f(x)\, dx$$



Similarly, we see that

$$R_n = a_{n+1} + a_{n+2} + \cdots \ge \int_n^\infty f(x)\, dx$$

Example 12.3.4. (a) Approximate the sum of the series $\sum 1/n^3$ by using the sum of the first 10 terms. Approximate the error involved in the approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

Example 12.3.5.

$$\int_n^\infty \frac{1}{x^3}\, dx = \lim_{t\to\infty} \left[-\frac{1}{2x^2}\right]_n^t = \lim_{t\to\infty} \left(-\frac{1}{2t^2} + \frac{1}{2n^2}\right) = \frac{1}{2n^2}$$

(a)

$$\sum_{n=1}^\infty \frac{1}{n^3} \approx \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{10^3} \approx 1.1975$$

According to the remainder estimate, we have

$$R_{10} \le \int_{10}^\infty \frac{1}{x^3}\, dx = \frac{1}{2(10)^2} = \frac{1}{200}$$

So the size of the error is at most 0.005.

(b) Accuracy to within 0.0005 means that we have to find a value of n such that $R_n \le 0.0005$. Since

$$R_n \le \int_{10}^\infty \frac{1}{x^3}\, dx = \frac{1}{2n^2}$$

We want $$\frac{1}{2n^2} \leq 0.0005$$

Solving this inequality, we get

$$n^2 > \frac{1}{0.001} = 1000 \quad \text{or} \quad n > \sqrt{1000} \approx 31.6$$

We need 32 terms to ensure accuracy to within 0.0005.

If we add s_n to each side of the inequality of the Remainder Estimate for the Integral Test, we get

Definition 12.3.4.

$$s_n + \int_{n+1}^{\infty} f(x)\ dx \leq s \leq s_n + \int_{n}^{\infty} f(x)\ dx$$

because $s_n + R_n = s$. These inequalities give a lower bound and an upper bound for s. They provide a more accurate approximation than the partial sum s_n does.

Example 12.3.6. Use the improved remainder estimate with $n = 10$ to estimate the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$.

Solution.

$$s_{11} + \int_{11}^{\infty} \frac{1}{x^3}\ dx \leq s \leq s_{10} + \int_{10}^{\infty} \frac{1}{x^3}\ dx$$

We know from the previous example that

$$\int_{n}^{\infty} \frac{1}{x^3}\ dx = \frac{1}{2n^2}$$

so $$s_{11} + \frac{1}{2(11)^2} \leq s \leq s_{10} + \frac{1}{2(10)^2}$$

Using $s_{10} \approx 1.197532$, we get

$$1.201664 \leq s \leq 1.202532$$

If we approximate s by the midpoint of this interval, then the error is at most half the length of the interval, so

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2021 \quad \text{with error} < 0.0005$$

We get a much better estimate with this method than the estimate $s \approx s_n$ in the previous example. Also, we only had to use 10 terms to get the error smaller than 0.0005 instead of 32 terms.

12.4 The Comparison Tests

In comparison tests, the idea is to compare a given series with a series that is know to be convergent or divergent.

Definition 12.4.1 (The Comparison Test). Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.

In other words,

(i) If we have a series whose terms are *smaller* than those of a known *convergent* series, then our series is also convergent.

(ii) If we have a series whose terms are *larger* than those of a known *divergent* series, then our series is also divergent.

Proof. Let

$$s_n = \sum_{i=1}^{n} a_i \qquad t_n = \sum_{i=1}^{n} b_i \qquad t = \sum_{n=1}^{n} b_n$$

(i) **Convergence**

The sequences $\{s_n\}$ and $\{t_n\}$ are increasing ($s_{n+1} = s_n + a_{n+1} \ge s_n$) because both series have positive terms. Also $t_n \to t$, so $t_n \le t$ for all n. This means that $\{s_n\}$ is increasing and bounded above and therefore converges by the Monotonic Sequence Theorem. Thus, $\sum a_n$ converges.

(ii) **Divergence**

If $\sum b_n$ is divergent, then $t \to \infty$ (since $\{t_n\}$ is increasing). BUt $a_i \ge b_i$ so $s_n \ge t_n$. Thus, $s_n \to \infty$. Therefore, $\sum a_n$ diverges.

Example 12.4.1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges or diverges.

Solution. As n gets larger, the dominant term in the denominator is $2n^2$, so we compare the given series with the series $\sum 5/(2n^2)$. Observe that

$$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$$

because the left side has a bigger denominator. We know the

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent because it is a constant times a p-series with $p = 2 > 1$. Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ is convergent by the Comparison Test.

NOTE Although the condition $a_n \leq b_n$ for $a_n \geq b_n$ in the Comparison Test is given for all n, we only need to verify it for $n \geq N$, where N is some fixed integer, because the convergence of a series is not affected by a finite number.

Definition 12.4.2 (**The Limit Comparison Test**). Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or diverge.

Proof. Let m and M be positive numbers such that $m < c < M$. Because a_n/b_n is close to c for a large n, there is an integer N such that

$$m < \frac{a_n}{b_n} < M \quad \text{when } n > N \quad \text{so}$$
$$mb_n < a_n < Mb_n \quad \text{when } n > N$$

We can conclude the following:

(i) If $\sum b_n$ converges, so does $\sum Mb_n$, so $\sum a_n$ converges by the Comparison Test.

(i) If $\sum b_n$ diverges, so does $\sum Mb_n$, so $\sum a_n$ diverges by the Comparison Test.

Example 12.4.2. Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

Solution. We use the limit comparison test with

$$a_n = \frac{1}{2^n - 1} \qquad b_n = \frac{1}{2^n}$$

and obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1(2^n - 1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - 1/2^n} = 1 > 0$$

Since this limit exists and $\sum 1/2^n$ is a convergent geometric series, the given series converges by the Limit Comparison Test.

Estimating Sums

We used the Comparison test to series $\sum a_n$ by comparison with $\sum b_n$. We can also use it to estimate the sum by compparing remaindeds. We continue to consider the remainder R_n and consider T_n for the comparison series $\sum b_n$ as the corresponding remainder.

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$$
$$T_n = t - t_n = b_{n+1} + b_{n+2} + \cdots$$

Since $a_n \leq b_n$, $R_n \leq T_n$.

Example 12.4.3. Use the sum of the first 100 terms to approximate the sum of the series $\sum 1/(n^3+1)$. Estimate the error involved in this approximation.

Solution. Since

$$\frac{1}{n^3+1} < \frac{1}{n^3}$$

the given series is convergent by the Comparison Test. Using the Remainder Estimate for the Integral Test in section 12.3 we found that

$$T_n \leq \int_n^\infty \frac{1}{x^3}\, dx = \frac{1}{2n^2}$$

Therefore, the remainder R_n for the given series satisfies

$$R_n \leq T_n \leq \frac{1}{2n^2}$$

With $n = 100$ we have

$$R_{100} \leq \frac{1}{2(100)^2} = 0.00005$$

Using a calculator, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^3+1} \approx \sum_{n=1}^{100} \frac{1}{n^3+1} \approx 0.6864538$$

with error less than 0.00005.

12.5 Alternating Series

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$
$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$$

We see from these examples that the nth term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n \qquad \text{or} a_n = (-1)^n b_n$$

where b_n is a positive number. (In fact, $b_n = |a_n|$).

Definition 12.5.1 (**The Alternating Series Test**). If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad (b_n > 0)$$

satisfies

$$\text{(i)}\ \ b_{n+1} \leq b_n \quad \text{for all } n$$

$$\text{(ii)}\ \ \lim_{n\to\infty} b_n = 0$$

then the series is convergent.

"The alternating series converges if its terms decrease toward 0 in absolute value".

Proof. We consider the even and odd partial sums separately.

We first consider *even* partial sums:

$$\begin{aligned} s_2 &= b_1 - b_2 \geq 0 && \text{since } b_2 \leq b_1 \\ s_4 &= s_2 + (b_3 - b_4) \geq s_2 && \text{since } b_4 \leq b_3 \\ \text{In general} \quad s_{2n} &= s_{2n-2} + (b_{2n-1} - b_{2n}) \geq s_{2n-2} && \text{since } b_{2n} \leq b_{2n-1} \end{aligned}$$

Thus $\qquad 0 \leq s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_{2n} \leq \cdots$

But we can also write

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}$$

Every term in brackets is positive, so $s_{2n} \leq b_1$ for all n. Therefore, the sequence $\{s_{2n}\}$ of even partial sums is increasing and bounded above. It is therefore convergent by the Monotonic Sequence Theorem. Let's call this limit s:

$$\lim_{n\to\infty} s_{2n} = s$$

Next we compute the limit of the *odd* partial sums:

$$\begin{aligned} \lim_{n\to\infty} s_{2n+1} &= \lim_{n\to\infty} (s_{2n} + b_{2n+1}) \\ &= \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} \\ &= s + 0 \\ &= s \end{aligned}$$

Since both partial sums converge to s, we have $\lim_{n\to\infty} s_{2n} = s$ and so the series is convergent.

Example 12.5.1. The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies

$$\text{(i)} \;\; b_{n+1} \le b_n \qquad \text{because} \qquad \frac{1}{n+1} < \frac{1}{n}$$

$$\text{(ii)} \;\; \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

so the series is convergent by the Alternating Series Test

Example 12.5.2. The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is alternating but

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \lim_{n\to\infty} \frac{3}{4 - \dfrac{1}{n}} = \frac{3}{4} \neq 0$$

Instead, we look at the nth term of the series:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n 3n}{4n-1}$$

The limit does not exist, so the series diverges by the Test for Divergence.

Estimating Sums

Theorem 12.5.1 (Alternating Series Estimation Theorem). If $s = \sum (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

$$\text{(i)} \;\; 0 \le b_{n+1} \le b_n \qquad \text{and} \qquad \lim_{n\to\infty} b_n = 0$$

then
$$|R_n| = |s - s_n| \le b_{n+1}$$

Proof. We know from the proof of the Alternating Series Test that s lies between any two consecutive partial sums s_n and s_{n+1}. It follows that

$$|s - s_n| \le |s_{n+1} - s_n| = b_{n+1}$$

"The size of the error is smaller than b_{n+1}, which is the absolute value of the first neglected term" (valid only for alternating series that satisfy the Alternating Series Estimation, not other theorems.

Example 12.5.3. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places (By definition, $0! = 1$).

Solution. We first observe that the series is convergent by the Alternating Series Test because

$$\text{(i)} \quad \frac{1}{(n+1)!} = \frac{1}{n!(n+1)} \leq \frac{1}{n!}$$

$$\text{(ii)} \quad 0 < \frac{1}{n!} < \frac{1}{n} \to 0 \quad \text{so} \quad \frac{1}{n!} \to 0 \text{ as } n \to \infty$$

To get a feel for how many terms we need to use in our approximation, let's write out the first few terms of the series:

$$\begin{aligned} s &= \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots \\ &= 1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120} + \tfrac{1}{720} - \tfrac{1}{5040} + \cdots \end{aligned}$$

Notice that $\qquad b_7 = \frac{1}{5040} < \frac{1}{5000} = 0.0002$

and $\qquad s_n = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.369056$

By the Alternating Series Estimation Theorem, we know that

$$|s - s_6| \leq b_7 < 0.0002$$

This error of less than 0.0002 does not affect the third decimal place, so we have

$$s \approx 0.368$$

correct to three decimal places.

12.6 Absolute Convergence and the Ratio and Root Tests

Given any series $\sum a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute value of the terms of the original series.

Definition 12.6.1. A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Notice that if $\sum a_n$ is a series with positive terms, then $\sum |a_n| = \sum a_n$ and so absolute convergence is the same as convergence in this case.

Example 12.6.1. The series

$$\sum_{n\to\infty}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent because

$$\sum_{n\to\infty}^{\infty} \left|\frac{(-1)^{n-1}}{n^2}\right| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is a convergent p-series ($p = 2$).

Definition 12.6.2. A series $\sum a_n$ is **conditionally convergent** if it is convergent but not absolutely convergent.

Definition 12.6.3. If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Proof. Observe that the inequality

$$0 \le a_n + |a_n| \le 2|a_n|$$

is true because $|a_n|$ is either a_n or $-a_n$. If $\sum a_n$ is absolutely convergent, then $\sum |a_n|$ is convergent, $\sum 2|a_n|$. Therefore, by the comparison test, $\sum(a_n + |a_n|)$ is convergent. Then

$$\sum a_n = \sum(a_n + |a_n|) - \sum |a_n|$$

is the difference of the two convergent series and is therefore convergent.

Definition 12.6.4. The Ratio Test

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive (no conclusion can be drawn about the convergence or divergence of $\sum a_n$).

Proof.

(i) The idea is to compare the given series with a convergent geometric series. Since $L > 1$, we can choose a number r such that $L < r < 1$. Since

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \quad \text{and} \quad L < r$$

the ratio $|a_{n+1}/a_n|$ will eventually be less than r; this means that there exists an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| < r \quad \text{whenever } n \geq N$$

or, equivalently,

$$a_{n+1} < r|a_n| \quad \text{whenever } n \geq N$$

Putting n successively equal to $N,\ N+1,\ N+2, \ldots$ in the previous equation, we obtain

$$\begin{aligned} |a_{N+1}| &< |a_N|r \\ |a_{N+2}| &< |a_{N+1}|r < |a_N|r^2 \\ |a_{N+3}| &< |a_{N+2}|r < |a_N|r^3 \end{aligned}$$

and, in general,

$$|a_{N+k}| < |a_N|r^k \qquad \text{for all } k \geq 1$$

Now the series

$$\sum_{k=1}^{\infty} |a_N|r^k = |a_N|r + |a_N|r^2 + |a_N|r^3 + \cdots$$

is convergent because it is a geometric series with $0 < r < 1$. So the previous inequality , together with the Comparison Test, shows that the series

$$\sum_{n=N+1}^{\infty} |a_n| = \sum_{k=1}^{\infty} |a_{N+k}| = |a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \cdots$$

is also convergent. It follows that the series $\sum_{n=1}^{\infty} |a_n|$ is also convergent. Therefore, $\sum a_n$ is absolutely convergent.

(ii) If $|a_{n+1}/a_n| \to L > 1$ or $|a_{n+1}/a_n| \to \infty$, then the ratio $|a_{n+1}/a_n|$ will eventually be greater than 1. This means that there exists an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \quad \text{whenever } n \geq N$$

This means that $|a_{n+1}/a_n| > |a_n|$ whenever $n \geq N$ and so

$$\lim_{n\to\infty} a_n \neq 0$$

Therefore $\sum a_n$ diverges by the Test for Divergence.

(iii) The Ratio Test gives no information if $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$. For instance,

for the convergent series $\sum 1/n^2$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \frac{1}{\left(1+\frac{1}{n}\right)^2} \to 1 \quad \text{as } n \to \infty$$

whereas for the divergent series $\sum 1/n$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to 1 \quad \text{as } n \to \infty$$

Therefore, if $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$, the series $\sum a_n$ might converge or diverge. In this case, the Ratio Test fails and we must use some other test.

Example 12.6.2. Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n} for absolute convergence.$

Solution. We use the Ratio Test with $a_n = (-1)^n n^3/3^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}(n+1)^3}{3(n+1)}}{\frac{(-1)^n n^3}{3^n}}\right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$
$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \to \frac{1}{3} < 1$$

The given series is absolutely convergent by the Ratio Test and therefore convergent.

Definition 12.6.5. The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root Test is inconclusive.

NOTE If $L = 1$ in the Ratio Test, don't try the Root Test because L will also be 1.

Example 12.6.3. Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$

Solution.

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2+\frac{3}{n}}{3+\frac{2}{n}} \to \frac{2}{3} < 1$$

The given series converges by the Root Test.

12.7 Strategy for Testing Series

12.8 Power Series

12.9 Representation of Functions as Power Series

12.10 Taylor and Maclaurin Series

12.11 The Binomial Series

12.12 Applications of Taylor Polynomials