

Multivariable Calculus and Linear Algebra

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Part I

Multivariable Calculus

Chapter 11

Parametric Equations and Polar Coordinates

11.1 Curves Defined by Parametric Equations

Suppose that x and y are both given as functions of a third variable t (called a **parameter** by the equations)

$$x = f(t) \quad y = g(t)$$

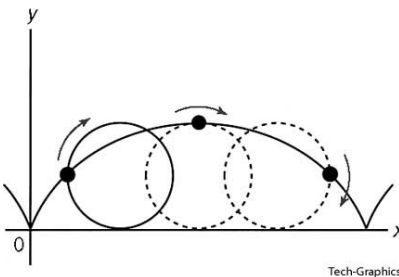
(called **parametric equations**). Each value of t determines a point (x,y) . As t changes, $(x,y) = (f(t), g(t))$ changes and traces out a curve C , which is called a **parametric curve**. The direction of the arrows on curve C show the change in the position of the equation as t increases.

We can also restrict t to a finite interval. In general, the curve with parametric equations

$$x = f(t) \quad y = g(t) \quad a \leq t \leq b$$

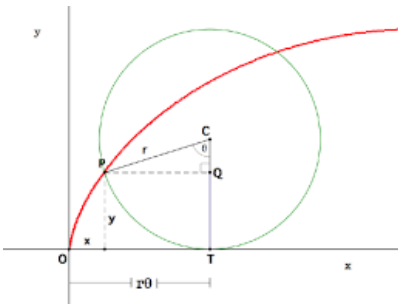
has **initial point** $(f(a), g(a))$ and **terminal point** $(f(b), g(b))$.

The Cycloid



Example 11.1.1. A circle with radius r rolls along the x -axis. The curve traced out by a point P on the circumference of the circle is called a **cycloid**. Find parametric equations for the cycloid.

Solution. We will use the angle of rotation θ as the parameter ($\theta = 0$ when P is at the origin).



Suppose the circle has rotated θ radians. Using the figure, the distance it has rolled from the origin is

$$|OT| = \text{arc } PT = r\theta$$

because P starts at the origin. Therefore, the center of the circle is $C(r\theta, r)$. Let the coordinates of P be (x, y) . Then from the figure,

$$x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$$

$$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$$

Definition 11.1.1. Parametric equations of the cycloid are

$$x = r(\theta - \sin\theta) \quad y = r(1 - \cos\theta)$$

11.2 Calculus with Parametric Curves

11.3 Polar Coordinates

11.4 Areas and Lengths in Polar Coordinates

11.5 Conic Sections

11.6 Conic Sections in Polar Coordinates

Chapter 12

Infinite Sequences and Series

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12.7 Strategy for Testing Series

12.8 Power Series

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12.10 Taylor and Maclaurin Series

12.11 The Binomial Series

12.12 Applications of Taylor Polynomials

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Partial Derivatives

15.1 Functions of Several Variables

15.2 Limits and Continuity

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15.4 Tangent Planes and Linear Approximations

15.5 The Chain Rule

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15.7 Maximum and Minimum Values

15.8 Lagrange Multipliers

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Multiple Integrals

16.1 Double Integrals over Rectangles

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16.3 Double Integrals over General Regions

16.4 Double Integrals in Polar Coordinates

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16.9 Change of Variables in Multiple Integrals

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Vector Calculus

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17.3 THE Fundamental Theorem for Line Integrals

17.4 Green's Theorem

17.5 Curl and Divergence

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17.8 Stokes' Theorem

17.9 The Divergence Theorem

17.10 Summary

Chapter 18

Second-Order Differential Equations

18.1 Second-Order Linear Equations

18.2 Nonhomogenous Linear Equations

18.3 Applications of Second-Order Differential Equations

18.4 Series Solutions

Part II

Linear Algebra

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Vectors

1.1 The Geometry and Algebra of Vectors

1.2 Length and Angle: The Dot Product

1.3 Lines and Planes

1.4 Code Vectors and Modular Systems

Chapter 2

Systems of Linear Equations

- 2.1 Introduction to Systems of Linear Equations
- 2.2 Direct Methods for Solving Linear Systems
- 2.3 Spanning Sets and Linear Independence
- 2.4 Applications
- 2.5 Iterative Method for Solving Linear Systems

Chapter 3

Matrices

3.1 Matrix Operations

3.2 Matrix Algebra

3.3 The Inverse of a Matrix

3.4 The LU Factorization

3.5 Subspaces, Basis, Dimension, and Rank

3.6 Introduction to Linear Transformations

3.7 Applications

Chapter 4

Eigenvalues and Eigenvectors

- 4.1 Introduction to Eigenvalues and Eigenvectors
- 4.2 Determinants
- 4.3 Eigenvalues and Eigenvectors of $n \times n$ Matrices
- 4.4 Similarity and Diagonalization
- 4.5 Iterative Methods for Computing Eigenvalues
- 4.6 Applications and the Perron-Frobenius Theorem

Chapter 5

Orthogonality

5.1 Orthogonality in \mathbb{R}^n

5.2 Orthogonal Complements and Orthogonal Projections

5.3 The Gram-Schmidt Process and the QR Factorization

5.4 Orthogonal Diagonalization of Symmetric Matrices

5.5 Applications

Chapter 6

Vector Spaces

6.1 Vector Spaces and Subspaces

6.2 Linear Independence, Basis, and Dimension

6.3 Change of Basis

6.4 Linear Transformation

6.5 The Kernel and Range of a Linear Transformation

6.6 The Matrix of a Linear Transformation

6.7 Applications

Chapter 7

Distance and Approximation

7.1 Inner Product Spaces

7.2 Norms and Distance Function

7.3 Least Squares Approximation

7.4 The Singular Value Decomposition

7.5 Applications