12.4 The Comparison Tests

In comparison tests, the idea is to compare a given series with a series that is know to be convergent or divergent.

Definition 12.4.1 (The Comparison Test). Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

In other words,

(i) If we have a series whose terms are *smaller* than those of a known *convergent* series, then our series is also convergent.

(ii) If we have a series whose terms are *larger* than those of a known *divergent* series, then our series is also divergent.

Proof. Let

$$s_n = \sum_{i=1}^{n} a_i \qquad t_n = \sum_{i=1}^{n} b_i \qquad t = \sum_{n=1}^{n} b_n$$

(i) **Convergence**

The sequences $\{s_n\}$ and $\{t_n\}$ are increasing $(s_{n+1} = s_n + a_{n+1} \geq s_n)$ because both series have positive terms. Also $t_n \to t$, so $t_n \leq t$ for all n. This means that $\{s_n\}$ is increasing and bounded above and therefore converges by the Monotonic Sequence Theorem. Thus, $\sum a_n$ converges.

(ii) **Divergence**

If $\sum b_n$ is divergent, then $t \to \infty$ (since $\{t_n\}$ is increasing). BUt $a_i \geq b_i$ so $s_n \geq t_n$. Thus, $s_n \to \infty$. Therefore, $\sum a_n$ diverges.

Example 12.4.1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges or diverges.

Solution. As n gets larger, the dominant term in the denominator is $2n^2$, so we compare the given series with the series $\sum 5/(2n^2)$. Observe that

$$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$$

because the left side has a bigger denominator. We know the

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent because it is a constant times a p-series with $p = 2 > 1$. Therefore, $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ is convergent by the Comparison Test.

NOTE Although the condition $a_n \leq b_n$ for $a_n \geq b_n$ in the Comparison Test is given for all n, we only need to verify it for $n \geq N$, where N is some fixed integer, because the convergence of a series is not affected by a finite number.

Definition 12.4.2 (The Limit Comparison Test). Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or diverge.

Proof. Let m and M be positive numbers such that $m < c < M$. Because a_n/b_n is close to c for a large n, there is an integer N such that

$$\begin{aligned} m < \frac{a_n}{b_n} < M \quad &\text{when } n > N \quad \text{so} \\ mb_n < a_n < Mb_n \quad &\text{when } n > N \end{aligned}$$

We can conclude the following:

(i) If $\sum b_n$ converges, so does $\sum Mb_n$, so $\sum a_n$ converges by the Comparison Test.

(i) If $\sum b_n$ diverges, so does $\sum Mb_n$, so $\sum a_n$ diverges by the Comparison Test.

Example 12.4.2. Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

Solution. We use the limit comparison test with

$$a_n = \frac{1}{2^n - 1} \qquad b_n = \frac{1}{2^n}$$

and obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1(2^n - 1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - 1/2^n} = 1 > 0$$

Since this limit exists and $\sum 1/2^n$ is a convergent geometric series, the given series converges by the Limit Comparison Test.

Estimating Sums

We used the Comparison test to series $\sum a_n$ by comparison with $\sum b_n$. We can also use it to estimate the sum by compparing remaindeds. We continue to consider the remainder R_n and consider T_n for the comparison series $\sum b_n$ as the corresponding remainder.

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$$
$$T_n = t - t_n = b_{n+1} + b_{n+2} + \cdots$$

Since $a_n \leq b_n$, $R_n \leq T_n$.

Example 12.4.3. Use the sum of the first 100 terms to approximate the sum of the series $\sum 1/(n^3+1)$. Estimate the error involved in this approximation.

Solution. Since

$$\frac{1}{n^3+1} < \frac{1}{n^3}$$

the given series is convergent by the Comparison Test. Using the Remainder Estimate for the Integral Test in section 12.3 we found that

$$T_n \leq \int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$$

Therefore, the remainder R_n for the given series satisfies

$$R_n \leq T_n \leq \frac{1}{2n^2}$$

With $n = 100$ we have

$$R_{100} \leq \frac{1}{2(100)^2} = 0.00005$$

Using a calculator, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^3+1} \approx \sum_{n=1}^{100} \frac{1}{n^3+1} \approx 0.6864538$$

with error less than 0.00005.