Spectral Koopman-Hopf Formula

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Abstract— **We discover the spectral Koopman Hopf formula for solving finite-time optimal control problems with constraints on the control input. The Hopf formula provides a state space parallelizable optimization-based solution for the computation of optimal value function. However, this formula only applies to optimal control problems where the underlying Hamiltonian is state-independent. The state-independent Hamiltonian severely restricts the applicability of the Hopf formula. We extend the applicability of the Hopf formula to the general class of control-affine nonlinear systems where the Hamiltonian function is not state-independent. We use spectral properties of the Koopman operator associated with the uncontrolled system to transform the state-dependent Hamiltonian to a state-independent one. The attractive feature of the Hopf formula is it transforms the curse of dimensionality problem associated with solving the Hamilton Jacobi equation to the so-called *curse of complexity* problem. We show that the proposed spectral Koopman Hopf formula also does not suffer from the curse of dimensionality, thereby preserving the complexity of the Hopf formula. This preservation of complexity is made possible by proposing the use of path-integral formula for the computation of Koopman principal eigenfunctions. We provide necessary and sufficient conditions for the applicability of path-integral formula. We present simulation results to verify the developed framework in the paper.**

I. Introduction

The solution to the finite-time optimal control problem for a system with nonlinear dynamics involves solving the Hamilton Jacobi partial differential equation (HJ PDE). With constraints on the control input, the HJ PDE admits a viscosity solution used for determining the optimal control [1]. Solving the HJ PDE for its viscosity solution is computationally challenging and suffers from the curse of dimensionality problem. Hence, the brute-force approach based on the spatial discretization of the HJ PDE is mainly restricted to solving optimal control problems for systems with lower dimensional state space [].

An alternative approach to the brute-force spatial discretization for solving the HJ PDE is the Hopf formula [2]. The Hopf formula provides the solution to the HJ PDE in the form of a space parallelizable optimization problem. In particular, using the Hopf formula, the optimal value function, $V(\mathbf{x}, t)$, at any point, say $\mathbf{x}$, in the state space, can be obtained as a solution to an optimization problem and is independent of the value of the, $V(\mathbf{y}, t)$ at any other point, $\mathbf{y}$, in the state space. However, the Hopf formula only applies when the underlying Hamiltonian in the optimal control problem is state-independent. The state-independent Hamiltonian imposes severe restrictions on the class of control systems where the Hopf formula can be applied. For example, the particular class of control system includes linear systems. This paper's main goal and contribution is to broaden the applicability of the Hopf formula to a class of control systems where the Hamiltonian is not state-independent. This is made possible using the spectral theory of the Koopman operator.

The Koopman theory has emerged as a powerful tool for the data-driven analysis and synthesis of nonlinear systems [3]–[9]. The Koopman operator provides a linear lifting of nonlinear system dynamics in the function space. Data-driven approaches for the finite-dimensional approximation of the Koopman operator and its spectrum are successfully applied for the analysis of the dynamical system [4], [10]–[12]. However, applying Koopman theory for synthesizing and controlling a dynamical system is challenged as the Koopman lifting of control affine dynamical system is bilinear [5]. In [13], a linear approximation of the bilinear Koopman model in the lifted functional space is used to transform a nonlinear control system into a large-dimensional linear control system. The Hopf formula is applied to the linear model in the lifted space. This paper proposes a fundamentally different and more rigorous approach to discovering the Koopman-based Hopf formula based on the spectral properties of the Koopman operator.

Following are the main contributions of this paper. We discover the Koopman-based Hopf formula based on the spectral theory of the Koopman operator. In contrast to the popular use of the Koopman operator for the high-dimensional linear lifting of nonlinear systems, we use n-*principal eigenfunctions* (where n is the state space dimension) of the Koopman operator associated with the uncontrolled system to transform the Hamiltonian to state-independent Hamiltonian. The state-independent Hamiltonian in the spectral Koopman coordinates is used to provide lower and upper bounds to the viscosity solution of the HJ PDE using the spectral Koopman-based Hopf formula. The Hopf formula's attractive feature is that it transforms the curse of dimensionality problem associated with solving HJ PDE into a curse of complexity problem. Hence, it is necessary that the spectral Koopman-Hopf formula also preserves this complexity. We show that this is precisely the case and that the new spectral Koopman-Hopf formula also does not suffer from curse of dimensionality but curse of complexity problem. This is made possible by proposing the use of path-integral formula for the computation of Koopman principal eigenfunctions. The path-integral formula essentially involves computing integral along a trajectory to

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determine a value of an eigenfunction at point $\mathbf{x}$ in the state space. In [14], we proposed the use of path-integral formula based on the solution of linear PDE. The main contribution of this paper towards path-integral formula for the computation of Koopman principal eigenfunction is we provide necessary and sufficient conditions independent of the linear PDE solution for the applicability of path-integral formula. The path-integral formula is instrumental in proving that the proposed spectral Koopman-Hopf formula also suffer from the curse complexity but not curse of dimensionality problem.

The organization of the paper is as follows. In Section II, we discuss preliminaries on Koopman spectrum and viscosity solution of HJ PDE. The main results of this paper are proved in Section III. The results on path integral approach for computing Koopman principal eigenfunctions and spectral Koopman-Hopf formula is proved in Section IV. The simulations results and concluding remark are in Sections VI and VII respectively.

II. Preliminaries

A. Spectral theory of Koopman operator

In this section, we provide a brief overview of existing results on the spectral theory of the Koopman operator. For more details on this topic, refer to [3], [15]. We present the preliminaries on the Koopman theory for a general dynamical system defined on a open set $\mathcal{X} \subseteq \mathbb{R}^n$ of the form:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{X}. \tag{1}$$

Definition 1: [Koopman Operator] The Koopman operator $\mathbb{U}_t$ corresponding to system (1) acting on the function $\psi : \mathcal{X} \to \mathbb{C}$ is defined as

$$[\mathbb{U}_t \psi](\mathbf{x}) = \psi(\mathbf{s}_t(\mathbf{x})). \tag{2}$$

The infinitesimal generator of the Koopman operator is given by

$$\lim_{t \to 0} \frac{(\mathbb{U}_t - I)\psi}{t} = \frac{\partial \psi}{\partial \mathbf{x}} \mathbf{F}(\mathbf{x}) =: \mathcal{K}_{\mathbf{F}} \psi, \quad t \geq 0. \tag{3}$$

Furthermore, for any $\mathcal{C}^1$ function ψ with $u(\mathbf{x}, t) = [\mathbb{U}_t \psi](\mathbf{x})$ satisfies following Koopman PDE

$$\frac{\partial u}{\partial t} = \mathcal{K}_{\mathbf{F}} u \tag{4}$$

Definition 2 (Eigenvalues and Eigenfunctions of Koopman): A function $\psi_\lambda(\mathbf{x}) \in \mathcal{C}^1$ is said to be an eigenfunction of the Koopman operator associated with eigenvalue λ if

$$[\mathbb{U}_t \psi_\lambda](\mathbf{x}) = e^{\lambda t} \psi_\lambda(\mathbf{x}). \tag{5}$$

Using the Koopman generator, the (5) can be written as

$$\frac{\partial \psi_\lambda}{\partial \mathbf{x}} \mathbf{F}(\mathbf{x}) = \lambda \psi_\lambda(\mathbf{x}). \tag{6}$$

Equations (5) and (6) provide a general definition of the Koopman spectrum. However, the spectrum can be defined over finite time or over a subset of the state space. The spectrum of interest to us in this paper could be well-defined over the subset of the state space.

Definition 3 (Open Eigenfunction [3]): Let $\psi_\lambda : \mathbf{C} \to \mathbb{C}$, where $\mathbf{C} \subset \mathcal{X}$ is not an invariant set. Let $\mathbf{x} \in \mathbf{C}$, and $\tau \in (\tau^-(\mathbf{x}), \tau^+(\mathbf{x})) = I_{\mathbf{x}}$, a connected open interval such that $\tau(\mathbf{x}) \in \mathbf{C}$ for all $\tau \in I_{\mathbf{x}}$. If

$$[\mathbb{U}_\tau \psi_\lambda](\mathbf{x}) = \psi_\lambda(\mathbf{s}_\tau(\mathbf{x})) = e^{\lambda \tau} \psi_\lambda(\mathbf{x}), \quad \forall \tau \in I_{\mathbf{x}}. \tag{7}$$

Then $\psi_\lambda(\mathbf{x})$ is called the open eigenfunction of the Koopman operator family $\mathbb{U}_t$, for $t \in \mathbb{R}$ with eigenvalue λ.
If $\mathbf{C}$ is a proper invariant subset of $\mathcal{X}$ in which case $I_{\mathbf{x}} = \mathbb{R}$ for every $\mathbf{x} \in \mathbf{C}$, then ψ_λ is called the subdomain eigenfunction. If $\mathbf{C} = \mathcal{X}$ then ψ_λ will be the ordinary eigenfunction associated with eigenvalue λ as in (5).

The open eigenfunctions as defined above can be extended from $\mathbf{C}$ to a larger reachable set when $\mathbf{C}$ is open based on the construction procedure outlined in [3, Definition 5.2, Lemma 5.1]. Let $\mathcal{P}$ be that larger domain.

The eigenvalues of the linearization of the system dynamics at the origin, i.e., $\frac{\partial \mathbf{F}}{\partial \mathbf{x}}(0)$, will form the eigenvalues of the Koopman operator [3, Proposition 5.8]. Our interest will be in constructing the corresponding eigenfunctions, defined over the domain $\mathcal{P}$. We will refer to these eigenfunctions as *principal eigenfunctions* [3]. For the case when the matrix $\frac{\partial \mathbf{F}}{\partial \mathbf{x}}(0)$ has multiple eigenvalues at λ with algebraic multiplicity not equal to geometric multiplicity, one can define *generalized principal eigenfunctions*. For example let λ be the eigenvalue with algebraic multiplicity m and geometric multiplicity one then generalized principal eigenfunctions, $\psi_\lambda^k(\mathbf{x})$ for $k = 1, \ldots, m$ with eigenvalue λ will satisfy

$$[\mathbb{U}_\tau \psi_\lambda^1](\mathbf{x}) = e^{\lambda \tau} \psi_\lambda^1(\mathbf{x}), \;\; [\mathbb{U}_\tau \psi_\lambda^k](\mathbf{x}) = e^{\lambda \tau} \psi_\lambda^k(\mathbf{x}) + t e^{\lambda \tau} \psi_\lambda^{k-1} \tag{8}$$

expressed in the differential form as

$$\frac{\partial \psi_\lambda^1}{\partial \mathbf{x}} \mathbf{F}(\mathbf{x}) = \lambda \psi_\lambda^1(\mathbf{x}), \;\; \frac{\partial \psi_\lambda^k}{\partial \mathbf{x}} \mathbf{F}(\mathbf{x}) = \lambda \psi_\lambda^k(\mathbf{x}) + \psi_\lambda^{k-1}(\mathbf{x}) \tag{9}$$

for $k = 2, \ldots, m$.

The principal eigenfunctions can be used as a change of coordinates in the linear representation of a nonlinear system and draw a connection to the famous Hartman-Grobman theorem on linearization and Poincare normal form [16]. The principal eigenfunctions will be defined over a proper subset $\mathcal{P}$ of the state space $\mathcal{X}$ (called subdomain eigenfunctions) or over the entire $\mathcal{X}$ [3, Lemma 5.1, Corollary 5.1, 5.2, and 5.8].

B. Viscosity Solution of Hamilton Jacobi PDE and Hopf Formula

Consider a control dynamical system of the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}, t) \tag{10}$$

where $\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^q$ are the states and control input respectively. The objective is to minimize the cost

functional of the form

$$\min_{\mathbf{u}\in\mathcal{U}} \mathcal{C}(\mathbf{x},\mathbf{u}(\cdot),t) = \min_{\mathbf{u}\in\mathcal{U}} J(\mathbf{x}(T)) + \int_t^T \frac{1}{2}\mathbf{u}^\top \mathbf{R}\mathbf{u} d\tau,$$
$$\text{s.t. } \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x},\mathbf{u},t)$$
$$\mathcal{U} := \{\mathbf{u}\in\mathbb{R}^q : \|\mathbf{Q}^{-1}\mathbf{u}\|_\infty \le 1\}, \quad t\in[0,T] \tag{11}$$

where, we assume $\mathbf{Q} > 0$ is some positive definite matrix. The terminal cost $J : \mathbb{R}^n \to \mathbb{R}$ is assumed to be convex function of $\mathbf{x}$. Let $V : \mathcal{X}\times\mathbb{R} :\to \mathbb{R}$ be the optimal value function i.e.,

$$V(\mathbf{x},t) = \inf_{\mathbf{u}\in\mathcal{U}} \mathcal{C}(\mathbf{x},\mathbf{u}(\cdot),t) \tag{12}$$

Applying Bellman's principle of optimality to this time-varying value function V leads to the following well known theorem.

Theorem 1: [17] Under the assumptions that $J(\mathbf{x})$ is bounded and uniformly continuous, the value function V defined in (12) is the viscosity solution to the following Hamilton-Jacobi (HJ) Partial Differential Equation (PDE),

$$\frac{\partial V}{\partial t} + H\left(\mathbf{x}, \frac{\partial V}{\partial \mathbf{x}}, t\right) = 0, \quad \mathcal{X}\times[t,T]$$
$$V(\mathbf{x},T) = J(\mathbf{x}), \quad \mathbf{x}\in\mathcal{X} \tag{13}$$

where the Hamiltonian $H : \mathcal{X}\times\mathcal{X}\times[t,T]\to\mathbb{R}$ is defined

$$H(\mathbf{x},\mathbf{p},t) = \min_{\mathbf{u}\in\mathcal{U}} \mathbf{p}^\top \mathbf{F}(\mathbf{x},\mathbf{u},t) + \frac{1}{2}\mathbf{u}^\top\mathbf{R}\mathbf{u}. \tag{14}$$

The HJ PDE is difficult to solve and suffers from the curse of dimensionality, where the computational complexity grows exponentially with respect to the state space dimension. Hence, computational approaches are restricted to systems with lower-dimensional state space models.

The curse of dimensionality problem for solving the HJ PDE is transformed to the so-called *curse of complexity* problem for the special case when the Hamiltonian H in (14) is state-independent using *Hopf-formula* [1]. The case when the Hamiltonian is independent of the state $\mathbf{x}$ i.e., $H(\mathbf{x},\mathbf{p},t) = H(\mathbf{p},t)$ in (14) the corresponding HJ PDE with boundary condition will be

$$\frac{\partial V}{\partial t} + H\left(\frac{\partial V}{\partial \mathbf{x}}, t\right) = 0, \quad \mathcal{X}\times[t,T]$$
$$V(\mathbf{x},T) = J(\mathbf{x}), \quad \mathbf{x}\in\mathcal{X} \tag{15}$$

The Hopf-formula was proposed in in [1] and was proved to be the viscosity solution in [2], [18], [19]. We have following theorem from [18] for the Hopf-formula for the case when Hamiltonian is state-independent but time-dependent.

Theorem 2: Under the assumption that $J(\mathbf{x})$ is convex and Lipchitz, and that $H(\mathbf{p},t)$ is pseudoconvex in $\mathbf{p}$ and satisfies (B.i-B.iii) in [18], then the minimax-viscosity solution of (15) is given by the following Hopf formula

$$V(\mathbf{x},t) = \sup_{\mathbf{p}\in\mathbb{R}^n}\left(\mathbf{p}^\top\mathbf{x} + \int_t^T H(\mathbf{p},\tau)d\tau - J^\star(\mathbf{p})\right) \tag{16}$$

where $J^\star(\mathbf{p})$ is the Fenchel-conjugate of $J(\mathbf{x})$ and is given by

$$J^\star(\mathbf{p}) = \sup_{\mathbf{x}\in\mathbb{R}^n}\left(\mathbf{p}^\top\mathbf{x} - J(\mathbf{x})\right) \tag{17}$$

The min-max viscosity and viscosity solutions are equivalent when $H(\mathbf{p},\tau)$ is either convex or concave in $\mathbf{p}$ for $\tau\in[0,t]$ [18], [20].

III. Comparison Principle, Koopman Spectrum and Hopf Formula

A. Koopman Spectrum and Hopf Formula

The state-dependent Hamiltonian is a severe restriction and limits the applicability of the Hopf formula to a small class of control systems. The restricted class of systems includes linear systems. The main contribution of this paper is to extend the applicability of the Hopf formula to a broader class of nonlinear control-affine systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \tag{18}$$

where $\mathbf{x}\in\mathcal{X}\subseteq\mathbb{R}^n$ and $\mathbf{u}\in\mathbb{R}^q$ are the state and the control input respectively. We make following assumption on the vector fields. The optimal control problem for the control affine system is stated as follows.

$$\inf_{\mathbf{u}\in\mathcal{U}} \mathcal{C}(\mathbf{x},\mathbf{u}(\cdot),t) = \inf_{\mathbf{u}\in\mathcal{U}} J(\mathbf{x}(T)) + \int_t^T \mathbf{u}^\top(\tau)\mathbf{R}\mathbf{u}(\tau)d\tau,$$
$$\text{s.t. } \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}, \quad t\in[0,T] \tag{19}$$

where, $\mathbf{g} = (\mathbf{g}_1,\ldots,\mathbf{g}_p)\in\mathbb{R}^{n\times p}$, $\mathcal{U} := \{\mathbf{u}\in\mathbb{R}^q : \|\mathbf{Q}^{-1}\mathbf{u}\|_\infty \le 1\}$ as defined in (**??**). We make following assumption on the system vector field.

Assumption 1: We assume that the equilibrium point at the origin for the uncontrolled system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is hyperbolic i.e., $\mathbf{A} := \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(0)$ has no eigenvalues on the imaginary axis. Let λ_k for $k = 1,\ldots,n$ be the eigenvalues of the matrix $\mathbf{A}$. Using the results of [3, Proposition 5.8], it follows that the λ_k are also the eigenvalues of the Koopman operator associated with the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with principal generalized eigenfunctions $\phi_k(\mathbf{x})\in\mathcal{C}^1$. Let

$$\Phi(\mathbf{x}) = (\phi_1(\mathbf{x}),\ldots,\phi_n(\mathbf{x}))^\top, \tag{20}$$

be the vector of eigenfunctions in real form and $\Lambda = \text{diag}\{\lambda_k\}$ be the matrix in real Jordan canonical form i.e.,

$$\frac{\partial\Phi}{\partial\mathbf{x}}\mathbf{f}(\mathbf{x}) = \Lambda\Phi(\mathbf{x}) \tag{21}$$

Using results of [3, Lemma 5.1, Corollary 5.1, 5.2, and 5.8], it follows that Φ is a $\mathcal{C}^1$ diffeomorphism and hence admit a inverse Φ^{-1}.

Assumption 2: With no loss of generality, in the rest of the paper, we will assume that the positive matrix $\mathbf{R}$ in the cost function is diagonal. Because, if not then one can also perform change of coordinates on the control input. As the matrix $\mathbf{R}$ is positive, we know that there exists a orthogonal matrix $\mathbf{T}$ such $\mathbf{R} = \mathbf{T}^\top\mathbf{D}\mathbf{T}$, where $\mathbf{D}$ is a diagonal matrix consisiting of eigenvalues of $\mathbf{R}$. The transformation on the control input can be defined as $\bar{\mathbf{u}} = \mathbf{T}\mathbf{u}$ with $\mathbf{u} = \mathbf{T}^\top\bar{\mathbf{u}}$.

Remark 1: In the rest of the paper, we will assume that the eigenfunctions $\Phi(\mathbf{x})$ are well-defined in the domain $\mathcal{P} \subseteq \mathcal{X}$. The domain $\mathcal{P}$ will also limit the validity and the approximation of the viscosity solution of the HJ PDE obtained using the proposed spectral Koopman approach.

Following the results of Theorem 1, we know that the solution to the optimal control problem (19) is provided by the viscosity solution of the HJ PDE (13). One of the main results of this paper will show how to recover the viscosity solution of the HJ PDE (13) usig Hopf formula. The first main results of this paper is proved under the following assumption.

Assumption 3: We assume that there exists constant matrix $\mathbf{B} \in \mathbb{R}^{n\times q}$ such that

$$\frac{\partial \Phi}{\partial \mathbf{x}}\mathbf{g} = \mathbf{B} \tag{22}$$

for all $\mathbf{x} \in \mathcal{X}$.

Remark 2: The above assumption is restrictive and subsequently we relax this assumption. However, this assumption is true for the case of linear system thereby allowing us to recover the results involving linear systems as the special case of our proposed approach.

We introduce some notations before presenting the first main results of this paper. Consider the following minimization problem.

$$\min_{\mathbf{u}\in\mathcal{U}} \mathbf{p}^\top (\mathbf{f} + \mathbf{g}\mathbf{u}) + \frac{1}{2}\mathbf{u}^\top \mathbf{R}\mathbf{u}. \tag{23}$$

It is clear that the solution to the unconstrained minimization problem i.e., with $\mathbf{u} \in \mathbb{R}^p$ is of the form

$$\mathbf{u}^\star = -\mathbf{p}^\top \mathbf{g}. \tag{24}$$

With the constraints on the control input, i.e., $\mathbf{u} \in \mathcal{U}$, the optimal control input will be of the form

$$\mathbf{u}^\star = \mathcal{P}_{\mathcal{U}}(-\mathbf{p}^\top \mathbf{g}) \tag{25}$$

where $\mathcal{P}_{\mathcal{U}}(\cdot)$ is the projection map on the set $\mathcal{U}$.

Theorem 3: Consider the optimal control problem (19) satisfying Assumptions 1 and 3. Then the viscosity solution $V(\mathbf{x}, t)$ to the optimal control problem (19) in domain $\mathcal{P}$ is given by

$$V(\mathbf{x}, t) = \bar{V}(e^{-\Lambda t}\Phi(\mathbf{x}), t) \tag{26}$$

where

$$\bar{V}(\mathbf{X}, t) = \sup_{\mathbf{P}\in\mathbb{R}^n} \left(\mathbf{P}^\top \mathbf{X} + \int_t^T \bar{H}(\mathbf{P}, \tau)d\tau - \bar{J}^\star(\mathbf{P}) \right) \tag{27}$$

$$\mathbf{X} = e^{-\Lambda t}\Phi(\mathbf{x}), \quad \mathbf{x} = \Phi^{-1}(e^{\Lambda t}\mathbf{X})$$

$$J(\mathbf{x}(T)) = J(\Phi^{-1}(e^{\Lambda T}\mathbf{X}(T)))$$

$$\bar{J}(\mathbf{X}) = J(\Phi^{-1}(e^{\Lambda T}\mathbf{X})), \quad \bar{J} = J \circ \Phi^{-1} \circ e^{\Lambda T}$$

$$J(\mathbf{x}) = J(\Phi^{-1}(e^{\Lambda t}\mathbf{X}))$$

$\bar{J}^\star(\mathbf{P})$ is the Fenchel-conjugate of $\bar{J}(\mathbf{X}) = J(\Phi^{-1}(e^{\Lambda T}\mathbf{X}))$ and is given by

$$\bar{J}^\star(\mathbf{P}) = \sup_{\mathbf{X}\in\mathbb{R}^n} \left(\mathbf{P}^\top \mathbf{X} - \bar{J}(\mathbf{X}) \right) \tag{28}$$

and $\Phi(\mathbf{x})$ and Λ are as defined in (20)-(21) and

$$\begin{aligned}\bar{H}(\mathbf{X}, \mathbf{P}, t) = \mathbf{P}^\top e^{-\Lambda t}\mathbf{B}\mathcal{P}_{\mathcal{U}}\left(-\mathbf{P}^\top e^{-\Lambda t}\mathbf{B}\right) \\ + \mathcal{P}_{\mathcal{U}}\left(-\mathbf{P}^\top e^{-\Lambda t}\mathbf{B}\right)^\top \mathbf{R}\mathcal{P}_{\mathcal{U}}\left(-\mathbf{P}^\top e^{-\Lambda t}\mathbf{B}\right)\end{aligned} \tag{29}$$

Proof: Following the results of Theorem 1, we know that the viscosity solution $V(\mathbf{x}, t)$ satisfy the HJ PDE (13) with the Hamiltonian given by

$$H(\mathbf{x}, \mathbf{p}) = \min_{\mathbf{u}\in\mathcal{U}} \mathbf{p}^\top(\mathbf{f} + \mathbf{g}\mathbf{u}) = \mathbf{p}^\top \mathbf{f} + \mathbf{p}^\top \mathbf{g}\mathbf{u} + \frac{1}{2}\mathbf{u}^\top \mathbf{R}\mathbf{u} \tag{30}$$

Minimizing w.r.t $\mathbf{u} \in \mathcal{U}$, we obtain

$$u^\star = \mathcal{P}_{\mathcal{U}}\left(-\mathbf{R}^{-1}\mathbf{g}^\top \mathbf{p}\right) \tag{31}$$

where $\mathcal{P}_{\mathcal{U}}(\cdot)$ denotes the projection map on the set $\mathcal{U}$. Substituting for optimal $u^\star$, we get

$$H(\mathbf{x}, \mathbf{p}) = \mathbf{p}^\top (\mathbf{f} + \mathbf{g}\mathbf{u}^\star) + \frac{1}{2}(\mathbf{u}^\star)^\top \mathbf{R}\mathbf{u}^\star. \tag{32}$$

Now consider the following canonical change of coordinates $(\mathbf{x}, \mathbf{p}) \to (\mathbf{X}, \mathbf{P})$ obtained using the generating function of the form [21]

$$W(\mathbf{x}, \mathbf{P}, t) = \mathbf{P}^\top e^{-\Lambda t}\Phi(\mathbf{x}) \tag{33}$$

and hence

$$\mathbf{X} = \frac{\partial W}{\partial \mathbf{P}} = e^{-\Lambda t}\Phi(\mathbf{x}), \quad \mathbf{p} = \frac{\partial W}{\partial \mathbf{x}} = \frac{\partial \Phi}{\partial \mathbf{x}}^\top e^{-\Lambda^\top t}\mathbf{P}$$

$$\mathbf{P} = e^{\Lambda^\top t}\left(\frac{\partial \Phi}{\partial \mathbf{x}}^\top\right)^{-1}\mathbf{p}$$

The optimal control in the new coordinates following Assumption 3 can be written as

$$u^\star = \mathcal{P}_{\mathcal{U}}\left(-\mathbf{P}^\top e^{-\Lambda t}\frac{\partial \Phi}{\partial \mathbf{x}}\mathbf{g}\right) = \mathcal{P}_{\mathcal{U}}\left(-\mathbf{P}^\top e^{-\Lambda t}\mathbf{B}\right)$$

The Hamiltonian in the new coordinates is given by

$$\begin{aligned}\bar{H}(\mathbf{X}, \mathbf{P}, t) &= H(\mathbf{x}(\mathbf{X}, \mathbf{P}), \mathbf{p}(\mathbf{X}, \mathbf{P})) + \frac{\partial W}{\partial t} \\ &= \mathbf{P}^\top e^{-\Lambda t}\mathbf{B}\mathcal{P}_{\mathcal{U}}\left(-\mathbf{P}^\top e^{-\Lambda t}\mathbf{B}\right) \\ &+ \mathcal{P}_{\mathcal{U}}\left(-\mathbf{P}^\top e^{-\Lambda t}\mathbf{B}\right)^\top \mathbf{R}\mathcal{P}_{\mathcal{U}}\left(-\mathbf{P}^\top e^{-\Lambda t}\mathbf{B}\right)\end{aligned} \tag{34}$$

Notice that the Hamiltonian is independent of the new state $\mathbf{X}$. Now consider the following optimal control problem

$$\begin{aligned}&\inf_{\mathbf{u}\in\mathcal{U}} \bar{\mathcal{C}}(\mathbf{X}, \mathbf{u}(\cdot), t) = \inf_{\mathbf{u}\in\mathcal{U}} \bar{J}(\mathbf{X}(T)) + \int_t^T \frac{1}{2}\mathbf{u}^\top \mathbf{R}\mathbf{u}d\tau, \quad t \in [0, T] \\ &\text{s.t. } \dot{\mathbf{X}} = e^{-\Lambda t}\mathbf{B}\mathbf{u}\end{aligned} \tag{35}$$

with boundary condition $\bar{J}(\mathbf{X}) := J(\Phi^{-1}(e^{\Lambda T}\mathbf{X}))$. It follows that the Hamiltonian for the optimal control problem in new $\mathbf{X} = e^{-\Lambda t}\Phi(\mathbf{x})$ will be of the form (34) and

hence state-independent. Hence, following the results of Theorem 2, the viscosity solution, $\bar{V}(\mathbf{X},t)$, is given by Hopf formula (27). Since, the optimal control problem in the original $\mathbf{x}$ coordinates (19) and the transformed problem in the Koopman principal eigenfunction coordinates (35) are related by time varying change of transformation of the form $\mathbf{X} = e^{-\Lambda t}\Phi(\mathbf{x})$, the optimal value function, $\bar{V}(\mathbf{X},t)$ can be used to obtained the optimal value function, $V(\mathbf{x},t)$ as

$$V(\mathbf{x},t) = \bar{V}(e^{-\Lambda t}\Phi(\mathbf{x}),t)$$

■

Remark 3: For the special case of a linear system, the Hamiltonian is state-dependent to begin with, but it can be made state-independent using time-varying changes of coordinates along the system solution. These known linear system results will arise as a special case of the above theorem. The principal generalized Koopman eigenfunction for linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ are of the form, $\Phi(\mathbf{x}) = \mathbf{W}^\top\mathbf{x}$, where $\mathbf{W}^\top$ are the generalized left eigenvector of $\mathbf{A}$ associated with the eigenmatrix Λ in real Jordan canonical form i.e., $\mathbf{W}^\top\mathbf{A} = \Lambda\mathbf{W}$. So, the change of coordinates is given by $\mathbf{X} = e^{-\Lambda t}\mathbf{W}^\top\mathbf{x}$. It then follows that the Assumption 3 is satisfied by linear system as both the $\frac{\partial\Phi}{\partial\mathbf{x}} = \mathbf{W}^\top$ and $\mathbf{g}(\mathbf{x})$ are constant matrices.

The Assumption (3) is in general very restrictive. For the next main result of this article we relax this assumption by using lower and upper bounds as follows.

Assumption 4: We assume that there exist symmetric positive semi-definite matrices $\mathbf{D}_u^i \in \mathbb{R}^{n\times n}$ and $\mathbf{D}_l^i \in \mathbb{R}^{n\times n}$ for all $i \in \{1,2,\cdots,m\}$ such that

$$\mathbf{D}_u^i \leq \frac{\partial\Phi}{\partial\mathbf{x}}\mathbf{g}_i\mathbf{g}_i^\top\frac{\partial\Phi}{\partial\mathbf{x}}^\top \leq \mathbf{D}_l^i. \tag{36}$$

for all $\mathbf{x} \in \mathcal{P}$, where $\mathbf{g} = \begin{bmatrix}\mathbf{g}_1 & \cdots & \mathbf{g}_p\end{bmatrix} \in \mathbb{R}^{n\times p}$.

Remark 4: For the case when $\mathbf{g}_i = \mathbf{B}_i$ is constant, then the upper and lower bounds can be obtained as follows.

We now propose the following comparison Lemma, which is used for proving the second main results of the paper.

Lemma 4: Consider the optimal control problem (11) and the associated HJ PDE (13). Let

1) $V_l(\mathbf{x},t) \in \mathcal{C}^1$ satisfies the following inequalities

$$\frac{\partial V_l}{\partial t} + H_l\left(\mathbf{x}, \frac{\partial V_l}{\partial\mathbf{x}}, t\right) \geq 0 \tag{37}$$

$$H_l(\mathbf{x},\mathbf{p},t) \leq H(\mathbf{x},\mathbf{p},t)$$

$$V_l(\mathbf{x}(T),T) \leq J(\mathbf{x}) \tag{38}$$

then we have $V_l(\mathbf{x},t) \leq V(\mathbf{x},t)$.

2) $V_u(\mathbf{x},t) \in \mathcal{C}^1$ satisfies the following inequalities

$$\frac{\partial V_u}{\partial t} + H_u\left(\mathbf{x}, \frac{\partial V_u}{\partial\mathbf{x}}, t\right) \leq 0 \tag{39}$$

$$H_u(\mathbf{x},\mathbf{p},t) \geq H(\mathbf{x},\mathbf{p},t)$$

$$V_u(\mathbf{x}(T),T) \geq J(\mathbf{x}) \tag{40}$$

then we have $V_u(\mathbf{x},t) \geq V(\mathbf{x},t)$.

We differ the proof of this Lemma to the Appendix. The key takeaway from Lemma 4 is that, given the bounds on the Hamiltonian H it is possible to find bounds on the value function V. Obtaining the Hamiltonian bounds (H_l/H_u) for which the HJ inequalities can be efficiently solved to find good approximations of the value function (V_l/V_u) now becomes the main challenge. We introduce the following Lemma which helps us obtain the bounds on the Hamiltonian.

We now use the results of Lemma 4 to prove one of the main results of this paper.

Theorem 5: Consider the optimal control problem (11) and the associated HJ PDE (13). Define

$$H_l(\mathbf{x},\mathbf{p},t) := \mathbf{p}^\top\mathbf{f} + \min_{\mathbf{u}\in\mathcal{U}}\left\{\mathbf{p}^\top\left(\frac{\partial\Phi}{\partial\mathbf{x}}\right)^{-1}\mathbf{D}_l^{\frac{1}{2}}\mathbf{u} + \frac{1}{2}\mathbf{u}^\top\mathbf{R}\mathbf{u}\right\} \tag{41}$$

and

$$H_u(\mathbf{x},\mathbf{p},t) := \mathbf{p}^\top\mathbf{f} + \min_{\mathbf{u}\in\mathcal{U}}\left\{\mathbf{p}^\top\left(\frac{\partial\Phi}{\partial\mathbf{x}}\right)^{-1}\mathbf{D}_u^{\frac{1}{2}}\mathbf{u} + \frac{1}{2}\mathbf{u}^\top\mathbf{R}\mathbf{u}\right\} \tag{42}$$

Then under Assumption 4, we have following lower and upper bounds to the viscosity solution $V(\mathbf{x},t)$ to the HJ PDE (13) characterizing the solution to the optimal control problem (19). The following bounds are applicable in the domain $\mathcal{P}$, where the eigenfunctions of the Koopman operator are well defined.

$$V_l(\mathbf{x},t) \leq V(\mathbf{x},t) \leq V_u(\mathbf{x},t) \tag{43}$$

Please refer to Appendix for the proof. We have following Theorem for the computation of the optimal cost function using the Hopf formula applied in Koopman principal eigenfunction coordinates.

Theorem 6: Let $\mathcal{P}$ be the domain where the principal eigenfunctions of the Koopman operator associated with uncontrolled system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ are well defined. We have following expression for the value function $V_{l(u)}$ satisfying the PDEs (37) (39) for all point $\mathbf{x} \in \mathcal{P}$.

$$V_{u(l)}(\mathbf{x},t) = \bar{V}_{u(l)}(e^{-\Lambda t}\Phi(\mathbf{x}),t) \tag{44}$$

where

$$\bar{V}_{u(l)}(\mathbf{X},t) = \sup_{\mathbf{P}\in\mathbb{R}^n}\left(\mathbf{P}^\top\mathbf{X} + \int_t^T \bar{H}_{u(l)}(\mathbf{P},\tau)d\tau - \bar{J}_{u(l)}^\star(\mathbf{P})\right) \tag{45}$$

$\bar{J}_{u(l)}^\star(\mathbf{P})$ is the Fenchel-conjugate of $\bar{J}_{u(l)}(\mathbf{X}) = J_{u(l)}(\Phi^{-1}(e^{\Lambda T}\mathbf{X}))$ and is given by

$$\bar{J}_{u(l)}^\star(\mathbf{P}) = \sup_{\mathbf{X}\in\mathbb{R}^n}\left(\mathbf{P}^\top\mathbf{X} - \bar{J}_{u(l)}(\mathbf{X})\right). \tag{46}$$

Proof: We prove the results for the V_u, the proof for V_l will follows along similar lines. Now consider the following canonical change of coordinates $(\mathbf{x},\mathbf{p}) \to (\mathbf{X},\mathbf{P})$ obtained using the generating function of the form [21]

$$W(\mathbf{x},\mathbf{P},t) = \mathbf{P}^\top e^{-\Lambda t}\Phi(\mathbf{x}) \tag{47}$$

and hence

$$\mathbf{X} = \frac{\partial W}{\partial \mathbf{P}} = e^{-\Lambda t}\Phi(\mathbf{x}), \quad \mathbf{p} = \frac{\partial W}{\partial \mathbf{x}} = \frac{\partial \Phi}{\partial \mathbf{x}}^{\top} e^{-\Lambda^{\top} t}\mathbf{P}$$

$$\mathbf{P} = e^{\Lambda^{\top} t}\left(\frac{\partial \Phi}{\partial \mathbf{x}}^{\top}\right)^{-1}\mathbf{p}$$

In this new coordinates the Hamiltonian H_u can be written as

$$\begin{aligned}\bar{H}_u(\mathbf{X},\mathbf{P},t) &= H_u(\mathbf{x}(\mathbf{X},\mathbf{P}),\mathbf{p}(\mathbf{X},\mathbf{P})) + \frac{\partial W}{\partial t} \\ &= \min_{\mathbf{u}\in\mathcal{U}}\left\{\mathbf{P}^{\top}e^{-\Lambda t}\mathbf{D}_l^{\frac{1}{2}}\mathbf{u} + \frac{1}{2}\mathbf{u}^{\top}\mathbf{R}\mathbf{u}\right\}\end{aligned}$$

since

$$\frac{\partial W}{\partial t} = -\mathbf{P}^{\top}\Lambda e^{-\Lambda t}\Phi(\mathbf{x}) = -\mathbf{P}^{\top}\Lambda\mathbf{X}.$$

One can observe that this is a state-independent Hamiltonian, therefore using the result of Theorem 3 we get

$$\bar{V}_u(\mathbf{X},t) = \sup_{\mathbf{P}\in\mathbb{R}^n}\left(\mathbf{P}^{\top}\mathbf{X} + \int_t^T \bar{H}_u(\mathbf{P},\tau)d\tau - \bar{J}_u^{\star}(\mathbf{P})\right).$$

Here the value function $\bar{V}_u(\mathbf{X},t)$ in the new coordinate $\mathbf{X} = e^{-\Lambda t}\Phi(\mathbf{x})$ is related to the value function in the original coordinates by the relation (44).

■

IV. Eigenfunction Computation

The analytical results developed in the previous section extend the applicability of the Hopf formula to a class of systems for which the Hamiltonian is not state-independent. However, the approach involves knowing or computing the principal eigenfunctions of the Koopman operator associated with the uncontrolled dynamical system. We also understand that the Hopf formula's main attraction is to transform the curse of dimensionality problem related to solving the HJ PDE to the curse of complexity problem. So, extending the applicability of the Hopf formula to a more general class of systems using Koopman eigenfunctions, one has to be careful to ensure that the curse of dimensionality is not shifting to computing the Koopman eigenfunctions. In this section, we show that this is not the case. In particular, we propose using a method based on the path-integral formula for computing Koopman eigenfunctions. The path-integral formula for computing the Koopman eigenfunctions shares some common features with the Hopf formula. The Hopf formula involves solving a convex optimization problem to determine the value of optimal value function for each point $\mathbf{x}$ in the state space, similarly to compute the value of eigenfunction at point $\mathbf{x}$ we require to evaluate an integral along a path generating by dynamical system starting from the initial condition $\mathbf{x}$. The main results presented in this section on the computation of Koopman principal eigenfunctions is an extension of results from [14] on the path-integral formula for computing Koopman eigenfunctions. In particular, we present complete set of results providing necessary and sufficient conditions for the convergence of path-integral formula for system with stable, anti-stable, and saddle type equilibrium points.

A. Path-Integral Formula for Eigenfunctions Computation

Consider the following decomposition of the nonlinear system into linear and nonlinear parts.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{F}_n(\mathbf{x}) \tag{48}$$

where $\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(0)$ and $\mathbf{F}_n(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - \mathbf{A}\mathbf{x}$. We make following assumption on the linearization of the system at the origin equilibrium point.

Assumption 5: The linear matrix $\mathbf{A}$ is assumed to be hyperbolic, i.e., no eigenvalues on the imaginary axis.

Following is the first main result of this section.

Theorem 7: Consider the dynamical system (48) satisfying Assumption 5. Assume that

$$\lim_{t\to\infty} e^{-\lambda t}\mathbf{w}_{\lambda}^{\top}\mathbf{F}_n(\mathbf{s}_t(\mathbf{x})) = 0 \tag{49}$$

for all $\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^n$, where, λ is the eigenvalue of matrix $\mathbf{A}$, $\mathbf{w}_{\lambda}$ is the left eigenvector of $\mathbf{A}$ with eigenvalue λ i.e., $\mathbf{w}_{\lambda}^{\top}\mathbf{A} = \lambda\mathbf{w}_{\lambda}^{\top}$, and $\mathbf{s}_t(\mathbf{x})$ is the flow of dynamical system (48). Then the principal eigenfunction corresponding to the eigenvalue λ is given by following formula.

$$\phi_{\lambda}(\mathbf{x}) = \mathbf{w}_{\lambda}^{\top}\mathbf{x} + \int_0^{\infty} e^{-\lambda t}\mathbf{w}_{\lambda}^{\top}\mathbf{F}_n(\mathbf{s}_t(\mathbf{x}))dt \tag{50}$$

where $\mathbf{w}_{\lambda}^{\top}\mathbf{x}$ is the linear part and $h(\mathbf{x}) = \int_0^{\infty} e^{-\lambda t}\mathbf{w}_{\lambda}^{\top}\mathbf{F}_n(\mathbf{s}_t(\mathbf{x}))dt$ is the nonlinear part of the principal eigenfunctions and satisfies $\frac{\partial h}{\partial \mathbf{x}}(0) = 0$.

Proof: We will prove that

$$\frac{\partial \phi_{\lambda}}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) = \lambda\phi_{\lambda}(\mathbf{x}) \tag{51}$$

We have,

$$\frac{\partial \phi_{\lambda}}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) = \mathbf{w}^{\top}\mathbf{f}(\mathbf{x}) + \int_0^{\infty} e^{-\lambda t}\frac{\partial}{\partial \mathbf{x}}[\mathbf{w}_{\lambda}^{\top}\mathbf{F}_n(\mathbf{s}_t(\mathbf{x}))]dt\mathbf{f}(\mathbf{x}) \tag{52}$$

Using the Definition of the Koopman operator (Definition 1) and Koopman PDE (4), we can write

$$\mathbf{w}_{\lambda}^{\top}\mathbf{F}_n(\mathbf{s}_t(\mathbf{x})) = [\mathbb{U}_t\mathbf{w}_{\lambda}^{\top}\mathbf{F}](\mathbf{x})$$

$$\frac{d}{dt}[\mathbb{U}_t\mathbf{w}_{\lambda}^{\top}\mathbf{F}_n(\mathbf{x})] = \frac{\partial}{\partial \mathbf{x}}[\mathbb{U}_t\mathbf{w}_{\lambda}^{\top}\mathbf{F}_n](\mathbf{x})\mathbf{f}(\mathbf{x})$$

We have using condition (49)

$$\begin{aligned}&\int_0^{\infty} e^{-\lambda t}\frac{\partial}{\partial \mathbf{x}}[\mathbf{w}_{\lambda}^{\top}\mathbf{F}_n(\mathbf{s}_t(\mathbf{x}))]dt\mathbf{f}(\mathbf{x}) \\ &= \int_0^{\infty} e^{-\lambda t}\frac{d}{dt}[\mathbb{U}_t\mathbf{w}_{\lambda}^{\top}\mathbf{F}_n(\mathbf{x})]dt \\ &= e^{-\lambda t}[\mathbb{U}_t\mathbf{w}_{\lambda}^{\top}\mathbf{F}_n](\mathbf{x})|_{t=0}^{\infty} + \lambda\int_0^{\infty} e^{-\lambda t}[\mathbb{U}_t\mathbf{w}_{\lambda}^{\top}\mathbf{F}_n](\mathbf{x})dt \\ &= -\mathbf{w}_{\lambda}^{\top}\mathbf{F}_n(\mathbf{x}) + \lambda\int_0^{\infty} e^{-\lambda t}[\mathbb{U}_t\mathbf{w}_{\lambda}^{\top}\mathbf{F}_n](\mathbf{x})dt\end{aligned}$$

Now using the decomposition (48) $\mathbf{w}_{\lambda}^{\top}\mathbf{f}(\mathbf{x}) = \mathbf{w}_{\lambda}^{\top}\mathbf{A}\mathbf{x} + \mathbf{w}_{\lambda}^{\top}\mathbf{F}_n(\mathbf{x})$, we obtain the desired result (51)

■

The condition (49) has to do with the growth rate of the linear and nonlinear part of the system dynamics, where $e^{-\lambda t}\mathbf{w}^\top$ and $\mathbf{F}_n(\mathbf{s}_t(\mathbf{x}))$ captures the linear and nonlinear part of the system dynamics respectively.

In the following Lemma we provide easily verifiable sufficient condition for the condition (49) to hold true. This sufficient condition applies to the case when the system has stable or anti-stable equilibrium point.

Lemma 8: Let the origin $\mathbf{x} = 0$ be an asymptotically stable equilibrium point of (48) with domain of attraction $\mathcal{D}$. Let λ_i for $i = 1, \ldots, n$ be the eigenvalues of $\mathbf{A}$ and satisfy

$$-\mathrm{Re}\lambda_i + 2\mathrm{Re}\lambda_{max} < 0 \tag{53}$$

where λ_{max} be the eigenvalue closest to the imaginary axis in the left half plane. Then (49) is true and the path integral formula is applicable for all point $\mathbf{x} \in \mathcal{D}$.

Proof: Since the equilibrium point at the origin is assumed to be asymptotically stable we have $\lim_{t\to\infty} \mathbf{s}_t(\mathbf{x}) = 0$ for all point $\mathbf{x} \in \mathcal{D}$ (the domain of attraction). Now $\mathbf{F}_n(\mathbf{x})$ is the purely nonlinear part of the vector field and hence for a small neighborhood $\|\mathbf{x}\| \leq \epsilon$ around the origin there exists a constant $c_{F_n} > 0$ such that

$$|\mathbf{w}_\lambda^\top \mathbf{F}_n(\mathbf{x})| \leq c_{F_n}\|\mathbf{x}\|^2$$

Hence, we have

$$|\mathbf{w}_\lambda^\top \mathbf{F}_n(\mathbf{s}_t(\mathbf{x}))| \leq c_{F_n}\|\mathbf{s}_t(\mathbf{x})\|^2$$

Since the equilibrium point is assumed to be hyperbolic, in the small neighborhood $\|\mathbf{x}\| \leq \epsilon$, using Hartman Grobman theorem [] there exists a near identity change of coordinates of the form

$$\mathbf{z} = \mathbf{x}+\mathbf{d}(\mathbf{x}) = \mathbf{D}(\mathbf{x}) \iff \mathbf{x} = \mathbf{D}^{-1}(\mathbf{z}) = \mathbf{z}+\bar{\mathbf{d}}(\mathbf{z}), \tag{54}$$

with $\mathbf{d}(\mathbf{x})$ and $\bar{\mathbf{d}}(\mathbf{z})$ purely nonlinear such that the nonlinear system (48) is transformed into linear system i.e.,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{F}_n(\mathbf{x}) \xrightarrow{\mathbf{z}=\mathbf{D}(\mathbf{x})} \dot{\mathbf{z}} = \mathbf{A}\mathbf{z} \tag{55}$$

and hence

$$\begin{aligned}\mathbf{s}_t(\mathbf{x}) &= \mathbf{D}^{-1}(e^{\mathbf{A}t}\mathbf{D}(\mathbf{x})) = \mathbf{D}^{-1}(e^{\mathbf{A}t}(\mathbf{x}+\mathbf{d}(\mathbf{x})))\\ &= e^{\mathbf{A}t}\mathbf{x} + e^{\mathbf{A}t}\mathbf{d}(\mathbf{x}) + \bar{\mathbf{d}}(e^{\mathbf{A}t}\mathbf{x} + e^{\mathbf{A}t}\mathbf{d}(\mathbf{x}))\end{aligned} \tag{56}$$

In the above, we have used (54) for $\mathbf{D}^{-1}$. Since $\bar{\mathbf{d}}(\mathbf{z})$ is purely nonlinear, for $\|\mathbf{x}\| \leq \epsilon$, we have following estimates

$$\|\bar{\mathbf{d}}(\mathbf{z})\| \leq c_{\bar{d}}\|\mathbf{z}\|^2, \quad \|\mathbf{d}(\mathbf{x})\| \leq c_d\|\mathbf{x}\|^2. \tag{57}$$

for some positive constants $c_{\bar{d}}$ and c_d. Now combining Eqs. (56) and (57) and the fact that $\|\mathbf{x}\| \leq \epsilon$, we obtain

$$\|\mathbf{s}_t\| \leq c_s e^{\mathrm{Re}(\lambda_{max})t} \implies \|\mathbf{s}_t\|^2 \leq c_s^2 e^{2\mathrm{Re}(\lambda_{max})t} \tag{58}$$

where λ_{max} is the eigenvalue closed to the imaginary axis and c_s is function of $c_{\bar{d}}, c_d$, and ϵ. Hence,

$$\lim_{t\to\infty} e^{-\lambda t}\mathbf{w}_\lambda^\top \mathbf{F}_n(\mathbf{s}_t(\mathbf{x})) \leq \lim_{t\to\infty} c_s^2 c_{F_n} e^{-\lambda t} e^{2\mathrm{Re}(\lambda_{max})t}$$

which is zero following (53). ■

So Eq. (53) provides a verifiable condition for the computation of principal eigenfunction using the path integral formula (62). The eigenfunction computation for the anti-stable case i.e., the case when all the eigenvalues of matrix $\mathbf{A}$ are in the right half plane follows from the results of Lemma 8 by time-reversing the vector field. This follows because if $\phi_\lambda(\mathbf{x})$ is the eigenfunction corresponding to eigenvalue λ for system (48) then $\phi_\lambda(\mathbf{x})$ is also the eigenfunction corresponding to eigenvalue $-\lambda$ for the time-reversed vector field as

$$\frac{\partial \phi_\lambda}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) = \lambda\phi_\lambda \implies \frac{\partial \phi_\lambda}{\partial \mathbf{x}}(-\mathbf{f}(\mathbf{x})) = -\lambda\phi_\lambda. \tag{59}$$

For system with saddle-type equilibrium point, the condition (49) can be satisfied for the case when the eigenvalue has real part positive i.e., $\mathrm{Re}(\lambda) > 0$ and the nonlinear part of the vector field $\mathbf{F}_n(\mathbf{x})$ is bounded. This will allow us to use the path-integral formula for the computation of eigenfunctions with positive real part. For eigenfunctions corresponding to eigenvalues with negative real part for saddle-type equilibrium point one can time-reverse the vector field to compute the eigenfunction.

Example 1:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \frac{1}{d(\mathbf{x})}\begin{pmatrix}6(x_1+\sin x_2) + 2\cos x_2(\frac{3x_2}{2} - \sin x_1)\\ 4\cos x_1(x_1+\sin x_2) - 2(\frac{3x_2}{2} - \sin x_1)\end{pmatrix} \tag{60}$$

where $d(\mathbf{x}) = 2\cos x_1 \cos x_2 + 3$. It can be verified that the origin is the equilibrium point and the linearization of the system at the origin has eigenvalues of $\lambda_1 = -1$ and $\lambda_2 = 2$. Hence, the equilibrium point is of saddle-type. For this example the condition (49) is satisfied. The eigenfunction corresponding to eigenvalue $\lambda = 2$ can be computed using Eq. (62). Similarly, the eigenfunction corresponding to eigenvalue $\lambda = -1$ can be computed again using (62) applied to time-reversed vector field (60). The eigenfunction for this system can be computed analytically are of the form

$$\begin{bmatrix}\phi_{\lambda_1}\\ \phi_{\lambda_2}\end{bmatrix} = \begin{bmatrix}\sin x_1 - 1.5x_2\\ x_1 + \sin x_2\end{bmatrix} \tag{61}$$

In Fig. 1, we observe that the eigenfunctions obtained using the path-integral formula (62) matches with the analytically expression of the eigenfunction.

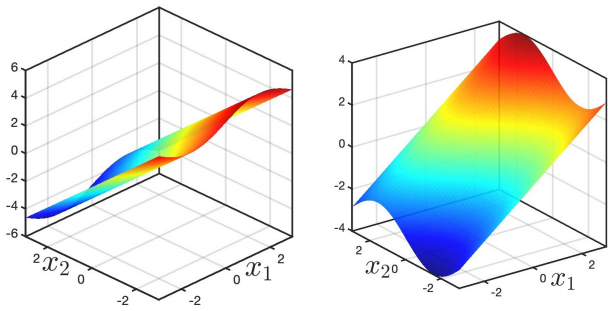
The formula in Eq. (62) can be written for vector of eigenfunction $\Phi(\mathbf{x}) = (\phi_{\lambda_1}, \ldots, \phi_{\lambda_n})^\top$ as

$$\phi_\lambda(\mathbf{x}) = \mathbf{W}^\top\mathbf{x} + \int_0^\infty e^{-\Lambda t}\mathbf{W}^\top \mathbf{F}_n(\mathbf{s}_t(\mathbf{x}))dt \tag{62}$$

where $\mathbf{W} \in \mathbb{R}^{n\times n}$ is a matrix consisting of left eigenvectors of $\mathbf{A}$ and Λ is the matrix of eigenvalues of $\mathbf{A}$ in real Jordan canonical form.

V. Spectral Koopman-Hopf Formula

We can combine the path integral and the Hopf formula to determine the viscosity solution at point $\mathbf{x}$. The following equation combining path integral formula (**??**) and Hopf



(a) ϕ_{λ_1} (b) ϕ_{λ_2}

Fig. 1: Eigenfunctions using path-integral formula

formula (16) can be viewed as a *spectral Koopman-Hopf formula* for the computation or approximation of the viscosity solution.

$$V(\mathbf{x},t) = \sup_{\mathbf{P}\in\mathbb{R}^n}\left\{\mathbf{P}^\top\left(e^{-\Lambda t}\left[\mathbf{W}^\top\mathbf{x}+\int_0^\infty e^{-\Lambda\tau}\mathbf{W}^\top\mathbf{F}_n(\mathbf{s}_\tau(\mathbf{x}))d\tau\right]\right) + \int_t^T \bar{H}(\mathbf{P},\tau)d\tau - \bar{J}^\star(\mathbf{P})\right\}. \tag{63}$$

The Hamiltonian $\bar{H}(\mathbf{P},\tau)$ in the above equation could either be of the form (**??**) or (**??**) depending upon whether Assumption (**??**) or (36) is satisfied. If Assumption (36) is satisfied, then Eq. (63) provides an upper or lower bound to the true viscosity solution of the HJ PDE. Note that the formula (63) clearly demonstrates that the spectral Koopman Hopf formula does not suffer from the curse of dimensionality problem. The curse of dimensionality problem is transformed into the curse of complexity problem in the spectral Koopman Hopf formula.

VI. Simulations

A. One Dimensional Analytical Example

Our first example is that of a scalar system. Consider a scalar control system of the form

$$\dot{x} = x - x^3 + (1-x^2)^{\frac{3}{2}}u \tag{64}$$

with cost function

$$C(x,u(\cdot)) = J(x(1)) + \int_t^1 \mathcal{I}_{\mathcal{U}}(u)d\tau, \quad t\in[0,1]$$

with $J(x) = \frac{1}{2}\frac{x^2}{1-x^2}$. The Hopf-formula cannot be applied to this system as the system is not linear and hence the resulting Hamiltonian will be state dependent. However, the principal eigenfunction of the Koopman operator corresponding to uncontrolled system $\dot{x} = x - x^3$ with eigenvalue $\lambda = 1$ admits an analytical form as

$$\phi_1(x) = \frac{x}{\sqrt{1-x^2}}, \quad x\in(-1,1) = \mathcal{P}.$$

Consider the following change of coordinates $X = e^{-t}\phi_1(x) = e^{-1}\frac{x}{\sqrt{1-x^2}}$. In the new coordinates, we have the following optimal control problem

$$\min_{u(\cdot)} C(X,u(\cdot)) = \bar{J}(X) + \int_t^1 \mathcal{I}_{\mathcal{U}}(u)d\tau$$
$$\text{s.t. } \dot{X} = e^{-t}u \tag{65}$$

where $\bar{J}(X) = \frac{1}{2}(e^1X)^2$. Using the FL-transform we get

$$J^*(P) = \sup_X\left(XP - \frac{1}{2}(e^1X)^2\right),$$

which obtains supremum at $X^* = e^{-2}P$, resulting in $J^*(P) = \frac{1}{2}e^{-2}P^2$. Restricting $u\in[-u_m,u_m]$, we can write the Hamiltonian as

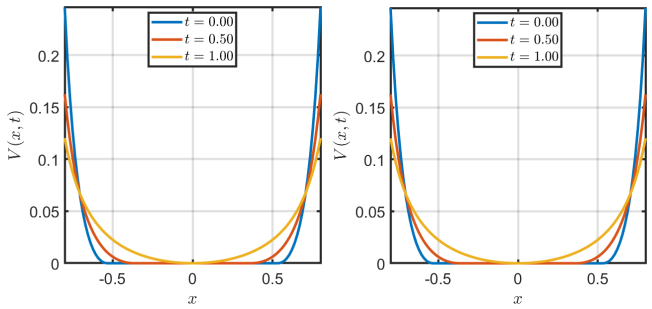
$$\bar{H}(P,t) = -u_mPe^{-t}(\text{sgn}(Pe^{-t})) = -u_m|Pe^{-t}| \tag{66}$$

$$\int_t^T \bar{H}(P,\tau)d\tau = -u_m|P|\left(e^{-t}-e^{-T}\right). \tag{67}$$

Therefore, (27) simplifies to

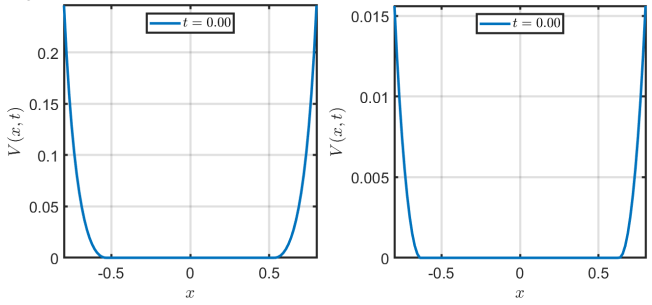
$$\bar{V}(X,t) = \sup_{P\in\mathbb{R}}\left\{P^\top X - u_m|P|\left(e^{-t}-e^{-T}\right) - \frac{e^{-2}}{2}P^2\right\}. \tag{68}$$

The solution of (68) with $T = 1$ and $u_m = 1$ for $t = 0, 0.5, 1$ is shown in Fig. 2a. Next we use the spectral Koopman Hopf formula (63) to obtain the value functions $V(x,t)$. This formula combines the classical Hopf type optimization formulation for the value function calculation with the path integral based spectral computation from the Koopman theory. For the system in (64) we can verify that $D_i^- = D_i^+ = 1$, where D_i^-, D_i^+ are as given in (36). Therefore, the underestimate and the overestimate of the value function coincide and we obtain the actual value function using the spectral Koopman Hopf formula. The value functions $V(x,t)$ obtained using the spectral Koopman Hopf formula at $t = 0, 0.5, 1$ are shown in Fig. 2b. We can observe that the actual value functions (Fig. 2a) agree very well with the ones obtained the spectral Koopman Hopf formula (Fig. 2b). Since the terminal cost is the only term that contributes in the cost function, the value function changes based on the terminal state at T only. Therefore, if a state $x(t)$ can be driven to the origin within time $T-t$, the corresponding value function will be $V(x,t) = 0$. Using this understanding, one can observe that the set $\{x|V(x,t) = 0\}$ for any $t\in[0,T]$ is the set of all the states that can be driven to the origin from time $T-t$ to T. We further compare for $t = 0$, the spectral Koopman Hopf formula value function with the ones obtained using linearization of the system. Fig. 2c shows the spectral Koopman Hopf formula value function. As previously discussed, for this system the value function obtained using the spectral Koopman Hopf formula gives the true value function. Fig. 2d show the linearization value function. Looking at the 0-level set of the linearization value function we can see that for states near the edges of the 0-level set, $V(x,0) = 0$ even if the state can not driven to 0 within the time interval $[0,1]$. Therefore, it is easy to see that the value function values obtained using linearization



(a) Value function obtained using the analytical form of the eigenfunction

(b) Value function obtained using spectral Koopman Hopf formula



(c) Value function using eigenfunction transformation

(d) Value function using linearization

Fig. 2: Value function comparison (1-D system)

are smaller than the actual values, i.e., linearization provides an underestimate of the value function.

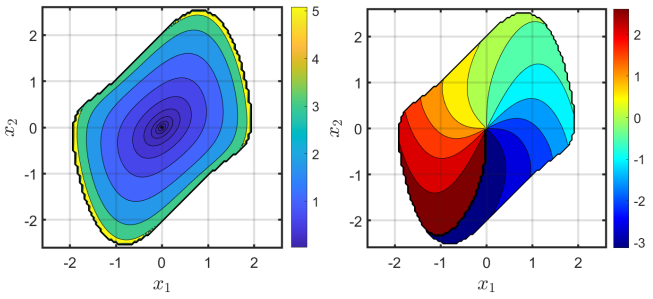
B. Vanderpol Oscillator

Our second example is that of a Vanderpol Oscillator.

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \mu(1 - x_1^2)x_2 - x_1 + u \tag{69}$$

For this system (with $\mu = 1$) the origin is an unstable equilibrium point and is surrounded by stable limit cycle. The equilibrium point at the origin for this system is anti-stable and hence the results from the Lemma 8 can be applied to the time reserved system. Linearization gives us a complex-conjugate pair of unstable eigenvalues and the corresponding complex-conjugate pair of eigenfunctions. Using this information in the path-integral formula we obtain a complex-conjugate pair of nonlinear eigenfunctions for the subdomain enclosed by the limit cycle. The absolute values and the angles of the obtained eigenfunction at various points in the domain are shown in Fig. 3.

Consider the optimization problem (11) subject to (69). Using the real (ϕ_r) and imaginary (ϕ_i) parts of the principal eigenfunction we get $\dot{\mathbf{X}} = e^{-t\Lambda}\frac{\partial \Phi}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})u$ where, $\Lambda = \begin{bmatrix} 0.5 & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 0.5 \end{bmatrix}$, $\mathbf{g}(\mathbf{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\Phi = \begin{bmatrix} \phi_r \\ \phi_i \end{bmatrix}$. The Hamiltonian for $u \in [-u_m, u_m]$ is $\bar{H}(\mathbf{X}, \mathbf{P}, t) = -u_m |e^{-t\Lambda}\frac{\partial \Phi}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})|$. This Hamiltonian can be bounded by two state independent Hamiltonians as $\bar{H}_-(\mathbf{P}, t) \leq \bar{H}(\mathbf{X}, \mathbf{P}, t) \leq \bar{H}_+(\mathbf{P}, t)$ by replacing the $\frac{\partial \Phi}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})$ with $\mathbf{B}_u$ and $\mathbf{B}_l$ respectively such that $\mathbf{B}_l\mathbf{B}_l^\top \leq \frac{\partial \Phi}{\partial \mathbf{x}}\mathbf{g}\mathbf{g}^\top\frac{\partial \Phi}{\partial \mathbf{x}}^\top \leq \mathbf{B}_u\mathbf{B}_u^\top$. Solving the optimal



(a) Absolute value

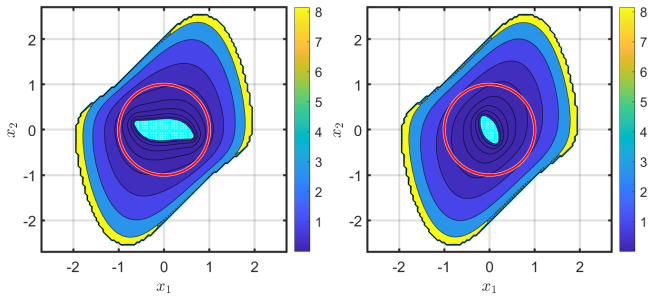
(b) Angle

Fig. 3: Van der Pol system: Eigenfunction

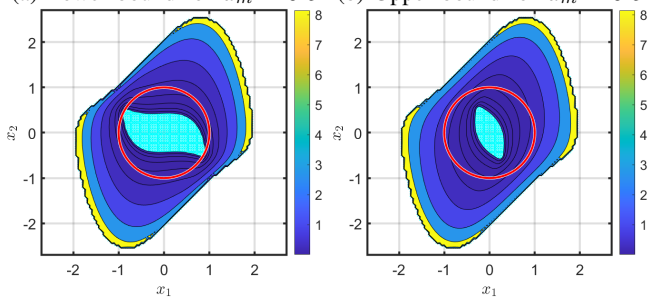
control problem (35) with $\mathbf{B}_u$ gives a lower bound $\bar{V}_-(\mathbf{X}, t)$ on the optimal value function. Similarly, solving the optimal control problem (35) with $\mathbf{B}_l$ gives an upper bound $\bar{V}_+(\mathbf{X}, t)$ on the optimal value function. In the unit ball around the origin (indicated by the red circle in Fig. 4), we obtain $\mathbf{B}_l = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$ and $\mathbf{B}_u = \begin{bmatrix} 1.63 & 1.21 \end{bmatrix}^\top$. The value of B_l is obtained by approximating the eigenfunction by only the linear part, i.e., $B_l = \mathbf{W}^\top\mathbf{g}$, where $\mathbf{W}$ is the matrix of realified left eigenvectors of the autonomous part of (69). The value of B_u is obtained by a sampling based optimization method. A large number of points are uniformly sampled in the unit ball around the origin and a feasibility problem is written for the sampled points using a semi-definite program with the constraint $\frac{\partial \Phi}{\partial \mathbf{x}}\mathbf{g}\mathbf{g}^\top\frac{\partial \Phi}{\partial \mathbf{x}}^\top \leq \mathbf{B}_u\mathbf{B}_u^\top$ for each sampled point. Setting $T = 1$ and $u_m = 0.5$, we solve for the bounds of the value function. Fig. 4b shows the upper bound $\bar{V}_+(\mathbf{X}, t)$ and the Fig. 4a shows the lower bound $\bar{V}_-(\mathbf{X}, t)$ respectively. The system has an unstable equilibrium point at the origin. The light blue region around the origin shows the set of states where the value function attains zero value, indicating that the states in the respective regions can be stabilized to the origin at time T. $\bar{V}_+(\mathbf{X}, t = 0) = 0$ provides an underestimate and $\bar{V}_-(\mathbf{X}, t = 0) = 0$ gives an overestimate of the region of the original system that can be driven to origin.

VII. Conclusion

We introduce the spectral Koopman Hopf formula for solving finite-time optimal control problems with input constraints. The spectral Koopman Hopf formula extends the applicability of the Hopf formula to control-affine nonlinear systems with state-dependent Hamiltonians by leveraging the spectral properties of the Koopman operator. The Hopf formula transforms the curse of dimensionality into a manageable complexity problem. Our method maintains the complexity benefits of the Hopf formula using a path-integral computation of Koopman eigenfunctions. This offers a parallelizable optimization-based solution for computing the optimal value function. We also provide an approximation framework to obtain under and over-estimates of the value function for the cases in which the optimal control problem can not be reduced to a form where the spectral Koopman Hopf formula is applicable directly.



(a) Lower bound for $u_m = 0.5$ (b) Upper bound for $u_m = 0.5$



(c) Lower bound for $u_m = 1.0$ (d) Upper bound for $u_m = 1.0$

Fig. 4: Van der Pol: Value function bounds ($t = 0, T = 1$)

APPENDIX

Proof: [Lemma 4] **Lower bound:** Consider the total change in V_l w.r.t. t along an optimal trajectory $\mathbf{x}^*(t)$ of $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}, t)$ obtained by solving for V.

$$\begin{aligned}\frac{dV_l}{dt} &= \frac{\partial V_l}{\partial t} + \frac{\partial V_l}{\partial x}\mathbf{F}(\mathbf{x}^*, \mathbf{u}^*, t) \\ &= \frac{\partial V_l}{\partial t} + H\left(\mathbf{x}^*, \frac{\partial V_l}{\partial x}, t\right) - h(\mathbf{u}^*). \end{aligned} \tag{70}$$

The second equality follows from the definition of the Hamiltonian (14). Using the condition $H(\mathbf{x}, \mathbf{p}, t) \geq H_l(\mathbf{x}, \mathbf{p}, t)$ in (70), we get

$$\frac{dV_l}{dt} \geq \frac{\partial V_l}{\partial t} + H_l\left(\mathbf{x}^*, \frac{\partial V_l}{\partial x}, t\right) - h(\mathbf{u}^*) \geq -h(\mathbf{u}^*). \tag{71}$$

Here, the last inequality follows from (40). From (13) and (14) we get

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + H\left(\mathbf{x}^*, V_x, t\right) - h(\mathbf{u}^*) = -h(\mathbf{u}^*). \tag{72}$$

Consider $E_l(\mathbf{x}(t), t) := V_l(\mathbf{x}(t), t) - V(\mathbf{x}(t), t)$, the difference between the two functions. Using (71) and (72) we get,

$$\frac{dE_l}{dt} = \frac{dV_l}{dt} - \frac{dV}{dt} \geq 0. \tag{73}$$

At the terminal time T, $E_l(\mathbf{x}(T), T) = V_l(\mathbf{x}(T), T) - V(\mathbf{x}(T), T) \leq 0$, this implies $E_l(\mathbf{x}(t), t) = V_l(\mathbf{x}(t), t) - V(\mathbf{x}(t), t) \leq 0$ for all $t \leq T$. Since, from (73) we see that the difference E_l can only increase with time.

Upper bound: Consider the total change in V_u w.r.t. t along an optimal trajectory $\mathbf{x}^*(t)$ of $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}, t)$ obtained by solving for V.

$$\begin{aligned}\frac{dV_u}{dt} &= \frac{\partial V_u}{\partial t} + \frac{\partial V_u}{\partial x}\mathbf{F}(\mathbf{x}^*, \mathbf{u}^*, t) \\ &= \frac{\partial V_u}{\partial t} + H\left(\mathbf{x}*, \frac{\partial V_u}{\partial x}, t\right) - h(\mathbf{u}^*). \end{aligned} \tag{74}$$

The second equality follows from the definition of the Hamiltonian (14). Using the condition $H(\mathbf{x}, \mathbf{p}, t) \leq H_u(\mathbf{x}, \mathbf{p}, t)$ in (74), we get

$$\frac{dV_u}{dt} \leq \frac{\partial V_u}{\partial t} + H_u\left(\mathbf{x}^*, \frac{\partial V_u}{\partial x}, t\right) - h(\mathbf{u}^*) \leq -h(\mathbf{u}^*). \tag{75}$$

Here, the last inequality follows from (**??**). Consider $E_u(\mathbf{x}(t), t) := V_u(\mathbf{x}(t), t) - V(\mathbf{x}(t), t)$, the difference between the two functions. Using (75) and (72) we get,

$$\frac{dE_u}{dt} = \frac{dV_u}{dt} - \frac{dV}{dt} \leq 0. \tag{76}$$

At the terminal time T, $E_l(\mathbf{x}(T), T) = V_l(\mathbf{x}(T), T) - V(\mathbf{x}(T), T) \geq 0$, this implies $E_l(\mathbf{x}(t), t) = V_l(\mathbf{x}(t), t) - V(\mathbf{x}(t), t) \geq 0$ for all $t \leq T$. Since, from (76) we see that the difference E_u can only decrease with time. ■

Proof: [Proof of Theorem 5] We will only prove the result for the single input case, the multi-input case will follow from the single input case following Assumption 2 on the matrix $\mathbf{R}$ being diagonal. For the single input case we will use the notations of $\mathbf{u} = u$ and $\mathbf{R} = r$. Consider the lower bound case, we have $\mathcal{U} = [-u_m, u_m]$ and

$$\begin{aligned} H(\mathbf{x}, \mathbf{p}) &= \mathbf{p}^\top \mathbf{f} + \min_{u \in \mathcal{U}} \mathbf{p}^\top \mathbf{g} u + \frac{1}{2} r u^2 \\ H_l(\mathbf{x}, \mathbf{p}) &= \mathbf{p}^\top \mathbf{f} + \min_{u \in \mathcal{U}} \mathbf{p}^\top \Phi_{\mathbf{x}}^{-1} \mathbf{D}_l^{\frac{1}{2}} u + \frac{1}{2} r u^2. \end{aligned}$$

First we want to show that $H_l(\mathbf{x}, \mathbf{p}) \leq H(\mathbf{x}, \mathbf{p})$. Since we are dealing with the constrained optimization, we will have to account for the cases when the constraints are active and inactive.

i) Constraints are inactive for both H and H_l: The optimal values will be

$$\begin{aligned} H(\mathbf{x}, \mathbf{p}) &= \mathbf{p}^\top \mathbf{f} - \frac{1}{2} \mathbf{p}^\top \mathbf{g} r^{-1} \mathbf{g}^\top \mathbf{p} \\ H_l(\mathbf{x}, \mathbf{p}) &= \mathbf{p}^\top \mathbf{f} - \frac{1}{2} \mathbf{p}^\top \Phi_{\mathbf{x}}^{-1} \left(\mathbf{D}_l^{\frac{1}{2}}\right) r^{-1} \left(\mathbf{D}_l^{\frac{1}{2}}\right)^\top \Phi_{\mathbf{x}}^{-\top} \mathbf{p} \end{aligned} \tag{77}$$

Using Assumption 4, we get $H_l(\mathbf{x}, \mathbf{p}) \leq H(\mathbf{x}, \mathbf{p})$.

ii) Constraints are active for both H and H_l: This implies $\left|-r^{-1}\mathbf{g}^\top \mathbf{p}\right| \geq u_m$ and $\left|-r^{-1}\left(\mathbf{D}_l^{\frac{1}{2}}\right)^\top \Phi_{\mathbf{x}}^{-\top} \mathbf{p}\right| \geq u_m$. We can express the optimizing arguments for H and H_l as $u^* = \text{sign}(-r^{-1}\mathbf{g}^\top \mathbf{p}) u_m$ and $u_l^* = \text{sign}\left(-r^{-1}\left(\mathbf{D}_l^{\frac{1}{2}}\right)^\top \Phi_{\mathbf{x}}^{-\top} \mathbf{p}\right) u_m$ respectively. We get

$$\begin{aligned} H(\mathbf{x}, \mathbf{p}) &= \mathbf{p}^\top (\mathbf{f} + \mathbf{g} u^*) + \frac{1}{2} r u_m^2 \\ H_l(\mathbf{x}, \mathbf{p}) &= \mathbf{p}^\top \left(\mathbf{f} + \Phi_{\mathbf{x}}^{-1} \mathbf{D}_l^{\frac{1}{2}} u_l^*\right) + \frac{1}{2} r u_m^2. \end{aligned} \tag{78}$$

To verify that $H_l(\mathbf{x},\mathbf{p}) \leq H(\mathbf{x},\mathbf{p})$, we can check if

$$\mathbf{p}^\top \Phi_{\mathbf{x}}^{-1} \mathbf{D}_l^{\frac{1}{2}} \operatorname{sign}\left(-r^{-1}\left(\mathbf{D}_l^{\frac{1}{2}}\right)^\top \Phi_{\mathbf{x}}^{-\top}\mathbf{p}\right) \leq \mathbf{p}^\top \mathbf{g} \operatorname{sign}(-r^{-1}\mathbf{g}^\top \mathbf{p}). \quad (79)$$

We can observe from the setup of the optimization problem that $H_l(\mathbf{x},\mathbf{p}) - \mathbf{p}^\top \mathbf{f} \leq 0$ and $H(\mathbf{x},\mathbf{p}) - \mathbf{p}^\top \mathbf{f} \leq 0$ (since $0 \in \mathcal{U}$), therefore both the quantities in inequality (79) are negative and we can write $H_l(\mathbf{x},\mathbf{p}) \leq H(\mathbf{x},\mathbf{p})$ if

$$\left|\mathbf{p}^\top \Phi_{\mathbf{x}}^{-1} \mathbf{D}_l^{\frac{1}{2}} \operatorname{sign}\left(-r^{-1}\left(\mathbf{D}_l^{\frac{1}{2}}\right)^\top \Phi_{\mathbf{x}}^{-\top}\mathbf{p}\right)\right| \geq \left|\mathbf{p}^\top \mathbf{g} \operatorname{sign}(-r^{-1}\mathbf{g}^\top \mathbf{p})\right|. \quad (80)$$

Using Assumption 4, we can write

$$r^{-1}\mathbf{p}^\top \Phi_{\mathbf{x}}^{-1}\left(\mathbf{D}_l^{\frac{1}{2}}\right)\left(\mathbf{D}_l^{\frac{1}{2}}\right)^\top \Phi_{\mathbf{x}}^{-\top}\mathbf{p} \geq r^{-1}\mathbf{p}^\top \mathbf{g}\mathbf{g}^\top \mathbf{p}$$
$$\implies r^{-1}\left|\mathbf{p}^\top \Phi_{\mathbf{x}}^{-1}\left(\mathbf{D}_l^{\frac{1}{2}}\right)\right|^2 \geq r^{-1}\left|\mathbf{p}^\top \mathbf{g}\right|^2. \quad (81)$$

Therefore, the inequality (80) holds true under Assumption 4, implying $H_l(\mathbf{x},\mathbf{p}) \leq H(\mathbf{x},\mathbf{p})$.

iii) Constraint is active for H_l but inactive for H: We have $u^* = -r^{-1}\mathbf{g}^\top \mathbf{p} \leq u_m$ and $u_l^* = \operatorname{sign}\left(-r^{-1}\left(\mathbf{D}_l^{\frac{1}{2}}\right)^\top \Phi_{\mathbf{x}}^{-\top}\mathbf{p}\right) u_m$. Using (77) and (78) we can check that $H_l(\mathbf{x},\mathbf{p}) \leq H(\mathbf{x},\mathbf{p})$, if

$$\mathbf{p}^\top \Phi_{\mathbf{x}}^{-1} \mathbf{D}_l^{\frac{1}{2}} \operatorname{sign}\left(-r^{-1}\left(\mathbf{D}_l^{\frac{1}{2}}\right)^\top \Phi_{\mathbf{x}}^{-\top}\mathbf{p}\right) u_M \leq -\frac{1}{2} r u_m^2 - \frac{1}{2}\mathbf{p}^\top \mathbf{g} r^{-1}\mathbf{g}^\top \mathbf{p}.$$

We observe that

$$\frac{1}{2}\mathbf{p}^\top \mathbf{g} r^{-1}\mathbf{g}^\top \mathbf{p} = \frac{1}{2}\left(-\mathbf{p}^\top \mathbf{g} r^{-1}\right) r \left(-r^{-1}\mathbf{g}^\top \mathbf{p}\right) = \frac{1}{2} r (u^*)^2 \leq \frac{1}{2} r u_m^2$$
$$\implies -\frac{1}{2} r u_m^2 - \frac{1}{2}\mathbf{p}^\top \mathbf{g} r^{-1}\mathbf{g}^\top \mathbf{p} \geq -r u_m^2.$$

Therefore we can say that $H_l(\mathbf{x},\mathbf{p}) \leq H(\mathbf{x},\mathbf{p})$ if

$$\mathbf{p}^\top \Phi_{\mathbf{x}}^{-1} \mathbf{D}_l^{\frac{1}{2}} \operatorname{sign}\left(-r^{-1}\left(\mathbf{D}_l^{\frac{1}{2}}\right)^\top \Phi_{\mathbf{x}}^{-\top}\mathbf{p}\right) u_m \leq -r u_m^2.$$

Since, $H_l(\mathbf{x},\mathbf{p}) - \mathbf{p}^\top \mathbf{f} \leq 0$, $H_l(\mathbf{x},\mathbf{p}) \leq H(\mathbf{x},\mathbf{p})$ if

$$\left|\mathbf{p}^\top \Phi_{\mathbf{x}}^{-1} \mathbf{D}_l^{\frac{1}{2}} \operatorname{sign}\left(-r^{-1}\left(\mathbf{D}_l^{\frac{1}{2}}\right)^\top \Phi_{\mathbf{x}}^{-\top}\mathbf{p}\right) u_m\right| \geq r u_m^2 \implies \left|\mathbf{p}^\top \Phi_{\mathbf{x}}^{-1} \mathbf{D}_l^{\frac{1}{2}} r^{-1}\right| \geq u_m.$$

Since the last inequality holds true under the assumption that the constraint is active for H_l, we get $H_l(\mathbf{x},\mathbf{p}) \leq H(\mathbf{x},\mathbf{p})$.

From (81) it is clear that $|u_l^*| \geq |u^*|$, therefore it is not possible to have a case where the constraint is active for H but inactive for H_l.

To show the upper bound i.e., $H(\mathbf{x},\mathbf{p}) \leq H_u(\mathbf{x},\mathbf{p})$ with

$$H_u(\mathbf{x},\mathbf{p}) = \mathbf{p}^\top \mathbf{f} + \min_{u\in\mathcal{U}} \mathbf{p}^\top \Phi_{\mathbf{x}}^{-1} \mathbf{D}_u^{\frac{1}{2}} u + \frac{1}{2} r u^2.$$

Using the left inequality from Assumption 4, we can show that by using $\mathbf{D}_u^{\frac{1}{2}}$ in place of $\mathbf{D}_l^{\frac{1}{2}}$ the inequalities in (79), (80) and (81) are reversed. Therefore following the same arguments as the lower bound case, we can prove the upper bound $H(\mathbf{x},\mathbf{p}) \leq H_u(\mathbf{x},\mathbf{p})$.

■

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