Lecture 6

Nonlinear Optimization: Portfolio Optimization

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Portfolio Optimization: basic elements

n assets, investment allocation x_i $(1 \leq i \leq n)$

$$x_1 + \cdots + x_n = 1, \quad x_i \geq 0, \quad i = 1, ..., n.$$

- return from asset i: r_i $(i = 1, ..., n)$,

$$\text{total return:} \quad R(\mathbf{x}) = x_1 r_1 + ... + x_n r_n.$$

expected return of the portfolio: $E[r_i] = \mu_i$ $(i = 1, ..., n)$,

$$\text{total expected return:} \quad E[R(\mathbf{x})] = x_1\mu_1 + ... + x_n\mu_n = \mu^T \mathbf{x}$$

- covariance between asset i and j:

$$\sigma_{ij} = cov(r_i, r_j) = E\left[(r_i - E[r_i])(r_j - E[r_j])\right], \quad (\sigma_{ii}^2 = \sigma_i^2)$$

covariance matrix:

$$\Sigma = \begin{pmatrix} \sigma_1^2 & .... & \sigma_{1n} \\ .... & ... & ... \\ \sigma_{n1} & ... & \sigma_n^2 \end{pmatrix}$$

variance of the portfolio:

$$var(R(\mathbf{x})) = \mathbf{x}^T \Sigma \mathbf{x} = \sum_{ij} x_i x_j \sigma_{ij}.$$

Estimation of Mean Return

constant investment:
start from 100, no reinvestment; new capital to cover shortfall.

year	1	2	3
return	20%	-25%	100%
cash flow	20	-25	100

self-financed investment:
start from 100, reinvest all profits; no new capital.

year	1	2	3
return	20%	-25%	100%
position	120	90	180

$$\mu_a = \frac{20\% - 25\% + 100\%}{3} = 31.6\% \; (= (95/100)/3)$$

$$\mu_g = (1.2 \times 0.75 \times 2)^{1/3} - 1 = 21\% \; (= (180/100)^{1/3} - 1)$$

given a series of observations of past returns, $r_1,\ldots,r_T$,

- arithmetic mean

$$\mu_a = \frac{r_1 + ... + r_T}{T}$$

- geometric mean

$$\mu_g = [(1 + r_1) \times .... \times (1 + r_T)]^{1/T} - 1$$

Three Typical Mean-Variance Models

- optimizes weighted sum

$$\max_{\mathbf{x}\geq 0}\left\{x_1\mu_1+....+x_n\mu_n-\delta\frac{\mathbf{x}^T\mathbf{\Sigma}\mathbf{x}}{2}\right\}\quad(\delta\text{: risk tolerance})$$
$$\text{s.t.}\quad x_1+....+x_n=1,\quad x_1,...,x_n\geq 0.$$

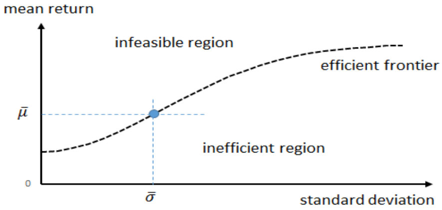
- minimize volatility under mean return constraint

$$\min_{\mathbf{x}\geq 0}\left\{\frac{1}{2}\mathbf{x}^T\mathbf{\Sigma}\mathbf{x}\right\}$$
$$\text{s. t.}\quad x_1\mu_1+....+x_n\mu_n\geq\bar{\mu}\qquad(\bar{\mu}\text{ : return target})$$
$$x_1+....+x_n=1,\quad x_1,...,x_n\geq 0.$$

- maximize mean return under variance constraint

$$\max_{\mathbf{x}\geq 0}\{x_1\mu_1+....+x_n\mu_n\}$$
$$\text{s.t.}\quad \mathbf{x}^T\mathbf{\Sigma}\mathbf{x}\leq\bar{\sigma}^2\qquad(\bar{\sigma}^2\text{ : risk tolerance}).$$
$$x_1+....+x_n=1,\quad x_1,...,x_n\geq 0.$$

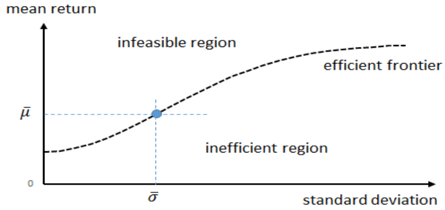
Efficient Frontier and Equivalence (I)



A point on the efficient frontier with $\sigma = \bar{\sigma}$ must be an optimal solution to

$$\begin{aligned} & \max_{\mathbf{x} \geq 0} \{x_1\mu_1 + ... + x_n\mu_n\} \\ \text{s.t.} \quad & \mathbf{x}^T \Sigma \mathbf{x} \leq \bar{\sigma}^2, \\ & x_1 + ... + x_n = 1, \quad x_1, ..., x_n \geq 0. \end{aligned}$$

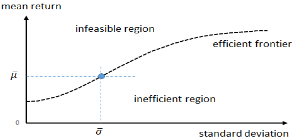
Efficient Frontier and Equivalence (II)



A point on the efficient frontier with $\mu = \bar{\mu}$ must be an optimal solution to

$$\begin{aligned} & \min_{\mathbf{x} \geq 0} \left\{ \frac{1}{2} \mathbf{x}^T \Sigma \mathbf{x} \right\} \\ \text{s. t.} \quad & x_1 \mu_1 + ... + x_n \mu_n \geq \bar{\mu}, \\ & x_1 + ... + x_n = 1, \quad x_1, ..., x_n \geq 0. \end{aligned}$$

Efficient Frontier and Equivalence



suppose we pick any $\bar{\mu}$,

$$\min_{\mathbf{x}\geq 0}\left\{\frac{1}{2}\mathbf{x}^T\Sigma\mathbf{x}\right\} \quad \text{s.t.} \quad x_1\mu_1+....+x_n\mu_n\geq\bar{\mu}, ...$$

and the objective value $1/2(\mathbf{x}^*)^T\Sigma\mathbf{x}^*=(\sigma^*)^2$.

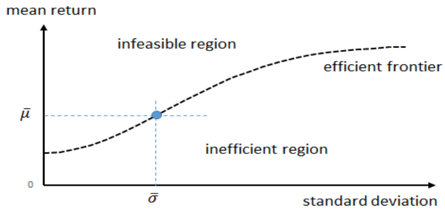
now use σ^* as $\bar{\sigma}$ to solve

$$\max_{\mathbf{x}\geq 0}\{x_1\mu_1+....+x_n\mu_n\} \quad \text{s.t.} \quad \mathbf{x}^T\Sigma\mathbf{x}\leq(\sigma^*)^2, ...$$

and the objective value $x_1^{**}\mu_1+\cdots+x_n^{**}\mu_n=\mu^*$

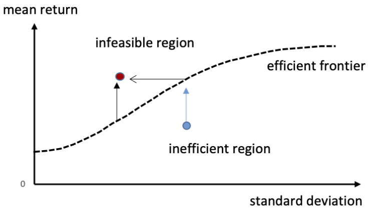
can $\mu^*<\bar{\mu}$? can $\mu^*>\bar{\mu}$?

Efficient Frontier



maximum mean return that can be achieved when the volatility level is below some threshold, or minimum volatility that can be achieved when the mean return exceeds a threshold

Efficient Frontier and Model 1



$$\max_{\mathbf{x}\geq 0}\left\{x_1\mu_1 + ... + x_n\mu_n - \delta\frac{\mathbf{x}^T\Sigma\mathbf{x}}{2}\right\}$$

$$x_1 + ... + x_n = 1, \quad x_1 \geq 0, ..., x_n \geq 0.$$

1. The solution cannot above the efficient frontier: it would mean there is a higher return at the same variance or a lower variance at the same return.
2. The solution cannot below the efficient frontier: it would mean the objective value can still improve.

Equivalence of Optimization Models (I)

model:

$$\min_{\mathbf{x}\geq 0}\left\{1/2\sum_{i,j}\sigma_{ij}x_ix_j\right\}$$
$$x_1\mu_1+\cdots+x_n\mu_n\geq\bar{\mu},$$
$$\boxed{x_1+....+x_n=1,}$$
$$x_i\geq 0,\quad i=1,...,n.$$

standard form:

$$\max_{\mathbf{x}\geq 0}\left\{-1/2\sum_{i,j}\sigma_{ij}x_ix_j\right\}$$
$$\bar{\mu}-x_1\mu_1-....-x_n\mu_n\leq 0,$$
$$\boxed{x_1+....+x_n-1\leq 0,}$$
$$\boxed{1-x_1-....-x_n\leq 0,}$$
$$-x_i\leq 0,\quad i=1,....,n.$$

$$\mathcal{L}=-1/2\sum_{i,j}\sigma_{ij}x_ix_j-\lambda_\delta(\bar{\mu}-x_1\mu_1-\cdots-x_n\mu_n)$$
$$\boxed{-\bar{\lambda}_1(x_1+...+x_n-1)-\bar{\lambda}_2(1-x_1-....-x_n)}+\sum_{i=1}\lambda_ix_i$$
$$=-1/2\sum_{i,j}\sigma_{ij}x_ix_j-\lambda_\delta(\bar{\mu}-x_1\mu_1-\cdots-x_n\mu_n)\boxed{+\bar{\lambda}(1-x_1-....-x_n)}+\sum_{i=1}\lambda_ix_i.$$

$$\lambda_\delta,\bar{\lambda}_1,\bar{\lambda}_2\geq 0,\lambda_i\geq 0\ (i=1,....,n),\ \text{and}\ \bar{\lambda}=\bar{\lambda}_1-\bar{\lambda}_2\ (<,=,>0).$$

Equivalence of Optimization Models (II)

Repeat: Lagrangian ($\lambda_\delta \geq 0$, $\lambda_i \geq 0$ $(i = 1, ..., n)$, $\bar{\lambda}$):

$$\mathcal{L} = -\frac{1}{2}\sum_{i,j}\sigma_{ij}x_i x_j - \lambda_\delta(\bar{\mu} - x_1\mu_1 \cdots - x_n\mu_n) + \bar{\lambda}(1 - x_1 - .... - x_n) + \sum_{i=1}\lambda_i x_i.$$

KKT condition:

$$-\sum_{j=1}^{n}\sigma_{ij}x_j + \lambda_\delta\mu_i - \bar{\lambda} + \lambda_i = 0, \; i = 1, \cdots, n,$$

$$\bar{\lambda}(1 - x_1 - .... - x_n) = 0, \quad x_1 + \cdots + x_n = 1,$$

$$\lambda_i x_i = 0, \quad x_i \geq 0, \quad \lambda_i \geq 0, \quad i = 1, \cdots, n,$$

$$\lambda_\delta(x_1\mu_1 + \cdots + x_n\mu_n - \bar{\mu}) = 0, \quad x_1\mu_1 + \cdots + x_n\mu_n \geq \bar{\mu}, \quad \lambda_\delta \geq 0.$$

The solution to the above x_i^* is an optimal solution for

$$\min_{\mathbf{x}\geq 0}\left\{\frac{1}{2}\sum_{i,j}\sigma_{ij}x_i x_j : x_1\mu_1 + ... + x_n\mu_n \geq \bar{\mu}, x_1 + ... + x_n = 1.\right\}$$

Equivalence of Optimization Models (III)

Now suppose that we minimize

$$\max_{\mathbf{x}\geq 0}\left\{x_1\mu_1 + ... + x_n\mu_n - \delta\frac{\mathbf{x}^T\boldsymbol{\Sigma}\mathbf{x}}{2}\right\}$$
$$\text{subj. to}\quad x_1 + ... + x_n = 1,\quad x_1 \geq 0, ..., x_n \geq 0.$$

Lagrangian is

$$\mathcal{L} = x_1\mu_1 + ... + x_n\mu_n - \delta\frac{\mathbf{x}^T\boldsymbol{\Sigma}\mathbf{x}}{2} + \bar{\lambda}'(1 - x_1 - .... - x_n) + \sum_{i=1}^{n}\lambda_i' x_i.$$

KKT conditions

$$\mu_i - \delta\sum_{j=1}^{n}\sigma_{ij}x_j - \bar{\lambda}' + \lambda_i' = 0,\ \ i = 1, \cdots, n,$$

$$\bar{\lambda}'(1 - x_1 - .... - x_n) = 0,\quad x_1 + \cdots + x_n = 1,$$

$$\lambda_i' x_i = 0,\quad x_i \geq 0,\quad \lambda_i' \geq 0,\quad i = 1, \cdots, n.$$

Equivalence of Optimization Models (IV)

KKT conditions of model I (max. mean-δ variance):

$$\mu_i - \delta \sum_{j=1}^{n} \sigma_{ij} x_j - \bar{\lambda}' + \lambda_i' = 0, \; i = 1, \cdots, n,$$

$$\bar{\lambda}'(1 - x_1 - .... - x_n) = 0, \quad x_1 + \cdots + x_n = 1,$$

$$\lambda_i' x_i = 0, \quad x_i \geq 0, \quad \lambda_i' \geq 0, \quad i = 1, \cdots, n.$$

KKT Conditions of Model II (minimizing variance):

$$-\sum_{j=1}^{n} \sigma_{ij} x_j + \lambda_\delta \mu_i - \bar{\lambda} + \lambda_i = 0, \; i = 1, \cdots, n,$$

$$\bar{\lambda}(1 - x_1 - .... - x_n) = 0, \quad x_1 + \cdots + x_n = 1,$$

$$\lambda_i x_i = 0, \quad x_i \geq 0, \quad \lambda_i \geq 0, \quad i = 1, \cdots, n,$$

$$\lambda_\delta (x_1 \mu_1 + \cdots + x_n \mu_n - \bar{\mu}) = 0, \quad x_1 \mu_1 + \cdots + x_n \mu_n \geq \bar{\mu}, \quad \lambda_\delta \geq 0.$$

What if we let

$$\delta = 1/\lambda_\delta, \quad \bar{\lambda}' = \bar{\lambda}/\lambda_\delta, \quad \lambda_i' = \lambda_i/\lambda_\delta, \; i = 1, \cdots, n.$$

Equivalence of Optimization Models (V)

If $\lambda_\delta = 0$, then the KKT conditions of Model II become:

$$-\sum_{j=1}^{n} \sigma_{ij} x_j^* - \bar{\lambda} + \lambda_i = 0, \; i = 1, \cdots, n,$$

$$\bar{\lambda}(1 - x_1^* - .... - x_n^*) = 0 \quad x_1^* + \cdots + x_n^* \leq 1,$$

$$\lambda_i x_i^* = 0, \quad x_i^* \geq 0, \quad \lambda_i \geq 0, \quad i = 1, \cdots, n,$$

$$x_1^* \mu_1 + \cdots + x_n^* \mu_n \geq \bar{\mu}.$$

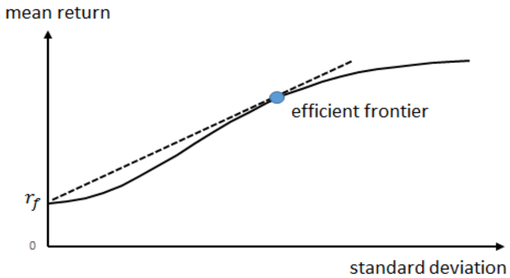
which is the same KKT conditions for the problem:

$$\min_{\mathbf{x} \geq 0} \left\{ \frac{1}{2} \sum_{i,j} \sigma_{ij} x_i x_j : x_1 + ... + x_n = 1, \quad x_i \geq 0, \quad i = 1, ..., n. \right\}$$

for model I, this means $\delta = 1/\lambda_\delta = \infty$ in the objective function:

$$\max_{\mathbf{x} \geq 0} \left\{ x_1 \mu_1 + ... + x_n \mu_n - \delta \frac{\mathbf{x}^T \Sigma \mathbf{x}}{2} \right\}$$

Efficient Frontier and Sharpe Ratio



(r_f: risk-free return)

each point on this curve correspond to a solution to an optimal solution of any of the three optimization problems (with a special choice of $\bar{\mu}$, $\bar{\sigma}$, and δ).

There is a special point on the curve. How to find it?

$$h(\mathbf{x}) = \frac{\boldsymbol{\mu}^T \mathbf{x} - r_f}{\sqrt{\mathbf{x}^T \Sigma \mathbf{x}}}.$$

Sharpe Ratio: Problem Formulation:

$$\max_{\mathbf{x}} \left\{ \frac{\boldsymbol{\mu}^T \mathbf{x} - r_f}{\sqrt{\mathbf{x}^T \Sigma \mathbf{x}}} \right\}.$$

$$\text{subject to}: \quad x_1 + ... + x_n = 1,$$

$$x_i \geq 0, \quad i = 1, ..., n.$$

The objective function looks complicated, need some simplification:

$$\min_{\mathbf{x}} \left\{ \frac{\sqrt{\mathbf{x}^T \Sigma \mathbf{x}}}{\boldsymbol{\mu}^T \mathbf{x} - r_f} \right\}.$$

Still difficult to handle directly.

Sharpe Ratio: Solution Techniques

suppose that x_i^* $(i = 1, ..., n)$ maximizes the Sharpe Ratio.

define κ such that

$$\mu_1 x_1^* + ... + \mu_n x_n^* - r_f = 1/\kappa.$$

We do not know values of x_i^* $(i = 1, ..., n)$ and κ, but they exist, and $\kappa > 0$.

We can rewrite the above as

$$\begin{aligned}\mu^T \mathbf{x}^* - r_f &= \mu_1 x_1^* + ... + \mu_n x_n^* - (x_1^* + \cdots + x_n^*) r_f \\ &= (\mu_1 - r_f) x_1^* + ... + (\mu_n - r_f) x_n^* = 1/\kappa\end{aligned}$$

so

$$(\mu_1 - r_f)(\kappa x_1^*) + ... + (\mu_n - r_f)(\kappa x_n^*) = 1.$$

let

$$y_i^* = \kappa x_i^*, \quad i = 1, ..., n.$$

then

$$(\mu_1 - r_f) y_1^* + ... + (\mu_n - r_f) y_n^* = 1.$$

and

$$\frac{\sqrt{(\mathbf{x}^*)^T \Sigma \mathbf{x}^*}}{\boldsymbol{\mu}^T \mathbf{x}^* - r_f} = \frac{\sqrt{(\mathbf{y}^*)^T \Sigma \mathbf{y}^*}/\kappa}{[(\mu_1 - r_f) y_1^* + ... + (\mu_n - r_f) y_n^*]/\kappa} = \sqrt{(\mathbf{y}^*)^T \Sigma \mathbf{y}^*}$$

Maximum Sharpe Ratio

Repeat:

$$\frac{\sqrt{(\mathbf{x}^*)^T\Sigma\mathbf{x}^*}}{\boldsymbol{\mu}^T\mathbf{x}^* - r_f} = \sqrt{(\mathbf{y}^*)^T\Sigma\mathbf{y}^*}, \qquad (\mu_1 - r_f)y_1^* + ... + (\mu_n - r_f)y_n^* = 1.$$

because $\mathbf{y}^* = \kappa\mathbf{x}^*$:

$$\begin{aligned} x_1^* + ... + x_n^* = 1 \quad &\longrightarrow \quad y_1^* + ... + y_n^* = \kappa, \\ x_i^* \geq 0 \quad &\longrightarrow \quad y_i^* \geq 0 \ (i = 1, ..., n). \end{aligned}$$

So we can determine $\mathbf{y}^*$ and κ by:

$$\min_{\mathbf{y} \geq 0, \kappa \geq 0} \left\{ \mathbf{y}^T \Sigma \mathbf{y} \right\}$$

$$\text{subject to} \qquad (\mu_1 - r_f)y_1 + ... + (\mu_n - r_f)y_n = 1, \quad y_1 + ... + y_n = \kappa.$$

The optimal solution to the original problem is

$$x_i^* = y_i^* / (y_1^* + \cdots + y_n^*), \quad i = 1, ..., n.$$

and the Sharpe ratio is $\frac{1}{\sqrt{(\mathbf{y}^*)^T\Sigma\mathbf{y}^*}}$.

Deviation in Return

$$\min_{\mathbf{x}\geq 0}\left\{\frac{1}{2}\mathbf{x}^T\Sigma\mathbf{x}\right\} \quad \text{subject to:} \quad x_1\mu_1+....+x_n\mu_n\geq\bar{\mu}, \quad x_1+....+x_n=1.$$

objective value (portfolio variance): $\mathbf{x}^T\Sigma\mathbf{x}=\sum_{i,j}\sigma_{ij}x_ix_j$.

where

$$\sigma_{ij}=\mathbf{E}[(r_i-\mu_i)(r_j-\mu_j)]$$

so

$$\mathbf{x}^T\Sigma\mathbf{x}=\mathbf{E}\left[\sum_{i,j}(r_i-\mu_i)(r_j-\mu_j)x_ix_j\right]=\mathbf{E}\left[\left(\sum_{i=1}^{n}(r_i-\mu_i)x_i\right)^2\right].$$

deviation of the portfolio return from its mean

$$U_{\mathbf{x}}=\sum_{i=1}^{n}(r_i-\mu_i)x_i \quad \longrightarrow \quad \mathbf{x}^T\Sigma\mathbf{x}=\mathbf{E}[U_{\mathbf{x}}^2]$$

mean and variance of this deviation

$$\mathbf{E}[U_{\mathbf{x}}]=0 \text{ (because } \mathbf{E}[r_i]=\mu_i) \qquad var(U_{\mathbf{x}})=\mathbf{E}[U_{\mathbf{x}}^2]=\sigma_x^2$$

Variance for Mean Absolute Variation (MAD)

- the deviation of the actual portfolio return from its mean:

$$U_{\mathbf{x}} = (r_1 - \mu_1)x_1 + ... + (r_n - \mu_n)x_n.$$

- minimizing $\mathbf{E}[U_{\mathbf{x}}]$ is meaningless because it is 0.
- minimizing portfolio variance is to minimize $\mathbf{E}[U_{\mathbf{x}}^2]$

what about minimizing $\mathbf{E}[|U_{\mathbf{x}}|]$ (Mean Absolute Deviation (MAD))?

if all returns r_i $(i = 1, \cdots, n)$ are normally distributed, then $U_{\mathbf{x}}$ is normally distributed with mean 0 and standard deviation $\sigma_{\mathbf{x}}$.

$$f(u) = \frac{1}{\sqrt{2\pi}\sigma_{\mathbf{x}}} e^{-\frac{u^2}{2\sigma_{\mathbf{x}}^2}}$$

$$\mathbf{E}\left[|U_{\mathbf{x}}|\right] = \frac{1}{\sqrt{2\pi}\sigma_{\mathbf{x}}} \int_{-\infty}^{\infty} |u| e^{-\frac{u^2}{2\sigma_{\mathbf{x}}^2}} du = \frac{2}{\sqrt{2\pi}\sigma_{\mathbf{x}}} \int_{0}^{\infty} u e^{-\frac{u^2}{2\sigma_{\mathbf{x}}^2}} du$$
$$= \frac{\sqrt{2}}{\sqrt{\pi}\sigma_{\mathbf{x}}} \left(-\sigma_x^2 e^{-\frac{u^2}{2\sigma_{\mathbf{x}}^2}} \Big|_0^{\infty} \right) = \sqrt{\frac{2}{\pi}}\sigma_{\mathbf{x}}$$

in this case, minimizing the variance is the equivalent to minimizing MAD

Minimizing MAD

assume return is normally distributed.

there are T samples of asset returns i: r_{it}, $t = 1, \cdots, T$, μ_i is the expected return $(i = 1, \cdots, n)$

to choose the optimal investment amounts $\mathbf{x}$, you can:

- let

$$\hat{\sigma}_{ij} = \frac{1}{T}\sum_{t=1}^{T}(r_{it} - \mu_i)(r_{jt} - \mu_j)$$

  and minimize the (estimated) portfolio variance:

$$\min_{\mathbf{x}} \left\{ \sum_{ij} \hat{\sigma}_{ij} x_i x_j \right\}$$

- or minimize MAD

$$\min_{\mathbf{x}} \left\{ \sum_{t=1}^{T} \left| \sum_{i=1}^{n} (r_{it} - \mu_i) x_i \right| \right\}$$

under the same contraints

$$\mu_1 x_1 + \cdots + \mu_n x_n \geq \bar{\mu}, \quad x_1 + \cdots + x_n = 1, \quad x_1 \geq 0, \cdots, x_n \geq 0.$$

Techniques for MAD Minimization

how to handle the MAD objective function:

$$\sum_{t=1}^{T}\left|\sum_{i=1}^{n}(r_{it}-\mu_i)x_i\right|?$$

1. for each $t = 1, \cdots, T$, introduce new variables y_t and z_t;
2. change the objective function to

$$\min\left\{\sum_{t=1}^{T}(y_t+z_t)\right\}$$

3. add constraints: for $t = 1, \cdots, T$:

$$y_t - z_t = \sum_{i=1}^{n}(r_{it}-\mu_i)x_i, \quad y_t \geq 0, \quad z_t \geq 0,$$

MAD Minimization Problem

$$\min_{\mathbf{y}\geq 0, \mathbf{z}\geq 0, \mathbf{x}} \left\{ \sum_{t=1}^{T} (y_t + z_t) \right\} \quad \text{subject to: } y_t - z_t = \sum_{i=1}^{n} (r_{it} - \mu_i) x_i, t = 1, \cdots, T, \cdots$$

suppose that $\mathbf{x}$ is chosen, how to choose y_t and z_t optimally?

- when $\sum_{i=1}^{n} (r_{it} - \mu_i) x_i \geq 0$, it is optimal to let

$$y_t = \sum_{i=1}^{n} (r_{it} - \mu_i) x_i \quad \text{and} \quad z_t = 0 \quad (\text{because } y_t + z_t = 2z_t + \sum_{i=1}^{n} (r_{it} - \mu_i) x_i).$$

when $\sum_{i=1}^{n} (r_{it} - \mu_i) x_i < 0$, it is optimal to let

$$y_t = 0 \quad \text{and} \quad z_t = -\sum_{i=1}^{n} (r_{it} - \mu_i) x_i \quad (\text{because } y_t + z_t = 2y_t - \sum_{i=1}^{n} (r_{it} - \mu_i) x_i).$$

in either case

$$y_t + z_t = \left| \sum_{i=1}^{n} (r_{it} - \mu_i) x_i \right|$$

Limitation of Mean-Variance Model

Two investment opportunities: return on investment 1 is either 10% or 30% with equal probability. Return on investment 2 is guaranteed at 10%. ($\sigma_1 > \sigma_2 = 0$).

what is the optimal allocation of your budget between the two investments?

let x be the % allocated to investment 1, so the % in 2 is $1 - x$

- can this optimal allocation be the solution of

$$\max_{x \geq 0} \left\{ \mu_1 x + \mu_2 (1 - x) \middle| \sigma_1^2 x^2 \leq \bar{\sigma}^2 \right\}, \quad \bar{\sigma} = \sigma_1 / 2.$$

- can the optimal allocation be the solution of:

$$\max_{x \geq 0} \left\{ \mu_1 x + \mu_2 (1 - x) - \frac{\delta}{2} \sigma_1^2 x^2 \right\}, \quad \delta = 0.2 / \sigma_1^2.$$

- can this optimal allocation be the solution:

$$\min_{x \geq 0} \left\{ \frac{1}{2} \sigma_1^2 x^2 \middle| \mu_1 x + \mu_2 (1 - x) \geq \bar{\mu} \right\}, \quad \bar{\mu} = 10\%.$$

Stochastic Dominance

Portfolio return X_1 stochastically dominates (first-order) return X_2 if

$$\Pr(X_1 \geq x) \geq \Pr(X_2 \geq x) \quad \text{for any } x$$

In the previous example, there are two possibilities $x = 10\%$ or $x = 30\%$:

$$\begin{aligned} & \Pr(X_1 \geq 10\%) && = 1 = \Pr(X_2 \geq 10\%) \\ \text{and} \qquad & \Pr(X_1 \geq 30\%) && = 0.5 > \Pr(X_2 \geq 30\%). \end{aligned}$$

Does that mean we should use the mean-variance model?

If the return of a portfolio follows the Normal distribution, then the return of a portfolio constructed by optimizing the mean-variance model cannot be stochastically dominated.

The Case with Normal Distribution

Import fact: If X is Normally distributed random variable: $X \sim \mathcal{N}(\mu, \sigma)$, then

$$\Pr(X \geq x) = \Pr\left(X_0 \geq \frac{x-\mu}{\sigma}\right) \quad X_0 \sim \mathcal{N}(0,1).$$

- a portfolio is constructed by optimizing a mean-variance model, return X_1, with mean μ_1 and standard deviation σ_1.
- any other portfolio: return X_2 with mean μ_2 and standard deviation σ_2.

mean var → max retn → or min var → to be optimal

$$\mu_1 > \mu_2 \quad \text{or} \quad \sigma_1 < \sigma_2.$$

ตัวใดตัวหนึ่งเท่านั้นที่จะเป็นจริง

If $X_1 \sim \mathcal{N}(\mu_1, \sigma_1)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2)$, can we always find some x such that: ได้

std   std ที่ μ_2, σ_2

$$\Pr(X_1 \geq x) = \Pr\left(X_0 \geq \frac{x-\mu_1}{\sigma_1}\right) \geq \Pr\left(X_0 \geq \frac{x-\mu_2}{\sigma_2}\right) = \Pr(X_2 \geq x)?$$

True พร้อมกันไม่ได้ ไม่ make sense

↳ ทำไมมองแบบว่า X_1 ไม่ dominated

Dominance and Mean-Variance Model under Normal Distribution

- optimizing the mean-variance model, portfolio return $X_1 \sim \mathcal{N}(\mu_1, \sigma_1)$.
- any other portfolio: return $X_2 \sim \mathcal{N}(\mu_2, \sigma_2)$.

If $\boxed{\sigma_1 < \sigma_2}$, let

$$x = a + \mu_1 + \frac{\sigma_1(\mu_1 - \mu_2)}{\sigma_2 - \sigma_1} = a + \mu_2 + \frac{\sigma_2(\mu_1 - \mu_2)}{\sigma_2 - \sigma_1}, \quad \boxed{a < 0.}$$

so

$$\frac{x - \mu_1}{\sigma_1} = \frac{a}{\sigma_1} + \frac{\mu_1 - \mu_2}{\sigma_2 - \sigma_1} < \frac{a}{\sigma_2} + \frac{\mu_1 - \mu_2}{\sigma_2 - \sigma_1} = \frac{x - \mu_2}{\sigma_2}.$$

smaller

then

NN x

$$\Pr(X_1 \geq x) = \Pr\left(X_0 \geq \frac{x - \mu_1}{\sigma_1}\right) > \Pr\left(X_0 \geq \frac{x - \mu_2}{\sigma_2}\right) = \Pr(X_2 \geq x).$$

Prob($X_0 \geq$ smaller) > Pr ($X_0 \geq$ ~)

If $\boxed{\mu_1 > \mu_2}$, let $x = \mu_1$,

$$\Pr(X_1 \geq \mu_1) = \Pr(X_0 \geq 0) > \Pr\left(X_0 \geq \frac{\mu_1 - \mu_2}{\sigma_2}\right) = \Pr(X_2 \geq \mu_1).$$

Black-Litterman Model: Motivation

Mean-variance model

$$\max_{\mathbf{x}} \{\mathbf{E}[R(\mathbf{x})] - \delta \mathrm{var}(R(\mathbf{x})) | C\mathbf{x} \leq \mathbf{d}\}$$

where

$$\mathbf{E}[R(\mathbf{x})] = \mu_1 x_1 + \cdots \mu_n x_n, \quad \mathrm{var}(R(\mathbf{x})) = \mathbf{x}^T \Sigma \mathbf{x} = \sum_{i,j} x_i x_j \sigma_{ij},$$

$$\boldsymbol{\mu} \doteq (\mu_1, \cdots, \mu_n) \text{ (mean return)}$$

- Solution is sensitive to values of the expected return ($\boldsymbol{\mu}$).
- Covariance matrix Σ can be estimated with conventional means while estimating the expected return $\boldsymbol{\mu}$ needs more effort.
- Need to include personal view into the estimated values.

Private Opinion

- Example 1: return of stock 1 will outperform that of stock 2 by 5%

$$\mu_1 - \mu_2 = 5\%.$$

μ_1 better than μ_2 ?

- Example 2: average market return will be around 3%

$$\frac{\mu_1 + \cdots + \mu_N}{N} = 3\%$$

- Example 3: the P/E ratio of stock 3 will be around 14:

$$\mu_3 = \frac{14 \times \text{earning}}{\text{current price}} - 1.$$

- Let

$$P = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1/N & 1/N & 1/N & \cdots & 1/N \\ 0 & 0 & 1 & \cdots & 0 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 5\% \\ 3\% \\ \frac{14\times\text{earning}}{\text{current price}} - 1 \end{pmatrix}.$$

$$P\boldsymbol{\mu} = \mathbf{q},$$

Estimate the Expected Returns by Black-Litterman

$$\mu_i = \mathbf{E}[r_i] + v_i, \qquad i = 1, \cdots, N.$$

i.e., expected returns are random variables.
market equilibrium return

$$\mathbf{E}[\mathbf{r}] \doteq (\mathbf{E}[r_1], \cdots, \mathbf{E}[r_n])$$

Assume that $v \sim \mathcal{N}(\mathbf{0}, \tau\Sigma)$ (τ: a small constant):

opinion (ours estimate)

real market tells us

The value of $\boldsymbol{\mu}$ is estimated by

If 2 asset

Error^2

$$\frac{(\mu_1 - E[r_1])^2}{\tau\sigma_1^2} + \frac{(\mu_2 - E[r_2])^2}{\tau\sigma_2^2}$$

$$\min_{\boldsymbol{\mu}} \{(\boldsymbol{\mu} - \mathbf{E}[\mathbf{r}])^T (\tau\Sigma)^{-1} (\boldsymbol{\mu} - \mathbf{E}[\mathbf{r}])\}$$

subject to: $\boxed{P\boldsymbol{\mu} = \mathbf{q}.}$

small constant

Σ: variance metrix of mean (not actual price)

How?!

The solution is

$$\boldsymbol{\mu}^* = \mathbf{E}[\mathbf{r}] + (\tau\Sigma)P^T(P\tau\Sigma P^T)^{-1}(\mathbf{q} - P\mathbf{E}[\mathbf{r}]).$$

Solution to Black-Litterman Model

- Lagrangian: let $\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} - \mathbf{E}[\mathbf{r}]$ and $\tilde{\mathbf{q}} = \mathbf{q} - P\mathbf{E}[\mathbf{r}]$.

must use kkt

$$\mathcal{L}(\boldsymbol{\mu}, \boldsymbol{\lambda}) = \tilde{\boldsymbol{\mu}}^T(\tau\Sigma)^{-1}\tilde{\boldsymbol{\mu}} + \boldsymbol{\lambda}(P\tilde{\boldsymbol{\mu}} - \tilde{\mathbf{q}})$$

- KKT condition: diff → how to diff ??

$$2(\tau\Sigma)^{-1}\tilde{\boldsymbol{\mu}} = P^T\boldsymbol{\lambda}, \quad \text{and} \quad P\tilde{\boldsymbol{\mu}} = \tilde{\mathbf{q}}.$$

still don't know

- Derivation: from the first KKT condition:

$$\tilde{\boldsymbol{\mu}} = (\tau\Sigma)P^T\boldsymbol{\lambda}/2.$$

get λ

apply to the second KKT condtion:

$$P(\tau\Sigma)P^T\boldsymbol{\lambda} = 2\tilde{\mathbf{q}} \quad \text{i.e.,} \quad \boldsymbol{\lambda} = 2[P(\tau\Sigma)P^T]^{-1}\tilde{\mathbf{q}}$$

use the above to eliminate $\boldsymbol{\lambda}$:

$$\tilde{\boldsymbol{\mu}} = (\tau\Sigma)P^T\boldsymbol{\lambda}/2 = (\tau\Sigma)P^T[P(\tau\Sigma)P^T]^{-1}\tilde{\mathbf{q}}$$

i.e.,

$$\boldsymbol{\mu}^* = \mathbf{E}[\mathbf{r}] + (\tau\Sigma)P^T(P\tau\Sigma P^T)^{-1}(\mathbf{q} - P\mathbf{E}[\mathbf{r}]).$$

How to determine $\mathbf{E}[\mathbf{r}]$: CAPM

- Question: will adding a new asset i improve the Sharpe Ratio?
- Question in math:

Asset 1 invest Asset 2 or market rest

$$\max_{x \geq 0} \left\{ \hat{R}(x) \doteq \frac{\mathbf{E}[r_i]x + \mathbf{E}[r_m](1-x) - r_f}{\sqrt{\mathrm{var}(r_i x + r_m(1-x))}} \right\} \implies x^* > 0?$$

r_i return of asset i; r_m: equilibrium return; r_f: risk-free return.

- Answer:

$$\boxed{\frac{d\hat{R}(x)}{dx}\Big|_{x=0} > 0?}$$

- The inequality holds if and only if

$$\mathbf{E}[r_i] > r_f + \frac{\mathrm{cov}(r_i, r_m)}{\mathrm{var}(r_m)}(\mathbf{E}[r_m] - r_f).$$

Deriving $d\hat{R}(x)/dx$

howto to take diff

$$g(x) = r_i x + r_m(1-x) = r_m + (r_i - r_m)x, \quad \hat{R}(x) \doteq \frac{\mathbf{E}[g(x)] - r_f}{\sqrt{\mathrm{var}(g(x))}}.$$

$$\frac{d\hat{R}(x)}{dx} = \frac{\mathbf{E}[r_i] - \mathbf{E}[r_m]}{\sqrt{\mathrm{var}(g(x))}} - \frac{1}{2}\frac{\mathbf{E}[g(x)] - r_f}{\sqrt{(\mathrm{var}(g(x)))^3}}\frac{d\mathrm{var}(g(x))}{dx}$$

since

$$\mathrm{var}(g(x)) = \mathbf{E}\left[(g(x) - \mathbf{E}[g(x)])^2\right]$$

$$\frac{d\mathrm{var}(g(x))}{dx} = 2\mathbf{E}[(g(x) - \mathbf{E}[g(x)]) * ((r_i - \mathbf{E}[r_i]) - (r_m - \mathbf{E}[r_m]))]$$

at $x = 0$, $g(x) = r_m$, so

$$\frac{d\mathrm{var}(g(x))}{dx} = 2\left(\mathrm{cov}(r_i, r_m) - \mathrm{var}(r_m)\right).$$

so

$$\frac{d\hat{R}(x)}{dx} = \frac{1}{\sqrt{\mathrm{var}(r_m)}}\left(\mathbf{E}[r_i] - r_f - \frac{\mathrm{cov}(r_i, r_m)}{\mathrm{var}(r_m)}(\mathbf{E}[r_m] - r_f)\right).$$

Equilibrium Expected Return (Market View)

reason

- more buy of stock i if:

asset i

buy i

$$\mathbf{E}[r_i] > r_f + \frac{\operatorname{cov}(r_i, r_m)}{\operatorname{var}(r_m)}(\mathbf{E}[r_m] - r_f) \implies \frac{d\hat{R}}{dx}|_{x=0} > 0.$$

price goes up, return goes down.

- more sell of stock i if:

people sell i

$$\mathbf{E}[r_i] < r_f + \frac{\operatorname{cov}(r_i, r_m)}{\operatorname{var}(r_m)}(\mathbf{E}[r_m] - r_f) \implies \frac{d\hat{R}}{dx}|_{x=0} < 0.$$

price goes down, return goes up.

- at equilibrium

CAPM model

$$\mathbf{E}[r_i] = r_f + \frac{\operatorname{cov}(r_i, r_m)}{\operatorname{var}(r_m)}(\mathbf{E}[r_m] - r_f) \implies \frac{d\hat{R}}{dx}|_{x=0} = 0.$$