Lecture 4

LP Application: Asset Pricing and Arbitrage

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Example 1: Pricing of a Call Option

call option on a stock

- current price: $S_0 = 40$.
- future price: either rise to $S_1 = 80$ or fall to $S_2 = 20$.
- strike price 50.
- complete market, no tax, no transaction cost, zero interest rate.

replication argument

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call payoff

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3. use the solution to determine positions on cash and stock:

$$x = -10, y = 0.5$$

borrow 10 and buy 0.5 share of the stock.

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3. use the solution to determine positions on cash and stock:

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borrow 10 and buy 0.5 share of the stock.

4. the cost of building these positions is the price for the option:

buying 0.5 share at 40

cost → $$0.5 \times 40 - 10 = 10.$$ ← fair price

get 10 from borrowing

Opportunity I:

call option on a stock with strike price 50:

- current price: $S_0 = 40$.
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we shall buy

if option price $\boxed{p < 10}$ how do you make a guaranteed profit from it.

decision now:

- option: buy or short?

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future balance:

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future balance:

- $S_1 = 80$: option (30), stock (-40), debt (10).

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future balance:

- $S_1 = 80$: option (30), stock (-40), debt (10).
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Opportunity II

call option on a stock with strike price 50:

- current price: $S_0 = 40$.
- future price: rise to $S_1 = 80$ or fall to $S_2 = 20$.

if option price $p > 10$, how do you make a guaranteed profit from it.

decision now:

- option: buy or short?

Opportunity II

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above fair price

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- $S_1 = 80$: option (-30), stock (40), debt (-10).

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Generalization

- a set of n financial assets.

  e.g., {cash, stock, option}, indexed by 1, 2, and 3 respectively.
- c_i: current price of asset i $(i = 1, ..., n)$

  e.g., $c_1 = 1,\ c_2 = 40,\ c_3 = p$ (price of the option).
- v_i^s: future value of asset i $(i = 1, ..., n)$ in scenario s.

  e.g., two possible scenarios, h(igh) and l(ow)

$$(v_1^h, v_2^h, v_3^h) = (1, 80, 30) \text{ and } (v_1^l, v_2^l, v_3^l) = (1, 20, 0).$$

decision: take position x_i in asset i $(i = 1, ..., n)$

- $x_i > 0$: long (buy, lend); $x_i < 0$: short (borrow, sell).
- total cost of building the position:

$$c_1 x_1 + ... + c_n x_n.$$

- future income in scenario s:

$$v_1^s x_1 + ... + v_n^s x_n.$$

LP and Type-A Arbitrage

Type-A Arbitrage: strictly negative cost to build the initial position and break-even in each of all future scenarios.

using an LP to detect type-A arbitrage opportunities:

$$\min_{x_1,\ldots,x_n} \{c_1 x_1 + \ldots + c_n x_n\}$$
$$\text{subject to} \qquad v_1^s x_1 + \ldots + v_n^s x_n \geq 0 \text{ for all } s.$$

(c_i: current asset price, v_i^s: future asset value in scenario s, x_i: positions)

- the LP always has a feasible solution

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$$x_1 = x_2 = \ldots. = x_n = 0.$$

- type-A arbitrage opportunity: a feasible solution $(x_1^*, \ldots, x_n^*)$ such that

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- the LP always has a feasible solution

$$x_1 = x_2 = \dots = x_n = 0.$$

- type-A arbitrage opportunity: a feasible solution $(x_1^*, \dots, x_n^*)$ such that

$$c_1 x_1^* + \dots + c_n x_n^* < 0.$$

$\hookrightarrow$ optimal $= -\infty$

Example 1: Case with No Arbitrage Opportunity

In the example above, LP simplifies to

$$\min_{x_1,x_2,x_3} \{x_1 + 40x_2 + px_3\}$$

$$\text{subject to} \qquad \begin{cases} x_1 + 80x_2 + 30x_3 & \geq 0, \\ x_1 + 20x_2 & \geq 0. \end{cases}$$

if $p = 10$, then

risk neutral prob

$$\begin{aligned} x_1 + 40x_2 + px_3 &= x_1 + 40x_2 + 10x_3 \\ &= \frac{1}{3}(x_1 + 80x_2 + 30x_3) + \frac{2}{3}(x_1 + 20x_2) \\ &\geq 0. \end{aligned}$$

so the optimal solution

$$x_1^* + 40x_2^* + 10x_3^* = 0,$$

no arbitrage opportunity.

Example 1: Cases with Type-A Arbitrage Opportunity

$$\min_{x_1, x_2, x_3} \{x_1 + 40x_2 + px_3\}$$

$$\text{subject to} \qquad \begin{cases} x_1 + 80x_2 + 30x_3 & \geq 0, \\ x_1 + 20x_2 & \geq 0. \end{cases}$$

observation: for any given x_2,

$$x_1 = -20x_2, x_3 = -2x_2$$

satisfy both constraints, and give rise to the objective function

$$(20 - 2p)x_2$$

- if $p < 10$,

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  $$x_2 < 0, \ x_1 = -20x_2 > 0, \text{ and } x_3 = -2x_2 > 0,$$

  i.e., short stock, lend cash, and buy option.

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$$(20 - 2p)x_2$$

- if $p < 10$,
$$x_2 < 0,\ x_1 = -20x_2 > 0,\ \text{and } x_3 = -2x_2 > 0,$$
i.e., short stock, lend cash, and buy option.
- if $p > 10$,
$$x_2 > 0,\ x_1 = -20x_2 < 0,\ \text{and } x_3 = -2x_2 < 0,$$
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Question: what is the optimal value of x_2? what is the optimal objective value?

Example 2

call option on a stock

- current price: $S_0 = 40$.
- future price: rise to $S_1 = 80$ or fall to $S_2 = 20$.
- price of the option: 10.
- complete market, no tax, no transaction cost, and zero interest

using a replicate argument to set the strike price

1. instead of buying/selling an option, hold x amount of cash and y shares of the stock to replicate current cost

$$x + 40y = 10$$

2. the position should also replicate future outcome of the option.

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$K \geq 80$

Atm call = 0

$$\begin{aligned} x + 80y &= 0 \\ x + 20y &= 0 \end{aligned}$$

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$$K \geq 80 \qquad\qquad 0 < K \leq 20$$

$$\begin{aligned} x + 80y &= 0 \\ x + 20y &= 0 \end{aligned} \qquad\qquad \begin{aligned} x + 80y &= 80 - K \\ x + 20y &= 20 - K \end{aligned}$$

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$K \geq 80$ $\qquad$ $0 < K \leq 20$ $\qquad$ $20 < K < 80$

$$\begin{aligned} x + 80y &= 0 \\ x + 20y &= 0 \end{aligned} \qquad \begin{aligned} x + 80y &= 80 - K \\ x + 20y &= 20 - K \end{aligned}$$

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- future price: rise to $S_1 = 80$ or fall to $S_2 = 20$.
- price of the option: 10.
- complete market, no tax, no transaction cost, and zero interest

using a replicate argument to set the strike price

1. instead of buying/selling an option, hold x amount of cash and y shares of the stock to replicate current cost
$$x + 40y = 10$$
2. the position should also replicate future outcome of the option.

$K \geq 80$	$0 < K \leq 20$	$20 < K < 80$
$\begin{aligned} x + 80y &= 0 \\ x + 20y &= 0 \end{aligned}$	$\begin{aligned} x + 80y &= 80 - K \\ x + 20y &= 20 - K \end{aligned}$	$\begin{aligned} x + 80y &= 80 - K \\ x + 20y &= 0 \end{aligned}$

Example 2: Arbitrage-Free Strike Price

1. if $K \geq 80$:

$$x + 40y = 10, \quad x + 80y = 0, \quad x + 20y = 0,$$

$$\longrightarrow \quad \text{no solution}$$

Example 2: Arbitrage-Free Strike Price

1. if $K \geq 80$:

$$x + 40y = 10, \quad x + 80y = 0, \quad x + 20y = 0,$$
$$\longrightarrow \quad \text{no solution}$$

contradict

2. if $0 < K \leq 20$:

$$x + 40y = 10, \quad x + 80y = 80 - K, \quad x + 20y = 20 - K,$$
$$\longrightarrow \quad x = -30, \quad y = 1, \quad K = 30$$

Example 2: Arbitrage-Free Strike Price

1. if $K \geq 80$:

$$x + 40y = 10, \quad x + 80y = 0, \quad x + 20y = 0,$$
$$\longrightarrow \quad \text{no solution}$$

2. if $0 < K \leq 20$:

$$x + 40y = 10, \quad x + 80y = 80 - K, \quad x + 20y = 20 - K,$$
$$\longrightarrow \quad x = -30, \quad y = 1, \quad K = 30$$

3. if $20 < K < 80$:

$$x + 40y = 10, \quad x + 80y = 80 - K, \quad x + 20y = 0,$$
$$\longrightarrow \quad x = -10, \quad y = 0.5, \quad K = 50$$

The only arbitrage-free strike price is: $K = 50$.

Arbitrage Opportunity

if strike price $20 < K < 50$:

- now: buy option (-10), lend cash (-10), and short $1/2$ share (20).
- future:
  1. $S_1 = 80$: option $(80 - K)$, debt (10), and $1/2$ share (-40).
     debt - short

     profit: $80 - K + 10 - 40 > 50 - K > 0.$
  2. $S_2 = 20$: collect debt (10), return $1/2$ share (-10),

Arbitrage Opportunity

if strike price $20 < K < 50$:

- now: buy option (-10), lend cash (-10), and short $1/2$ share (20).
- future:
  1. $S_1 = 80$: option $(80 - K)$, debt (10), and $1/2$ share (-40).
     $$\text{profit:} \quad 80 - K + 10 - 40 > 50 - K > 0.$$
  2. $S_2 = 20$: collect debt (10), return $1/2$ share (-10),

if strike price $50 < K < 80$:

- now: short an option (10), borrow cash (10), buy $1/2$ share (-20).
- future:
  1. $S_1 = 80$: option $-(80 - K)$, debt (-10), $1/2$ share (40).
     $$\text{profit:} \quad -(80 - K) - 10 + 40 = K - 50 > 0$$
  2. $S_2 = 20$: debt (-10) and $1/2$ share (10).

Type-B Arbitrage

Type-B arbitrage: no cost to build the initial position, at least break-even in all future scenarios, and strictly profitable in some cases.

using LP to detect type-B arbitrage opportunities:

$$\begin{aligned} & \max_{x_1,\dots,x_n} \left\{ \sum_s (v_1^s x_1 + \dots + v_n^s x_n) \right\} \\ \text{subject to} \quad & c_1 x_1 + c_2 x_2 + \dots + c_n x_n \le 0, \\ & v_1^s x_1 + \dots + v_n^s x_n \ge 0 \quad \text{for all } s. \end{aligned}$$

(c_i: current asset price, v_i^s: future asset value in scenario s, x_i: positions)

- the LP always has a feasible solution

$$x_1 = x_2 = \dots = x_n = 0$$

- type-B arbitrage opportunity: a feasible solution $(x_1^*, \dots, x_n^*)$ such that

$$\sum_s (v_1^s x_1^* + \dots + v_n^s x_n^*) > 0.$$

Type-B Arbitrage: Example 2

$p = \$10$ and $20 < K < 80$, the LP becomes

$$\max_{x_1, x_2, x_3} \{2x_1 + 100x_2 + (80 - K)x_3\}$$

$$\text{subject to} \qquad \begin{cases} x_1 + 40x_2 + 10x_3 & \leq 0, \\ x_1 + 80x_2 + (80 - K)x_3 & \geq 0, \\ x_1 + 20x_2 & \geq 0. \end{cases}$$

Type-B Arbitrage: Example 2

$p = \$10$ and $20 < K < 80$, the LP becomes

$$\max_{x_1, x_2, x_3} \{2x_1 + 100x_2 + (80 - K)x_3\}$$

$$\text{subject to} \qquad \begin{cases} x_1 + 40x_2 + 10x_3 & \leq 0, \\ x_1 + 80x_2 + (80 - K)x_3 & \geq 0, \\ x_1 + 20x_2 & \geq 0. \end{cases}$$

we can write the objective function as

$$\begin{aligned} & 2x_1 + 100x_2 + (80 - K)x_3 \\ = \quad & 3(x_1 + 40x_2 + 10x_3) - (x_1 + 20x_2) + (50 - K)x_3 \end{aligned}$$

Type-B Arbitrage: Example 2

Lecture 4

Qiong Wang

$p = \$10$ and $20 < K < 80$, the LP becomes

$$\max_{x_1, x_2, x_3} \{2x_1 + 100x_2 + (80 - K)x_3\}$$

$$\text{subject to} \quad \begin{cases} x_1 + 40x_2 + 10x_3 & \leq 0, \\ x_1 + 80x_2 + (80 - K)x_3 & \geq 0, \\ x_1 + 20x_2 & \geq 0. \end{cases}$$

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กำจัด*

if $K = 50$:

$$\begin{aligned} & 2x_1 + 100x_2 + (80 - K)x_3 \\ = \; & 3(x_1 + 40x_2 + 10x_3) - (x_1 + 20x_2) \leq 0 \qquad \text{(why?)} \end{aligned}$$

initial cost

so the best objective value is zero, no arbitrage opportunity.

Example 2: Cases with Type-B Arbitrage Opportunity

$$\max_{x_1,x_2,x_3} \{2x_1 + 100x_2 + (80-K)x_3\}$$

$$\text{subject to} \quad \begin{cases} x_1 + 40x_2 + 10x_3 & \leq 0 \\ x_1 + 80x_2 + (80-K)x_3 & \geq 0, \\ x_1 + 20x_2 & \geq 0. \end{cases}$$

- let $x_1 = -20x_2$ and $x_3 = -2x_2$, the obj becomes

$$2x_1 + 100x_2 + (80-K)x_3 = (50-K)x_3 = 2(K-50)x_2$$

- all constraints are satisfied if

$$(K-50)x_2 \geq 0 \quad \text{(why?)}$$

- strictly positive profit (arbitrage opportunity):
  - if $K > 50$ (buy stock, borrow cash, and short option):

  $$x_2 > 0, x_1 = -20x_2 < 0, x_3 = -2x_2 < 0.$$

  - if $K < 50$ (do exactly the opposite):

  $$x_2 < 0, x_1 = -20x_2, x_3 = -2x_2.$$

Quotes of SP500 Options at CBOE

Do we have an arbitrage opportunity?

Calls — SEPTEMBER 2018 (EXPIRATION: 09/28)

Strike	Last	Net	Bid	Ask	Vol	Int
SPXW1828I2870-E	34.65	+2.40	31.80	32.30	19	1149
SPXW1828I2875-E	29.65	-0.09	28.50	29.00	12	13859
SPXW1828I2880-E	27.85	+1.85	25.30	25.80	62	1063
SPXW1828I2885-E	24.00	+0.40	22.30	22.70	167	2503

Puts — SEPTEMBER 2018 (EXPIRATION: 09/28)

Strike	Last	Net	Bid	Ask	Vol	Int
SPXW1828U2870-E	21.50	-6.05	21.50	21.90	39	1175
SPXW1828U2875-E	23.30	-5.38	23.20	23.60	441	5768
SPXW1828U2880-E	24.52	-6.98	24.90	25.40	31	1621
SPXW1828U2885-E	26.15	-10.17	26.90	27.40	2	201

Calls — OCTOBER 2018 (EXPIRATION: 10/01)

Strike	Last	Net	Bid	Ask	Vol	Int
SPXW1801J2870-E	36.97	0.0	32.90	33.50	0	9
SPXW1801J2875-E	30.90	+1.82	29.60	30.20	10	26
SPXW1801J2880-E	30.00	+3.80	26.50	27.00	1	51
SPXW1801J2885-E	24.90	+2.20	23.40	23.90	101	514

Puts — OCTOBER 2018 (EXPIRATION: 10/01)

Strike	Last	Net	Bid	Ask	Vol	Int
SPXW1801V2870-E	29.60	0.0	22.40	23.00	0	77
SPXW1801V2875-E	24.11	-4.85	24.00	24.70	130	99
SPXW1801V2880-E	30.90	0.0	25.80	26.50	0	134
SPXW1801V2885-E	31.00	0.0	27.80	28.40	0	69

Calls — OCTOBER 2018 (EXPIRATION: 10/03)

Strike	Last	Net	Bid	Ask	Vol	Int
SPXW1803J2870-E	35.60	0.0	34.70	35.40	0	33
SPXW1803J2875-E	31.95	0.0	31.40	32.10	0	119
SPXW1803J2880-E	27.90	0.0	28.30	28.80	0	23
SPXW1803J2885-E	25.85	0.0	25.30	25.80	0	21

Puts — OCTOBER 2018 (EXPIRATION: 10/03)

Strike	Last	Net	Bid	Ask	Vol	Int
SPXW1803V2870-E	25.65	-2.85	24.00	24.60	1	51
SPXW1803V2875-E	25.13	-6.07	25.60	26.20	10	317
SPXW1803V2880-E	25.89	-10.11	27.40	28.10	16	277
SPXW1803V2885-E	27.65	-0.60	29.40	30.00	2	35

Example

Suppose that we only buy and sell the following four European call options:

Calls SEPTEMBER 2018 (EXPIRATION: 09/19)

Strike	Last	Net	Bid	Ask	Vol	Int
SPXW18191I2870-E	23.40	0.0	22.90	23.40	0	83
SPXW18191I2875-E	23.00	+3.00	19.70	20.10	1	695
SPXW18191I2880-E	17.20	-3.80	16.60	17.00	20	280
SPXW18191I2885-E	15.70	+0.89	13.80	14.30	130	330

- all expire on 09/19/2018

  strike prices are 2870, 2875, 2880, 2885.

Future Values of the Portfolio

S: possible future prices of the index on 09/19/2018, $S \geq 0$.

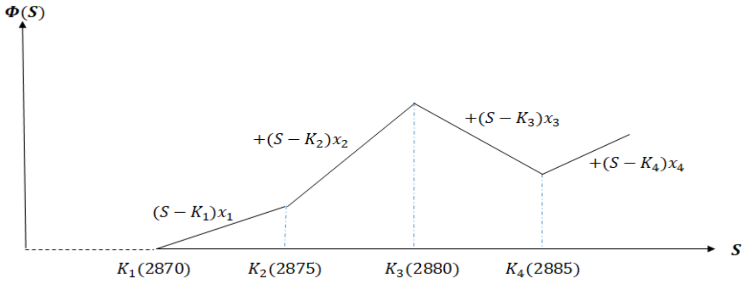
Choose positions (x_1, x_2, x_3, x_4) to keep $\Phi(S) \geq 0$ for all S

$$
\begin{aligned}
2870 < S \leq 2875 : & \\
\Phi(S) &= (S - 2870)x_1 \\
2875 < S \leq 2880 : & \\
\Phi(S) &= (S - 2870)x_1 + (S - 2875)x_2 \\
2880 < S \leq 2885 : & \\
\Phi(S) &= (S - 2870)x_1 + (S - 2875)x_2 + (S - 2880)x_3 \\
S \geq 2885 : & \\
\Phi(S) &= (S - 2870)x_1 + (S - 2875)x_2 \\
&\quad + (S - 2880)x_3 + (S - 2885)x_4
\end{aligned}
$$

.

Future Value at Different Price Levels

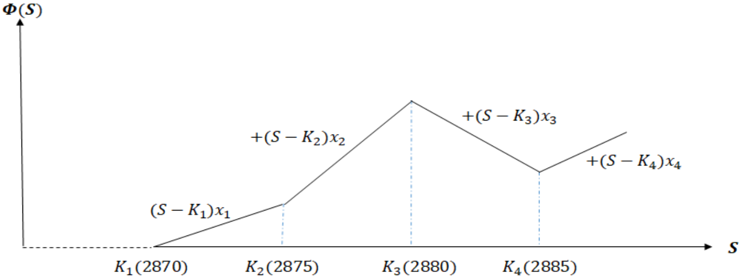
Given x_1, x_2, x_3, x_4:



How to keep $\Phi(S)$ nonnegative for all S?

Future Value at Different Price Levels

Given x_1, x_2, x_3, x_4:

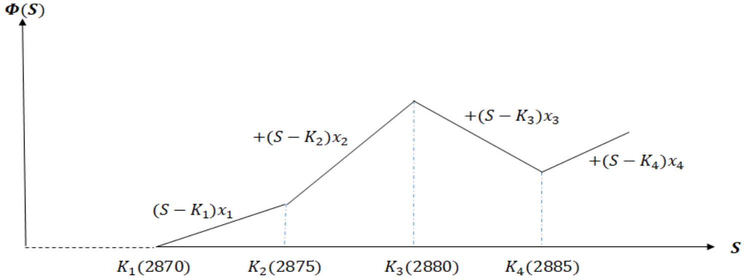


How to keep $\Phi(S)$ nonnegative for all S?

$$\Phi(K_1) \geq 0, \ \Phi(K_2) \geq 0, \ \Phi(K_3) \geq 0, \ \Phi(K_4) \geq 0.$$

Future Value at Different Price Levels

Given x_1, x_2, x_3, x_4:



How to keep $\Phi(S)$ nonnegative for all S?

$$\Phi(K_1) \geq 0, \;\; \Phi(K_2) \geq 0, \;\; \Phi(K_3) \geq 0, \;\; \Phi(K_4) \geq 0.$$

and

$$\Phi(K_4 + 1) - \Phi(K_4) \geq 0.$$

No Loss Constraints

$$\Phi(S) = (S - K_1)^+ x_1 + (S - K_2)^+ x_2 + (S - K_3)^+ x_3 + (S - K_4)^+ x_4.$$

$$K_1 = 2870, \quad K_2 = 2875, \quad K_3 = 2880, \quad K_4 = 2885.$$

Only need five points of S to control the entire curve:

No Loss Constraints

$$\Phi(S) = (S - K_1)^+ x_1 + (S - K_2)^+ x_2 + (S - K_3)^+ x_3 + (S - K_4)^+ x_4.$$

$$K_1 = 2870, \quad K_2 = 2875, \quad K_3 = 2880, \quad K_4 = 2885.$$

Only need five points of S to control the entire curve:

1. $S = K_1, \quad \Phi(K_1) \geq 0 \quad \longrightarrow \quad$ trivial,

No Loss Constraints

$$\Phi(S) = (S - K_1)^+ x_1 + (S - K_2)^+ x_2 + (S - K_3)^+ x_3 + (S - K_4)^+ x_4.$$

$$K_1 = 2870, \quad K_2 = 2875, \quad K_3 = 2880, \quad K_4 = 2885.$$

Only need five points of S to control the entire curve:

1. $S = K_1, \quad \Phi(K_1) \geq 0 \quad \longrightarrow \quad$ trivial,
2. $S = K_2, \quad \Phi(K_2) \geq 0 \quad \longrightarrow \quad 5x_1 \geq 0,$

No Loss Constraints

$$\Phi(S) = (S - K_1)^+ x_1 + (S - K_2)^+ x_2 + (S - K_3)^+ x_3 + (S - K_4)^+ x_4.$$

$$K_1 = 2870, \quad K_2 = 2875, \quad K_3 = 2880, \quad K_4 = 2885.$$

Only need five points of S to control the entire curve:

1. $S = K_1, \quad \Phi(K_1) \geq 0 \quad \longrightarrow \quad$ trivial,
2. $S = K_2, \quad \Phi(K_2) \geq 0 \quad \longrightarrow \quad 5x_1 \geq 0,$
3. $S = K_3, \quad \Phi(K_3) \geq 0 \quad \longrightarrow \quad 10x_1 + 5x_2 \geq 0,$

No Loss Constraints

$$\Phi(S) = (S - K_1)^+ x_1 + (S - K_2)^+ x_2 + (S - K_3)^+ x_3 + (S - K_4)^+ x_4.$$

$$K_1 = 2870, \quad K_2 = 2875, \quad K_3 = 2880, \quad K_4 = 2885.$$

Only need five points of S to control the entire curve:

1. $S = K_1, \quad \Phi(K_1) \geq 0 \quad \longrightarrow \quad$ trivial,
2. $S = K_2, \quad \Phi(K_2) \geq 0 \quad \longrightarrow \quad 5x_1 \geq 0,$
3. $S = K_3, \quad \Phi(K_3) \geq 0 \quad \longrightarrow \quad 10x_1 + 5x_2 \geq 0,$
4. $S = K_4, \quad \Phi(K_4) \geq 0 \quad \longrightarrow \quad 15x_1 + 10x_2 + 5x_3 \geq 0,$

No Loss Constraints

$$\Phi(S) = (S - K_1)^+ x_1 + (S - K_2)^+ x_2 + (S - K_3)^+ x_3 + (S - K_4)^+ x_4.$$

$$K_1 = 2870, \quad K_2 = 2875, \quad K_3 = 2880, \quad K_4 = 2885.$$

Only need five points of S to control the entire curve:

1. $S = K_1, \quad \Phi(K_1) \geq 0 \quad \longrightarrow \quad$ trivial,
2. $S = K_2, \quad \Phi(K_2) \geq 0 \quad \longrightarrow \quad 5x_1 \geq 0,$
3. $S = K_3, \quad \Phi(K_3) \geq 0 \quad \longrightarrow \quad 10x_1 + 5x_2 \geq 0,$
4. $S = K_4, \quad \Phi(K_4) \geq 0 \quad \longrightarrow \quad 15x_1 + 10x_2 + 5x_3 \geq 0,$
5. $S = K_4 + 1, \quad \Phi(K_4 + 1) - \Phi(K_4) \geq 0 \quad \longrightarrow \quad x_1 + x_2 + x_3 + x_4 \geq 0.$

Data,Model, and Results

Calls SEPTEMBER 2018 (EXPIRATION: 09/19)

Strike	Last	Net	Bid	Ask	Vol	Int
SPXW1819I2870-E	23.40	0.0	22.90	23.40	0	83
SPXW1819I2875-E	23.00	+3.00	19.70	20.10	1	695
SPXW1819I2880-E	17.20	-3.80	16.60	17.00	20	280
SPXW1819I2885-E	15.70	+0.89	13.80	14.30	130	330

$$\begin{aligned}
\min_{x_1,x_2,x_3,x_4} \quad & \{23.4x_1 + 23x_2 + 17.2x_3 + 15.7x_4\} \\
\text{subject to:} \quad & 5x_1 \geq 0, \\
& 10x_1 + 5x_2 \geq 0, \\
& 15x_1 + 10x_2 + 5x_3 \geq 0, \\
& x_1 + x_2 + x_3 + x_4 \geq 0.
\end{aligned}$$

arbitrage opportunity, e.g.,

$$x_1 > 0, \;\; x_2 = -2x_1, x_3 = x_1, \;\; x_4 = 0.$$

More Realistically.....

Calls

SEPTEMBER 2018 (EXPIRATION: 09/19)

Strike	Last	Net	Bid	Ask	Vol	Int
SPXW181912870-E	23.40	0.0	22.90	23.40	0	83
SPXW181912875-E	23.00	+3.00	19.70	20.10	1	695
SPXW181912880-E	17.20	-3.80	16.60	17.00	20	280
SPXW181912885-E	15.70	+0.89	13.80	14.30	130	330

- x_i^+: number of options to buy, $x_i^+ \geq 0$,
  x_i^-: number of options to sell, $x_i^- \geq 0$.
- objective function (cost of building your positions)

$$23.4x_1^+ - 22.9x_1^- + 20.10x_2^+ - 19.70x_2^- + 17x_3^+ - 16.6x_3^- + 14.3x_4^+ - 13.8x_4^-$$

LP Formulation in the Presence of Bid-Ask Spread

x_i^+: number of options to buy,
x_i^-: number of options to sell.

minimize:

$$23.4x_1^+ - 22.9x_1^- + 20.10x_2^+ - 19.70x_2^- + 17x_3^+ - 16.6x_3^- + 14.3x_4^+ - 13.8x_4^-$$

subject to:

$$\begin{aligned}
&5(x_1^+ - x_1^-) \geq 0, \\
&10(x_1^+ - x_1^-) + 5(x_2^+ - x_2^-) \geq 0, \\
&15(x_1^+ - x_1^-) + 10(x_2^+ - x_2^-) + 5(x_3^+ - x_3^-) \geq 0, \\
&(x_1^+ - x_1^-) + (x_2^+ - x_2^-) + (x_3^+ - x_3^-) + (x_4^+ - x_4^-) \geq 0, \\
&x_i^+ \geq 0, \;\; x_i^- \geq 0, \;\; i = 1, 2, 3, 4.
\end{aligned}$$

Primal-Dual Transformation

what is the dual model of our LP?

$$\begin{aligned} & \min_{x_1,\dots,x_n} \{c_1 x_1 + \dots + c_n x_n\}, \\ \text{subject to} \quad & v_1^s x_1 + \dots + v_n^s x_n \geq 0, \;\; s = 1, \dots, S. \end{aligned}$$

where S is the set of all possible scenarios.

observe that the LP is somewhat different from the standard format because

- the right-hand side of every constraints is zero.
- there is no non-negativity requirement on $x_1, \dots, x_n$, so we need to rewrite the LP (doubling variables) as

$$\begin{aligned} & \min_{x_1^+, x_1^-, \dots, x_n^+, x_n^-} \left\{ c_1(x_1^+ - x_1^-) + \dots + c_n(x_n^+ - x_n^-) \right\}. \\ \text{subject to} \quad & v_1^s(x_1^+ - x_1^-) + \dots + v_n^s(x_n^+ - x_n^-) \geq 0, \;\; s = 1, \dots, S. \\ & x_1^+ \geq 0, x_1^- \geq 0, \dots\dots, x_n^+ \geq 0, x_n^- \geq 0. \end{aligned}$$

Dual LP of Example 1

following this approach, LP for our example 1 (p:option price)

$$\min_{x_1,x_2,x_3} \{x_1 + 40x_2 + px_3\}$$

$$\begin{aligned} \text{subject to} \qquad & x_1 + 80x_2 + 30x_3 \geq 0, \\ & x_1 + 20x_2 \qquad\quad \geq 0. \end{aligned}$$

is transformed into

$$\min_{x_i^+,x_i^-,i=1,2,3} \left\{(x_1^+ - x_1^-) + 40(x_2^+ - x_2^-) + p(x_3^+ - x_3^-)\right\}$$

$$\begin{aligned} \text{subject to} \qquad & (x_1^+ - x_1^-) + 80(x_2^+ - x_2^-) + 30(x_3^+ - x_3^-) \geq 0, && (1) \\ & (x_1^+ - x_1^-) + 20(x_2^+ - x_2^-) \qquad\qquad\qquad\quad \geq 0, && (2) \\ & x_i^+ \geq 0, \ x_i^- \geq 0, \ i = 1, 2, 3. \end{aligned}$$

dual LP: variable and objective

- y_1 and y_2 dual variables associated with (**??**) and (**??**) respectively,
- objective function:

$$\max_{y_1,y_2}\{0 \times y_1 + 0 \times y_2\}$$

Dual Solution of Example 1

dual LP: constraints:

1. associated with x_1^+ and x_1^-:

$$y_1 + y_2 \leq 1 \text{ and } -y_1 - y_2 \leq -1 \quad \longrightarrow y_1 + y_2 = 1.$$

2. associated with x_2^+ and x_2^-:

$$80y_1 + 20y_2 \leq 40 \text{ and } -80y_1 - 20y_2 \leq -40 \longrightarrow 80y_1 + 20y_2 = 40.$$

3. associated with x_3^+ and x_3^-:

$$30y_1 \leq p \text{ and } -30y_1 \leq -p \longrightarrow 30y_1 = p.$$

Dual LP:

$$\max_{y_1, y_2}\{0\}$$

$$\text{subject to} \qquad \left.\begin{array}{rl} y_1 + y_2 & = 1 \\ 80y_1 + 20y_2 & = 40 \end{array}\right\} \longrightarrow y_1 = 1/3, y_2 = 2/3,$$

$$30y_1 = p,$$

$$y_1 \geq 0, \;\; y_2 \geq 0.$$

- every feasible solution is an optimal solution of this LP.
- feasible solution exists only if $p = 10$, the price that excludes arbitrage.

Risk Neutral Probabilities

$$\min_{x_1,x_2,x_3} \{x_1 + 40x_2 + px_3\}$$

$$\begin{cases} x_1 + 80x_2 + 30x_3 \geq 0, \\ \\ x_1 + 20x_2 \qquad\quad \geq 0. \end{cases}$$

0 (no arbitrage) if and only if

$$p = 10.$$

$$\max_{y_1,y_2} \{0\}$$

$$\begin{cases} y_1 + y_2 = 1, \\ 80y_1 + 20y_2 = 40, \\ 30y_1 = p, \\ y_1 \geq 0, \ \ y_2 \geq 0. \end{cases}$$

the problem is feasible if and only if

$$p = 10.$$

if (y_1, y_2) is feasible, then

1. $y_1 \geq 0$, $y_2 \geq 0$, and $y_1 + y_2 = 1$.
2. stock and option prices equal assets' weighted (by y_1 and y_2) future values:
$$40 = 80y_1 + 20y_2 \text{ and } 10 = 30y_1.$$
3. from complementary slackness condition
$$y_1(x_1 + 80x_2 + 30x_3) = 0 \text{ and } y_2(x_1 + 20x_2) = 0$$

what do we call y_1 and y_2 in asset pricing theory?

First Fundamental Theorem of Asset Pricing

- given

  n: types of assets;
  c_i: current price;
  S: the set of all future scenarios;
  v_i^s: future asset value in scenario s.

- no arbitrage opportunity if and only if there exists risk neutral probabilities

  $$y_1 \geq 0, ..., y_S \geq 0,$$

  where for each asset $i = 1, ..., n$,

  $$y_1 v_i^1 + ... + y_S v_i^S = c_i$$

  (when the asset is cash, the equation specializes to $y_1 + ... + y_S = 1$).

Arbitrage and LP Duality Theory

$$\min_{x_1,\dots,x_n} \{c_1x_1 + ... + c_nx_n\},$$

$$\text{subject to} \qquad v_1^s x_1 + ... + v_n^s x_n \geq 0, \;\; s = 1, ..., S.$$

dual problem

$$\max_{\mathbf{y}}\{0\}$$

subject to: for $i = 1, \cdots, n$,

$$\sum_s v_i^s y_s \leq c_i, \quad -\sum_s v_i^s y_s \leq -c_i \quad \longrightarrow \quad \sum_s v_i^s y_s = c_i.$$

- weak and strong duality:

  $$0 \leq c_1x_1 + ... + c_nx_n$$

  either

  $$c_1x_1^* + \cdots + c_nx_n^* = 0.$$

  or one LP is unbounded and the other is infeasible.
- Complementary slackness condition: if the solution exists, then

  $$y_s^*(v_1^s x_1^* + \cdots + v_n^s x_n^*) = 0$$