S.P.

Homework 4: Due on Thursday, September 28, 50 points

Problem 1: (12 points) Suppose that you have the opportunity to buy a commodity for \$1 per unit in one market and resell it for $\$(1 + b(y))$ per unit in another market. Here y is the amount you buy and sell and

$$b(y) = \frac{1}{(100 + y)^2}.$$

Questions:

1. (6 points) Determine the amount to buy and sell to maximize your profit.
2. (6 points) Now suppose that you can buy at most $\bar{y}$ units in the first market, so your profit maximization is constrained by

   $$y \leq \bar{y}.$$

   Determine the value of the Lagrangian multiplier attached to that constraint for two cases: $\bar{y} = 75$ and $\bar{y} = 150$.

1) Maximize profit

sell price

buy price

$$P(y) = \text{Profit} = y \times (\underline{1 + b(y)} - 1)$$

$$= y\, b(y)$$

$$= \frac{y}{(100+y)^2}$$

$$\left(\frac{y}{(100+y^2)}\right)' = 0$$

$$\frac{-y+100}{(y+100)^3} = 0 \quad ; \; y \geq 0$$

$$\boxed{y = 100}$$

← $y = 100$ to maximize profit

*

2)

case ①: max $P(y)$

$$y \le 75 \rightarrow \boxed{\overset{\lambda}{7}5-y \ge 0} \quad g(x)$$

$$\max \; P(y) + \underbrace{\lambda(75-y)}_{}{}^{=0}$$

$$\lambda \ge 0 \, , \; y \ge 0$$

$$g(x) = \frac{y}{(100+y)^2} + \lambda(75-y)$$

$$= \frac{y}{(100+y)^2} + \lambda 75 - \lambda y =$$

$$F_y = \frac{-y+100}{(100+y)^3} + 0 - \lambda = 0$$

$$\frac{-y+100}{(100+y)^3} = \lambda \rightarrow \boxed{\lambda = \frac{-75+100}{(100+75)^3}}$$

$$F_\lambda = 0 + 75 - y = 0$$

$$\boxed{\lambda = \frac{25}{175^3} \ge 0} \text{ optimal} \qquad \boxed{y = 75} \text{ optimal sol}$$

$$\lambda = \frac{25}{(175)^3} \ge 0 \; * \text{ optimal}$$

Case 2 $\quad y \le 150$

$$\max \; P(y) + \lambda(150 - y)$$

$$L(y,\lambda) = \frac{y}{(100+y)^2} + \lambda(150 - y)$$

$\therefore \; \lambda \geq 0,$

$$F_y = \frac{-y+100}{(100+y)^3} + 0 - \lambda = 0$$

$$F_y = 0 + 150 - y \geq 0$$

$y = 150$ ← feasible

but optmal is 100 so constrant not active *

$$\lambda = \frac{-y+100}{(100+y)^3} \rightarrow \boxed{\lambda = \frac{-50}{(250)^3}} \therefore \boxed{\lambda < 0}$$

not optimal *

try with constrant not active will get optimal sol when $\boxed{y = 100}$ *

* so $\boxed{\lambda_2 = 0}$ #

Problem 2: (16 points) Suppose that you have a budget of 50 million, and opportunities to invest in three different projects. The return (excluding the invested amount) from each project is

$$r_1 = \frac{10x_1}{1+x_1}, r_2 = \sqrt{x_2}, \text{and } r_3 = 10(1-e^{-x_3}).$$

where x_i is the amount invested in project i $(i = 1, 2, 3)$.

Questions:

1. (4 points) Formulate a nonlinear optimization model to maximize profit.
2. (4 points) Develop the Lagrangian of the problem and KKT conditions.
3. (4 points) Let λ be the Lagrange multiplier attached to the budget constraint
$$x_1 + x_2 + x_3 \leq 50.$$
Use KKT conditions to express the optimal values of x_i $(i = 1, 2, 3)$ as functions of λ (notice by these conditions, all $x_i > 0$ at the optimum).
4. (4 points) Solve the KKT conditions to determine the value of λ first, and then use this value to determine the optimal investments.

1) Maximize $r_1 + r_2 + r_3$

s.t. $x_1 + x_2 + x_3 \leq 50$ million

$x_i \geq 0$

2) $L(x_1, x_2, x_3, \lambda)$

$\nabla L = 0$

$\lambda \geq 0$

$x_i \geq 0, \forall i$

3) $L(x_1, x_2, x_3, \lambda) = \dfrac{10x_1}{1+x_1} + \sqrt{x_2} + 10(1-e^{-x_3}) + \lambda(50 - (x_1+x_2+x_3))$

$\nabla L = 0 \rightarrow F_{x_1} = \dfrac{10}{(1+x_1)^2} - \lambda = 0 \rightarrow \dfrac{10}{(1+x_1)^2} = \lambda \rightarrow \dfrac{10}{\lambda} = (1+x_1)^2$

$\pm\sqrt{\dfrac{10}{\lambda}} = 1 + x_1$

$F_{x_2} = \dfrac{1}{2\sqrt{x_2}} - \lambda = 0 \rightarrow \dfrac{1}{2\lambda} = \sqrt{x_2} \rightarrow \boxed{\dfrac{1}{4\lambda^2} = x_2}$ in λ terms $\boxed{\pm\sqrt{\dfrac{10}{\lambda}} - 1 = x_1}$

$F_{x_3} = 10e^{-x_3} - \lambda = 0 \rightarrow \dfrac{\lambda}{10} = e^{-x_3} \rightarrow \dfrac{10}{\lambda} = e^{x_3} \rightarrow \boxed{\ln\left(\dfrac{10}{\lambda}\right) = x_3}$

$F_\lambda \rightarrow \boxed{50 = x_1 + x_2 + x_3}$

find λ

4) $50 = \left(\pm\sqrt{\dfrac{10}{\lambda}} - 1\right) + \left(\dfrac{1}{4\lambda^2}\right) + \ln\left(\dfrac{10}{\lambda}\right)$

using Newton rapson

optimal solution

got $\lambda = 0.084$ 7/0

$x_1 = 9.90$ $x_2 = 35.32$ $x_3 = 4.77$

```python
import math

def f(lamb):
    return math.sqrt(10 / lamb) - 1 + 1 / (4 * lamb ** 2) + math.log(10 / lamb) - 50

# Define the derivative of the function
def df(lamb):
    return -5 / (2 * lamb ** 1.5) - 1 / (2 * lamb ** 3) - 1 / lamb

# Initial guess
lamb = 0.1  # This is just a guess, it might need to be adjusted

# Iteration count
max_iter = 10000000

# Tolerance
tol = 1e-12

for i in range(max_iter):
    lamb_new = lamb - f(lamb) / df(lamb)  # Newton-Raphson update

    # Check for convergence
    if abs(lamb_new - lamb) < tol:
        break

    lamb = lamb_new
else:
    print("The method did not converge")


x1 = math.sqrt(10/lamb_new)-1
x2 = 1/(4*lamb_new**2)
x3=math.log(10/lamb_new)

print("Lambda: ", lamb_new)
print(f'Optimal investments are x1 = {x1}, x2 = {x2}, x3 = {x3}')
```

0.0s

```
Lambda:  0.08413203492281647
Optimal investments are x1 = 9.90232949385604, x2 = 35.31971754327805, x3 = 4.777952962867916
```

Problem 3: (12 points) Let there be n assets and K future scenarios. Let $\pi_{i,k}$ be the return of asset i ($i = 1, ..., n$) in scenario k, which is associated with the risk-neutral probabilities p_k ($k = 1, ..., K$). The "theoretical" price of asset i is

$$p_1\pi_{i,1} + ... + p_K\pi_{i,K}, \; i = 1, ..., n.$$

Suppose that the current prices of these assets are s_i ($i = 1, \cdots, n$). You want to estimate values of p_k ($k = 1, ..., K$) by minimizing the sum of the squared difference between the theoretical price and actual prices s_i ($i = 1, ..., n$).

1. (4 points) Formulate this problem as a non-linear constrained optimization model.
2. (3 points) Explain why the optimal objective value is 0 when there is no arbitrage opportunity.
3. (5 points) Use KKT conditions to show that if there is no arbitrage opportunity, then Lagrangian multipliers attached to constraints

$$p_1 + ... + p_K = 1 \text{ and } p_k \geq 0, \; k = 1, ..., K,$$

must all be 0.

$\Sigma(\ )$

1) $P = \sum_{i=1}^{n} \left(\sum_{k=1}^{k} p_k \cdot \pi_{i,k} - S_i \right)^2$

≥ 0

s.t. $\sum_{k=1}^{K} p_k = 1$, $p_k \geq 0 \; \forall k = 1, .., K$

2) if there no arbitrage the expected return and the actual market price should be the same and the optimal value should be 0.

3) KKT

$$L = \sum_{i=1}^{n} \left(\sum_{k=1}^{K} p_k \cdot \pi_{i,k} - S_i \right)^2 + \lambda \left(\sum_{k=1}^{K} p_k - 1 \right) + \sum_{k=1}^{K} \bar{\lambda}_i \cdot p_k$$

kkt condition

1) $\boxed{\nabla L = 0}$

2) $\bar{\lambda}_k \geq 0, \forall k$

3) $\bar{\lambda}_k \times p_k \geq 0, \forall k$

4) $\sum_{k=1}^{k} (p_k - 1) = 0$

we can see that all of the constraintt is not binding and not incentivize so both λ and $\bar{\lambda}_i \; \forall k$ are equal to 0.

since $\min \{ L \} = 0$ then $\lambda, \bar{\lambda}_i \; \forall i = 0$

Problem 4: (10 points) Consider Kelly's problem with n assets, and p_i and b_i ($1 \le i \le n$) are probabilities of winning and returns respectively. Assume that

$$\sum_{i=1}^{S} \frac{1}{1+b_i} > 1.$$

Use KKT condition to show that $x_0^* > 0$, i.e., it is optimal leave a fraction of you budget not invested (hint: you may try contradiction: suppose that $x_0^* = 0$, then what happens to KKT conditions?).

$$L = \sum_{i=1}^{n} p_i \ln[x_0 + (1+b_i)x_i] - \bar{\lambda}\left(x_0 + \sum_{i=1}^{n} x_i - 1\right) + \sum_{i=0}^{n} \lambda_i x_i$$

$$\frac{\partial L}{\partial x_0} = \sum_{i=1}^{n}\left(\frac{p_i}{x_0 + (1+b_i)x_i}\right) - \bar{\lambda} + \lambda_0 = 0$$

$$\frac{\partial L}{\partial x_i} = \frac{p_i(1+b_i)}{x_0 + (1+b_i)x_i} - \bar{\lambda} + \lambda_i = 0$$

$$x_0 + \sum_{i=1}^{n} x_i \le 1$$

know $\displaystyle\sum_{i=1}^{n} \frac{1}{1+b_i} > 1$, assume $x_0 = 0$

$$\frac{\partial L}{\partial x_i} = \frac{p_i}{(1+b_i)x_i} = \frac{\bar{\lambda} - \lambda_i}{(1+b_i)}$$

$x_0^* = 0 \rightarrow x_i > 0$

$\lambda_i = 0$

$$\frac{\partial L}{\partial x_0} = \bar{\lambda}\underbrace{\left(\sum_{i=1}^{n} \frac{1}{1+b_i} - 1\right)}_{>0} = -\underbrace{\lambda_0}_{\text{must } > 0} + \cancel{\sum_{i=1}^{n} \frac{\lambda_i}{1+b_i}}^{0}$$

from $\boxed{\lambda_0 \geq 0}$

Contradiction

x_0 cannot equal to 0

then $x_0 > 0$