Homework 4: Due on Thursday, September 28, 50 points

Problem 1: (12 points) Suppose that you have the opportunity to buy a commodity for \$1 per unit in one market and resell it for \$$(1 + b(y))$ per unit in another market. Here y is the amount you buy and sell and

$$b(y) = \frac{1}{(100 + y)^2}.$$

Questions:

1. (6 points) Determine the amount to buy and sell to maximize your profit.
2. (6 points) Now suppose that you can buy at most $\bar{y}$ units in the first market, so your profit maximization is constrained by

   $$y \leq \bar{y}.$$

   Determine the value of the Lagrangian multiplier attached to that constraint for two cases: $\bar{y} = 75$ and $\bar{y} = 150$.

Problem 2: (16 points) Suppose that you have a budget of 50 million, and opportunities to invest in three different projects. The return (excluding the invested amount) from each project is

$$r_1 = \frac{10x_1}{1 + x_1}, r_2 = \sqrt{x_2}, \text{and } r_3 = 10(1 - e^{-x_3}).$$

where x_i is the amount invested in project i ($i = 1, 2, 3$).

Questions:

1. (4 points) Formulate a nonlinear optimization model to maximize profit.
2. (4 points) Develop the Lagrangian of the problem and KKT conditions.
3. (4 points) Let λ be the Lagrange multiplier attached to the budget constraint

   $$x_1 + x_2 + x_3 \leq 50.$$

   Use KKT conditions to express the optimal values of x_i ($i = 1, 2, 3$) as functions of λ (notice by these conditions, all $x_i > 0$ at the optimum).
4. (4 points) Solve the KKT conditions to determine the value of λ first, and then use this value to determine the optimal investments.

Problem 3: (12 points) Let there be n assets and K future scenarios. Let $\pi_{i,k}$ be the return of asset i ($i = 1, ..., n$) in scenario k, which is associated with the risk-neutral probabilities p_k ($k = 1, ..., K$). The "theoretical" price of asset i is

$$p_1\pi_{i,1} + ... + p_K\pi_{i,K}, \ i = 1, ..., n.$$

Suppose that the current prices of these assets are s_i $(i = 1, \cdots, n)$. You want to estimate values of p_k $(k = 1, ..., K)$ by minimizing the sum of the squared difference between the theoretical price and actual prices s_i $(i = 1, ..., n)$.

1. (4 points) Formulate this problem as a non-linear constrained optimization model.
2. (3 points) Explain why the optimal objective value is 0 when there is no arbitrage opportunity.
3. (5 points) Use KKT conditions to show that if there is no arbitrage opportunity, then Lagrangian multipliers attached to constraints

$$p_1 + ... + p_K = 1 \text{ and } p_k \geq 0, \ k = 1, ..., K,$$

   must all be 0.

Problem 4: (10 points) Consider Kelly's problem with n assets, and p_i and b_i $(1 \leq i \leq n)$ are probabilities of winning and returns respectively. Assume that

$$\sum_{i=1}^{S} \frac{1}{1 + b_i} > 1.$$

Use KKT condition to show that $x_0^* > 0$, i.e., it is optimal leave a fraction of you budget not invested (hint: you may try contradiction: suppose that $x_0^* = 0$, then what happens to KKT conditions?).