IE524 HW3 Solution

1 Problem 1

$$\begin{array}{ll} \max & Z = 2x_1 + 7x_2 + 4x_3 \\ \text{s.t.} & x_1 + 2x_2 + 1x_3 \leq 10 \\ & 3x_1 + 3x_2 + 2x_3 \leq 10 \\ & x_1,\, x_2,\, x_3 \geq 0 \end{array}$$

1.1

The dual problem is

$$\begin{array}{ll} \min & Y = 10y_1 + 10y_2 \\ \text{s.t.} & y_1 + 3y_2 \geq 2 \\ & 2y_1 + 3y_2 \geq 7 \\ & y_1 + 2y_2 \geq 4 \\ & y_1,\, y_2 \geq 0 \end{array}$$

1.2

The optimal value of dual problem is $Y^* = \frac{70}{3}$. Applying *Weak Duality Theorem*, we have $Z^* \leq Y^* = \frac{70}{3} < 24$.

1.3

The optimal dual solution is $y^* = (0,\, \frac{7}{3})^T$ and $A^T y^* - b = (5,\, 0,\, \frac{2}{3})$. Using *Complementary Slackness*, we have $x_1^* = x_3^* = 0$ and $x_2^* = \frac{10}{3}$, $Z = \frac{70}{3} = Y$.

2 Problem 2

$$\begin{array}{lll} (P) & \min & 2x_1 + 5x_2 + 9x_3 \\ & \text{s.t.} & x_1 + 3x_2 + 2x_3 \geq 5 \\ & & x_1 + 5x_2 + 4x_3 \geq 12 \\ & & x_1,\, x_2,\, x_3 \geq 0 \end{array}$$

2.1

The dual problem is

$$\begin{array}{lll} (D) & \max & 5y_1 + 12y_2 \\ & \text{s.t.} & y_1 + y_2 \leq 2 \\ & & 3y_1 + 5y_2 \leq 5 \\ & & 2y_1 + 4y_2 \leq 9 \\ & & y_1,\, y_2 \geq 0 \end{array}$$

2.2

We can see that $y = (0, 1)^T$ is a dual feasible solution so $p^* \geq d^* \geq 12 > 10$.

2.3

Only $3y_1 + 5y_2$ can be tight at the optimum. Because if $y_1 + y_2 = 2$, then

$$3y_1 + 5y_2 \geq 3(y_1 + y_2) = 6.$$

Similarly, if $2y_1 + 4y_2 = 9$, then

$$3y_1 + 5y_2 \geq \frac{5}{2}(y_1 + 2y_2) = \frac{45}{4} > 9.$$

2.4

Using conclusion in 3 and *Complementary Slackness*, we know that $x_1^* = x_3^* = 0$. So $x_2^* = \frac{12}{5}$ and $p^* = 12$.

2.5

It's easy to see that $y = (0, 1)^T$ is a dual feasible solution so

$$12 = p^* \geq d^* \geq b^T y = 12,$$

which means *Strong Duality Theorem* is satisfied and the solution must be optimal.

3 Problem 3

The answer to this problem is not unique. You can choose your own current price and future price set. Here is just an example.

$$\text{Current Price: } S_0, \quad \text{Future Price: } 10,\ 40,\ 50$$

3.1 No Arbitrage Opportunity

Using *First Fundamental Theorem of Asset Pricing*, we know that there is no arbitrage opportunity if and only if there exists risk neutral probabilities $y_{1\sim 3} \geq 0$ such that

$$\begin{aligned}
&y_1 + y_2 + y_3 = 1 \\
&10y_1 + 40y_2 + 50y_3 = S_0 \\
&10y_2 + 20y_3 = 4 \\
&20y_2 + 30y_3 = 6
\end{aligned}$$

This linear system has solution only when $S_0 = 90$ and $y_1 = 0.8$, $y_2 = 0$, $y_3 = 0.2$.

3.2 With Arbitrage Opportunity

Using the same set of prices, when $S_0 \neq 90$, the primal LP will be

$$\begin{array}{ll}
\min & x_1 + S_0 x_2 + 4x_3 + 6x_4 \\
\text{s.t.} & x_1 + 10x_2 + 0x_3 + 0x_4 \geq 0 \\
& x_1 + 40x_2 + 10x_3 + 20x_4 \geq 0 \\
& x_1 + 50x_2 + 20x_3 + 30x_4 \geq 0
\end{array}$$

When all inequality constraints are tight, we have

$$x_1 = -10x_2,\ x_3 = 10x_2,\ x_4 = -20x_2$$

and the objective function will be $(S_0 - 90)x_2$.

- If $S_0 < 90$, $x_2 > 0$, $x_1 < 0$, $x_3 > 0$, $x_4 < 0$, *i.e.*, we should sell stock, lend money, buy option1 and sell option2.
- If $S_0 > 90$, $x_2 < 0$, $x_1 > 0$, $x_3 < 0$, $x_4 > 0$, *i.e.*, we should buy stock, borrow money, sell option1 and buy option2.

4 Problem 4

Here are the decision variables:

No.	Options	Buy(Ask)	Sell(Bid)	Net
1	Call4485	x_1^+	x_1^-	x_1
2	Call4490	x_2^+	x_2^-	x_2
3	Call4495	x_3^+	x_3^-	x_3
4	Put4485	x_4^+	x_4^-	x_4
5	Put4490	x_5^+	x_5^-	x_5
6	Put4495	x_6^+	x_6^-	x_6

Suppose the future price is S, the value will be

$$\phi(S) = (S-4485)^+x_1 + (S-4490)^+x_2 + (S-4495)^+x_3 + (4485-S)^+x_4 + (4490-S)^+x_5 + (4495-S)^+x_6$$

To keep $\phi(S) \geq 0$, $\forall S \geq 0$, we need

$$\begin{aligned}
\phi(4484) - \phi(4485) &= x_4 + x_5 + x_6 \geq 0 \\
\phi(4485) &= 5x_5 + 10x_6 \geq 0 \\
\phi(4490) &= 5x_1 + 5x_6 \geq 0 \\
\phi(4495) &= 10x_1 + 5x_2 \geq 0 \\
\phi(4496) - \phi(4495) &= x_1 + x_2 + x_3 \geq 0
\end{aligned}$$

The primal LP problem will be

$$\begin{array}{ll}
\min & 23.4x_1^+ - 23x_1^- + 20.6x_1^+ - 20.3x_1^- + 18x_1^+ - 17.7x_1^- \\
 & +19x_1^+ - 18.7x_1^- + 21.3x_1^+ - 20.9x_1^- + 23.8x_1^+ - 23.3x_1^- \\
\text{s.t.} & 5(x_5^+ - x_5^-) + 10(x_6^+ - x_6^-) \geq 0 \\
 & 5(x_1^+ - x_1^-) + 5(x_6^+ - x_6^-) \geq 0 \\
 & 10(x_1^+ - x_1^-) + 5(x_2^+ - x_2^-) \geq 0 \\
 & (x_1^+ - x_1^-) + (x_2^+ - x_2^-) + (x_3^+ - x_3^-) \geq 0 \\
 & (x_4^+ - x_4^-) + (x_5^+ - x_5^-) + (x_6^+ - x_6^-) \geq 0 \\
 & x_i^+,\, x_i^- \geq 0,\ \forall 1 \leq i \leq 6.
\end{array}$$

The optimal value is 0, so there is no arbitrage opportunity. It can be attained by using any software.