Lecture 3

Linear Programming: Duality Theory

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Example:Formulation

Model 1

$$\max_{x_1,x_2,x_3} \{x_1 + 4x_2 + 2x_3\}$$

$$\begin{cases} x_1 + 2x_2 + x_3 \le 8, \\ \qquad x_2 - 2x_3 \le 3, \\ x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0. \end{cases}$$

Model 2

$$\min_{y_1,y_2} \{8y_1 + 3y_2\}$$

$$\begin{cases} y_1 \ge 1, \\ 2y_1 + y_2 \ge 4, \\ y_1 - 2y_2 \ge 2, \\ y_1 \ge 0,\ y_2 \ge 0. \end{cases}$$

- no. of variables $\longleftrightarrow$ no. of constraints.
- max $\longrightarrow$ min and $\le$ constraints $\longrightarrow$ $\ge$ constraints
- objective $\longleftrightarrow$ RHS of constraints
- column of constraints $\longleftrightarrow$ rows of constraints

Example: Comparing Objective Values

Model 1

$$\max_{x_1, x_2} \{x_1 + 4x_2\}$$

$$\begin{cases} x_1 + 2x_2 \leq 8, \\ \quad\quad\;\; x_2 \leq 3, \\ x_1 \geq 0, \;\; x_2 \geq 0. \end{cases}$$

Model 2

$$\min_{y_1, y_2} \{8y_1 + 3y_2\}$$

$$\begin{cases} y_1 \quad\quad\; \geq 1, \\ 2y_1 + y_2 \geq 4, \\ y_1 \geq 0, \;\; y_2 \geq 0. \end{cases}$$

- What is the optimal solution to Model 1?

$$x_1^* = 2, \;\; x_2^* = 3, \;\; x_1^* + 4x_2^* = 14.$$

- What is the optimal solution to Model 2?

$$y_1^* = 1, \;\; y_2^* = 2, \;\; 8y_1^* + 3y_2^* = 14.$$

Example: Constraints Change in the Primal LP

$$\max_{x_1, x_2} \left\{ x_1 + 4x_2 \middle| \text{subj. to: } x_1 + 2x_2 \le 8, x_2 \le 3, x_1 \ge 0, x_2 \ge 0 \right\}$$

optimal solution: $x_1^* = 2,\ x_2^* = 3,\ x_1^* + 4x_2^* = 14$.

what if we change the first constraint

$$\max_{x_1, x_2} \left\{ x_1 + 4x_2 \middle| \text{subj. to: } x_1 + 2x_2 \le 9, x_2 \le 3, x_1 \ge 0, x_2 \ge 0 \right\}$$

optimal solution: $x_1^* = 3,\ x_2^* = 3,\ x_1^* + 4x_2^* = 15$.

what if we change the second constraint

$$\max_{x_1, x_2} \left\{ x_1 + 4x_2 \middle| \text{subj. to: } x_1 + 2x_2 \le 8,\ x_2 \le 4, x_1 \ge 0, x_2 \ge 0 \right\}$$

optimal solution: $x_1^* = 0,\ x_2^* = 4,\ x_1^* + 4x_2^* = 16$.

remember the optimal solution to the dual problem

$$y_1^* = 1, y_2^* = 2$$

Example: Constraints Change in the Dual LP

$$\min_{y_1, y_2} \left\{ 8y_1 + 3y_2 \middle| \text{subj. to: } y_1 \geq 1, 2y_1 + y_2 \geq 4, y_1 \geq 0, y_2 \geq 0 \right\}$$

optimal solution: $y_1^* = 1, \; y_2^* = 2, \; 8y_1^* + 3y_2^* = 14$.

what if we change the first constraint

$$\min_{y_1, y_2} \left\{ 8y_1 + 3y_2 \middle| \text{subj. to: } \cancel{y_1 \geq 1}, 2y_1 + y_2 \geq 4, y_1 \geq 0, y_2 \geq 0 \right\}$$

optimal solution: $y_1^* = 0, \; y_2^* = 4, \; 8y_1^* + 3y_2^* = 12$.

what if we change the second constraint

$$\min_{y_1, y_2} \left\{ 8y_1 + 3y_2 \middle| \text{subj. to: } y_1 \geq 1, 2y_1 + y_2 \geq 3, y_1 \geq 0, y_2 \geq 0 \right\}$$

optimal solution: $y_1^* = 1, \; y_2^* = 1, \; 8y_1^* + 3y_2^* = 11$.

remember the optimal solution to the primal LP

$$x_1^* = 2, x_2^* = 3$$

The Primal LP (General Formulation)

$$\max_{x_1,\ldots,x_n} \{c_1x_1 + ... + c_nx_n\}$$

$$\begin{cases} a_{11}x_1 + ... + a_{1n}x_n \le b_1, \\ \quad .... \\ a_{m1}x_1 + ... + a_{mn}x_n \le b_m, \\ x_1 \ge 0, ..., x_n \ge 0. \end{cases}$$

n products to be built from m assets

- c_i: price of product i ($i = 1, ..., n$).
- b_j: amount of asset j available ($j = 1, ..., m$).
- a_{ji}: amount of asset j ($j = 1, ..., m$) used by product i ($i = 1, ..., n$).
- x_i (decision): amount of product i to build ($i = 1, ..., n$)?

Question: what is the minimum price for selling all assets?

The Dual LP

y_j the unit price of asset j ($y_j \geq 0, j = 1, \cdots, m$), the total payment:

$$b_1 y_1 + ... + b_m y_m.$$

- $(a_{11}, ..., a_{m1})$ can be used to build product 1:

$$a_{11} y_1 + ... + a_{m1} y_m \geq c_1,$$

- similarly, $(a_{1i}, ..., a_{mi})$ can be used to build product i, so

$$a_{1i} y_1 + ... + a_{mi} y_m \geq c_i, \quad i = 1, ...n.$$

The dual LP:

$$\min_{y_1, \cdots, y_m} \{b_1 y_1 + ... + b_m y_m\}$$

$$\begin{cases} a_{11} y_1 + ... + a_{m1} y_m \geq c_1, \\ \qquad \qquad .... \\ a_{1n} y_1 + ... + a_{mn} y_m \geq c_n, \\ y_1 \geq 0, ..., y_m \geq 0. \end{cases}$$

Standard LP Formulations

max vs. min, variable vs. constraints, $\leq$ vs. $\geq$, row vs. column:

Formulation 1 (Primal)

$$\max_{x_1,\ldots,x_n} \{c_1x_1 + \ldots. + c_nx_n\}$$

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n \leq b_1, \\ \quad \ldots\ldots.. \\ a_{m1}x_1 + \ldots + a_{mn}x_n \leq b_m, \\ x_1 \geq 0, \ldots., x_n \geq 0. \end{cases}$$

Formulation 2 (Dual)

$$\min_{y_1,\ldots,y_m} \{b_1y_1 + \ldots. + b_my_m\}$$

$$\begin{cases} a_{11}y_1 + \ldots + a_{m1}y_m \geq c_1, \\ \quad \ldots\ldots.. \\ a_{1n}y_1 + \ldots + a_{mn}y_m \geq c_n, \\ y_1 \geq 0, \ldots., y_m \geq 0. \end{cases}$$

Example:

$$\max_{x_1,x_2,x_3} \{x_1 + 4x_2 + 2x_3\}$$

$$\begin{cases} x_1 + 2x_2 + x_3 \leq 8, \\ \qquad x_2 - 2x_3 \leq 3, \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0. \end{cases}$$

$$\min_{y_1,y_2} \{8y_1 + 3y_2\}$$

$$\begin{cases} y_1 \qquad \geq 1, \\ 2y_1 + y_2 \geq 4, \\ y_1 - 2y_2 \geq 2, \\ y_1 \geq 0,\ y_2 \geq 0. \end{cases}$$

Example: Formulate a Dual LP

3 var

Primal LP $\min_{y_1,y_2,y_3} \{5y_1 - 3y_2 + 9y_3\}$

(1) change for var

4 constraint

$$\begin{cases} y_1 + 2y_2 + 7y_3 & \geq 4, \\ -2y_1 + 6y_3 & \geq 1, \\ -5y_2 & \geq -3, \\ y_1 + y_3 & \geq 6, \\ y_1 \ \geq 0, \ y_2 \ \geq 0, \ y_3 & \geq 0. \end{cases}$$

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 0 & 6 \\ 0 & -5 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Dual LP:

- decision variables: x_1, x_2, x_3, x_4.
- objective function: $\max_{x_1,x_2,x_3,x_4} \{4x_1 + x_2 - 3x_3 + 6x_4\}$

4 cof

3 constraint

$\geq \to \leq$

coef =

transpose

- constraints: $$\begin{cases} x_1 - 2x_2 \qquad + x_4 \leq 5, \\ 2x_1 \qquad - 5x_3 \qquad \leq -3, \\ 7x_1 + 6x_2 \qquad + x_4 \leq 9, \\ x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0, \ x_4 \geq 0. \end{cases}$$

$$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 6 & -5 & 0 \\ 7 & 6 & 0 & 1 \end{bmatrix}$$

Simple Case 1

primal LP

$$z = \max_{x}\{2x\}$$

$$\begin{cases} x \le 3, \\ x \le 5, \\ x \ge 0. \end{cases}$$

2 const

2 var

dual LP

$$u = \min_{y_1, y_2}\{3y_1 + 5y_2\}$$

$$\begin{cases} y_1 + y_2 \ge 2, \\ y_1 \ge 0, \\ y_2 \ge 0. \end{cases}$$

- what is the optimal solution of the primal LP, x^*?
  what is the optimal solution of the dual LP, (y_1^*, y_2^*)?
- what is the value of z and what is the value of u?
  what are the values of

$$y_1^*(3 - x^*),\ \ y_2^*(5 - x^*),\ \text{ and } x^*(y_1^* + y_2^* - 2)?$$

  $(x^* = 3,\ y_1^* = 2,\ y_2^* = 0)$.
- if x is **feasible** solution of the primal LP and (y_1, y_2) is a feasible solutions of the dual LPs, how does $2x$ compare with $3y_1 + 5y_2$?

Simple Case 2

primal LP

$$u = \min_{y}\{6y\}$$

$$\begin{cases} y \geq 3, \\ y \geq 5, \\ y \geq 0. \end{cases}$$

dual LP

$$z = \max_{x_1, x_2}\{3x_1 + 5x_2\}$$

$$\begin{cases} x_1 + x_2 \leq 6, \\ x_1 \geq 0, \\ x_2 \geq 0. \end{cases}$$

- if (x_1, x_2) and y are feasible solutions of the primal and dual LPs,

$$6y \geq 3x_1 + 5x_2.$$

- the two LPs have the same optimal objective value, i.e.,

$$u = z.$$

- if (x_1^*, x_2^*) and y^* are optimal solutions of the dual and primal LPs

$$x_1^*(y^* - 3) = x_2^*(y^* - 5) = y^*(6 - x_1^* - x_2^*) = 0.$$

Weak Duality

if $(x_1, ..., x_n)$ satisfy the primal constraints:

$$\begin{cases} a_{11}x_1 + ... + a_{1n}x_n \leq b_1, \\ \qquad \text{.................,} \\ a_{m1}x_1 + ... + a_{mn}x_n \leq b_m, \\ x_1 \geq 0, ..., x_n \geq 0. \end{cases}$$

and $(y_1, ..., y_m)$ satisfy the dual constraints:

$$\begin{cases} a_{11}y_1 + ... + a_{m1}y_m \geq c_1, \\ \qquad \text{............,} \\ a_{1n}y_1 + ... + a_{mn}y_m \geq c_n, \\ y_1 \geq 0, ..., y_m \geq 0. \end{cases}$$

then $$b_1y_1 + ... + b_my_m \geq c_1x_1 + ... + c_nx_n.$$

intuition: to buy a portfolio of assets, a viable offer of payment can never be lower than the maximum value that these assets can generate

Proof of Weak Duality (Special Case)

primal LP

$$\max_{x_1, x_2} \{x_1 + 4x_2\}$$

$$\begin{cases} x_1 + 2x_2 \leq 8, \\ \quad\quad x_2 \leq 3, \\ x_1 \geq 0, \ x_2 \geq 0. \end{cases}$$

dual LP

$$\min_{y_1, y_2} \{8y_1 + 3y_2\}$$

$$\begin{cases} y_1 \quad\quad \geq 1, \\ 2y_1 + y_2 \geq 4, \\ y_1 \geq 0, \ y_2 \geq 0. \end{cases}$$

$$\begin{aligned} x_1 + 2x_2 \leq 8 \quad &\longrightarrow \quad (x_1 + 2x_2)y_1 \leq 8y_1 \\ x_2 \leq 3 \quad &\longrightarrow \quad x_2 y_2 \leq 3y_2 \end{aligned}$$

add them together

$$x_1 y_1 + (2y_1 + y_2)x_2 \leq 8y_1 + 3y_2.$$

and notice that

$$y_1 \geq 1 \quad 2y_1 + y_2 \geq 4 \quad \longrightarrow \quad 8y_1 + 3y_2 \geq x_1 + 4x_2.$$

Proof of Weak Duality (General Case)

$(x_1, ..., x_n)$ feasible for

$$\begin{cases} a_{11}x_1 + ... + a_{1n}x_n \leq b_1, \\ \quad .... \\ a_{m1}x_1 + ... + a_{mn}x_n \leq b_m, \\ x_1 \geq 0, ..., x_n \geq 0. \end{cases}$$

$(y_1, ..., y_m)$ feasible for

$$\begin{cases} a_{11}y_1 + ... + a_{m1}y_m \geq c_1, \\ \quad .... \\ a_{1n}y_1 + ... + a_{mn}y_m \geq c_n, \\ y_1 \geq 0, ..., y_m \geq 0. \end{cases}$$

- multiply constraint j of the Primal LP with y_j and sum over j:

$$\begin{aligned} & (a_{11}x_1 + ... + a_{1n}x_n)y_1 + ... + (a_{m1}x_1 + ... + a_{mn}x_n)y_m. \\ \leq \quad & b_1y_1 + ... + b_my_m \end{aligned}$$

- multiply constraint i of the Dual LP with x_i and sum over i:

$$\begin{aligned} & (a_{11}y_1 + ... + a_{m1}y_m)x_1 + ... + (a_{1n}y_1 + ... + a_{mn}y_m)x_n. \\ \geq \quad & c_1x_1 + ... + c_nx_n \end{aligned}$$

- observe that two inequalities have the same left-hand side

$$c_1x_1 + ... + c_nx_n \leq \sum_{j=1}^{m}\sum_{i=1}^{n} a_{ji}x_iy_j \leq b_1y_1 + ... + b_my_m.$$

Special Case

what if the optimal value of the primal LP is unbounded example:

$$\max_{x_1, x_2}\{2x_1 + x_2\}$$

$$\text{s.t.} \qquad x_1 - x_2 \le 3, \quad x_1 - 2x_2 \le 5,$$
$$x_1 \ge 0,\ x_2 \ge 0.$$

its dual LP

$$\min_{y_1, y_2}\{3y_1 + 5y_2\}$$

$$\text{s.t.} \qquad y_1 + y_2 \ge 2, \quad -y_1 - 2y_2 \ge 1,$$
$$y_1 \ge 0,\ y_2 \ge 0.$$

1. what is the solution to the dual LP?
2. what is the intuition?

The dual/primal LP is infeasible if the primal/dual LP is unbounded (can you prove the statement by Weak Duality).

Strong Duality

$(x_1^*, ..., x_n^*)$ optimizes

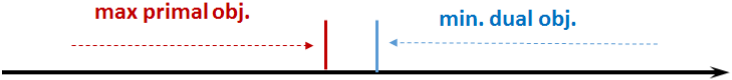
$$\max_{x_1,\dots,x_n} \{c_1x_1 + ... + c_nx_n\}$$

$$\begin{cases} a_{11}x_1 + ... + a_{1n}x_n \le b_1, \\ \quad .... \\ a_{m1}x_1 + ... + a_{mn}x_n \le b_m, \\ x_1 \ge 0, ..., x_n \ge 0. \end{cases}$$

$(y_1^*, ..., y_m^*)$ optimizes

$$\min_{y_1,\dots,y_m} \{b_1y_1 + ... + b_my_m\}$$

$$\begin{cases} a_{11}y_1 + ... + a_{m1}y_m \ge c_1, \\ \quad .... \\ a_{1n}y_1 + ... + a_{mn}y_m \ge c_n, \\ y_1 \ge 0, ..., y_m \ge 0. \end{cases}$$



<u>Strong Duality:</u> when both LPs attain their optimal solutions,

$$c_1x_1^* + ... + c_nx_n^* = b_1y_1^* + ... + b_my_m^*,$$

i.e., max. profit = min. payment.

Complementary Slackness:Primal LP

each primal constraint is associated with a dual variable,

$$\max_{x_1,\dots,x_n} \{c_1x_1 + ... + c_nx_n\}$$

$$\begin{cases} a_{11}x_1 + ... + a_{1n}x_n \leq b_1, & -------> \ y_1, \\ \qquad\qquad .... & \\ a_{m1}x_1 + ... + a_{mn}x_n \leq b_m, & -------> \ y_m. \\ x_1 \geq 0, ..., x_n \geq 0. & \end{cases}$$

<u>Complementary Slackness Condition:</u>

$$(b_j - a_{j1}x_1^* - .... - a_{jn}x_n^*)y_j^* = 0, \quad j = 1, ..., m.$$

in words: if resource j is not fully utilized (slack), i.e.,

$$b_j - a_{j1}x_1^* - .... - a_{jn}x_n^* > 0,$$

then its shadow price

$$y_j^* = 0, \quad 1 \leq j \leq m.$$

Complementary Slackness: Dual LP

each dual constraint is associated with a primal variable

$$\min_{y_1,\dots,y_m} \{b_1 y_1 + \dots. + b_m y_m\}$$

$$\begin{cases} a_{11}y_1 + \dots + a_{m1}y_m \geq c_1, & -------> \ x_1, \\ \qquad \dots\dots\dots & \\ a_{1n}y_1 + \dots + a_{mn}y_m \geq c_n, & -------> \ x_n. \\ y_1 \geq 0, \dots., y_m \geq 0. & \end{cases}$$

Complementary Slackness Condition:

$$(a_{1i}y_1^* + \dots + a_{mi}y_m^* - c_i)x_i^* = 0, \quad i = 1, \dots., n.$$

in words: if product i's total resource cost exceeds its market value,

$$a_{1i}y_1^* + \dots + a_{mi}y_m^* - c_i > 0,$$

then its optimal quantity:

$$x_i^* = 0, \ \ 1 \leq i \leq n.$$

Complementary Slackness: General

optimal solution $(x_1^*, ..., x_n^*)$

$$\max_{x_1,\ldots,x_n} \{c_1 x_1 + ... + c_n x_n\}$$

$$\begin{cases} a_{11}x_1 + ... + a_{1n}x_n \leq b_1, \\ \quad .... \\ a_{m1}x_1 + ... + a_{mn}x_n \leq b_m, \\ x_1 \geq 0, ..., x_n \geq 0. \end{cases}$$

optimal solution $(y_1^*, ..., y_m^*)$

$$\min_{y_1,\ldots,y_m} \{b_1 y_1 + ... + b_m y_m\}$$

$$\begin{cases} a_{11}y_1 + ... + a_{m1}y_m \geq c_1, \\ \quad .... \\ a_{1n}y_1 + ... + a_{mn}y_m \geq c_n, \\ y_1 \geq 0, ..., y_m \geq 0. \end{cases}$$

complementary slackness:

$$\begin{aligned} y_j^*(b_j - a_{j1}x_1^* - .... - a_{jn}x_n^*) &= 0, \ j = 1, ..., m, \\ x_i^*(a_{1i}y_1^* + ... + a_{mi}y_m^* - c_i) &= 0, \ i = 1, ..., n. \end{aligned}$$

Proof of Complementary Slackness (Special Case)

primal LP

$$\max_{x_1, x_2} \{x_1 + 4x_2\}$$

$$\begin{cases} x_1 + 2x_2 \leq 8, \\ \qquad\; x_2 \leq 3, \\ x_1 \geq 0, \;\; x_2 \geq 0. \end{cases}$$

dual LP

$$\min_{y_1, y_2} \{8y_1 + 3y_2\}$$

$$\begin{cases} y_1 \qquad\; \geq 1, \\ 2y_1 + y_2 \geq 4, \\ y_1 \geq 0, \;\; y_2 \geq 0. \end{cases}$$

$$\begin{aligned} (8 - x_1^* - 2x_2^*)y_1^* \geq 0, \quad & (3 - x_2^*)y_2^* \geq 0, \\ (y_1^* - 1)x_1^* \geq 0, \quad & (2y_1^* + y_2^* - 4)x_2^* \geq 0. \end{aligned} \tag{1}$$

add all four terms together:

$$(8y_1^* + 3y_2^*) - (x_1^* + 4x_2^*),$$

by the Strong Duality, this sum =0.
since every term in (**??**) is nonnegative, they all have to be 0.

Optional: Proof of Complementary Slackness (1)

$$\max_{x_1,\ldots,x_n} \{c_1x_1 + ... + c_nx_n\} \qquad \min_{y_1,\ldots,y_m} \{b_1y_1 + ... + b_my_m\}$$

$$\begin{cases} a_{11}x_1 + ... + a_{1n}x_n \le b_1, \\ \qquad .... \\ a_{m1}x_1 + ... + a_{mn}x_n \le b_m, \\ x_1 \ge 0, ..., x_n \ge 0. \end{cases} \qquad \begin{cases} a_{11}y_1 + ... + a_{m1}y_m \ge c_1, \\ \qquad .... \\ a_{1n}y_1 + ... + a_{mn}y_m \ge c_n, \\ y_1 \ge 0, ..., y_m \ge 0. \end{cases}$$

rewrite constraints as

$$\begin{cases} b_1 - a_{11}x_1 - ... - a_{1n}x_n \ge 0, \\ \qquad .... \\ b_m - a_{m1}x_1 - .... - a_{mn}x_n \ge 0, \\ x_1 \ge 0, ..., x_n \ge 0. \end{cases} \qquad \begin{cases} a_{11}y_1 + ... + a_{m1}y_m - c_1 \ge 0 \\ \qquad .... \\ a_{1n}y_1 + ... + a_{mn}y_m - c_n \ge 0, \\ y_1 \ge 0, ..., y_m \ge 0. \end{cases}$$

so

$$\begin{aligned} y_1(b_1 - a_{11}x_1 - ... - a_{1n}x_n) + ... + y_m(b_m - a_{m1}x_1 - .... - a_{mn}x_n) &\ge 0, \\ x_1(a_{11}y_1 + ... + a_{m1}y_m - c_1) + ... + x_n(a_{1n}y_1 + ... + a_{mn}y_m - c_n) &\ge 0. \end{aligned}$$

Optional: Proof of Complementary Slackness (II)

$(x_1^*, ..., x_n^*)$, $(y_1^*, ..., y_m^*)$: optimal solutions of the primal and dual LPs.

from the last side:

$$\begin{aligned}
y_1^*(b_1 - a_{11}x_1^* - ... - a_{1n}x_n^*) + ... + y_m^*(b_m - a_{m1}x_1^* - .... - a_{mn}x_n^*) &\geq 0, \\
x_1^*(a_{11}y_1^* + ... + a_{m1}y_m^* - c_1) + ... + x_n^*(a_{1n}y_1^* + ... + a_{mn}y_m^* - c_n) &\geq 0.
\end{aligned}$$

the left hand side of the two equations:

$$\begin{aligned}
& y_1^*(b_1 - a_{11}x_1^* - ... - a_{1n}x_n^*) + ... + y_m^*(b_m - a_{m1}x_1^* - .... - a_{mn}x_n^*) \\
+ \quad & x_1^*(a_{11}y_1^* + ... + a_{m1}y_m^* - c_1) + ... + x_n^*(a_{1n}y_1^* + ... + a_{mn}y_m^* - c_n) \\
= \quad & (b_1 y_1^* + ... + b_m y_m^*) - (c_1 x_1^* + ... + c_n x_n^*) \\
= \quad & 0 \qquad \text{(by strong duality)}
\end{aligned}$$

so

$$\begin{aligned}
y_j^*(b_j - a_{j1}x_1^* - ... - a_{jn}x_n^*) &= 0, \quad j = 1, ..., m, \\
x_i^*(a_{1i}y_1^* + ... + a_{mj}y_m^* - c_i) &= 0, \quad i = 1, ..., n.
\end{aligned}$$

Example

a shortcut to solve the following LP:

$$\min_{y_1, y_2, y_3} \{4y_1 + 6y_2 + 18y_3\}$$

$$\begin{cases} y_1 + 3y_3 \geq 3, \\ y_2 + 2y_3 \geq 5, \\ y_1 \geq 0, \;\; y_2 \geq 0, \;\; y_3 \geq 0. \end{cases}$$

what is the dual to this LP? $\max\limits_{x_1, x_2} \{3x_1 + 5x_2\}$

$$\begin{cases} x_1 \leq 4, \\ x_2 \leq 6, \\ 3x_1 + 2x_2 \leq 18, \\ x_1 \geq 0, x_2 \geq 0. \end{cases}$$

the optimal solution is intuitively obvious (why?)

$$x_1^* = 2, \;\; x_2^* = 6, \;\; 3x_1^* + 5x_2^* = 36.$$

Example (continue)

$$\max_{x_1,x_2} \{3x_1 + 5x_2\}$$

$$\begin{cases} x_1 \leq 4, \\ x_2 \leq 6, \\ 3x_1 + 2x_2 \leq 18, \\ x_1 \geq 0, x_2 \geq 0. \end{cases}$$

$$\min_{y_1,y_2,y_3} \{4y_1 + 6y_2 + 18y_3\}$$

$$\begin{cases} y_1 + 3y_3 \geq 3, \\ y_2 + 2y_3 \geq 5, \\ y_1 \geq 0, \ \ y_2 \geq 0, \ \ y_3 \geq 0. \end{cases}$$

$x_1^* = 2, \ x_2^* = 6, \ 3x_1^* + 5x_2^* = 36.$

From values of x_1^* and x_2^*, it is immediate that:

$$y_1^* = 0, \, 3y_3^* = 3, \ y_2^* + 2y_3^* = 5,$$

so $\qquad y_1^* = 0, \ y_2^* = 3, \ y_3^* = 1 \ obj = 4y_1^* + 6y_2^* + 18y_3^* = 36.$

we can also verify Strong Duality

$$3x_1^* + 5x_2^* = 4y_1^* + 6y_2^* + 18y_3^* = 36.$$