Lecture 7

Integer Linear Programming (ILP)

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October 02, 2023

Integer Programming Models

$$\max_{\mathbf{x}}\{c_1x_1 + ... + c_nx_n\}$$

$$\text{subject to :} \quad A\mathbf{x} \le \mathbf{b},$$

$$x_i \in \mathcal{Z} \text{ for some or all } i \ (\mathcal{Z} \text{ is a set of integers}).$$

- difference from LP only in the last (integrality) constraint.
- mixed ILP (MILP): only some variables need to be integers.
- it is possible to have a nonlinear objective function or constraints, but linear models are already hard enough.

for financial applications, 0-1 IP (binary variables) is particularly useful.

example: $\mathcal{Z} = \{0, 1\}$, x_i is a binary variable ($x_i = 1$ means to include stock i in your portfolio and $x_i = 0$ means not to include.)

Example of MILP: Index Fund

construct an index fund by selecting from a set of stocks $\{1, \ldots, n\}$:

r_{it}: return on stock i in year t; M_t: market return in year t; ϵ: tolerance.

$$x_i : \% \text{ to invest in stock } i, \quad y_i = \begin{cases} 1 & \text{if stock } i \text{ is picked} \\ 0 & \text{otherwise} \end{cases}, \quad i = 1, \ldots, n.$$

$$\begin{aligned} & \min_{\mathbf{x},\mathbf{y}} \{y_1 + \ldots + y_n\} \\ \text{subject to:} \quad & r_{1t}x_1 + \ldots + r_{nt}x_n \geq M_t - \epsilon, \quad t = 1, \ldots, T. \\ & x_1 + \ldots + x_n = 1, \\ & 0 \leq x_i \leq y_i, \; y_i \in \{0, 1\}, \quad i = 1, \ldots, n, \end{aligned}$$

- optimize other objective functions

$$\min_{\mathbf{x},\mathbf{y}} \{c_1 y_1 + \ldots + c_n y_n\}, \quad c_i : \text{cost of including stock } i \; (1 = 1, \ldots, n).$$

- set a limit on the number of a subset of stocks to be included:

$$\sum_{i \in \mathcal{S}'} y_i \leq m \quad \text{or} \quad \sum_{i \in \mathcal{S}'} y_i \geq m'.$$

Modeling Logical Requirements

consider a set of 10 stocks, $x_i \in \{0, 1\}$ ($i = 1, \ldots, 10$): ← choose or not choosing

- exactly one of stocks 4, 5, 6, 7 must be included — only choose 1 from a stock

$$x_4 + x_5 + x_6 + x_7 = 1.$$

- if stock 1 is included then stock 2 cannot be included

$$x_1 + x_2 \leq 1.$$

- if stock 3 is included, then stock 4 must be included

~~$x_3 + x_4 = 2$~~ ??!

include x 3 → x_4

$$x_3 \leq x_4.$$

- if stock 8 is included, then either stock 9 or stock 10 must be included

$$x_8 \leq x_9 + x_{10}.$$

$x_1 + x_2 \geq x_8 + x_9$

- if stock 1 or 2 is included, then neither stock 8 nor 9 should be included

$$x_8 + x_9 \leq 2(1 - x_1), \;\; x_8 + x_9 \leq 2(1 - x_2).$$

$x_1 = 0$

Index Fund Example

Constructing an index fund: there are N ($N \geq 100$) stocks to consider.

Let y_i be a binary variable to determine whether stock i is included ($y_i = 1$) or not ($y_i = 0$), $i = 1, \cdots, N$.

1. stock 1 cannot be selected if any of stocks $m, \cdots, N$ is selected.

$$\sum_{i=m}^{N} y_i \leq (N - m + 1)(1 - y_1)$$

2. Class 1 stocks are $1, \cdots, 30$ and Class 2 stocks are $N - 29, \cdots, N$. If any Class 1 stock is selected, then select exactly 10 Class 2 stocks

   let $\bar{y} \in \{0, 1\}$:

$$\bar{y} \geq \frac{1}{30} \sum_{i=1}^{30} y_i \quad \text{and} \quad \bar{y} \leq \sum_{i=1}^{30} y_i.$$

   $\bar{y} = 1$ if and only if a Class 1 stock is selected.

$$10\bar{y} \leq \sum_{i=N-29}^{N} y_i \leq 30 - 20\bar{y}.$$

Sequential Investment Problem

You have \$20m that can be invested in one of the two projects

1. project 1:

	amount	return
round 1	\$10m	12%
round 2	\$10m	7%

2. project 2:

	amount	return
round 1	\$7m	4%
round 2	\$6m	−2%
round 3	\$7m	22%

- all profits and losses are counted at the end, i.e., no reinvestment.
- can invest in a round only if having fully invested in all previous rounds.

Question: which project(s) to invest in? up to which round?

MILP of Sequential Investment Problem

x_1, x_2: amounts of round 1 and 2 investment for project 1.

$\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$: amounts of round 1, 2, and 3 investment for project 2.

$$\max_{\mathbf{x},\mathbf{y}}\{0.12x_1 + 0.07x_2 + 0.04\hat{x}_1 - 0.02\hat{x}_2 + 0.22\hat{x}_3\}$$

subject to:

$$x_1 + x_2 + \hat{x}_1 + \hat{x}_2 + \hat{x}_3 \leq 20,$$

$$x_1, x_2, \hat{x}_1, \hat{x}_2, \hat{x}_3 \geq 0.$$

$y_1 \in \{0, 1\}$: complete round 1 investment in project 1.

$$x_1 \leq 10, \quad y_1 \leq x_1/10, \quad x_2 \leq 10y_1,$$

$\hat{y}_1 \in \{0, 1\}$: complete round 1 investments in project 2.

$$\hat{x}_1 \leq 7, \ \hat{y}_1 \leq \hat{x}_1/7, \ \hat{x}_2 \leq 6\hat{y}_1,$$

$\hat{y}_2 \in \{0, 1\}$: complete round 2 investments in project 2.

$$\hat{y}_2 \leq \hat{x}_2/6, \quad \hat{x}_3 \leq 7\hat{y}_2.$$

Brute Force: for small-size problems

- a budget of 20 million, four candidate assets, which assets to invest in?

asset	required investment	profit
1	7	8
2	10	12
3	6	6
4	4	4

- integer programming formulation

$$
\begin{aligned}
& \max_{\mathbf{x}} \{8x_1 + 12x_2 + 6x_3 + 4x_4\} \\
\text{s.t.} \quad & 7x_1 + 10x_2 + 6x_3 + 4x_4 \leq 20, \\
& x_1, x_2, x_3, x_4 \in \{0, 1\}.
\end{aligned}
$$

by enumerating all (16) possible choices: invest in assets 2, 3, 4. total investment is 20 (within the budget), and total profit is 22.

will not work for more realistic problems. e.g., if there are 100 assets, then the number of possible choices:

$$2^{100} = 1.3 \times 10^{30}.$$

Reducing Solution Space by Fixing Variables

example: a MILP has only binary variables and the following constraints:

$$\begin{array}{rcccl} 4x_1 + 3x_2 + 3x_3 & \leq & 3 & \longrightarrow & x_1 = 0, \\ x_1 - 2x_2 + 4x_3 & \geq & 2 & \longrightarrow & x_3 = 1, \\ 4x_1 - 2x_2 + x_3 & \leq & -1 & \longrightarrow & x_2 = 1. \end{array}$$

fixing one binary variable cuts the solution space by half.

sequential fixing (a lucky case:)

$$\begin{array}{ll} & \max_{\mathbf{x}} \{3x_1 + 5x_2 - 2x_3 + x_4\} \\ \text{s.t.} & x_1 + x_2 + 2x_4 \leq 1, \\ & x_1 - 2x_2 + x_4 \geq 0, \\ & x_2 + x_3 \geq 1, \\ & x_1 + 2x_2 - x_3 \geq 0, \\ & x_1, x_2, x_3, x_4 \in \{0, 1\}. \end{array}$$

$$\begin{array}{rcl} x_1 + x_2 + 2x_4 \leq 1 & \to & x_4^* = 0, \\ x_1 - 2x_2 \geq 0 & \to & x_2^* = 0, \\ x_3 \geq 1 & \to & x_3^* = 1, \\ x_1 - 1 \geq 0 & \to & x_1^* = 1. \end{array}$$

$$(x_1^*, x_2^*, x_3^*, x_4^*) = (1, 0, 1, 0) \text{ and } 3x_1^* + 5x_2^* - 2x_3^* + x_4^* = 1.$$

Rounding LP Solution: Generally Not a Good Idea

- IP Model

$$\max_{\mathbf{x}}\{2x_1 + x_2\}$$
$$\text{s. t.} \quad 3x_1 + 2x_2 \le 5.5,$$
$$x_1, x_2 \ge 0,$$
$$x_1, x_2 \quad \text{integers.}$$

$$x_1^* = 1, x_2^* = 1, \quad 2x_1^* + x_2^* = 3.$$

- LP Model

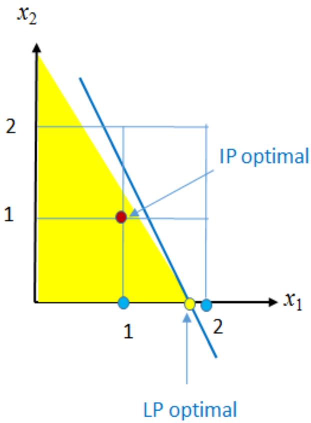
$$\max_{\mathbf{x}}\{2x_1 + x_2\}$$
$$\text{s. t.} \quad 3x_1 + 2x_2 \le 5.5,$$
$$x_1, x_2 \ge 0.$$

$$x_1^* = 11/6, \; x_2^* = 0.$$

rounding down:

$$x_1^* = 1, \; x_2^* = 0, \; 2x_1^* + x_2^* = 2.$$

rounding up: $x_1^* = 2$, infeasible.



Another Example

$$
\begin{aligned}
& \max\{20x_1 + 10x_2 + 10x_3\} \\
\text{subject to} \quad & 2x_1 + 20x_2 + 4x_3 \leq 15 \\
& 6x_1 + 20x_2 + 4x_3 = 20 \\
& x_1, x_2, x_3 \geq 0 \text{ integer}
\end{aligned}
$$

Without the integrality constraint, the problem can be simplify to ($x_2 = 0$):

$$
\begin{aligned}
& \max\{20x_1 + 10x_3\} \\
\text{subject to} \quad & 2x_1 + 4x_3 \leq 15 \\
& 6x_1 + 4x_3 = 20 \\
& x_1, x_3 \geq 0
\end{aligned}
$$

The objective can be simplify to

$$\max\{20x_1 + 10(5 - 3/2x_1)\} = \max\{50 + 5x_1\},$$

so the optimal solution is

$$x_3 = 0, x_1 = 10/3.$$

no integer solution can be obtained by rounding the fractional LP solution.

Branch and Bound:Basic Insights

$$
\begin{aligned}
Z^* &= \max_{\mathbf{x}} \{r_1x_1 + r_2x_2 + ... + r_nx_n\} \\
\text{subject to:} \quad & A\mathbf{x} \le \mathbf{b}, \quad x_1, ..., x_n \in \{0, 1\}.
\end{aligned}
$$

- "LP relaxation":

$$
\begin{aligned}
z^* &= \max_{\mathbf{x}} \{r_1x_1 + r_2x_2 + ... + r_nx_n\} \qquad (1) \\
\text{subject to:} \quad & A\mathbf{x} \le \mathbf{b}, \; 0 \le x_i \le 1, \; i = 1, ..., n.
\end{aligned}
$$

  so a *feasible* binary vector $\mathbf{x}$ is optimal if

$$r_1x_1 + .. + r_nx_n = z^*.$$

- if no such binary values: fix x_1 at $x_1 = 0$ and $x_1 = 1$ and solve (1) respectively,

$$z^*(\{0\}) \le z^* \text{ and } z^*(\{1\}) \le z^*,$$

  if we can find a *feasible* binary vector $\mathbf{x}$:

$$r_1x_1 + ... + r_nx_n \ge z^*(\{i\}),$$

  then no need to consider any solution with $x_1 = i$ $(i = 0, 1)$.
- if such a vector cannot be found, fix the next variable...

Branch and Bound: Illustration

$$
\begin{aligned}
Z^* &= \max_{\mathbf{x}}\{16x_1 + 22x_2 + 12x_3 + 8x_4\} \\
\text{subject to:} \quad & 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14,\ \ x_1, x_2, x_3, x_4 \in \{0, 1\}
\end{aligned}
$$

LP relaxation solution ($0 \leq x_1, x_2, x_3, x_4 \leq 1$):

$$x_1^* = 1 \quad x_2^* = 1 \quad x_3^* = 0.5, \quad z^* = 44$$

let $x_1 = 0$ and allow the rest of variables to vary between 0 and 1:

$$
\begin{aligned}
z^*(\{0\}) &= \max_{\mathbf{x}}\{22x_2 + 12x_3 + 8x_4\} \\
\text{subject to:} \quad & 7x_2 + 4x_3 + 3x_4 \leq 14, \ \ 0 \leq x_2, x_3, x_4 \leq 1.
\end{aligned}
$$

the optimal solution is

$$(x_2^*, x_3^*, x_4^*) = (1, 1, 1) \text{ so } \quad z^*(\{0\}) = 42 \leq Z^*$$

- 42 is the lower bound of the ILP function (since it is feasible to reach), but an upper bound for LP solutions with $x_1 = 0$.
- we only need to explore solutions with $x_1 = 1$.

Branch and Bound: Illustration

$$
\begin{aligned}
Z^* \quad &= \quad \max_{\mathbf{x}}\{16x_1 + 22x_2 + 12x_3 + 8x_4\} \\
\text{subject to:} \quad & \quad 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14, \;\; x_1, x_2, x_3, x_4 \in \{0, 1\}
\end{aligned}
$$

LP relaxation when $x_1 = 1$:

$$
\begin{aligned}
z^*(\{1\}) \quad &= \quad 16 + \max_{\mathbf{x}}\{22x_2 + 12x_3 + 8x_4\} \\
\text{subject to:} \quad & \quad 7x_2 + 4x_3 + 3x_4 \leq 9, \quad 0 \leq x_2, x_3, x_4 \leq 1.
\end{aligned}
$$

the optimal solution:

$$
(x_2^*, x_3^*, x_4^*) = (1, 1/2, 0), \quad \text{so} \quad z^*(\{1\}) = 44 \geq z^*(\{0\}) = 42.
$$

- 44 is an upper bound on the objective of the ILP.
- it cannot be reached by fixing $x_1 = 0$ (the upper bound there is 42).
- and we do not know if it can be reached by fixing $x_1 = 1$.

Branch and Bound: Illustration

$$
\begin{aligned}
Z^* &= \max_{\mathbf{x}}\{16x_1 + 22x_2 + 12x_3 + 8x_4\} \\
\text{s. t.} \quad & 5x_1 + 7x_2 + 4x_3 + 3x_4 \le 14, \;\; x_1, x_2, x_3, x_4 \in \{0, 1\}
\end{aligned}
$$

- fixing $x_1 = 0$: a feasible solution with $z^*(\{0\}) = 42$.
- fixing $x_1 = 1$: an upper bound $z^*(\{1\}) = 44$.

continue to explore the branch of $x_1 = 1$

$x_2 = 0$

$$z^*(\{1,0\}) = 16 + \max_{\mathbf{x}}\{12x_3 + 8x_4\}$$

subject to:

$$4x_3 + 3x_4 \le 9, \quad 0 \le x_3, x_4 \le 1.$$

LP solution:

$$x_3^* = 1, \;\; x_4^* = 1, \;\; z^*(\{1,0\}) = 36.$$

abandon this branch $(x_1 = 1, \; x_2 = 0)$.

$x_2 = 1$

$$z^*(\{1,1\}) = 38 + \max_{\mathbf{x}}\{12x_3 + 8x_4\}$$

subject to:

$$4x_3 + 3x_4 \le 2, \quad 0 \le x_3, x_4 \le 1.$$

LP solution:

$$x_3^* = 1/2, \;\; x_4^* = 0, \;\; z^*(\{1,1\}) = 44.$$

explore this branch $(x_1 = 1, \; x_2 = 1)$.

Branch and Bound: Illustration

$$
\begin{aligned}
Z^* &= \max_{\mathbf{x}}\{16x_1 + 22x_2 + 12x_3 + 8x_4\} \\
\text{s. t.} \quad & 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14, \; x_1, x_2, x_3, x_4 \in \{0, 1\}
\end{aligned}
$$

- fixing $x_1 = 0$: feasible solution $z^*(\{0\}) = 42$ (keep).
- fixing $x_1 = 1$, $x_2 = 0$: upper bound $z^*(\{1, 0\}) = 36$ (abandon).
- fixing $x_1 = 1$, $x_2 = 1$: upper bound $z^*(\{1, 1\}) = 44$ (continue).

$x_3 = 0$

$$z^*(\{1,1,0\}) = 38 + \max_{\mathbf{x}}\{8x_4\}$$

subject to : $3x_4 \leq 2, \; 0 \leq x_4 \leq 1.$

LP solution:

$x_4^* = 2/3, z^*(\{1, 1, 0\}) = 43.33.$

$x_3 = 1$

$$z^*(\{1,1,1\}) = 50 + \max_{\mathbf{x}}\{8x_4\}$$

subject to: $3x_4 \leq -2, \; 0 \leq x_4 \leq 1.$

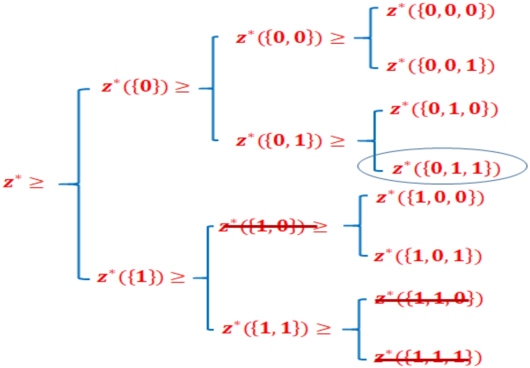
LP infeasible (abandon this branch)

to continue on branch $(1, 1, 0)$: $z^*(\{1, 1, 0, 0\}) = 38$ and $(1, 1, 0, 1)$ infeasible.

so we keep $(0, 1, 1, 1)$ $(z^*(\{0, 1, 1, 1\}) = 42)$ as the optimal solution.

Branch and Bound: Illustration

$$
\begin{aligned}
Z^* &= \max_{\mathbf{x}}\{c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4\} \\
\text{s.t.} \quad & a_{j1}x_1 + a_{j2}x_2 + a_{j3}x_3 + a_{j4}x_4 \le b_j \ (j = 1, .., J), \ \ x_1, x_2, x_3, x_4 \in \{0, 1\}.
\end{aligned}
$$



no need to search any branch from $z^*(...)$ if you can find a binary solution $\mathbf{x}'$:

$$c_1x_1' + c_2x_2' + c_3x_3' + c_4x_4' \ge z^*(...).$$

Branch & Bound: An Application

Risk-return model with minimum transaction level:

- mean return $\boldsymbol{\mu}$, covariance matrix Σ, return target $\bar{r}$.
- for each stock i, the purchase amount is either 0 or no less l_i.

$$\begin{aligned} &\min_{\mathbf{x}}\{\mathbf{x}^T\Sigma\mathbf{x} | \boldsymbol{\mu}\cdot\mathbf{x} \geq \bar{r},\ \mathbf{x} \geq 0,\} \\ &\text{if } x_i > 0,\ \text{then } x_i \geq l_i. \end{aligned} \tag{2}$$

steps:

1. solve (2) without the second constraint, i.e.,
$$\min_{\mathbf{x}}\{\mathbf{x}^T\Sigma\mathbf{x} | \boldsymbol{\mu}\cdot\mathbf{x} \geq \bar{r},\ \mathbf{x} \geq 0\} \tag{3}$$
2. if $x_i^* \geq l_i$ for all $x_i^* > 0$, then the optimal solution is found.
3. otherwise choose a stock i where $0 < x_i < l_i$, and branch out two problems:
   - change (3) by adding constraint $x_i = 0$.
   - change (3) by adding constraint $x_i \geq l_i$.

   for each branch, repeat the above (with updated problem) until the optimal solution of the branch is found or dominated by another branch.