

Lecture 6

Qiong Wang

Modeling Issues

mean return
optimization models

Applications

Application 1:
Sharpe Ratio
Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model

Lecture 6

Nonlinear Optimization: Portfolio Optimization

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Portfolio Optimization: basic elements

Lecture 6

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Black-
Litterman
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n assets, investment allocation x_i ($1 \leq i \leq n$)

$$x_1 + \dots + x_n = 1, \quad x_i \geq 0, \quad i = 1, \dots, n.$$

- return from asset i : r_i ($i = 1, \dots, n$),

$$\text{total return: } R(\mathbf{x}) = x_1 r_1 + \dots + x_n r_n.$$

expected return of the portfolio: $E[r_i] = \mu_i$ ($i = 1, \dots, n$),

$$\text{total expected return: } E[R(\mathbf{x})] = x_1 \mu_1 + \dots + x_n \mu_n = \mu^T \mathbf{x}$$

- covariance between asset i and j :

$$\sigma_{ij} = \text{cov}(r_i, r_j) = E[(r_i - E[r_i])(r_j - E[r_j])], \quad (\sigma_{ii}^2 = \sigma_i^2)$$

covariance matrix:

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \dots & \sigma_{1n} \\ \dots & \dots & \dots \\ \sigma_{n1} & \dots & \sigma_n^2 \end{pmatrix}$$

variance of the portfolio:

$$\text{var}(R(\mathbf{x})) = \mathbf{x}^T \Sigma \mathbf{x} = \sum_{ij} x_i x_j \sigma_{ij}.$$

Estimation of Mean Return

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Modeling Issues

mean return
optimization models

Applications

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Sharpe Ratio
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Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model

constant investment:

start from 100, no reinvestment; new capital to cover shortfall.

year	1	2	3
return	20%	-25%	100%
cash flow	20	-25	100

self-financed investment:

start from 100, reinvest all profits; no new capital.

year	1	2	3
return	20%	-25%	100%
position	120	90	180

$$\mu_a = \frac{20\% - 25\% + 100\%}{3} = 31.6\% (= (95/100)/3)$$

$$\mu_g = (1.2 \times 0.75 \times 2)^{1/3} - 1 = 21\% (= (180/100)^{1/3} - 1)$$

given a series of observations of past returns, r_1, \dots, r_T ,

- arithmetic mean

$$\mu_a = \frac{r_1 + \dots + r_T}{T}$$

- geometric mean

$$\mu_g = [(1 + r_1) \times \dots \times (1 + r_T)]^{1/T} - 1$$

Three Typical Mean-Variance Models

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Modeling Issues

mean return

optimization models

Applications

Application 1:
Sharpe Ratio

Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
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Model

- optimizes weighted sum

$$\begin{aligned} \max_{\mathbf{x} \geq 0} \quad & \left\{ x_1 \mu_1 + \dots + x_n \mu_n - \delta \frac{\mathbf{x}^T \Sigma \mathbf{x}}{2} \right\} \quad (\delta: \text{risk tolerance}) \\ \text{s.t.} \quad & x_1 + \dots + x_n = 1, \quad x_1, \dots, x_n \geq 0. \end{aligned}$$

- minimize volatility under mean return constraint

$$\begin{aligned} \min_{\mathbf{x} \geq 0} \quad & \left\{ \frac{1}{2} \mathbf{x}^T \Sigma \mathbf{x} \right\} \\ \text{s.t.} \quad & x_1 \mu_1 + \dots + x_n \mu_n \geq \bar{\mu} \quad (\bar{\mu}: \text{return target}) \\ & x_1 + \dots + x_n = 1, \quad x_1, \dots, x_n \geq 0. \end{aligned}$$

- maximize mean return under variance constraint

$$\begin{aligned} \max_{\mathbf{x} \geq 0} \quad & \{ x_1 \mu_1 + \dots + x_n \mu_n \} \\ \text{s.t.} \quad & \mathbf{x}^T \Sigma \mathbf{x} \leq \bar{\sigma}^2 \quad (\bar{\sigma}^2: \text{risk tolerance}). \\ & x_1 + \dots + x_n = 1, \quad x_1, \dots, x_n \geq 0. \end{aligned}$$

all same

Efficient Frontier and Equivalence (I)

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Modeling Issues

mean return

optimization models

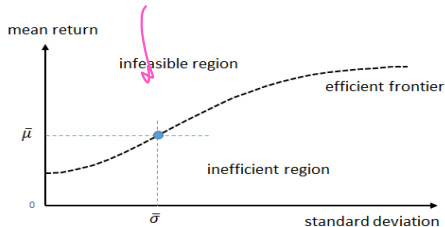
Applications

Application 1:
Sharpe Ratio

Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model



A point on the efficient frontier with $\sigma = \bar{\sigma}$ must be an optimal solution to

$$\begin{aligned} & \max_{\mathbf{x} \geq 0} \{x_1 \mu_1 + \dots + x_n \mu_n\} \\ \text{s.t.} \quad & \mathbf{x}^T \Sigma \mathbf{x} \leq \bar{\sigma}^2, \\ & x_1 + \dots + x_n = 1, \quad x_1, \dots, x_n \geq 0. \end{aligned}$$

Efficient Frontier and Equivalence (II)

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Modeling Issues

mean return

optimization models

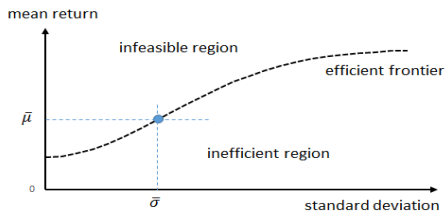
Applications

Application 1:
Sharpe Ratio

Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model



A point on the efficient frontier with $\mu = \bar{\mu}$ must be an optimal solution to

$$\begin{aligned} \min_{\mathbf{x} \geq 0} \quad & \left\{ \frac{1}{2} \mathbf{x}^T \Sigma \mathbf{x} \right\} \\ \text{s. t.} \quad & x_1 \mu_1 + \dots + x_n \mu_n \geq \bar{\mu}, \\ & x_1 + \dots + x_n = 1, \quad x_1, \dots, x_n \geq 0. \end{aligned}$$

Efficient Frontier and Equivalence

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Modeling Issues

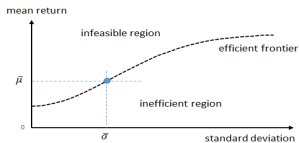
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optimization models

Applications

Application 1:
Sharpe Ratio
Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model



suppose we pick any $\bar{\mu}$,

$$\min_{\mathbf{x} \geq 0} \left\{ \frac{1}{2} \mathbf{x}^T \Sigma \mathbf{x} \right\} \quad \text{s.t.} \quad x_1 \mu_1 + \dots + x_n \mu_n \geq \bar{\mu}, \dots$$

and the objective value $\frac{1}{2}(\mathbf{x}^*)^T \Sigma \mathbf{x}^* = (\sigma^*)^2$.

now use σ^* as $\bar{\sigma}$ to solve

$$\max_{\mathbf{x} \geq 0} \{x_1 \mu_1 + \dots + x_n \mu_n\} \quad \text{s.t.} \quad \mathbf{x}^T \Sigma \mathbf{x} \leq (\sigma^*)^2, \dots$$

and the objective value $x_1^{**} \mu_1 + \dots + x_n^{**} \mu_n = \mu^*$

can $\mu^* < \bar{\mu}$? ~~cannot~~ because \rightarrow we maximize μ
can $\mu^* > \bar{\mu}$? ~~cannot~~ \rightarrow we are already at optimal frontier

Efficient Frontier

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Modeling Issues

mean return

optimization models

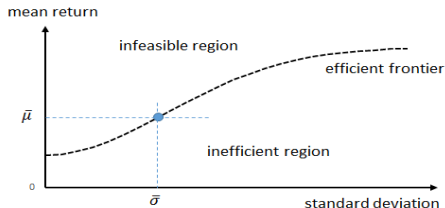
Applications

Application 1:
Sharpe Ratio

Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model



maximum mean return that can be achieved when the volatility level is below some threshold, or minimum volatility that can be achieved when the mean return exceeds a threshold

Efficient Frontier and Model 1

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Modeling Issues

mean return

optimization models

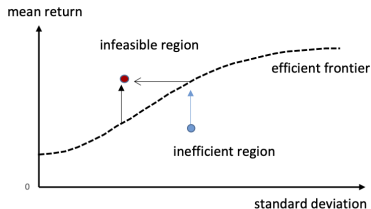
Applications

Application 1:
Sharpe Ratio

Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model



*risk taken
adj. term*

$$\max_{\mathbf{x} \geq 0} \left\{ x_1 \mu_1 + \dots + x_n \mu_n - \underbrace{\delta}_{\text{risk taken adj. term}} \frac{\mathbf{x}^T \Sigma \mathbf{x}}{2} \right\}$$
$$x_1 + \dots + x_n = 1, \quad x_1 \geq 0, \dots, x_n \geq 0.$$

- 1 The solution cannot be above the efficient frontier: it would mean there is a higher return at the same variance or a lower variance at the same return.
- 2 The solution cannot be below the efficient frontier: it would mean the objective value can still improve.

Equivalence of Optimization Models (I)

Lecture 6

Qiong Wang

Modeling Issues

mean return

optimization models

Applications

Application 1:
Sharpe Ratio

Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model

model:

$$\min_{\mathbf{x} \geq 0} \left\{ 1/2 \sum_{i,j} \sigma_{ij} x_i x_j \right\}$$

$$x_1 \mu_1 + \dots + x_n \mu_n \geq \bar{\mu},$$

$$\boxed{x_1 + \dots + x_n = 1,}$$

$$x_i \geq 0, \quad i = 1, \dots, n.$$

standard form:

$$\max_{\mathbf{x} \geq 0} \left\{ -1/2 \sum_{i,j} \sigma_{ij} x_i x_j \right\}$$

$$\bar{\mu} - x_1 \mu_1 - \dots - x_n \mu_n \leq 0,$$

$$\boxed{x_1 + \dots + x_n - 1 \leq 0,}$$

$$\boxed{1 - x_1 - \dots - x_n \leq 0,}$$

$$-x_i \leq 0, \quad i = 1, \dots, n.$$

$$\mathcal{L} = -1/2 \sum_{i,j} \sigma_{ij} x_i x_j - \lambda_\delta (\bar{\mu} - x_1 \mu_1 - \dots - x_n \mu_n)$$

$$\boxed{-\bar{\lambda}_1 (x_1 + \dots + x_n - 1) - \bar{\lambda}_2 (1 - x_1 - \dots - x_n)} + \sum_{i=1} \lambda_i x_i$$

$$= -1/2 \sum_{i,j} \sigma_{ij} x_i x_j - \lambda_\delta (\bar{\mu} - x_1 \mu_1 - \dots - x_n \mu_n) \boxed{+\bar{\lambda}(1 - x_1 - \dots - x_n)} + \sum_{i=1} \lambda_i x_i.$$

$$\lambda_\delta, \bar{\lambda}_1, \bar{\lambda}_2 \geq 0, \lambda_i \geq 0 \quad (i = 1, \dots, n), \text{ and } \boxed{\bar{\lambda} = \bar{\lambda}_1 - \bar{\lambda}_2} \quad (<, =, > 0).$$

← combine

Equivalence of Optimization Models (II)

Lecture 6

Qiong Wang

Modeling Issues

mean return

optimization models

Applications

Application 1:

Sharpe Ratio

Application 2:

Solving

Mean-Variance

Model by LP

Limitations

Black-
Litterman
Model

Repeat: Lagrangian ($\lambda_\delta \geq 0$, $\lambda_i \geq 0$ ($i = 1, \dots, n$), $\bar{\lambda}$):

$$\mathcal{L} = -\frac{1}{2} \sum_{i,j} \sigma_{ij} x_i x_j - \lambda_\delta (\bar{\mu} - x_1 \mu_1 - \dots - x_n \mu_n) + \bar{\lambda} (1 - x_1 - \dots - x_n) + \sum_{i=1}^n \lambda_i x_i.$$

KKT condition:

$$-\sum_{j=1}^n \sigma_{ij} x_j + \lambda_\delta \mu_i - \bar{\lambda} + \lambda_i = 0, \quad i = 1, \dots, n,$$

$$\bar{\lambda} (1 - x_1 - \dots - x_n) = 0, \quad x_1 + \dots + x_n = 1,$$

$$\lambda_i x_i = 0, \quad x_i \geq 0, \quad \lambda_i \geq 0, \quad i = 1, \dots, n,$$

$$\lambda_\delta (x_1 \mu_1 + \dots + x_n \mu_n - \bar{\mu}) = 0, \quad x_1 \mu_1 + \dots + x_n \mu_n \geq \bar{\mu}, \quad \lambda_\delta \geq 0.$$

The solution to the above x_i^* is an optimal solution for

$$\min_{x \geq 0} \left\{ \frac{1}{2} \sum_{i,j} \sigma_{ij} x_i x_j : x_1 \mu_1 + \dots + x_n \mu_n \geq \bar{\mu}, x_1 + \dots + x_n = 1. \right\}$$

Equivalence of Optimization Models (III)

Lecture 6

Qiong Wang

Modeling
Issues

mean return

optimization models

Applications

Application 1:
Sharpe Ratio

Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model

Now suppose that we minimize

$$\begin{aligned} \max_{\mathbf{x} \geq 0} \quad & \left\{ x_1 \mu_1 + \dots + x_n \mu_n - \delta \frac{\mathbf{x}^T \Sigma \mathbf{x}}{2} \right\} \\ \text{subj. to} \quad & x_1 + \dots + x_n = 1, \quad x_1 \geq 0, \dots, x_n \geq 0. \end{aligned}$$

Lagrangian is

$$\mathcal{L} = x_1 \mu_1 + \dots + x_n \mu_n - \delta \frac{\mathbf{x}^T \Sigma \mathbf{x}}{2} + \bar{\lambda}'(1 - x_1 - \dots - x_n) + \sum_{i=1}^n \lambda'_i x_i.$$

KKT conditions

$$\mu_i - \delta \sum_{j=1}^n \sigma_{ij} x_j - \bar{\lambda}' + \lambda'_i = 0, \quad i = 1, \dots, n,$$

$$\bar{\lambda}'(1 - x_1 - \dots - x_n) = 0, \quad x_1 + \dots + x_n = 1,$$

$$\lambda'_i x_i = 0, \quad x_i \geq 0, \quad \lambda'_i \geq 0, \quad i = 1, \dots, n.$$

Equivalence of Optimization Models (IV)

Lecture 6

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Modeling
Issues

mean return

optimization models

Applications

Application 1:
Sharpe Ratio

Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model

KKT conditions of model I (max. mean- δ variance):

$$\mu_i - \delta \sum_{j=1}^n \sigma_{ij} x_j - \bar{\lambda}' + \lambda'_i = 0, \quad i = 1, \dots, n,$$

$$\bar{\lambda}'(1 - x_1 - \dots - x_n) = 0, \quad x_1 + \dots + x_n = 1,$$

$$\lambda'_i x_i = 0, \quad x_i \geq 0, \quad \lambda'_i \geq 0, \quad i = 1, \dots, n.$$

KKT Conditions of Model II (minimizing variance):

$$-\sum_{j=1}^n \sigma_{ij} x_j + \lambda_\delta \mu_i - \bar{\lambda} + \lambda_i = 0, \quad i = 1, \dots, n,$$

$$\bar{\lambda}(1 - x_1 - \dots - x_n) = 0, \quad x_1 + \dots + x_n = 1,$$

$$\lambda_i x_i = 0, \quad x_i \geq 0, \quad \lambda_i \geq 0, \quad i = 1, \dots, n,$$

$$\lambda_\delta (x_1 \mu_1 + \dots + x_n \mu_n - \bar{\mu}) = 0, \quad x_1 \mu_1 + \dots + x_n \mu_n \geq \bar{\mu}, \quad \lambda_\delta \geq 0.$$

What if we let

Handwritten notes:
 - $\lambda_\delta \neq 0$ (with arrow pointing to λ_δ in the equation below)
 - λ_δ is circled in pink, with a note "everything is same \rightarrow equivalent"
 - λ_δ is also circled in pink, with a note "everything is same \rightarrow equivalent"
 - λ_i is circled in pink, with a note "everything is same \rightarrow equivalent"

$$\delta = 1/\lambda_\delta, \quad \bar{\lambda}' = \bar{\lambda}/\lambda_\delta, \quad \lambda'_i = \lambda_i/\lambda_\delta, \quad i = 1, \dots, n.$$

Equivalence of Optimization Models (V)

Lecture 6

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Modeling Issues

mean return

optimization models

Applications

Application 1:

Sharpe Ratio

Application 2:

Solving

Mean-Variance

Model by LP

Limitations

Black-Litterman Model

If $\lambda_\delta = 0$, then the KKT conditions of Model II become:

$$-\sum_{j=1}^n \sigma_{ij} x_j^* - \bar{\lambda} + \lambda_i = 0, \quad i = 1, \dots, n,$$

$$\bar{\lambda}(1 - x_1^* - \dots - x_n^*) = 0 \quad x_1^* + \dots + x_n^* \leq 1,$$

$$\lambda_i x_i^* = 0, \quad x_i^* \geq 0, \quad \lambda_i \geq 0, \quad i = 1, \dots, n,$$

$$x_1^* \mu_1 + \dots + x_n^* \mu_n \geq \bar{\mu}.$$

which is the same KKT conditions for the problem:

$$\min_{\mathbf{x} \geq 0} \left\{ \frac{1}{2} \sum_{i,j} \sigma_{ij} x_i x_j : x_1 + \dots + x_n = 1, \quad x_i \geq 0, \quad i = 1, \dots, n. \right\}$$

for model I, this means $\delta = 1/\lambda_\delta = \infty$ in the objective function:

$$\max_{\mathbf{x} \geq 0} \left\{ x_1 \mu_1 + \dots + x_n \mu_n - \delta \frac{\mathbf{x}^T \Sigma \mathbf{x}}{2} \right\}$$

Efficient Frontier and Sharpe Ratio

Lecture 6

Qiong Wang

Modeling Issues

mean return
optimization models

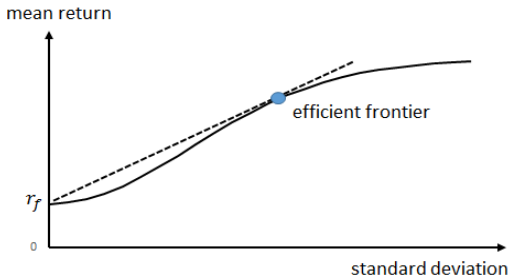
Applications

Application 1:
Sharpe Ratio

Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model



(r_f : risk-free return)

each point on this curve correspond to a solution to an optimal solution of any of the three optimization problems (with a special choice of $\bar{\mu}$, $\bar{\sigma}$, and δ).

There is a special point on the curve. How to find it?

Sharpe ratio

$$h(\mathbf{x}) = \frac{\boldsymbol{\mu}^T \mathbf{x} - r_f}{\sqrt{\mathbf{x}^T \boldsymbol{\Sigma} \mathbf{x}}}.$$

Sharpe Ratio: Problem Formulation:

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Modeling Issues

mean return
optimization models

Applications

Application 1:
Sharpe Ratio

Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model

$$\max_{\mathbf{x}} \left\{ \frac{\boldsymbol{\mu}^T \mathbf{x} - r_f}{\sqrt{\mathbf{x}^T \boldsymbol{\Sigma} \mathbf{x}}} \right\}.$$

subject to : $x_1 + \dots + x_n = 1,$

$$x_i \geq 0, \quad i = 1, \dots, n.$$

The objective function looks complicated, need some simplification:

$$\min_{\mathbf{x}} \left\{ \frac{\sqrt{\mathbf{x}^T \boldsymbol{\Sigma} \mathbf{x}}}{\boldsymbol{\mu}^T \mathbf{x} - r_f} \right\}.$$

Still difficult to handle directly.

Sharpe Ratio: Solution Techniques

Lecture 6

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Modeling Issues

mean return
optimization models

Applications

Application 1:
Sharpe Ratio

Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model

suppose that x_i^* ($i = 1, \dots, n$) maximizes the Sharpe Ratio.

define κ such that

$$\mu_1 x_1^* + \dots + \mu_n x_n^* - r_f = 1/\kappa$$

We do not know values of x_i^* ($i = 1, \dots, n$) and κ , but they exist, and $\kappa > 0$.

We can rewrite the above as

$$\begin{aligned} \mu^T \mathbf{x}^* - r_f &= \mu_1 x_1^* + \dots + \mu_n x_n^* - (x_1^* + \dots + x_n^*) r_f \\ &= (\mu_1 - r_f) x_1^* + \dots + (\mu_n - r_f) x_n^* = 1/\kappa \end{aligned}$$

so

$$(\mu_1 - r_f) (\kappa x_1^*) + \dots + (\mu_n - r_f) (\kappa x_n^*) = 1.$$

let

$$\text{Let } y_i^* = \kappa x_i^*, \quad i = 1, \dots, n.$$

then

$$(\mu_1 - r_f) y_1^* + \dots + (\mu_n - r_f) y_n^* = 1.$$

and

$$\frac{\sqrt{(\mathbf{x}^*)^T \Sigma \mathbf{x}^*}}{\mu^T \mathbf{x}^* - r_f} = \frac{\sqrt{(\mathbf{y}^*)^T \Sigma \mathbf{y}^* / \kappa}}{[(\mu_1 - r_f) y_1^* + \dots + (\mu_n - r_f) y_n^*] / \kappa} = \sqrt{(\mathbf{y}^*)^T \Sigma \mathbf{y}^*}$$

invest d optim -

if $\kappa \leq 0$

port $\leq r_f$

$\kappa x_i^* = \text{invest}$

choose x_i in order choose y_i

covariance

$$y_i^* = \kappa x_i^*$$

Maximum Sharpe Ratio

Lecture 6

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Modeling
Issues

mean return
optimization models

Applications

Application 1:
Sharpe Ratio

Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model

Repeat:

$$\frac{\sqrt{(\mathbf{x}^*)^T \Sigma \mathbf{x}^*}}{\mu^T \mathbf{x}^* - r_f} = \sqrt{(\mathbf{y}^*)^T \Sigma \mathbf{y}^*}, \quad (\mu_1 - r_f)y_1^* + \dots + (\mu_n - r_f)y_n^* = 1.$$

because $\mathbf{y}^* = \kappa \mathbf{x}^*$:

$$\begin{aligned} x_1^* + \dots + x_n^* = 1 &\longrightarrow y_1^* + \dots + y_n^* = \kappa, \\ x_i^* \geq 0 &\longrightarrow y_i^* \geq 0 \quad (i = 1, \dots, n). \end{aligned}$$

So we can determine \mathbf{y}^* and κ by:

$$\begin{aligned} &\min_{\mathbf{y} \geq 0, \kappa \geq 0} \{ \mathbf{y}^T \Sigma \mathbf{y} \} && \text{easier to solve} \\ &\text{subject to} && \downarrow \\ &(\mu_1 - r_f)y_1 + \dots + (\mu_n - r_f)y_n = 1, && y_1 + \dots + y_n = \kappa. \end{aligned}$$

The optimal solution to the original problem is

$$x_i^* = y_i^* / (y_1^* + \dots + y_n^*), \quad i = 1, \dots, n.$$

and the Sharpe ratio is $\frac{1}{\sqrt{(\mathbf{y}^*)^T \Sigma \mathbf{y}^*}}$.

Deviation in Return

Lecture 6

Qiong Wang

Modeling Issues

mean return
optimization models

Applications

Application 1:
Sharpe Ratio
Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model

$$\min_{\mathbf{x} \geq 0} \left\{ \frac{1}{2} \mathbf{x}^T \Sigma \mathbf{x} \right\} \quad \text{subject to:} \quad x_1 \mu_1 + \dots + x_n \mu_n \geq \bar{\mu}, \quad x_1 + \dots + x_n = 1.$$

objective value (portfolio variance): $\mathbf{x}^T \Sigma \mathbf{x} = \sum_{i,j} \sigma_{ij} x_i x_j$.

where

$$\sigma_{ij} = \mathbf{E}[(r_i - \mu_i)(r_j - \mu_j)]$$

so

$$\mathbf{x}^T \Sigma \mathbf{x} = \mathbf{E} \left[\sum_{i,j} (r_i - \mu_i)(r_j - \mu_j) x_i x_j \right] = \mathbf{E} \left[\left(\sum_{i=1}^n (r_i - \mu_i) x_i \right)^2 \right].$$

deviation of the portfolio return from its mean

$$U_{\mathbf{x}} = \sum_{i=1}^n (r_i - \mu_i) x_i \quad \longrightarrow \quad \mathbf{x}^T \Sigma \mathbf{x} = \mathbf{E}[U_{\mathbf{x}}^2]$$

mean and variance of this deviation

$$\mathbf{E}[U_{\mathbf{x}}] = 0 \quad (\text{because } \mathbf{E}[r_i] = \mu_i) \quad \text{var}(U_{\mathbf{x}}) = \mathbf{E}[U_{\mathbf{x}}^2] = \sigma_{\mathbf{x}}^2$$

Variance for Mean Absolute Variation (MAD)

Lecture 6

Qiong Wang

Modeling Issues

mean return
optimization models

Applications

Application 1:
Sharpe Ratio

Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model

- the deviation of the actual portfolio return from its mean:

$$U_x = (r_1 - \mu_1)x_1 + \dots + (r_n - \mu_n)x_n.$$

- minimizing $E[U_x]$ is meaningless because it is 0.
- minimizing portfolio variance is to minimize $E[U_x^2]$

what about minimizing $E[|U_x|]$ (Mean Absolute Deviation (MAD))?

if all returns r_i ($i = 1, \dots, n$) are normally distributed, then U_x is normally distributed with mean 0 and standard deviation σ_x .

$$\begin{aligned} f(u) &= \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{u^2}{2\sigma_x^2}} \quad \text{im} \\ E[|U_x|] &= \frac{1}{\sqrt{2\pi}\sigma_x} \int_{-\infty}^{\infty} |u| e^{-\frac{u^2}{2\sigma_x^2}} du = \frac{2}{\sqrt{2\pi}\sigma_x} \int_0^{\infty} u e^{-\frac{u^2}{2\sigma_x^2}} du \quad \text{no abs} \\ &= \frac{\sqrt{2}}{\sqrt{\pi}\sigma_x} \left(-\sigma_x^2 e^{-\frac{u^2}{2\sigma_x^2}} \right) \Big|_0^{\infty} = \sqrt{\frac{2}{\pi}} \sigma_x \end{aligned}$$

in this case, minimizing the variance is the equivalent to minimizing MAD

Minimizing MAD

Lecture 6

Qiong Wang

Modeling Issues

mean return

optimization models

Applications

Application 1:

Sharpe Ratio

Application 2:

Solving

Mean-Variance

Model by LP

Limitations

Black-

Litterman

Model

assume return is normally distributed.

there are T samples of asset returns i : r_{it} , $t = 1, \dots, T$, μ_i is the expected return ($i = 1, \dots, n$)

to choose the optimal investment amounts \mathbf{x} , you can:

- let

$$\hat{\sigma}_{ij} = \frac{1}{T} \sum_{t=1}^T (r_{it} - \mu_i)(r_{jt} - \mu_j)$$

and minimize the (estimated) portfolio variance:

$$\min_{\mathbf{x}} \left\{ \sum_{ij} \hat{\sigma}_{ij} x_i x_j \right\}$$

- or minimize MAD

$$\min_{\mathbf{x}} \left\{ \sum_{t=1}^T \left| \sum_{i=1}^n (r_{it} - \mu_i) x_i \right| \right\}$$

↓ Same

under the same constraints

$$\mu_1 x_1 + \dots + \mu_n x_n \geq \bar{\mu}, \quad x_1 + \dots + x_n = 1, \quad x_1 \geq 0, \dots, x_n \geq 0.$$

Techniques for MAD Minimization

Lecture 6

how to handle the MAD objective function:

$$\sum_{t=1}^T \left| \sum_{i=1}^n (r_{it} - \mu_i) x_i \right| ?$$

- 1 for each $t = 1, \dots, T$, introduce new variables y_t and z_t ;
- 2 change the objective function to

$$\min \left\{ \sum_{t=1}^T (y_t + z_t) \right\}$$

divide

- 3 add constraints: for $t = 1, \dots, T$:

constraint \rightarrow $y_t - z_t = \sum_{i=1}^n (r_{it} - \mu_i) x_i, \quad y_t \geq 0, \quad z_t \geq 0,$

definition

so: $y_t + z_t = 2z_t + \sum_{i=1}^n (r_{it} - \mu_i) x_i$ and $y_t + z_t = 2y_t - \sum_{i=1}^n (r_{it} - \mu_i) x_i.$

Qiong Wang

Modeling
Issues

mean return
optimization models

Applications

Application 1:
Sharpe Ratio

Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model

MAD Minimization Problem

Lecture 6

Qiong Wang

Modeling
Issues

mean return
optimization models

Applications

Application 1:
Sharpe Ratio

Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model

$$\min_{y \geq 0, z \geq 0, x} \left\{ \sum_{t=1}^T (y_t + z_t) \right\} \quad \text{subject to: } y_t - z_t = \sum_{i=1}^n (r_{it} - \mu_i) x_i, t = 1, \dots, T, \dots$$

Handwritten notes: $z_t = y_t - z_t$ (blue), $y_t = \sum_{i=1}^n (r_{it} - \mu_i) x_i + z_t$ (pink)

how this formulation work?

for given x:

- when $\sum_{i=1}^n (r_{it} - \mu_i) x_i \geq 0$, it is optimal that

$$y_t = \sum_{i=1}^n (r_{it} - \mu_i) x_i \quad \text{and} \quad z_t = 0.$$

Handwritten notes: "positive" (pink), "and" (red), "min" (pink), "2z_t" (pink), "z_t must equal to 0" (pink)

- when $\sum_{i=1}^n (r_{it} - \mu_i) x_i < 0$, it is optimal that

$$y_t = 0 \quad \text{and} \quad z_t = -\sum_{i=1}^n (r_{it} - \mu_i) x_i.$$

Handwritten notes: "if" (red), "min" (blue), "y_t + y_t - z_t" (blue), "neg" (pink), "positive" (red), "also posn" (pink)

in either case

$$y_t + z_t = \left| \sum_{i=1}^n (r_{it} - \mu_i) x_i \right|$$

Limitation of Mean-Variance Model: Stochastic Dominance

Lecture 6

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Modeling Issues

mean return

optimization models

Applications

Application 1:
Sharpe Ratio

Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model

Two investment opportunities: return on investment 1 is either 10% or 30%.
return on investment 2 is guaranteed at 10%. (so $\sigma_1 > 0$, $\sigma_2 = 0$, $\sigma_{12} = 0$).

what is the optimal allocation of your budget between the two investments?

let x be the % allocated to investment 1, so the % in 2 is $1 - x$

- can this optimal allocation be the solution of

$$\max_{x \geq 0} \left\{ \mu_1 x + \mu_2 (1 - x) \mid \text{s. t. } \sigma_1^2 x^2 \leq \bar{\sigma}^2, \dots \right\}$$

- can the optimal allocation be the solution of:

$$\max_{x \geq 0} \left\{ \mu_1 x + \mu_2 (1 - x) - \frac{\delta}{2} \sigma_1^2 x^2 \mid \text{s.t.} \dots \dots \dots \right\}.$$

- can this optimal allocation be the solution:

$$\min_{x \geq 0} \left\{ \frac{1}{2} \sigma_1^2 x^2 \mid \text{s.t. } \mu_1 x + \mu_2 (1 - x) \geq \bar{\mu} \right\}.$$

Black-Litterman Model: Motivation

Lecture 6

Qiong Wang

Modeling Issues

mean return
optimization models

Applications

Application 1:
Sharpe Ratio
Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-Litterman Model

Mean-variance model

$$\max_{\mathbf{x}} \{ \mathbf{E}[R(\mathbf{x})] - \delta \text{var}(\mathbf{x}) \mid C\mathbf{x} \leq \mathbf{d} \}$$

where

$$\mathbf{E}[R(\mathbf{x})] = \mu_1 x_1 + \cdots + \mu_n x_n, \quad \text{var}(\mathbf{x}) = \mathbf{x}^T \Sigma \mathbf{x} = \sum_{i,j} x_i x_j \sigma_{ij},$$

$\boldsymbol{\mu} \doteq (\mu_1, \dots, \mu_n)$ (mean return)

$\mathbf{E}[\mathbf{r}] \doteq (\mathbf{E}[r_1], \dots, \mathbf{E}[r_n])$ (market equilibrium return)

- The solution is sensitive to values of the expected return ($\boldsymbol{\mu}$).
- Covariance matrix Σ can be estimated with conventional means while estimating the expected return $\boldsymbol{\mu}$ needs more effort.
- The need to include personal view into the estimated values.

Starting Point: CAPM

Lecture 6

Qiong Wang

Modeling Issues

mean return
optimization models

Applications

Application 1:
Sharpe Ratio

Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model

- Question: will adding a new asset i improve the Sharpe Ratio?
- Question in math:

$$\max_{x \geq 0} \left\{ \hat{R}(x) \doteq \frac{\mathbf{E}[r_i]x + \mathbf{E}[r_m](1-x) - r_f}{\sqrt{\text{var}(r_i x + r_m(1-x))}} \right\} \implies x^* > 0?$$

r_i return of asset i ; r_m : equilibrium return; r_f : risk-free return.

- Answer:

$$\left. \frac{d\hat{R}(x)}{dx} \right|_{x=0} > 0?$$

- The inequality holds if and only if

$$\mathbf{E}[r_i] > r_f + \frac{\text{cov}(r_i, r_m)}{\text{var}(r_m)} (\mathbf{E}[r_m] - r_f).$$

Deriving $d\hat{R}(x)/dx$

Lecture 6

Qiong Wang

Modeling Issues

mean return
optimization models

Applications

Application 1:
Sharpe Ratio
Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model

$$g(x) = r_i x + r_m(1 - x) = r_m + (r_i - r_m)x, \quad \hat{R}(x) \doteq \frac{\mathbf{E}[g(x)] - r_f}{\sqrt{\text{var}(g(x))}}.$$

$$\frac{d\hat{R}(x)}{dx} = \frac{\mathbf{E}[r_i] - \mathbf{E}[r_m]}{\sqrt{\text{var}(g(x))}} - \frac{1}{2} \frac{\mathbf{E}[g(x)] - r_f}{\sqrt{(\text{var}(g(x)))^3}} \frac{d\text{var}(g(x))}{dx}$$

since

$$\text{var}(g(x)) = \mathbf{E} \left[(g(x) - \mathbf{E}[g(x)])^2 \right]$$

$$\frac{d\text{var}(g(x))}{dx} = 2\mathbf{E}[(g(x) - \mathbf{E}[g(x)]) * ((r_i - \mathbf{E}[r_i]) - (r_m - \mathbf{E}[r_m]))]$$

at $x = 0$, $g(x) = r_m$, so

$$\frac{d\text{var}(g(x))}{dx} = 2(\text{cov}(r_i, r_m) - \text{var}(r_m)).$$

so

$$\frac{d\hat{R}(x)}{dx} = \frac{1}{\sqrt{\text{var}(r_m)}} \left(\mathbf{E}[r_i] - r_f - \frac{\text{cov}(r_i, r_m)}{\text{var}(r_m)} (\mathbf{E}[r_m] - r_f) \right).$$

Equilibrium Expected Return (Market View)

Lecture 6

Qiong Wang

Modeling Issues

mean return
optimization models

Applications

Application 1:
Sharpe Ratio

Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model

- more buy of stock i if:

$$\mathbf{E}[r_i] > r_f + \frac{\text{cov}(r_i, r_m)}{\text{var}(r_m)} (\mathbf{E}[r_m] - r_f) \implies \left. \frac{d\hat{R}}{dx} \right|_{x=0} > 0.$$

price goes up, return goes down.

- more sell of stock i if:

$$\mathbf{E}[r_i] < r_f + \frac{\text{cov}(r_i, r_m)}{\text{var}(r_m)} (\mathbf{E}[r_m] - r_f) \implies \left. \frac{d\hat{R}}{dx} \right|_{x=0} < 0.$$

price goes down, return goes up.

- at equilibrium

$$\mathbf{E}[r_i] = r_f + \frac{\text{cov}(r_i, r_m)}{\text{var}(r_m)} (\mathbf{E}[r_m] - r_f) \implies \left. \frac{d\hat{R}}{dx} \right|_{x=0} = 0.$$

Private Opinion

Lecture 6

Qiong Wang

Modeling Issues

mean return
optimization models

Applications

Application 1:
Sharpe Ratio
Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model

- Example 1: return of stock 1 will outperform that of stock 2 by 5%

$$\mu_1 - \mu_2 = 5\%.$$

- Example 2: average market return will be around 3%

$$\frac{\mu_1 + \cdots + \mu_N}{N} = 3\%$$

- Example 3: the P/E ratio of stock 3 will be around 14:

$$\mu_3 = \frac{14 \times \text{earning}}{\text{current price}} - 1.$$

- Let

$$P = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1/N & 1/N & 1/N & \cdots & 1/N \\ 0 & 0 & 1 & \cdots & 0 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} 5\% \\ 3\% \\ \frac{14 \times \text{earning}}{\text{current price}} - 1 \end{pmatrix}$$

$$P\mu = \mathbf{q},$$

Estimate the Expected Returns by Black-Litterman: Perfect Private Opinion

Lecture 6

Qiong Wang

Modeling Issues

mean return
optimization models

Applications

Application 1:
Sharpe Ratio
Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-
Litterman
Model

$$\mu_i = \mathbf{E}[r_i] + v_i, \quad i = 1, \dots, N.$$

i.e., expected returns are random variables.

Assume that $v \sim \mathbf{N}(\mathbf{0}, \tau \Sigma)$ (τ : a small constant):

The value of μ is estimated by

$$\min_{\mu} \{(\mu - \mathbf{E}[r])^T (\tau \Sigma)^{-1} (\mu - \mathbf{E}[r])\}$$

$$\text{subject to: } P\mu = \mathbf{q}.$$

The solution is

$$\mu^* = \mathbf{E}[r] + (\tau \Sigma) P^T (P \tau \Sigma P^T)^{-1} (\mathbf{q} - P \mathbf{E}[r]).$$

Derivation of Black-Litterman: Perfect Information

Lecture 6

Qiong Wang

Modeling Issues

mean return

optimization models

Applications

Application 1:
Sharpe Ratio

Application 2:
Solving
Mean-Variance
Model by LP

Limitations

Black-Litterman Model

- Lagrangian: let $\tilde{\mu} = \mu - \mathbf{E}[r]$ and $\tilde{\mathbf{q}} = \mathbf{q} - P\mathbf{E}[r]$.

$$\mathcal{L}(\mu, \lambda) = \tilde{\mu}^T (\tau \Sigma)^{-1} \tilde{\mu} + \lambda (P\tilde{\mu} - \tilde{\mathbf{q}})$$

- KKT condition:

$$2(\tau \Sigma)^{-1} \tilde{\mu} = P^T \lambda, \quad \text{and} \quad P\tilde{\mu} = \tilde{\mathbf{q}}.$$

- Derivation: from the first KKT condition:

$$\tilde{\mu} = (\tau \Sigma) P^T \lambda / 2.$$

apply to the second KKT condition:

$$P(\tau \Sigma) P^T \lambda = 2\tilde{\mathbf{q}} \quad \text{i.e.,} \quad \lambda = 2[P(\tau \Sigma) P^T]^{-1} \tilde{\mathbf{q}}$$

use the above to eliminate λ :

$$\tilde{\mu} = (\tau \Sigma) P^T \lambda / 2 = (\tau \Sigma) P^T [P(\tau \Sigma) P^T]^{-1} \tilde{\mathbf{q}}$$

i.e.,

$$\mu^* = \mathbf{E}[r] + (\tau \Sigma) P^T (P \tau \Sigma P^T)^{-1} (\mathbf{q} - P\mathbf{E}[r]).$$