Lecture 6

Nonlinear Optimization: Portfolio Optimization

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Portfolio Optimization: basic elements

n assets, investment allocation x_i $(1 \leq i \leq n)$

$$x_1 + \cdots + x_n = 1, \quad x_i \geq 0, \quad i = 1, ..., n.$$

- return from asset i: r_i $(i = 1, ..., n)$,

$$\text{total return:} \quad R(\mathbf{x}) = x_1 r_1 + ... + x_n r_n.$$

expected return of the portfolio: $E[r_i] = \mu_i$ $(i = 1, ..., n)$,

$$\text{total expected return:} \quad E[R(\mathbf{x})] = x_1\mu_1 + ... + x_n\mu_n = \mu^T\mathbf{x}$$

- covariance between asset i and j:

$$\sigma_{ij} = cov(r_i, r_j) = E\left[(r_i - E[r_i])(r_j - E[r_j])\right], \quad (\sigma_{ii}^2 = \sigma_i^2)$$

covariance matrix:

$$\Sigma = \begin{pmatrix} \sigma_1^2 & .... & \sigma_{1n} \\ .... & ... & ... \\ \sigma_{n1} & ... & \sigma_n^2 \end{pmatrix}$$

variance of the portfolio:

$$var(R(\mathbf{x})) = \mathbf{x}^T\Sigma\mathbf{x} = \sum_{ij} x_i x_j \sigma_{ij}.$$

Estimation of Mean Return

constant investment:
start from 100, no reinvestment; new capital to cover shortfall.

year	1	2	3
return	20%	-25%	100%
cash flow	20	-25	100

self-financed investment:
start from 100, reinvest all profits; no new capital.

year	1	2	3
return	20%	-25%	100%
position	120	90	180

$$\mu_a = \frac{20\% - 25\% + 100\%}{3} = 31.6\% \; (= (95/100)/3)$$

$$\mu_g = (1.2 \times 0.75 \times 2)^{1/3} - 1 = 21\% \; (= (180/100)^{1/3} - 1)$$

given a series of observations of past returns, $r_1, \ldots, r_T$,

- arithmetic mean

$$\mu_a = \frac{r_1 + ... + r_T}{T}$$

- geometric mean

$$\mu_g = [(1 + r_1) \times .... \times (1 + r_T)]^{1/T} - 1$$

Three Typical Mean-Variance Models

- optimizes weighted sum

$$\max_{\mathbf{x}\geq 0}\left\{x_1\mu_1+....+x_n\mu_n-\delta\frac{\mathbf{x}^T\Sigma\mathbf{x}}{2}\right\}\quad(\delta:\text{ risk tolerance})$$
$$\text{s.t.}\quad x_1+....+x_n=1,\quad x_1,...,x_n\geq 0.$$

- minimize volatility under mean return constraint

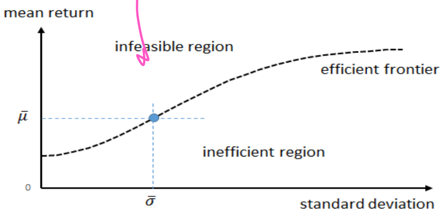
all same

$$\min_{\mathbf{x}\geq 0}\left\{\frac{1}{2}\mathbf{x}^T\Sigma\mathbf{x}\right\}$$
$$\text{s. t.}\quad x_1\mu_1+....+x_n\mu_n\geq\bar{\mu}\qquad(\bar{\mu}:\text{ return target})$$
$$x_1+....+x_n=1,\quad x_1,...,x_n\geq 0.$$

- maximize mean return under variance constraint

$$\max_{\mathbf{x}\geq 0}\{x_1\mu_1+....+x_n\mu_n\}$$
$$\text{s.t.}\quad \mathbf{x}^T\Sigma\mathbf{x}\leq\bar{\sigma}^2\qquad(\bar{\sigma}^2:\text{ risk tolerance}).$$
$$x_1+....+x_n=1,\quad x_1,...,x_n\geq 0.$$

Efficient Frontier and Equivalence (I)



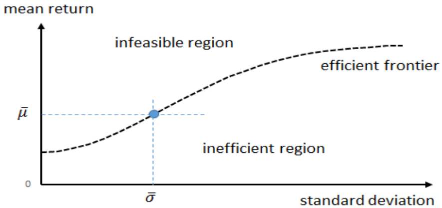
A point on the efficient frontier with $\sigma = \bar{\sigma}$ must be an optimal solution to

$$\max_{\mathbf{x} \geq 0} \{x_1\mu_1 + ... + x_n\mu_n\}$$

$$\text{s.t.} \quad \mathbf{x}^T \Sigma \mathbf{x} \leq \bar{\sigma}^2,$$

$$x_1 + ... + x_n = 1, \quad x_1, ..., x_n \geq 0.$$

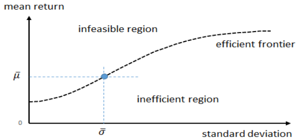
Efficient Frontier and Equivalence (II)



A point on the efficient frontier with $\mu = \bar{\mu}$ must be an optimal solution to

$$\begin{aligned} &\min_{\mathbf{x}\geq 0} \left\{ \frac{1}{2}\mathbf{x}^T \Sigma \mathbf{x} \right\} \\ \text{s. t.} \quad & x_1\mu_1 + ... + x_n\mu_n \geq \bar{\mu}, \\ & x_1 + ... + x_n = 1, \quad x_1, ..., x_n \geq 0. \end{aligned}$$

Efficient Frontier and Equivalence



suppose we pick any $\bar{\mu}$,

$$\min_{\mathbf{x}\geq 0}\left\{\frac{1}{2}\mathbf{x}^T\Sigma\mathbf{x}\right\} \quad \text{s.t.} \quad x_1\mu_1+....+x_n\mu_n\geq\bar{\mu}, ...$$

and the objective value $1/2(\mathbf{x}^*)^T\Sigma\mathbf{x}^* = (\sigma^*)^2$.

so $\mu^* \geq \bar{\mu}$

now use σ^* as $\bar{\sigma}$ to solve

maximize

$$\max_{\mathbf{x}\geq 0}\{x_1\mu_1+....+x_n\mu_n\} \quad \text{s.t.} \quad \mathbf{x}^T\Sigma\mathbf{x}\leq(\sigma^*)^2, ...$$

optimum

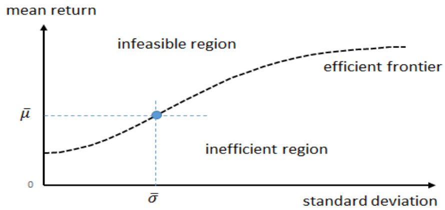
and the objective value $x_1^{**}\mu_1+\cdots+x_n^{**}\mu_n=\mu^*$

cannot → because → we maximize μ

can $\mu^*<\bar{\mu}$? can $\mu^*>\bar{\mu}$?

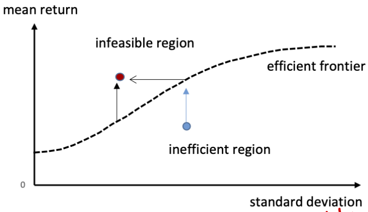
cannot → we are already at optimal frontier

Efficient Frontier



maximum mean return that can be achieved when the volatility level is below some threshold, or minimum volatility that can be achieved when the mean return exceeds a threshold

Efficient Frontier and Model 1



risk taker adj. term

$$\max_{\mathbf{x}\geq 0}\left\{x_1\mu_1+....+x_n\mu_n-\delta\frac{\mathbf{x}^T\Sigma\mathbf{x}}{2}\right\}$$

$$x_1+....+x_n=1,\quad x_1\geq 0,...,x_n\geq 0.$$

1. The solution cannot above the efficient frontier: it would mean there is a higher return at the same variance or a lower variance at the same return.
2. The solution cannot below the efficient frontier: it would mean the objective value can still improve.

Equivalence of Optimization Models (I)

model:

$$\min_{\mathbf{x}\geq 0}\left\{1/2\sum_{i,j}\sigma_{ij}x_ix_j\right\}$$

$$x_1\mu_1+\cdots+x_n\mu_n\geq\bar{\mu},$$

$$\boxed{x_1+....+x_n=1,}$$

$$x_i\geq 0,\quad i=1,...,n.$$

standard form:

$$\max_{\mathbf{x}\geq 0}\left\{-1/2\sum_{i,j}\sigma_{ij}x_ix_j\right\}$$

$$\bar{\mu}-x_1\mu_1-....-x_n\mu_n\leq 0,$$

$$\boxed{x_1+....+x_n-1\leq 0,}$$

$$\boxed{1-x_1-....-x_n\leq 0,}$$

$$-x_i\leq 0,\quad i=1,....,n.$$

$$\mathcal{L}=-1/2\sum_{i,j}\sigma_{ij}x_ix_j-\lambda_\delta(\bar{\mu}-x_1\mu_1-\cdots-x_n\mu_n)$$

$$\boxed{-\bar{\lambda}_1(x_1+...+x_n-1)-\bar{\lambda}_2(1-x_1-....-x_n)}+\sum_{i=1}\lambda_ix_i$$

$$=-1/2\sum_{i,j}\sigma_{ij}x_ix_j-\lambda_\delta(\bar{\mu}-x_1\mu_1-\cdots-x_n\mu_n)\boxed{+\bar{\lambda}(1-x_1-....-x_n)}+\sum_{i=1}\lambda_ix_i.$$

combine

$$\lambda_\delta,\bar{\lambda}_1,\bar{\lambda}_2\geq 0,\lambda_i\geq 0\ (i=1,....,n),\text{ and }\bar{\lambda}=\bar{\lambda}_1-\bar{\lambda}_2\ (<,=,>0).$$

Equivalence of Optimization Models (II)

Repeat: Lagrangian ($\lambda_\delta \geq 0$, $\lambda_i \geq 0$ ($i = 1, ..., n$), $\bar{\lambda}$):

$$\mathcal{L} = -\frac{1}{2}\sum_{i,j}\sigma_{ij}x_i x_j - \lambda_\delta(\bar{\mu} - x_1\mu_1 \cdots - x_n\mu_n) + \bar{\lambda}(1 - x_1 - .... - x_n) + \sum_{i=1}\lambda_i x_i.$$

KKT condition:

$F_{x_i} =$

$$-\sum_{j=1}^{n}\sigma_{ij}x_j + \lambda_\delta\mu_i - \bar{\lambda} + \lambda_i = 0, \; i = 1, \cdots, n,$$

$$\bar{\lambda}(1 - x_1 - .... - x_n) = 0, \quad x_1 + \cdots + x_n = 1,$$

$$\lambda_i x_i = 0, \quad x_i \geq 0, \quad \lambda_i \geq 0, \quad i = 1, \cdots, n,$$

$$\lambda_\delta(x_1\mu_1 + \cdots + x_n\mu_n - \bar{\mu}) = 0, \quad x_1\mu_1 + \cdots + x_n\mu_n \geq \bar{\mu}, \quad \lambda_\delta \geq 0.$$

The solution to the above x_i^* is an optimal solution for

$$\min_{\mathbf{x}\geq 0}\left\{\frac{1}{2}\sum_{i,j}\sigma_{ij}x_i x_j : x_1\mu_1 + ... + x_n\mu_n \geq \bar{\mu}, x_1 + ... + x_n = 1.\right\}$$

Equivalence of Optimization Models (III)

Now suppose that we minimize

$$\max_{\mathbf{x}\geq 0}\left\{x_1\mu_1+....+x_n\mu_n-\delta\frac{\mathbf{x}^T\Sigma\mathbf{x}}{2}\right\}$$
$$\text{subj. to}\quad x_1+....+x_n=1,\quad x_1\geq 0,...,x_n\geq 0.$$

Lagrangian is

$$\mathcal{L}=x_1\mu_1+....+x_n\mu_n-\delta\frac{\mathbf{x}^T\Sigma\mathbf{x}}{2}+\bar{\lambda}'(1-x_1-....-x_n)+\sum_{i=1}^n\lambda_i'x_i.$$

KKT conditions

$$\mu_i-\delta\sum_{j=1}^n\sigma_{ij}x_j-\bar{\lambda}'+\lambda_i'=0,\ \ i=1,\cdots,n,$$

$$\bar{\lambda}'(1-x_1-....-x_n)=0,\quad x_1+\cdots+x_n=1,$$

$$\lambda_i'x_i=0,\quad x_i\geq 0,\quad \lambda_i'\geq 0,\quad i=1,\cdots,n.$$

Equivalence of Optimization Models (IV)

KKT conditions of model I (max. mean-δ variance):

$$\mu_i - \delta \sum_{j=1}^{n} \sigma_{ij} x_j - \bar{\lambda}' + \lambda_i' = 0,\ \ i = 1, \cdots, n,$$

$$\bar{\lambda}'(1 - x_1 - .... - x_n) = 0, \quad x_1 + \cdots + x_n = 1,$$

$$\lambda_i' x_i = 0, \quad x_i \geq 0, \quad \lambda_i' \geq 0, \quad i = 1, \cdots, n.$$

KKT Conditions of Model II (minimizing variance):

$$-\sum_{j=1}^{n} \sigma_{ij} x_j + \lambda_\delta \mu_i - \bar{\lambda} + \lambda_i = 0,\ \ i = 1, \cdots, n,$$

$$\bar{\lambda}(1 - x_1 - .... - x_n) = 0, \quad x_1 + \cdots + x_n = 1,$$

ที่ $\lambda_\delta \neq 0$ เพราะ เอา constraint นี้ออก

$$\lambda_i x_i = 0, \quad x_i \geq 0, \quad \lambda_i \geq 0, \quad i = 1, \cdots, n,$$

$$\lambda_\delta(x_1\mu_1 + \cdots + x_n\mu_n - \bar{\mu}) = 0, \quad x_1\mu_1 + \cdots + x_n\mu_n \geq \bar{\mu}, \quad \lambda_\delta \geq 0.$$

แค่เอามาหาร ตลอด ด้วย λ_δ — everything's same ↳ equivalent

What if we let $\lambda_\delta \neq 0$

$$\delta = 1/\lambda_\delta, \quad \bar{\lambda}' = \bar{\lambda}/\lambda_\delta, \quad \lambda_i' = \lambda_i/\lambda_\delta, \ \ i = 1, \cdots, n.$$

Equivalence of Optimization Models (V)

If $\lambda_\delta = 0$, then the KKT conditions of Model II become:

$$-\sum_{j=1}^{n} \sigma_{ij} x_j^* - \bar{\lambda} + \lambda_i = 0, \;\; i = 1, \cdots, n,$$

$$\bar{\lambda}(1 - x_1^* - .... - x_n^*) = 0 \quad x_1^* + \cdots + x_n^* \leq 1,$$

$$\lambda_i x_i^* = 0, \quad x_i^* \geq 0, \quad \lambda_i \geq 0, \quad i = 1, \cdots, n,$$

$$x_1^* \mu_1 + \cdots + x_n^* \mu_n \geq \bar{\mu}.$$

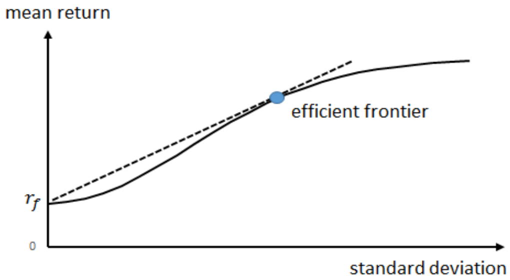
which is the same KKT conditions for the problem:

$$\min_{\mathbf{x} \geq 0} \left\{ \frac{1}{2} \sum_{i,j} \sigma_{ij} x_i x_j : x_1 + ... + x_n = 1, \quad x_i \geq 0, \quad i = 1, ..., n. \right\}$$

for model I, this means $\delta = 1/\lambda_\delta = \infty$ in the objective function:

$$\max_{\mathbf{x} \geq 0} \left\{ x_1 \mu_1 + ... + x_n \mu_n - \delta \frac{\mathbf{x}^T \Sigma \mathbf{x}}{2} \right\}$$

Efficient Frontier and Sharpe Ratio



(r_f: risk-free return)

each point on this curve correspond to a solution to an optimal solution of any of the three optimization problems (with a special choice of $\bar{\mu}$, $\bar{\sigma}$, and δ).

Sharpe ratio

There is a special point on the curve. How to find it?

$$h(\mathbf{x}) = \frac{\boldsymbol{\mu}^T \mathbf{x} - r_f}{\sqrt{\mathbf{x}^T \Sigma \mathbf{x}}}.$$

Sharpe Ratio: Problem Formulation:

$$\max_{\mathbf{x}} \left\{ \frac{\boldsymbol{\mu}^T\mathbf{x} - r_f}{\sqrt{\mathbf{x}^T\Sigma\mathbf{x}}} \right\}.$$

$$\text{subject to}: \quad x_1 + ... + x_n = 1,$$

$$x_i \geq 0, \quad i = 1, ..., n.$$

The objective function looks complicated, need some simplification:

$$\min_{\mathbf{x}} \left\{ \frac{\sqrt{\mathbf{x}^T\Sigma\mathbf{x}}}{\boldsymbol{\mu}^T\mathbf{x} - r_f} \right\}.$$

Still difficult to handle directly.

Sharpe Ratio: Solution Techniques

suppose that x_i^* $(i = 1, ..., n)$ maximizes the Sharpe Ratio.

define κ such that

inverse of optimum

$$\mu_1 x_1^* + ... + \mu_n x_n^* - r_f = 1/\kappa.$$

We do not know values of x_i^* $(i = 1, ..., n)$ and κ, but they exist, and $\kappa > 0$.

if $\kappa < 0$
port $< r_f$

We can rewrite the above as

vector

$$\begin{aligned}\mu^T \mathbf{x}^* - r_f &= \mu_1 x_1^* + ... + \mu_n x_n^* - (x_1^* + \cdots + x_n^*) r_f \\ &= (\mu_1 - r_f) x_1^* + ... + (\mu_n - r_f) x_n^* = 1/\kappa\end{aligned}$$

so

$$(\mu_1 - r_f)(\kappa x_1^*) + ... + (\mu_n - r_f)(\kappa x_n^*) = 1.$$

$\kappa = x^{*-1}$ = inverse

let

Let

$$\boxed{y_i^* = \kappa x_i^*,} \quad i = 1, ..., n.$$

then

choose x^* is like choosing y^*

$$(\mu_1 - r_f) y_1^* + ... + (\mu_n - r_f) y_n^* = 1.$$

and

convex

$$\boxed{y^* = \kappa x_i^*}$$

$$\frac{\sqrt{(\mathbf{x}^*)^T \Sigma \mathbf{x}^*}}{\boldsymbol{\mu}^T \mathbf{x}^* - r_f} = \frac{\sqrt{(\mathbf{y}^*)^T \Sigma \mathbf{y}^*}/\cancel{\kappa}}{\underbrace{[(\mu_1 - r_f) y_1^* + ... + (\mu_n - r_f) y_n^*]}_{1}/\cancel{\kappa}} = \sqrt{(\mathbf{y}^*)^T \Sigma \mathbf{y}^*}$$

Maximum Sharpe Ratio

Repeat:

$$\frac{\sqrt{(\mathbf{x}^*)^T\Sigma\mathbf{x}^*}}{\boldsymbol{\mu}^T\mathbf{x}^* - r_f} = \sqrt{(\mathbf{y}^*)^T\Sigma\mathbf{y}^*}, \qquad (\mu_1 - r_f)y_1^* + ... + (\mu_n - r_f)y_n^* = 1.$$

because $\mathbf{y}^* = \kappa\mathbf{x}^*$:

$$\underbrace{\begin{array}{rcl} x_1^* + ... + x_n^* = 1 & \longrightarrow & y_1^* + ... + y_n^* = \kappa, \\ x_i^* \geq 0 & \longrightarrow & y_i^* \geq 0 \ (i = 1, ..., n). \end{array}}$$

So we can determine $\mathbf{y}^*$ and κ by:

easier to solve ✓

$$\min_{\mathbf{y}\geq 0, \kappa \geq 0} \left\{\mathbf{y}^T\Sigma\mathbf{y}\right\}$$

$$\text{subject to} \qquad \underbrace{(\mu_1 - r_f)y_1 + ... + (\mu_n - r_f)y_n = 1,} \quad y_1 + ... + y_n = \kappa.$$

The optimal solution to the original problem is

$$x_i^* = y_i^*/(y_1^* + \cdots + y_n^*), \quad i = 1, ..., n.$$

and the Sharpe ratio is $\frac{1}{\sqrt{(\mathbf{y}^*)^T\Sigma\mathbf{y}^*}}$.

Deviation in Return

$$\min_{\mathbf{x}\geq 0}\left\{\frac{1}{2}\mathbf{x}^T\Sigma\mathbf{x}\right\} \quad \text{subject to:} \quad x_1\mu_1+....+x_n\mu_n\geq\bar{\mu}, \quad x_1+....+x_n=1.$$

objective value (portfolio variance): $\mathbf{x}^T\Sigma\mathbf{x}=\sum_{i,j}\sigma_{ij}x_ix_j$.

where

$$\sigma_{ij}=\mathbf{E}[(r_i-\mu_i)(r_j-\mu_j)]$$

so

$$\mathbf{x}^T\Sigma\mathbf{x}=\mathbf{E}\left[\sum_{i,j}(r_i-\mu_i)(r_j-\mu_j)x_ix_j\right]=\mathbf{E}\left[\left(\sum_{i=1}^{n}(r_i-\mu_i)x_i\right)^2\right].$$

deviation of the portfolio return from its mean

$$U_{\mathbf{x}}=\sum_{i=1}^{n}(r_i-\mu_i)x_i \quad \longrightarrow \quad \mathbf{x}^T\Sigma\mathbf{x}=\mathbf{E}[U_{\mathbf{x}}^2]$$

mean and variance of this deviation

$$\mathbf{E}[U_{\mathbf{x}}]=0 \text{ (because } \mathbf{E}[r_i]=\mu_i) \qquad var(U_{\mathbf{x}})=\mathbf{E}[U_{\mathbf{x}}^2]=\sigma_x^2$$

Variance for Mean Absolute Variation (MAD)

- the deviation of the actual portfolio return from its mean:

$$U_{\mathbf{x}} = (r_1 - \mu_1)x_1 + \ldots.. + (r_n - \mu_n)x_n.$$

- minimizing $\mathbf{E}[U_{\mathbf{x}}]$ is meaningless because it is 0.
- minimizing portfolio variance is to minimize $\mathbf{E}[U_{\mathbf{x}}^2]$

what about minimizing $\mathbf{E}[|U_{\mathbf{x}}|]$ (Mean Absolute Deviation (MAD))?

if all returns r_i $(i = 1, \cdots, n)$ are normally distributed, then $U_{\mathbf{x}}$ is normally distributed with mean 0 and standard deviation $\sigma_{\mathbf{x}}$.

$$f(u) = \frac{1}{\sqrt{2\pi}\sigma_{\mathbf{x}}} e^{-\frac{u^2}{2\sigma_{\mathbf{x}}^2}}$$

$$\mathbf{E}[|U_{\mathbf{x}}|] = \frac{1}{\sqrt{2\pi}\sigma_{\mathbf{x}}} \int_{-\infty}^{\infty} |u| e^{-\frac{u^2}{2\sigma_{\mathbf{x}}^2}} du = \frac{2}{\sqrt{2\pi}\sigma_{\mathbf{x}}} \int_0^{\infty} u e^{-\frac{u^2}{2\sigma_{\mathbf{x}}^2}} du$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}\cancel{\sigma_{\mathbf{x}}}} \left(-\sigma_x^2 e^{-\frac{u^2}{2\sigma_{\mathbf{x}}^2}} \Big|_0^{\infty} \right) = \sqrt{\frac{2}{\pi}} \sigma_{\mathbf{x}}$$

0

into abs

in this case, minimizing the variance is the equivalent to minimizing MAD

Minimizing MAD

assume return is normally distributed.

there are T samples of asset returns i: r_{it}, $t = 1, \cdots, T$, μ_i is the expected return ($i = 1, \cdots, n$)

to choose the optimal investment amounts $\mathbf{x}$, you can:

- let

$$\hat{\sigma}_{ij} = \frac{1}{T}\sum_{t=1}^{T}(r_{it} - \mu_i)(r_{jt} - \mu_j)$$

  and minimize the (estimated) portfolio variance:

$$\min_{\mathbf{x}} \left\{ \sum_{ij} \hat{\sigma}_{ij} x_i x_j \right\}$$

same

- or minimize MAD

$$\min_{\mathbf{x}} \left\{ \sum_{t=1}^{T} \left| \sum_{i=1}^{n} (r_{it} - \mu_i) x_i \right| \right\}$$

under the same contraints

$$\mu_1 x_1 + \cdots + \mu_n x_n \geq \bar{\mu}, \quad x_1 + \cdots + x_n = 1, \quad x_1 \geq 0, \cdots, x_n \geq 0.$$

Techniques for MAD Minimization

how to handle the MAD objective function:

$$\sum_{t=1}^{T}\left|\sum_{i=1}^{n}(r_{it}-\mu_i)x_i\right| ?$$

1. for each $t = 1, \cdots, T$, introduce new variables y_t and z_t;
2. change the objective function to

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$$\min\left\{\sum_{t=1}^{T}(y_t + z_t)\right\}$$

3. add constraints: for $t = 1, \cdots, T$:

definition

constrant

$$y_t - z_t = \sum_{i=1}^{n}(r_{it}-\mu_i)x_i, \quad y_t \geq 0, \quad z_t \geq 0,$$

so: $\quad y_t + z_t = 2z_t + \sum_{i=1}^{n}(r_{it}-\mu_i)x_i \quad$ and $\quad y_t + z_t = 2y_t - \sum_{i=1}^{n}(r_{it}-\mu_i)x_i.$

MAD Minimization Problem

$z_t = y_t - \sum$ ___

$$\min_{\mathbf{y}\geq 0, \mathbf{z}\geq 0, \mathbf{x}} \left\{ \sum_{t=1}^{T} (y_t + z_t) \right\} \quad \text{subject to: } y_t - z_t = \sum_{i=1}^{n} (r_{it} - \mu_i) x_i,\ t = 1, \cdots, T, \cdots$$

how this formulation work?

$y_t = \sum_{i=1}^{n} (r_{it} - \mu_i) x_i + z_t$

for given $\mathbf{x}$:

no

- when $\sum_{i=1}^{n} (r_{it} - \mu_i) x_i \geq 0$, it is optimal that

and = min($\sum$ ___) + 2 z_t

$$y_t = \sum_{i=1}^{n} (r_{it} - \mu_i) x_i \quad \text{and} \quad z_t = 0.$$

positive

z_t must equal to 0

- when $\sum_{i=1}^{n} (r_{it} - \mu_i) x_i < 0$, it is optimal that

if

min] must = 0

$y_t + y_t - \sum$ ___

$$y_t = 0 \quad \text{and} \quad z_t = -\sum_{i=1}^{n} (r_{it} - \mu_i) x_i.$$

neg

positive

also positive

in either case

$$y_t + z_t = \left| \sum_{i=1}^{n} (r_{it} - \mu_i) x_i \right|$$

Limitation of Mean-Variance Model

Two investment opportunities: return on investment 1 is either 10% or 30% with equal probability. Return on investment 2 is guaranteed at 10%. ($\sigma_1 > \sigma_2 = 0$).

what is the optimal allocation of your budget between the two investments?

let x be the % allocated to investment 1, so the % in 2 is $1 - x$

- can this optimal allocation be the solution of

$$\max_{x \geq 0} \left\{ \mu_1 x + \mu_2 (1 - x) \middle| \sigma_1^2 x^2 \leq \bar{\sigma}^2 \right\}, \quad \bar{\sigma} = \sigma_1 / 2.$$

- can the optimal allocation be the solution of:

$$\max_{x \geq 0} \left\{ \mu_1 x + \mu_2 (1 - x) - \frac{\delta}{2} \sigma_1^2 x^2 \right\}, \quad \delta = 0.2/\sigma_1^2.$$

- can this optimal allocation be the solution:

$$\min_{x \geq 0} \left\{ \frac{1}{2} \sigma_1^2 x^2 \, \middle| \mu_1 x + \mu_2 (1 - x) \geq \bar{\mu} \right\}, \quad \bar{\mu} = 10\%.$$

Stochastic Dominance

Portfolio return X_1 stochastically dominates (first-order) return X_2 if

$$\Pr(X_1 \geq x) \geq \Pr(X_2 \geq x) \quad \text{for any } x$$

In the previous example, there are two possibilities $x = 10\%$ or $x = 30\%$:

$$\begin{aligned} & \Pr(X_1 \geq 10\%) && = 1 = \Pr(X_2 \geq 10\%) \\ \text{and} \qquad & \Pr(X_1 \geq 30\%) && = 0.5 > \Pr(X_2 \geq 30\%). \end{aligned}$$

Does that mean we should use the mean-variance model?

If the return of a portfolio follows the Normal distribution, then the return of a portfolio constructed by optimizing the mean-variance model cannot be stochastically dominated.

The Case with Normal Distribution

Import fact: If X is Normally distributed random variable: $X \sim \mathcal{N}(\mu, \sigma)$, then

$$\Pr(X \geq x) = \Pr\left(X_0 \geq \frac{x-\mu}{\sigma}\right) \quad X_0 \sim \mathcal{N}(0,1).$$

- a portfolio is constructed by optimizing a mean-variance model, return X_1, with mean μ_1 and standard deviation σ_1.
- any other portfolio: return X_2 with mean μ_2 and standard deviation σ_2.

mean var → max retn → or min var → to be optimal

$$\mu_1 > \mu_2 \quad \text{or} \quad \sigma_1 < \sigma_2.$$

ตัวใดตัวหนึ่งเท่านั้นที่จะเป็นจริง

If $X_1 \sim \mathcal{N}(\mu_1, \sigma_1)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2)$, can we always find some x such that: ได้

std   std ที่ μ_2, σ_2

$$\Pr(X_1 \geq x) = \Pr\left(X_0 \geq \frac{x-\mu_1}{\sigma_1}\right) \geq \Pr\left(X_0 \geq \frac{x-\mu_2}{\sigma_2}\right) = \Pr(X_2 \geq x)?$$

True พร้อมกันไม่ได้

ไม่ make sense

↳ ถ้าไม่มีจะแปลว่า X_1 ไม่ dominated

Dominance and Mean-Variance Model under Normal Distribution

- optimizing the mean-variance model, portfolio return $X_1 \sim \mathcal{N}(\mu_1, \sigma_1)$.
- any other portfolio: return $X_2 \sim \mathcal{N}(\mu_2, \sigma_2)$.

If $\boxed{\sigma_1 < \sigma_2}$, let

$$x = a + \mu_1 + \frac{\sigma_1(\mu_1 - \mu_2)}{\sigma_2 - \sigma_1} = a + \mu_2 + \frac{\sigma_2(\mu_1 - \mu_2)}{\sigma_2 - \sigma_1}, \quad \boxed{a < 0.}$$

so

$$\frac{x - \mu_1}{\sigma_1} = \frac{a}{\sigma_1} + \frac{\mu_1 - \mu_2}{\sigma_2 - \sigma_1} < \frac{a}{\sigma_2} + \frac{\mu_1 - \mu_2}{\sigma_2 - \sigma_1} = \frac{x - \mu_2}{\sigma_2}.$$

2 1

smaller

then

N x

$$\Pr(X_1 \geq x) = \Pr\left(X_0 \geq \frac{x - \mu_1}{\sigma_1}\right) > \Pr\left(X_0 \geq \frac{x - \mu_2}{\sigma_2}\right) = \Pr(X_2 \geq x).$$

Prob($X_0 \geq$ smaller) > P~($X_0 \geq$ ~)

If $\boxed{\mu_1 > \mu_2}$, let $x = \mu_1$,

$$\Pr(X_1 \geq \mu_1) = \Pr(X_0 \geq 0) > \Pr\left(X_0 \geq \frac{\mu_1 - \mu_2}{\sigma_2}\right) = \Pr(X_2 \geq \mu_1).$$

Black-Litterman Model: Motivation

Mean-variance model

$$\max_{\mathbf{x}}\{\mathbf{E}[R(\mathbf{x})] - \delta \mathrm{var}(R(\mathbf{x})) | C\mathbf{x} \leq \mathbf{d}\}$$

where

$$\mathbf{E}[R(\mathbf{x})] = \mu_1 x_1 + \cdots \mu_n x_n, \quad \mathrm{var}(R(\mathbf{x})) = \mathbf{x}^T \Sigma \mathbf{x} = \sum_{i,j} x_i x_j \sigma_{ij},$$

$$\boldsymbol{\mu} \doteq (\mu_1, \cdots, \mu_n) \text{ (mean return)}$$

- Solution is sensitive to values of the expected return ($\boldsymbol{\mu}$).
- Covariance matrix Σ can be estimated with conventional means while estimating the expected return $\boldsymbol{\mu}$ needs more effort.
- Need to include personal view into the estimated values.

Private Opinion

- Example 1: return of stock 1 will outperform that of stock 2 by 5%

$$\mu_1 - \mu_2 = 5\%.$$

- Example 2: average market return will be around 3%

$$\frac{\mu_1 + \cdots + \mu_N}{N} = 3\%$$

- Example 3: the P/E ratio of stock 3 will be around 14:

$$\mu_3 = \frac{14 \times \text{earning}}{\text{current price}} - 1.$$

- Let

$$P = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1/N & 1/N & 1/N & \cdots & 1/N \\ 0 & 0 & 1 & \cdots & 0 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 5\% \\ 3\% \\ \frac{14 \times \text{earning}}{\text{current price}} - 1 \end{pmatrix}.$$

$$P\boldsymbol{\mu} = \mathbf{q},$$

Estimate the Expected Returns by Black-Litterman

$$\mu_i = \mathbf{E}[r_i] + v_i, \qquad i = 1, \cdots, N.$$

i.e., expected returns are random variables.
market equilibrium return

$$\mathbf{E}[\mathbf{r}] \doteq (\mathbf{E}[r_1], \cdots, \mathbf{E}[r_n])$$

Assume that $v \sim \mathcal{N}(\mathbf{0}, \tau\Sigma)$ (τ: a small constant):

opinion (ours estimate) real market tells us

The value of $\boldsymbol{\mu}$ is estimated by

If 2 asset "Yi" Error^2

$\frac{(\mu_1 - E[r_1])^2}{\tau \sigma_1^2} + \frac{(\mu_2 - E[r_2])^2}{\tau \sigma_2^2}$

$$\min_{\boldsymbol{\mu}} \{(\boldsymbol{\mu} - \mathbf{E}[\mathbf{r}])^T (\tau\Sigma)^{-1} (\boldsymbol{\mu} - \mathbf{E}[\mathbf{r}])\}$$

subject to: $\boxed{P\boldsymbol{\mu} = \mathbf{q}.}$

small constant

Σ: variance metrix of mean (not actual price)

How?!

The solution is

$$\boldsymbol{\mu}^* = \mathbf{E}[\mathbf{r}] + (\tau\Sigma)P^T(P\tau\Sigma P^T)^{-1}(\mathbf{q} - P\mathbf{E}[\mathbf{r}]).$$

Solution to Black-Litterman Model

- Lagrangian: let $\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} - \mathbf{E}[\mathbf{r}]$ and $\tilde{\mathbf{q}} = \mathbf{q} - P\mathbf{E}[\mathbf{r}]$.

must use kkt

$$\mathcal{L}(\boldsymbol{\mu}, \boldsymbol{\lambda}) = \tilde{\boldsymbol{\mu}}^T(\tau\Sigma)^{-1}\tilde{\boldsymbol{\mu}} + \boldsymbol{\lambda}(P\tilde{\boldsymbol{\mu}} - \tilde{\mathbf{q}})$$

diff → how to diff ??

- KKT condition:

$$2(\tau\Sigma)^{-1}\tilde{\boldsymbol{\mu}} = P^T\boldsymbol{\lambda}, \quad \text{and} \quad P\tilde{\boldsymbol{\mu}} = \tilde{\mathbf{q}}.$$

still don't know

- Derivation: from the first KKT condition:

$$\tilde{\boldsymbol{\mu}} = (\tau\Sigma)P^T\boldsymbol{\lambda}/2.$$

apply to the second KKT condtion:

get λ

$$P(\tau\Sigma)P^T\boldsymbol{\lambda} = 2\tilde{\mathbf{q}} \quad \text{i.e.,} \quad \boldsymbol{\lambda} = 2[P(\tau\Sigma)P^T]^{-1}\tilde{\mathbf{q}}$$

use the above to eliminate $\boldsymbol{\lambda}$:

$$\tilde{\boldsymbol{\mu}} = (\tau\Sigma)P^T\boldsymbol{\lambda}/2 = (\tau\Sigma)P^T[P(\tau\Sigma)P^T]^{-1}\tilde{\mathbf{q}}$$

i.e.,

$$\boldsymbol{\mu}^* = \mathbf{E}[\mathbf{r}] + (\tau\Sigma)P^T(P\tau\Sigma P^T)^{-1}(\mathbf{q} - P\mathbf{E}[\mathbf{r}]).$$

How to determine $\mathbf{E}[\mathbf{r}]$: CAPM

- Question: will adding a new asset i improve the Sharpe Ratio?
- Question in math:

Asset 1 invest Asset 2 or market rest

$$\max_{x \geq 0} \left\{ \hat{R}(x) \doteq \frac{\mathbf{E}[r_i]x + \mathbf{E}[r_m](1-x) - r_f}{\sqrt{\text{var}(r_i x + r_m(1-x))}} \right\} \implies x^* > 0?$$

r_i return of asset i; r_m: equilibrium return; r_f: risk-free return.

- Answer:

$$\boxed{\frac{d\hat{R}(x)}{dx}|_{x=0} > 0?}$$

- The inequality holds if and only if

$$\mathbf{E}[r_i] > r_f + \frac{\text{cov}(r_i, r_m)}{\text{var}(r_m)}(\mathbf{E}[r_m] - r_f).$$

Deriving $d\hat{R}(x)/dx$

how to take diff

$$g(x) = r_i x + r_m(1-x) = r_m + (r_i - r_m)x, \quad \hat{R}(x) \doteq \frac{\mathbf{E}[g(x)] - r_f}{\sqrt{\mathrm{var}(g(x))}}.$$

$$\frac{d\hat{R}(x)}{dx} = \frac{\mathbf{E}[r_i] - \mathbf{E}[r_m]}{\sqrt{\mathrm{var}(g(x))}} - \frac{1}{2}\frac{\mathbf{E}[g(x)] - r_f}{\sqrt{(\mathrm{var}(g(x)))^3}}\frac{d\mathrm{var}(g(x))}{dx}$$

since

$$\mathrm{var}(g(x)) = \mathbf{E}\left[(g(x) - \mathbf{E}[g(x)])^2\right]$$

$$\frac{d\mathrm{var}(g(x))}{dx} = 2\mathbf{E}[(g(x) - \mathbf{E}[g(x)]) * ((r_i - \mathbf{E}[r_i]) - (r_m - \mathbf{E}[r_m]))]$$

at $x = 0$, $g(x) = r_m$, so

$$\frac{d\mathrm{var}(g(x))}{dx} = 2\left(\mathrm{cov}(r_i, r_m) - \mathrm{var}(r_m)\right).$$

so

$$\frac{d\hat{R}(x)}{dx} = \frac{1}{\sqrt{\mathrm{var}(r_m)}}\left(\mathbf{E}[r_i] - r_f - \frac{\mathrm{cov}(r_i, r_m)}{\mathrm{var}(r_m)}(\mathbf{E}[r_m] - r_f)\right).$$

Equilibrium Expected Return (Market View)

reason

- more buy of stock i if:

asset i

buy i

$$\mathbf{E}[r_i] > r_f + \frac{\text{cov}(r_i, r_m)}{\text{var}(r_m)}(\mathbf{E}[r_m] - r_f) \implies \frac{d\hat{R}}{dx}|_{x=0} > 0.$$

price goes up, return goes down.

- more sell of stock i if:

people sell i

$$\mathbf{E}[r_i] < r_f + \frac{\text{cov}(r_i, r_m)}{\text{var}(r_m)}(\mathbf{E}[r_m] - r_f) \implies \frac{d\hat{R}}{dx}|_{x=0} < 0.$$

price goes down, return goes up.

- at equilibrium

CAPM model

$$\mathbf{E}[r_i] = r_f + \frac{\text{cov}(r_i, r_m)}{\text{var}(r_m)}(\mathbf{E}[r_m] - r_f) \implies \frac{d\hat{R}}{dx}|_{x=0} = 0.$$