

Homework 3: Due on September 21, Thursday, 50 points

Problem 1: (9 points) Consider the following LP:

$$\text{Maximize } Z = 2x_1 + 7x_2 + 4x_3,$$

subject to

$$\begin{aligned}x_1 + 2x_2 + x_3 &\leq 10, \\3x_1 + 3x_2 + 2x_3 &\leq 10, \\ \text{and } x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0.\end{aligned}$$

Questions:

1. (3 points) Construct the dual problem for this LP.
2. (3 points) Prove that the optimal value of Z cannot exceed 24 by applying the Weak Duality Theorem.
3. (3 points) Solve the dual LP by inspection and apply the Complementary Slackness Condition to determine the optimal solution to the primal LP. Apply the Strong Duality Theorem to verify your results.

Problem 2 (15 points): Consider the following Linear Programming Model:

$$\begin{aligned}\min_{x_1, x_2, x_3} \quad & \{2x_1 + 5x_2 + 9x_3\} \\ \text{subject to: } & x_1 + 3x_2 + 2x_3 \geq 5, \\ & x_1 + 5x_2 + 4x_3 \geq 12, \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.\end{aligned}$$

Questions:

1. (3 points) Formulate the dual problem of this LP
2. (3 points) Find a feasible solution of the dual (maximization) problem and then use Weak Duality Theorem to show that the optimal solution of the original (minimization) problem cannot be less than 10.
3. (3 points) In the dual (maximization) problem, only one constraint (excluding non-negativity constraints) can be at equality at the optimum. Which one should it be, show your reasoning.
4. (3 points) Use the answer to 3) and the Complementary Slackness Condition to find the optimal solutions to the original (minimization) problem.
5. (3 points) Use the Strong Duality Theorem to show that the solution you find must be optimal.

Problem 3: (10 points) There are two call options with strike prices \$30 and \$20 on the same underlying asset with the same maturity date. They are traded at \$4 and \$6 (both buying and selling) at the moment. You can also change your cash position by borrowing or lending. Assume that the interest rate is 0.

Questions: Assume three future scenarios and make up two sets of prices of the asset under these scenarios such that:

1. (5 points): the first set of prices provides an arbitrage opportunity. Formulate an LP for detecting the opportunity and present a solution that exploits the opportunity.
2. (5 points): the second set of prices does not allow arbitrage. Formulate an LP for calculating the risk neutral probability and determine values of these probabilities.

Problem 4: (16 points) The table below shows bid/ask prices of call/put options of SPX with three different strike price. These options are listed on CBOE and expire on September 15, 2023.

options	Call		Put	
SPX4485.000	23	23.4	18.7	19
SPX4490.000	20.3	20.6	20.9	21.3
SPX4495.000	17.7	18	23.3	23.8

Question: Suppose that you only trade these 6 options. Both buying at the ask price and selling at the bid price are allowed. Formulate and solve a linear program model to determine whether these options provide you with an arbitrage opportunity.