# Lecture 4

## LP Application: Asset Pricing and Arbitrage

Qiong Wang

Department of Industrial and Enterprise Systems Engineering
University of Illinois at Urbana-Champaign

September 11- 13, 2023

# Example 1: Pricing of a Call Option



call option on a stock

- current price: $S_0 = 40$.
- future price: either rise to $S_1 = 80$ or fall to $S_2 = 20$.
- strike price 50.
- complete market, no tax, no transaction cost, zero interest rate.

what is the right price for this option?

1. hold $x$ dollars of cash and $y$ shares of the stock
2. apply the replication argument on the quantities:

$$x + 80y = 30 \qquad \text{and} \qquad x + 20y = 0.$$

3. use the solution to determine positions on cash and stock:

$$x = -10, y = 0.5$$

   borrow 10 and buy 0.5 share of the stock.

4. the cost of building these positions is the price for the option:

$$0.5 \times 40 - 10 = 10.$$

# Opportunity I:



call option on a stock with strike price 50:

- current price: $S_0 = 40$.
- future price: rise to $S_1 = 80$ or fall to $S_2 = 20$.

if option price $p < 10$, how do you make a guaranteed profit from it.

decision now:

- option: buy or short? answer: buy $(-p)$,
- stock: buy or short? answer: short half a share (20),
- cash: lend or borrow? answer: lend $(-10)$.

how much do you get from the above? $-p + 20 - 10 = 10 - p$.

future balance:

- $S_1 = 80$: option (30), stock $(-40)$, debt (10).
- $S_2 = 20$: option (0), stock $(-10)$, debt (10).

# Opportunity II



call option on a stock with strike price 50:

- current price: $S_0 = 40$.
- future price: rise to $S_1 = 80$ or fall to $S_2 = 20$.

if option price $p > 10$, how do you make a guaranteed profit from it.

decision now:

- option: buy or short? answer: short ($p$),
- stock: buy or short? answer: buy half a share ($-20$),
- cash: lend or borrow? answer: borrow (10).

how much do you get from the above? $p - 20 + 10 = p - 10$.

future balance:

- $S_1 = 80$: option ($-30$), stock (40), debt ($-10$).
- $S_2 = 20$: option (0), stock (10), debt ($-10$).

# Generalization

- a set of $n$ financial assets.

  e.g., {cash, stock, option}, indexed by 1, 2, and 3 respectively.
- $c_i$: current price of asset $i$ ($i = 1, ..., n$)

  e.g., $c_1 = 1,\ c_2 = 40,\ c_3 = p$ (price of the option).
- $v_i^s$: future value of asset $i$ ($i = 1, ..., n$) in scenario $s$.

  e.g., two possible scenarios, $h$(igh) and $l$(ow)

$$(v_1^h, v_2^h, v_3^h) = (1, 80, 30) \text{ and } (v_1^l, v_2^l, v_3^l) = (1, 20, 0).$$

decision: take position $x_i$ in asset $i$ ($i = 1, ..., n$)

- $x_i > 0$: long (buy, lend); $x_i < 0$: short (borrow, sell).
- total cost of building the position:

$$c_1 x_1 + ... + c_n x_n.$$

- future income in scenario $s$:

$$v_1^s x_1 + ... + v_n^s x_n.$$

# LP and Type-A Arbitrage



**Type-A Arbitrage**: strictly negative cost to build the initial position and break-even in each of all future scenarios.

using an LP to detect type-A arbitrage opportunities:

$$\min_{x_1,\ldots,x_n} \{c_1 x_1 + \ldots + c_n x_n\}$$
$$\text{subject to} \qquad v_1^s x_1 + \ldots + v_n^s x_n \geq 0 \text{ for all } s.$$

($c_i$: current asset price, $v_i^s$: future asset value in scenario $s$, $x_i$: positions)

- the LP always has a feasible solution

$$x_1 = x_2 = \ldots. = x_n = 0.$$

- type-A arbitrage opportunity: a feasible solution $(x_1^*, \ldots, x_n^*)$ such that

$$c_1 x_1^* + \ldots + c_n x_n^* < 0.$$

# Example 1: Case with No Arbitrage Opportunity



In the example above, LP simplifies to

$$\min_{x_1, x_2, x_3} \{x_1 + 40x_2 + px_3\}$$

$$\text{subject to} \quad \begin{cases} x_1 + 80x_2 + 30x_3 & \geq 0, \\ x_1 + 20x_2 & \geq 0. \end{cases}$$

if $p = 10$, then

$$\begin{aligned} x_1 + 40x_2 + px_3 &= x_1 + 40x_2 + 10x_3 \\ &= \frac{1}{3}(x_1 + 80x_2 + 30x_3) + \frac{2}{3}(x_1 + 20x_2) \\ &\geq 0. \end{aligned}$$

so the optimal solution

$$x_1^* + 40x_2^* + 10x_3^* = 0,$$

no arbitrage opportunity.

# Example 1: Cases with Type-A Arbitrage Opportunity



$$\min_{x_1, x_2, x_3} \{x_1 + 40x_2 + px_3\}$$

$$\text{subject to} \quad \begin{cases} x_1 + 80x_2 + 30x_3 & \geq 0, \\ x_1 + 20x_2 & \geq 0. \end{cases}$$

observation: for any given $x_2$,

$$x_1 = -20x_2, x_3 = -2x_2$$

satisfy both constraints, and give rise to the objective function

$$(20 - 2p)x_2$$

- if $p < 10$,

  $$x_2 < 0, \; x_1 = -20x_2 > 0, \; \text{and } x_3 = -2x_2 > 0,$$

  i.e., short stock, lend cash, and buy option.
- if $p > 10$,

  $$x_2 > 0, \; x_1 = -20x_2 < 0, \; \text{and } x_3 = -2x_2 < 0,$$

  i.e., buy stock, borrow cash, sell option.

Question: what is the optimal value of $x_2$? what is the optimal objective value?

# Example 2

call option on a stock

- current price: $S_0 = 40$.
- future price: rise to $S_1 = 80$ or fall to $S_2 = 20$.
- price of the option: 10.
- complete market, no tax, no transaction cost, and zero interest

using a replicate argument to set the strike price

1. instead of buying/selling an option, hold $x$ amount of cash and $y$ shares of the stock to replicate current cost

$$x + 40y = 10$$

2. the position should also replicate future outcome of the option.

| $K \geq 80$ | $0 < K \leq 20$ | $20 < K < 80$ |
|---|---|---|
| $\begin{aligned} x + 80y &= 0 \\ x + 20y &= 0 \end{aligned}$ | $\begin{aligned} x + 80y &= 80 - K \\ x + 20y &= 20 - K \end{aligned}$ | $\begin{aligned} x + 80y &= 80 - K \\ x + 20y &= 0 \end{aligned}$ |

# Example 2: Arbitrage-Free Strike Price

1. if $K \geq 80$:

$$\begin{aligned} & x + 40y = 10, \quad x + 80y = 0, \quad x + 20y = 0, \\ \longrightarrow \quad & \text{no solution} \end{aligned}$$

2. if $0 < K \leq 20$:

$$\begin{aligned} & x + 40y = 10, \quad x + 80y = 80 - K, \quad x + 20y = 20 - K, \\ \longrightarrow \quad & x = -30, \quad y = 1, \quad K = 30 \end{aligned}$$

3. if $20 < K < 80$:

$$\begin{aligned} & x + 40y = 10, \quad x + 80y = 80 - K, \quad x + 20y = 0, \\ \longrightarrow \quad & x = -10, \quad y = 0.5, \quad K = 50 \end{aligned}$$

The only arbitrage-free strike price is: $K = 50$.

# Arbitrage Opportunity



if strike price $20 < K < 50$:

- now: buy option $(-10)$, lend cash $(-10)$, and short $1/2$ share $(20)$.
- future:
  1. $S_1 = 80$: option $(80 - K)$, debt $(10)$, and $1/2$ share $(-40)$.
  $$\text{profit:} \quad 80 - K + 10 - 40 > 50 - K > 0.$$
  2. $S_2 = 20$: collect debt $(10)$, return $1/2$ share $(-10)$,

if strike price $50 < K < 80$:

- now: short an option $(10)$, borrow cash $(10)$, buy $1/2$ share $(-20)$.
- future:
  1. $S_1 = 80$: option $-(80 - K)$, debt $(-10)$, $1/2$ share $(40)$.
  $$\text{profit:} \quad -(80 - K) - 10 + 40 = K - 50 > 0$$
  2. $S_2 = 20$: debt $(-10)$ and $1/2$ share $(10)$.

# Type-B Arbitrage

**Type-B arbitrage**: no cost to build the initial position, at least break-even in all future scenarios, and strictly profitable in some cases.

using LP to detect type-B arbitrage opportunities:

$$\begin{array}{ll} & \max_{x_1,\dots,x_n} \left\{ \sum_s (v_1^s x_1 + .... + v_n^s x_n) \right\} \\ \text{subject to} & c_1 x_1 + c_2 x_2 + .... + c_n x_n \leq 0, \\ & v_1^s x_1 + ... + v_n^s x_n \geq 0 \quad \text{for all } s. \end{array}$$

($c_i$: current asset price, $v_i^s$: future asset value in scenario $s$, $x_i$: positions)

- the LP always has a feasible solution

$$x_1 = x_2 = .... = x_n = 0$$

- type-B arbitrage opportunity: a feasible solution $(x_1^*, ..., x_n^*)$ such that

$$\sum_s (v_1^s x_1^* + .... + v_n^s x_n^*) > 0.$$

# Type-B Arbitrage: Example 2



$p = \$10$ and $20 < K < 80$, the LP becomes

$$\max_{x_1, x_2, x_3} \{2x_1 + 100x_2 + (80 - K)x_3\}$$

$$\text{subject to} \quad \begin{cases} x_1 + 40x_2 + 10x_3 & \leq 0, \\ x_1 + 80x_2 + (80 - K)x_3 & \geq 0, \\ x_1 + 20x_2 & \geq 0. \end{cases}$$

we can write the objective function as

$$\begin{aligned} & 2x_1 + 100x_2 + (80 - K)x_3 \\ = \quad & 3(x_1 + 40x_2 + 10x_3) - (x_1 + 20x_2) + (50 - K)x_3 \end{aligned}$$

if $K = 50$:

$$\begin{aligned} & 2x_1 + 100x_2 + (80 - K)x_3 \\ = \quad & 3(x_1 + 40x_2 + 10x_3) - (x_1 + 20x_2) \leq 0 \quad \text{(why?)} \end{aligned}$$

so the best objective value is zero, no arbitrage opportunity.

# Example 2: Cases with Type-B Arbitrage Opportunity



$$\max_{x_1,x_2,x_3} \{2x_1 + 100x_2 + (80-K)x_3\}$$

$$\text{subject to} \qquad \begin{cases} x_1 + 40x_2 + 10x_3 & \leq 0 \\ x_1 + 80x_2 + (80-K)x_3 & \geq 0, \\ x_1 + 20x_2 & \geq 0. \end{cases}$$

- let $x_1 = -20x_2$ and $x_3 = -2x_2$, the obj becomes

$$2x_1 + 100x_2 + (80-K)x_3 = (50-K)x_3 = 2(K-50)x_2$$

- all constraints are satisfied if

$$(K-50)x_2 \geq 0 \quad \text{(why?)}$$

- strictly positive profit (arbitrage opportunity):
  - if $K > 50$ (buy stock, borrow cash, and short option):
  $$x_2 > 0, x_1 = -20x_2 < 0, x_3 = -2x_2 < 0.$$
  - if $K < 50$ (do exactly the opposite):
  $$x_2 < 0, x_1 = -20x_2, x_3 = -2x_2.$$

# Quotes of SP500 Options at CBOE



Do we have an arbitrage opportunity?

**Calls** SEPTEMBER 2018 (EXPIRATION: 09/28)

| Strike | Last | Net | Bid | Ask | Vol | Int |
|---|---|---|---|---|---|---|
| SPXW1828I2870-E | 34.65 | +2.40 | 31.80 | 32.30 | 19 | 1149 |
| SPXW1828I2875-E | 29.65 | -0.09 | 28.50 | 29.00 | 12 | 13859 |
| SPXW1828I2880-E | 27.85 | +1.85 | 25.30 | 25.80 | 62 | 1063 |
| SPXW1828I2885-E | 24.00 | +0.40 | 22.30 | 22.70 | 167 | 2503 |

**Puts** SEPTEMBER 2018 (EXPIRATION: 09/28)

| Strike | Last | Net | Bid | Ask | Vol | Int |
|---|---|---|---|---|---|---|
| SPXW1828U2870-E | 21.50 | -6.05 | 21.50 | 21.90 | 39 | 1175 |
| SPXW1828U2875-E | 23.30 | -5.38 | 23.20 | 23.60 | 441 | 5768 |
| SPXW1828U2880-E | 24.52 | -6.98 | 24.90 | 25.40 | 31 | 1621 |
| SPXW1828U2885-E | 26.15 | -10.17 | 26.90 | 27.40 | 2 | 201 |

**Calls** OCTOBER 2018 (EXPIRATION: 10/01)

| Strike | Last | Net | Bid | Ask | Vol | Int |
|---|---|---|---|---|---|---|
| SPXW1801J2870-E | 36.97 | 0.0 | 32.90 | 33.50 | 0 | 9 |
| SPXW1801J2875-E | 30.90 | +1.82 | 29.60 | 30.20 | 10 | 26 |
| SPXW1801J2880-E | 30.00 | +3.80 | 26.50 | 27.00 | 1 | 51 |
| SPXW1801J2885-E | 24.90 | +2.20 | 23.40 | 23.90 | 101 | 514 |

**Puts** OCTOBER 2018 (EXPIRATION: 10/01)

| Strike | Last | Net | Bid | Ask | Vol | Int |
|---|---|---|---|---|---|---|
| SPXW1801V2870-E | 29.60 | 0.0 | 22.40 | 23.00 | 0 | 77 |
| SPXW1801V2875-E | 24.11 | -4.85 | 24.00 | 24.70 | 130 | 99 |
| SPXW1801V2880-E | 30.90 | 0.0 | 25.80 | 26.50 | 0 | 134 |
| SPXW1801V2885-E | 31.00 | 0.0 | 27.80 | 28.40 | 0 | 69 |

**Calls** OCTOBER 2018 (EXPIRATION: 10/03)

| Strike | Last | Net | Bid | Ask | Vol | Int |
|---|---|---|---|---|---|---|
| SPXW1803J2870-E | 35.60 | 0.0 | 34.70 | 35.40 | 0 | 33 |
| SPXW1803J2875-E | 31.95 | 0.0 | 31.40 | 32.10 | 0 | 119 |
| SPXW1803J2880-E | 27.90 | 0.0 | 28.30 | 28.80 | 0 | 23 |
| SPXW1803J2885-E | 25.85 | 0.0 | 25.30 | 25.80 | 0 | 21 |

**Puts** OCTOBER 2018 (EXPIRATION: 10/03)

| Strike | Last | Net | Bid | Ask | Vol | Int |
|---|---|---|---|---|---|---|
| SPXW1803V2870-E | 25.65 | -2.85 | 24.00 | 24.60 | 1 | 51 |
| SPXW1803V2875-E | 25.13 | -6.07 | 25.60 | 26.20 | 10 | 317 |
| SPXW1803V2880-E | 25.89 | -10.11 | 27.40 | 28.10 | 16 | 277 |
| SPXW1803V2885-E | 27.65 | -0.60 | 29.40 | 30.00 | 2 | 35 |

# Example

Suppose that we only buy and sell the following four European call options:

**Calls** SEPTEMBER 2018 (EXPIRATION: 09/19)

| Strike | Last | Net | Bid | Ask | Vol | Int |
|---|---|---|---|---|---|---|
| SPXW181912870-E | 23.40 | 0.0 | 22.90 | 23.40 | 0 | 83 |
| SPXW181912875-E | 23.00 | +3.00 | 19.70 | 20.10 | 1 | 695 |
| SPXW181912880-E | 17.20 | -3.80 | 16.60 | 17.00 | 20 | 280 |
| SPXW181912885-E | 15.70 | +0.89 | 13.80 | 14.30 | 130 | 330 |

- all expire on 09/19/2018

  strike prices are 2870, 2875, 2880, 2885.

# Future Values of the Portfolio



$S$: possible future prices of the index on 09/19/2018, $S \geq 0$.

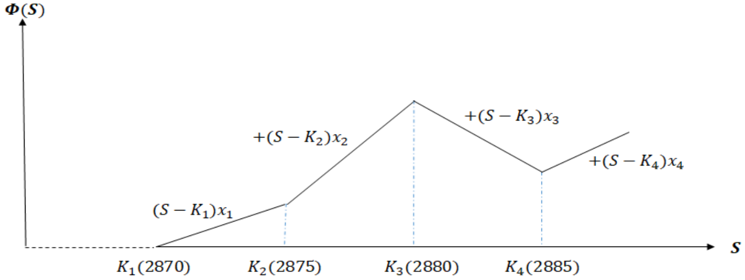
Choose positions $(x_1, x_2, x_3, x_4)$ to keep $\Phi(S) \geq 0$ for all $S$

$$
\begin{aligned}
2870 < S \leq 2875: & \\
\Phi(S) &= (S-2870)x_1 \\
2875 < S \leq 2880: & \\
\Phi(S) &= (S-2870)x_1 + (S-2875)x_2 \\
2880 < S \leq 2885: & \\
\Phi(S) &= (S-2870)x_1 + (S-2875)x_2 + (S-2880)x_3 \\
S \geq 2885: & \\
\Phi(S) &= (S-2870)x_1 + (S-2875)x_2 \\
& \quad + (S-2880)x_3 + (S-2885)x_4
\end{aligned}
$$

.

# Future Value at Different Price Levels

Given $x_1, x_2, x_3, x_4$:




How to keep $\Phi(S)$ nonnegative for all $S$?

$$\Phi(K_1) \geq 0, \;\; \Phi(K_2) \geq 0, \;\; \Phi(K_3) \geq 0, \;\; \Phi(K_4) \geq 0.$$

and

$$\Phi(K_4 + 1) - \Phi(K_4) \geq 0.$$

# No Loss Constraints



$$\Phi(S) = (S-K_1)^+ x_1 + (S-K_2)^+ x_2 + (S-K_3)^+ x_3 + (S-K_4)^+ x_4.$$

$$K_1 = 2870, \quad K_2 = 2875, \quad K_3 = 2880, \quad K_4 = 2885.$$

Only need five points of $S$ to control the entire curve:

1. $S = K_1, \quad \Phi(K_1) \geq 0 \quad \longrightarrow \quad$ trivial,
2. $S = K_2, \quad \Phi(K_2) \geq 0 \quad \longrightarrow \quad 5x_1 \geq 0,$
3. $S = K_3, \quad \Phi(K_3) \geq 0 \quad \longrightarrow \quad 10x_1 + 5x_2 \geq 0,$
4. $S = K_4, \quad \Phi(K_4) \geq 0 \quad \longrightarrow \quad 15x_1 + 10x_2 + 5x_3 \geq 0,$
5. $S = K_4 + 1, \quad \Phi(K_4+1) - \Phi(K_4) \geq 0 \quad \longrightarrow \quad x_1 + x_2 + x_3 + x_4 \geq 0.$

# Data,Model, and Results

**Calls** — SEPTEMBER 2018 (EXPIRATION: 09/19)

| Strike | Last | Net | Bid | Ask | Vol | Int |
|---|---|---|---|---|---|---|
| SPXW1819I2870-E | 23.40 | 0.0 | 22.90 | 23.40 | 0 | 83 |
| SPXW1819I2875-E | 23.00 | +3.00 | 19.70 | 20.10 | 1 | 695 |
| SPXW1819I2880-E | 17.20 | -3.80 | 16.60 | 17.00 | 20 | 280 |
| SPXW1819I2885-E | 15.70 | +0.89 | 13.80 | 14.30 | 130 | 330 |

$$\begin{aligned}
\min_{x_1, x_2, x_3, x_4} \quad & \{23.4x_1 + 23x_2 + 17.2x_3 + 15.7x_4\} \\
\text{subject to:} \quad & 5x_1 \geq 0, \\
& 10x_1 + 5x_2 \geq 0, \\
& 15x_1 + 10x_2 + 5x_3 \geq 0, \\
& x_1 + x_2 + x_3 + x_4 \geq 0.
\end{aligned}$$

arbitrage opportunity, e.g.,

$$x_1 > 0, \;\; x_2 = -2x_1, x_3 = x_1, \;\; x_4 = 0.$$

# More Realistically.....

## Calls

SEPTEMBER 2018 (EXPIRATION: 09/19)

| Strike | Last | Net | Bid | Ask | Vol | Int |
|---|---|---|---|---|---|---|
| SPXW1819I2870-E | 23.40 | 0.0 | 22.90 | 23.40 | 0 | 83 |
| SPXW1819I2875-E | 23.00 | +3.00 | 19.70 | 20.10 | 1 | 695 |
| SPXW1819I2880-E | 17.20 | -3.80 | 16.60 | 17.00 | 20 | 280 |
| SPXW1819I2885-E | 15.70 | +0.89 | 13.80 | 14.30 | 130 | 330 |

- $x_i^+$: number of options to buy, $x_i^+ \geq 0$,
  $x_i^-$: number of options to sell, $x_i^- \geq 0$.
- objective function (cost of building your positions)

$$23.4x_1^+ - 22.9x_1^- + 20.10x_2^+ - 19.70x_2^- + 17x_3^+ - 16.6x_3^- + 14.3x_4^+ - 13.8x_4^-$$

# LP Formulation in the Presence of Bid-Ask Spread



$x_i^+$: number of options to buy,
$x_i^-$: number of options to sell.

minimize:

$$23.4x_1^+ - 22.9x_1^- + 20.10x_2^+ - 19.70x_2^- + 17x_3^+ - 16.6x_3^- + 14.3x_4^+ - 13.8x_4^-$$

subject to:

$$\begin{aligned}
&5(x_1^+ - x_1^-) \geq 0, \\
&10(x_1^+ - x_1^-) + 5(x_2^+ - x_2^-) \geq 0, \\
&15(x_1^+ - x_1^-) + 10(x_2^+ - x_2^-) + 5(x_3^+ - x_3^-) \geq 0, \\
&(x_1^+ - x_1^-) + (x_2^+ - x_2^-) + (x_3^+ - x_3^-) + (x_4^+ - x_4^-) \geq 0, \\
&x_i^+ \geq 0,\ x_i^- \geq 0,\ i = 1, 2, 3, 4.
\end{aligned}$$

# Primal-Dual Transformation



what is the dual model of our LP?

$$\begin{aligned} & \min_{x_1,\dots,x_n} \{c_1 x_1 + \dots + c_n x_n\}, \\ \text{subject to} \quad & v_1^s x_1 + \dots + v_n^s x_n \geq 0, \ s = 1, \dots, S. \end{aligned}$$

where $S$ is the set of all possible scenarios.

The LP is somewhat different because

- The right-hand side of every constraint is zero.
- $x_1, \dots, x_n$ are not necessary non-negative, rewrite the LP (splitting variables):

$$\begin{aligned} & \min_{x_1^+, x_1^-, \dots, x_n^+, x_n^-} \left\{ c_1(x_1^+ - x_1^-) + \dots + c_n(x_n^+ - x_n^-) \right\}. \\ \text{subject to} \quad & v_1^s(x_1^+ - x_1^-) + \dots + v_n^s(x_n^+ - x_n^-) \geq 0, \ s = 1, \dots, S. \\ & x_1^+ \geq 0, x_1^- \geq 0, \dots.., x_n^+ \geq 0, x_n^- \geq 0. \end{aligned}$$

# Dual LP of Example 1



applying the formulation to Example 1 ($p$: option price)

$$\min_{x_1,x_2,x_3} \{x_1 + 40x_2 + px_3\}$$

$$\begin{aligned} \text{subject to} \qquad & x_1 + 80x_2 + 30x_3 \geq 0, \\ & x_1 + 20x_2 \qquad\quad \geq 0. \end{aligned}$$

is transformed into

$$\min_{x_i^+, x_i^-, i=1,2,3} \left\{(x_1^+ - x_1^-) + 40(x_2^+ - x_2^-) + p(x_3^+ - x_3^-)\right\}$$

$$\begin{aligned} \text{subject to} \qquad & (x_1^+ - x_1^-) + 80(x_2^+ - x_2^-) + 30(x_3^+ - x_3^-) \geq 0, & (1) \\ & (x_1^+ - x_1^-) + 20(x_2^+ - x_2^-) \geq 0, & (2) \\ & x_i^+ \geq 0,\ x_i^- \geq 0,\ i = 1, 2, 3. \end{aligned}$$

dual LP: variable and objective

- $y_1$ and $y_2$ dual variables associated with (1) and (2) respectively,
- objective function:

$$\max_{y_1,y_2}\{0 \times y_1 + 0 \times y_2\}$$

# Dual Solution of Example 1

dual LP: constraints

1. associated with $x_1^+$ and $x_1^-$:
$$y_1 + y_2 \leq 1 \text{ and } -y_1 - y_2 \leq -1 \quad \longrightarrow y_1 + y_2 = 1.$$
2. associated with $x_2^+$ and $x_2^-$:
$$80y_1 + 20y_2 \leq 40 \text{ and } -80y_1 - 20y_2 \leq -40 \longrightarrow 80y_1 + 20y_2 = 40.$$
3. associated with $x_3^+$ and $x_3^-$:
$$30y_1 \leq p \text{ and } -30y_1 \leq -p \longrightarrow 30y_1 = p.$$

Dual LP:

$$\max_{y_1, y_2}\{0\}$$

$$\text{subject to} \qquad \left.\begin{array}{rl} y_1 + y_2 & = 1 \\ 80y_1 + 20y_2 & = 40 \end{array}\right\} \longrightarrow y_1 = 1/3, y_2 = 2/3,$$
$$30y_1 = p,$$
$$y_1 \geq 0, \;\; y_2 \geq 0.$$

- every feasible solution is an optimal solution of this LP.
- feasible solution exists only if $p = 10$, the arbitrage-free price.

# Risk Neutral Probabilities

$$\min_{x_1,x_2,x_3} \{x_1 + 40x_2 + px_3\}$$

$$\begin{cases} x_1 + 80x_2 + 30x_3 \geq 0, \\ \\ x_1 + 20x_2 \geq 0. \end{cases}$$

0 (no arbitrage) if and only if

$$p = 10.$$

$$\max_{y_1,y_2}\{0\}$$

$$\begin{cases} y_1 + y_2 = 1, \\ 80y_1 + 20y_2 = 40, \\ 30y_1 = p, \\ y_1 \geq 0, \ y_2 \geq 0. \end{cases}$$

the problem is feasible if and only if

$$p = 10.$$

if $(y_1, y_2)$ is feasible, then

1. $y_1 \geq 0$, $y_2 \geq 0$, and $y_1 + y_2 = 1$.
2. stock and option prices equal assets' weighted (by $y_1$ and $y_2$) future values:
$$40 = 80y_1 + 20y_2 \quad \text{and} \quad 10 = 30y_1.$$
3. from complementary slackness condition
$$y_1(x_1 + 80x_2 + 30x_3) = 0 \text{ and } y_2(x_1 + 20x_2) = 0$$

what do we call $y_1$ and $y_2$ in asset pricing theory?

# First Fundamental Theorem of Asset Pricing



- given

  $n$: types of assets;
  $c_i$: current price;
  $S$: the set of all future scenarios;
  $v_i^s$: future asset value in scenario $s$.

- no arbitrage opportunity if and only if there exists risk neutral probabilities

$$y_1 \geq 0, \ldots, y_S \geq 0,$$

  where for each asset $i = 1, \ldots, n$,

$$y_1 v_i^1 + \ldots + y_S v_i^S = c_i$$

  (when the asset is cash, the equation specializes to $y_1 + \ldots + y_S = 1$).

# Arbitrage and LP Duality Theory



$$\min_{x_1,\ldots,x_n} \{c_1 x_1 + \ldots + c_n x_n\},$$

$$\text{subject to} \qquad v_1^s x_1 + \ldots + v_n^s x_n \geq 0, \;\; s = 1, \ldots, S.$$

dual problem

$$\max_{\mathbf{y}} \{0\}$$

subject to: for $i = 1, \cdots, n$,

$$\sum_s v_i^s y_s \leq c_i, \quad -\sum_s v_i^s y_s \leq -c_i \quad \longrightarrow \quad \sum_s v_i^s y_s = c_i.$$

- weak and strong duality:

  $$0 \leq c_1 x_1 + .... + c_n x_n$$

  either

  $$c_1 x_1^* + \cdots + c_n x_n^* = 0.$$

  or one LP is unbounded and the other is infeasible.
- Complementary slackness condition: if the solution exists, then

  $$y_s^* (v_1^s x_1^* + \cdots + v_n^s x_n^*) = 0$$

# Dual Problem of Example 2

$$\max_{x_1,x_2,x_3} \{2x_1 + 100x_2 + (80-K)x_3\}$$

$$\begin{aligned} x_1 + 40x_2 + 10x_3 \le 0 \quad &\Rightarrow \quad x_1 + 40x_2 + 10x_3 \\ x_1 + 80x_2 + (80-K)x_3 \ge 0 \quad &\Rightarrow \quad -x_1 - 80x_2 - (80-K)x_3 \le 0 \\ x_1 + 20x_2 \ge 0 \quad &\Rightarrow \quad -x_1 - 20x_2 \le 0 \end{aligned}$$

The dual LP

$$\min_{y_0,y_1,y_2} \{0\}$$

$$\begin{aligned} y_0 - y_1 - y_2 = 2 \quad &\Rightarrow \quad \frac{1+y_1}{y_0} + \frac{1+y_2}{y_0} = 1 \\ 40y_0 - 80y_1 - 20y_2 = 100 \quad &\Rightarrow \quad 80\frac{1+y_1}{y_0} + 20\frac{1+y_2}{y_0} = 40 \\ 10y_0 - (80-K)y_1 = (80-K) \quad &\Rightarrow \quad (80-K)\frac{1+y_1}{y_0} = 10 \\ y_0 \ge 0,\ y_1 \ge 0,\ y_2 \ge 0 \quad &\Rightarrow \quad \frac{1+y_1}{y_0} \ge 0,\ \frac{1+y_2}{y_0} \ge 0. \end{aligned}$$

# Risk Neutral Probability



$$\frac{1+y_1}{y_0}+\frac{1+y_2}{y_0}=1, \quad 80\frac{1+y_1}{y_0}+20\frac{1+y_2}{y_0}=40, \quad (80-K)\frac{1+y_1}{y_0}=10,$$

and

$$\frac{1+y_1}{y_0}\geq 0 \quad \text{and} \quad \frac{1+y_2}{y_0}\geq 0.$$

Let

$$\tilde{y}_1=\frac{1+y_1}{y_0} \quad \text{and} \quad \tilde{y}_2=\frac{1+y_2}{y_0}.$$

Then

$$\tilde{y}_1+\tilde{y}_2=1, \quad 80\tilde{y}_1+20\tilde{y}_2=40, \quad (80-K)\tilde{y}_1=10, \quad \tilde{y}_1\geq 0, \quad \tilde{y}_2\geq 0.$$

- Solve the first two equations

$$\tilde{y}_1=\frac{1}{3} \quad \text{and} \quad \tilde{y}_2=\frac{2}{3}.$$

- the equations are feasible only if

$$K=50 \qquad \text{(arbitrage free strike price)}.$$

# General Case for Type B Arbitrage





$$\begin{aligned} & \max_{x_1,\dots,x_n} \left\{ \sum_{s=1}^{S} (v_1^s x_1 + .... + v_n^s x_n) \right\} \\ \text{subject to} \quad & c_1 x_1 + c_2 x_2 + .... + c_n x_n \le 0, \\ & -v_1^s x_1 - ... - v_n^s x_n \le 0, \quad s = 1, \cdots, S \end{aligned}$$

Dual LP

$$\min_{y_0, y_1, \cdots, y_S} \{0\}$$

$$\begin{aligned} y_0 - y_1 - y_2 - .... - y_S = S, \quad & (\text{cash: } c_1 = v_1^1 = ... = v_1^S = 0) \\ c_i y_0 - \sum_{s=1}^{S} y_i v_i^s = \sum_{s=1}^{S} v_i^S, \quad & i = 2, \cdots, n \\ y_0 \ge 0, \cdots, y_S \ge 0 & \end{aligned}$$

If the LP is feasible, then risk neutral probabilities are

$$\tilde{y}_1 = (1 + y_1)/y_0, \cdots, \tilde{y}_S = (1 + y_S)/y_0, \ (y_0 = S + y_1 + ... + y_S).$$