IE524-Section A Exam

October 12, 2022

Exam time (**100 minutes**): 3:00-4:40pm.
Three problems, 100 points.
Open book, open notes, calculators are allowed.
No laptop, phone, tablet, or any other device that can communicate.

Problem 1 (40 points): Consider a case with three assets: cash, stock, and a call option on that stock. There are two future scenarios for the stock price,\$10 in scenario 1 and \$30 in scenario 2. The strike price of the option is \$15. The current price of the stock is \$18 and that of the option is \$6. Interest income is negligible, so the value of the cash remains to be \$1 in all cases.

1. (10 points) Formulate a linear program (LP) for detecting Type A arbitrage opportunity.
2. (10 points) Formulate a linear program (LP) for detecting Type B arbitrage opportunity.
3. (10 points) Choose one of the two LPs to formulate a dual problem for determining risk-neutral probabilities. Show the probabilities exist, so this case has no arbitrage opportunity.
4. (10 points) Change one of the future prices of the stock so that there will be an arbitrage opportunity and explain how (buy or short stock/option) you can exploit this opportunity.

Problem 2: (30 points) Allocating an investment in two assets, a financial manager solves the following nonlinear optimization model to minimize the Sharpe Ratio.

$$\begin{aligned} & \min_{y_1, y_2, \kappa} \left\{ y_1^2 + y_1 y_2 + 2y_2^2 \right\} \\ \text{subject to: } & y_1 + y_2 = \kappa \\ & 2y_1 + 4y_2 = 1 \\ & y_1 \geq 0,\ y_2 \geq 0,\ \kappa \geq 0. \end{aligned}$$

Assume the risk-free return is 1 (meaning one percent, for simplicity we drop % everywhere in the return).

1. (8 points) Determines values of mean returns of the two assets (μ_1,μ_2) and the covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}.$$

2. (12 points) Formulate the Lagrangian and the KKT condition.

3. (10 points) In the optimal solution, $\kappa = 5/16$. Use this value and the above formulation to determine the optimal values of y_1^* and y_2^*, and the optimal allocation into the two assets (x_1,x_2), and the original Sharpe Ratio (note: not the inverse of the ratio).

Problem 3: (30 points) A manager is creating an index fund. There are N stocks to choose from. To determine how which stocks to buy and how large is the fraction of the chosen stock in the fund, the manager solves the following integer linear program.

$$\min_{\mathbf{x},\mathbf{y},\bar{y}} \left\{ \sum_{i=1}^{N} c_i x_i + \sum_{i=1}^{N} k_i y_i \right\}$$

subject to

$$\begin{aligned}
&\sum_{i=1}^{N} \mu_i x_i \geq \eta. \\
&x_1 + \cdots + x_N = 1. && (1) \\
&y_1 + \cdots + y_N \leq n_0. \\
&x_i \leq y_i, \quad i = 1, \cdots, N. && (2) \\
&\sum_{i=1}^{10} y_i \leq \sum_{i=20}^{N} y_i. \\
&y_1 + y_2 + y_3 + y_4 \leq 4\bar{y}. \\
&\bar{y} \leq y_1 + y_2 + y_3 + y_4. \\
&y_N \leq 1 - \bar{y}. \\
&0 \leq x_i \leq 1, \quad 1 \leq i \leq N. && (3) \\
&\bar{y}, y_1, \cdots, y_N = 0 \text{ or } 1. && (4)
\end{aligned}$$

Here c_i, k_i, μ_i ($i = 1, \cdots, N$), and η are positive values; n_0 is a positive integer, and N is the total number of stocks. The rests are decision variables, where $y_1, \cdots, y_N$ and $\bar{y}$ are all binary variables. The manager wants to construct the fund with the minimum total buying cost and satisfaction of all conditions presented as constraints in the above formulation.

1. (6 points) If you want to buy 10 shares of stock 20, what will be the cost, to be expressed by notations of input parameters.

2. (24 points) Besides (1), (2), (3), and (4), which are standard conditions (e.g., all the fractions should add up to 1). The rest constraints specify four special conditions for this particular problem. Describe these four conditions in short sentences, 6 points for each correct description.