# Stamatics Assignment

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June 2023

- ## 1 Problem 1

**Problem:** $N$ letters are to be put in $N$ separate envelopes, assuming an envelope can hold only a single letter. What is the probability that at least one letter is in the correct envelope?

**Solution:**

Let's consider the number of derangements of $N$ letters, which is the number of permutations in which no letter is in its correct envelope. We denote the number of derangements of $N$ objects as $D(N)$.

The total number of permutations of $N$ letters is $N!$ (factorial).

We know, D(N):-

$$D(N) = N!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^N \frac{1}{N!}\right)$$

Clearly,
At least one is in the correct envelope = N! - D(N)

So, The probability that at least one letter is in the correct envelope is the complement of the probability that all letters are in the wrong envelopes. Therefore, the desired probability $P(\text{at least one letter in the correct envelope})$ can be calculated as:

$$P(\text{at least one letter in the correct envelope}) = 1 - \frac{D(N)}{N!}$$

Now, let's calculate the probability for $N = 50$:-

$$P(\text{at least one letter in the correct envelope}) \approx 1 - \frac{D(50)}{50!}$$

## • 2 Problem 2

**Problem:** You have three identical presents. The good gift has 1000 dollars, while the other two presents have nothing. The host of the party asks you to select a present. If you select the good gift, you keep it. You select Present 1. But the host opens the second present and it has nothing. The host knows where the money is and always reveals the blank prize. Also, assume that when we choose the prize gift, the host chooses one of the blank gifts with equal probability. You are given the option to switch your guess to the third present. What are your expected winnings if you switch?

**Solution:**

Let's analyze the possible scenarios and calculate the expected winnings for each scenario.

**Scenario 1:** The good gift is in Present 1 (chosen by you).

In this scenario, if you switch your guess to Present 3, you will lose and your winnings will be 0.

**Scenario 2:** The good gift is in Present 2 (not chosen by you).

In this scenario, the host will always open Present 3 to reveal nothing. If you switch your guess to Present 3, you will win 1000 dollars.

**Scenario 3:** The good gift is in Present 3 (not chosen by you).

In this scenario, the host will always open Present 2 to reveal nothing. If you switch your guess to Present 2, you will win 1000 dollars.

The probability of each scenario is $\frac{1}{3}$ since each present is equally likely to contain the good gift.

Therefore, the expected winnings if you switch your guess can be calculated as:

$$\text{Expected Winnings (Switch)} = \left(\frac{1}{3} \times 0\right) + \left(\frac{1}{3} \times 1000\right) + \left(\frac{1}{3} \times 1000\right) = \frac{2000}{3} \approx 666.67$$

Hence, your expected winnings if you switch your guess to the third present are approximately 666.67 dollars.

- ## 3 Problem-9

**Prove that convolution of two distribution functions is also a distribution function**

**Proof:**

Let $F$ and $G$ be two distribution functions. We want to show that their convolution $H = F * G$ is also a distribution function.

**Step 1: Existence**

To prove that $H$ is a distribution function, we need to show that it satisfies the following properties:

1. $H$ is non-decreasing.
2. $\lim_{x \to -\infty} H(x) = 0$.
3. $\lim_{x \to \infty} H(x) = 1$.

**Property 1: Non-decreasing**

Let $x_1 < x_2$. We have:

$$H(x_2) - H(x_1) = \int_{-\infty}^{\infty} F(x_2 - y)G(y)\,dy - \int_{-\infty}^{\infty} F(x_1 - y)G(y)\,dy$$

Using the substitution $z = x_2 - y$ in the first integral, we obtain:

$$H(x_2) - H(x_1) = \int_{-\infty}^{\infty} F(z)G(x_2 - z)\,dz - \int_{-\infty}^{\infty} F(x_1 - y)G(y)\,dy$$

Since $F$ and $G$ are distribution functions, they are non-decreasing. Therefore, $F(z) \geq F(x_1 - y)$ and $G(x_2 - z) \geq G(y)$ for all $z$ and $y$. Hence, we have:

$$H(x_2) - H(x_1) \geq 0$$

which shows that $H$ is non-decreasing.

**Property 2:** $\lim_{x \to -\infty} H(x) = 0$

For any fixed $x$, we can rewrite $H(x)$ as:

$$H(x) = \int_{-\infty}^{\infty} F(x - y)G(y)\,dy$$

Since $F$ and $G$ are distribution functions, we know that $\lim_{x\to-\infty} F(x-y) = 0$ and $\lim_{y\to\infty} G(y) = 0$. Therefore, the integrand $F(x-y)G(y)$ tends to 0 as $x$ tends to $-\infty$. Hence, we have:

$$\lim_{x\to-\infty} H(x) = \lim_{x\to-\infty} \int_{-\infty}^{\infty} F(x-y)G(y)\,dy = 0$$

**Property 3:** $\lim_{x\to\infty} H(x) = 1$

For any fixed $x$, we can rewrite $H(x)$ as:

$$H(x) = \int_{-\infty}^{\infty} F(x-y)G(y)\,dy$$

Since $F$ and $G$ are distribution functions, we know that $\lim_{x\to\infty} F(x-y) = 1$ and $\lim_{y\to-\infty} G(y) = 1$. Therefore, the integrand $F(x-y)G(y)$ tends to 1 as $x$ tends to $\infty$. Hence, we have:

$$\lim_{x\to\infty} H(x) = \lim_{x\to\infty} \int_{-\infty}^{\infty} F(x-y)G(y)\,dy = 1$$

**Step 2: Continuity**

To prove that $H$ is continuous, we need to show that it is right-continuous. Let $x_n \downarrow x$ be a sequence of numbers decreasing to $x$. We have:

$$H(x) - H(x_n) = \int_{-\infty}^{\infty} F(x-y)G(y)\,dy - \int_{-\infty}^{\infty} F(x_n-y)G(y)\,dy$$

Using the same argument as in Property 1, we can show that $H(x) - H(x_n) \geq 0$ for all $n$, which implies that $H(x) \geq H(x_n)$ for all $n$. Therefore, $H$ is right-continuous.

**Conclusion**

We have shown that $H$ satisfies all the properties of a distribution function. Therefore, the convolution $H = F * G$ is also a distribution function.