Exercise :-

1.1 Question # 01 :-

For the given matrix/vector pairs, compute the following quantities: a_{ii}, $a_{ij}a_{ij}$, $a_{ij}a_{jk}$, $a_{ij}b_j$, $a_{ij}b_ib_j$, b_ib_j, b_ib_i - for each case point out whether the result is a scaler, vector or matrix. Note that $a_{ij}b_j$ is actually the matrix product $[a][b]$. while $a_{ij}a_{jk}$ is the product $[a][a]$:

(2) $a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$, $b_i = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$$a_{ij} = \begin{bmatrix} \overset{a_{11}}{1} & \overset{a_{12}}{1} & \overset{a_{13}}{1} \\ \overset{a_{21}}{0} & \overset{a_{22}}{4} & \overset{a_{23}}{2} \\ \overset{a_{31}}{0} & \overset{a_{32}}{1} & \overset{a_{33}}{1} \end{bmatrix}$$

⊙ $a_{ii} = a_{11} + a_{22} + a_{33}$

$= 1 + 4 + 1 = 6$ (scaler)

$a_{ii} = 6$

⊙ $a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$

$= 1 + 1 + 1 + 0 + 16 + 4 + 0 + 1 + 1 = 25$ (scaler)

$a_{ij}a_{ij} = 25$

⊙ $a_{ij}a_{jk} = \begin{bmatrix} a_{11}a_{11}+a_{12}a_{21}+a_{13}a_{31} & a_{11}a_{12}+a_{12}a_{22}+a_{13}a_{32} & a_{11}a_{13}+a_{12}a_{23}+a_{13}a_{33} \\ a_{21}a_{11}+a_{22}a_{21}+a_{23}a_{31} & a_{21}a_{12}+a_{22}a_{22}+a_{23}a_{32} & a_{21}a_{13}+a_{22}a_{23}+a_{23}a_{33} \\ a_{31}a_{11}+a_{32}a_{21}+a_{33}a_{31} & a_{31}a_{12}+a_{32}a_{22}+a_{33}a_{32} & a_{31}a_{13}+a_{32}a_{23}+a_{33}a_{33} \end{bmatrix}$

$= \begin{bmatrix} 1.1+1.0+1.0 & 1.1+1.4+1.1 & 1.1+1.2+1.1 \\ 0.1+4.0+2.0 & 0.1+4.4+2.1 & 0.1+4.2+2.1 \\ 0.1+1.0+1.0 & 0.1+1.4+1.1 & 0.1+1.2+1.1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 6 & 4 \\ 0 & 18 & 10 \\ 0 & 5 & 3 \end{bmatrix}$ (matrix)

⊙ $a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3$

$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+0+2 \\ 0+0+4 \\ 0+0+2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ (vector)

(i) $a_{ij} b_i b_j = a_{11} b_1 b_1 + a_{12} b_1 b_2 + a_{13} b_1 b_3 + a_{21} b_2 b_1 + a_{22} b_2 b_2$

$+ a_{23} b_2 b_3 + a_{31} b_3 b_1 + a_{32} b_3 b_2 + a_{33} b_3 b_3$

$= 1 + 0 + 2 + 0 + 0 + 0 + 0 + 0 + 4 = 7$ (scaler)

(i) $b_i b_j = \begin{bmatrix} b_1 b_1 & b_1 b_2 & b_1 b_3 \\ b_2 b_1 & b_2 b_2 & b_2 b_3 \\ b_3 b_1 & b_3 b_2 & b_3 b_3 \end{bmatrix}$

$= \begin{bmatrix} 1 \cdot 1 & 1 \cdot 0 & 1 \cdot 2 \\ 0 \cdot 1 & 0 \cdot 0 & 0 \cdot 2 \\ 2 \cdot 1 & 2 \cdot 0 & 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}$ (matrix)

(i) $b_i b_i = b_1 b_1 + b_2 b_2 + b_3 b_3$

$= 1 + 0 + 4 = 5$ (scaler)

(b) $a_{ij} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$, $b_i = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

(i) $a_{ii} = a_{11} + a_{22} + a_{33}$

$= 1 + 2 + 2 = 5$ (scaler)

(i) $a_{ij} a_{ij} = a_{11} a_{11} + a_{12} a_{12} + a_{13} a_{13} + a_{21} a_{21} + a_{22} a_{22} + a_{23} a_{23}$

$+ a_{31} a_{31} + a_{32} a_{32} + a_{33} a_{33}$

$= 1 + 4 + 0 + 0 + 4 + \cancel{4} + 1 + 0 + 16 + 4 = 30$ (scalor)

(i) $a_{ij} a_{jk} = a_{i1}(a_{1k}) + a_{i2} a_{2k} + a_{i3} a_{3k}$

$= \begin{bmatrix} 1 & 6 & 2 \\ 0 & 8 & 4 \\ 0 & 16 & 8 \end{bmatrix}$ (matrix)

(i) $a_{ij} b_j = a_{i1} b_1 + a_{i2} b_2 + a_{i3} b_3$

$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2 + 2 + 0 \\ 0 + 2 + 1 \\ 0 + 4 + 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix}$ (vector)

2) ⊙ $a_{ij} b_i b_j = a_{11} b_1 b_1 + a_{12} b_1 b_2 + a_{13} b_1 b_3 + a_{21} b_2 b_1 + a_{22} b_2 b_2 + a_{23} b_2 b_3 + a_{31} b_3 b_1 + a_{32} b_3 b_2 + a_{33} b_3 b_3$

$= 4+4+0+0+2+1+0+4+2 = 17$ (scaler)

⊙ $b_i b_j = \begin{bmatrix} b_1 b_1 & b_1 b_2 & b_1 b_3 \\ b_2 b_1 & b_2 b_2 & b_2 b_3 \\ b_3 b_1 & b_3 b_2 & b_{3\,3} \end{bmatrix}$

$= \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ (matrix)

⊙ $b_i b_i = b_1 b_1 + b_2 b_2 + b_3 b_3$

$= 4+1+1 = 6$ (scaler)

(C) $a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}$, $b_i = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

⊙ $a_{ii} = a_{11} + a_{22} + a_{33}$

$= 1+0+4 = 5$ (scaler)

⊙ $a_{ij} a_{ij} = a_{11} a_{11} + a_{12} a_{12} + a_{13} a_{13} + a_{21} a_{21} + a_{22} a_{22} + a_{23} a_{23} + a_{31} a_{31} + a_{32} a_{32} + a_{33} a_{33}$

$= 1+1+1+1+0+4+0+1+16 = 25$ (scaler)

⊙ $a_{ij} a_{jk} = [a_{i1} a_{1k} + a_{i2} a_{2k} + a_{i3} a_{3k})$

$= \begin{bmatrix} 2 & 2 & 7 \\ 1 & 3 & 9 \\ 1 & 4 & 18 \end{bmatrix}$ (matrix)

⊙ $a_{ij} b_j = a_{i1} b_1 + a_{i2} b_2 + a_{i3} b_3$

$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 1+1+0 \\ 1+0+0 \\ 0+1+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ (vector)

$$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$$
$$= 1+1+0+1+0+0+0+0+0 = 3 \text{ (scaler)}$$

$$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ (matrix)}$$

$$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3$$
$$= 1+1+0 = 2 \text{ (scaler)}$$

(d) $a_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 3 & 0 \end{bmatrix}, \quad b_i = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$a_{ii} = a_{11} + a_{22} + a_{33}$$
$$= 1 + 2 + 0 = 3 \text{ (scaler)}$$

$$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$$
$$= 1+0+0+0+4+1+0+9+0$$
$$= 15 \text{ (scaler)}$$

$$a_{ij}a_{jk} = a_{i1}a_{1k} + a_{i2}a_{2k} + a_{i3}a_{3k}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 2 \\ 0 & 6 & 3 \end{bmatrix}$$

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1+0+0 \\ 0+0+1 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

3) ⊙ $a_{ij} b_i b_j = a_{11} b_1 b_1 + a_{12} b_1 b_2 + a_{13} b_1 b_3 + a_{21} b_2 b_1 + a_{22} b_2 b_2 + a_{23} b_2 b_3 + a_{31} b_3 b_1 + a_{32} b_3 b_2 + a_{33} b_3 b_3$

$= 1. + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0$

$= 1$ (scaler).

⊙ $b_i b_j = \begin{bmatrix} b_1 b_1 & b_1 b_2 & b_1 b_3 \\ b_2 b_1 & b_2 b_2 & b_2 b_3 \\ b_3 b_1 & b_3 b_2 & b_3 b_3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ (matrix).

⊙ $b_i b_i = b_1 b_1 + b_2 b_2 + b_3 b_3$

$= 1 + 0 + 1 = 2$ (scaler)

1.2. Question # 02:-

Use the decomposition result to express a_{ij} from Exercise 1.1 in terms of sum of symmetric & antisymmetric matrices. Verify that $a_{(ij)}$ and $a_{[ij]}$ satisfy the conditions given in the last paragraph of section 1.2.

(a) $a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

$a_{ji} = (a_{ij})^t = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

$(a_{ij} + a_{ji}) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 1 \\ 1 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix}$

$(a_{ij} - a_{ji}) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 1 \\ 1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

$$a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & +1 & +1 \\ -1 & 0 & +1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 4 & 3/2 \\ 1/2 & 3/2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & +1/2 & +1/2 \\ -1/2 & 0 & +1/2 \\ -1/2 & -1/2 & 0 \end{bmatrix}$$

$$a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$a(ij)$ and $a[ji]$ satisfy the appropriate conditions

(b) $a_{ij} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$ $\quad a_{ji} = (a_{ij})^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

$$a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$$

$$= \frac{1}{2}\left[\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}\right] + \frac{1}{2}\left[\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}\right]$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2/2 & 1 & 0 \\ 1 & 2 & 5/2 \\ 0 & 5/2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -3/2 \\ 0 & 3/2 & 0 \end{bmatrix}$$

$$a_{ij} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$$

$a(ij)$ & $a[ji]$ satisfy the appropriate condition.

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4) (c) $a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}$

$$a_{ji} = (a_{ij})^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$a_{ij} = \frac{1}{2}\left[a_{ij} + a_{ji}\right] + \frac{1}{2}\left(a_{ij} - a_{ji}\right)$$

$$= \frac{1}{2}\left[\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 2 & 4 \end{pmatrix}\right] + \frac{1}{2}\left[\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 2 & 4 \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1/2 \\ 1 & 0 & 3/2 \\ 1/2 & 3/2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \\ -1/2 & -1/2 & 0 \end{bmatrix}$$

$$a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

clearly $a(ij)$ & $a[ij]$ satisfy the approximate condition.

(d) $a_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 3 & 0 \end{bmatrix}$

$$a_{ji} = (a_{ij})^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

$$a_{ij} = \frac{1}{2}\left(a_{ij} + a_{ji}\right) + \frac{1}{2}\left(a_{ij} - a_{ji}\right)$$

$$= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 3 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix}\right] + \frac{1}{2}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 3 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 4 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}$$

$$a_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 3 & 0 \end{bmatrix}$$

clearly $a(ij)$ & $a[ij]$ satisfy approximate condition.

1.3 # Question # 03

If a_{ij} is symmetric and b_{ij} is antisymmetric prove in general that the product $a_{ij} \cdot b_{ij}$ is zero. Verify this result for specific case by using the symmetric & antisymmetric terms from Exercise 1.2.

General form:-

$$a_{ij} b_{ij} = -a_{ji} b_{ji}$$ by using A-symmetric property

- Antisymmetric $a_{ij} b_{ij} = -a_{ji} b_{ji}$
- Symmetric $a_{ij} b_{ij} = a_{ji} b_{ji}$

$$a_{ij} b_{ij} \mp a_{ji} b_{ji}$$

$$a_{ij} b_{ij} = -a_{ij} b_{ij}$$ by using symmetry property

$$a_{ij} b_{ij} + a_{ij} b_{ij} = 0$$

$$2 a_{ij} b_{ij} = 0$$

$$a_{ij} b_{ij} = 0$$

(a) $a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

from question 1.2 (a)

$$a_{(ij)}\, a_{[ij]} = \frac{1}{4} tr \left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^t \right)$$

$$= \frac{1}{4} tr \left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \right)$$

$$= \frac{1}{4} tr \begin{bmatrix} 0+1+1 & -2+0+0 & -2+(-1)+0 \\ 0+8+3 & -1+0+3 & -1+(-8)+0 \\ 0+3+2 & -1+0+1 & -1+(-3)+0 \end{bmatrix}$$

$$= \frac{1}{4} tr \begin{bmatrix} 2 & -2 & -3 \\ 11 & 2 & -9 \\ 5 & 0 & -4 \end{bmatrix}$$

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$2+2-4=0$

$$= \frac{1}{4}[0] = 0$$

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5) (b) $q_{(ij)}\, q_{[ij]} = \frac{1}{4}\, tr\left[\begin{pmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{pmatrix}\begin{pmatrix} 0 & -2 & 0 \\ +2 & 0 & +3 \\ 0 & -3 & 0 \end{pmatrix}\right]$

$= \frac{1}{4}\, tr\left[\begin{pmatrix} 0+4+0 & -4+0+0 & 0+6+0 \\ 0+(+8)+0 & -4+0-15 & 0+12+0 \\ 0+(+10)+(0) & 0+0-12 & 0+15-0 \end{pmatrix}\right]$

$= \frac{1}{4}\, tr\begin{bmatrix} 4 & -4 & 6 \\ 8 & -19 & 12 \\ 10 & -12 & 15 \end{bmatrix}$ trace $= 4-19+15$ $= 0$

$= \frac{1}{4}[0] = 0$

(c) $q_{(ij)} \cdot q_{[ji]} = \frac{1}{4}\left[trac\left[\begin{pmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{pmatrix}\begin{pmatrix} 0 & 6 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}\right]\right]$

$= \frac{1}{4}\, trac\begin{bmatrix} 0+0-1 & 0+0+(-1) & 2+2+0 \\ 0+0-3 & 0+0+(-3) & 2+0+0 \\ 0+0+7 & 0+0+(-8) & 1+3+0 \end{bmatrix}$

$= \frac{1}{4}\, tra\begin{bmatrix} -1 & -1 & 4 \\ -3 & -3 & 2 \\ 7 & -8 & 4 \end{bmatrix}$

$= \frac{1}{4}(0) = 0$

(d) $q_{(ij)} \cdot q_{[ji]} = \frac{1}{4}\, tra\left[\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 4 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}\right]$

$= \frac{1}{4}\, tra\begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+0+8 & 0+0+(-8) \\ 0+0+0 & 0+0+0 & 0-8+0 \end{bmatrix}$

$= \frac{1}{4}\, tra\begin{bmatrix} 0 & 0 & 0 \\ 0 & 8 & -8 \\ 0 & 0 & -8 \end{bmatrix}$

$= \frac{1}{4}[0] = 0$

1.4 **Question # 04:-**

Explicitly verify the following properties of Kronecker delta

$$\delta_{ij} a_j = a_i$$
$$\delta_{ij} a_{jk} = a_{ik}$$

$$\Rightarrow \delta_{ij} a_j = \delta_{11} a_1 + \delta_{12} a_2 + \delta_{13} a_3$$

$$= \begin{bmatrix} \overset{1}{\cancel{\delta_{11}}} a_1 + \overset{0}{\cancel{\delta_{12}}} a_2 + \overset{0}{\cancel{\delta_{13}}} a_3 \\ \overset{0}{\cancel{\delta_{21}}} a_1 + \overset{1}{\cancel{\delta_{22}}} a_2 + \overset{0}{\cancel{\delta_{23}}} a_3 \\ \overset{0}{\cancel{\delta_{31}}} a_1 + \overset{0}{\cancel{\delta_{32}}} a_2 + \overset{1}{\cancel{\delta_{33}}} a_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_i \qquad \therefore \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$\Rightarrow \delta_{ij} a_{jk} = a_{ik}$$

$$= \begin{bmatrix} \overset{1}{\cancel{\delta_{11}}} a_{11} + \overset{0}{\cancel{\delta_{12}}} a_{21} + \overset{0}{\cancel{\delta_{13}}} a_{31} & \overset{1}{\cancel{\delta_{11}}} a_{12} + \overset{0}{\cancel{\delta_{12}}} a_{22} + \overset{0}{\cancel{\delta_{13}}} a_{32} & \overset{1}{\cancel{\delta_{11}}} a_{13} + \overset{0}{\cancel{\delta_{12}}} a_{23} + \overset{0}{\cancel{\delta_{13}}} a_{33} \\ \overset{0}{\cancel{\delta_{21}}} a_{11} + \overset{1}{\cancel{\delta_{22}}} a_{21} + \overset{0}{\cancel{\delta_{23}}} a_{31} & \overset{0}{\cancel{\delta_{21}}} a_{12} + \overset{1}{\cancel{\delta_{22}}} a_{22} + \overset{0}{\cancel{\delta_{23}}} a_{32} & \overset{0}{\cancel{\delta_{21}}} a_{13} + \overset{1}{\cancel{\delta_{22}}} a_{23} + \overset{0}{\cancel{\delta_{23}}} a_{33} \\ \overset{0}{\cancel{\delta_{31}}} a_{11} + \overset{0}{\cancel{\delta_{32}}} a_{21} + \overset{1}{\cancel{\delta_{33}}} a_{31} & \overset{0}{\cancel{\delta_{31}}} a_{12} + \overset{0}{\cancel{\delta_{32}}} a_{22} + \overset{1}{\cancel{\delta_{33}}} a_{32} & \overset{0}{\cancel{\delta_{31}}} a_{13} + \overset{0}{\cancel{\delta_{32}}} a_{23} + \overset{1}{\cancel{\delta_{33}}} a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{ij}$$

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1.5 Question #05

Formally expand the expression for the determinant and justify that either index notation form yields a result for specific case by using the symmetric and anti symmetric terms from Exercise 1.9.

$$\det(a_{ij}) = \varepsilon_{ijk}\, a_{1i} a_{2j} a_{3k}$$

$$= \overset{1}{\cancel{\varepsilon_{123}}} a_{11} a_{22} a_{33} + \overset{1}{\cancel{\varepsilon_{231}}} a_{12} a_{23} a_{31} + \overset{1}{\cancel{\varepsilon_{312}}} a_{13} a_{21} a_{32}$$

$$+ \overset{-1}{\cancel{\varepsilon_{132}}} a_{11} a_{23} a_{32} + \overset{-1}{\cancel{\varepsilon_{321}}} a_{13} a_{22} a_{31} + \overset{-1}{\cancel{\varepsilon_{213}}} a_{12} a_{21} a_{33}$$

$$= a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{13} a_{22} a_{31} - a_{12} a_{21} a_{33}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

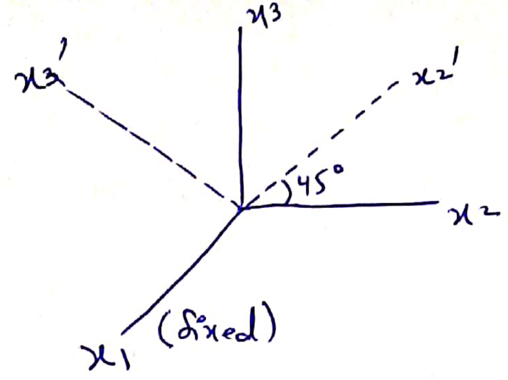
$$\varepsilon_{ijk} = \begin{cases} 1 & \text{even permutation} \\ -1 & \text{odd permutation} \\ 0 & \end{cases}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{13})$$

$$\det(a_{ij}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

1.6 Question #06:-

Determine the components of vector b_i and matrix a_{ij} given in Ex 1.1 in a new co-ordinate system found through a rotation of 45° ($\pi/4$) about x_1-axis. The rotation directions follow the positive sense in example 1.2.



Vector :-
$a_i' = Q_{ip} a_p$
matrix :-
$a_{ij}' = Q_{ip} Q_{jp} a_{pq}$

$$Q_{ij} = \begin{bmatrix} \cos(x_1', x_1) & \cos(x_1', x_2) & \cos(x_1', x_3) \\ \cos(x_2', x_1) & \cos(x_2', x_2) & \cos(x_2', x_3) \\ \cos(x_3', x_1) & \cos(x_3', x_2) & \cos(x_3', x_3) \end{bmatrix}$$

$$= \begin{bmatrix} \cos 0° & \cos 90° & \cos 90 \\ \cos 90° & \cos 45° & \cos 45° \\ \cos 90° & \cos 135° & \cos 45 \end{bmatrix}$$

$$Q_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

from Exercise 1

(a) $b_j = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $\quad a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

$$a_i' = Q_{ij} b_j$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+0+0 \\ 0+0+\sqrt{2} \\ 0+0+\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jp} a_{pq}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^t$$

$$= \begin{bmatrix} 1 & \sqrt{2} & -\sqrt{2} \\ 0 & 4.5 & -1.5 \\ 0 & 1.5 & -0.5 \end{bmatrix}$$

$\therefore a_{ij}' = Q_{ip} Q_{jq} a_{pq}$
$= \underline{Q_{ip} a_{pq}} \, Q_{jq}$
$= Q_{iq} Q_{jq}$
$= Q_{iq} (Q_{jq})^t$
$= Q_{iq} Q_{qj}$
$a_{ij}' = Q_{ij}$

17 **Question # 07:-**

The most general form of fourth order isotropic tensor can be expressed by

$$\alpha \delta_{ij}\delta_{kl} + \beta \delta_{ik}\delta_{jl} + \gamma \delta_{il}\delta_{jk}$$

where α, β & γ are arbitrary constants verify that this form remain the same under the general transformation given by (15.1).

$$\alpha' \delta_{ij}' \delta_{kl}' + \beta' \delta_{ik}' \delta_{jl}' + \gamma' \delta_{il}' \delta_{jk}'$$

$$= \alpha' Q_{im}Q_{jn}Q_{kp}Q_{lq} + \beta' Q_{im}Q_{kp}Q_{jn}Q_{lp} + \gamma' Q_{im}Q_{lq} Q_{jn}Q_{kp}$$

$$= Q_{im}Q_{jn}Q_{kp}Q_{lq}(\alpha \delta_{mn}\delta_{pq} + \beta \delta_{mp}\delta_{nq} + \gamma \delta_{mq}\delta_{np})$$

$$= \alpha\, Q_{im}Q_{jm}Q_{kp}Q_{lp} + \beta\, Q_{im}Q_{jn}Q_{km}Q_{ln} + \gamma\, Q_{lm}Q_{kn}Q_{jn}Q_{im}$$

$$= \alpha \delta_{ij}\delta_{kl} + \beta \delta_{ik}\delta_{jl} + \gamma \delta_{il}\delta_{jk}.$$

Question # 08:-

Show that the second order tensor $a\delta_{ij}$ where a is an arbitrary constant, retains its form under any transformation Q_{ij}. This form is then an isotropic second-order tensor.

Second order tensor.

$$a'\delta_{ij} = Q_{ip}Q_{jq}\, a\delta_{pq}$$

$$= a\, Q_{ip}\delta_{pq}Q_{jq}$$

$$= a\, \delta_{iq} Q_{jq}^{(T)}$$

$$= a\delta_{iq} Q_{qj}$$

$$a_{ij}' = a\delta_{ij}$$

7) from Exercise 1 (b)

$$b_j = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad a_{ij} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$$

$$b_i' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0+0 \\ 0+1/\sqrt{2}+1/\sqrt{2} \\ 0-1/\sqrt{2}+1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{2}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jq} a_{pq}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \sqrt{2} & -\sqrt{2} \\ 0 & 4.5 & -1.5 \\ 0 & 1.5 & -0.5 \end{bmatrix}$$

from Exercise 1 (c).

$$b_j = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$b_i' = Q_{ij} b_j$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0+0 \\ 0+1/\sqrt{2}+0 \\ 0-1/\sqrt{2}+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jp} a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

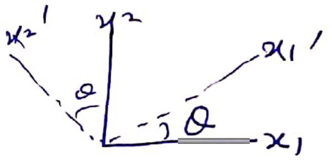
$$= \begin{bmatrix} 1 & \sqrt{2} & 0 \\ 1/\sqrt{2} & 3.5 & 2.5 \\ 1/\sqrt{2} & 3/2 & 1/2 \end{bmatrix}$$

8) Question # 07:-

Consider the twodimensional co-ordinate transformati shown - - - transformation matrix for This case is given by

$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

if $b_i = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, $a_{ij} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$ are compoents - - - polar co-ordinate system.



$$Q_{ij} = \begin{bmatrix} \cos(x_1', x_1) & \cos(x_1', x_2) \\ \cos(x_2', x_1) & \cos(x_2', x_2) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \cos(90-\theta) \\ \cos(90+\theta) & \cos\theta \end{bmatrix}$$

$\because \cos \alpha-\beta = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\cos 90-\theta = \cancel{\cos 90^\circ}\cos\theta + \sin 90^\circ \sin\theta$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$\therefore \cos\alpha+\beta = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\cos 90+\theta = \cancel{\cos 90^\circ \cos\theta} - \sin 90 \sin\theta$

$$b_i' = Q_{ij} b_j = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1\cos\theta + b_2\sin\theta \\ -b_1\sin\theta + b_2\cos\theta \end{bmatrix}$$

$$a_{ij} = Q_{ip} Q_{jq} a_{pq}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\cos^2\theta + (a_{12}-a_{21})\sin\theta\cos\theta + a_{22}\sin^2\theta & a_{12}\cos^2\theta - (a_{11}-a_{22})\sin\theta\cos\theta - a_{21}\sin^2\theta \\ a_{21}\cos^2\theta - (a_{11}-a_{22})\sin\theta\cos\theta - a_{12}\sin^2\theta & a_{11}\sin^2\theta - (a_{12}+a_{21})\sin\theta\cos\theta + a_{22}\cos^2\theta \end{bmatrix}$$

Question # 10:-

For the fourth order isotropic tensor given in Ex 1.9 show that if $\beta = \gamma$ then the tensor will have the following symmetry $C_{ijkl} = C_{klij}$.

$$C_{ijkl} = \alpha\, \delta_{ij}\delta_{kl} + \beta\, \delta_{ik}\,\delta_{jl} + \gamma\, \delta_{il}\,\delta_{jk}$$

$$= \alpha\, \delta_{ij}\delta_{kl} + \beta\left(\delta_{ik}\,\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

$$= \alpha\, \delta_{kl}\,\delta_{ij} + \beta\left(\delta_{ki}\,\delta_{lj} + \delta_{kj}\delta_{li}\right)$$

$$= C_{klij}.$$