5.6 Exponential Growth / Decay Model.

The function

$$\boxed{Q(t) = Q_0 e^{kt}} \quad , 0 \leq t < \infty$$

describes · exponential growth if $k > 0$
· exponential decay if $k < 0$

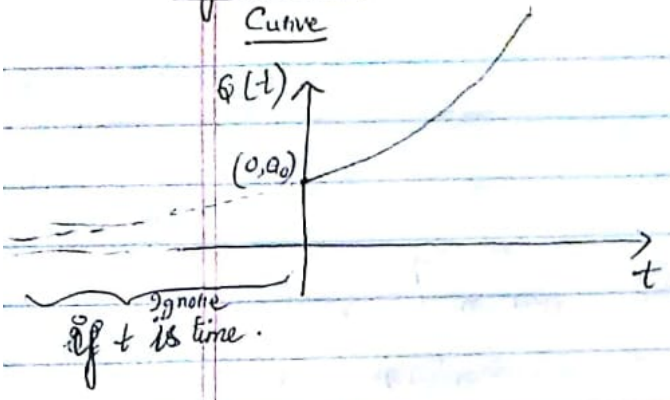
k is called the · growth constant if $k > 0$
· decay constant if $k < 0$.

$Q(0) = Q_0 e^{k \cdot 0} = Q_0 e^0 = Q_0$.

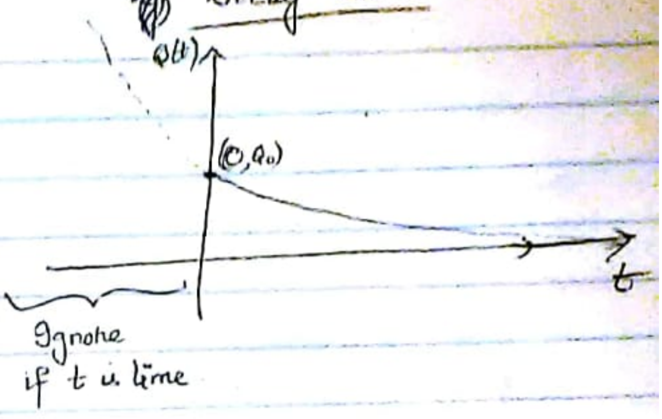
Thus Q_0 is the initial amount (may be initial population of bacteria, initial amount of radioactive material, etc.).

- Growth models are ~~usually~~ used to describe growth of population, bacteria, etc.
- Decay models are ~~usually~~ used to describe decay of radioactive materials, depreciation of ~~expensive~~ electronic goods, etc

Growth Model Curve



Decay Curve



· $\boxed{\text{Note that}}$. $Q(t) = Q_0 e^{kt}$

$\Rightarrow Q'(t) = k\,Q_0\,e^{kt} = k\,Q(t)$

Thus, rate of change is proportional to the size with constant of proportion (k) · positive for exponential growth

· negative for exp. decay.

[e.g.] Suppose it is ~~estimated~~ expected that in t years from now, population (in thousands) of Charlottesville will obey growth model.

$$Q(t) = 200e^{kt}$$

The population of Charlottesville 10 years from now is estimated to be 240 (thousands).

(a) Find k

(b) What is the current population?

(c) What is the estimated population 20 years later?

(d) When ~~the rate of change of~~ does population become 400?

(e) Find rate of change after 2 years.

Soln a) $240 = Q(10) = 200e^{10k}$

$$\Rightarrow \frac{24}{20} = \frac{6}{5} = e^{10k}$$

$$\Rightarrow 1.2 = e^{10k}$$

$$\Rightarrow 10k = \ln(1.2)$$

$$\Rightarrow k = \frac{1}{10}\ln(1.2)$$

(b) Current population

$= Q(0)$

$= 200 e^{k \cdot 0} = 200$ (thousands)

(c) $Q(20) = 200 e^{20k}$

$= 200 e^{20 \cdot \frac{1}{10} \ln(1.2)}$

$= 200 e^{2 \ln(1.2)}$ (thousands)

Ans

(d) $400 = 200e^{kt}$

$\Rightarrow \ln 2 = kt \quad \Rightarrow \quad t = \frac{1}{k}\ln 2 = \frac{1}{\frac{1}{10}\ln(1.2)}\ln 2$

$= 10\,\frac{\ln(2)}{\ln(1.2)}$

So, population becomes 400 after $10\,\frac{\ln 2}{\ln(1.2)}$ years.

(e) Need $Q'(2)$.

$Q'(t) = 200k\,e^{kt}$.

$Q'(2) = 200 \cdot \frac{1}{10\ln(1.2)}\,e^{2k}$. Ans.

Ex: C-14 is a radioactive material that decays exponentially. Skeletal remains of the so-called Pittsburgh man (PM), unearthed in Pennsylvania, had lost 82 % of the Carbon-14 they originally contained. The half-life of Carbon-14 is 5770 years. Determine the approximate age of the bones.

Soln:

Let $Q(t)$ be the amount of radioactive material present t years after (PM) died.

$$Q(t) = Q_0 e^{kt} \quad (k<0)$$

$$Q(5770) = \frac{1}{2} Q_0.$$

$$\frac{1}{2} Q_0 = Q_0 e^{5770k} \Rightarrow \ln\frac{1}{2} = 5770k$$

$$\Rightarrow \frac{1}{5770} \ln\left(\frac{1}{2}\right) = k$$

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82% is lost, so 18% remains

$$\frac{18}{100} Q_0 = Q_0 e^{kt}$$

$$\Rightarrow \ln(.18) = kt$$

$$\Rightarrow t = \frac{1}{k} \ln(0.18)$$

$$= \frac{5770}{\ln\left(\frac{1}{2}\right)} \ln(0.18)$$

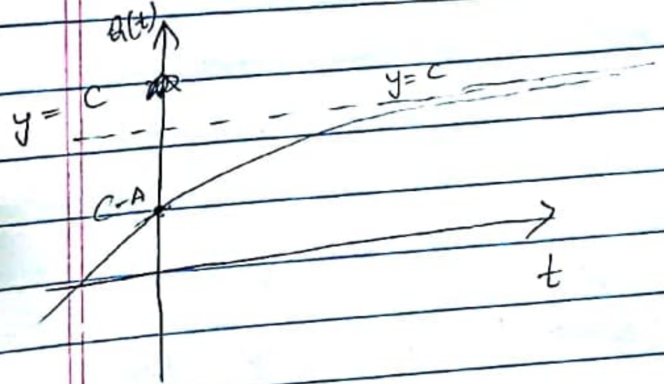
$$\boxed{t = \frac{5770 \, . \ln(0.18)}{\ln\left(\frac{1}{2}\right)}}$$

So, age of bone is ↑ Ans

• Some other exponential Models.

ⓐ **Learning Curves** : Describes certain types of learning processes.

$$Q(t) = C - Ae^{-kt} \quad \text{where } C>0,\ A>0,\ k>0.$$



• $Q(0) = C - A$

• $\lim\limits_{t\to\infty} Q(t) = \lim\limits_{t\to\infty}\left(C - Ae^{-kt}\right)$

$= \lim\limits_{t\to\infty} C - A \lim\limits_{t\to\infty} e^{-kt}$

$= C - A\cdot(0)$

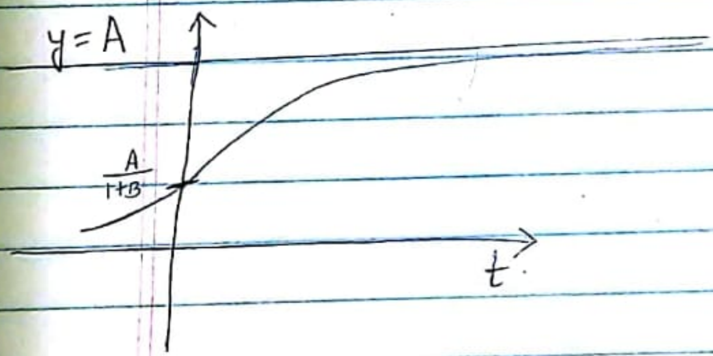
$= C.$

e.g.
Q(t) may describe the productivity of worker (say # of pizza one can make per hour) t months after employment ; in this scenario, C represents full potential of worker.

(b) Logistic Model : a.k.a. 'restricted' growth model'

$$Q(t) = \frac{A}{1+Be^{-kt}}$$

(A, B, k are positive constants).



e.g. $Q(t)$ could represent

– the number of rabbits present in a farm with a fixed size (say 1000 rabbits max)

In this scenario, $A = 1000$ which represents the environmental limit.