1.

$$\frac{dy}{dt} = \sqrt{y},\ y(1) = 0.$$

The function $f(t,y) = \sqrt{y}$ has $\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}}$. Notice that f is continuous on $(-\infty,\infty)\times[0,\infty)$ and $\frac{\partial f}{\partial y}$ is only continuous on $(-\infty,\infty)\times(0,\infty)$. There is no open rectangle containing the initial condition point $(1,0)$ on which f and/or $\partial f/\partial y$ are continuous, so neither the existence or uniqueness of a solution is guaranteed (on some interval containing 1) by the theorem.

Note however that the equilibrium solution $y = 0$ solves the IVP.

2.

Show that the initial value problem

$$t\frac{dy}{dt} = 2y,\ y(0) = 0$$

has infinitely many solutions. Note that the differential equation is linear. Why does this example not contradict Theorem 2.4.4?

Separating variables and solving gives the family of solutions $y = ct^2$. The initial condition $y(0) = 0$ is satisfied with any choice of the constant c, and the initial value problem has infinitely many solutions. In standard form the DE is $\frac{dy}{dt} - \frac{2}{t}y = 0$. The coefficient function $-\frac{2}{t}$ fails to be continuous on any open interval containing $t = 0$, so the theorem does not apply.

3. Find two different solutions to the initial value problem

$$4\frac{dy}{dt} = 5y^{1/5},\ y(1) = 0$$

for $t \geq 1$. Why doesn't this contradict Theorem 2.4.5?

Separating variables and integrating gives $y^{4/5} = t + c$ or $y = (t+c)^{5/4}$. With $y(1) = 0$ we choose $c = -1$, so that $y = (t-1)^{5/4}$ is a solution to the IVP. But $y = 0$ is a second solution to the same IVP.

If $f(t,y) = \frac{5}{4}y^{1/5}$, then $\frac{\partial f}{\partial y} = \frac{1}{4}y^{-4/5}$, and $\frac{\partial f}{\partial y}$ is not continuous on any open rectangle containing $(1,0)$.

4.

Consider the differential equation

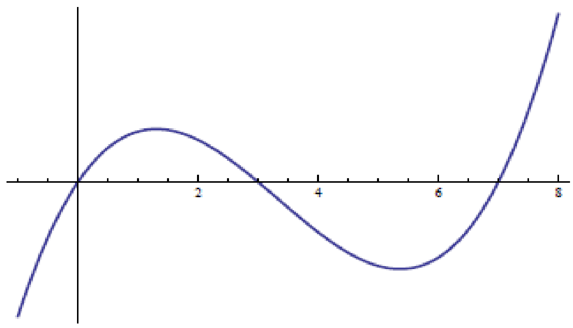
$$\frac{dy}{dt} = f(y)$$

where the graph of $f(y)$ is as shown below.

(a) What are the equilibrium solutions?

(b) Sketch the phase line for $\frac{dy}{dt}$.

(c) Using the phase line, sketch some solution curves in the ty-plane.



(a) From the graph, it appears that the equilibrium solutions are $y = 0$, $y = 3$, and $y = 7$.

(b)

(c)