One-sided limits

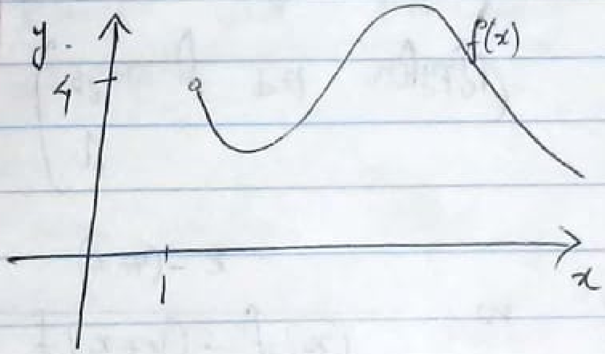
We say $\lim_{x \to a^+} f(x) = L$ (resp. $\lim_{x \to a^-} f(x) = L$)

if :

as x gets very close to a from the right (resp., left) but $x \neq a$, all values of $f(x)$ get very close to L. (Note that L can be ∞ or $-\infty$ too).

If there is no such L, we say $\lim_{x \to a^\pm} f(x)$ does not exist (or DNE)

$\boxed{\text{e.g.}}$



$f(1)$ is not defined (we mark it by a circle).

$f : (1, \infty) \to \mathbb{R}$:

- However, $\lim_{x \to 1^+} f(x) = 4$ (as the diagram show

Here, $\lim_{x \to 1^-} f(x)$ does not make sense

as domain of f is $(1, \infty)$.

Thus, " $\lim_{x \to 1^-} f(x)$ " is meaningless.

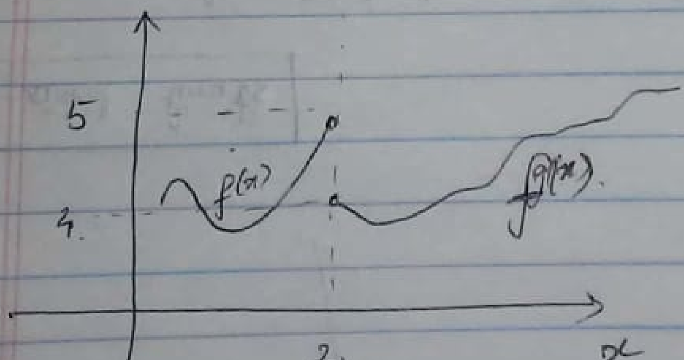
We write $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ only when we can

associate a precise meaning to it — e.g. $x \to a^+$ means

a is being approached from the right.

Thus, if we want to evaluate $f(x)$ as we approach a from the right, f has to be defined on the right of a.!

e.g.



Here,

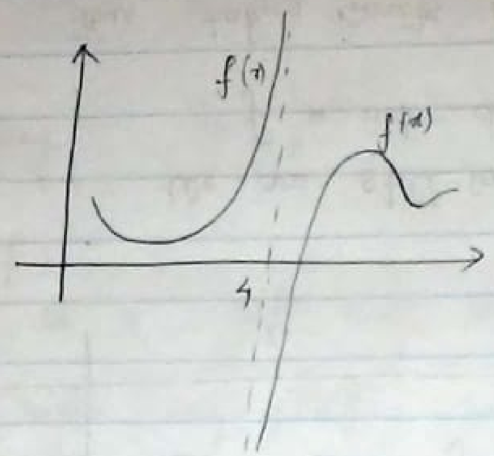
$\lim_{x \to 2^-} f(x) = 5$

$\lim_{x \to 2^+} f(x) = 4.$

We are still in a situation where f is not defined at $x = 2$.

Thus, taking limits is not at all the same as evaluating the function.

e.g



$$\lim_{x \to 4^+} f(x) = -\infty$$

$$\lim_{x \to 4^-} f(x) = \infty$$

Two-sided limits

We say $\lim_{x \to a} f(x) = L$ if $\lim_{x \to a^+} f(x) = L$ and

$\lim_{x \to a^-} f(x) = L$.

If the 2 one-sided limits are not equal, then.

$\lim_{x \to a} f(x)$ DNE

'i.e.' we approach 'a' from any direction, the 'f(x)'-values get very close to a single quantity L. (may be ∞ or $-\infty$)

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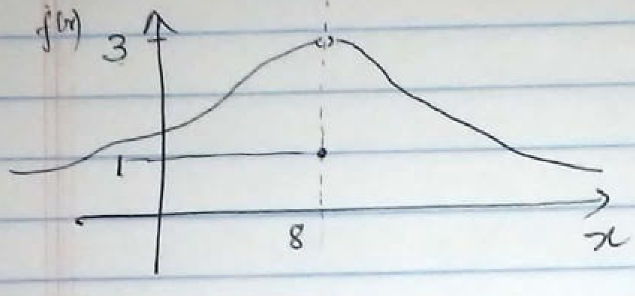
Note that the value of f at a given point 'a' ('f' might not even be defined at a) is irrelevant for finding out the limit.

e.g.



Here

$$\lim_{x \to 8^-} f(x) = 3$$

$$\lim_{x \to 8^+} f(x) = 3.$$

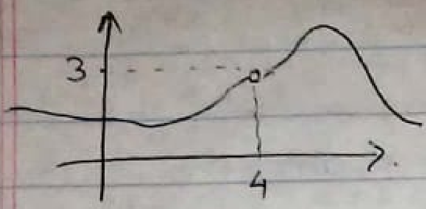
$$f(8) = 1.$$

Thus, $\lim_{x \to 8} f(x) = 3$. But $f(8) = 1$. So, it is irrelevant what the value of f at that point is, when we are bothered about finding one-sided / two-sided limits.

Evaluation will be relevant when we talk about 'continuity' of a function. We will see this in Sec. 2-5.

e.g

(i)

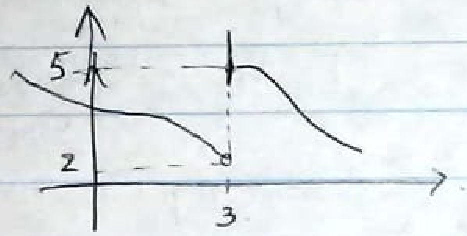


$\lim_{x \to 4^-} f(x) = 3$

$\lim_{x \to 4^+} f(x) = 3$

So, $\lim_{x \to 4} f(x) = 3$.

(ii)

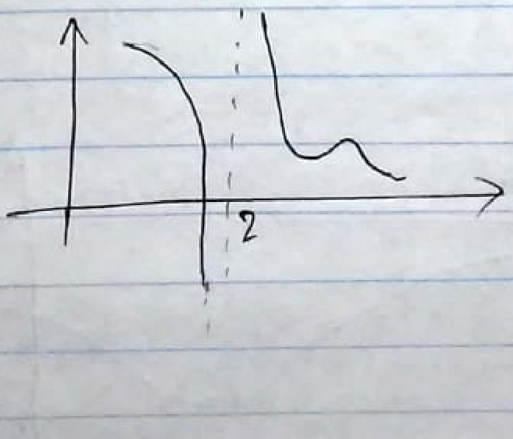


$\lim_{x \to 3^-} f(x) = 2$

$\lim_{x \to 3^+} f(x) = 5$

Thus, $\lim_{x \to 3} f(x)$ DNE.

(iii)

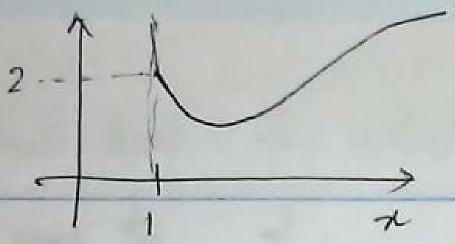


$\lim_{x \to 2^-} f(x) = -\infty$

$\lim_{x \to 2^+} f(x) = \infty$.

So, $\lim_{x \to 2} f(x)$ DNE.

Note



Thus domain of f is $[1, \infty)$.

So, '$\lim_{x \to 1^-} f(x)$' makes no sense.

But we can talk about $\lim_{x \to 1^+} f(x) = 2$. In fact, here $f(1) = 2$ also as is clear from the picture.

Basic Facts

- $\lim_{x \to a} x = a$. [Draw the graph $y = x$. It is clear].

- $\lim_{x \to a} c = c$. [limit of a constant function is that constant itself].

So, $\lim_{x \to 8} \left(2^{100}\right) = 2^{100}$.

Theorem

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ (i.e. both limits exist; L & M are some real numbers; cannot be ∞ or $-\infty$).

Then.

① $\lim_{x\to a} [f(x)]^r = \left(\lim_{x\to a} f(x)\right)^r = L^r$, r is a positive constant.

② $\lim_{x\to a}\left(c\, f(x)\right) = c\left(\lim_{x\to a} f(x)\right) = cL$, $c \in \mathbb{R}$.

③ $\lim_{x\to a}\left(f(x) \pm g(x)\right) = \left(\lim_{x\to a} f(x)\right) \pm \left(\lim_{x\to a} g(x)\right) = L \pm M$

④ $\lim_{x\to a}\left(f(x)\, g(x)\right) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = LM$.

⑤ $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$, provided $M \neq 0$.

(These limit laws also hold if 'a' is replaced by a^+ or a^-).

Let us use these to evaluate certain limits. Also, note how to write down solutions.

i) Evaluate $\lim_{x \to 2} (x^3 - 54x)$

$\boxed{\text{Sol}^{\underline{n}}}$ Using ③, $\lim_{x \to 2} (x^3 - 54x)$

$$= \lim_{x \to 2} x^3 - \lim_{x \to 2} 54x$$

$$= \left(\lim_{x \to 2} x\right)^3 - 54\left(\lim_{x \to 2} x\right) \quad [\text{Using ① \& ②}]$$

$$= 2^3 - 54.2$$

$$= 8 - 108 = -100.$$

[We did not have to evaluate one-sided limits here as the limit laws directly applied].

(ii) Use limit laws to calculate the limit of $\frac{\sqrt[3]{x}}{x^2+1}$ as x goes to 2.

Soln: $\lim_{x\to 2} \frac{\sqrt[3]{x}}{x^2+1}$. Note that $\lim_{x\to 2}(x^2+1)$

$$= \lim_{x\to 2} x^2 + \lim_{x\to 2} 1$$

$$= \left(\lim_{x\to 2} x\right)^2 + 1$$

(limit of a constant function)

$$= 2^2 + 1$$

$$= 5 \neq 0.$$

Thus, rule (5) applies.

$$\lim_{x\to 2}\left(\frac{\sqrt[3]{x}}{x^2+1}\right) = \frac{\lim_{x\to 2}\sqrt[3]{x}}{\lim_{x\to 2}(x^2+1)} = \frac{\left(\lim_{x\to 2} x\right)^{1/3}}{5}$$

$$= \frac{2^{1/3}}{5}$$

Ans.

Note / Trick

If $f(x)$ is a combination of $+, -, \times, \div$ of power of x, then $\lim_{x \to a} f(x) = f(a)$, i.e. just plug in $x = a$, as long as $f(a)$ is defined.

e.g. (i) Suppose $f(x) = 5x^3 - 2x + 3$.

It's a polynomial, so f is defined over all real numbers.

Thus, $\lim_{x \to 1} f(x) = 5(1)^3 - 2(1) + 3$ $= 5 - 2 + 3 = 6.$ (using the above note; you can also do it using the pvs laws).

(ii) $f(x) = \dfrac{x^2 - 4}{x - 2}$.

As long as $x_0 \neq 2$, then $f(x_0)$ is defined.

Thus, $\lim_{x \to 3} f(x) = f(3) = \dfrac{3^2 - 4}{3 - 2} = \dfrac{5}{1} = 5$

(We actually used ~~law~~ rule 5 here)

However, $\lim_{x \to 2} f(x)$ cannot be found out simply by plugging in.

We note that $f(x) = \dfrac{x^2-4}{x-2} = \dfrac{(x+2)(x-2)}{(x-2)}$

$= x+2$.

Now, $\lim_{x \to 2} f(x) = \lim_{x \to 2} (x+2) = (2+2) = 4$.

Here, there was 'cancellation' which helped us. We'll develop newer and newer tools which will help to compute limit - 'cancellation' is one of them.

$\boxed{\text{Natural Question}}$ So why bother about 1-sided limits?

The following example shows its use.

• Say $f(x) = \begin{cases} x^2+5 & x<1 \\ 3 & x=1. \\ \sqrt[3]{x-9} & x>1. \end{cases}$ piecewise defined function.

It is clear from the definition, that $x=1$ is a point of interest.

[Q] Evaluate the limit of f as x tends to 1. ~~If it DNE~~ Clearly specify why, whether your answer is a real number, ∞, $-\infty$ or DNE.

$\boxed{\text{Sol}^n}$. (We can't just evaluate $\lim_{x\to 1} f(x)$ as f is defined differently on $(-\infty, 1)$ & $(1, \infty)$. Thus, naturally, we look at ~~both~~ one-sided limits).

Now, the limit laws also hold for one-sided limits.

So, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2+5) = (1)^2+5 = 6.$

(remember, we can simply evaluate using the "Trick")

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \left(\sqrt[3]{x-9}\right)$$

$$= \left(\lim_{x\to 1^+} (x-9)\right)^{1/3} = (-8)^{1/3} = -2$$

Thus, $\lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x)$.

So, $\boxed{\lim_{x \to 1} f(x) \text{ DNE}}$.

$\square$.

A Few More Examples

i) Evaluate $\lim_{x \to 3} \dfrac{x^2 + 9x + 1}{x^2 + 5x + 7}$.

(Soln): If $f(x) = \dfrac{x^2 + 9x + 1}{x^2 + 5x + 7}$, then $f(3)$ is defined.
(it's a rational function; denominator does not vanish at 3).

So, $\lim_{x \to 3} f(x) = f(3)$

$$= \frac{3^2 + 9.3 + 1}{3^2 + 5.3 + 7}$$

$$= \frac{37}{31}.$$

Alternately, Using Rule 5, (Since $\lim_{x \to 3} (x^2 + 5x + 7) \neq 0$),

$$\lim_{x \to 3} f(x) = \frac{\lim_{x \to 3} (x^2 + 9x + 1)}{\lim_{x \to 3} (x^2 + 5x + 7)} = \frac{3^2 + 9.3 + 1}{3^2 + 5.3 + 7} \text{ (using 1, 2, 3, 4)}$$

$$= \frac{37}{31}.$$

Ans

(ii) $\lim_{x \to 2^+} \left(\frac{x^2 + \sqrt{x+2}}{x+1} \right)$

Note that $\lim_{x \to 2^+} (x+1) = 3$, so rule (5) applies.

Thus $\lim_{x \to 2^+} \left(\frac{x^2 + \sqrt{x+2}}{x+1} \right) = \frac{\lim_{x \to 2^+} \left(x^2 + \sqrt{x+2}\right)}{\lim_{x \to 2^+} (x+1)}$

$$= \frac{\lim_{x \to 2^+} x^2 + \lim_{x \to 2^+} \left(\sqrt{x+2}\right)}{\left(\lim_{x \to 2^+} x\right) + 1}$$

$$= \frac{4 + \sqrt{2+2}}{2+1} = \frac{4+2}{3} = 2.$$

(It's good to write down in details some of the solutions, so that the 'Limit Rules' (1)-(5) are something you get very used to).

$\boxed{\text{Remark}}$ Observe that I am writing "$\lim_{x \to a}$" at the appropriate places, in front of appropriate functions; also note the distribution of limit over numerator & denominator is also very clear; the 'bar' for the fraction is just beside the equality sign

e.g. $= \frac{\lim_{x \to 2^+} \left(x^2 + \sqrt{x+2}\right)}{\lim_{x \to 2^+} (x+1)}$.

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Writing is crucial for limit problems. A lot of students forget to write "$\lim\limits_{x \to \cdots}$" in some steps; or keep writing "$\lim\limits_{x \to \cdots}$" even after evaluating; e.g.

$$\lim_{x \to 2} x^2 \neq \lim_{x \to 2} 2^2$$

Once you evaluate, drop the "$\lim\limits_{x \to}$" part.

So, $\lim\limits_{x \to 2} x^2 = 2^2 = 4$.

Do not lose marks over these writing issues. A lot of students lose points in this way.