Indeterminate forms "$\frac{0}{0}$"

"$\frac{0}{0}$" indeterminate form is a limit of the form

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) \quad \text{where} \quad \lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

So, Limit law (5) fails; you cannot say.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \times \text{ here as it gives } "\frac{0}{0}".$$

- We ~~can~~ secretly evaluate limits for both numerator and denominator in our heads, to determine whether it's an indeterminate form or not. but DO NOT WRITE $\frac{0}{0}$ ever. Mathematically, it makes no sense.

e.g. Is this an indeterminate form?

$$\lim_{x \to 2} \left(\frac{2x^2 - 5x + 2}{x^2 - 4} \right)$$

$\boxed{\text{Soln.:}}$ As x goes to 2, numerator goes to 0,
" " " " ", denominator goes to 0.

So, indeed it is an indeterminate form.

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[Note I never wrote $\frac{0}{0}$ in the soln; but definitely that is what the given expression looks like in my head!]

So, now we learn as to how to tackle indeterminate forms.

$\boxed{\text{Tricks}}$ (i) Usually, there will be cancellation of terms once you factorize the numerator and denominator.

ii) Sometimes you will have to rationalize ~~the denominator~~ and then cancel terms out. again using factorization.

iii). L'Hospital's rule also helps. but YOU ARE NOT ALLOWED TO USE L'HOSPITAL'S RULE IN MIDTERM I.

~~(iv) Sometimes, one-sided limits will help you out too.~~

e.g. $\lim_{x\to 2} \frac{4(x^2-4)}{x-2}$.

(Soln) (You secretly have already ~~thought~~ realized that this is a "$\frac{0}{0}$" form).

$$\lim_{x\to 2} \frac{4(x^2-4)}{x-2} = \lim_{x\to 2} \left(\frac{4(x-2)(x+2)}{(x-2)}\right)$$

$$= \lim_{x\to 2} \left(4\,(x+2)\right) = 4.(2+2) = 16.$$

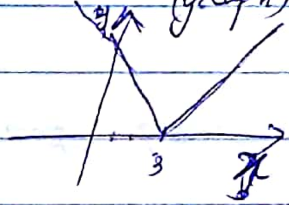
□

We might have to use. ~~absolute value functions~~ one-sided limits too sometimes.

$\boxed{Q}$ Calculate $\displaystyle\lim_{x\to 3} \frac{|x-3|}{x^2-2x-3}$.

$\boxed{\text{Sol}^n}$: (Again you have secretly realized it's a "$\frac{0}{0}$" form).

Recall $|x-3| = \begin{cases} x-3 & x \geq 3. \\ -(x-3) & x \leq 3. \end{cases}$ (Graph).



(As soon as you see the division $x \geq 3$, $x \leq 3$, you immediately know that you have to use One-sided limits).

$$\lim_{x\to 3^+} \frac{x-3}{x^2-2x-3} = \lim_{x\to 3^+} \frac{x-3}{(x-3)(x+1)}$$

$$= \lim_{x\to 3^+} \frac{1}{x+1} = \frac{1}{4}.$$

$$\lim_{x\to 3^-} \frac{-(x-3)}{x^2-2x-3} = \lim_{x\to 3^-} \frac{-(x-3)}{(x-3)(x+1)} = \lim_{x\to 3^-} \frac{-1}{x+1} = -\frac{1}{4}.$$

Thus, $\displaystyle\lim_{x\to 3} \frac{|x-3|}{x^2-2x-3}$ DNE.

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[Remark: Observe very carefully as to how the solution is written. I write "$\lim\limits_{x\to\cdots}$" wherever needed and stop writing it as soon as I evaluate. When writing down solutions, even for Webassign scratch work, develop the habit of writing like this. Your scores will get affected by your writing, remember this.]

(Back)

$\boxed{Q}$. Find $\displaystyle\lim_{x\to 2} \frac{2x^2-5x+2}{x+2}$.

$\boxed{Sol^n}$ (It is not an indeterminate form; $\lim\limits_{x\to 2}(x+2) = 4$)

So, Rule (5) applies.

$$\lim_{x\to 2} \frac{2x^2-5x+2}{x+2} = \frac{\lim\limits_{x\to 2}(2x^2-5x+2)}{\lim\limits_{x\to 2}(x+2)}$$

$$= \frac{0}{4} = 0.$$

$\square$

(It's always important that you 'check' (secretly / actually) $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ $\lim\limits_{x\to a} f(x)$ $\lim\limits_{x\to a} g(x)$ the limits of each of the numerator & denominator, as you start working on such problems).

Limits

$\frac{\text{(Non-zero)}}{\text{(Zero)}}$ form :

[Q] $\lim_{x \to 2} \frac{x^2+1}{2-x}$.

Soln) You have realized that this is a "$\frac{5}{0}$" form when evaluated. How do we go about answering this?

Ans: We use one-sided limits.

Note that as x goes to 2 from the left. the numerator goes to 5 whereas denominator goes to 0 from the positive side.

So, $\frac{x^2+1}{2-x}$ becomes larger and larger.

Hence, $\lim_{x \to 2^-} \frac{x^2+1}{2-x} = \infty$.

What happens when x tends to 2 from the right?

· Numerator $\to 5$

The denominator $\to 0$ but now from the negative side (as $2-x < 0$ since $x > 2$ approaching 2 from right).

Hence, $\dfrac{x^2+1}{2-x}$ becomes smaller and smaller (more and more negative)

So, $\lim\limits_{x \to 2^-} \dfrac{x^2+1}{2-x} = -\infty$.

Thus, $\lim\limits_{x \to 2} \dfrac{x^2+1}{2-x}$ DNE.

Remark This example also illustrates the power of one-sided limits!

• Example where rationalization is required.

Q. $\lim\limits_{h \to 0} \dfrac{\sqrt{1+h}-1}{h} = ?$

Sol^n^ (You've realized, it's a "$\frac{0}{0}$" form).

$$\frac{\sqrt{1+h}-1}{h} = \frac{(\sqrt{1+h}-1)(\sqrt{1+h}+1)}{h(\sqrt{1+h}+1)}$$

$$= \frac{(\sqrt{1+h})^2-(1)^2}{h(\sqrt{1+h}+1)}.$$

$$= \frac{1+h-1}{h(\sqrt{1+h}+1)} = \frac{h}{h(\sqrt{1+h}+1)} = \frac{1}{\sqrt{1+h}+1}.$$

Thus, $\lim\limits_{h \to 0} \dfrac{\sqrt{1+h}-1}{h} = \lim\limits_{h \to 0} \dfrac{1}{\sqrt{1+h}+1} = \dfrac{1}{\sqrt{1+0}+1} = \dfrac{1}{2}$. $\square$

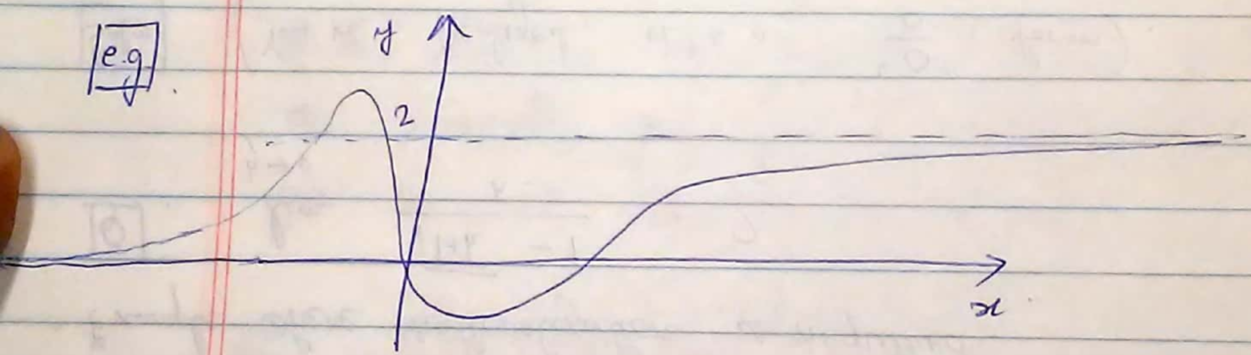
Limits at Infinity

We say $\lim_{x \to \infty} f(x) = L$ $\left(\boxed{\text{OR}}\ \lim_{x \to +\infty} f(x) = L\right)$ if, as

x gets very positive ($\boxed{\text{OR}}$ negative), all values of

$f(x)$ get very close to a value L.

$\boxed{\text{e.g}}$.



$\lim_{x \to \infty} f(x) = 2$; $\lim_{x \to -\infty} f(x) = 0$.

The dotted straight line $(y=2)$, as well as the x-axis $(y=0)$ are called "horizontal asymptotes" of f.

$\boxed{\text{Remark}}$ When asked to find horizontal asymptotes of some function f, you're supposed to find out

$\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

e.g. Find horizontal asymptote of $\frac{2x^2}{1+x^2}$

Soln. $\lim_{x\to\infty} \frac{2x^2}{1+x^2} = \lim_{x\to\infty} \frac{\frac{2x^2}{x^2}}{\frac{1}{x^2}+\frac{x^2}{x^2}}$ (Dividing numerator and denominator by the highest power of x in the denominator).

$= \lim_{x\to\infty} \frac{2}{1+\frac{1}{x^2}}$

$= \frac{2}{1+0} = 2.$

So, $y=2$ is a horizontal asymptote.

$\lim_{x\to-\infty} \frac{2x^2}{1+x^2} = \lim_{x\to-\infty} \frac{2}{1+\frac{1}{x^2}}$ (same trick)

$= \frac{2}{1+0} = 2.$

Thus, $y=2$ is the horizontal asymptote.

$\square$.

We'll talk more about asymptotes later but this trick of dividing numerator and denominator by the highest power of x in the denominator, is what will help us to evaluate limits at infinity.

$\boxed{\text{Basic facts}}$ $\boxed{\text{Suppose } n>0}$

- $\lim\limits_{x \to \infty} x^n = \infty$ if $n>0$

- $\lim\limits_{x \to -\infty} x^n = \begin{cases} \infty & \text{if } n \text{ is even} \\ -\infty & \text{if } n \text{ is odd} \end{cases}$

- If $f(x)$ is a polynomial, then.

$\lim\limits_{x \to \infty} f(x)$ is the same as the limit of the dominating term (usually the highest power of x).

$\boxed{\text{e.g.}}$

- If $n<0$,

  . $\lim\limits_{x \to \infty} x^n = 0$

  . $\lim\limits_{x \to -\infty} x^n = 0$.

$\boxed{\text{Why?}}$ Since $n<0$, let $n=-m$, where $m>0$.

So, $x^n = x^{-m} = \frac{1}{x^m}$.

Thus, $\lim\limits_{x \to \infty} \frac{1}{x^m} = 0$.

$\lim\limits_{x \to -\infty} \frac{1}{x^m} = 0$.

In other words,

$$\lim_{x \to \pm\infty} \frac{1}{x^n} = 0 \quad \text{whenever} \quad n>0.$$

$\boxed{Q}$. Let $f(x) = (-5x^3 - 4x^2 + 2)$

Does $\lim_{x \to -\infty} f(x)$ exist? If yes, what is it?

$\boxed{\text{Soln}}$. $-5x^3$ is the dominating term.

Thus, $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (-5x^3)$

$\cancel{= -5 \lim_{x \to -\infty} x^3 = -5(-\infty) = \infty}$

$= \infty$.

$\boxed{\text{Secretly}}$.

$\left(\begin{array}{l} x^3 \to -\infty. \\ \text{So, } -x^3 \to \infty \\ \text{So, } -5x^3 \to \infty. \end{array} \right)$

[Note: I directly wrote ∞ but secretly realized what is happening. We do this because "$5.\infty$" or "$-(-\infty)$" – these expressions are not mathematically precise. So, secretly do this sort of scratch work but do not write such expressions].

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$\boxed{Q}$ $\lim_{x \to \infty} \dfrac{3x^3 - 2x - 2}{-2x^3 + 5x^2 - 1}$

[Remark: If you ~~plug in ∞ in~~ evaluate numerator & denominator, you get '$\frac{\infty}{-\infty}$' expression; this will become important when we study L'Hospital's Rule]

$\boxed{\text{Sol}^{\underline{n}}}$ (Divide up & down by the highest power of x (∄) in the denominator).

$$\lim_{x \to \infty} \frac{3x^3 - 2x - 2}{-2x^3 + 5x^2 - 1}$$

$$= \lim_{x \to \infty} \frac{\frac{3x^3}{x^3} - \frac{2x}{x^3} - \frac{2}{x^3}}{-\frac{2x^3}{x^3} + \frac{5x^2}{x^3} - \frac{1}{x^3}} = \lim_{x \to \infty} \left(\frac{3 - \frac{2}{x} - \frac{2}{x^3}}{-2 + \frac{5}{x} - \frac{1}{x^3}} \right)$$ (Note crucially where I am writing the limit).

$$= \frac{\lim_{x \to \infty} \left(3 - \frac{2}{x} - \frac{2}{x^3}\right)}{\lim_{x \to \infty} \left(-2 + \frac{5}{x} - \frac{1}{x^3}\right)}$$ (since rule ⑤ applies).

$$= \frac{3 - 0 - 0}{-2 + 0 - 0}$$

$$= -\frac{3}{2}.$$

□.

[Q]. Find $\lim\limits_{x \to \infty} \dfrac{2x^3 - 3x^2 + 1}{x^2 + 2x + 4}$.

Ans. (divide up & down by. x^2)

$$\lim_{x \to \infty} \frac{2x^3 - 3x^2 + 1}{x^2 + 2x + 4}$$

$$= \lim_{x \to \infty} \frac{\frac{2x^3}{x^2} - \frac{3x^2}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{2x}{x^2} + \frac{4}{x^2}} = \lim_{x \to \infty} \frac{2x - 3 + \frac{1}{x^2}}{1 + \frac{2}{x} + \frac{4}{x^2}}.$$

Thus. $\cancel{\dfrac{\lim\limits_{x \to \infty} \left(2x - 3 + \frac{1}{x^2}\right)}{\lim\limits_{x \to \infty} \left(1 + \frac{2}{x} + \frac{4}{x^2}\right)}}$ ~~(Rule 5 ap~~

(Rule 5 does not apply here as limit of the numerator is ∞. but we required 'L' to be a ~~real~~ finite number remember?)

[However, secretly you ~~have~~ realized, if you distribute limits over numerator & denominator, you get "$\frac{\infty}{1}$"

Thus, you can directly write. ∞].

i.e. $\lim\limits_{x \to \infty} \left(\dfrac{2x - 3 + \frac{1}{x^2}}{1 + \frac{2}{x} + \frac{4}{x^2}} \right) = \infty$.

[Be very careful while writing solutions with limits! Don't lose points]

Continuity

~~Suppose $f(x)$~~

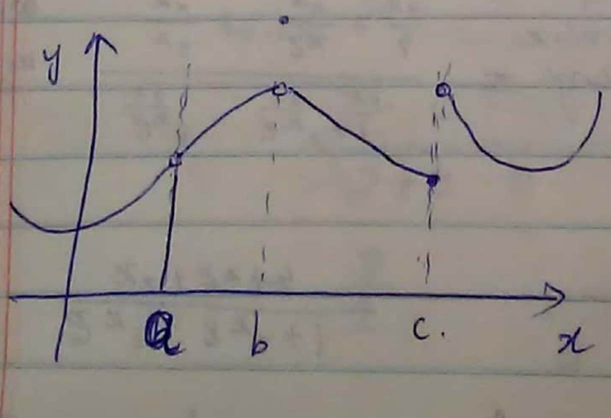
A function f is continuous at a number $= b$, if the following hold:

i) $f(b)$ is defined

ii) $\lim_{x \to b} f(x)$ exists. (ie. $\lim_{x \to b^-} f(x) = \lim_{x \to b^+} f(x)$)

(iii) $\lim_{x \to b} f(x) = f(b)$

[In terms of graphs, there should not be any 'breaks' or 'jumps' at the point $x = b$]



- f is not continuous at a because it is not defined.

- f is not continuous at b. because $\lim_{x \to b} f(x) \neq f(b)$.

- f is not continuous at $x = c$ since $\lim_{x \to c} f(x)$ DNE.