STUDENT NAME: ______________________________

INSTRUCTOR: ______________________________

Please sign the pledge:
On my honor as a student, I have neither given nor received aid on this exam.

Directions

Answer each question in the space provided. Please write clearly and legibly. ***Show all of your work—your work must justify your answer, and clearly identify your final answer. No books, notes or calculators are allowed.***

For instructor use only

Page	Points	Score
2	10	
3	18	
4	10	
5	14	
6	16	
7	10	
8	10	
9	12	
Total:	100	

Some Formulas You May Find Useful

Description	**Formula**
Pythagoras' Theorem (right triangle with legs x, y, hypothenuse h)	$h^2 = x^2 + y^2$
Area of the surface of a sphere of radius R	$4\pi R^2$
Volume of a rectangular box, base xy, height h	xyh
Surface area of a rectangular box, base xy, height h, no top	$2xh + 2yh + xy$
Volume of a cylinder of radius r and height h	$\pi r^2 h$
Compound Interest: $A(t)$ accumulated amount after t years, P is principal, r nominal annual interest rate, m number of conversion periods per year	$A(t) = P\left(1 + \frac{r}{m}\right)^{mt}$

1. [10 pts] Let C be the curve in the xy-plane given by the following equation.

$$x^2y + y^3 = 2$$

(a) Use the method of implicit differentiation to find $\dfrac{dy}{dx}$.

Solution: Differentiate both sides of the equation with respect to x, respect the product and chain rules, and solve for $\dfrac{dy}{dx}$.

$$\begin{aligned}
\frac{d}{dx}\big(x^2y + y^3 &= 2\big) \\
2xy + x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} &= 0 \\
&\Rightarrow \\
x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} &= -2xy \\
\frac{dy}{dx}\big(x^2 + 3y^2\big) &= -2xy \\
&\Rightarrow \\
\frac{dy}{dx} &= \frac{-2xy}{x^2+3y^2}.
\end{aligned}$$

(b) Find an equation for the line tangent to C at the point $(1,1)$.

Solution: Use the formula for $\dfrac{dy}{dx}$ found in part (a) to find the slope of the line tangent to C at $(1,1)$ by plugging in $x = 1$ and $y = 1$:

$$\begin{aligned}
\left.\frac{dy}{dx}\right|_{(1,1)} &= \frac{-2(1)(1)}{(1)^2 + 3(1)^2} \\
&= \frac{-2}{1+3} \\
&= -\frac{1}{2}.
\end{aligned}$$

So the tangent line has slope $-\dfrac{1}{2}$ and goes through the point $(1,1)$. Thus, using point-slope form, an equation of the line tangent to C at the point $(1,1)$ is

$$y - 1 = -\frac{1}{2}(x-1).$$

2. [5 pts] Suppose that $f(x) = \sqrt{x + \sqrt{x^2+4}}$. Find $f'(x)$. (No need to simplify.)

Call $u = x + \sqrt{x^2+4}$ and $v = x^2 + 4$. Applying the chain rule twice we obtain that

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \left(\frac{d\sqrt{u}}{du}\right)\left(\frac{d\left(x+\sqrt{x^2+4}\right)}{dx}\right) = \frac{1}{2\sqrt{u}}\left(1 + \frac{d\sqrt{v}}{dv}\frac{dv}{dx}\right) = \frac{1}{2\sqrt{x+\sqrt{x^2+4}}}\left(1+\frac{x}{\sqrt{x^2+4}}\right)$$

Alernatively,

$$\begin{aligned} f'(x) = \frac{d}{dx}\left[\left(x+\sqrt{x^2+4}\right)^{1/2}\right] &= \frac{1}{2}\left(x+\sqrt{x^2+4}\right)^{-1/2}\frac{d}{dx}\left[x+(x^2+4)^{1/2}\right] \\ &= \frac{1}{2}\left(x+\sqrt{x^2+4}\right)^{-1/2}\left(1+\frac{1}{2}(x^2+4)^{-1/2}(2x)\right) \end{aligned}$$

3. Some Chapter 5 Problems

(a) [4 pts] Find a and b, so that $f(x) = ae^{bx}$ satisfies $f(0) = 2$ and $f(3) = 6$.

Using the formula for $f(x)$ we have $f(0) = ae^0 = a$ and this must equal 2 so we obtain that

$$a = 2$$

Therefore $f(x) = 2e^{bx}$ and since $f(3) = 6$ we must solve $2e^{3b} = 6$ which is the same as $e^{3b} = 3$. Applying ln to both sides of the equation we have $3b = \ln 3$ so

$$b = \frac{\ln 3}{3}$$

(b) [4 pts] Solve the following equation for x: $2^{\log_3 x} = \frac{1}{4}$. (Simplify your answer.)

We write $2^{\log_3 x} = \frac{1}{4}$ as $2^{\log_3 x} = 2^{-2}$ which gives the equation $\log_3 x = -2$ and therefore

$$x = 3^{-2} = \frac{1}{9}$$

(c) [5 pts] How long will it take \$10000 to grow to \$18000 if the investment earns an interest rate of 10% per year compounded monthly? You need not simplify your answer (an expression you'd plug into a calculator).

Using the equation $A = P\left(1+\frac{r}{m}\right)^{mt}$ with $A = 18000$, $P = 10000$, $r = 0.1$ and $m = 12$ we need that we must solve the equation

$$18000 = 10000\left(1+\frac{0.1}{12}\right)^{12t}$$

which is equivalent to

$$\frac{9}{5} = \left(1+\frac{0.1}{12}\right)^{12t}$$

applying ln to both sides of the equation we obtain

$$\ln\left(\frac{9}{5}\right) = 12t\ln\left(1+\frac{0.1}{12}\right)$$

so that

$$t = \frac{\ln\left(\frac{9}{5}\right)}{12\ln\left(1+\frac{0.1}{12}\right)}$$

4. [10 pts] Based on customer trials and surveys, CV8 Theater decides that a child's size serving of popcorn should be 32 in^3. CV8 will serve the popcorn in rectangular boxes with square base and no top. Assuming that it is to hold 32 in^3, what dimensions for the box should the theater choose in order to minimize the box's surface area (thereby minimizing its cost). You must fully justify your claim that you've found the dimensions minimizing the surface area (using calculus, of course).

Minimize surface area A of box.

$A = 4xh + x^2$.

$V = x^2 h = 32$ so $h = \frac{32}{x^2}$

Thus, $A = \frac{128}{x} + x^2$, a function we wish to minimize over $(0, \infty)$.

We have, $A' = 2x - \frac{128}{x^2} = 0$.

$x = 4$ is the only critical number of A.

Either by the sign chart or by 2nd derivative test $A'' = 2 + \frac{256}{x^3} > 0$ (for all x,in $(0, \infty)$, we see that $x = 4$ (in) and $h = 2$ (in) yield the minimum surface area (which is $A(4) = 48\ in^2$)

5. (a) [3 pts] Why must $f(t) = t^3 - 12t^2 + 1$ have both an absolute maximum value and an absolute minimum value over the interval $[-1, 2]$?

   The function f is a polynomial and thus it is continuous on the closed interval $[-1, 2]$. Therefore by the Extreme Value Theorem f has both and absolute maximum and absolute minimum value on $[-1, 2]$.

   (b) [7 pts] Compute the absolute maximum value and absolute minimum value of $f(t) = t^3 - 12t^2 + 1$ over $[-1, 2]$. (Show all work in order to justify your answer.)

   5b Solution:
   $f'(t) = 3t^2 - 24t = 3t(t - 8)$ which means that the critical values of f are at $0, 8$.
   8 is not in the interval $[-1, 2]$ so we can ignore it. If we compute the function value at the critical points and endpoints we get

$$f(-1) = -12 \qquad f(0) = 1 \qquad f(2) = -39$$

   Comparing these values we see that 1 is the largest and -39 is the smallest so f has an absolute maximum value of 1 at $x = 0$ and an absolute minimum value of -39 at $x = 2$.

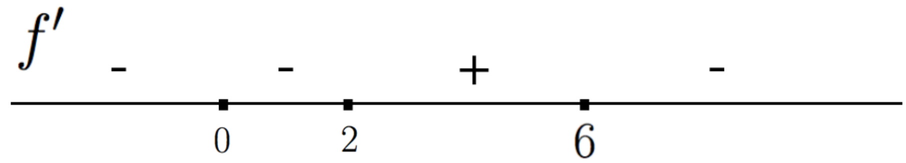
6. [4 pts] Simplify $\sqrt[3]{\left(\sqrt{2xy^2}\right)^{12}}$ completely, writing it using only positive exponents, a single coefficient, and using each variable at most once once in your expression. (Assume x and y are positive)

6 Solution:

$$\sqrt[3]{\left(\sqrt{2xy^2}\right)^{12}} = \left(\left(\left(2xy^2\right)^{1/2}\right)^{12}\right)^{1/3} = (2xy^2)^2 = 4x^2y^4$$

7. The function $f(x) = \frac{1}{x(x-6)^2}$ has derivative $f'(x) = \frac{3(2-x)}{(x-6)^3x^2}$.

   (a) [5 pts] Identify the intervals on which f is increasing and the intervals on which f is decreasing.



   f is increasing on $(2, 6)$
   f is decreasing on $(-\infty, 0), (0, 2)$, and $(6, \infty)$

   (b) [3 pts] Find relative maxima and relative minima of f (if any).

   $f(x)$ has relative minimum $\dfrac{1}{32}$ at $x = 2$
   $f(x)$ does not have relative maximum

8. [8 pts] Where is the graph of $f(x) = x^4 - 4x^3$ concave up? Where is it concave down? Also find points of inflection on the graph of f (if any).

   Solution (8 points):
   $f'(x) = 4x^3 - 12x^2$
   $f''(x) = 12x^2 - 24x = 12x(x - 2)$
   Sign of f':

   $+ \qquad - \qquad +$
   $- - - - | - - - - | - - - -$
   $0 \qquad 2$

   f is concave up on $(-\infty, 0)$ and $(2, \infty)$ and concave down on $(0, 2)$ with inflection points at $x = 0$ and $x = 2$.

9. [10 pts] A cylinder's height is increasing at the rate of 1 inch per minute while its radius is decreasing at the rate of 1 inch per minute. Find the rate of change in the volume of this cylinder at the instant when its radius is 10 inches and its height is 8 inches. Is the volume increasing or decreasing?

Solution

We are given that $r = 10$ in, $h = 8$ in, $\dfrac{dr}{dt} = -1$ in/min, and $\dfrac{dh}{dt} = 1$ in/min. We differentiate both sides of the volume equation with respect to time (using product rule!):

$$\begin{aligned} V &= \pi r^2 h \\ \frac{dV}{dt} &= \pi\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right) \\ \frac{dV}{dt} &= \pi(10^2)(1) + 2\pi(10)(8)(-1) \\ \frac{dV}{dt} &= \pi(100 - 160) \\ \frac{dV}{dt} &= -60\pi \text{ in}^3/\text{min} \end{aligned}$$

$\dfrac{dV}{dt}$ is negative, so the volume is decreasing!

10. [10 pts] Sketch the graph of a function f with the following properties.

(a) The domain of $f(x)$ is $(-\infty, 3) \cup (3, \infty)$ and f is continuous on its domain.

(b) y-intercept -1 and x-intercepts -2 and 1

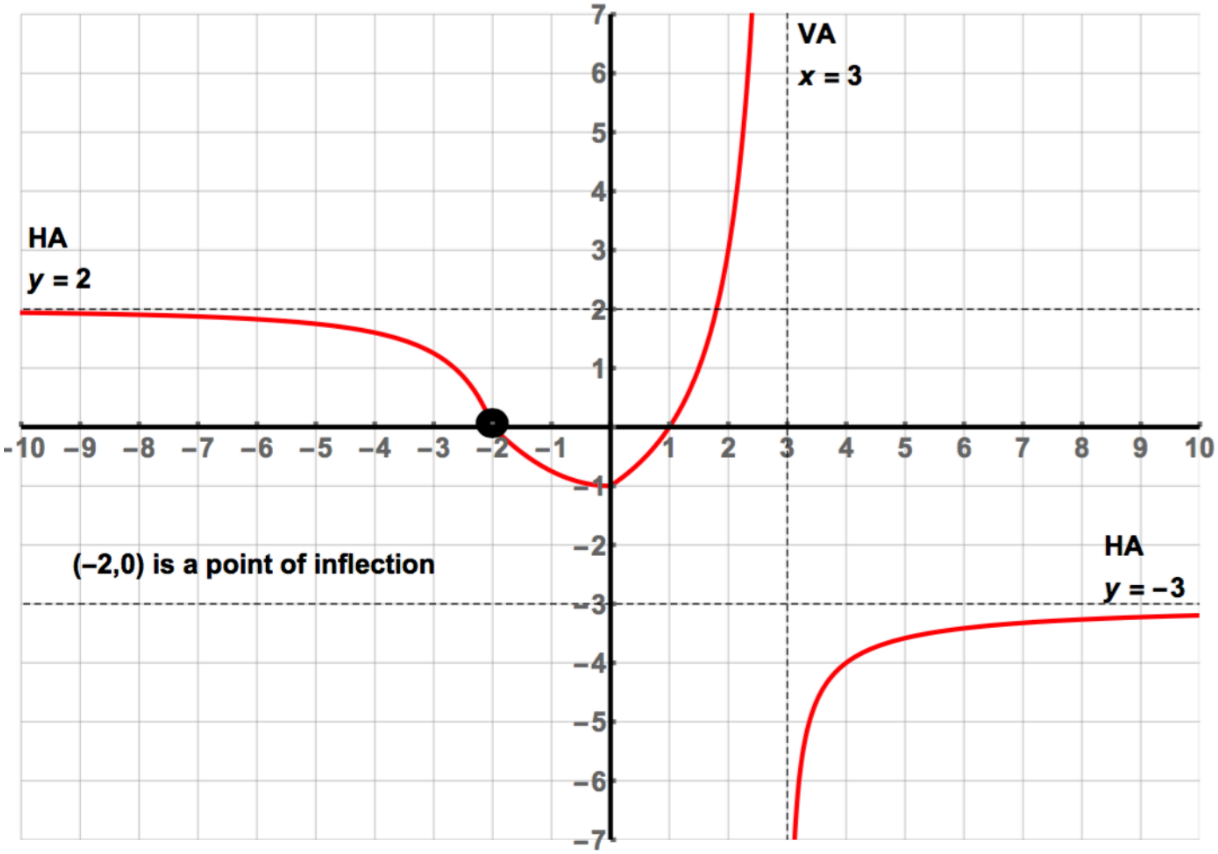
(c) $x = 3$ is a vertical asymptote

(d) $\lim_{x\to\infty} f(x) = -3$ and $\lim_{x\to-\infty} f(x) = 2$

(e) $f'(x) > 0$ for $0 < x < 3$ and for $3 < x$ and $f'(x) < 0$ for $x < 0$.

(f) $f''(x) > 0$ for $-2 < x < 3$ and $f''(x) < 0$ for $x < -2$ and for $x > 3$.

Be sure to include and label in your graph all asymptotes as well as points of inflection (if any).



11. [12 pts] Multiple-Choice: Circle the correct response.

(a) What is the maximum number of *horizontal asymptotes* that the graph of a function can have?

(a) one
(b) two
(c) three
(d) as many as we want—there is no maximium
(e) zero

Correct answer is (b)

(b) What is the maximum number of *vertical asymptotes* that the graph of a function can have?

(a) one
(b) two
(c) three
(d) as many as we want—there is no maximum
(e) zero

Correct answer is (d)

(c) Which of the following statements (I)–(III) are true:

(I) if $f(x)$ has a critical number at $x = 0$ then $f(x)$ has either a relative minimum or a relative maximum at $x = 0$.
(II) If $f(x)$ is continuous on (a, b) then $f(x)$ has an absolute maximum on (a, b).
(III) If $f''(a) = 0$ then $(a, f(a))$ is an inflection point of the graph of $f(x)$.

(a) III only
(b) I, II and III
(c) I and III
(d) II and III
(e) none

Correct answer is (e)

(d) For what range of values of a will $f(x) = (\log_{\frac{1}{2}} a)^x$ be a decreasing function?

(a) $0 < a < 1$
(b) $a > 1$
(c) $1 < a < 3/2$
(d) $0 < a < 1/2$
(e) $1/2 < a < 1$

Correct answer is (e). **Solution:** We know that $f(x) = b^x$ is decreasing provided that $0 < b < 1$. Thus we want

$$0 < \log_{1/2} a < 1.$$

Applying the exponential function with base 1/2 to both sides, keeping in mind that it's a decreasing function, we obtain

$$1 > a > 1/2, \text{which is (e).}$$

Alternatives: consideration of $a = (1/2)^{1/2}$ immediately leads to the correct choice (e). One can solve the problem pleasantly by a process of elimination. It's trivial that $a \neq 1/2$, which eliminates (a). The observation that $\log_{1/2} a$ is negative for $a > 1$ eliminates (b) and (c). Consideration of $a = 1/4$ eliminates (d).