<u>Some Problems</u>

1. Find the general solution to
$$y''' - 4y' = t$$
using annihilating operators.

Solution: If $L = (D^3 - 4D)$, then the above equation is $L(y) = t$.

Step 1: We first look at the complementary equation $y''' - 4y' = 0$. This has characteristic polynomial $r^3 - 4r = 0$ obtained by substituting $y(t) = e^{rt}$. It has roots $r = 0, 2, -2$. So the solution to this looks like $y_c(t) = c_1 + c_2e^{2t} + c_3e^{-2t}$.

Step 2: Note that $D^2(t) = 0$. So D^2 serves as an annihilating polynomial for t.

Step 3: Note that $D^2L(y) = D^2(t)$ (since $L(y) = t$), and this is equal to 0. Thus we now have $D^2(D^3 - 4D)(y) = 0$.

Substituting $y = e^{rt}$ again, we have $y^2(y^3 - 4y) = 0$. This shows that $y = 0, 0, 0, 2, -2$. (roots are shown with multiplicity)

Thus the general solution at this step is $c_1 + c_2e^{2t} + c_3e^{-2t} + c_4t + c_5t^2$. (Note that we write it in the form so that the solution found out at the first step remains intact i.e. I am not changing the constants there; this is simply for our convenience)

Step 4: Now we need to have $L(c_1 + c_2e^{2t} + c_3e^{-2t} + c_4t + c_5t^2) = t$. The LHS of this equation is
$L(c_1 + c_2e^{2t} + c_3e^{-2t}) + L(c_4t + c_5t^2)$. The first term is zero, by what we have shown in Step 1. So we only need to consider the second term.

$$(D^3 - 4D)(c_4t + c_5t^2) = t$$
$$\implies D^3(c_4t) + D^3(c_5t^2) - 4D(c_4t) - 4D(c_5t^2) = t$$
$$\implies 0 + 0 - 4c_4 - 8c_5 = t$$
$$\implies -4c_4 + (-8c_5 - 1)t = 0.$$

Now $\{1, t\}$ is linearly independent. So we must have $c_4 = 0, c_5 = -\frac{1}{8}$. Thus, $y_p(t) = -\frac{1}{8}t^2$ is a particular solution.

(Note that Steps 2,3,4 are meant to give the particular solution $y_p(t)$. This is the part other than the part we found out in Step 1. Always make note of this fact: from Step 1, we have already found a part of the solution called $y_c(t)$. In Step 3, we come across a solution $y_c(t) + y_p(t)$. Step 4 is about finding what this $y_p(t)$ is. So we have to simply look at the $y_p(t)$ part only in Step 4 (i.e. $L(y_p(t)) =?$ whatever is given on the RHS of the original equation) as $L(y_c(t) = 0$ as in Step 1.)

Conclusion: The general solution is $y(t) = c_1 + c_2e^{2t} + c_3e^{-2t} - \frac{1}{8}t^2$. ■

2. Find the general solution to $y''' - 4y' = 3\cos(t)$.

Solution:

Step 1: $y''' - 4y' = 0$. We have $(D^3 - 4D)(y) = 0$. General solution is $y_c(t) = c_1 + c_2e^{2t} + c_3e^{-2t}$ as above.

Step 2: $D^2 + 1$ annihilates $3\cos t$. (In general investigate or keep the table in the book memorized.) $D(3\cos(t) = -3\sin(t)$. So $D^2(3\cos(t)) = -3\cos t$. Thus, we have $(D^2 + 1))(3\cos(t)) = 0$.

Step 3: We have $(D^2 + 1)(D^3 - 4D)(y) = 0$. The characteristic polynomial we get is

$$(r^2 + 1)r(r^2 - 4) = 0$$

. The roots are $r = i, -i, 0, 2, -2$.

Thus the general solution has the form $c_1 + c_2e^{2t} + c_3e^{-2t} + c_4\cos(t) + c_5\sin(t)$.

The last two terms now constitute $y_p(t)$.

Step 4: We need

$$(D^3 - 4D)(y_p(t)) = 3\cos t.$$

i.e.

$$\begin{aligned}
(D^3 - 4D)(c_4\cos(t) + c_5\sin(t)) &= 3\cos(t)\\
\implies D^3(c_4\cos(t)) + D^3(c_5\sin(t)) - 4D(c_4\cos(t)) - 4D(c_5\sin(t)) &= 3\cos t\\
\implies c_4\sin(t) - c_5\cos(t) + 4c_4\sin(t) - 4c_5\cos(t) &= 3\cos(t)\\
\implies 5c_4\sin(t) + (-5c_5 - 3)\cos(t) &= 0.
\end{aligned}$$

Thus by linear independence of $\cos(t)$ and $\sin(t)$, we have $c_4 = 0,\ c_5 = -\frac{3}{5}$.

Hence, $y_p(t) = -\frac{3}{5}\sin(t)$.

Conclusion: The general solution is $y(t) = c_1 + c_2e^{2t} + c_3e^{-2t} - \frac{3}{5}\sin(t)$. ■

3. Solve $y''' - 4y' = t + \cos(t)$.

Solution:

We solve this by breaking the problem into $y''' - 4y' = t$ and $y''' - 4y' = \cos(t)$ and solving these two.

Note that in both of these the solution $y_c(t)$ remains the same for the homogeneous part.

We obtain $y_{p_1}(t), y_{p_2}(t)$ respectively for the two equations.

And the answer is given by $y(t) = y_c(t) + y_{p_1}(t) + y_{p_2}(t)$. By what we have solved in the last two problems, we have

$$y(t) = c_1 + c_2e^{2t} + c_3e^{-2t} - \frac{1}{8}t^2 - \frac{3}{5}\sin(t).$$

■

4. Solve $y'' + 3y' + 2y = \cos(t)$ using the method of annihilating operators with complex coefficients.

Solution: $\cos(t)$ is the real part of e^{it}. Thus we have to look at $L(y) = e^{it}$ where. $L = D^2 + 3D + 2$. After solving for the particular solution in this case, we have to look at its real part. Then adding this to $y_c(t)$ works. [**Note that while solving for $y_c(t)$ if you get complex roots, then convert them to the usual $\cos(t), \sin(t)$ parts**].

Step 1: We solve $(D^2 + 3D + 2)(y) = 0$ first. We substitute $y = e^{zt}$ where z can be a real/complex number. The characteristic polynomial is

$$z^2 + 3z + 2 = 0$$

which has roots $z = -1, -2$. Thus $y_c(t) = c_1e^{-t} + c_2e^{-2t}$.

Step 2: We have to find an annihilating operator for e^{it}. $D - i$ works.

Step 3: We look at $(D - i)(D^2 + 3D + 2)(y) = 0$. Again substituting e^{zt}, the characteristic polynomial is

$$(z - i)(z^2 + 3z + 2) = 0$$

and hence the roots are $z = i, -1, -2$. So the general solution has to look like $y(t) = c_1e^{-t} + c_2e^{-2t} + c_3e^{it}$. (**Note that here we do not separate e^{it} out into $\cos t, \sin t$ parts yet**) As usual, $y_p(t) = c_3e^{it}$.

Step 4: As we have seen, we only need to investigate

$$(D^2 + 3D + 2)[c_3e^{it}] = e^{it}.$$

THis upon simplification gives $-c_3e^{it} + 3ic_3e^{it} + 2c_3e^{it} = e^{it}$. Thus we get $c_3 = \frac{1}{1+3i} = \frac{1-3i}{10}$. (The later is obtained by multiplying the numerator and denominator of $\frac{1}{1+3i}$ by $1 - 3i$. Thus $y_p(t) = \frac{1-3i}{10}e^{it}$. †

We need to look at its real part which will give us $y_p(t)$ for our original equation.

Write e^{it} as $\cos(t) + i\sin(t)$. Multiply everything out to find that the real part is $c_1e^{-t} + c_2e^{-2t} + \frac{1}{10}\cos(t) + \frac{3}{10}\sin(t)$.

Conclusion: The general solution to $y'' + 3y' + 2y = \cos(t)$, is given by
$y(t) = c_1e^{-t} + c_2e^{-2t} + \frac{1}{10}\cos(t) + \frac{3}{10}\sin(t)$. ■

5. Solve $y'' + 3y' + 2y = \sin(t)$ by method of complex annihilators.

Solution: Again note that $\sin(t)$ is the imaginary part of e^{it}. So solve the equation $y''+3y'+2y = e^{it}$. Get the particular solution, then look at the imaginary part of this solution. It works.

We follow the above problem till Step 4 †.

After that we need to look at the imaginary part. Do the same thing: $e^{it} = \cos(t) + i\sin(t)$ and multiply everything out to get the imaginary part. This is our desired particular solution.

Add this to $y_c(t)$ and get the general solution. ■

• In general, if you are given $r\cos(t)$ or $r\sin(t)$ on the RHS, we have to look at re^{it} and follow the above methods.