Here we will use:

(i) Mathematical modelling to set it up; we shall find the domain too

(ii) Optimization techniques learnt previously (e.g. E.V.T., increase/decrease intervals, etc..)

Guidelines:

1. Assign a letter to each variable in the problem. Draw and label if possible.
2. Find an expression for the quantity to be optimized.
3. Use the conditions given to write the quantity found in Step 2 into a function of one variable. Note any restrictions to be placed on the domain of the function from physical considerations of the problem.
4. Optimize using known techniques.
5. Write answer in a sentence; don't forget units!

Problem 1: Find the dimensions of a rectangle with a perimeter of 100 ft that has the largest possible area.

Solution: (Draw a figure if needed!) Let x, y be respectively the length and breadth of the rectangle (in ft). Thus area $A = xy$. Also, we have $2x + 2y = 100 \implies x + y = 50 \implies y = 50 - x$. Thus, $A(x) = x(50 - x)$. Domain: we need $A(x) \geq 0$. Thus, $0 \leq x \leq 50$. (note we are allowing the case when $A(x) = 0$ by taking into account $x = 0, 50$. We are doing this solely for convenience in optimizing; different situations will require different such improvisations). So, we need to optimize $A(x)$ over $[0, 50]$. We apply closed interval method.

$A'(x) = 50 - 2x$. A' exists everywhere as A is a polynomial. So only critical points on $(0, 50)$ are obtained from: $A'(x) = 0 \implies 50 - 2x = 0 \implies x = 25$. Since $A(0) = A(50) = 0, A(25) = 25.25 = 625$, we conclude that area is maximized when $x = 25$ ft, $y = 25$ ft.

Problem 2: When a person coughs, the trachea(windpipe) contracts, allowing air to be expelled at a maximum velocity. It can be shown that during a cough, the velocity v of airflow is given by the function

$$v = f(r) = kr^2(R - r)$$

where r is the trachea's radius (in cm) during a cough, R is the trachea's normal radius (in cm), and k is a positive constant that depends on the length of the trachea. Find the radius r for which the velocity of airflow is greatest.

Solution: (Anybody who wants to draw a figure, good luck!) We need to maximize $f(r)$ on the interval $[0, R]$. (Maximum radius is R and since it contracts, the minimum is 0.) Since f' exists everywhere as f is a polynomial in r, only critical point on $(0, R)$ is obtained from:

$$f'(r) = 0 \implies 2kr(R - r) + kr^2(-1) = 0$$

$$\implies 2krR - 2kr^2 - kr^2 = 0 \implies 2krR - 3kr^2 = 0 \implies kr(-3r + 2R) = 0 \implies r = \frac{2}{3}R$$

($r \neq 0$ as we are choosing from $(0, R)$). Since, $f(0) = 0$, $f(R) = 0$, $f\Big(\frac{2}{3}R\Big) = k(\frac{4}{9}R)(\frac{1}{3}R) = \frac{4k}{27}R^3$, we conclude that velocity is greatest when the radius is $\frac{2}{3}R$ cm.

Now we look at an example where we cannot apply Closed Interval Test but 'rough curve sketch'/sign chart of f' to conclude about absolute extrema.

Problem 3: Determine the minimum cost of constructing a rectangular box under the following constraints:

a) The rectangle which makes up the top of the box has one side length twice as long as the other side length.

b)The volume of the box is 144 ft^3.

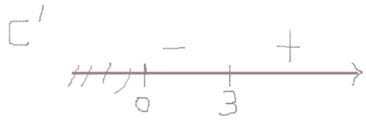
c) The top and base cost \$1 per ft^2.

d)The other sides cost \$0.50 per ft^2.

Solution: Let $x, 2x$ be the lengths of the base, y be the height (in ft). (Draw the diagram) We need to minimize cost C.

$144 = 2x^2y \implies y = \dfrac{72}{x^2}$. The total cost is $C = 2(2x)(x)1 + 2(2x)(y)(\frac{1}{2}) + 2(x)(y)(\frac{1}{2}) = 4x^2 + 3xy$. Thus, $C(x) = 4x^2 + 3x\dfrac{72}{x^2} = 4x^2 + \dfrac{216}{x}$. Need to minimize $C(x)$. Domain: $(0, \infty)$.

$C'(x) = 8x - \dfrac{216}{x^2} = \dfrac{8x^3 - 216}{x^2}$. So, $C'(x) = 0 \implies x = 3$. Thus $C'(x)$ is discontinuous at 0. So sign chart of $C'(x)$ looks like



which shows that there is a relative minima at x= 3, which in this case becomes the absolute minima as C is decreasing on $(0, 3)$ and increasing on $(3, \infty)$. Thus the minimum cost in dollars is $C(3) = 4.9 + \dfrac{216}{3} = 36 + 72 = 108$. $\boxed{108\ \$}$.

Problem 4: You are designing a tablet screen. For protection, the top and bottom of the screen must each be 1 inch away from the top and bottom of the tablet. The left and right sides of the screen must each be 2 inches away from the left and right sides of the tablet. The perimeter of the tablet must be 40 inches. Find the dimensions that will result in the maximum area for the screen.

Solution: Let x, y (in inches) be the length and height of the screen respectively. (Draw the diagram!)

• Length of Tablet: • Height of Tablet: • Relation between x, y:

• Area of screen in terms of x, y: • Area in terms of x: $A(x) =$

• Domain:

• Optimize:

Problem 5: An open bucket in the form of a right circular cylinder is to be constructed with a capacity of 1 ft^3. Find the radius and height of the cylinder if the amount of material used is minimal. [Hint: Surface area of the vertical part of the cylinder $2\pi(radius) \times height$]

Solution: Let r, l in ft be the radius and height respectively. Draw the diagram. We need to minimize

• Surface area $A =$ ____________________. • Relationship between r, l: $1 = \pi r^2 l \implies l = \dfrac{1}{\pi r^2}$. So, $A(r) =$ ____________________. • Domain is __________.

• Optimize: