# 6.2 Integration by Substitution (or $u$-substitution)

## 6.2.1 What is integration by substitution?

Recall chain-rule: $\frac{d}{dx}(f(g(x)) = f'(g(x))g'(x)$. Thus,

$$\int f'(g(x))g'(x)dx = f(g(x)) + C.$$

The spirit here is to use the above rule of integration and but perform a substitution first ($u = g(x)$) to make the integration easier.

**Example 1.** *Find* $\int 2x(x^2+1)^5 \ dx$.

**Solution:**

- Let $u = (x^2+1)$.
- $\frac{du}{dx} = 2x \implies du = 2xdx$.
- Substitute: $\int 2x(x^2+1)^5 dx = \int u^5 du$.
- Integrate: $\int u^5 du = \frac{u^6}{6} + C$.
- Revert to original variable: $\frac{u^6}{6} + C = \frac{(x^2+1)^6}{6} + C$.

Thus, $\int 2x(x^2+1)^5 dx = \frac{(x^2+1)^6}{6} + C$.

### 6.2.2 General Procedure For Integration by Substitution

1. Make a smart choice of $u = g(x)$.

   • Rules of thumb:

   ★ Identify the inner function of a composition.

   ★ Best if $g'(x)$ appears as being multiplied with the rest of the integrand. (see the previous example for instance)

   ★ Last resort: whatever that simplifies the integral.

2. $\dfrac{du}{dx} = g'(x) \implies du = g'(x)dx$ OR you can also do, $dx = \dfrac{du}{g'(x)}$. (I suggest you stick to the second one)

3. Substitute $u = g(x), dx = \dfrac{du}{g'(x)}$ into the original integration to convert it into an integration of something with $u$ as the variable.

4. Integrate.

5. Once integrated, revert back to the original variable by putting $u = g(x)$.

   ★ If (3) and (4) fail, go back to (1) and try a different $u$.

### 6.2.3 Basic Examples

**Example 2.** *Evaluate* $\displaystyle\int (4x^3 + 8x)\sqrt{x^4 + 4x^2}\ dx$

**Solution:** Let $u = x^4 + 4x^2$. So, $\dfrac{du}{dx} = (4x^3 + 8x) \implies dx = \dfrac{du}{4x^3 + 8x}$.

$$\int (4x^3+8x)\sqrt{x^4+4x^2}dx = \int (4x^3+8x)\sqrt{u}\frac{du}{4x^3+8x} = \int \sqrt{u}du = \frac{2u^{\frac{3}{2}}}{3} + C = \frac{2(x^4+4x^2)^{\frac{3}{2}}}{3} + C.$$

**Example 3.** *Evaluate* $\displaystyle\int (t^3 + 1)e^{t^4+4t}\ dt.$

**Solution:** Let $u = t^4 + 4t$. $\dfrac{du}{dt} = 4t^3 + 4 \implies dt = \dfrac{du}{4(t^3+1)}$. Thus,

$$\int (t^3+1)e^{t^4+4t}\ dt = \int (t^3+1)e^u\ \frac{du}{4(t^3+1)} = \frac{1}{4}\int e^u\ du = \frac{e^u}{4} + C = \frac{e^{t^4+4t}}{4} + C.$$

**Example 4.** *Evaluate* $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

**Solution:** Let $u = 1-4x^2$. $\dfrac{du}{dx} = -8x \implies dx = \dfrac{du}{-8x}$.

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = \int \frac{x}{\sqrt{u}}\frac{du}{(-8x)} = -\frac{1}{8}\int u^{-\frac{1}{2}}\,du = -\frac{1}{8}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = -\frac{1}{4}(1-4x^2)^{\frac{1}{2}} + C.$$

### 6.2.4 Advanced Examples

1. *Sometimes the choice of* $u$ *is less obvious.*

   **Example 5.** *Evaluate* $\displaystyle\int \frac{6x+3}{x^2+x}\,dx$.

   **Solution:** Let $u = x^2+x$. Thus, $dx = \dfrac{du}{2x+1}$.

   $$\int \frac{6x+3}{x^2+x}\,dx = \int \frac{3(2x+1)}{u}\frac{du}{2x+1} = \int 3\int \frac{du}{u} = 3\ln(|u|) + C = 3\ln(|x^2+x|) + C.$$

   The same line of thought also helps us choose $u$ correctly in calculating the following integral:

   $$\int \frac{\ln x}{x}\,dx,$$

2. *Sometimes we have to try harder at the stage of substitution to actually eliminate* <u>*ALL*</u> *the $x$ in the integrand.*

**Example 6.** *Evaluate* $\displaystyle\int \frac{2x^3}{(x^2+2)^4}\,dx$.

**Solution:** $u = x^2+2$. $dx = \dfrac{du}{2x}$.

$$\int \frac{2x^3}{(x^2+2)^4}\,dx = \int \frac{2x^3}{u^4}\,\frac{du}{2x} = \int \frac{x^2}{u^4}\,du = \int \frac{u-2}{u^4}\,du = \int \frac{1}{u^3}du - 2\int \frac{1}{u^4}du = \frac{u^{-2}}{-2} - 2\frac{u^{-3}}{-3} + C$$

$$= -\frac{1}{2(x^2+2)} + \frac{2}{3(x^2+2)^3} + C.$$

- The same line of thought also helps for calculating

$$\int x(x+1)^{20}\,dx, \qquad \int x^3(x^2+2)^{\frac{5}{2}}\,dx.$$

3. *Sometimes we have to apply two different substitutions.*

**Example 7.** *Find* $\displaystyle\int (e^{-3x} + \frac{e^x}{e^x+3})\,dx$.

**Solution:** $\displaystyle\int (e^{-3x} + \frac{e^x}{e^x+3})\,dx = \int e^{-3x}\,dx + \int \frac{e^x}{e^x+3}\,dx$. Let $u = -3x, v = e^x+3$. Then $du = -3\,dx,\ dv = e^x\,dx$.

$$\int e^{-3x}\,dx + \int \frac{e^x}{e^x+3}\,dx = \int e^u\,\frac{du}{-3} + \int \frac{e^x}{v}\frac{dv}{e^x} = -\frac{1}{3}\int e^u\,du + \int \frac{dv}{v}$$

$$= -\frac{1}{3}e^u + \ln(|v|) + C = -\frac{e^{-3x}}{3} + \ln(|e^x+3|) + C.$$