STUDENT NAME: ____________________

INSTRUCTOR: ____________________

Please sign the pledge:
On my honor as a student, I have neither given nor received aid on this exam.

Directions

Answer each question in the space provided. Please write clearly and legibly. ***Show all of your work—your work must justify your answer, and clearly identify your final answer. No books, notes, or electronic devices of any kind may be used during the exam period. You must simplify results of function evaluations when it is possible to do so. For example,*** $4^{3/2}$ ***should be evaluated*** (***replaced by*** 8)**.**

For instructor use only

Page	Points	Score
2	13	
3	12	
4	12	
5	10	
6	14	
7	15	
8	13	
9	11	
Total:	100	

1. Domain & Range: You need not show any work for parts (a) and (b) below.

   (a) [3 pts] Find the domain of the function $f(x) = \dfrac{x^2+2}{x^2-1}$. Write your answer using interval notation.

   $f(x) = \frac{x^2+2}{x^2-1} = \frac{x^2+2}{(x-1)(x+1)}$ is not defined at 1 and -1 because these numbers make the denominator of f equal to 0. Thus the domain of f is

   $$(-\infty, -1) \cup (-1, 1) \cup (1, \infty).$$

   (b) [2 pts] Let g be a function whose range is $[2, 7]$. Let h be the function defined by $h(x) = 5g(x)$. What is the range of $h(x)$? Write your answer in interval notation.

   The range of h is $[10, 35]$ is set of numbers resulting from multiplying each number in $[2, 7]$ by 5.

2. [4 pts] Let f and g be two linear functions defined by $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$.

   (a) Find a formula for $(g \circ f)(x)$, which should show that $g \circ f$ is also a linear function.

   $(g \circ f)(x) = g(f(x)) = m_2(f(x)) + b_2 = m_2(m_1x + b_1) + b_2 = m_1m_2x + (m_2b_1 + b_2)$.

   (b) What is the slope of the line $y = (g \circ f)(x)$? m_1m_2

   (c) What is the y-intercept of the line $y = (g \circ f)(x)$? $m_2b_1 + b_2$.

3. [4 pts] A stone is dropped into a calm lake, creating a circular ripple that travels outward in such a way that the radius of the circular ripple in feet, t seconds after the stone's impact, is $2t$.

   (a) Express the *area* A enclosed by the ripple as function of time t after impact (where A is measured in square feet and t is measured in seconds)

   $$A(t) = \underline{\pi(2t)^2 = 4\pi t^2}$$

   (b) Find the rate of change in area A with respect to time t; that is, find $\frac{dA}{dt}$.

   $$\frac{dA}{dt} = \frac{d}{dt}\left[4\pi t^2\right] = 4\pi\frac{d}{dt}\left[t^2\right] = 4\pi(2t) = 8\pi t.$$

4. [12 pts] Evaluate the following limits or write DNE if the limit does not exist. Your work must justify your answer; moreover, your work must be well organized.

(a) $\lim_{x\to 1} \dfrac{x^2-1}{x^2-4}$

$$\lim_{x\to 1} \frac{x^2-1}{x^2-4} = \frac{0}{-3} = 0$$

(b) $\lim_{x\to 2} \dfrac{\sqrt{x^2-3}-1}{x-2}$

$$\begin{aligned}
\lim_{x\to 2} \frac{\sqrt{x^2-3}-1}{x-2} &= \lim_{x\to 2} \left(\frac{\sqrt{x^2-3}-1}{x-2} \cdot \frac{\sqrt{x^2-3}+1}{\sqrt{x^2-3}+1} \right) \\
&= \lim_{x\to 2} \frac{x^2-3-1}{(x-2)(\sqrt{x^2-3}+1)} \\
&= \lim_{x\to 2} \frac{x^2-4}{(x-2)(\sqrt{x^2-3}+1)} \\
&= \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(\sqrt{x^2-3}+1)} \\
&= \lim_{x\to 2} \frac{(x+2)}{(\sqrt{x^2-3}+1)} \\
&= \frac{4}{\sqrt{4-3}+1} \\
&= 2.
\end{aligned}$$

(c) $\lim_{x\to 3^-} \dfrac{3x-x^2}{|x-3|}$

$$\begin{aligned}
\lim_{x\to 3^-} \frac{3x-x^2}{|x-3|} &= \lim_{x\to 3^-} \frac{3x-x^2}{-(x-3)} \\
&= \lim_{x\to 3^-} \frac{x(3-x)}{-(x-3)} \\
&= \lim_{x\to 3^-} \frac{x(3-x)}{3-x} \\
&= \lim_{x\to 3^-} x \\
&= 3.
\end{aligned}$$

5. [4 pts] The population of catfish in Lake Anna t years after 1995 is modeled by

$$p(t) = \frac{5t^2 + 6t}{t^2 - 8t + 4},$$

where $p(t)$ is measured in hundreds of catfish. In the long run (i.e. as $t \to \infty$) how many catfish will populate the lake? You must show your limit computation to receive full credit.

$$\begin{aligned} \lim_{t\to\infty} p(t) &= \lim_{t\to\infty} \frac{5t^2 + 6t}{t^2 - 8t + 4} \\ &= \lim_{t\to\infty} \frac{5 + 6/t}{1 - 8/t + 4/t^2} \\ &= 5. \end{aligned}$$

Thus, the long-run catfish population, according to the model, is 500 catfish.

6. [8 pts] (a) State precisely what it means for a function $f(x)$ to be continuous at $x = a$.

$f(a)$ is defined; $\lim_{x\to a} f(x)$ exists; $\lim_{x\to a} f(x) = f(a)$.

(b) Let

$$f(x) = \begin{cases} 2x^2 - 1 & \text{if } x \le -1 \\ 4x - 11 & \text{if } -1 < x < 3 \\ \frac{\sqrt{x+1}}{x-1} & \text{if } x \ge 3 \end{cases}$$

Is $f(x)$ continuous at $x = 3$? Justify your answer using the definition of continuity you stated in response to (a).

Note $f(3) = \frac{\sqrt{3+1}}{3-1} = \frac{\sqrt{4}}{2} = 1$ is defined. Moreover, $\lim_{x\to 3} f(x)$ exists and equals 1 because the left- and right-hand limits of f at 3 are both equal to 1:

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (4x - 11) = 1 \quad \text{and} \quad \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{\sqrt{x+1}}{x-1} = \frac{4}{2} = 1.$$

Finally

$$\lim_{x\to 3} f(x) = 1 = f(3).$$

Thus f is continuous at 3—all three conditions for continuity at 3 are satisfied.

7. [10 pts] (a) Complete the following "limit definition" of the derivative: the derivative of f at x, denoted $f'(x)$ is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \qquad \text{provided the limit exists.}$$

(b) Let $f(x) = \frac{1}{2x-3}$. *Use the definition of derivative you just stated* to find an expression for $f'(x)$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{2(x+h)-3} - \frac{1}{2x-3}}{h} \\
&= \lim_{h\to 0} \frac{\frac{2x-3}{(2(x+h)-3)(2x-3)} - \frac{2(x+h)-3}{(2(x+h)-3)(2x-3)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{2x-3-(2(x+h)-3)}{(2(x+h)-3)(2x-3)}}{h} \\
&= \lim_{h\to 0} \frac{-2h}{h(2(x+h)-3)(2x-3)} \\
&= \lim_{h\to 0} \frac{-2}{(2(x+h)-3)(2x-3)} \\
&= \frac{-2}{(2x-3)^2}
\end{aligned}$$

8. [8 pts] Dr. Snowworthy has constructed mathematical models for the amounts of snowfall in Duluth, MN and in Buffalo, NY. According to her model, the amount of snowfall in Duluth, in inches, t months after December 1st, $1 \le t \le 4$, is predicted to be

$$D(t) = 13 + 12t^2 - 2t^3 \quad \text{and in Buffalo is predicted to be} \quad B(t) = 10 + 2t + 9t^2 - t^3.$$

According to Dr. Snowworthy's model, is there a time t in the interval $[0, 4]$ such that the amount of snow fallen in Duluth will equal the amount fallen in Buffalo? Carefully justify your answer.

We are to determine whether there is a time t in $[0, 4]$ for which $D(t) = B(t)$ or, equivalently, whether there is a time t in $[0, 4]$ for which $D(t) - B(t) = 0$. Set $h(t) = D(t) - B(t) = 3 - 2t + 3t^2 - t^3$. Note $h(0) = 3$ and $h(4) = 3 - 8 + 48 - 64 = -21$. Because h is continuous on the closed interval [0, 4] (it's a polynomial) and 0 is between $h(0) = 3$ and $h(4) = -21$, by the IVT there is a number c in $[0, 4]$ such that $h(c) = 0$. By definition of h, we have $D(c) = B(c)$. Thus there is a time c in $[0, 4]$ for which Snowworthy's model says the amount of snowfall in Duluth will equal that in Buffalo.

9. [6 pts] Find an equation of the line tangent to the graph of $f(x) = x^3 - 2x^2 + 3$ at the point $(1, 2)$ on the graph.

$f'(x) = 3x^2 - 4x$. Slope of the tangent is $f'(1) = 3(1)^2 - 4(1) = -1$. Thus an equation of the tangent is

$$y - 2 = -1(x - 1).$$

10. [15 pts] Find the derivatives of the following functions. You don't need to simplify your answers.

(a) $g(x) = \dfrac{x^3 + 5x + 2}{4x^2 + 7x}$

$$\begin{aligned} g'(x) &= \frac{(4x^2 + 7x)\frac{d}{dx}[x^3 + 5x + 2] - (x^3 + 5x + 2)\frac{d}{dx}[4x^2 + 7x]}{(4x^2 + 7x)^2} \\ &= \frac{(4x^2 + 7x)(3x^2 + 5) - (x^3 + 5x + 2)(8x + 7)}{(4x^2 + 7x)^2} \end{aligned}$$

(b) $f(x) = \left(x^5 - 7x^3 + 4\right)^{3/4}$

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\left(x^5 - 7x^3 + 4\right)^{3/4}\right] \\ &= (3/4)\left(x^5 - 7x^3 + 4\right)^{-1/4} \frac{d}{dx}[x^5 - 7x^3 + 4] \\ &= (3/4)\left(x^5 - 7x^3 + 4\right)^{-1/4} \left(5x^4 - 21x^2\right) \end{aligned}$$

(c) $g(x) = \left(x^2 + 9\right)\sqrt{4 + \dfrac{1}{x^2}}$

$$\begin{aligned} g'(x) &= \left(x^2 + 9\right)\frac{d}{dx}\left[\left(4 + x^{-2}\right)^{1/2}\right] + \left(\sqrt{4 + \frac{1}{x^2}}\right)\frac{d}{dx}\left[\left(x^2 + 9\right)\right] \\ &= \left(x^2 + 9\right)(1/2)\left(4 + x^{-2}\right)^{-1/2}\left(-2x^{-3}\right) + \left(\sqrt{4 + \frac{1}{x^2}}\right)(2x) \end{aligned}$$

11. [6 pts] Find the equations of all horizontal tangent lines to the graph of $f(x) = x^3 - 12x$.

The solutions of $f'(x) = 0$; that is,

$$(*) \qquad 3x^2 - 12 = 0$$

will be x coordinates of points on the graph where tangents are horizontal (slope 0). Note $(*)$ can be written

$$3(x+2)(x-2) = 0,$$

which shows $(*)$ has solutions $x = 2$, and $x = -2$. The points where tangents are horizontal are $(2, f(2)) = (2, -16)$ and $(-2, f(-2)) = (-2, 16)$. The horizontal tangent lines are $y = -16$ and $y = 16$.

12. [7 pts] Not satisfied with his salary as a Calculus instructor, Dr. Diffenbach decides to invest in the stock market. After careful consideration, he decides to invest in "Soda Pop Corporation" (SPC). Instead of buying a lot of shares at once, Dr. Diffenbach buys different amounts of shares over a long period of time. Let $f(t)$ represent the number of SPC shares Dr. Diffenbach owns at time t months (after start of investing). Assume the price per share of the stock of SPC at time t is $g(t)$ dollars.

(a) What does the function $h(t) = f(t)g(t)$ represent? (*Hint.* Think of the units)

The dollar value of the shares of SPC that Dr. D holds t months after start of investing.

(b) Assume that both f and g are differentiable functions at $t = 4$ and that $f(4) = 3000$, $f'(4) = 400$, $g(4) = 20$, and $g'(4) = 2$. Find $h'(4)$.

$h'(t) = \frac{d}{dt}[f(t)g(t)] = f(t)g'(t) + g(t)f'(t)$.

$h'(4) = f(4)g'(4) + g(4)f'(4) = 3000(2) + 20(400) = 6000 + 8000 = 14,000$ (dollars/month)

(c) What does $h'(4)$ represent?

The rate of change with respect to time of the dollar value of Dr. D's shares of SPC $t = 4$ months after the start of investing; the value of Dr. D's holding in SPC is increasing at \$14000 per month after $t = 4$ months.

13. [4 pts] Let f and g be differentiable functions such that

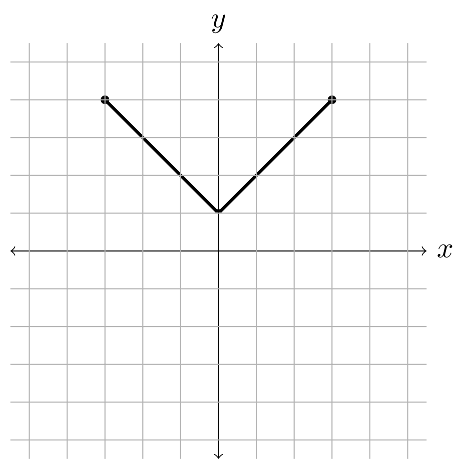
$$f(-1) = 3, \quad f'(-1) = 2, \quad f'(3) = 1, \quad g(-1) = 3, \quad g'(-1) = 3, \quad g'(3) = -11$$

If $h(x) = f\big(g(x) + 4x\big)$, what is $h'(-1)$?

$h'(x) = f'(g(x) + 4x)(g'(x) + 4)$. Thus

$$h'(-1) = f'(g(-1) + 4(-1))(g'(-1) + 4) = f'(3 - 4)(3 + 4) = f'(-1)(7) = 2(7) = 14.$$

14. [4 pts] On the coordinate grid below sketch a graph of a function f with domain $[-3, 3]$ such that (i) f is continuous on $[-3, 3]$, (ii) $f(-3) = 4$, (iii) $f(0) = 1$, and (iv) $f'(0)$ does not exist.



The preceding is one of many possible examples.

15. [3 pts] If $f(x) = x^{5/3}$, what is $f'(27)$? Fully simplify your answer.

$f'(x) = \frac{5}{3}x^{2/3}$; thus

$$f'(27) = \frac{5}{3}(27)^{2/3} = \frac{5}{3}(3^3)^{2/3} = \frac{5}{3}3^2 = 15.$$