Math 1210 - Limits Handout

Definition. *We say the function $f(x)$ has limit L, a unique real number, as x approaches a (a point "near" the domain of f), if the value of $f(x)$ is as close as we like (but not necessarily equal) to L for all x sufficiently close to, but not equal to, a.*

Notation: $\lim_{x \to a} f(x) = L$

It is possible that no such L exists, i.e. $f(x)$ does not approach one unique finite value as x gets close to a, then the limit does not exist, abbreviated DNE. However, the following two notions are sometimes useful:

Definition. *The function $f(x)$ has limit infinity as x approaches a, if the value of $f(x)$ is greater than any given number for all x sufficiently close to, but not equal to, a.*

Notation: $\lim_{x \to a} f(x) = \infty$

Similarly we can define limits equaling negative infinity:

Definition. *The function $f(x)$ has limit negative infinity as x approaches a if the value of $f(x)$ is less than than any given number for all x sufficiently close to, but not equal to, a.*

Notation: $\lim_{x \to a} f(x) = -\infty$

Note that it does not matter if $f(x)$ is defined at $x = a$, and even if $f(a)$ is defined, it need not equal the limit L. In fact,

Proposition. *If $f(x) = g(x)$ for all $x \neq a$, then*

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

In fact, it suffices for $f(x)$ to equal to $g(x)$ for all x "near" (but not equal to) a.

It also is useful to have the notions of one-sided limits:

Definition. *The function $f(x)$ has right-hand limit L, a unique real number, as x approaches a from the right, or from above, if the value of $f(x)$ is as close as we like (but not necessarily equal) to L for all x sufficiently close to, but strictly greater than, a.*

Notation: $\lim_{x \to a^+} f(x) = L$ *or* $\lim_{x \downarrow a} f(x) = L$

Likewise, we can define:

Definition. *The function f has left-hand limit L, a unique real number, as x approaches a from the left, or from below, if the value of $f(x)$ is as close as we like (but not necessarily equal) to L for all x sufficiently close to, but strictly greater than, a.*

Notation: $\lim_{x \to a^-} f(x) = L$ *or* $\lim_{x \uparrow a} f(x) = L$

We can define the one-sided limit being infinity or negative infinity in an analogous way as in the two-sided case.

One-sided limits are especially useful when dealing with piecewise functions like absolute value. The following proposition ties together the notions of one- and two-sided limits:

Proposition. *Suppose $f(x)$ is defined for all x sufficiently close to (but not necessary at) a. Then*

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} = L$$

If one (or both) of the one-sided limits does not exist, then the two-sided limit does not exist. If both of the one-sided limits exist, but they have different values, then the two-sided limit does not exist.

Some simple facts:

Proposition. *For any real number a and constant c*

1. $\lim\limits_{x\to a} c = c$.
2. $\lim\limits_{x\to a} x = a$.

These also hold for one-sided limits and limits at infinity.

Our main tool for evaluating limits is the following theorem:

Theorem (Laws of Limits)**.** *If $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$, i.e. both limits exist and are finite, then*

1. *For any constant c, $\lim\limits_{x\to a} c \cdot f(x) = c\left[\lim\limits_{x\to a} f(x)\right] = c \cdot L$.*
2. *For any power r, if L^r is defined, then $\lim\limits_{x\to a}[f(x)]^r = \left[\lim\limits_{x\to a} f(x)\right]^r = L^r$. If L^r is not defined, $\lim\limits_{x\to a}[f(x)]^r$ DNE or is $\pm\infty$.*
3. *$\lim\limits_{x\to a}[f(x) \pm g(x)] = \left[\lim\limits_{x\to a} f(x)\right] \pm \left[\lim\limits_{x\to a} g(x)\right] = L \pm M$.*
4. *$\lim\limits_{x\to a}[f(x) \cdot g(x)] = \left[\lim\limits_{x\to a} f(x)\right] \cdot \left[\lim\limits_{x\to a} g(x)\right] = L \cdot M$.*
5. (a) *If $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M \neq 0$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{L}{M}$*

   (b) *If $\lim\limits_{x\to a} f(x) \neq 0$ and $\lim\limits_{x\to a} g(x) = 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ DNE, or is $\pm\infty$.*

All the same rules work for one-sided limits and limits at infinity.

Note that this theorem says nothing about what happens in the case when $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$. In this case, $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ is said to be in $\frac{0}{0}$-indeterminate form. In these cases, the limit may or may not exist and we cannot say anything about the possible values of the limit.

The general approach to limits in $\frac{0}{0}$ indeterminate form is to try factoring the numerator and denominator, and then canceling like terms, until you can apply the above theorem. Sometimes other techniques like conjugation can be useful.

The next important use of limits is talking about the long term behavior of functions. Thus we introduce the notion of limits at infinity:

Definition. *The function $f(x)$ has limit L, a unique real number, as x increases without bound, or as x approaches infinity, if the value of $f(x)$ can be made as close to (but not necessarily equal to) L as we like for all x sufficiently large in absolute value and positive.*
Notation: $\lim\limits_{x\to\infty} f(x) = L$

Likewise, we can talk about limits at negative infinity:

Definition. *The function $f(x)$ has limit L, a unique real number, as x decreases without bound, or as x approaches negative infinity, if the value of $f(x)$ can be made as close to (but not necessarily equal to) L for all x sufficiently large in absolute value and negative.*
Notation: $\lim\limits_{x\to-\infty} f(x) = L$

We can also define limits at infinity equaling positive or negative infinity, analogously to before. For instance, $\lim\limits_{x\to\infty} f(x) = \infty$ means that $f(x)$ is eventually greater than any given number for all x sufficiently large in absolute value postive.

Limits at infinity satisfy all of the same Laws of Limits in Theorem 0.1.

Here is an important fact we will need about limits at infinity:

Proposition. *For all $r > 0$,*

$$\lim_{x\to\infty} \frac{1}{x^r} = 0 \qquad \textit{and} \qquad \lim_{x\to-\infty} \frac{1}{x^r} = 0$$

provided that $\frac{1}{x^r}$ is defined.

For instance, $\lim\limits_{x\to-\infty} \dfrac{1}{\sqrt{x}}$ DNE since $\sqrt{x}$ is undefined for negative x.

You will often be asked to evaluate limits at positive or negative infinity for rational functions, i.e. quotients of polynomial functions. The general approach is to divide through the numerator and denominator by the highest power of x (or the applicable variable) which appears in the denominator. Then you can use the previous proposition along with the properties of limits to evaluate the limit at infinity.

Dealing with limits at infinity often requires these intuitive ideas about functions growing without bound trumping any finite value:

Proposition.

1. *If $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = M$ a finite number, then $\lim\limits_{x\to\infty} [f(x) + g(x)] = \infty$.*
2. *If $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = M > 0$ a finite positive number, then $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \infty$. If $M < 0$ is negative, the limit will be negative infinity.*

There are of course analogous results for when x approaches negative infinity, or if $f(x)$ decreases without bound.

Using our notions of limits, we can classify our first class of "nice" functions - those with graphs with no jumps or holes. We start with defining continuity at a single point:

Definition. *A function $f(x)$ is continuous at $x = a$ if*

1. *$f(a)$ is defined, i.e. a is in the domain of f.*
2. *$\lim_{x \to a} f(x)$ exists (and is finite).*
3. *$\lim_{x \to a} f(x) = f(a)$*

If $f(x)$ is not continuous at $x = a$, we say $f(x)$ is *discontinuous* at $x = a$. We say a function f is continuous on an interval (a, b) if f is continuous at $x = c$ for all c in the interval (a, b). If f is continuous on the whole real line, we usually just say f is continuous. The graph of a continuous function can be draw without lifting one's writing utensil off of the paper.

Using the properties of limits, it is easy to show the following:

Theorem. *Properties of Continuous Functions:*

1. *The constant function $f(x) = c$ is continuous (on the whole real line).*
2. *The identity function $f(x) = x$ is continuous (on the whole real line).*

If $f(x)$ and $g(x)$ are functions which are continuous at $x = a$, then,

1. *For r a real number, $[f(x)]^r$ is continuous at $x = a$ when $[f(a)]^r$ is defined and discontinuous otherwise.*
2. *$f \pm g$ is continuous at $x = a$.*
3. *$f \cdot g$ is continuous at $x = a$.*
4. *$\dfrac{f}{g}$ is continuous at $x = a$ if $g(a) \neq 0$ and discontinuous otherwise.*

From the above theorem, it is easy to see that

Corollary.

1. *Polynomials are continuous (on the whole real line).*
2. *Rational functions (quotients of polynomials) are continuous at all x such that $g(x) \neq 0$.*

We like continuous functions since they make evaluating limits trivial: if $f(x)$ is continuous at $x = a$, all you need to do to calculate $\lim_{x \to a} f(x)$ is evaluate $f(a)$. Another potent property of continuous functions: The Intermediate Value Theorem.

Theorem (Intermediate Value Theorem)**.** *If $f(x)$ is a continuous function on a closed interval (i.e. including the endpoints) $[a, b]$, and M is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = M$.*

The Intermediate Value Theorem, or the IVT for short, is very useful in proving the existance of solutions to equations. A common application: if $f(x)$ is continuous on $[a, b]$, and $f(a) < 0 < f(b)$, then $f(x)$ has a zero in (a, b). That is, there is a number c with $a < c < b$ with $f(c) = 0$.