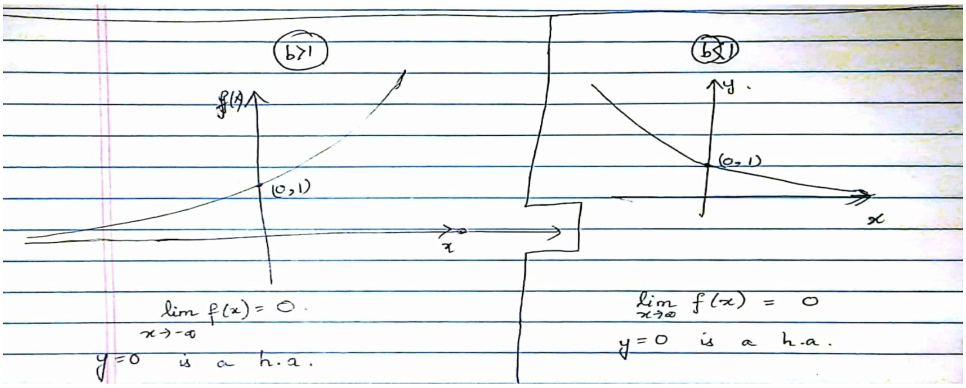
5.1 Exponential Functions

5.1.1 What are exponential functions?

The function defined by $f(x) = b^x$ $(b > 0, b \neq 1)$ is called an exponential function with base b and exponent x.

5.1.2 Graphs of exponential functions

- Graphs:



- Summary – properties of exponential functions:

  (a) Domain: $(-\infty, \infty)$

  (b) Range: $(0, \infty)$

  (c) Passes through $(0, 1)$.

  (d) Continuous on $(-\infty, \infty)$.

  (e) Increasing on $(-\infty, \infty)$ if $b > 1$; decreasing on (∞, ∞) if $b < 1$.

5.1.3 Laws of Exponents

Let $a, b > 0$. Let x, y be real numbers. Then

1. $b^x.b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$
4. $(ab)^x = a^x b^x$
5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Example 1. $(64^{\frac{4}{3}})^{-\frac{1}{2}} = 64^{-\frac{4.1}{3.2}} = 64^{-\frac{2}{3}} = \dfrac{1}{64^{\frac{2}{3}}} = \dfrac{1}{(64^{\frac{1}{3}})^2} = \dfrac{1}{4^2} = \dfrac{1}{16}$.

Example 2. $(16.81)^{-\frac{1}{4}} = 16^{-\frac{1}{4}}.81^{-\frac{1}{4}} = \frac{1}{16^{\frac{1}{4}}}.\frac{1}{81^{\frac{1}{4}}} = \frac{1}{2}.\frac{1}{3} = \frac{1}{6}$

Example 3. *Solve the equation* $8 \cdot 2^{2x} = \frac{1}{4^{4x+1}}$.

Solution: The trick is to bring everything to the same base and then compare exponents.

$$2^3.2^{2x} = 2^{-2(4x+1)} \implies 2^{3+2x} = 2^{-8x-2}$$

So we have

$$3 + 2x = -8x - 2 \implies 10x = -5 \implies x = -\frac{1}{2}.$$

5.1.4 The base e

The number e can be defined to be

$$e := \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^{\frac{1}{x}} \approx 2.71828...$$

It is an irrational number. $f(x) = e^x$ is a special exponential function.

Differentiation of $e^{f(x)}$:

We have

$$\frac{d}{dx}(e^x) = e^x.$$

Thus, if f is a differentiable function, then using chain rule we have

$$\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)}f'(x).$$

Example 4. *Compute* f' *where* $f(x) = x^3(e^{\sqrt{2x}} + 5)^{20}$.

Solution:

$$\begin{aligned} f'(x) &= 3x^2(e^{\sqrt{2x}} + 5)^{20} + x^3(20(e^{\sqrt{2x}} + 5)^{19})e^{\sqrt{2x}}\sqrt{2}\frac{1}{2\sqrt{x}} \\ &= x^2(e^{\sqrt{2x}} + 5)^{19}\left[3(e^{\sqrt{2x}} + 5) + 10\sqrt{2}\frac{e^{\sqrt{2x}}}{\sqrt{x}}\right]. \end{aligned}$$

Remark: Now we that we can differentiate exponential functions: relative max/min, critical points, absolute max/min, inflection points, concavity, finding tangent lines: you are ready to face all such questions!