Related Rates

We often have two or more quantities that change with time, and that are related through an implicitly equation. If we know some of the unknown quantities, and how fast some of the unknowns are changing, we can often use implicit differentiation to deduce the remaining pieces of information.

Strategy for Related Rate Problems.

- Picture: Draw a picture.
- Unknowns: Give names to the unknown quantities, and label them on the picture.
- Knowns: Give the known quantities names, and label them on the picture.
- Equation: Write an equation relating the unkown and unknown quantities.
- Differentiate: Use implicit or explicit differentiation, to find the zeros, extrema, etc.

Example. A particle is moving through the xy-plane along the curve determined by the equation $x^2 - 2xy - y^2 = 7$. When the particle is at the point $(2, -1)$, it has a vertical velocity of -1. What is its horizontal velocity at this point?

- The x and y coordinates depend on time t: $x = x(t)$, $y = y(t)$.
- We know $\frac{dy}{dt}|_{(2,-1)} = -1$. We wish to find $\frac{dx}{dt}$.
- $\frac{d}{dt}\left[[x(t)]^2 - 2x(t)y(t) - [y(t)]^2\right] = \frac{d}{dt}7$
- $2x(t) \cdot x'(t) - [2x'(t) \cdot y(t) + 2x(t) \cdot y'(t)] - 2y(t) \cdot y'(t) = 0.$
- $(2)(2)x'(t) - [2x'(t) \cdot (-1) + 2(2)(-1)] - 2(-1)(-1) = 0 \Rightarrow 6x'(t) = -2 \Rightarrow x'(t) = -\frac{1}{3}.$

Example. A rock is thrown into a still pond and causes a circular ripple. If the radius of the ripple is increasing at a rate of 2 feet per second, haw fast is the area enclosed by the ripple changing when the radius is 10 feet?

- (Draw picture.) $r =$ radius of ripple. $A =$ area enclosed by the ripple.
- $A = \pi r^2$, both r and A depend on the time t.
- $\frac{d}{dt}A = \frac{d}{dt}\left[\pi r^2\right] \Rightarrow \frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$.
- When $r = 10$, $\frac{dr}{dt} = 2$. At this instant, we have $\frac{dA}{dt} = 2\pi(10) \cdot (2) = 40\pi$ sqft./second.

Example. A 10-foot ladder is placed against a vertical wall. Suppose the bottom slides away from the wall at a constant rate of 3 feet per second. How fast is the top sliding down the wall (negative rate) when the bottom is 6 feet from the wall?

- Picture:
- $L = 10$ ladder length, $D =$ ladder bottom distance from wall, $H =$ height of ladder tip.
- $D^2 + H^2 = L^2 = 100 \Rightarrow \frac{d}{dt}\left[D^2 + H^2\right] = \frac{d}{dt}100 \Rightarrow 2D \cdot \frac{dD}{dt} + 2H \cdot \frac{dH}{dt} = 0$.
- When $D = 6$, $\frac{dD}{dt} = 3$ and $H = \sqrt{L^2 - D^2} = \sqrt{100 - 36} = 8$.
- Then $2(6)(3) + 2(8)\frac{dH}{dt} = 0 \Rightarrow \frac{dH}{dt} = -\frac{36}{16} = -\frac{9}{4}$ feet per second.

Example. A street lamp is on top of a 20 ft. pole. A person who is 5 feet tall walks away from the pole at a rate of 5 feet per second. At what rate is the tip of the person's shadow moving away from the lamppost when he is 20 feet from the pole?

- Draw picture:
- $y = 20$ height of pole. $x =$ distance of person from pole. $s =$ length of shadow. $z =$ distance of shadow tip from pole. $s = z - x$.