Let $f(x)$ be a function. We want to graph it! Here's how to do this in detail:

(1) Describe the **domain** of $f(x)$.

(2) Find the y-intercept of the graph, namely $f(0)$, provided $f(0)$ exists—thus the graph passes through $(0, f(0)$; find the x-intercept(s) if feasible (these will be solutions to $f(x) = 0$). Note some graphs have no x-intercepts (such as that of $f(x) = x^2 + 1$.)

(3) Determine the **end behavior** of $f(x)$. That is, compute the limits

$$\lim_{x\to+\infty} f(x) \qquad \text{and} \qquad \lim_{x\to-\infty} f(x),$$

or state that they do not exist. If they don't exist, you may want to note whether the y-values approach $+\infty$ or $-\infty$.

(4) Find the **horizontal** and **vertical asymptotes** of $f(x)$. A function $f(x)$ has a horizontal asymptote $y = L$ if $\lim\limits_{x\to+\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$. A function $f(x)$ has a vertical asymptote $x = a$ if $\lim\limits_{x\to a^+} f(x) = \pm\infty$ or $\lim\limits_{x\to a^-} f(x) = \pm\infty$.

(5) Determine the intervals on which $f(x)$ is **increasing** and **decreasing** i.e. draw the sign line for f'.

(6) Find and classify the **relative extrema** of $f(x)$.

(7) Determine the intervals on which $f(x)$ is **concave up** and **concave down** i.e. draw the sign line for f''.

(8) Find the **inflection points** of $f(x)$.

(9) I find it useful now to create a new number line in which I record all the points found in the previous two sign charts and, on each interval between these, sketch a little , , or to record the increasing/decreasing behavior AND concavity simultaneously. Let us call this the *sketch summary* line.

(10) Sketch the graph. It always is the best to start by drawing the asymptotes using dotted lines, then the intercepts, max/min.

Some notes:

- According to Tan, a rational function $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, has a vertical asymptote at $x = a$ if $p(a) \neq 0$ and $q(a) = 0$. Thus to find the VAs of the graph of $\dfrac{p(x)}{q(x)}$, where p, q are polynomials, you can first cancel any common factors and then any numbers that still make the denominator 0 will correspond to vertical asymptotes. This should agree with your findings when solving the limits $\lim\limits_{x\to a^+} \dfrac{p(x)}{q(x)}$ and $\lim\limits_{x\to a^-} \dfrac{p(x)}{q(x)}$ using factor-cancel.

Okay, let's do it!

Problem 1. Sketch $f(x) = \dfrac{2x^2+5}{4-x^2}$.

(1) The domain of $f(x)$ is:
$(-\infty,-2)\cup(-2,2)\cup(2,\infty)$.

(2) x-intercepts: $2x^2+5\neq 0$; so, no $x-$intercept.
y-intercept: $f(0)=\dfrac{5}{4}$. So, $(0,\dfrac{5}{4})$ is the y-intercept.

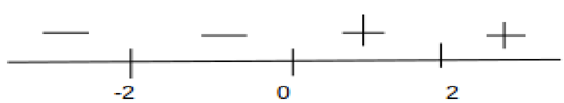
(3) Compute:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to\infty}\frac{2+\frac{5}{x^2}}{\frac{4}{x^2}-1} = -2 \qquad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{2+\frac{5}{x^2}}{\frac{4}{x^2}-1} = -2.$$

Hence the horizontal asymptote(s) is/are $y=-2$

(4) Does $f(x)$ have any vertical asymptotes? If so, list them with them justification. (Remember to cancel out any common factors when thinking about/finding vertical asymptotes):
$4-x^2=0 \implies x=\pm 2$. Numerator does not vanish at $x=\pm 2$. Note that $\lim_{x\to-2^+} f(x) = \infty$, $\lim_{x\to-2^-} f(x) = -\infty$. So vertical asymptotes are $x=2$ and $x=-2$.

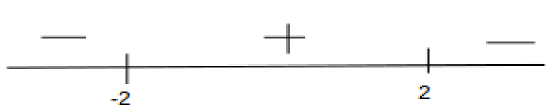
(5) $f'(x) = \dfrac{26x}{(4-x^2)^2}$.



$f(x)$ is increasing on: $(0,2),(2,\infty)$. $f(x)$ is decreasing on: $(-\infty,-2),(-2,0)$.

(6) Critical points: $x=0$ relative maxima: None
relative minima: @ $x=0$. Local minimum value: $f(0)=\dfrac{5}{4}$

(7) $f''(x) = \dfrac{26(3x^2+4)}{(4-x^2)^3}$.

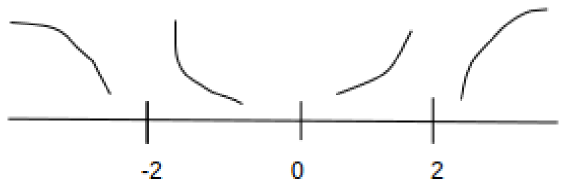


$f(x)$ is concave up on: $(-2,2)$

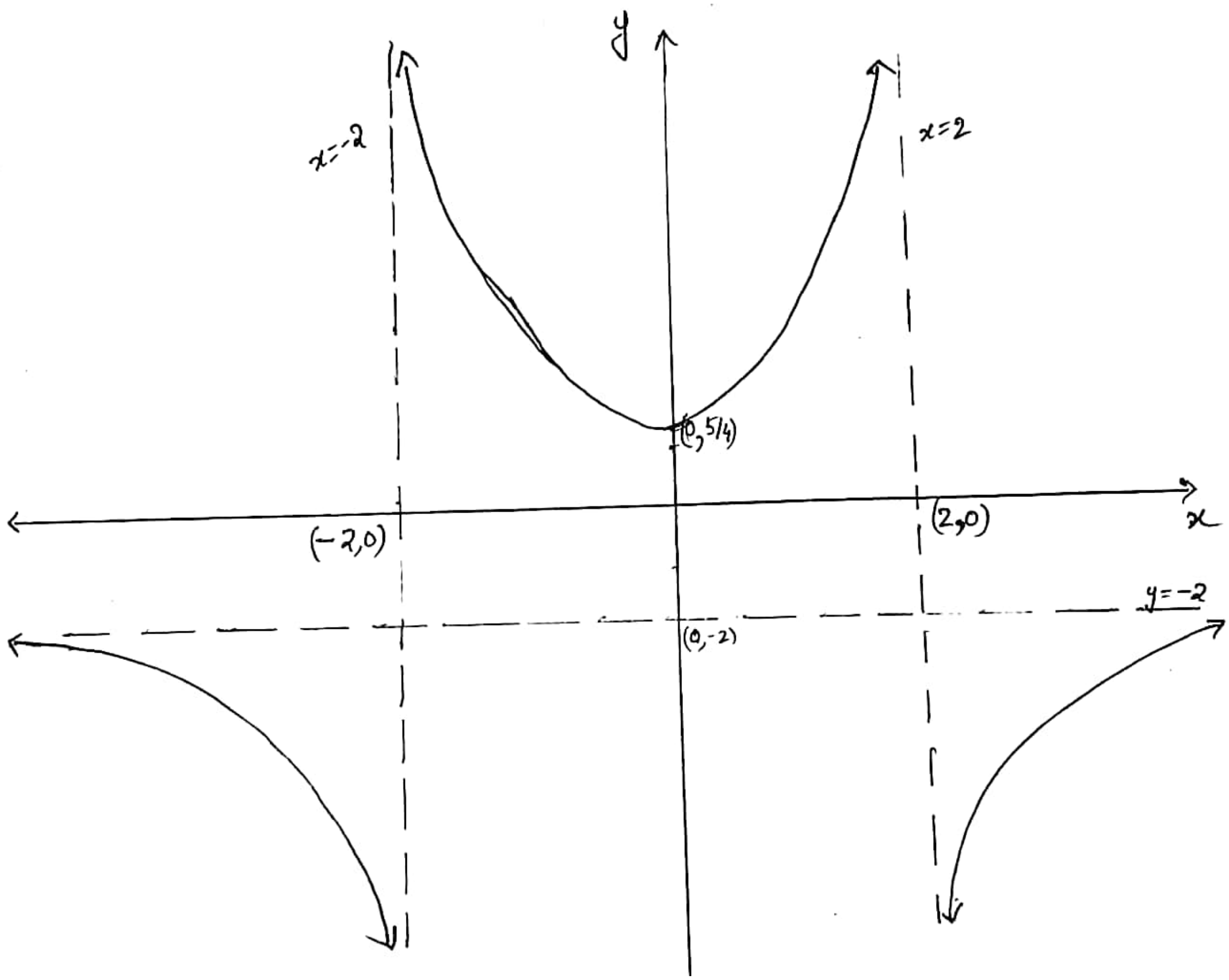
$f(x)$ is concave down on: $(-\infty,-2),(2,\infty)$

(8) Inflection points are: None

(9) Sketch Summary Line:



(10) Sketch.



Problem 2. Sketch the graph of a function, which is continuous on its domain, with the given properties. **Always Label any asymptotes (dotted lines on the graph) and any relevant x or y-coordinates**. If no such function exists, explain why.

(a) Domain: $(-\infty, 1) \cup (1, \infty)$.

(b) $f(-2) = -3, f(0) = 4$, and $f(3) = 0$.

(c) $\lim_{x\to 1^-} f(x) = \infty, \lim_{x\to 1^+} f(x) = -\infty$.

(d) $\lim_{x\to -\infty} f(x) = 0, \lim_{x\to \infty} f(x) = 2$.

(e) f' is negative on $(-\infty, -2)$ and f' is positive on $(-2, 1)$ and $(1, \infty)$.

(f) f is concave down on $(-\infty, -4)$ and $(1, \infty)$, and f is concave up on $(-4, 1)$.

Solution: