Discussions on I-Ulrich Modules

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Based on joint work with Hailong Dao and Prashanth Sridhar
(both at University of Kansas)

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- Talk about the connection to the category of Reflexive modules.

All of the results in this talk are available at On Reflexive and I-Ulrich modules over curve singularities (mainly, Sec 4).

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How abundant are these modules?

Motivation Continued

We found that the equality $xM = IM$ for certain modules $M \in \mathrm{CM}(R)$ appears in many situations related to our investigation where I is a height one ideal with principal reduction x, in a one-dimensional local Cohen-Macaulay ring.

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So, we make a more general study of such modules with respect to any regular(i.e. height 1) ideal I and we shall see that the category of such modules has nice properties.

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Let $(R, \mathfrak{m}, k)$ be a one-dimensional Cohen-Macaulay local ring with infinite residue field k and total ring of fractions K. Let $\bar{R}$ denote the integral closure of R. All modules M considered are finitely generated.

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Thus all regular ideals have a principal reduction.

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I-Ulrich Modules

Let I be a height 1 ideal. We say that $M \in \mathrm{CM}(R)$ is *I-Ulrich* if $e_I(M) = \ell(M/IM)$, where $e_I(M)$: Hilbert Samuel multiplicity of M with respect to I and $\ell(\cdot)$ denotes length.

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• Note that if $M \cong N$ in $\mathrm{CM}(R)$, then the same isomorphism takes IM to IN, so $\ell(M/IM) = \ell(N/IN)$ for any ideal I and so I-Ulrich condition is preserved under isomorphism i.e. $\mathrm{Ul}_I(R)$ makes sense!

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In fact, high enough powers of every ideal is Ulrich with respect to itself.

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- $B(I) \subset \bar{R}$.
- $B(I) = B(I^n)$ for all $n \geq 1$.
- If x is a principal reduction of I, then it is well-known that

$$B(I) = R\left[\frac{I}{x}\right].$$

Theorem (Main Properties)

Let R be a one-dimensional Cohen-Macaulay local ring and I is regular. Suppose that $x \in I$ is a principal reduction and $M \in \mathrm{CM}(R)$. TFAE:

1. M *is* I*-Ulrich.*
2. $IM = xM$.
3. $IM \subseteq xM$.
4. $IM \cong M$.
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3. $IM \subseteq xM$.
4. $IM \cong M$.
5. $M \in \mathrm{CM}(B(I))$.
6. M is I^n-Ulrich for all $n \geq 1$.
7. M is I^n-Ulrich for infinitely many n.
8. M is I^n-Ulrich for some $n \geq 1$.

Proof Cntd: (3) is equivalent to $\frac{I}{x}M \subseteq M$. In other words (3) $\implies M \in \mathrm{CM}(B(I))$, since $R[I/x] = B(I)$. This also shows that (5) $\implies$ (3). Since $B(I) = B(I^n)$ for any $n \geq 1$, we have (5) $\implies$ (6). Clearly (6) $\implies$ (7) $\implies$ (8). Assume (8). We have $\ell(M/I^nM) = e_{I^n}(M) = ne_I(M)$. Note that for each i, I^iM is in $\mathrm{CM}(R)$ and hence we get for each i,

$$\ell(I^iM/I^{i+1}M) \leq \ell(I^iM/xI^iM) = e_I(I^iM) = e_I(M).$$

Thus, equality must occur for each i; in particular, it occurs for $i = 0$, which shows that M is I-Ulrich. $\square$

- Note that I is I-Ulrich simply says that $I^2 = xI$ i.e. I is stable, a concept heavily used in Lipman's work on Arf rings.

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Without any assumption on the existence of a principal reduction (relaxing to finite residue field), the following still holds:

Theorem

Let R be a one-dimensional Cohen-Macaulay local ring. Let I be a regular ideal and $M \in \mathrm{CM}(R)$. The following are equivalent:

1. $IM \cong M$.
2. *M is I-Ulrich.*
3. *M is I^n-Ulrich for all $n \geq 1$.*
4. *M is I^n-Ulrich for infinitely many n.*
5. *M is I^n-Ulrich for some $n \geq 1$.*
6. $M \in \mathrm{CM}(B(I))$.

Corollary

Let I be a regular ideal. Then R is I-Ulrich if and only if I is principal.

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Let R have a canonical ideal ω_R. Then R is ω_R-Ulrich if and only if R is Gorenstein.

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Proof.

Minimal multiplicity iff $\mathfrak{m}^2 = x\mathfrak{m}$. □

The case of Birational Extensions

Remark

Note that if $M \in \mathrm{CM}(R)$ is I-Ulrich, the proof of the main theorem showed that the action of $B(I)$ on M extends the action of R on M. (Recall, $\frac{I}{x}M \subset M$.)

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Birational Extension

We say that an extension $f : R \to S$ is *birational* if $S \subset K$. Equivalently $K = \mathrm{Frac}(S)$.

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In particular, if $M \subseteq K$, multiplication in K gives an action of $B(I)$ on M.

Birational Extension

We say that an extension $f : R \to S$ is *birational* if $S \subset K$. Equivalently $K = \mathrm{Frac}(S)$. Also such an f induces a bijection on the sets of minimal primes of S and R and f_P is an isomorphism at all minimal primes P of R.

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Let $R \subseteq S$ be a finite birational extension of rings. Then S is I-Ulrich if and only if $B(I) \subseteq S$.

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We saw that $\mathrm{Ul}_I(R) = \mathrm{CM}(B(I))$. So there is the multiplication action of $B(I)$ on S, induced from the multiplication in K, i.e. $B(I)S \subseteq S$.

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Corollary

Let I be a regular ideal. If $\bar{R}$ is a finitely generated R-module, then $\bar{R}$ and the conductor ideal $\mathfrak{c} := R :_K \bar{R}$ (largest common ideal of R and $\bar{R}$) are I-Ulrich.

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In fact, $\mathfrak{c} :_K \mathfrak{c} = \bar{R}$ and hence $B(\mathfrak{c}) = \bar{R}$: another way of proving the above.

More Properties

Proposition

Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathrm{CM}(R)$. If B is I-Ulrich then so are A, C.

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B is I-Ulrich if and only if I kills the middle module, but if that's the case then I kills the other two as well. □

Corollary

Let $M \in \mathrm{Ul}_I(R)$. *For any* $f \in M^*, \mathrm{Im}(f) \in \mathrm{Ul}_I(R)$.

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$0 \to J \cap L \to J \oplus L \to J + L \to 0$. □

Corollary (Ulrich Lattice)

The set of I-Ulrich ideals is a lattice under addition and intersection.

Largest Element in Ulrich Lattice

In fact, one can show that the largest element in this lattice is $b(I) := R :_K B(I)$.

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- If $M \in \mathrm{Ul}_I(R)$, then *the trace ideal of* M also is I-Ulrich,

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- If $M \in \mathrm{Ul}_I(R)$, then *the trace ideal of* M also is I-Ulrich,
- Any trace ideal obtained as above is in $b(I)$.

Recall that the trace ideal of M is $\mathrm{tr}(M) := \sum_{f \in M^*} f(M)$. By previous slide, clearly, $\mathrm{tr}(M) \in \mathrm{Ul}_I(R)$.

Proposition

If $M \in \mathrm{Ul}_I(R)$, then $\mathrm{Hom}_R(M, N) \in \mathrm{Ul}_I(R)$ for any module $N \in \mathrm{CM}(R)$. In particular, $M^*, M^{**} \in \mathrm{Ul}_I(R)$.

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PROOF: Note that there is an embedding

$$\mathrm{Hom}_R(M,N) \otimes_R R/xR \to \mathrm{Hom}_R(M/xM, N/xN)$$

and the latter is killed by I since $M \in \mathrm{Ul}_I(R)$. This shows that $\mathrm{Hom}_R(M,N) \otimes_R R/xR$ is killed by I and this finishes the proof. $\square$

A Finiteness Result

Theorem

Let $\mathfrak{c} := R :_K \bar{R}$ be the conductor and I be a regular ideal. If $\mathfrak{c} \cong I^s$ for some s, then $\mathrm{Ul}_I(R) = \mathrm{CM}(\overline{R})$. If furthermore R is complete and reduced, then $\mathrm{Ul}_I(R)$ has finite type.

Proposition

Assume that I is a regular ideal. Let $S = \mathrm{End}_R(I)$ (which is a *birational extension* of R). If M is I-Ulrich, then

$$\mathrm{Hom}_R(M, I) \cong \mathrm{Hom}_R(M, S).$$

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We have an exact sequence

$$0 \to L \to I \otimes M \to IM \to 0$$

where L has finite length.

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The first is isomorphic to $\mathrm{Hom}_R(M, I)$ as $IM \cong M$, and by Hom-tensor adjointness, the second is isomorphic to

$$\mathrm{Hom}_R(M, \mathrm{Hom}_R(I, I)) = \mathrm{Hom}_R(M, S).$$

□

Corollary

Assume that R has a canonical ideal ω_R. The following are equivalent:

1. $M \in \mathrm{Ul}_{\omega_R}(R)$
2. $\mathrm{Hom}_R(M, \omega_R) \cong \mathrm{Hom}_R(M, R)$.

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PROOF: (1) $\Longrightarrow$ (2) by previous slide and the fact that $\mathrm{End}_R(\omega_R) = R$.

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Conversely, using Hom-Tensor adjointness, statement (2) is the same as

$$\mathrm{Hom}_R(M,\omega_R)\cong\mathrm{Hom}_R(M,\mathrm{Hom}_R(\omega_R,\omega_R))\cong\mathrm{Hom}_R(\omega_R\otimes_R M,\omega_R)\cong\mathrm{Hom}_R(\omega_R M,\omega_R).$$

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Hence taking $\mathrm{Hom}_R(-, \omega_R)$ and using duality, we get

$$M \cong \omega_R M$$

and this finishes the proof.

Connection with $\mathrm{Ref}(R)$

Theorem

Assume that R has a canonical ideal ω_R and $M \in \mathrm{Ul}_{\omega_R}(R)$. Then $M \in \mathrm{Ref}(R)$.

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Previous slide shows that $M^* \cong M^\vee$, where $M^* = \mathrm{Hom}_R(M, R)$ and $M^\vee = \mathrm{Hom}_R(M, \omega_R)$.

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$$M^{**} \cong M^{*\vee} \cong M^{\vee\vee} \cong M$$

as desired. $\square$

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Recall from our critical example that ω_R^n is ω_R-Ulrich for $n >> 0$. Now apply previous slide. $\square$

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Additional Information: In the article, we further do the following:

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Additional Information: In the article, we further do the following:

- use ω_R-Ulrich modules to classify reflexive birational extensions of R;

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- use ω_R-Ulrich modules to classify reflexive birational extensions of R;
- use I-Ulrich modules to classify reflexive Gorenstein birational extensions
- relate the trace ideal of an I-Ulrich module to the *core of* I.

Theorem

Suppose that R is a one-dimensional Cohen-Macaulay local ring with a canonical ideal ω_R. Let S be a module finite R-algebra such that S is a maximal Cohen-Macaulay module over R. The following are equivalent:

1. $\mathrm{CM}(S) \subset \mathrm{Ref}(R)$.
2. $\omega_S \in \mathrm{Ref}(R)$.
3. *S is ω_R-Ulrich as an R-module.*

Theorem

Suppose that R is a one-dimensional Cohen-Macaulay local ring with a canonical ideal ω_R. Let S be a finite birational extension of R such that $S \in \mathrm{Ref}(R)$. Let $I = R :_K S$ be the conductor of S in R. The following are equivalent:

1. *S is Gorenstein.*
2. *I is I-Ulrich and ω_R-Ulrich. That is $I \cong I^2 \cong I\omega_R$.*

Theorem

Suppose that R is a one-dimensional Cohen-Macaulay local ring with a canonical ideal ω_R. Let S be a finite birational extension of R such that $S \in \mathrm{Ref}(R)$. Let $I = R :_K S$ be the conductor of S in R. The following are equivalent:

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This extends Goto's theorem.

Corollary

Suppose that $(R, \mathfrak{m})$ is a one-dimensional Cohen-Macaulay local ring with a canonical ideal ω_R. Let $S = \mathrm{End}_R(\mathfrak{m})$. The following are equivalent:

1. *S is Gorenstein.*
2. *R has minimal multiplicity and is 'almost Gorenstein'.*

Theorem

Assume that the residue field of R is infinite. Let $M \in \operatorname{Ref}(R)$. The following are equivalent.

1. *M is I-Ulrich.*
2. $\operatorname{tr}(M) \subseteq b(I)$.
3. $\operatorname{tr}(M) \subseteq (x) :_R I$ *for some principal reduction x of I.*
4. $\operatorname{tr}(M) \subseteq (x) :_R I$ *for any principal reduction x of I.*
5. $\operatorname{tr}(M) \subseteq \operatorname{core}(I) :_R I$.


Thank
You