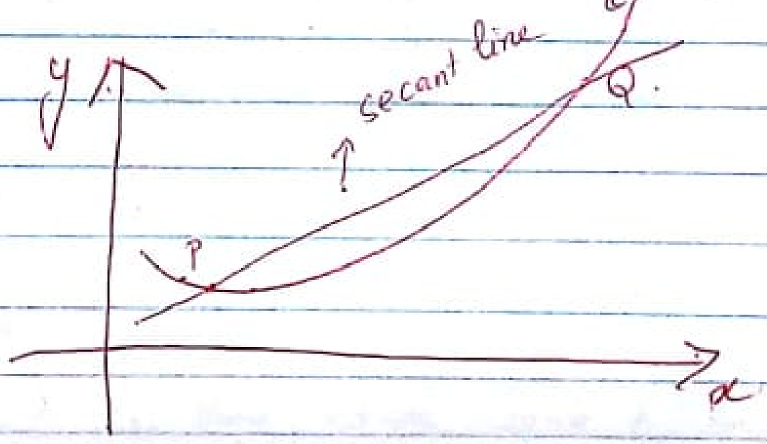
2.6 Derivative

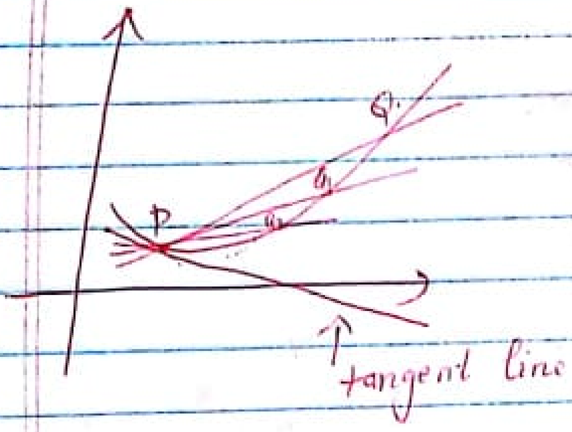
In an earlier note, I had motivated the use of limits to figure out how to get to a tangent line from secant lines. However, there was an error in the explanation which I will correct here.

To define the tangent line to a curve C at a point P on the curve, fix P and let Q be any point on C distinct from P.

The straight line passing through P and Q is called a secant line.

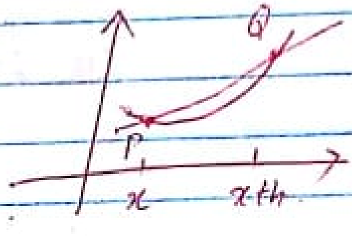


To get to the tangent line at P, we keep P fixed and start varying Q so that it moves closer towards P. The 'limiting case' of this varying Q keeping P fixed gives the tangent line at P.

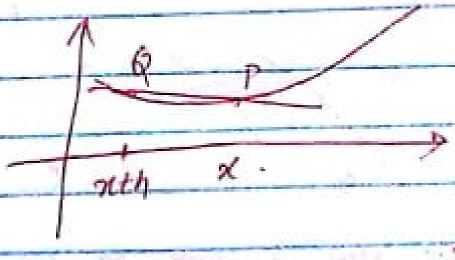


Suppose the curve C is the graph of a function f defined by $y = f(x)$.

Then P is described in terms of co-ordinates as $(x, f(x))$ whereas Q is $(x+h, f(x+h))$

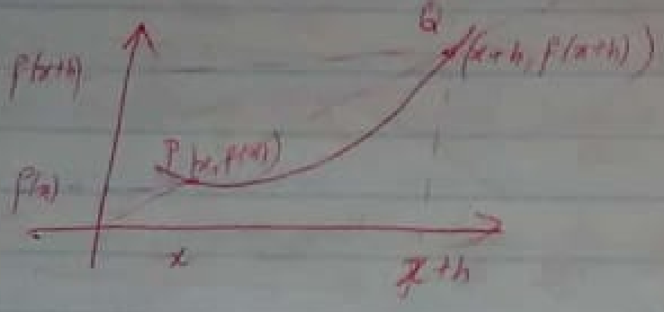


Note that h need not be positive. ← In this figure it is as P is to the left of Q.



← This figure is also possible. Evidently, if I write $(x+h, f(x+h))$ for Q, h must be negative.

Anyway, let us have this diagram



Slope of the secant line is $\dfrac{f(x+h) - f(x)}{x+h-x}$

$$= \frac{f(x+h) - f(x)}{h}$$

"Letting Q move closer towards P" translates to finding

$\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$: geometrically, it is now clear that this gives us the slope of the tangent line at $(x, f(x))$ provided the limit exists.

Definition: The slope of the secant line:

$$\frac{f(x+h)-f(x)}{h},$$

This is called the average rate of change of f over the interval. $[x, x+h]$.

[In general, if we have a function $f:[a,b]\to\mathbb{R}$, then

$\frac{f(b)-f(a)}{b-a}$ is called the average rate of change of f over $[a,b]$].

The slope of the tangent line at $(x, f(x))$:

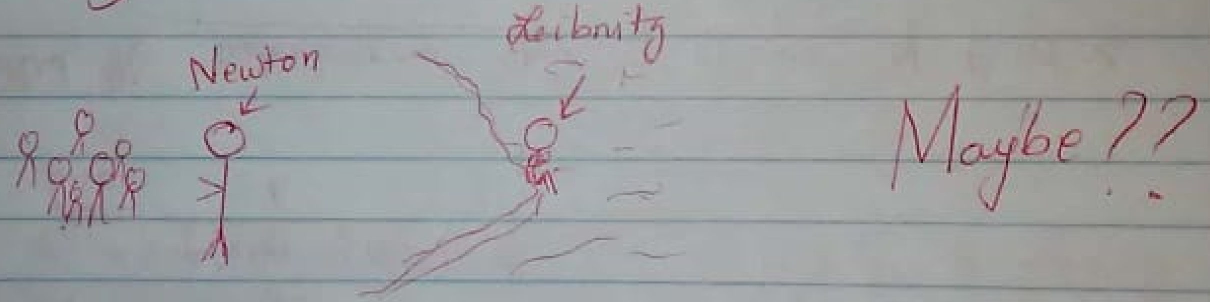
$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

is called the instantaneous rate of change of f at x

Thus, average rate of change is measured over a interval, whereas instantaneous rate of change is just observed at a point.

For example, if $f(x)$ is the position of a car at time x, then $\frac{f(x+h)-f(x)}{h}$ is the average velocity over $[x, x+h]$ whereas $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ is the velocity at time x.

Fun fact: This ~~simple~~ debate over 'average velocity' and 'velocity' led to the beginning of Calculus — I find it really weird sometimes ... Newton & Leibnitz more or less came up with the idea of Calculus at the same time. However, history remembers Newton more because of ... maybe his marketing skills?



Definition

~~The derivative~~

A function f is said to be differentiable at a point a in its domain if

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$ exists. (and is finite; so ∞ or $-\infty$ won't be counted).

(You might be asked to define this)

It is denoted as $f'(a)$ or $\left.\frac{d}{dx}(f(x))\right|_{x=a}$

In general, the derivative of a function f with respect to x is the function f' (read "f prime")

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$ [$f'(x)$ is the slope of the tangent line at $(x, f(x))$]

The domain of f' is the set of all x such that the above limit exists.

If $y = f(x)$, the common notations for derivative are.

- $D_x f(x)$ Read "d sub x of f of x"
- $\frac{dy}{dx}$
- y'

To find $f'(x)$.

(I) Compute $f(x+h)$

(II) Compute $f(x+h) - f(x)$

(III) Find $\frac{f(x+h) - f(x)}{h}$

(IV) Find $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Remark: $\frac{df}{dx}$ is often called the Leibnitz notation

$$\frac{df}{dx} = \frac{d}{dx}(f(x))$$

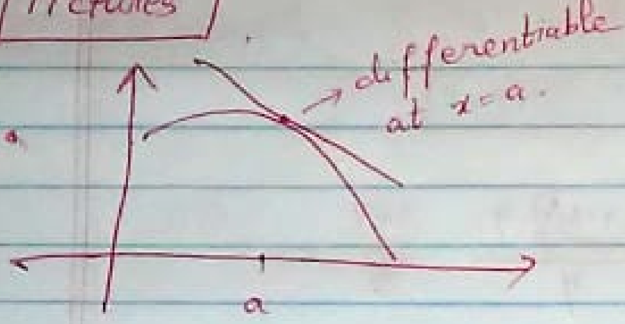
Important Fact

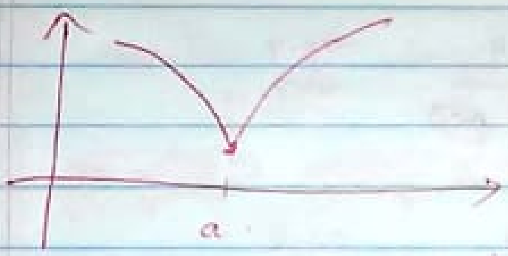
Remark If a function f is differentiable at $x = a$, then f is continuous at $x = a$

(Important fact for True/False Questions).

However, f is continuous at $x = a$, does not imply f is differentiable at $x = a$.

Pictures





not differentiable at a but continuous at $x = a$.

no tangent line.

$\boxed{\text{Example 4}}$ $\boxed{\text{Typical Question}}$

Let $f(x) = \sqrt{x}$

(a) Use the limit definition of derivative to find $f'(x)$ for any $x > 0$.

(b) Find the equation of the tangent line of $f(x)$ at $x = 4$.

$\boxed{\text{Sol}^{n}}$ (a)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

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(b) [Recall: $f'(4)$ is the slope of the tangent line at $(4, f(4))$

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4} \; ; \quad f(4) = \sqrt{4} = 2.$$

Thus, equation of tangent line of $f(x)$ at $x = 4$:

(We know this line has slope $\frac{1}{4}$ and passes through $(2, 2)$).

$$y - 2 = \frac{1}{4}(x - 2)$$

or, $\boxed{4y - 8 = x - 2}$

□

Remark Given a function f, if you are asked to find the equation of ~~the~~ tangent line to graph of f at some point $x = a$:

i) Find $f'(a)$ and $f(a)$. [$f'(a)$ gives you slope; and the tangent line passes through $(a, f(a))$]

ii) **Equation** $y - f(a) = f'(a)(x - a)$ [Point-slope form] $y - y_0 = m(x - x_0)$

Rules of Differentiation

① $\frac{d}{dx}(c) = 0$ (i.e. derivative of a constant function is 0)

Pf Let $f(x) = c$.

Then $\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h}$

$= \lim_{h \to 0} 0 = 0$.

e.g. $\frac{d}{dx}(\pi^2) = 0$

② If n is any real number, then

$\frac{d}{dx}(x^n) = x^{n-1}$

Pf Let us prove for $n = 3$.

Let $f(x) = x^3$.

$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{x^3 + h^3 + 3x^2h + 3xh^2 - x^3}{h}$

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$$= \lim_{h \to 0} \frac{h^3 + 3x^2h + 3xh^2}{h}$$

$$= \lim_{h \to 0} \left(\frac{h\,(h^2 + 3x^2 + 3xh)}{h} \right)$$

$$= \lim_{h \to 0} (h^2 + 3x^2 + 3xh) = 3x^2$$

Thus, $\frac{d}{dx}(x^3) = 3x^2$.

Can you prove for $n = 2$? We already proved this for $n = \frac{1}{2}$.

(In general, it is difficult to prove for an arbitrary n & we omit it)

(2) $\boxed{eg}$. $f(x) = x^{5/2}$.

$$\frac{d}{dx}(f(x)) = \frac{5}{2} x^{5/2 - 1} = \frac{5}{2} x^{\frac{3}{2}}.$$

(3) $\frac{d}{dx}(c f(x)) = c \frac{d}{dx}(f(x))$ where c is a constant

$\boxed{\text{Proof}}$ Exercise

(4) $\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{d}{dx}\left(f(x)\right) \pm \frac{d}{dx}\left(g(x)\right)$

$\boxed{\text{Proof}}$ Let's just do it for '+'.

Let $F(x) = f(x) + g(x)$

Then $F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$

$= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h}$

$= \lim_{h \to 0} \frac{(f(x+h) - f(x)) + (g(x+h) - g(x))}{h}$

$= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right]$

$= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h}\right] + \lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h}\right)$ (since, limit distributes over sum)

$= f'(x) + g'(x).$

$\boxed{eg}$ Let $f(x) = x^3 + 89x^2 + 2$

Then $f'(x) = \frac{d}{dx}(x^3 + 89x + 2)$

$= \frac{d}{dx}(x^3) + \frac{d}{dx}(89x) + \frac{d}{dx}(2)$

$= 3x^2 + 89\frac{d}{dx}(x) + 0$

$= 3x^2 + 89$

As you become more comfortable with derivatives, you directly write. $f'(x) = 3x^2 + 89$!

~~Let $f(x) = \frac{x^5}{3}$~~

(5) Product Rule ~~and Quotient Rule~~

$$\frac{d}{dx}\left(f(x)g(x)\right) = g(x)\frac{d}{dx}(f(x)) + f(x)\frac{d}{dx}(g(x))$$ [in Leibniz notation]

[i.e. $\left(f(x)g(x)\right)' = f'(x)g(x) + g'(x)f(x)$]

(6) ~~Quotient~~

$\boxed{e.g}$ $\frac{d}{dx}\left[\underbrace{(x^3+x)}_{f(x)}\underbrace{(x^5+1)}_{g(x)}\right]$

$= f'(x)\,g(x) + g'(x)\,f(x)$

$= (3x^2+1)(x^5+1) + 5x^4 \cdot (x^3+x)$

(6) $\boxed{\text{Quotient Rule}}$

$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \cancel{\frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}}$

$= \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{g(x)^2}$ $\left(\text{ie } \left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}\right)$