STUDENT NAME: ______________________________

INSTRUCTOR: ______________________________

Please sign the pledge:
On my honor as a student, I have neither given nor received aid on this exam.

Directions

Answer each question in the space provided. Please write clearly and legibly. ***Show all of your work—your work must justify your answers. Clearly identify your final answers. No books, notes, or electronic devices of any kind may be used during the exam period. You must simplify results of function evaluations when it is possible to do so. For example,*** $4^{3/2}$ ***should be evaluated*** $($***replaced by*** $8)$***.***

For instructor use only

Page	Points	Score
2	15	
3	13	
4	13	
5	11	
6	10	
8	8	
9	10	
10	10	
Total:	90	

1. [15 pts] Find derivatives of the following functions. Do not simplify your answers.

(a) $f(x) = x^5 + \ln(x^4 + 1) + e^2$

$$f'(x) = \frac{d}{dx}[x^5] + \frac{d}{dx}[\ln(x^4+1)] + \frac{d}{dx}[e^2] = 5x^4 + \frac{1}{x^4+1}(4x^3) + 0 = 5x^4 + \frac{4x^3}{x^4+1}.$$

(b) $g(x) = \dfrac{e^{x^3+5x}}{x^2+2}$

$$g'(x) = \frac{(x^2+2)\frac{d}{dx}\left[e^{x^3+5x}\right] - e^{x^3+5x}\frac{d}{dx}[x^2+2]}{(x^2+2)^2} = \frac{(x^2+2)e^{x^3+5x}(3x^2+5) - e^{x^3+5x}(2x)}{(x^2+2)^2}$$

(c) $p(x) = x^{2x+1}$

Note $p(x) = e^{(2x+1)\ln(x)}$. Thus

$$p'(x) = e^{(2x+1)\ln(x)}\frac{d}{dx}[(2x+1)\ln(x)] = e^{(2x+1)\ln(x)}\left((2x+1)\frac{1}{x} + \ln(x)(2)\right)$$

Alternative computation: Take ln of both sides of original equation, apply laws of logs, and then take the derivative:

$$\ln(p(x)) = \ln\left(x^{2x+1}\right)$$

Thus,

$$\ln(p(x)) = (2x+1)\ln(x).$$

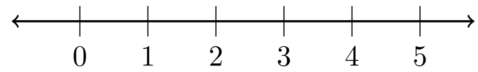
Taking the derivative of both sides, we have

$$\frac{1}{p(x)}p'(x) = (2x+1)(1/x) + \ln(x)(2);$$

hence

$$p'(x) = p(x)\left((2x+1)\frac{1}{x} + \ln(x)(2)\right) = x^{2x+1}\left((2x+1)\frac{1}{x} + \ln(x)(2)\right)$$

2. [5 pts] A particle is traveling along the coordinate line below; its position for $t \geq 0$ is given by $s(t) = \sqrt{3t + \sqrt{t^3}}$, where s is measured in centimeters and t is measured in seconds. Find the velocity of the particle at time $t = 1$. *Include units.*



Solution: Note that $s(t) = \left(3t + t^{3/2}\right)^{1/2}$. The velocity function v is given by

$$v(t) = s'(t) = \frac{1}{2}\left(3t + t^{3/2}\right)^{-1/2}\left(3 + \frac{3}{2}t^{1/2}\right).$$

Thus,

$$v(1) = \frac{1}{2}(4)^{-1/2}\left(3 + \frac{3}{2}\right) = \frac{1}{2}\frac{1}{4^{1/2}}\frac{9}{2} = \frac{9}{8}\ \frac{\text{cm}}{\text{sec}}.$$

3. [8 pts] (a) What theorem guarantees that the function $f(x) = x^3 + 3x^2 - 9x + 3$ has an absolute maximum value and an absolute minimum value on $[-4, 0]$?

Extreme Value Theorem

(b) Use the Closed Interval Method to find the absolute maximum value and the absolute minimum value of $f(x) = x^3 + 3x^2 - 9x + 3$ on $[-4, 0]$.

Solution: Observe that $f'(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3) = 3(x + 3)(x - 1)$, so that f has -3 and 1 as critical numbers but only $x = -3$ is in $(-4, 0)$. Consider the following outputs of f:

x	$f(x)$
-4	23
0	3
-3	30

Thus, by the Closed Interval Method, the absolute maximum value of f on $[-4, 0]$ is 30 and the absolute minimum value is 3.

4. [4 pts] Find all numbers a such that the function $f(x) = e^{ax}$ satisfies $f''(x) = f(x)$.

Solution: Observe that $f'(x) = \frac{d}{dx}[e^{ax}] = e^{ax}\frac{d}{dx}[ax] = e^{ax}(a) = ae^{ax}$. Thus, $f''(x) = a\frac{d}{dx}[e^{ax}] = ae^{ax}(a) = a^2e^{ax}$. Hence, $f''(x) = f(x)$ is the same as

$$a^2e^{ax} = e^{ax}$$

The preceding equation my be rewritten as

$$e^{ax}(a^2 - 1) = 0.$$

Since e^{ax} is never 0 (in fact, always positive), the preceding equation is zero if and only if $a = 1$ or $a = -1$.

5. [4 pts] Solve the equation $32^{x-1} = \dfrac{2}{64^x}$ for x.

Solution: $32^{x-1} = \dfrac{2}{64^x} \iff 32^{x-1}64^x = 2 \iff (2^5)^{x-1}(2^6)^x = 2^1 \iff 2^{5x-5}2^{6x} = 2^1 \iff 2^{11x-5} = 2^1 \iff 11x - 5 = 1 \iff x = \dfrac{6}{11}$

6. [5 pts] [TRUE / FALSE]. Circle your response. No work required. **Rubric:** No partial credit.

(a) **True False** If $x^2 + y^2 = r^2$, then $\ln(x) + \ln(y) = \ln(r)$.

False: This statement is false for many reasons. E.g., $x^2 + y^2 = r^2$ holds for $x = 1$, $y = 0$ and $r = 1$, but $\ln(x) + \ln(y)$, being equal to $\ln(1) + \ln(0)$, is not defined because $\ln(0)$ is not defined (and thus cannot equal $\ln(r) = \ln(1)$, The statement continues to be false even if we restrict x, y, and r to be positive—in the domain of ln. Consider $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$ and $r = 1$. Then $x^2+y^2 = r^2$, but $\ln(x) + \ln(y) = \ln(1/\sqrt{2}) + \ln(1/\sqrt{2}) < 0$, while $\ln(r) = \ln(1) = 0$.

(b) **True False** The domain of $f(x) = b^x$ for $b > 0$ and $b \neq 1$ is $(-\infty, \infty)$.

True: b^x is a real number for every real number x. The graph of $f(x) = b^x$ extends across the whole xy-plane.

(c) **True False** If $f''(2) = 0$, then $(2, f(2))$ is a point of inflection on the graph of f.

False: Consider $f(x) = (x - 2)^4$. You know that the concavity of the graph of f must change at a point of inflection. Thus, the presence of a possible point of inflection with x coordinate 3 would be indicated on the f'' sign line by a sign change at 2. The condition $f''(2) = 0$ tells you that 2 is on the f'' sign line, but does not tell you that f'' changes in sign at 2.

(d) **True False** If $f'(c) = 0$, then $f(x)$ has a relative min or max at $x = c$.

False: Consider $f(x) = x^3$ with $c = 0$.

(e) **True False** If $f''(x) < 0$ for all x in $(0, \infty)$ and $f'(1) = 0$, then $f(1)$ is the absolute maximum value of f on $(0, \infty)$.

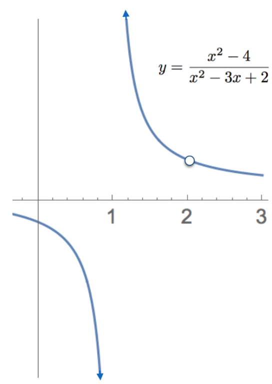
True: Because $f''(x) < 0$ for all x in $(0, \infty)$, the graph of f is concave down over the whole interval $(0, \infty)$. This means, in particular, that the tangent line to the graph of f at $(1, f(1))$ must lie above the graph. This tangent line, having slope 0, is the horizontal line $y = f(1)$. Because the line $y = f(1)$ lies above the graph $y = f(x)$ over $(0, \infty)$, we see that $f(x) \leq f(1)$ for all x in $(0, \infty)$; thus $f(1)$ is the absolute maximum value of f on $(0, \infty)$.

7. [3 pts] Fill-in the blank. No work required.

   (a) $\log_2\left(\frac{1}{8}\right) = \underline{\ -3\ }$. $\log_2\left(\frac{1}{8}\right) = \log_2\left(2^{-3}\right) = -3$.

   (b) $3^{\ln(1)} + \ln\left(\frac{1}{e}\right) = \underline{\ 0\ }$. $3^{\ln(1)} + \ln\left(\frac{1}{e}\right) = 3^0 + \ln(e^{-1}) = 1 + (-1) = 0$.

   (c) The graph of $f(x) = \ln(x)$ is concave $\underline{\text{ down }}$



8. [4 pts] Let $f(x) = \frac{x^2-4}{x^2-3x+2}$, a function whose graph is pictured at right.

   (a) Write down the vertical asymptote(s) of the graph of f. No work required.

   Solution: $x = 1$ is a vertical asymptote of the graph of f. We see this via the graph or by the theorem in the text concerning the vertical asymptotes of rational functions: for $x \neq 2$,

$$f(x) = \frac{x^2-4}{x^2-3x+2} = \frac{(x+2)(x-2)}{(x-2)(x-1)} = \frac{(x+2)}{x-1}.$$

   Because $(x+2)$ does not vanish at 1 while $x = 1$ does, we know that the graph of f has the vertical line $x = 1$ as a V.A.

   (b) For each vertical asymptote you wrote down in response to part (a), provide a justification of why it is a vertical asymptote using the definition of vertical asymptote.

Solution: $\lim_{x\to 1^+} f(x) = \infty$ (alternatively, $\lim_{x\to 1^-} f(x) = -\infty$ also provides a definition-based justification of the claim that $x = 1$ is a VA).

9. [4 pts] The growth rate of Escherichia coli, a common bacterium found in the human intestine, is proportional to its size. Under ideal laboratory conditions, when this bacterium is grown in a nutrient broth medium, the number of cells in a culture doubles approximately every 30 min. If the initial population is 50, determine the function $Q(t)$ that expresses the growth of the number of cells of this bacterium as a function of time t (in minutes).

Solution: Let $Q(t)$ be the quantity of bacteria present t minutes after there are 50 present. We know

$$(*) \qquad Q(t) = 50e^{kt}$$

for an appropriate growth constant k. We are told that $Q(30) = 100$. Thus, substituting $t = 30$ in $(*)$, we obtain,

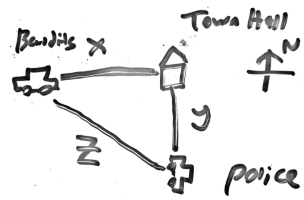
$$100 = 50e^{k(30)},$$

so that $2 = e^{k(30)}$, which, upon taking the ln of both sides, yields $\ln(2) = k(30)$. We have $k = \frac{\ln(2)}{30}$ and thus our model is

$$Q(t) = 50e^{\frac{\ln(2)}{30}t}.$$

10. [10 pts] A group of Canadian bandits have stolen several barrels of maple syrup and are driving east at a speed of 50 mi/h towards Town Hall. The Royal police are driving North towards Town Hall at a speed of 80 mi/h, attempting to intercept the bandits. Find the rate of change (with respect to time) of the distance between the bandits and the police at the instant when the police are 3 miles from Town Hall and Bandits are 4 miles from Town Hall.

1.



2.

$$\frac{dx}{dt} = -50\text{mi/h}, \frac{dy}{dt} = -80\text{mi/h}$$

Want to find $\frac{dz}{dt}$ at the instant when $x = 4$ and $y = 3$ (and $z = \sqrt{3^2 + 4^2} = 5$).

3. $x^2 + y^2 = z^2$
4. $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$, so that $x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$,
5. When $x = 4, y = 3$, and $z = 5$,

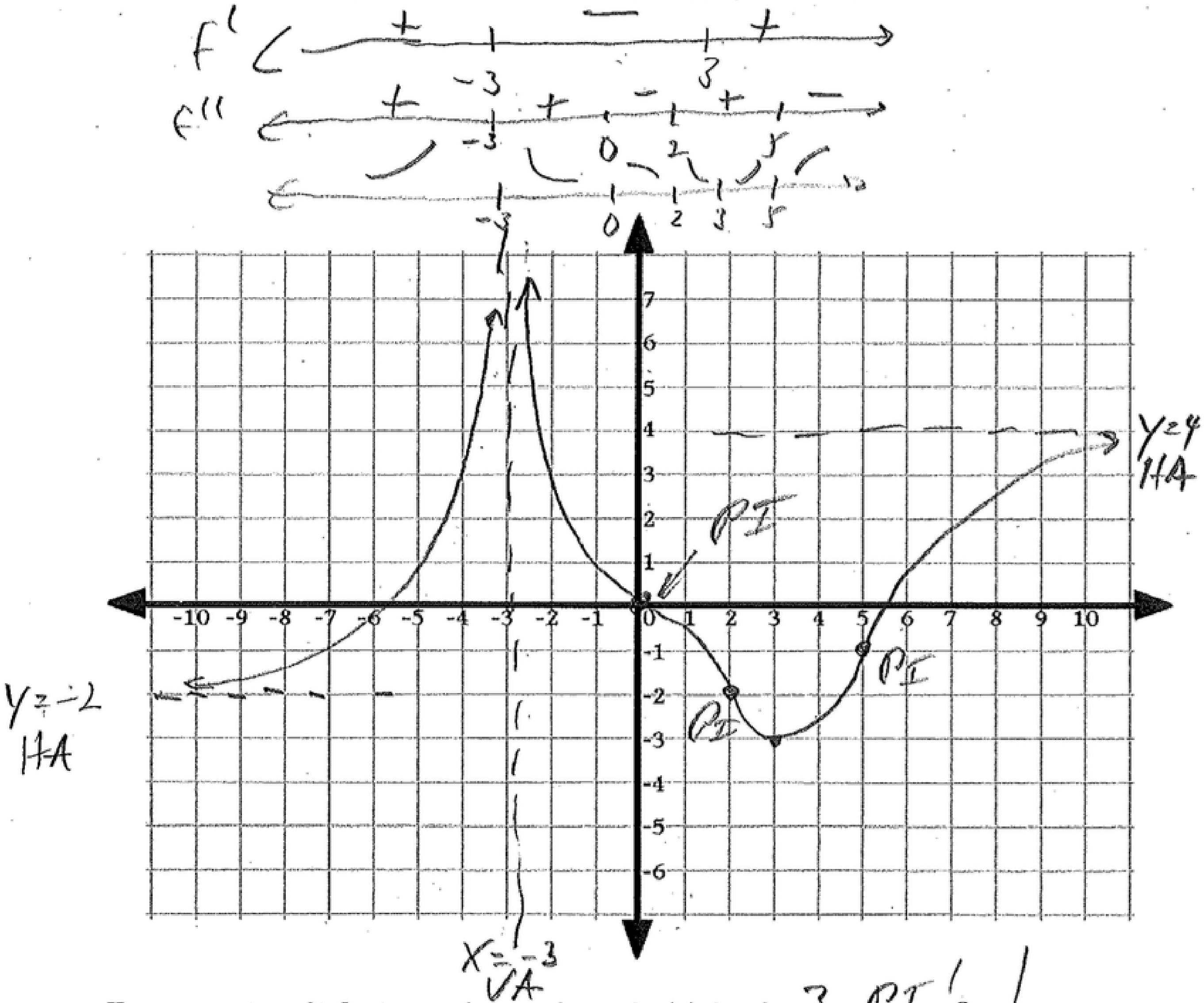
$$4(-50) + 3(-80) = 5\frac{dz}{dt},$$

and thus

$$\frac{dz}{dt} = \frac{-200 - 240}{5} = -\frac{440}{5} = -88\text{mi/h}$$

11. [10 pts] On the grid provided, sketch the graph of a function f that satisfies all of the conditions listed below. Then answer the fill-in-the-blank question that appear below your sketch.

1. The domain of f is $(-\infty, -3) \cup (-3, \infty)$ and f is continuous on its domain.
2. $f(0) = 0$; $f(3) = -3$
3. $x = -3$ is a vertical asymptote,
4. $\lim_{x\to\infty} f(x) = 4$ and $\lim_{x\to-\infty} f(x) = -2$,
5. $f'(x) > 0$ for x in $(-\infty, -3)$ and $(3, \infty)$, and $f'(x) < 0$ for x in $(-3, 3)$,
6. $f''(x) > 0$ for x in $(-\infty, -3)$, $(-3, 0)$, and $(2, 5)$. $f''(x) < 0$ for x in $(0, 2)$ and $(5, \infty)$.

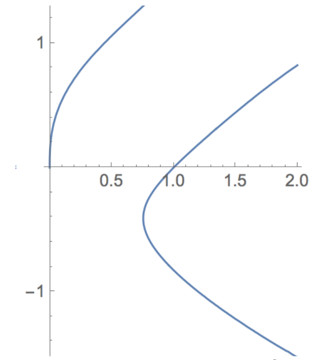


How many points of inflection are there on the graph of f above? 3 PI's !

11. [8 pts] Given that $xe^y = y^3 + x^2$.

(a) Find $\frac{dy}{dx}$ via implicit differentiation.

(b) Find an equation of the line tangent to the graph of $xe^y = y^3 + x^2$ at the point $(1, 0)$. Use the graph (only) to check your answer for plausibility.



Portion of Graph of $xe^y = y^3 + x^2$

(a)

$$\begin{aligned}
\frac{d}{dx}\left[xe^y\right] &= \frac{d}{dx}\left[y^3 + x^2\right] \\
x\frac{d}{dx}[e^y] + e^y\frac{d}{dx}[x] &= \frac{d}{dx}\left[y^3\right] + \frac{d}{dx}\left[x^2\right] \\
xe^y\frac{dy}{dx} + e^y(1) &= 3y^2\frac{dy}{dx} + 2x\frac{dx}{dx} \\
xe^y\frac{dy}{dx} + e^y &= 3y^2\frac{dy}{dx} + 2x \\
\left(xe^y - 3y^2\right)\frac{dy}{dx} &= 2x - e^y \\
\frac{dy}{dx} &= \frac{2x - e^y}{xe^y - 3y^2}.
\end{aligned}$$

(b) $\left.\frac{dy}{dx}\right|_{x=1,y=0} = \frac{2(1)-e^0}{1e^0-3(0)^2} = 1$. Thus, $y - 0 = 1(x - 1)$ is an equation of the tangent to the graph at $(1, 0)$. The slope value of 1 is consistent with the graph provided.

12. [5 pts] Find the horizontal asymptotes of the graph of $f(x) = \begin{cases} e^x & \text{if } x < 0 \\ \frac{x^4+1}{3x^4+2} & \text{if } x > 0 \end{cases}.$ Justify your answer using the definition of horizontal aysmptote.

Solution:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} e^x = 0$$

and

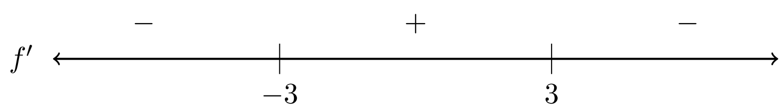
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^4+1}{3x^4+2} = \lim_{x\to\infty} \frac{1+\frac{1}{x^4}}{3+\frac{2}{x^4}} = \frac{1}{3}.$$

Thus, $y = 0$ and $y = \frac{1}{3}$ are horizontal aysymptotes of the graph of f.

13. [5 pts] Identify the intervals on which $f(x) = \dfrac{x}{x^2+9}$ is increasing and the intervals on which f is decreasing.

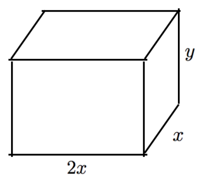
Solution: $f'(x) = \frac{(x^2+9)(1)-x(2x)}{(x^2+9)^2} = \frac{9-x^2}{(x^2+9)^2} = \frac{(3-x)(3+x)}{(x^2+9)^2}$.

Thus, we obtain the following f' sign line:



We see that f is decreasing on both $(-\infty, -3)$ and $(3, \infty)$ while f is increasing on $(-3, 3)$.

14. [10 pts] We are building a rectangular storage container with an open top (**no top**). The container is to have a volume of 8 cubic meters and its length should be twice its width. Materials for the sides cost \$4 per square meter; the bottom, however, is on sale at Home Depot, and so the material for it costs \$3 per square meter. Determine the dimensions of the container that costs the least to make. You must completely justify your answer; in particular, you must show that you have found the dimensions *minimizing* cost (using calculus).



Solution: Let x denote the width (in meters) of the box and y, its height (in meters). We seek to minimize cost C (in dollars) of the container.

$$C = 2x^2(3) + 6xy(4) = 6x^2 + 24xy.$$

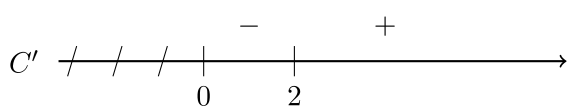
Because the containing is to have a volume of 8 m^3, we know $2x^2y = 8$, so that $y = \frac{4}{x^2}$ and we obtain

$$C(x) = 6x^2 + 24x\frac{4}{x^2} = 6x^2 + \frac{96}{x}.$$

We must have $x > 0$ and $y > 0$ to have a physical box; the constraint $y > 0$, expressed in terms of x becomes $\frac{4}{x^2}$ which is satisfied for any $x > 0$. Thus,there is a box (having volume 8 cubic meters) for every $x > 0$. Our pure math problem is to minimize $C(x) = 6x^2 + \frac{96}{x}$ over $(0, \infty)$.

$$C'(x) = 12x - \frac{96}{x^2} = \frac{12x^3 - 96}{x^2} = \frac{12(x^3 - 8)}{x^2}.$$

Thus, we have the following C' sign line:



Our C' sign lines shows that C decreases on $(0, 2)$ and increases on $(2, \infty)$. This, together with the continuity of C at 2, shows that $x = 2$ yields the absolute minimum value of C over $(0, \infty)$. The corresponding y value is $y = \frac{4}{2^2} = 1$. Thus our cost-minimizing dimensions should be 2 meters by 4 meters by 1 meter.