4.4 Optimization

Absolute Extrema of a function f

If $f(x) \leq f(c)$ (resp. $f(x) \geq f(c)$) for all x in the domain of f, then $f(c)$ is called the absolute maximum value of f. (resp. absolute minimum value).

Theorem (Extreme Value Theorem)

If a function f is continuous on a closed interval $[a,b]$, then f has both an absolute maximum value and an absolute minimum value on $[a,b]$.

Closed Interval Method: This gives us the algo for finding absolute max/min on a closed, bounded interval $[a,b]$. They always exists by the theorem above.

1. Find critical number that lie in open interval (a,b). (Next, we will treat these as well as a & b as crit. pts.)
2. Compute f at each critical number; compute $f(a)$, $f(b)$.
3. The largest out of these is absolute max.
The least " " " is absolute min.

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Eg: Find absolute extrema for $g(x) = x^3 + 3x^2 - 1$ over the interval $[-3, 1]$

Sol^n: $g'(x) = 3x^2 + 6x$; ~~note~~ g' exists everywhere as g is a polynomial.

$g'(x) = 0$

$\Rightarrow 3x(x+2) = 0 \Rightarrow x = 0, -2$.

Thus, critical points on $(-3, 1)$: $x = 0, -2$.

- $f(0) = 0 - 1$;
- $f(-2) = (-2)^3 + 3(-2)^2 - 1 = -8 + 12 - 1 = 3$
- $f(-3) = -27 + 27 - 1 = -1$
- $f(1) = 1 + 3 - 1 = 3$.

Thus,

- $(0, -1)$, $(-3, -1)$ — absolute minimum
- $(-2, 3)$, $(1, 3)$ — absolute maximum.

Note: -1 is the absolute min. value whereas graph of g has an absolute min. at $(0, -1)$ and $(-3, -1)$.

3 is the absolute max. value whereas graph of g has an absolute max at $(-2, 3)$ and $(1, 3)$.

② Find abs. max/min: $\sqrt{4-x^2}$.

$f(x) = \sqrt{4-x^2}$. Domain: $[-2, 2]$. So, abs. max/min exist!

$f'(x) = \dfrac{-x}{\sqrt{4-x^2}}$. So, 0 is the only critical point on $(-2, 2)$.

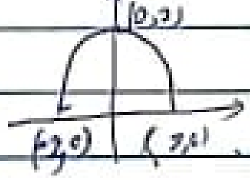
- $f(0) = 2$
- $f(-2) = 0$
- $f(2) = 0$.

Thus, abs. max. value is 2. (at $(0,2)$)

abs. min value $= 0$

@ points $(-2,0)$, $(2,0)$.

Note: this is what we found out in Example 4 of curve sketching



So, while sketching curves, if you find domain is a closed interval, then you have to apply the 'closed interval test' to find max/min to help you draw.

Some facts to keep in mind:

- We talked about Extreme Value Theorem (EVT). However, EVT is not always applicable as we may not have a closed interval:

eg Find the absolute extrema of $f(x) = \frac{2x}{x^2+4}$ over $[-1, \infty)$.

Soln: Here EVT can't be applied. But we can still solve this!

- $f'(x) = -\frac{2(x^2-4)}{(x^2+4)^2}$ (Check!)

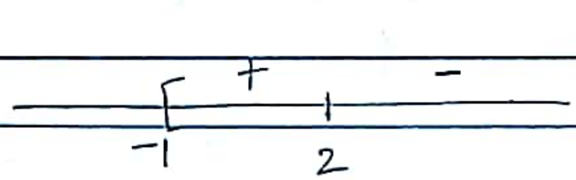
- f is continuous on $[-1, \infty)$

- ~~Sign chart of~~ $f'(x) = 0 \Rightarrow x = 2, -2$. But we need only $x = 2$

Only critical point is 2 on $(1, \infty)$

- $f(-1) = -\frac{2}{5}$; • $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{2x}{x^2+4} = 0$

$\boxed{f'}$



Thus, • f increases on $[-1, 2)$ becomes $\frac{4}{8}$ at $x = 2$

and then decreases on $(2, \infty)$

• f approaches 0 as $x \to \infty$. (it never goes below 0)

Thus, $(2, \frac{4}{8})$ is an absolute max.

$(-1, \frac{-2}{5})$ is an absolute minimum.

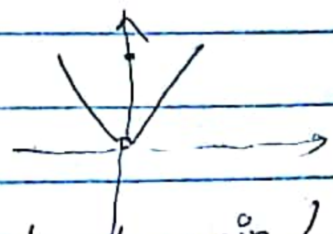
So, here we analyzed it differently using methods similar to curve sketching!

• State True/False.

(a) If f is defined on a closed interval $[a,b]$, then f has an absolute minimum.

Ans: False. [We're not given f is continuous.

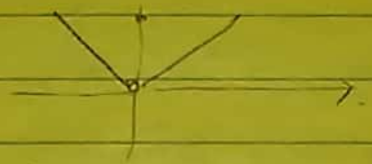
$$f(x) = \begin{cases} |x| & , x \neq 0 \\ 1 & x = 0 \end{cases}$$



No abs. min.!

(b) If f is not continuous on $[a, b]$, then f cannot have an absolute max. value.

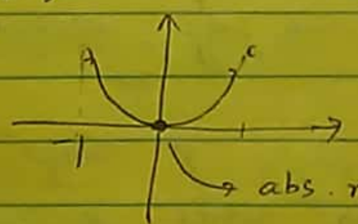
Ans: False.



Not continuous.
But 1 is an abs. max value.

(c) If f is continuous on an open interval (a, b), then f cannot have an absolute ~~maximum~~ minimum value.

Ans: False. $f(x) = x^2$ on $(-1, 1)$.



So, keep an open mind while tackling problems!