- Finding an Integrating factor.

An integrating factor for $\frac{dy}{dt} + p(t)y = g(t)$ is $\mu(t) = e^{\int p(t)dt}$.

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we have
$$\frac{d}{dt}(\mu(t)y) = \mu(t)g(t).$$
Integrating both sides of this equation with respect to t gives
$$\mu(t)y = \int \mu(t)g(t)dt + C$$
for arbitrary constant C. We can solve for y to get
$$\begin{aligned} y(t) &= \frac{1}{\mu(t)}\int \mu(t)g(t) + \frac{C}{\mu(t)} \\ &= e^{-\int p(t)dt}\int e^{\int p(t)dt}g(t)dt + Ce^{-\int p(t)dt}. \end{aligned}$$

- Given an intial value problem we will modify the above solution with suitable limits of integration. Let us look at an example:

$$\frac{dr}{ds} + \frac{5r}{s-3} = 2, \;\; r(2) = 1.$$

Solve it and give the interval of existence of your solution.

Solution:

An integrating factor is $(s-3)^5$. The DE has solution $(s-3)^5 r = (s-3)^6/3 + c$. Setting $s = 2, r = 1$ gives $c = -4/3$. Thus $r = (s-3)/3 - \frac{4}{3}(s-3)^{-5}$. The interval of existence is $(-\infty, 3)$.

- Another example:

$$\frac{dy}{dt} + (\cos t)\, y = t, \quad y(\pi/2) = 1.$$

Solution:

Using the integrating factor $e^{\sin t}$ we have $\frac{d}{dt}(ye^{\sin t}) = te^{\sin t}$, so that

$$ye^{\sin t} = \int_{\pi/2}^{t} ue^{\sin u} du + c.$$

Set $t = \pi/2, y = 1$ to determine $c = e$. Thus

$$y = e^{-\sin t} \int_{\pi/2}^{t} ue^{\sin u}\, du + e^{1-\sin t}.$$

Try solving out Problems 13,18 c, 19, 20, 21. Good practice.

- Definition of homogeneous DE.

$$\frac{dy}{dt} + p(t)y = 0,$$

- Separable Equations

$$xe^{x^2-y} = y\frac{dy}{dx}.$$

Solution:

We have

$$\int xe^{x^2}\, dx = \int ye^{y}\, dy.$$

Using integration by parts on the right-hand side we have the implicit solution

$$\frac{1}{2}e^{x^2} = ye^{y} - e^{y} + c.$$

- Note that Separation of Variables does indeed result in a loss of certain solutions.

For example, one can verify that

$$x\frac{dy}{dx} = \sqrt{1-y^2}.$$

has a solution

$$y = \sin(\ln|x| + c).$$

However, one can also verify that there are other solutions. For example take a look at Prob 22 (page 46).

- Read up Section 2.3 which just talk about some models and stuff.

On the other hand, let's solve the following problem to get a general solution:

If you follow Page 39 of your book, then upon solving

$$\frac{dy}{dx} = \frac{x^3}{y(1+x^4)}$$

we got the solution to be

$$\frac{y^2}{2} = \frac{1}{4}\ln(1+x^4) + C$$

Show that if $y = y(x)$ satisfies

$$\frac{dy}{dx} = \frac{x^3}{y(1+x^4)}$$

on an open interval (a, b), then the function

$$G(x) = \frac{y(x)^2}{2} - \frac{1}{4}\ln(1+x^4)$$

is constant on (a, b). Conclude that there is a constant C so that

$$\frac{y^2}{2} = \frac{1}{4}\ln(1+x^4) + C$$

for all x in (a, b).