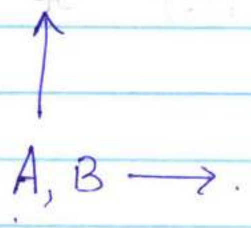
Initially, we will go through a few problems. which will hopefully be a review again.

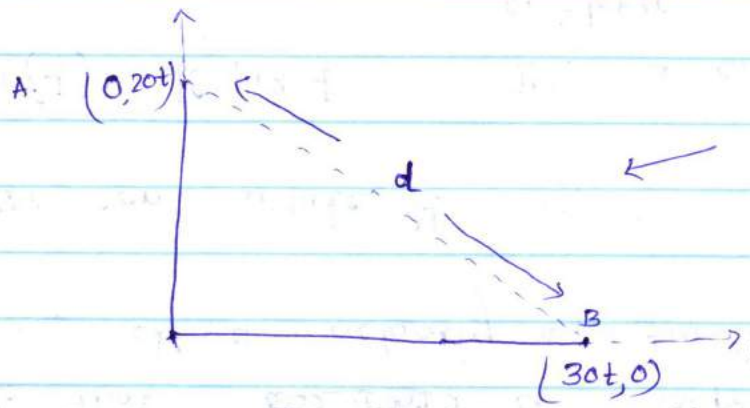
$\boxed{Q.1}$ Two ships leave port at the same time. Ship A sails north at a speed of 20 mph. while Ship B sails east at a speed of 30 mph.

ⓐ Find an expression in terms of time t (in hours) giving the distance between the two ships.

$\boxed{Ans:}$ Draw a diagram!

In t hours, A travels $20t$ miles north while B travels $30t$ miles east. How does it look on the Cartesian plane, assuming both started at $(0,0)$?





This is the diagram after time t. So, what's the distance.

$$d = \sqrt{(30t-0)^2 + (0-20t)^2}$$

$$= \sqrt{900t^2 + 400t^2} = \sqrt{1300}\, t$$

$$= 10\sqrt{13}\, t$$

Thus, this example illustrated the use of the distance formula.

Now, note that ~~d~~ d depends on t right? ~~..~~ So, d is actually a function of t.

Hence, we can write it as

$d(t) = 10\sqrt{13}\,t$. This is an expression of the distance between A and B after t hours. and the unit is miles. (it is distance!!)

(b) Find the distance after 2 hours.

$\boxed{\text{Ans}}$ Just find $d(2)$.

$$d(2) = 10\sqrt{13}\;2 = 20\sqrt{13}$$

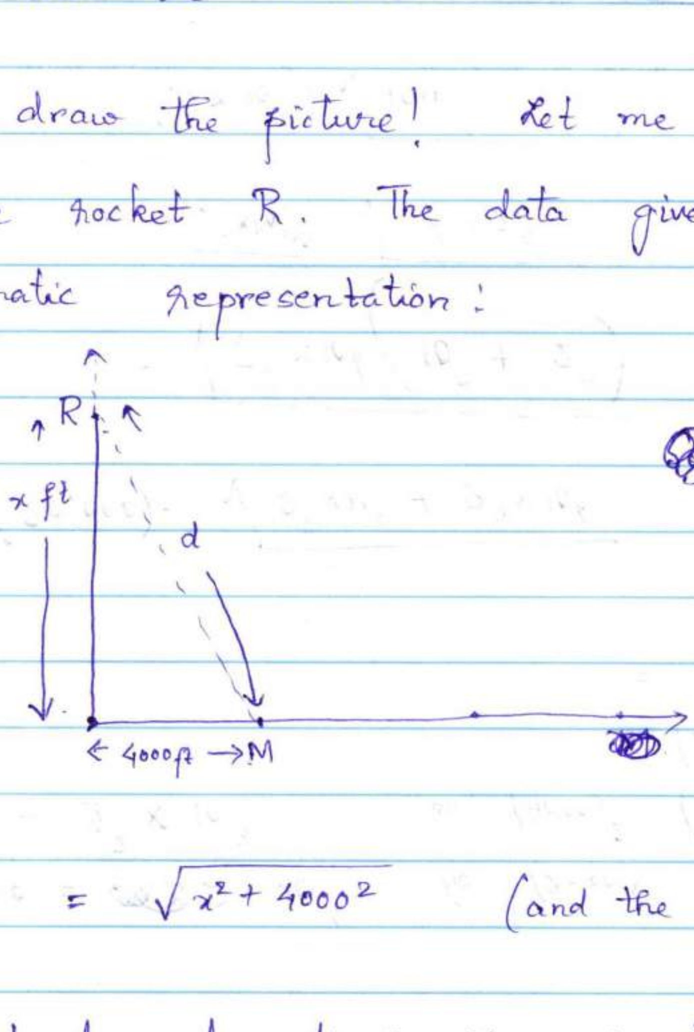
Hence, required answer is $20\sqrt{13}$ miles.

It's not just enough to know how to solve a problem; Writing matters immensely too. The pvs example illustrates the writing style we are expecting from you guys. Trust me, every one of you will improve with time.

Prob 2 At a distance of 4000 ft from the launch site, a spectator is observing a rocket being launched. Suppose the rocket lifts off vertically and reaches an altitude of x ft.

(a) Find an expression giving the distance between the spectator and the rocket

Ans. Again, draw the picture! Let me call the man M and the rocket R. The data given ~~indi~~ has the following diagrammatic representation:



So, $d = \sqrt{x^2 + 4000^2}$ (and the unit is ft).

Note here, that d depends on x; so it's a function of x.

Thus $d(x) = \sqrt{x^2 + 4000^2}$ (unit is ft).

(b) What is the distance between them when the rocket is ~~at~~ an altitude of 20,000 ft?

$\boxed{\text{Ans}}$ Need to find.

$$d(20{,}000) = \sqrt{(20{,}000)^2 + (4000)^2} \quad (\text{in ft})$$

$20{,}000 = 2 \times 10^4$. So, $(20{,}000)^2 = 2^2 \times 10^8$

$4000 = 2^2 \times 10^3$. So, $(4000)^2 = (2^2 \times 10^3)^2$

$$= (2^2)^2 \times (10^3)^2$$

$$= 2^4 \times 10^6.$$

Thus, $d(20{,}000) = \sqrt{2^2 \times 10^8 + 2^4 \times 10^6}$

$$= \sqrt{2^2 \times 10^6 (10^2 + 2^2)}$$

$$= \sqrt{2^2 \times (10^3)^2} \sqrt{10^2 + 2^2}$$

$$= (2 \times 10^3) \sqrt{104}.$$

So, the required distance is $2000\sqrt{104}$ ft. $\square$

This problem was also a bit about algebra. We don't need mindless calculations all the time!

• Working with powers of 10 does make calculations super easy most of the times.

• We used.
• $\sqrt{ab} = \sqrt{a}\sqrt{b}$ provided $a \geq 0, b \geq 0$. ($\sqrt{\text{negative number}}$ ✗).

• $(ab)^x = a^x b^x$. (this is the general rule of the prvs bullet).

• $a(b+c) = ab + ac$.

All of these are high school algebra. It's all in there, you just have to recall them from the correct places in your brains (:p)

Recall the following: Equation.

Vertical line : $x = a$ for some a.

Horizontal line : $y = a$ for some a

Point-Slope form : $y - y_1 = m(x - x_1)$. [Represents a straight line passing through (x_1, y_1) and having slope m].

Slope-Intercept Form: $y = mx + b$.

Q ⓐ Find the equation of a straight line passing through $(-2, 4)$, having slope $\frac{1}{3}$. ~~Express it in~~

ⓑ Express it in slope-intercept form.

ⓒ What is the x-intercept?

Ans: ⓐ Required equation is:

$$y-(4) = \frac{1}{3}(x-(-2))$$

or, $y-4 = \frac{1}{3}(x+2)$.

or, $3y-12 = x+2$

or, $\underbrace{3y-x-14}_{} = 0$.] Better to simplify.

→ this is called the general Equation of a line

$(Ax+By+C=0)$.

ⓑ $y-4 = \frac{1}{3}(x+2)$

or, $y = \frac{1}{3}x+\frac{2}{3}+4. = \frac{1}{3}x+\frac{2+12}{3}$

or, $\boxed{y = \frac{1}{3}x+\frac{14}{3}}$

This is the required Ans

(c) (To find x-intercept, simply put $y=0$;) ~~because it~~
~~is onto on the x axis~~

$$-4 = \frac{1}{3}(x+2)$$

$$\text{or,}\quad -12 = x+2$$

$$\text{or,}\quad x = -14.$$

Hence, x-intercept is at $(-14, 0)$.

$\square$

On WebAssign, you will find a lot of questions of this nature.

Recall functions.

Usually, we write $f: A \to B$.

- A is the domain.
- B is the codomain.
- $f(A) = \{ y \in B \mid f(a) = y \text{ for some } a \in A \}$.
  (also called image of f).

  is called the range of f.

So, $\underset{(\text{subset})}{f(A) \subseteq B}$.

Polynomial Function

It is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

This is called a degree n-polynomial.

$\boxed{\text{e.g}}$ $f(x) = 2x^2 + 5x + 97$. is a degree 2 polynomial.

(degree of a polynomial is simply the highest power of the variable).

$\boxed{Q}$ What is the degree of.

$$x^5 - 5 . ?$$

Ans: It is a 5-degree polynomial.

Polynomials are always functions. with domain $\mathbb{R}$. Sometimes we will restrict the domain to intervals (a, b) or $[a, b]$ or $[a, b)$. or $(a, b]$, etc.

e.g. $f(x) = x^5 - 5$.

We might be only interested in

$$f : [0, 1] \to \mathbb{R}.$$

This means that only inputs we choose for f will be from $[0, 1]$.

It will depend on the question being asked.

$\boxed{\text{eg}}$ $f(x) = \sqrt{x^2 - 4}$

Can f be treated as a function from

$$f : [0, 1] \to \mathbb{R}.$$

$\boxed{\text{Ans:}}$ No! We need $x^2 - 4 \geqslant 0$

or, $x^2 \geqslant 4$.

So, algebra shows that $x \geqslant 2$

OR $x \leqslant -2$.

So, ~~Between~~ on the interval $(-2, 2)$, f is not defined.

One needs to be careful about the domain always.

Often we are interested in knowing roots of polynomials.

[e.g.] Find roots of $2x^2 + 3x + 4$.

i.e. if $f(x) = 2x^2 + 3x + 4$. (always a function, because it's a polynomial)

we need to

find all x such that

$f(x) = 0$.

We will learn about a cool theorem which will tell us whether given an interval $[a, b]$, a function f has a root in that interval or not. All in due time.

[Note: Given any function, f, it always makes sense to talk about <u>root of a function</u>; simply find all x such that $f(x) = 0$].

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Returning to the Problem $2x^2 + 3x + 4$.

Remember Sreedharacharya's formula:

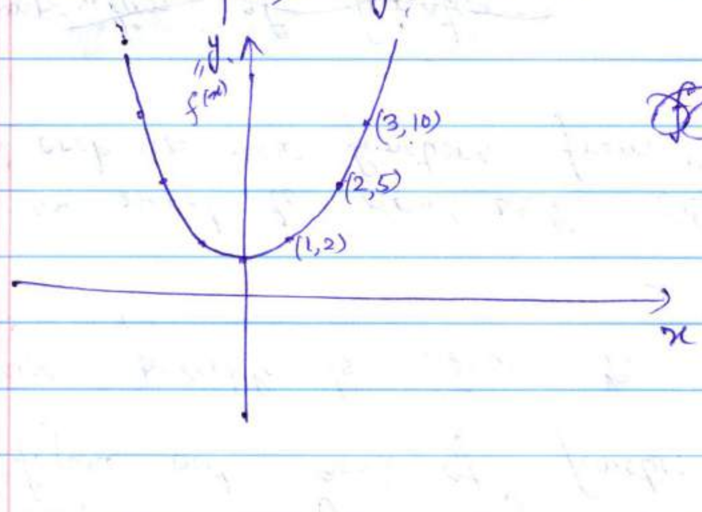
$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$ are the roots of ax^2+bx+c

You can apply it and get answers!

Now, remember we talked about graphs of functions (equations)?

$y = ax^2+bx+c \rightarrow$ This quadratic equation graphically is represented as a parabola.

For example, $y = x^2+1$. has the graph.



x^2+1 is a polynomial in x.

Also, vertical line test shows that y is given as a function of x.

$f(x) = x^2+1$ is a 2-degree polynomial.

Finding the roots of $f(x)$ means looking at the x-intercepts of the graph of f

(Whenever, it hits x-axis, $y=0$ i.e. $f(x)=0$).

~~Hence~~

Hence,

- find the roots of f.
- find the x-intercepts of the graph of f

} Same. Question.

(The example x^2+1 has no real roots; clear from picture; also try quadratic root formula)

We'll depend mostly on algebra to figure out roots of functions because it is not always possible to draw the graph.

Thus, we need to deal with algebra involving functions.
We'll cook up new functions from existing ones.

- ~~Composition of functions~~

- $f : A \to C$, $g : B \to C$.

Then $f+g : A \cap B \to C$.

$f-g : A \cap B \to C$.

$fg : A \cap B \to C$.

$$f/g = \frac{f(x)}{g(x)} : A \cap B \setminus \{x \in B \mid g(x) = 0\} \longrightarrow C.$$

That is addition, subtraction, product, quotients of functions are all well-defined provided we choose the domains very carefully.

$\boxed{\text{e.g.}}$. $f(x) = \sqrt{x^2 - 4}$

$g(x) = x + 3$.

Find $\dfrac{f(x)}{g(x)}$ and specify the domain.

Ans: $f : (-\infty, -2] \cup [2, \infty) \longrightarrow \mathbb{R}$ (as we saw in a previous example)

$g : \mathbb{R} \to \mathbb{R}$ (it's a linear polynomial).

Note $g(-3) = 0$. So, -3 is a root of g.

Thus, $\dfrac{f(x)}{g(x)} = \dfrac{\sqrt{x^2-4}}{x+3}$ on the domain. $\left((-\infty, -2] \cup [2, \infty)\right) \setminus \{-3\}$.

(we removed the point (-3) ~~under~~ from the domain of f)..

$\boxed{\text{Note}}$: $\boxed{\text{Rational functions}}$: Suppose f, g are polynomials. So, $f, g : \mathbb{R} \to \mathbb{R}$.

Then $h(x) = \dfrac{f(x)}{g(x)}$ ~~Generally, f.~~

is called a rational function.

In general, we just remove the zeros of g from $\mathbb{R}$ to get the domain for h.

However, sometimes, roots g cancel out roots of f.

e.g. $h(x) = \dfrac{x^2+5x+6}{x+3}$

$= \dfrac{(x+3)(x+2)}{(x+3)} = x+2.$

So, domain still remains $\mathbb{R}$. Beware of such tricky questions!

e.g. Suppose $f(x) = \sqrt{4+x}$
$g(x) = 2x+1.$

Find $f+g$, fg, f/g, specify domains.

Ans: • $(f+g)(x) := f(x) + g(x)$
$= \sqrt{4+x} + 2x+1$ on the domain $[-4, \infty)$.

• $fg(x) = (2x+1)\sqrt{4+x}$ on $[-4, \infty)$.

• $\dfrac{f(x)}{g(x)} = \dfrac{\sqrt{4+x}}{2x+1}$ on $[-4, \infty) \setminus \left\{-\frac{1}{2}\right\}$:

$= \left[-4, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$

Uses: Suppose a factory makes some materials.
at a cost price given by a function
$C(x)$ where x is the amount of material used.

Suppose the selling price is $S(x)$. What is the profit?

Ans: The profit is given by the function
$P(x) = S(x) - C(x)$.
Thus, these algebra are indeed useful!

Composition of Functions

Let f and g be functions. Then the composition of g and f is a new function denoted by

$$(g \circ f)(x) = g(f(x)).$$

The domain of $g \circ f$ is the set of all x in the domain of f such that $f(x)$ lies in the domain of g.

Diagramatically,

$$A \xrightarrow{f} B \xrightarrow{g} C.$$

$$g \circ f : A \to C.$$

~~But remember that $g \circ f$ may not be defined~~

~~e.g. $f(x) = \sqrt{x+1}$~~

~~$g(x_1) =$~~

e.g. $f(x) = \sqrt{x} + 1$, $x \geqslant 0$. (This is a way of representing functions)

(i.e. domain of f is $[0, \infty)$).

$g(x) = (x-1)^2$, $x \geqslant 1$.

Find $(g \circ f)(x)$.

Ans: Note that $f(x) \geqslant 1$. as $\sqrt{x} + 1 \geqslant 1$ when $x \geqslant 0$.

So, $g \circ f$ makes sense.

$$
\begin{aligned}
(g \circ f)(x) = g(f(x)) &= (f(x) - 1)^2 \\
&= (\sqrt{x} + 1 - 1)^2 \\
&= (\sqrt{x})^2 \\
&= x.
\end{aligned}
$$

and the domain is simply $[0, \infty)$.

$[0, \infty) \xrightarrow{f} [1, \infty) \xrightarrow{g} [0, \infty)$.

$g \circ f : [0, \infty) \to [0, \infty)$.

It always helps to have this sort of a diagram in mind.

So, we found out $(g \circ f)(x) = x$. This is a special case, we say g is a left inverse of f.

- Find $f \circ g$.

Note: $g(x) = (x-1)^2, \; x \geqslant 1$.

So, $g : [1, \infty) \to [0, \infty)$.

Thus, $f \circ g$ makes sense.

$$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} + 1$$
$$= \sqrt{(x-1)^2} + 1.$$
$$= (x-1) + 1$$

~~(since, $x \geqslant 1$, so enough to look at~~

$$= x.$$

Domain is $[1, \infty)$.

Hence, $f \circ g : [1, \infty) \to [1, \infty)$

So (Thus, here g is also a right-inverse of f. because $(f \circ g)(x) = x$.

So This situation is called 'f & g are inverses of each other'.

Some important functions:

(a) $\boxed{\text{Identity function}}$ $f(x) = x$. for all $x \in \text{Domain}(f)$.

We represent f by 'id' sometimes.

$\text{id} : \mathbb{R} \longrightarrow \mathbb{R}$.

$x \longmapsto x$.

(b). $\boxed{\text{Constant Function}}$ $f : A \to B$., $f(a) = b_0$ for all $a \in A$ for a fixed $b \in B$.

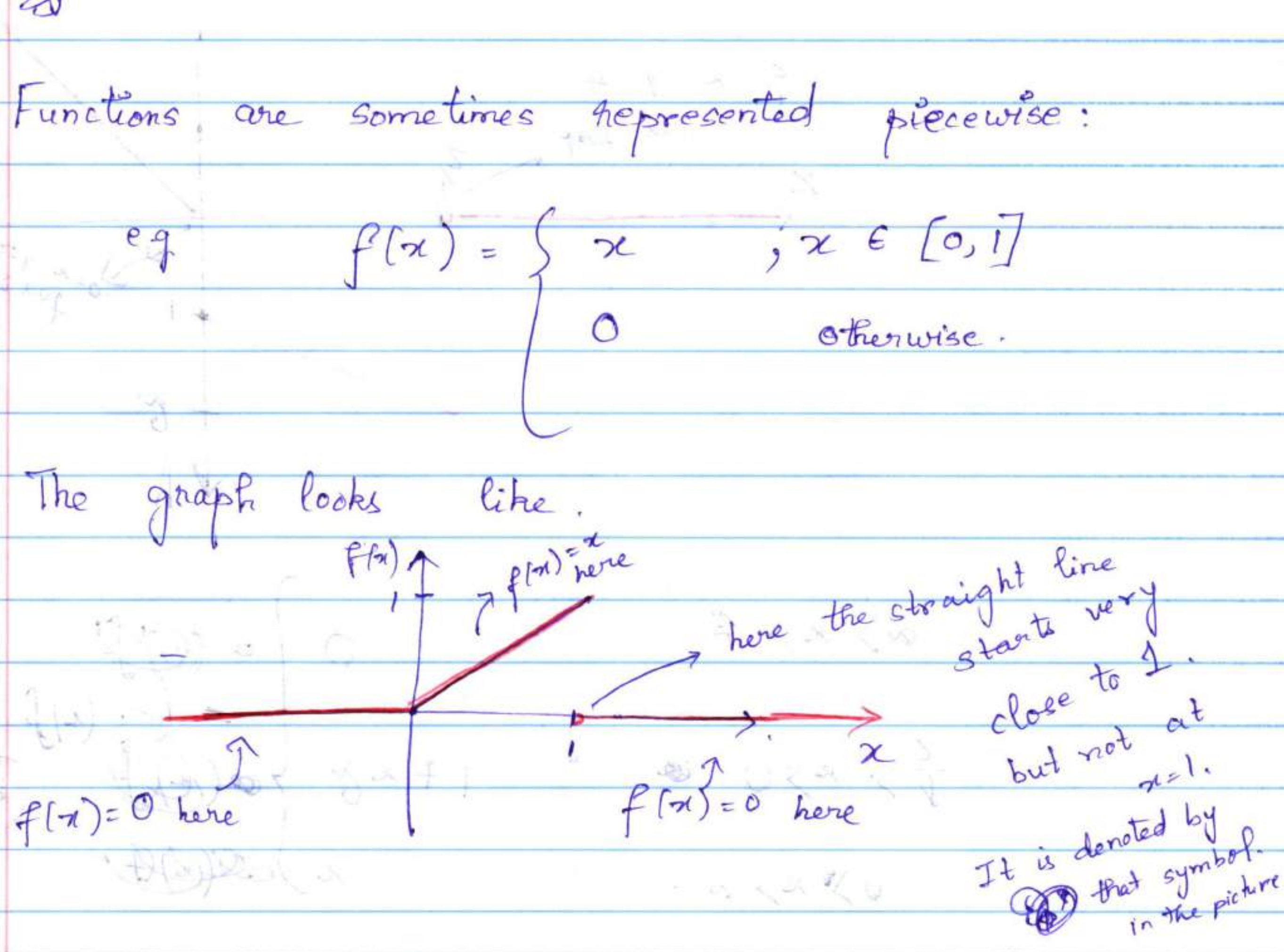
$\boxed{\text{e.g}}$ $f : \mathbb{R} \to \mathbb{R}$

$x \mapsto c$. i.e. $f(x) = c$ for some fixed $c \in \mathbb{R}$.

(c) Functions are sometimes represented piecewise:

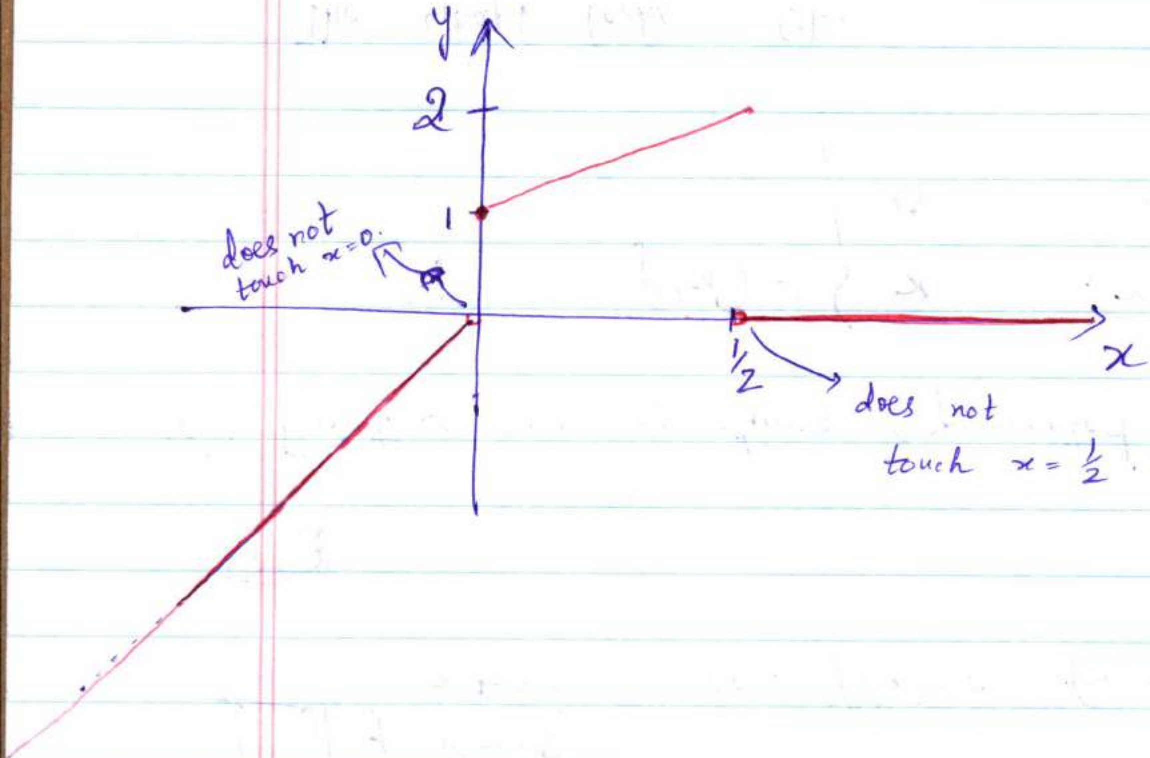
e.g $$f(x) = \begin{cases} x & ; x \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

The graph looks like.



(e.g.)

$$f(x) = \begin{cases} x, & -\infty < x \leq 0 \\ 2x+1, & 0 \leq x \leq \frac{1}{2} \\ 0 & \frac{1}{2} < x < \infty. \end{cases}$$



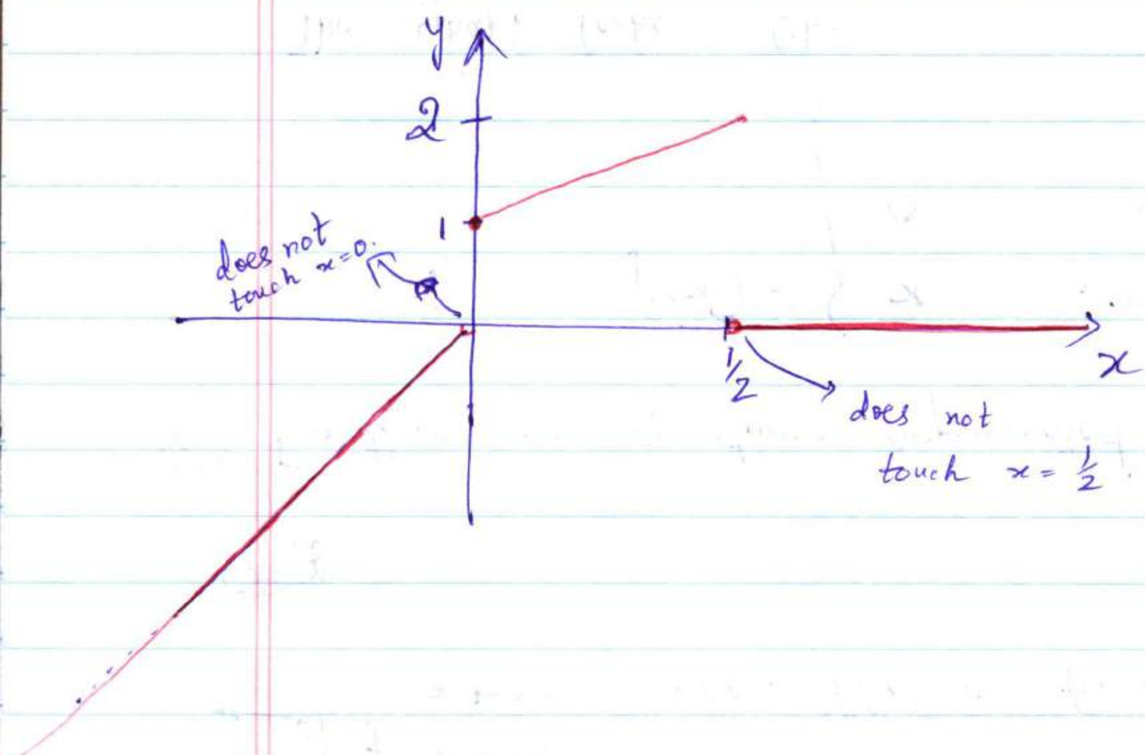
(d) A specific example of this is the 'absolute value' function:

$$f(x) := |x|$$

i.e. $$f(x) = \begin{cases} x & x \geq 0 \\ -x & x \leq 0. \end{cases}$$

(e.g)

$$f(x) = \begin{cases} x & , \quad -\infty < x \leq 0 \\ 2x+1 & , \quad 0 \leq x \leq \frac{1}{2} \\ 0 & \quad \frac{1}{2} < x < \infty . \end{cases}$$



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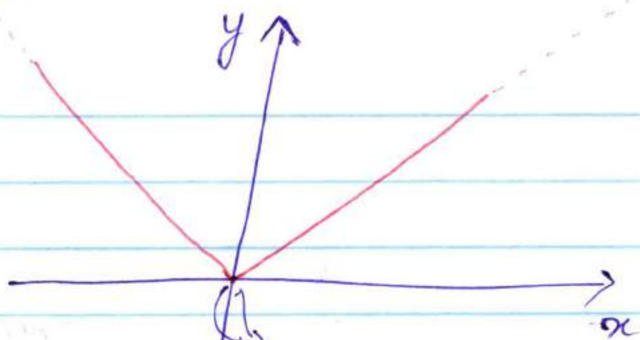
The graph is.



($x=0$ is included as both branches have the same value at $x=0$.)

Example (d) is very important.

We will often face problems regarding inequalities with absolute values.

Example: $|x-2| \geqslant 3$. What x satisfy this?

We have discussed this in a previous note. Again:

Suppose $x \geqslant 2$.

Then $|x-2| = x-2$.

We need $x-2 \geqslant 3$.

or, $x \geqslant 5$.

Thus, $x \in [5, \infty)$

Suppose $x \leqslant 2$.

Then $|x-2| = -(x-2)$.

We need, $-(x-2) \geqslant 3$.

or, $x-2 \leqslant -3$

or, $x \leqslant -1$.

So, $x \in [-\infty, -1]$

Hence, $x \in [-\infty, -1] \cup [5, \infty)$.

e.g. $|2x+3| \leq 5$. Solve for x.

Ans. If $2x+3 \geq 0$, then $|2x+3| = 2x+3$.

Thus, $2x+3 \leq 5$

or, $2x \leq 2$

or, $x \leq 1$.

If $2x+3 \leq 0$, then $|2x+3| = -(2x+3)$.

So, $-(2x+3) \leq 5$

or, $2x+3 \geq -5$

or, $2x \geq -8$

or, $x \geq -4$.

Thus, combining the two we have

$$-4 \leq x \leq 1. \text{ i.e. } x \in [-4, 1].$$

$\square$.

Refer to the prvs note for 'trick' involving such computations.

$\boxed{\text{Algebra Fact}}$ If $x^2 = a$ for $a \geq 0$.

then $x = \pm\sqrt{a}$.

i.e. $x = \sqrt{a}$ OR $x = -\sqrt{a}$. Thus, $|x| = \sqrt{a}$.

Similarly, if

$$x^2 - a \geq b. \qquad (a+b \geq 0)$$

or, $x^2 \geq a+b$.

or, $|x| \geq \sqrt{a+b}$.

So, $x \leq -\sqrt{a+b}$ OR $x \geq \sqrt{a+b}$. (do this on your own).