Let $f(x)$ be a function. We want to graph it! Here's how to do this in detail:

(1) Describe the **domain** of $f(x)$.

(2) Find the **x- and y-intercepts** of $f(x)$. Recall that a number x in the domain of $f(x)$ is an x-intercept if $f(x) = 0$. A number y in the range of $f(x)$ is a y-intercept if $f(0) = y$.

(3) Determine the **end behavior** of $f(x)$. That is, compute the limits

$$\lim_{x\to+\infty} f(x) \qquad \text{and} \qquad \lim_{x\to-\infty} f(x),$$

or state that they do not exist. If they don't exist, you may want to note whether the y-values approach $+\infty$ or $-\infty$.

(4) Find the **horizontal** and **vertical asymptotes** of $f(x)$. A function $f(x)$ has a horizontal asymptote L if $\lim_{x\to+\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$. A function $f(x)$ has a vertical asymptote at a number a if $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$.

(5) Determine the intervals on which $f(x)$ is **increasing** and **decreasing**.

(6) Find and classify the **relative extrema** of $f(x)$ using the critical point method.

(7) Determine the intervals on which $f(x)$ is **concave up** and **concave down**.

(8) Find the **inflection points** of $f(x)$ using the inflection point method.

(9) Plot all intercepts, critical points, inflection points and any other "interesting points" found in the previous steps. Then use the information about asymptotes, increasing/decreasing behavior and concavity to **sketch the graph** of $f(x)$.

Some notes:

- According to Tan, a rational function $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, has a vertical asymptote at $x = a$ if $p(a) \neq 0$ and $q(a) = 0$. This should agree with your findings when solving the limits $\lim_{x\to a^+} \dfrac{p(x)}{q(x)}$ and $\lim_{x\to a^-} \dfrac{p(x)}{q(x)}$ using factor-cancel. You can directly use this to state the vertical asymptotes. However, it is crucial to identify whether function is approaching ∞ or $-\infty$ as our sketch will depend on that.
- Polynomials never have vertical or horizontal asymptotes.
- A rational function can have at most one horizontal asymptote.
- Right before Step 9, I find it useful to create a new number line in which I record all critical points, inflection points and "interesting points" and, on each interval between these, record the increasing/decreasing behavior AND concavity simultaneously. This will be a good guide as you draw the final sketch in Step 9. You may even want to sketch a little , , or on each interval.

Okay, let's do it!

Problem 1. Sketch $f(x) = x^3 - 3x^2 - 24x + 32$.

(1) The domain of $f(x)$ is: $\mathbb{R}$.

(2) The x-intercepts of $f(x)$ are: on the intervals $[-5,-4]$, $[1,2]$, $[6,7]$

The y-intercepts of $f(x)$ are: $(0, 32)$.

(3) Compute:

$$\lim_{x\to+\infty} f(x) = \text{DNE } (\infty)$$

$$\lim_{x\to-\infty} f(x) = \text{DNE } (-\infty).$$

(4) Does $f(x)$ have any horizontal or vertical asymptotes? If so, list them: No. because f is a polynomial.

(5) Find the intervals on which $f(x)$ increases and decreases.

$f(x)$ is increasing on: $(-\infty,-2)$, $(4,\infty)$

$f(x)$ is decreasing on: $(-2, 4)$

$f'(x) = 3x^2 - 6x - 24$
$= 3(x^2 - 2x - 8) = 3(x-4)(x+2)$

$+$ -2 $-$ 4 $+$

(6) $f(x)$ has the following critical points: $4, -2$

$f(x)$ has the following relative maxima: $(-2, 60)$

$f(x)$ has the following relative minima: $(4, -48)$

$f(-2) = 60$

$f(4) = -48$

(7) Find the intervals on which $f(x)$ is concave up and concave down.

$f(x)$ is concave up on: $(1, \infty)$

$f(x)$ is concave down on: $(-\infty, 1)$

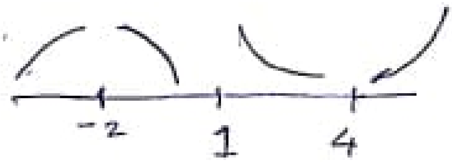
$f''(x) = 6x - 6$.

$x = 1$.

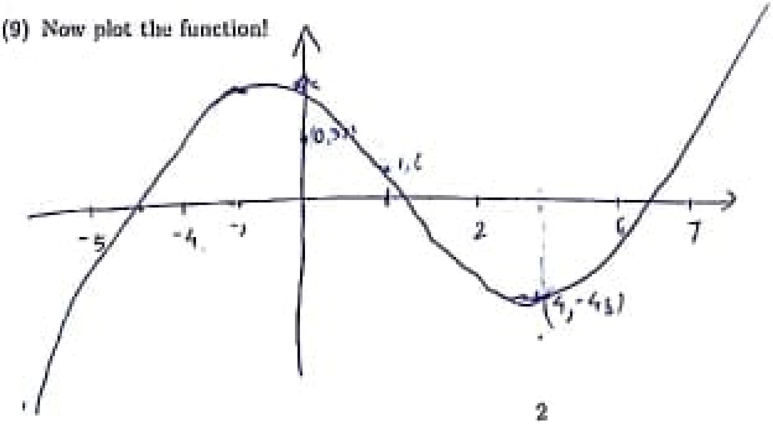
$-$ 1 $+$

(8) For the following x, $f''(x) = 0$ or DNE:

$f(x)$ has the following inflection points (list both x and y values): $(1, 6)$



(9) Now plot the function!



e.g. Sketch:

$$g(x) = \frac{2x^2+5}{4-x^2}$$

Soln.

- Domain $= (-\infty,-2) \cup (-2,2) \cup (2,\infty)$.

- $2x^2+5 \neq 0$ anywhere. So, there is no x-intercept;
- $g(0) = \frac{5}{4}$. Thus $\left(0, \frac{5}{4}\right)$ is the y-intercept.

- Now, $4-x^2=0 \Rightarrow x = \pm 2$. and $2x^2+5 \neq 0$. Thus,

  $x=2$ and $x=-2$ are vertical asymptotes of g. $\left[\lim_{x\to 2^-} f(x) = \infty,\ \lim_{x\to 2^+} f(x) = -\infty\right.$

- $\lim_{x\to\infty} \frac{2x^2+5}{4-x^2} = \lim_{x\to\infty} \frac{2+\frac{5}{x^2}}{\frac{4}{x^2}-1} = -2$;

  $\lim_{x\to-\infty} \frac{2x^2+5}{4-x^2} = -2$;

  Thus, $y=-2$ is the horizontal asymptote.

- $g'(x) = \frac{(4-x^2)4x + 2(2x^2+5)x}{(4-x^2)^2}$

$$= \frac{16x - 4x^3 + 4x^3 + 10x}{(4-x^2)^2} = \frac{26x}{(4-x^2)^2}.$$

$g'(x) = 0 \Rightarrow x = 0$. [2, -2 are not in domain]

Thus, $x=0$ is the critical point.

$$\underset{0}{\overset{-\quad\quad +}{\longleftarrow\!\!\!\longrightarrow}}$$

$g'(-1) < 0$

$g'(1) > 0$.

Thus, at $x = 0$, there is a relative minimum.

$$g''(x) = \frac{(4-x^2)^2\, 26 - 26x\left(2(4-x^2)(-2x)\right)}{(4-x^2)^4}$$

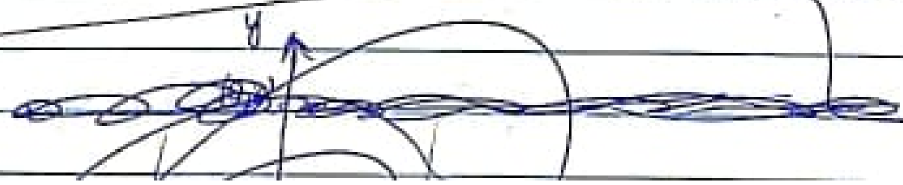
$$= \frac{(4-x^2)(26)\left[4-x^2+4x^2\right]}{(4-x^2)^4} = \frac{26(3x^2+4)}{(4-x^2)^3}$$

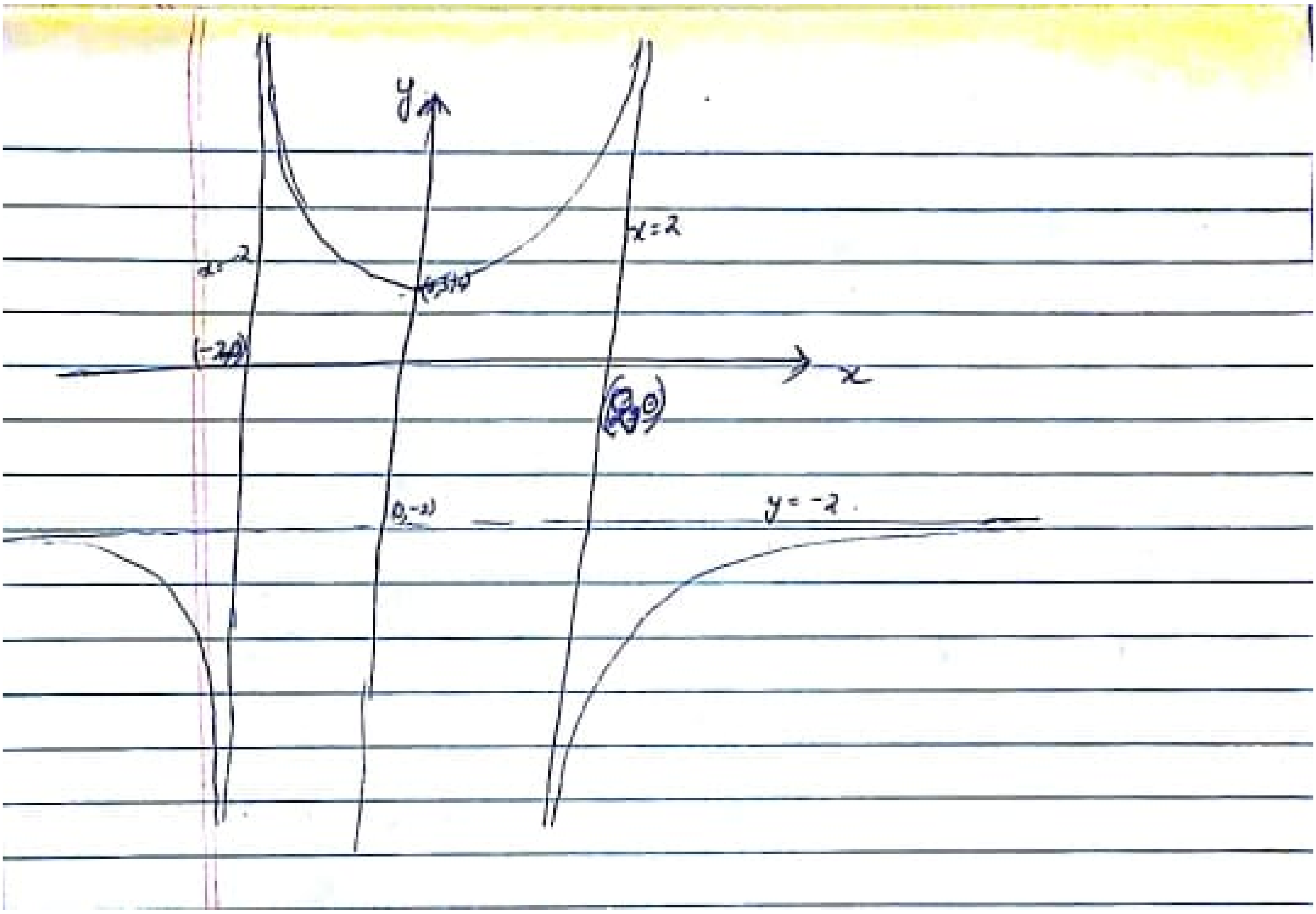
$g''(x) > 0$ for all $x \neq \pm 2$. (not in the domain).

No inflection point; always concave up for $x \in (-2, 2)$

~~concave~~ down for $x \in (2, \infty)$

$x \in (-\infty, -2)$





y
x
x=-2
x=2
(0,3/4)
(-3,0)
(3,0)
(0,-2)
y = -2

③ Sketch. $f(x) = x^{2/3}$

$$f'(x) = \frac{2}{3x^{1/3}}$$

$$f''(x) = \frac{-2}{9x^{4/3}}$$

- Domain: $\mathbb{R}$.
- Intercepts: $(0,0)$
- f' is discontinuous at $x = 0$.

$-$ $+$
0

- $\lim_{x \to +\infty} f(x) = \infty$
- $\lim_{x \to -\infty} f(x) = -\infty$.

- $f' \neq 0$ but f' DNE at 0.

So, 0 is the only critical pt.

No vertical asymptotes.

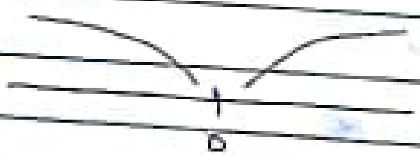
No horizontal asymptotes.

$-$ $+$
0

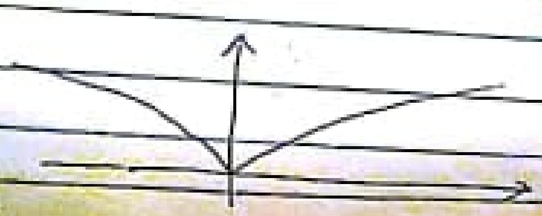
So, rel. min at $(0,0)$.

- $f'' < 0$ always. No inflection point.

f'' DNE at 0



Thus



(4) $f(x) = \sqrt{4-x^2}$. (Motivating example for Absolute Max/Min on closed intervals).

- Domain: $[-2, 2]$.
- So, no question of asymptotes.
- Intercepts: $(2,0)$, $(-2,0)$, $(0,2)$

- $f'(x) = \dfrac{-x}{\sqrt{4-x^2}}$; $f''(x) = \dfrac{\sqrt{4-x^2}\,(-1) - (-x)\frac{(-2x)}{2\sqrt{4-x^2}}}{(4-x^2)}$

$$= \frac{-(4-x^2) - x^2}{(4-x^2)\sqrt{4-x^2}} = \frac{-4}{(4-x^2)^{3/2}}$$

- f' DNE at $2, -2$ and $=0$ ~~at~~ $x=0$

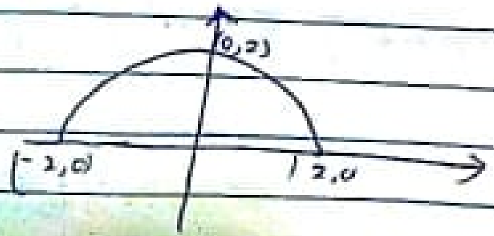
+ −
−2 0 2

- Crit. pts are 0, 2, −2.
- Rel. min. at $(-2,0)$, $(2,0)$ by plugging in simply. Can't apply derivative test here at boundary points
- Rel. max. at $(0,2)$

- f''

−2 0 2.

- No inflection point



semi-circle)

(4.4) Optimization

Absolute Extrema of a function f

If $f(x) \leq f(c)$ (resp. $f(x) \geq f(c)$) for all x in the domain of f, then $f(c)$ is called the absolute maximum value of f. (resp. absolute minimum value).

Theorem (Extreme Value Theorem)

If a function f is continuous on a closed interval $[a,b]$, then f has both an absolute maximum value and an absolute minimum value on $[a,b]$.

Closed Interval Method: This gives us the algo for finding absolute max/min on a closed, bounded interval $[a,b]$. They always exists by the theorem above.

1. Find critical number that lie in open interval (a,b).
(Next, we will treat these as well as a & b as crit. pts.)
2. Compute f at each critical number; compute $f(a)$, $f(b)$.
3. The largest out of these is absolute max.
The least " " " is absolute min.

Eg: Find absolute extrema for $g(x) = x^3 + 3x^2 - 1$ over the interval $[-3, 1]$.

Soln: $g'(x) = 3x^2 + 6x$; ~~note~~ g' exists everywhere as g is a polynomial.

$g'(x) = 0$

$\Rightarrow 3x(x+2) = 0 \Rightarrow x = 0, -2$.

Thus, critical points on $(-3, 1)$: $x = 0, -2$.

- $f(0) = 0 - 1$;
- $f(-2) = (-2)^3 + 3(-2)^2 - 1 = -8 + 12 - 1 = 3$
- $f(-3) = -27 + 27 - 1 = -1$
- $f(1) = 1 + 3 - 1 = 3$.

Thus,

- $(0, -1)$, $(-3, -1)$ – absolute minimum
- $(-2, 3)$, $(1, 3)$ – absolute maximum.

Note: -1 is the absolute min. value whereas graph of g has an absolute min. at $(0, -1)$ and $(-3, -1)$.

3 is the absolute max. value whereas graph of g has an absolute max at $(-2, 3)$ and $(1, 3)$.

② Find abs. max/min: $\sqrt{4-x^2}$.

$f(x) = \sqrt{4-x^2}$. Domain: $[-2, 2]$. So, abs. max/min exist!

$f'(x) = \dfrac{-x}{\sqrt{4-x^2}}$. So, 0 is the only critical point on $(-2, 2)$.

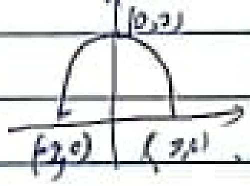
- $f(0) = 2$
- $f(-2) = 0$
- $f(2) = 0$.

Thus, abs. max. value is 2. (at $(0,2)$)

abs. min value $= 0$

@ points $(-2,0)$, $(2,0)$.

Note: this is what we found out in Example 4 of curve sketching



So, while sketching curves, if you find domain is a closed interval, then you have to apply the 'closed interval test' to find max/min to help you draw.

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