STUDENT NAME: ______________________________

INSTRUCTOR: ______________________________

Please sign the pledge:
On my honor as a student, I have neither given nor received aid on this exam.

Directions

Answer each question in the space provided. Please write clearly and legibly. ***Show all of your work—your work must justify your answer. Clearly identify your final answer. No books, notes, or electronic devices of any kind may be used during the exam period. You must simplify results of function evaluations when it is possible to do so. For example,*** $4^{3/2}$ ***should be evaluated*** (***replaced by*** 8)***. Your work on all limit problems must be well organized; in particular, for a limit that exists, a string of equalities must connect the original limit problem with the value of the limit.***

For instructor use only

Page	Points	Score
2	13	
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4	12	
5	8	
6	10	
7	13	
8	16	
9	9	
Total:	100	

1. [5 pts] Find the domain of the function $f(x) = \dfrac{\sqrt{x-1}}{x^2-x-6}$, expressing your answer in interval notation. Provide work showing how you arrived at your answer.

   Solution: In order for $\sqrt{x-1}$ to be real, we must have $x-1 \geq 0$; equivalently, $x \geq 1$. Thus, the domain of f is contained in $[1,\infty)$. However, no number x making the denominator equal to 0 is in the domain; the zeros of the the denominator are the roots of the equation $0 = x^2 - x - 6 = (x-3)(x+2)$. Thus the zeros of the denominator are 3 and -2. Hence, the domain is $[1,\infty)$ with 3 excluded:

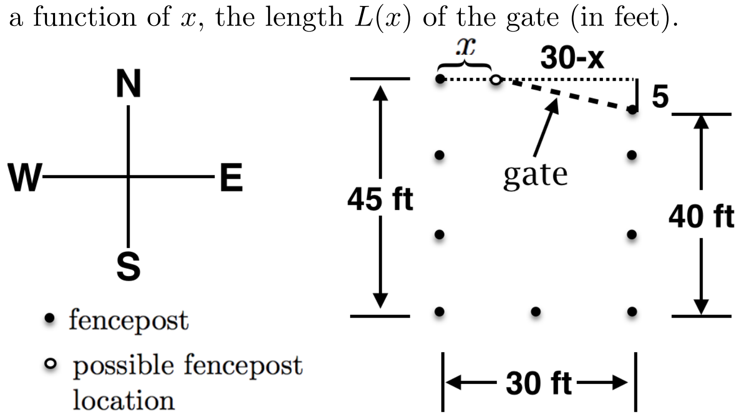
$$[1,3) \cup (3,\infty).$$

2. [4 pts] Find an equation of the line passing through the point $(-3,2)$ that is parallel to the line $-2x+3y+10=0$.

   Solution: The given line may be written in the form $y = \frac{2}{3}x - \frac{10}{3}$ and thus it has slope $\frac{2}{3}$. The line whose equation is requested has the same slope $\frac{2}{3}$ because it is parallel to the given line. Thus an equation of the line through the point $(-3,2)$ that is parallel to the line $-2x+3y+10=0$ is

$$y - 2 = \frac{2}{3}(x+3).$$

3. [4 pts] Sketching a design for a livestock enclosure (see below), a farmer has settled on positions for 9 of 10 fenceposts. The 10th fencepost will be set a distance x feet east of the northwestern corner post, $0 \leq x \leq 30$. A gate will span the gap between the 10th post and the northeastern-most post, as pictured. Express, as a function of x, the length $L(x)$ of the gate (in feet).



   Solution: As pictured above, the gate forms the hypotenuse of a right triangle whose vertical leg has length 5 feet and whose horizontal leg has length $30-x$ feet. Thus, by the Pythagorean Theorem;

$$L(x) = \sqrt{(30-x)^2 + 25}.$$

4. [4 pts] Steelworth Investment Bank is constantly buying and selling shares of AAPL stock. The number of shares of AAPL held by Steelworth Investment Bank at time t, measured in thousands, is modeled by

$$n(t) = \frac{8t^2 - t + 70}{6t + t^2 + 10}$$

How many shares of AAPL will Steelworth Investment Bank hold in the long run (i.e, as $t \to \infty$).

Solution: We have

$$\begin{aligned}\lim_{t\to\infty} n(t) &= \lim_{t\to\infty} \frac{8t^2 - t + 70}{6t + t^2 + 10} \\ &= \lim_{t\to\infty} \frac{(8t^2 - t + 70)\cdot \frac{1}{t^2}}{(6t + t^2 + 10)\cdot \frac{1}{t^2}} \\ &= \lim_{t\to\infty} \frac{8 - 1/t + 70/t^2}{6/t + 1 + 10/t^2} \\ &= \frac{8}{1} = 8.\end{aligned}$$

Thus, in the long run, Steelworth will hold 8000 shares of AAPL stock.

5. [10 pts] You need not show any work in completing this graph-based problem. Consider the function g graphed below right.

(a) Record the domain of g in interval notation.

(-3,3)

(b) Record the range of g in interval notation.

(-1.5, 5)

(c) At which numbers in its domain is the function g discontinuous?

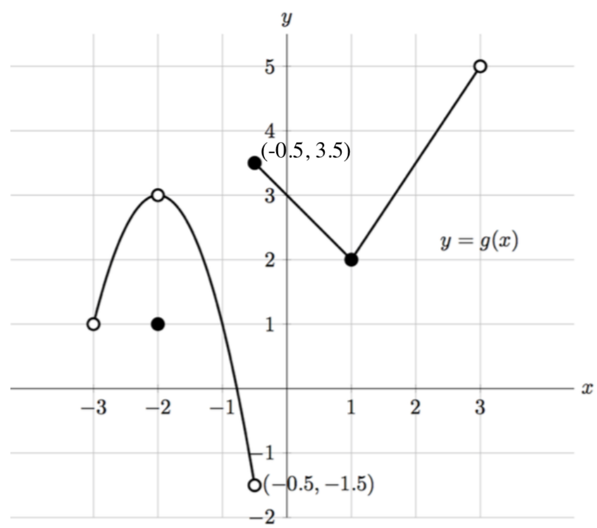
-2, -1/2

(d) At which numbers in its domain is the function g not differentiable?

-2, -1/2, 1

(e) Find $\lim_{x\to -2} g(g(x))$ if it exists.

$\lim_{x\to -2} g(g(x)) = \lim_{u\to 3^-} g(u) = 5$. (Note: You can find this limit experimentally—what happens when you substitute into $g(g(x))$ a value for x very close to -2 but not equal to -2?)



6. [5 pts] [TRUE/FALSE] Circle your response. No work required.

(a) If f is differentiable at $x = x_0$, then f is continuous at $x = x_0$. $\boxed{\text{True}}$ False

This is a theorem from Unit 1 of the course

(b) If f is continuous at $x = x_0$, then f is differentiable at $x = x_0$. True $\boxed{\text{False}}$

The absolute value function $f(x) = |x|$ is continuous at $x = 0$, but is not differentiable at $x = 0$. Same is true for $f(x) = x^{1/3}$.

(c) If $\lim_{x\to x_0^+} f(x) \neq \lim_{x\to x_0^-} f(x)$, then $f(x_0)$ is not defined. True $\boxed{\text{False}}$

Consider $f(x) = -1$ if $x < 0$ but $f(x) = 1$ if $x \geq 0$.

(d) For all real numbers $\sqrt{x^2} = x$. True $\boxed{\text{False}}$

$\sqrt{x^2} = x$ is false whenever $x < 0$; what's true in general is $\sqrt{x^2} = |x|$.

(e) $f(x) = \frac{x^2-1}{x-1}$ is not continuous at 1. $\boxed{\text{True}}$ False

f is not continuous at 1 because $f(1)$ is undefined.

7. [12 pts] Evaluate the following limits or write DNE if the limit does not exist. Your work must justify your answer; moreover, your work must be well organized. Work based on "l'Hospital's Rule" will receive no credit.

(a)

$$\lim_{x\to 4}\frac{x^2-5x+4}{x^2-16}=\lim_{x\to 4}\frac{(x-1)(x-4)}{(x+4)(x-4)}=\lim_{x\to 4}\frac{(x-1)}{(x+4)}=\frac{3}{8}.$$

(b)

$$\begin{aligned}\lim_{x\to 0^-}\frac{x+|x|}{x}&=\lim_{x\to 0^-}\frac{x+(-x)}{x}\\&=\lim_{x\to 0^-}\frac{0}{x}\\&=\lim_{x\to 0^-}0\\&=0.\end{aligned}$$

(c) $\displaystyle\lim_{x\to 0}\frac{x+1}{g(x)}$ where $g(x)=\begin{cases}-x+1 & x<0\\ \sqrt{x}+1 & x>0\end{cases}$

Solution 1: $\displaystyle\lim_{x\to 0^-}\frac{x+1}{g(x)}=\lim_{x\to 0^-}\frac{x+1}{-x+1}=1$ and $\displaystyle\lim_{x\to 0^+}\frac{x+1}{g(x)}=\lim_{x\to 0^+}\frac{x+1}{\sqrt{x}+1}=1;$

Thus, because the left-and right-hand limits agree, $\displaystyle\lim_{x\to 0^+}\frac{x+1}{g(x)}=1.$

Solution 2:

$$\begin{aligned}\lim_{x\to 0}\frac{x+1}{g(x)}&=\frac{\lim_{x\to 0}(x+1)}{\lim_{x\to 0}g(x)} && \text{(provided both limits on the right exist and } \lim_{x\to 0}g(x)\neq 0)\\&=\frac{1}{\lim_{x\to 0}g(x)} && \text{(provided } \lim_{x\to 0}g(x) \text{ exists and is nonzero).}\end{aligned}$$

We analyze $\lim_{x\to 0}g(x)$ via left- and right-hand limits:

$$\begin{aligned}\lim_{x\to 0^-}g(x)&=\lim_{x\to 0^-}(-x+1)=1\\ \lim_{x\to 0^+}g(x)&=\lim_{x\to 0^-}(\sqrt{x}+1)=1.\end{aligned}$$

Because the left- and right-hand limits have common value 1, we know $\lim_{x\to 0}g(x)=1$ and our earlier work now shows

$$\lim_{x\to 0}\frac{x+1}{g(x)}=\frac{1}{\lim_{x\to 0}g(x)}=\frac{1}{1}=1.$$

8. [8 pts] (a) Complete the following definition:

The function f is *continuous* at the number a provided that

(1) $f(a)$ is defined;

(2) $\lim_{x\to a} f(x)$ exists;

(3) $\lim_{x\to a} f(x) = f(a)$.

(b) Is there a value of c making the function $f(x) = \begin{cases} \frac{1}{2}x^2, & \text{if } x < 1 \\ c, & \text{if } x = 1 \\ \frac{2x}{x^2+1} & \text{if } x > 1 \end{cases}$ continuous at 1? Carefully justify your answer using the definition of continuity you just stated.

Solution: The first condition for continuity of f at 1 is satisfied (assuming c is assigned a value); $f(1) = c$. Is the second? That is, does

$$\lim_{x\to 1} f(x) \text{ exist?}$$

We analyze the left- and right-hand limits at 1:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{2}x^2 = \frac{1}{2}$$

while

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{2x}{x^2+1} = \frac{2}{2} = 1.$$

Thus, because the left- and right-hand limits differ, $\lim_{x\to 1} f(x)$ does not exist. The value of c has no effect on either of these one-sided limits at 1. Thus, there is no value of c for which f is continuous at 1; the second condition for continuity of f at 1 fails.

9. [10 pts] (a) Complete the following "limit definition" of the derivative: the derivative of f at x, denoted $f'(x)$ is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \quad \text{provided the limit exists}$$

(b) Let $f(x) = \sqrt{3x+1}$. *Use the definition of derivative you just stated* to find $f'(x)$. *No credit will be awarded for work that is not based on the definition of derivative.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3(x+h)+1}-\sqrt{3x+1}}{h} \\
&= \lim_{h\to 0} \left(\frac{\sqrt{3(x+h)+1}-\sqrt{3x+1}}{h} \cdot \frac{\sqrt{3(x+h)+1}+\sqrt{3x+1}}{\sqrt{3(x+h)+1}+\sqrt{3x+1}} \right) \\
&= \lim_{h\to 0} \frac{3(x+h)+1-(3x+1)}{h(\sqrt{3(x+h)+1}+\sqrt{3x+1})} \\
&= \lim_{h\to 0} \frac{3x+3h+1-3x-1}{h(\sqrt{3(x+h)+1}+\sqrt{3x+1})} \\
&= \lim_{h\to 0} \frac{3h}{h(\sqrt{3(x+h)+1}+\sqrt{3x+1})} \\
&= \lim_{h\to 0} \frac{3}{(\sqrt{3(x+h)+1}+\sqrt{3x+1})} \\
&= \frac{3}{(\sqrt{3(x+0)+1}+\sqrt{3x+1})} \\
&= \frac{3}{2\sqrt{3x+1}}
\end{aligned}$$

10. [7 pts] A Nuclear Regulatory Commission scientist projects that, for a certain reactor subjected to intermittent coolant-flow interruption, the temperature of the reactor core t hours after the initial interruption, $0 \leq t \leq 3$, is given by

$$f(t) = 3t^2 - t^3 + t + 1, \quad \text{in thousands of degrees Celsius.}$$

The temperature at which a core meltdown occurs is about 2.9 thousand degrees Celsius. Show there must be a time t in the modeling interval $[0, 3]$ at which the temperature of the reactor core equals the meltdown temperature 2.9. Your answer must be carefully justified.

Solution: Observe that $f(0) = 1$ while $f(3) = 4$. Because f is continuous on $[0, 3]$ (it's a polynomial) and 2.9 is between $f(0) = 1$ and $f(3) = 4$, the Intermediate-Value Theorem tells us that there must be a time c in $[0, 3]$ such that $f(c) = 2.9$.

11. [6 pts] Suppose that $p(x) = x\, g(f(x))$ and that $g(2) = 4, g'(2) = -1, f(3) = 2$, and $f'(3) = 5$. Find an equation of the line tangent to the graph of p at the point $(3, 12)$.

Solution: The slope m of the line tangent to the graph of p at $(3, 12)$ will be $p'(3)$. By the product and chain rules,

$$p'(x) = x\frac{d}{dx}[g(f(x))] + g(f(x))\frac{d}{dx}[x] = xg'(f(x))f'(x) + g(f(x)).$$

Thus, $m = p'(3) = 3g'(f(3))f'(3) + g(f(3)) = 3g'(2)f'(3) + g(2) = 3(-1)(5) + 4 = -11$. Hence,

$$y - 12 = -11(x - 3)$$

is an equation of the line tangent to the graph of p at $(3, 12)$.

12. [10 pts] Find derivatives of the following functions. **Do not simplify your answers.**

(a) $f(x) = \dfrac{x^3+4}{x^2+3x+\pi^2}$

$$\begin{aligned} f'(x) &= \frac{(x^2+3x+\pi^2)\frac{d}{dx}[x^3+4] - (x^3+4)\frac{d}{dx}[x^2+3x+\pi^2]}{(x^2+3x+\pi^2)^2} \\ &= \frac{(x^2+3x+\pi^2)3x^2 - (x^3+4)(2x+3)}{(x^2+3x+\pi^2)^2} \end{aligned}$$

(b) $g(x) = (x^2-20)\sqrt{5x-1}$

$$\begin{aligned} g'(x) &= (x^2-20)\frac{d}{dx}\left[(5x-1)^{1/2}\right] + \sqrt{5x-1}\frac{d}{dx}[x^2-20] \\ &= (x^2-20)\frac{1}{2}(5x-1)^{-1/2}\frac{d}{dx}[5x-1] + \sqrt{5x-1}(2x-0) \\ &= (x^2-20)\frac{1}{2}(5x-1)^{-1/2}(5) + \sqrt{5x-1}(2x) \end{aligned}$$

13. [6 pts] Let f be the function defined by $f(x) = \frac{1}{15}(x^3-23)^{5/2}$. Compute $f'(3)$. **Your final answer should be in the form of an integer.**

Solution:

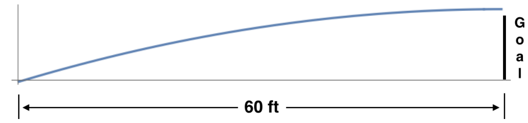
$$\begin{aligned} f'(x) &= \frac{1}{15}\cdot\frac{5}{2}(x^3-23)^{3/2}(3x^2) \\ &= \frac{1}{3\cdot 2}(x^3-23)^{3/2}(3x^2) \\ &= \frac{x^2}{2}(x^3-23)^{3/2} \end{aligned}$$

Therefore,

$$\begin{aligned} f'(3) = \frac{9}{2}(27-23)^{3/2} = \frac{9}{2}(4^{3/2}) &= \frac{9}{2}(4^{1/2})^3 \\ &= \frac{9}{2}\cdot 2^3 \\ &= 9\cdot 2^2 \\ &= 36 \end{aligned}$$

14. [9 pts] A ball is is kicked directly toward a goal as pictured below right.

The ball's initial velocity has a horizontal component a ft/sec and a vertical component b ft/sec. Here a and b are positive constants. The curve the ball follows after it is kicked is well modeled by



$$y = -\frac{16}{a^2}x^2 + \frac{b}{a}x, \quad \text{where } x \text{ and } y \text{ are measured in feet.}$$

(a) Find $y'(x) = \frac{dy}{dx}$, keeping in mind that a and b are constants.

$$\frac{dy}{dx} = -\frac{-32x}{a^2} + \frac{b}{a}.$$

(b) Find the point on the curve $y(x) = -\frac{16}{a^2}x^2 + \frac{b}{a}x$ at which the tangent line to the curve is horizontal. (The x-coordinate of this point depends on both a and b while the y-coordinate depends only on b.) *Also note that the ball attains its maximum height at this point—the maximum height depends only on b.*

Solution: We know that $y'(x)$ is the slope of the line tangent to the curve $y = -\frac{16}{a^2}x^2 + \frac{b}{a}x$ at the point $(x, y(x))$. We seek a point at which the tangent is horizontal—has slope zero—thus, we seek an x for which $y'(x) = 0$. Hence, we solve

$$\frac{-32x}{a^2} + \frac{b}{a} = 0,$$

obtaining $x = \frac{ab}{32}$. The corresponding y-coordinate is

$$y(ab/32) = \frac{-16}{a^2}\frac{a^2b^2}{32^2} + \frac{b}{a}\frac{ab}{32} = \frac{-16}{32^2}b^2 + \frac{b^2}{32} = \frac{b^2}{32}\left(-\frac{16}{32} + 1\right) = \frac{b^2}{64}.$$

Thus the point on the curve $y(x) = -\frac{16}{a^2}x^2 + \frac{b}{a}x$ at which the tangent line to the curve is horizontal is $\left(\frac{ab}{32}, \frac{b^2}{64}\right)$.

(c) Suppose the ordered pair you just found (yielding maximum height) is directly above the goal, having coordinates $(60, 9)$. Determine the corresponding initial-velocity components: a and b.

Solution: We seek positive numbers a and b such that $\left(\frac{ab}{32}, \frac{b^2}{64}\right) = (60, 9)$. Thus a and b satisfy

$$\frac{ab}{32} = 60 \quad \text{and} \quad \frac{b^2}{64} = 9.$$

The equation on the right has solutions $b = \pm 24$, but b is positive. Thus, $b = 24$ ft/sec. Substituting $b = 24$ into $\frac{ab}{32} = 60$ yields $a = 80$ ft/sec.