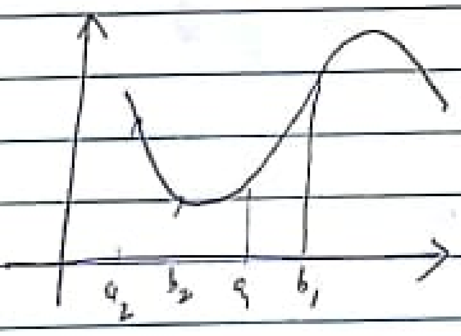
$\boxed{4.1}$ Applications of the First Derivative

- A function f is increasing on an interval (a,b) if for every two numbers x_1 and x_2 in (a,b),

  $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

- A function f in decreasing on an interval (a, b) if for every two numbers x_1 & x_2 in (a,b),

  $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.



- f is increasing on (a_1, b_1)
- f is decreasing on (a_2, b_2).

Recall that the derivative gives us rate of change of a function. This suggests, that if a function is differentiable, derivative should help assessing intervals on which f is increasing/ decreasing.

Theorem 1.

ⓐ If $f'(x) > 0$ for every x in (a, b), then f is increasing on (a, b)

ⓑ If $f'(x) < 0$ for every x in (a, b), then f is decreasing on (a, b).

ⓒ If $f'(x) = 0$ for every value x in (a, b), then f is constant on (a, b).

Here is an algorithm to figure where 'f' is increasing/decreasing

① Solve $f'(x) = 0$. Also, note where f' is discontinuous, and identify open intervals determined by these numbers.

② Select a test number c in each interval found in Step 1, and determine the sign of $f'(c)$ in that interval. Associate that sign to the interval.

ⓐ If $f'(c) > 0$, then f is increasing on that interval

ⓑ If $f'(c) < 0$, then f is decreasing on that interval.

[redacted]

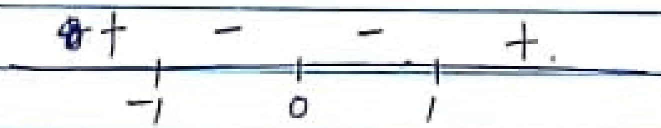
e.g. Find the intervals where $f(x) = x + \frac{1}{x}$ is increasing and where it is decreasing.

Solution: $f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$.

$f'(x) = 0$. So, $1 - \frac{1}{x^2} = 0$

$x = \pm 1$.

Also, f' is discontinuous at 0.



Choose

The signs come from the following:)

Let test points be $-2, -\frac{1}{2}, \frac{1}{2}, 2$.

$f'(-2) = \frac{4-1}{4} > 0$.

$f'(-\frac{1}{2}) = 1 - 4 < 0$.

$f'(\frac{1}{2}) = 1 - 4 < 0$.

$f'(2) = \frac{4-1}{4} > 0$

Thus, f is increasing on $(-\infty, -1), (1, \infty)$;

f is decreasing on $(-1, 0), (0, 1)$.

DO NOT WRITE $\boxed{(-1,0) \cup (0,1)}$ to denote interval where f is decreasing; use ',' instead.

Recall

Why? The definition of increasing function which does not involve derivative at all! ($f(x_1) > f(x_2)$ whenever $x_1 < x_2$).

If $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x$, then $f'(x) > 0$ on $(-\infty, 1)$, $(3, \infty)$

So, f is increasing on $(-\infty, 1)$

f is increasing on $(3, \infty)$.

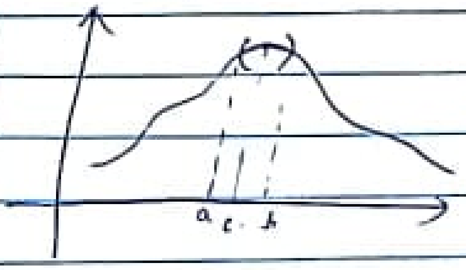
But f is not increasing on $(-\infty, 1) \cup (3, \infty)$.

because $f(.9) > f(3.1)$ inspite of $0.9 < 3.1$.

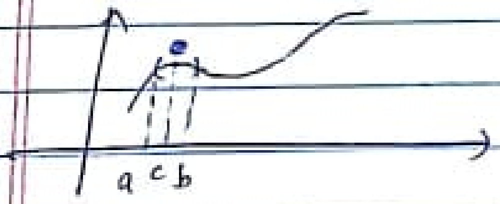
Relative Extrema.

- Relative Maximum

A function f has a relative maximum at $x=c$ if there exists an open interval (a,b) containing c such that $f(x) \leq f(c)$ for all x in (a,b)

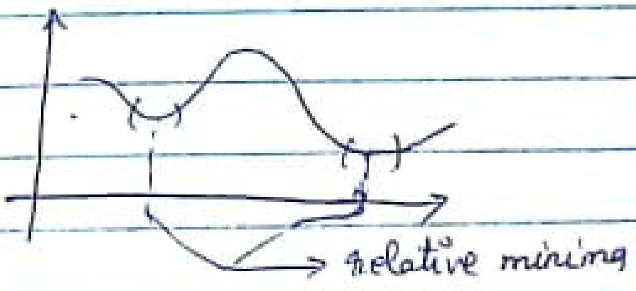


$f(c) \geqslant f(x)$ for all x in (a,b)



. $f(c) \geqslant f(x)$ for all x in (a,b).

- Relative Minimum: A function f has a relative minimum at $x=c$ if there exists an open interval (a,b) containing c such that $f(x) \geqslant f(c)$ for all x in (a,b).



Derivatives are again going to be crucial to figure out relative max/min.

Definition: Critical Number/Point of f.

The values of x in the domain of f where either $f'(x) = 0$ or where $f'(x)$ DNE, are called the critical points of f

Algorithm to find Relative Extrema of a continuous function.

① Determine critical points of f.

② Determine the sign of $f'(x)$ to the left and right of each critical number.

ⓐ If $f'(x)$ changes sign from positive to negative as we move across a crit. no. c, then f has a relative maximum at $x = c$.

ⓑ If $f'(x)$ changes sign from negative to positive as we move across a crit. pt. c, then f has a relative min at $x = c$.

ⓒ If $f'(x)$ does not change sign as we move ...., f does not have a relative extremum at $x = c$.

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Note: This is essentially doing the same thing as before (drawing sign chart & ..., except that we are taking into account the point where $f'(x)$ DNE instead of f' being discontinuous)

We are using the following :

~~[illegible] Find critical points~~

$\boxed{\text{Thm}}$. If f is continuous on (a, b), $c \in (a,b)$ and $f(c)$ is a local/(relative) extremum, then either $f'(c)=0$ or $f'(c)$ DNE.

Thus, relative extrema can occur only at critical points.

(IMPORTANT For True/False)

However, being a critical point does not always give a local extremum.

e.g. $f(x) = x^3+2$.

$f'(x) = 3x^2$. So, f' always exists and 0 is the only critical point. But $f(0)$ is not a local max/min.

The algorithm we just described is called

'<u>First - derivative Test for Local Extrema</u>'.

e.g. Find relative max/min of the function

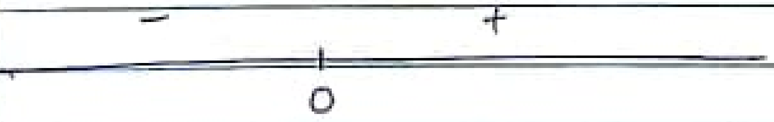
$f(x) = x^{2/3}$.

<u>Soln:</u> • f is defined everywhere

• $f'(x) = \frac{2}{3} x^{-1/3}$ So, $f'(x) = 0$ has no solⁿ.

However, f' does not exist at $x = 0$.

Thus, 0 is the only critical point.



$f'(1) = \frac{2}{3} > 0$

$f'(-1) = -\frac{2}{3} < 0$

So, at $\overset{x=0}{0}$ there is a relative minimum of f } Both steps are required in answer

The relative minimum value is $f(0) = 0$.

Important

e.g $f(x) = \frac{9}{x} + x$. Find critical points & locate min/max.

Soln: $f'(x) = -\frac{9}{x^2} + 1$.

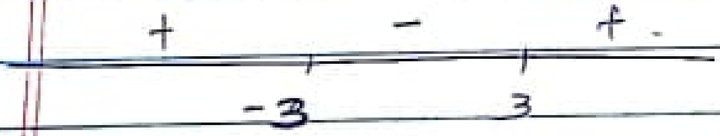
f' is not defined at 0 but 0 is not in the domain of f. So, $x = 0$ is not a critical point.

$f'(x) = 0$. $\Rightarrow -\frac{9}{x^2} + 1 = 0$

$\Rightarrow 1 = \frac{9}{x^2}$

$\Rightarrow x = \pm 3$

Thus, critical points are $x = 3, -3$.



$f'(-4) = -\frac{9}{16} + 1 > 0$.

$f'(1) = -9 + 1 < 0$

$f'(4) = -\frac{9}{16} + 1 > 0$.

Thus, from sign chart, we have. at $x = -3$, f has a relative max.

at $x = 3$ f has a relative min.

$f(-3) = \frac{9}{-3} - 3 = -6$ is a relative max value

$f(3) = \frac{9}{3} + 3 = 6$ is a " min. value.

e.g. $f(x) = x^3 - 12x^2 + 36x$. Find relative max/min.

Exercise

4.2 Applications of 2nd derivative

Note that provided f' is friendly, we can talk about f''. Hence we can apply same treatment to f' now as we did to f to figure out where f' is increasing/decreasing etc. etc.

Definition: Given a function $f(x)$, we say the graph of f is concave up (respectively, concave down) on the interval (a,b) if $f'(x)$ is increasing i.e. $f''(x) > 0$ on (a,b) (resp. $f'(x)$ is decreasing, i.e. $f''(x) < 0$ on (a,b)).

$f''(x)$	$f'(x)$	graph of f
+	increasing	concave up
−	decreasing	concave down

Definition

An inflection point $(a, f(a))$ of f is a point in the graph ~~domain~~ of f where concavity changes.

What happens at an inflection point?

Theorem

If $y = f(x)$ is continuous on (a, b) and has an inflection point at $x = c \in (a, b)$, then either $f''(c) = 0$ or $f''(c)$ DNE. ~~(i.e. it is sort of a point of relative max)~~

Note $f''(c) = 0$ does not imply that c is an inflection point. We have to check concavity change.

So, here's an algorithm:

Finding Inflection Points.

① Compute $f''(x)$.

② Solve $f''(x) = 0$ and also ~~g~~ find x such that $f''(x)$ DNE.

③ ~~Determine sign of $f''(x)$~~ Draw Sign Chart for $f''(x)$:

if there is a change in sign as we move across a particular point c, then $(c, f(c))$ is a inflection point.

e.g Find inflection point for $g(x) = \frac{1}{x-3}$

$g'(x) = -\frac{1}{(x-3)^2}$; $g''(x) = -\frac{2}{(x-3)^3}$

$g''(x) \neq 0$ for all x in the domain of g.

g'' is not defined at $x = 3$. but $x = 3$ is not in the domain of g.. So, 3 can't be included in our discussion.

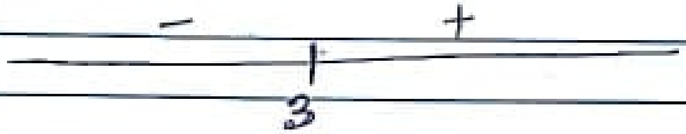
So, g has no inflection point.

[e.g.] Find inflection points for $f(x) = x^3 - 9x^2 + 24x - 10$.

Sol^n: $\cdot$ $f'(x) = 3x^2 - 18x + 24$

$\cdot$ $f''(x) = 6x - 18$.

So, $f''(x) = 0$ when $x = 3$. ; f'' exists everywhere.
(Thus 3 is the only candidate for checking)



$f''(2) = 12 - 18 < 0$
$f''(4) > 0$.

Thus, $(3, f(3))$ is an I.P. (inflection point).

Second-Derivative Test for Local Max/Min.

① Compute $f'(x)$, $f''(x)$.

② Compute all critical numbers of f at which $f'(x) = 0$.

③ Compute $f''(c)$ for such critical number c.
- ⓐ If $f''(c) < 0$, then f has a relative maximum at c.
- ⓑ If $f''(c) > 0$, then f has a relative minimum at c.
- ⓒ If $f''(c) = 0$ or $f''(c)$ DNE, then Test is inconclusive.

(e.g). Find relative max/min of the function:

$$f(x) = \frac{9}{x} + x.$$

<u>Soln</u>: $f'(x) = -\frac{9}{x^2} + 1.$ So, $f'(x) = 0$ at $x = 3, -3$.

$f''(x) = \frac{18}{x^3}$; $f''(3) > 0$ so, at $x = 3$ f has a local min with value $f(3) = 6$

$f''(-3) < 0$. So, at $x = -3$ f has a local max. with value -6.

First-Derivative Test is stronger than 2nd derivative Test.

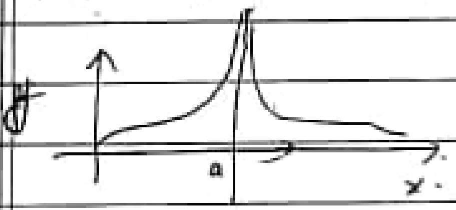
e.g. $f(x) = x^{2/3}$. 2nd-derivative test can't be applied!!

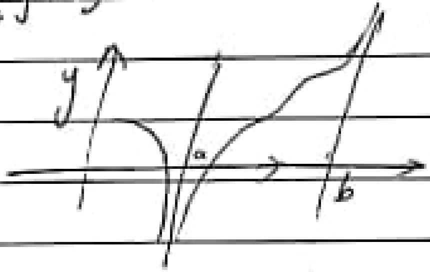
$\boxed{4.3}$ Curve-Sketching

We'll use the tools we've developed so far to graph functions now.

We need two more tools:

$\boxed{\text{Defn}}$ The vertical line $x=a$ is a vertical asymtote for the graph of $y=f(x)$ if $\lim_{x\to a^+} f(x)=\pm\infty$ or $\lim_{x\to a^-} f(x)=\pm\infty$

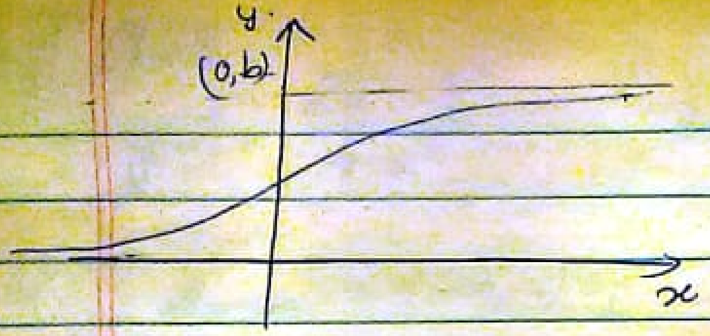




(Theorem) If $f(x)=\dfrac{p(x)}{q(x)}$ is a rational function,

and if $q(c)=0$, $p(c)\neq 0$, then $x=c$ is a vertical asymtote of f.

$\boxed{\text{Defn.}}$ The horizontal line $y=b$ is a horizontal asymptote for the graph of f if $\lim_{x\to\infty} f(x)=b$ or $\lim_{x\to-\infty} f(x)=b$



y=b & y=0 are horiz. asymp...

Facts: Polynomials of degree 1 or greater never have horizontal asymptotes as they go to -∞ or ∞ as x blows up or becomes extremely small.

- A rational function can have only one horizontal asymptote.

~~Graphing Strategy.~~

- The graph of a polynomial function never has a vertical or horizontal asymptote.

Graphing Technique.

① Determine domain of f.

② Find x & y intercepts ~~of~~ of f.

③ Determine the horizontal and vertical asymptotes of f.

④ Analyze $f'(x)$: find intervals where f is increasing, decreasing ; find critical points ; find relative extrema.

⑤ Analyze $f''(x)$: Find concavity ; inflection points.

⑥ Sketch the graph:
  ⓐ Draw asymptotes
  ⓑ Locate intercepts, local extrema & inflection points.
  ⓒ Fill in gaps using info from Steps 1-4.
  Wherever unsure, compute values at some points & connect dots.

$\boxed{\text{e.g}}$ Sketch

$$g(x) = \frac{2x^2+5}{4-x^2}$$

Soln. · Domain $= (-\infty,-2) \cup (-2,2) \cup (2,\infty)$.

- $2x^2+5 \neq 0$ anywhere. So, there is no x-intercept ;
  $g(0) = \frac{5}{4}$. Thus $\left(0, \frac{5}{4}\right)$ is the y-intercept.

- Now, $4-x^2=0 \Rightarrow x = \pm 2$ and. $2x^2+5 \neq 0$. Thus, $x=2$ and $x=-2$ are vertical asymptotes of g. $\left[\begin{array}{l} \lim_{x\to 2^-} f(x) = \infty \\ \lim_{x\to 2^+} f(x) = -\infty \end{array}\right.$

- $\lim_{x\to\infty} \frac{2x^2+5}{4-x^2} = \lim_{x\to\infty} \frac{2+\frac{5}{x^2}}{\frac{4}{x^2}-1} = -2$;
  $\lim_{x\to-\infty} \frac{2x^2+5}{4-x^2} = -2$. ;

  Thus, $y=-2$ is the horizontal asymptote.

- $$g'(x) = \frac{(4-x^2)4x + 2(2x^2+5)x}{(4-x^2)^2}$$

$$= \frac{16x - 4x^3 + 4x^3 + 10x}{(4-x^2)^2} = \frac{26x}{(4-x^2)^2}.$$

$g'(x) = 0 \Rightarrow x = 0$. [2, −2 are not in domain]

Thus, $x=0$ is the critical point.

$-$ | $+$
0

$g'(-1) < 0$
$g'(1) > 0$.

Thus, at $x = 0$, there is a relative minimum.

- $$g''(x) = \frac{(4-x^2)^2 26 - 26x(2(4-x^2)(-2x))}{(4-x^2)^4}$$

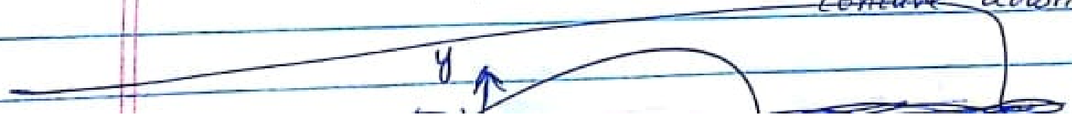
$$= \frac{(4-x^2)(26)[4-x^2+4x^2]}{(4-x^2)^4} = \frac{26(3x^2+4)}{(4-x^2)^3}$$

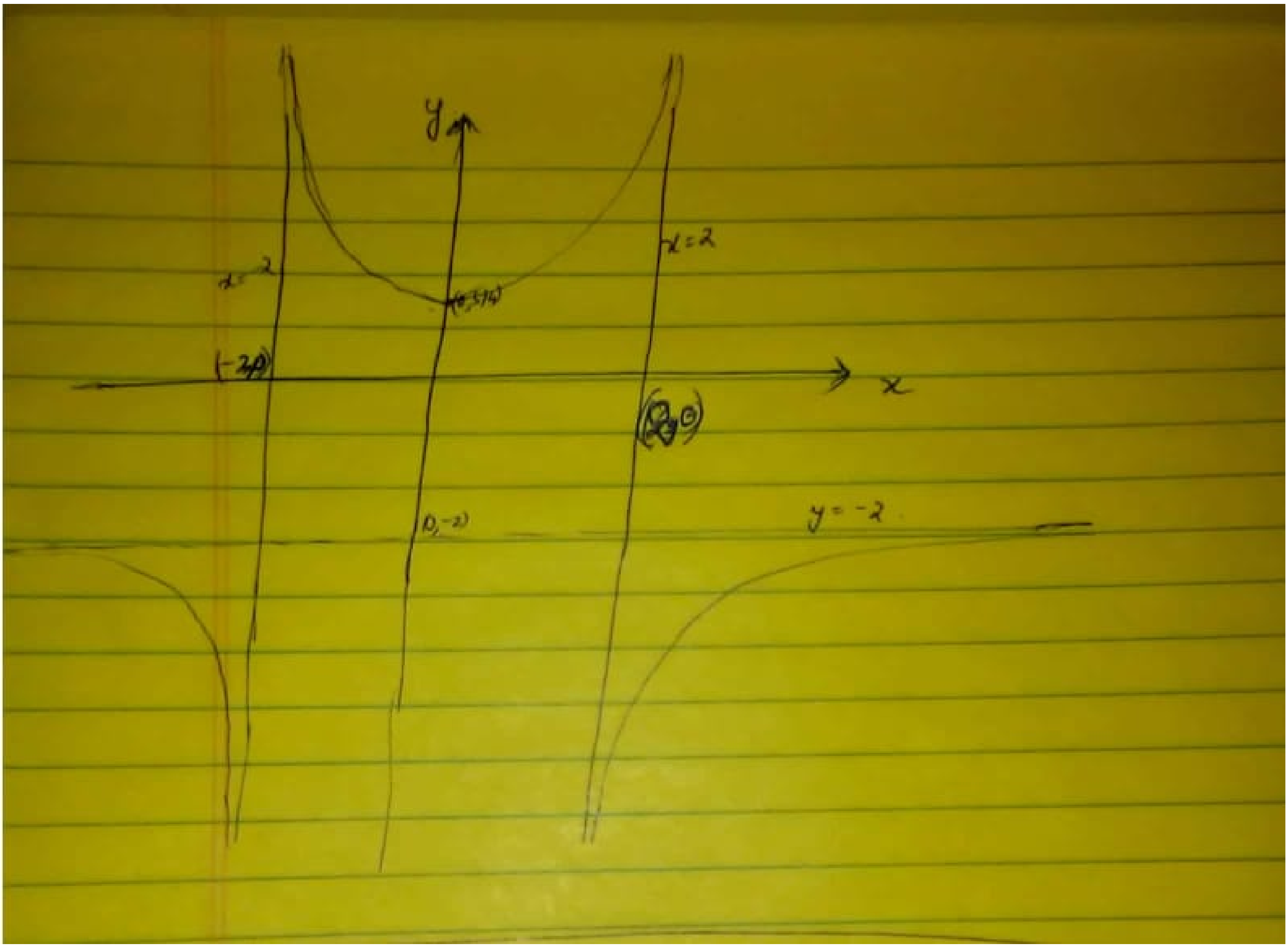
- $g''(x) > 0$ for all $x \neq \pm 2$. (not in the domain).

No inflection point.. ; always concave up for $x \in (-2, 2)$

Concave down for $x \in (2, \infty)$

$x \in (-\infty, -2)$.




y
x=-2
(0,3/4)
x=2
(-2,0)
x
(2,0)
(0,-2)
y=-2