5.2 Logarithmic Functions

5.2.1 What are logarithms and logarithmic functions?

- For $b > 0, b \neq 1$ we say $\log_b x = y$ if $x = b^y$.
- The function $f(x) = \log_b x$ is called the logarithmic function with base b.

So, we see that $x = b^{\log_b x}$ for all x in the domain of the logarithmic function.

Notation:

- $\log x$ means $\log_{10} x$
- $\ln x$ means $\log_e x$— this is called the **Natural Logarithm**.

Example 1. *Evaluate the following:*

(a) $\log_2 8$.

Solution: Let $\log_2 8 = x$. *So we have* $2^x = 8$ *which implies that* $x = 3$. *Hence,* $\log_2 8 = 3$.

(b) $\log_9 27$

Solution: Let $\log_9 27 = x$. *Thus,* $9^x = 27 \implies 3^{2x} = 27 \implies 2x = 3 \implies x = \frac{3}{2}$. *So,* $\log_9 27 = \frac{3}{2}$.

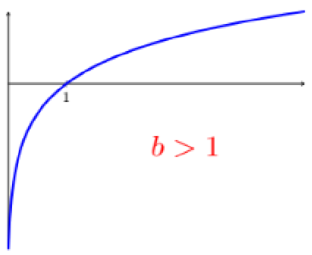
(c) $\ln e^2 = 2$ *(Check yourself!),*

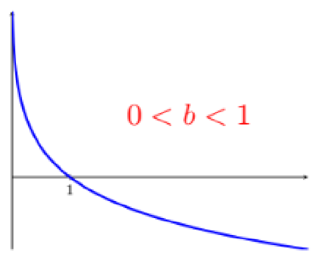
(d) $\ln 1 = 0$ *(Check yourself!) — this is a very important observation and we will use this often.*

(e) $\ln(-63)$ *is undefined as* $e^x > 0$ *for all real numbers* x .

Properties of Logarithmic Functions:

(1) The domain of $\log_b$ is $(0, \infty)$.
(2) The range of $\log_b$ is $(-\infty, \infty)$.
(3) The x-intercept of $\log_b$ is 1 because $b^0 = 1$.
(4) $\log_b$ is continuous.
(5) $\log_b$ is increasing on $(0, \infty)$ if $b > 1$ and is decreasing on $(0, \infty)$ if $0 < b < 1$.
(6) Its graph has $x = 0$ as a vertical asymptote.





Laws of Logarithms: If b, x, and y are positive real numbers and $b \neq 1$, then

(1) $\log_b(xy) = \log_b(x) + \log_b(y)$
(2) $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
(3) $\log_b(x^y) = y \log_b(x)$.
(4) $\log_b(1) = 0$
(5) $\log_b(b) = 1$

Example 2. *Rewrite* $2\ln a - 3\ln b$ *as the logarithm of a single quantity.*

Solution: $2\ln a - 3\ln b = \ln a^2 - \ln b^3 = \ln\left(\frac{a^2}{b^3}\right)$.

Example 3. *Expand and simplify* $\ln\frac{\sqrt{x}e^x}{(2+x^2)^2}$.

Solution:

$$\begin{aligned}\ln\frac{\sqrt{x}e^x}{(2+x^2)^2} &= \ln(\sqrt{x}) + \ln e^x - \ln(2+x^2)^2\\ &= \frac{1}{2}\ln x + x\ln e - 2\ln(2+x^2)\\ &= \frac{1}{2}\ln x + x - 2\ln(2+x^2)\end{aligned}$$

5.2.2 Logarithms and Exponentials

Logarithms and exponentials are inverse functions of each other: f, g are inverse functions of each other if $(f\circ g)(x) = x$ for all x in the domain of g; and $(g\circ f)(x) = x$ for all x in the domain of f.

Let $f(x) = \log_b(x)$, $g(x) = b^x$. Then $(f\circ g)(x) = \log_b(b^x) = x\log_b b = x$ for all $x\in(-\infty,\infty)$; $(g\circ f)(x) = b^{\log_b x} = x$ for all $x\in(0,\infty)$.

In particular, we have $\ln e^x = x$ for all $x\in(-\infty,\infty)$; $x = e^{\ln x}$ for all $x\in(0,\infty)$.

Example 4. *(a) Solve the equation* $5e^{3x} + 3 = 7$. *(b)* $4\ln(8+x) = 2\ln(16x+164)$

Solution a:

$5e^{3x} + 3 = 7$
$\implies 5e^{3x} = 4$
$\implies e^{3x} = \frac{4}{5}$
$\implies 3x = \ln(\frac{4}{5})$
$\implies x = \frac{1}{3}\ln(\frac{4}{5})$

Solution b:

Example 5. *Find the domain of* $f(x) = \ln x - \ln(2-x)$.

Solution:

5.4 Derivative of e^x and $\ln x$

We have already discussed the derivative of $e^{f(x)}$. It is $e^{f(x)}f'(x)$.

- $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ for all $x\in(-\infty,0)\cup(0,\infty)$.
- $\frac{d}{dx}(\ln x) = \frac{1}{x}$ for all $x\in(0,\infty)$.
- If f is a function that is always positive, then using chain rule, $\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}$.

Example 6. *Differentiate* $(\ln(x^2+5)^3)^5$. ***Solution:*** $5(3\ln(x^2+5))^4\frac{6x}{x^2+5}$.