Some Solved Problems

First Order Linear O.D.E. (Read Pages 29-31)

Example:

$$\frac{dy}{dt} + \frac{2}{t}y = t^2, y(1) = 2$$

Solution: The integrating factor is $e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = e^{\ln t^2} = t^2$. Thus we get

$$t^2\frac{dy}{dt} + 2t = t^4$$

$$\implies \frac{d}{dt}(t^2y) = t^4.$$

Now integrating we get, $t^2y = \frac{t^5}{5} + c$. Plugging in initial conditions, we have $1.2 = \frac{1}{5} + c$ which implies $c = \frac{9}{5}$. Thus, $y(t) = \frac{t^3}{5} + \frac{9}{5t^2}$.

■

First Order Linear O.D.E. Continued (Read Pages 31-32)

Example: $\frac{dy}{dx} + 2xy = 3, y(2) = 3$.

Solution: Here the integrating factor is $e^{\int 2xdx} = e^{x^2}$. Thus, we get,

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = 3e^{x^2}$$

$$\implies \frac{d}{dx}(ye^{x^2}) = 3e^{x^2}$$

Here the R.H.S. cannot be simply integrated so we will use fundamental theorem of calculus with base point as the initial x value. Thus, we have

$$ye^{x^2} = 3\int_2^x e^{t^2}dt + C.$$

Plugging in initial conditions we have

$$3e^4 = 3\int_2^2 e^{t^2}dt + C = 3.0 + C$$

$$\implies C = 3e^4$$

Thus our solution is

$$y(x) = e^{-x^2}(3\int_2^x e^{t^2}dt + 3e^4).$$

■

Separable Equations (Read pages 40-42)

The normal separable initial value problem is easy to understand. For example,

$$\frac{dy}{dt} = \frac{t^3 + t}{y^3}, y(1) = 2$$

can easily be solved as the integrand on both sides are easy to compute.

So instead we look at something where integrals on both sides are not easy and we apply the approach discussed on Page 42.

$$\frac{dy}{dt} = \frac{1}{3}y^{-2}\sin(t^2), y(1) = 2.$$

Separating variables we have

$$3y^2dy = \sin(t^2)dt$$

$$\implies 3\int_2^y s^2ds = \int_1^t \sin(u^2)du$$

$$\implies y^3 - 8 = \int_1^t \sin(u^2)du$$

$$\implies y^3 = 8 + \int_1^t \sin(u^2)du$$

$$\implies y = (8 + \int_1^t \sin(u^2)du)^{\frac{1}{3}}.$$

Thus

$$y(t) = (8 + \int_1^t \sin(u^2)du)^{\frac{1}{3}}.$$

■

Read Theorem 2.4.4

Example: $\ln(t+3)\frac{dy}{dt} + (\sin t)y = e^t, y(1) = 2$. For this we need to find the largest open interval containing initial value such that the given problem has a unique solution.

Solution: Notice that this is a first order linear O.D.E. SO we apply Theorem 2.4.4.

$$\frac{dy}{dt} + \frac{\sin t}{\ln(t+3)}y = \frac{e^t}{\ln(t+3)}$$

and we need the largest interval containing initial t-value (i.e. 1) on which both $\frac{\sin t}{\ln(t+3)}$ and $\frac{e^t}{\ln(t+3)}$ are continuous. We can see that $(-2, \infty)$ solves our purpose. ■

Read Theorem 2.4.5 and 2.4.6

Example 1: $\frac{dP}{dt} = P - 2P^2, P(0) = 1$. Again need to find some domain where existence and uniqueness are guaranteed if possible.

Solution: Note that this is separable indeed but is not a first order O.D.E. So we have to resort to Theorem 2.4.5. First find a rectangle which contains the point $(0, 1)$ (this is our initial point in this case) on which $P - P^2$ is continuous. Clearly, the rectangle $(-\infty, \infty) \times (-\infty, \infty)$ solves it. (the first rectangle is chosen for the choice of t, the second for the choice of $P(t)$). Thus Theorem 2.4.5 ensures the existence of a solution.

Now for uniqueness we have to look at $\frac{\partial}{\partial P}(P - 2P^2) = 1 - 4P$ and this is clearly continuous on our earlier chosen rectangle. Thus, there is some open interval containing $t = 0$ where there exists a unique solution. ■

Example 2:

$$\frac{dy}{dx} = \frac{x}{y^2 - 4}, \ y(1) = 2.$$

Solution: Here in the notation of the book $f(x, y) = \frac{x}{y^2-4}$. This is not a linear O.D.E. So we have to apply Theorem 2.4.5. First we need to find rectangle containing $(1, 2)$ on which f is continuous. One can see that on the rectangles $(-\infty, \infty) \times (-\infty, -2)$, $(-\infty, \infty) \times (-2, 2)$, $(-\infty, \infty) \times (2, \infty)$, f is continuous but they do not contain the initial point $(1, 2)$. Thus, existence cannot be guaranteed.

Now $\frac{\partial f}{\partial y} = \frac{-2xy}{(y^2-4)^2}$ is also continuous on the above rectangles but none of them contain the initial point. Hence no conclusion can be made from Theorem 2.4.5 in this case.

However had the initial condition been $y(2) = 1$, then we can see that both f and $\frac{\partial f}{\partial y}$ are continuous on $(-\infty, \infty) \times (-2, 2)$ which contains the point $(2, 1)$ in this case. Thus existence is guaranteed, and also uniqueness on some interval containing $x = 2$ is guaranteed by Theorem 2.4.5. ■

Read up Sections 2.4.-2.6 which talk about some population models, financial models, phase line

In case of the phase line, do understand what it means to have equilibrium solutions and limiting solutions. For example, given phase line, you have to conclude $\lim_{t\to\infty} y(t)$ and stuff like that.

Change of Variable

Homogeneous (Page 125)

Example: $\frac{dy}{dx} = \frac{x^2+y^2}{xy-x^2}$.

Solution: We have

$$\frac{dy}{dx} = \frac{1 + \frac{y^2}{x^2}}{\frac{y}{x} - 1}$$

CLearly this is homogeneous and hence we substitute $v = \frac{y}{x}$. So we have $xv = y$. Upon differentiating with respect to x, we have $x\frac{dv}{dx} + v = \frac{dy}{dx}$. Substitute this in the original equation,

$$x\frac{dv}{dx} + v = \frac{1+v^2}{v-1}$$

$$\implies x\frac{dv}{dx} = \frac{1+v^2}{v-1} - v$$

$$\implies x\frac{dv}{dx} = \frac{1+v}{1-v}$$

This is separable so now solve it. Then finally once we have our answer, we again change the variable v to $\frac{y}{x}$ and write out the answer in terms of y and x. ■

Bernoulli (Page 126)

Example:

$$\frac{dy}{dt} = \frac{-y}{t} + \frac{t-1}{2y}.$$

Solution: We have

$$\frac{dy}{dt} + \frac{1}{t}y = \frac{t-1}{2}y^{-1}.$$

Thus we have a Bernoulli equation with $b = -1$. So let $v = y^{1-(-1)} = y^2$. Sp, $\frac{dv}{dt} = 2y\frac{dy}{dt}$ Multiply original equation throughout by $y^{-(-1)}$ to get

$$y^{\frac{dy}{dt}} + \frac{1}{t}y^2 = \frac{t-1}{2}.$$

Changing everything to v, we have

$$\frac{1}{2}\frac{dv}{dt} + \frac{1}{t}v = \frac{t-1}{2}$$

$$\implies \frac{dv}{dt} + \frac{2}{t}v = t-1.$$

This is a first order linear O.D.E. with integrating factor e^{t^2}. It has solution

$$v = \frac{t^2}{4} - \frac{t}{3} + \frac{c}{t^2}$$

. Since $v = y^2$, we have

$$y^2 = \frac{t^2}{4} - \frac{t}{3} + \frac{c}{t^2}.$$

■

Change in Independent Variable (Page 127-128)

The basic fact here is the application of the chain rule.

Example: Make the change of independent variable $t = x^2$ in

$$\frac{dy}{dx} = 2x^3 + 4xy + 2x.$$

Solve the new equation, and then give the solution in terms of the original variable.

Solution: $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt}2x$. Rewrite the DE as $2t^{\frac{1}{2}}\frac{dy}{dt} = 2t^{\frac{3}{2}} + 4t^{\frac{1}{2}}y + 2t^{\frac{1}{2}}$ or $\frac{dy}{dt} = t + 2y + 1$. This is a linear equation with integrating factor e^{-2t}, and we have $y = -\frac{3}{4} - \frac{1}{2}t + ce^{2t}$ upon solving. Converting to original variables, we have

$$y(x) = -\frac{3}{4} - \frac{1}{2}x^2 + ce^{2x^2}.$$

■

Exact Equations (Read the squared portion on Page 137)

Example: Find a function $f(x)$ such that

$$y^2 \cos x + yf(x)\frac{dy}{dx} = 0$$

is exact. Solve the resulting equation.

Solution: We want

$$\frac{\partial}{\partial y}(y^2 \cos x) = \frac{\partial}{\partial x}(yf(x)),$$

so, $2y \cos x = yf'(x)$. This says that $f'(x) = 2\cos x$ and the choice $f(x) = 2\sin x + c$ makes the equation exact for any constant c. For convenience we choose $c = 0$.

Now we need a function $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = y^2 \cos x, \ \frac{\partial \phi}{\partial y} = 2y \sin x$$

and then our answer will be $\phi(x, y) = k$ for any constant k. (This gives us an implicit solution to the DE)

Integrating the first one with respect to dx, we have

$$\phi(x, y) = y^2 \sin x + h(y)$$

where h is a function of y. (We have this as we assumed y is constant during the integration, so $h(y)$ serves as the constant of integration here).

Now differentiate the obtained $\phi(x, y)$ with respect to y and equate it to the second requirement. Thus,

$$2y \sin x + h'(y) = 2y \sin x$$

and this shows that $h'(y) = 0$. Thus for convenience we choose $h(y) = 0$ -the zero function.

Hence our solution is $y^2 \sin x = k$ for any constant k. ■

Integrating factors to make equations exact (Pages 139-142)

Example: Show that ye^x is an integrating factor of

$$xy + y + 2x\frac{dy}{dx} = 0$$

and solve it.

Solution: First multiply throughout by ye^x to get $xy^2e^x + y^2e^x + 2xye^x \frac{dy}{dx} = 0$. Now our $M(x, y) = xy^2e^x + y^2e^x$ and $N(x, y) = 2xye^x$. We need to check exactness.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2xye^x + 2ye^x$$

and hence exact. Thus it is indeed an integrating factor.

Now we need a function $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = M(x, y) = xy^2e^x + y^2e^x, \ \frac{\partial \phi}{\partial y} = N(x, y) = 2xye^x$$

and then our answer will be $\phi(x, y) = k$ for any constant k.

Looking at the second requirement and integrating with respect to dy we have

$$\phi(x, y) = \int 2xe^x y dy = xe^x y^2 + f(x)$$

for some function $f(x)$. Then use the first requirement on ϕ to see that

$$xe^x y^2 + e^x y^2 + f'(x) = y^2 xe^x + y^2 e^x$$

and hence we can choose $f(x) = 0$. Thus the implicit solution is $xe^x y^2 = k$ for any constant k. ■

Differentials Page 142-147

Here the crucial facts are all the equations which are numbered on the mentioned pages.

Example 1: Solve the equation

$$(x^2 - y^2)dy = 2xydx$$

by treating y as the independent variable.

Solution: We need $\frac{dx}{dy}$. So from the equation we have

$$\frac{dx}{dy} = \frac{x^2 - y^2}{2xy} = \frac{x}{2y} - \frac{y}{2x}$$

This is a homogeneous equation, so substitute $v = \frac{x}{y}$ and solve accordingly and then change back again to initial variables y,x. ■

Example 2: Solve the initial value problem

$$ydx - xdy = x^3y^5(ydx + xdy),\ y(4) = \frac{1}{2}$$

by recognizing differentials.

Solution: Note that we need to be aware of the differentials in general as are written on Page 147. Let us follow the numbering in that chart on Page 147.

Divide the equation by y^2 and use Common Differential 2 to get

$$\frac{ydx - xdy}{y^2} = (xy)^3 d(xy).$$

Now using common differential 9 with $\phi(x, y) = xy, h'(t) = t^3$, this is

$$\frac{ydx - xdy}{y^2} = h'(xy)d(xy) = d(h(xy)) = d\Big(\frac{(xy)^4}{4}\Big).$$

Since the left side is $d\Big(\frac{x}{y}\Big)$, we have

$$d\Big(\frac{x}{y}\Big) = d\Big(\frac{(xy)^4}{4}\Big)$$

$$\implies d\Big(\frac{x}{y} - \frac{(xy)^4}{4}\Big) = 0$$

$$\implies \frac{x}{y} - \frac{(xy)^4}{4} = c.$$

Using initial conditions, we have $c = 4$. Thus an implicit solution to the IVP is

$$\frac{x}{y} = \frac{(xy)^4}{4} + 4.$$

■