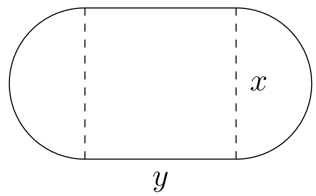
Applied Optimization

Problem 1. The port-side window on a submarine is to have the shape of a rectangle bordered by two semicircles (as the picture shows). Navy regulations dictate that the perimeter of the window must be 10 feet. Find the dimensions x and y that allow the greatest amount of light through.



Problem 2. You are building a wooden box for your grandfather's birthday. The square base will be made of mahogany, which costs 35 cents per square foot, while the sides are to be built from oak, which costs 15 cents per square foot. There is no top. For the box to measure 20 cubic feet in volume, what are the dimensions of the box that minimize cost?

Problem 3. A family wants to set up a campsite next to a river and surround it with fencing to keep the animals out at night. Using the river as one side, they will fence off the other three sides of a rectangular enclosure with 100 yards of fencing. The enclosure must also be able to fit a circular firepit somewhere, which measures 20 yards in diameter. Find the length and width of the rectangular campsite which maximizes the family's camping space. (Hint: Be careful what you choose for a *closed interval*.)

Problem 4. An Apple engineer is designing the screen for a new iPad. The top and bottom of the screen must each be 1 inch away from the top and bottom of the iPad's case. Likewise, the left and right sides of the screen must each be 2 inches away from the left and right sides of the case. In total, the case has a perimeter of 40 inches. Help the engineer determine the dimensions of the iPad with maximum screen area. (Hint: Draw it!)

Exponential Growth and Decay

Problem 5. A bacteria colony grows exponentially. Suppose the initial population is 1000 bacteria and that after 2 hours, there are 3500 bacteria.

(a) Find a formula for the population of the colony after t hours.

(b) How many bacteria are present after 5 hours?

(c) How long does it take the colony to reach 10 times its initial size?

Problem 6. Carbon-14 is used in carbon-dating of archaeological discoveries, such as ancient bones. An extinct Sumerian village's remains are found to have one-twentieth of the Carbon-14 originally present. If Carbon-14 has a half-life of 5730 years, how long has the village been extinct?

Related Rates

Problem 7. On a major league baseball diamond, the distance from home to first is 90 feet. (Since a baseball diamond is really a square, the rest of the side lengths are 90 feet as well.) Dexter Fowler hits a line drive and begins running towards first base at a speed of 24 feet per second.

(a) At what rate is Dexter's distance from second base decreasing when he is halfway to first base?

(b) At the same moment, what is the rate of change of his distance from third base?

Problem 8. A car salesperson uses the following model to predict how interest rates influence monthly sales of new cars:

$$C = \frac{150000}{\sqrt{r^2+5}} - \frac{4900r^2}{3},$$

where r is the interest rate (as a decimal) and C is the number of new cars sold per month.

(a) If the current interest rate is 4% and the interest rate is changing at 0.8% per month, how fast is the rate of car sales changing?

(b) Using this rate of change as an approximation tool, how many cars should the salesperson expect to be sold in the next month? (Hint: Linear approximation.)

Closed interval method

Problem 9. For each of the following, indicate whether the statement is **True** or **False**. If true, provide a brief justification for your response. If false, provide a specific counterexample.

(a) A function $f(x)$ defined on a closed interval $[a, b]$ must have an absolute minimum and an absolute maximum on the interval.

(b) A function $f(x)$ which is continuous on a closed interval $[a, b]$ must have an absolute minimum and an absolute maximum on the interval.

(c) If $f(x)$ is not continuous on $[a, b]$, then $f(x)$ does NOT have any absolute extrema on the interval.

Problem 10. The altitude of a rocket in feet after t seconds of flight is given by

$$f(t) = -t^3 + 54t^2 + 480t + 10.$$

(a) Assume the rocket takes off at time $t = 0$. Write a closed interval on which the rocket's flight takes place.

(b) Find all absolute maxima and minima of the rocket's altitude on the closed interval you found in part (a). Interpret these.

(c) Find any inflection points of $f(t)$. Interpret these.

Logarithmic Differentiation

Problem 11. Differentiate each of the following expressions. Some may require logarithmic differentiation, while others may not.

(a) $(\ln x)^x + x^{\ln x}$

(b) $10^x 5^{-x} 2^{-x}$

(c) $e^{\sqrt{\ln(x^2)}}$

(d) x^{2x+1}

(e) $\sqrt{x}^{\sqrt{x}} + e^e$

(f) $(3x^2 + 22)^{1/x}$

Problem 12. Find $f'(x)$ where

$$f(x) = (1 + x^2)^{\ln(1-x)}.$$

Problem 13.

i. For what values of a is $f(x) = \left(\log_{\frac{1}{2}} a\right)^x$ decreasing?

ii. For what values of a is $f(x) = (\log_3 a)^x$ decreasing? For which are they increasing?