1. Find a function $f(x)$ so that

$$y^2\cos x + yf(x)\frac{dy}{dx} = 0$$

is exact. Solve the resulting equation.

Soln:

We want

$$\frac{\partial}{\partial y}\left(y^2\cos x\right) = \frac{\partial}{\partial x}\left(yf(x)\right)$$

so that $2y\cos x = yf'(x)$. This says $f'(x) = 2\cos x$ and the choice $f(x) = 2\sin x + c$ makes the equation exact. We'll set $c = 0$ for convenience. A function $\varphi(x, y)$ satisfying $\frac{\partial\varphi}{\partial x} = y^2\cos x$ and $\frac{\partial\varphi}{\partial y} = 2y\sin x$ is $\varphi(x, y) = y^2\sin x$. Thus the DE has implicit solution $y^2\sin x = k$ for any constant $k$.

2.

$$\frac{dy}{dt} = \frac{-y}{t} + \frac{t-1}{2y}.$$

Soln:

Write the DE as

$$\frac{dy}{dt} + \frac{1}{t}y = \frac{t-1}{2}y^{-1}$$

to recognize it as a Bernoulli equation with $p(t) = \frac{1}{t}$, $q(t) = \frac{t-1}{2}$ and $b = -1$. Making the substitution $v = y^2$ converts the DE to

$$\frac{dv}{dt} + \frac{2}{t}v = t - 1.$$

This is a linear DE with integrating factor

$$e^{\int \frac{2}{t}\,dt} = e^{2\ln t} = t^2.$$

It has solution

$$v = \frac{t^2}{4} - \frac{t}{3} + \frac{c}{t^2}.$$

Since $v = y^2$, the original DE has implicit solution

$$y^2 = \frac{t^2}{4} - \frac{t}{3} + \frac{c}{t^2}.$$

3.

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy - x^2}.$$

Soln:

Write

$$\frac{x^2 + y^2}{xy - x^2} = \frac{1 + \frac{y^2}{x^2}}{\frac{y}{x} - 1}$$

and make the substitution $v = y/x$. The DE becomes

$$x\frac{dv}{dx} + v = \frac{1 + v^2}{v - 1}$$

which simplifies to

$$x\frac{dv}{dx} = \frac{1 + v}{v - 1}.$$

Separating variables gives

$$\int \left(\frac{v+1}{v+1} - \frac{2}{v+1}\right) dv = \int \frac{1}{x} dx$$

so that $v - 2\ln|v+1| = \ln|x| + c$. In terms of the original variables we have the implicit solution

$$\frac{y}{x} - 2\ln\left|\frac{y}{x} + 1\right| = \ln|x| + c.$$

4. Make the change of independent variable $t = x^2$ in the equation

$$\frac{dy}{dx} = 2x^3 + 4xy + 2x.$$

Solve the new equation, and then give the solution in terms of the original variables.

Soln:

With $t = x^2$ we have $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = 2x\frac{dy}{dt}$. Rewrite the DE as $2t^{1/2}\frac{dy}{dt} = 2t^{3/2}+4t^{1/2}y+2t^{1/2}$ or $\frac{dy}{dt} = t+2y+1$. This is a linear equation with integrating factor $e^{-2t}$, and we have $y = -\frac{3}{4} - \frac{1}{2}t + ce^{2t}$, or $y = -\frac{3}{4} - \frac{1}{2}x^2 + ce^{2x^2}$.

5. Solve the equation

$$(x^2 - y^2)dy = 2xydx$$

by thinking of $y$ as the independent variable, and $x$ as the dependent variable.

Soln: (c) With $x$ as the dependent variable, write the DE as

$$\frac{dx}{dy} = \frac{x^2 - y^2}{2xy}$$

or

$$\frac{dx}{dy} = \frac{x}{2y} - \frac{y}{2x}.$$

To solve this, make the substitution $v = \frac{x}{y}$ so that

$$v + y\frac{dv}{dy} = \frac{1}{2}v - \frac{1}{2}\frac{1}{v}.$$

This is separable, and after some algebra we have

$$\int \frac{1}{y} dy = -\int \frac{2v}{1+v^2} dv$$

so that

$$\ln|y| = -\ln(1+v^2) + c.$$

Rewriting the last line subsituting $v = \frac{x}{y}$ gives an implicit solution to the original DE.

6. If you are given that the equation

$$\sin x + y + f(x)\frac{dy}{dx} = 0$$

has $\mu(x) = x$ as an integrating factor, determine all possible choices for $f(x)$.

Soln:

We are told that $x\sin x + xy + xf(x)\frac{dy}{dx} = 0$ is exact, so

$$\frac{\partial}{\partial y}(x\sin x + xy) = \frac{\partial}{\partial x}(xf(x)).$$

This means that $x = x\frac{df}{dx} + f$, or $\frac{df}{dx} + \frac{1}{x}f = 1$. This is a linear DE with integrating factor $e^{\ln x} = x$. Multiplying by this integrating factor gives $\frac{d}{dx}(xf) = x$, so that $xf = x^2/2 + c$ and $f = \frac{x}{2} + \frac{c}{x}$.

7. Show that $ye^x$ is an I.F. of

$$xy + y + 2x\frac{dy}{dx} = 0$$ and solve it.

Soln:

Multiply by $ye^x$ to get $xy^2e^x + y^2e^x + 2xye^x\frac{dy}{dx} = 0$. We then have

$$\frac{\partial}{\partial y}\left(xy^2e^x + y^2e^x\right) = \frac{\partial}{\partial x}(2xye^x) = 2xye^x + 2ye^x$$

and the equation is now exact. Find $\varphi(x,y)$ with $\frac{\partial\varphi}{\partial x} = xy^2e^x + y^2e^x$ and $\frac{\partial\varphi}{\partial y} = 2xye^x$. Starting with the second of these two requirements, we learn that

$$\varphi(x,y) = \int 2xe^x y\,dy = xe^x y^2 + f(x)$$

for some function $f(x)$. Then use the first requirement on $\varphi$ to see that $xe^xy^2 + e^xy^2 + f'(x) = y^2xe^x + y^2e^x$, and so we may choose $f(x) = 0$. An implicit solution to the DE is $xe^xy^2 = c$.

8. By recognizing differentials, solve the initial value problem

$$y\,dx - x\,dy = x^3y^5(y\,dx + x\,dy), \quad y(4) = \frac{1}{2}.$$

Soln: Divide the given equation by $y^2$ and use Common Differential 2 to obtain

$$\frac{ydx - xdy}{y^2} = (xy)^3(ydx + xdy) = (xy)^3d(xy).$$

Thus,

$$d\left(\frac{x}{y}\right) = d\left(\frac{(xy)^4}{4}\right)$$

So we have,

$$\frac{x}{y} - \frac{(xy)^4}{4} = c.$$

Setting $x = 4, y = \frac{1}{2}$ gives $c = 4$, so that

$$\frac{x}{y} - \frac{(xy)^4}{4} = 4$$

is an implicit solution to the IVP.