If $f(x)$ and $g(x)$ are functions such that $f(x)$ is differentiable at $x$ and $g(x)$ is differentiable at $f(x)$, then the chain rule says that

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x).$$

**Problem 1.** Practice the chain rule by differentiating the following functions(no need to simplify):

(a) $\sqrt{x^3 + 2x}$

$$\frac{1}{2}(x^3+2x)^{-1/2}(3x^2+2)$$

(b) $(x^2 + x + 1)^{1000}$

$$1000(x^2+x+1)^{999}(2x+1)$$

(c) $(x^2 + x + 1)^{1000}$

$$1000(x^2+x+1)^{999}(2x+1)$$

(d) $(2x^4 - 1)^2(4x + 1)^5$

$$(4x+1)^5\left(2(2x^4-1)8x^3\right) + (2x^4-1)^2 5(4x+1)^4 4$$

(e) $\dfrac{1 - x}{(2x^2 + 7)^2}$

$$\frac{(2x^2+7)^2(-1) - (1-x)2(2x^2+7)4x}{(2x^2+7)^4}$$

(f) $\sqrt{(5x^2 + 2)^4 + 3}$

$$\frac{1}{2}\left((5x^2+2)^4+3\right)^{-1/2}\left(4(5x^2+2)^3 \cdot 10x\right)$$

(g) $\left(\dfrac{10x^2 + 3x}{x^3 - 4x^2 + 1}\right)^{3/2}$

$$\frac{3}{2}\left(\frac{10x^2+3x}{x^3-4x^2+1}\right)^{1/2}\left[\frac{(x^3-4x^2+1)(20x+3)-(10x^2+3x)(3x^2-8x)}{(x^3-4x^2+1)^2}\right]$$

(h) Show that $\frac{d}{dx}[f(cx)] = cf'(cx)$ where $f$ is differentiable, $c$ is a constant.

Let $g(x) = cx$. So, $f(cx) = f(g(x))$. Thus, $\frac{d}{dx}(f(c(x))) = f'(g(x)) \cdot g'(x)$

$= f'(cx)\, c$

(i) If $f$ is differentiable and $f(x) > 0$ for all $x$, then show that $\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$.

$$\frac{d}{dx}\left((f(x))^{1/2}\right) = \frac{1}{2} f(x)^{-1/2} f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$$

(j) Write $T/F$. If false write what the correct statement should be:

If $f$ is differentiable, then $\frac{d}{dx}\left[f\left(\frac{1}{x}\right)\right] = f'\left(\frac{1}{x}\right)$.

$$\frac{d}{dx}\left(f\left(\frac{1}{x}\right)\right) = f'\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right)$$

**Problem 2.** Let $f(x)$ and $g(x)$ be functions with the following values:

$$f(1) = -3 \qquad f'(1) = -\frac{2}{3}$$
$$f(4) = 4 \qquad f'(4) = -5$$
$$g(1) = 0 \qquad g'(1) = 0$$
$$g(4) = 1 \qquad g'(4) = \frac{1}{2}.$$

(a) Calculate $h'(4)$ when $h(x) = (f \circ g)(x)$.

$h'(x) = f'(g(x)) \cdot g'(x)$

$h'(4) = f'(g(4))\, g'(4) = f'(1) \cdot \left(\frac{1}{2}\right) = -\frac{2}{3} \cdot \frac{1}{2} = -\frac{1}{3}$

(b) Calculate $j'(1)$ when $j(x) = [f(x)]^3$.

$j'(x) = 3\, f(x)^2 \cdot f'(x)$ So, $j'(1) = 3\, (f(1))^2 f'(1) = 3(-3)^2 \left(-\frac{2}{3}\right) = -18$

(c) Calculate $k'(4)$ when $k(x) = (g \circ f)(x)$.

$k'(x) = g'(f(x)) \cdot f'(x)$ So, $k'(4) = g'(f(4)) \cdot f'(4) = g'(4)(-5) = \frac{1}{2} \cdot (-5) = \frac{-5}{2}$

**Problem 3.** The adiabatic law for a gas, the law that governs the behaviour of a gas that is expanding without gaining or losing heat is given by the equation

$$P(t)(V(t))^{\gamma} = k$$

where $k, \gamma$ are constants, $P(t), V(t)$ are the pressure and volume of the gas respectively at time $t$. Show that

$$\frac{1}{V}\frac{dV}{dt} = -\frac{1}{\gamma}\frac{1}{P}\frac{dP}{dt}.$$

[We often suppress the notation $V(t)$ to be just $V$ provided that we understand that it is still a function of $t$; e.g. $\frac{df}{dx}$ makes sense but what we actually mean is $\frac{d}{dx}(f(x))$. Keep this in mind and do not get confused with notations.]

$PV^{\gamma} = k$

$\frac{d}{dt}(PV^{\gamma}) = \frac{d}{dt}(k) = 0$

or, $P \frac{d}{dt}(V^{\gamma}) + V^{\gamma}\frac{dP}{dt} = 0$

or, $P.\left(\gamma V^{\gamma-1}\frac{dV}{dt}\right) + V^{\gamma}\frac{dP}{dt} = 0$

or, $\gamma P V^{\gamma-1}\frac{dV}{dt} = -V^{\gamma}\frac{dP}{dt}$

or, $\frac{V^{\gamma-1}}{V^{\gamma}}\frac{dV}{dt} = \frac{-1}{\gamma P}\frac{dP}{dt}$

or, $\boxed{\frac{1}{V}\frac{dV}{dt} = -\frac{1}{\gamma}\frac{1}{P}\frac{dP}{dt}}$