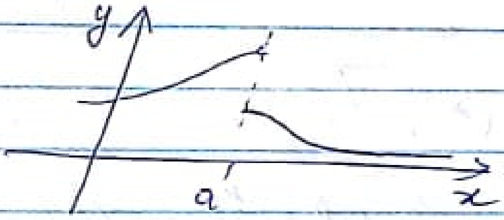
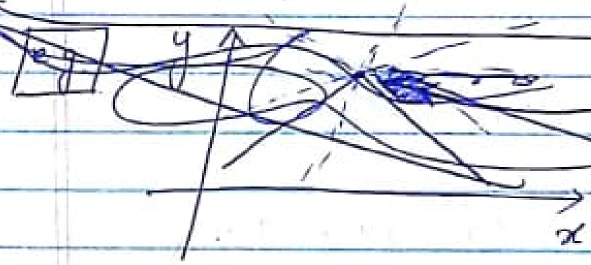
Recall There are ~~mainly~~ 3 situations when the derivative of a function does not exist at some point (say a). Pictorially,

(i) The function is not continuous at $x = a$.

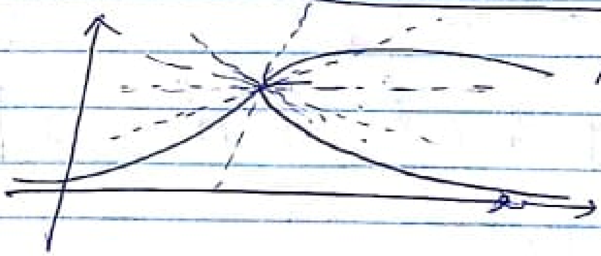
e.g



(ii) 'The ~~slope of the~~ tangent line' does not make sense. (This happens when there are usually 'sharp points' on the graph).



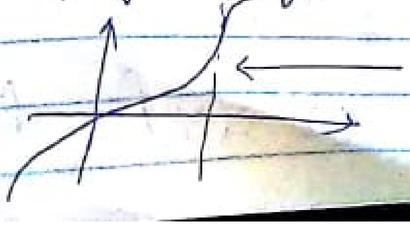
~~here there is no unique tangent line; so 'the tangent~~



At this point, there is no unique tangent line; "the tangent line" doesn't make sense.

(iii) If the slope of the tangent line is undefined.

e.g



the slope of the tangent line is undefined; i.e. when tangent lines are vertical the derivative does not exist.

e.g. $f(x) = \sqrt[3]{x}$. It is continuous at $x=0$.

$f'(x) = \frac{1}{3} x^{-\frac{2}{3}}$, So $f'(0)$ does not make sense.

Hence, f is not differentiable at $x=0$.

Remark: I thought it's good to have a discussion of these 3 cases again as they're:

i) important for understanding

ii) important for multiple choice / 'T/F' questions.

3.3 Chain-Rule.

Let $h(x) = g(f(x))$, where f and g are differentiable. Then h is differentiable and

$$\boxed{h'(x) = \frac{d}{dx}(h(x)) = \frac{d}{dx}\left(g(f(x))\right) = g'(f(x)) \cdot f'(x).}$$

Equivalently, if we write $\boxed{y = h(x) = g(u)}$ where $u = f(x)$,

then $$\boxed{\frac{dy}{dx} = \frac{dg}{du} \cdot \frac{du}{dx}}$$

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· Consequence of Chain-Rule.

$\boxed{\text{The General Power-Rule}}$

If f is differentiable and $h(x) = [f(x)]^n$, (n, a real number) then $h'(x) = n\,[f(x)]^{n-1}\, f'(x)$.

$\boxed{\text{Proof}}$ Let $g(x) = x^n$. So, $g'(x) = nx^{n-1}$. ... (*)

So, $h(x) = g(f(x))$

Thus, by Chain-rule, $h'(x) = g'(f(x)).f'(x)$.

~~Now, $g'(x) = nx^{n-1}$~~ $= n(f(x))^{n-1}.f'(x)$ (using (*)).

□

$\boxed{\text{e.g}}$ Suppose $f(1)=3$, $f'(1)=-2$, $f(3)=2$, $f'(3)=-3$,
and $g(1)=3$, $g'(1)=-1$, $g(3)=4$, $g'(3)=0$.

If $h(x) = f(x^2 g(x))$, find $h'(1)$.

$\boxed{\text{Soln}}$ $h'(x) = f'(x^2 g(x)).(2x\,g(x) + x^2 g'(x))$ (Using i) Chain rule ii) Product rule on the inner function)

So, $h'(1) = f'(1g(1))(2g(1) + 1g'(1))$
$= f'(3)(2.3 + (-1))$
$= (-3)(5) = -15$.

□

3.5 Higher Order derivatives.

Note that f' is itself a function and we can talk about

i) f' is continuous or not

ii) f' is differentiable or not.

This motivates the search for "higher order derivatives" i.e. repeatedly taking derivatives of a given function f.

Answering (i) implies checking

ⓐ $f'(a)$ is defined for all $a \in$ Domain

ⓑ $\lim_{x \to a} f'(x)$ exists.

ⓒ $\lim_{x \to a} f'(x) = f'(a)$

Answering (ii) needs checking

$$\lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} \quad \text{exists or not.}$$

If it exists, it is called the 2nd derivative of f with respect to x.

$\boxed{\text{Notation:}}$ $f''(x)$ OR $\frac{d^2}{dx^2}(f(x))$ or if $y = f(x)$, then $\frac{d^2y}{dx^2}$

Similarly we can talk about even higher derivatives.

- $f'''(x), f''''(x), f'''''(x)$

Instead of writing so many " ' "s. we write.

$f'''(x) = f^{(3)}(x)$; $f''''(x) = f^{(4)}(x)$ and so on.

[So, $f^{(0)}(x) = f(x)$ and $f^{(1)}(x) = f'(x)$]

Another notation is to take. $y = f(x)$.

Then $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}$ are standard notations

Most important Practical Example of Higher-Order of derivatives is the following:

Let $s(t)$ represent position of a particle in time t. Then. $s'(t) = \frac{d}{dt}(s(t))$ is the instantaneous rate of change of position at time t; in other words, $s'(t)$ is the velocity at time t.

$s''(t) = \frac{d^2}{dt^2}(s(t))$. is the rate of change of velocity, i.e. acceleration at time t.

$\boxed{\text{Rmk}}$ We often suppress the functional notation. $s(t)$ to be s, where it is understood, that s is a function of t. Thus, ~~acceleration~~ velocity becomes $\frac{ds}{dt}$, whereas $\frac{d^2s}{dt^2}$ becomes acceleration.

$a = \frac{d}{} \ldots \frac{d}{}(\frac{ds}{}) \; \frac{d^2s}{} \; \frac{d}{}$

e.g. A ball is thrown upwards from the roof of a building. Let height from ground at time t be given by

$s = -16t^2 + 24t + 120$. where s is in ft, t is in seconds.

(a) Find velocity & acceleration 3 seconds after it is thrown into air.

(b) Find height of the building.

(Sol^n. b) Height of the building is simply s evaluated at 0.

So, 120 ft.

(Sol^n. a) $v = \frac{ds}{dt} = -32t + 24$

So, velocity (in ft/s) 3 seconds after it is thrown is

$= -32 \times 3 + 24 = -72$.

So, the ball is falling downward at 72 ft/s.

$a = \frac{dv}{dt} = -32$.

Thus, the acceleration downward is constant $32\ ft/s^2$.