6.1 Antiderivatives, Rules of Integration

A function F is an <u>antiderivative</u> of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

e.g. $F(x) = x^3 + 1$ is an antiderivative of $f(x) = 3x^2$ as $F'(x) = 3x^2 = f(x)$

However, note that $F(x) = x^3 + C$ (C is arbitrary constant) is also an antiderivative of $f(x)$!

Theorem If $F(x)$ is an anti-derivative of $f(x)$, then <u>all</u> anti-derivatives of $f(x)$ must look like $G(x) = F(x) + C$ (C - arbitrary constant), no others.

The process of finding all anti-derivatives of a function is called <u>anti-differentiation</u> or "<u>integration</u>" (we will use this terminology)

Thus, from our discussion, we have.

"$\int f(x)\,dx = F(x) + C$" : Read as 'the indefinite integral of $f(x)$ is the family of functions $F(x)+C$.'

- $\int$: integral sign.

- $\int f(x)\,dx$
  $\hookrightarrow$ Integrand: (the thing between $\int$' and 'dx'; it is the function being integrated)

- The end result is called 'integral of f'.

<u>Warning</u>: Do not omit 'dx'. $\int f(x)$ makes no sense)

$\boxed{\text{Note:}}$ If the function is of 't' instead of 'x'.

i.e. $f(t)$, then we write $\int f(t)\,dt$

(Actually, t & x inside the integral are just "dummy variables")

e.g. $\int 1\,dx = x + C$ (Why? because

$\quad\quad \Vert \quad\quad\quad\quad\quad\quad\quad\quad$ if $F(x) = x + C$

$\int 1.dx \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ then $F'(x) = 1$).

Integrating again, we get

$$\int (x + C)\,dx = \frac{x^2}{2} + Cx + C_1 \quad (\text{Check!})$$

• Basic Integration Rules.

- $\int k\,dx = kx + C$ (k constant)
- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ (n is any constant except $n=-1$).
- $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln(|x|) + C.$ $(x \neq 0)$
- $\int e^x\,dx = e^x + C$
- $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$
- $\int k f(x)\,dx = k \int f(x)\,dx$ (where k is any constant)
- No general rule for $\int f(x)g(x)\,dx$, $\int \frac{f(x)}{g(x)}\,dx$, $\int f(g(x))\,dx$ ..

~~① Multi~~ ~~Let's look at a few indefinite integrals.~~

Example ① $\int \left(t^{-\frac{1}{2}} - e^t + \frac{1}{t}\right) dt.$

$$= \int t^{-1/2}\,dt - \int e^t\,dt + \int \frac{1}{t}\,dt$$

$$= \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - e^t + \ln(|t|) + C$$

$$= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} - e^t + \ln(|t|) + C$$

(we can combine all the constants into a single one!)

$$= 2\sqrt{t} - e^t + \ln(|t|) + C.$$

Ans

② $\int \frac{u-1}{u^2}\,du.$

$$= \int \left(\frac{u}{u^2} - \frac{1}{u^2}\right) du.$$

$$= \int \left(\frac{1}{u} - \frac{1}{u^2}\right) du.$$

$$= \int \frac{1}{u}\,du - \int \frac{1}{u^2}\,du$$

$$= \ln(|u|) - \frac{u^{-2+1}}{-2+1} + C$$

$$= \ln(|u|) + \frac{1}{u} + C.$$

□

$\boxed{\text{State T/F}}$

$$\int (x^2+1)^2\,dx = \frac{1}{3}(x^2+1)^3 + C.$$

Soln:- False. Let $F(x) = \frac{1}{3}(x^2+1)^3 + C$.

$$F'(x) = (x^2+1)\,2x \neq (x^2+1)^2.$$

So, don't always rush and integrate. Differentiation is still your friend!

$\boxed{\text{Initial Value Problem}}$

You will be given a specific condition for the anti-derivative to satisfy; we have to solve a "differential equation" (an equation involving derivatives) with some (initial) condition satisfied.

$\boxed{\text{e.g}}$ Let $f(x) = 3x^2 - 2x$. Find the anti-derivative $F(x)$ which also satisfies $F(2) = 3$.

$\boxed{\text{Sol}^n}$ $F(x) = \int f(x)\,dx = \int (3x^2 - 2x)\,dx = x^3 - x^2 + C$.

$3 = F(2) = 2^3 - 2^2 + C \Rightarrow C = -1$.

So, $\boxed{F(x) = x^3 - x^2 - 1}$.

$\boxed{\text{e.g}}$ Look at worksheet.