Continuity

~~Suppose $f(x)$~~

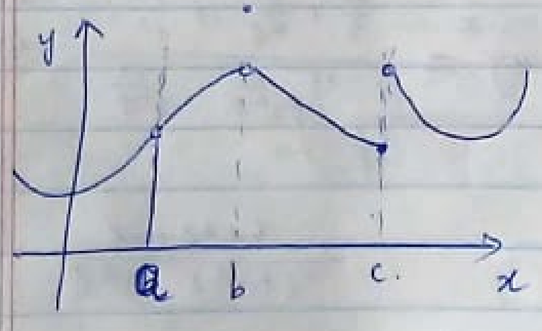
A function f is continuous at a number $= b$, if the following hold:

i) $f(b)$ is defined

ii) $\lim_{x \to b} f(x)$ exists. $\left(\text{i.e. } \lim_{x \to b^-} f(x) = \lim_{x \to b^+} f(x)\right)$

(iii) $\lim_{x \to b} f(x) = f(b)$

[In terms of graphs, there should not be any 'breaks' or 'jumps' at the point $x = b$]



- f is not continuous at a because it is not defined.
- f is not continuous at b. because $\lim_{x \to b} f(x) \neq f(b)$.
- f is not continuous at $x = c$ since $\lim_{x \to c} f(x)$ DNE.

The next example is a typical question that you can expect in your exams. So, as always, pay attention to the solution writing:

$\boxed{Q}$. Find the values of m and b that make the following function continuous:

$$f(x) = \begin{cases} 5 - x^2, & x \leq -1 \\ mx + b, & -1 < x < 1 \\ x^2 + 1 & 1 \leq x \end{cases}$$

$\boxed{\text{Soln}}$. At all $x \neq \pm 1$, f is continuous as it is a polynomial on each of the three intervals $(-\infty, -1]$, $(-1, 1)$, $[1, \infty)$.

We need to check at the points $x = 1$ and $x = -1$.

$\boxed{x=1}$. We need $f(-1)$ to make sense, $\lim_{x \to 1} f(x)$ to exist and equal to $f(-1)$.

- $f(-1) = 5 - (-1)^2 = 4$.
- $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (5 - x^2) = 4$
- $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (mx + b) = m(-1) + b = b - m$.

So, we need.

① . - - - . $b - m = 4$ for $\lim_{x \to -1} f(x)$ to exist and equal to $f(-1)$.

Next

$x=1$ • $f(1) = 1^2 + 1 = 2$.

• $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (mx + b) = m(1) + b = b + m$.

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 + 1) = 1^2 + 1 = 2$.

So, we need $b + m = 2$ . . . ②.

From ① and ②, we have

$$b - m = 4$$
$$b + m = 2.$$

Solving this, we get $b = 3$, $m = -1$

Thus, f is continuous when $b = 3$, $m = -1$

Ans

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$\boxed{\text{Tips}}$ The following things will be checked when your writing is graded for such a problem:

i) Writing limits at appropriate places & not forgetting to write it!!

ii) At the "branch point" (e.g. $x=-1$ or $x=1$) you have to clearly compute:

$\lim_{x\to 1^-} f(x)$ & $\lim_{x\to 1^+} f(x)$ & $f(1)$ and "observe whether they match"/ "make them match" depending on the question (e.g. in the last example we had to make them match to solve for b & m).

$\boxed{\text{Remark}}$ So, we ~~now~~ have the definition of a function to be continuous at a point in its domain. $\boxed{\text{Def}^n}$

Next obvious thing to do is to define what a "continuous function" is.

Def^n: A function f is called continuous on its domain if it is continuous at every point of its domain. ("in the sense of the previous definition").

Remark We actually intuitively used this definition when we solved the previous question, regarding m & b. (Right?).

If f is not continuous at some point in its domain, we say <u>f is discontinuous at that point.</u>

Important We will often deal with functions defined on finite intervals. (say, (a,b), $[a,b]$, etc.).

Here, continuity at end points will refer to only one-sided limits according to which side will be contextual.

e.g.

$f : [a,b] \longrightarrow \mathbb{R}$. f is continuous at a means

$f(a)$ is defined and

$\lim_{x \to a^+} f(x)$ exists and equals $f(a)$.

I hope this is pretty clear to you. (because $\lim_{x \to a^-} f(x)$ is not making sense here). ~~but is th~~

Even in such a scenario, one can 'casually say' $\lim_{x \to a} f(x)$ exists & is equal to $f(a)$. ~~but keep in mind that only~~ [illegible]

Properties of Continuous functions

1) The constant function $f(x) = c$ is continuous everywhere.

2) The identity function $f(x) = x$ is continuous everywhere.

If f and g are continuous at $x = a$, then

(3) $[f(x)]^n$, where n is a real number, is continuous at $x = a$ whenever it is defined at that number.

(4) $f \pm g$ is continuous at $x = a$.

(5) $f.g$ is continuous at $x = a$

(6) f/g is continuous at $x = a$ provided $g(a) \neq 0$.

Thus, using (1), (2), (3), (4) repeatedly, we have

(7) A polynomial function $y = P(x)$ is continuous at every point x.

Using (7) & (6), we have

(8) A rational function $R(x) = \frac{P(x)}{Q(x)}$ is continuous at every x where $q(x) \neq 0$..

Note It is important you guys understand/remember both the definition of continuity. A typical question often asked is:

"State precisely what it means for a function f to be continuous at $x = a$".

e.g. Where are the following functions continuous?

(a) $f(x) = x^5 - 6x^2 + 8x + 4$

Ans: It is a polynomial, so it is continuous on $\mathbb{R}$

(b) $h(x) = \dfrac{8x^{10} - 4x + 1}{x^2 + 1}$.

Ans: It is a rational function whose denominator does not have a zero. (or a root) So, again, it is continuous on $\mathbb{R}$.

(c) $h(x) = \dfrac{4x^3 - 3x^2 + 1}{x^2 - 3x + 2}$ $\left(= \dfrac{4x^3 - 3x^2 + 1}{(x-2)(x-1)} \right)$.

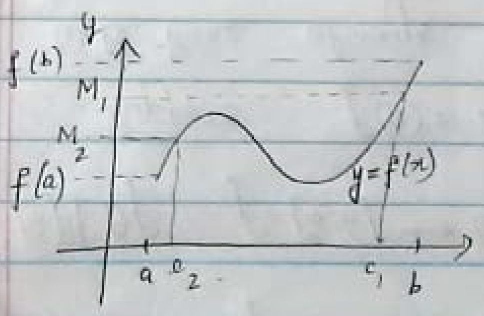
Ans: It's a rational function; roots of the denominator are $x = 2$, and $x = 1$. So, h is a continuous on $(-\infty, 1) \cup (1, 2) \cup (2, \infty)$.

Next, we move onto a result, which according to me, is the coolest result of this course.

$\boxed{\text{Intermediate Value Theorem}}$ (IVT) (Statement is extremely important)

If f is continuous on a closed interval $[a, b]$ and M is any number between $f(a)$ and $f(b)$, then there exists at least one c in $[a, b]$ such that $f(c) = M$.

(It just says that if I have a "continuous hammer" and I start hammering from $f(a)$ to $f(b)$, then all values between $f(a)$ and $f(b)$ have to be hammered at some point) (OR if you draw a figure without lifting the pen, then you actually move across all intermediate points while traversing from initial to final point).



OR

For any M between $f(a)$ & $f(b)$, there exists c in $[a, b]$, such that $f(c) = M$. (the figure depicts two such M_1 & M_2 and corresponding c_1 & c_2].

- It is important that f is continuous on the domain which is a closed interval. Otherwise, this will fail. So, do not fall for tricks which do not specify whether a given abstract function is continuous or not.

e.g. $\boxed{Q1}$ If $f : [a,b] \to \mathbb{R}$.

Suppose $f(a) = 2$, $f(b) = -1$.

Will there always exist ~~as as~~ some number c between a and b such that $f(c) = 0$?

Ans: No. We do not know whether ~~to~~ f is continuous or not.

$\boxed{Q2}$ What happens if f is continuous?

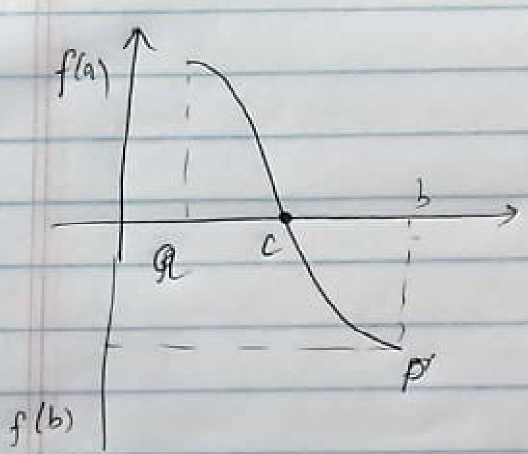
Ans: Yes. by IVT. f is continuous on $[a,b]$.

$f(a) = 2$; $f(b) = -1$ and $f(b) < 0 < f(a)$ (i.e. 0 lies between $f(a)$ and $f(b)$). So, IVT applies.

In fact, this $\boxed{Q2}$ is our next theorem!

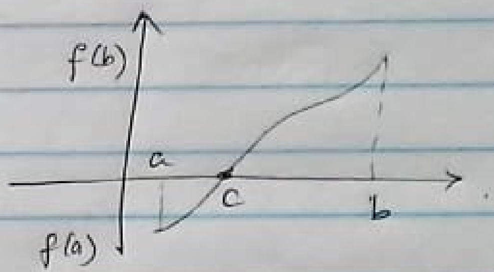
$\boxed{\text{Theorem}}$ Existence of Zeros of a Continuous Function.

If f is a continuous function on a closed interval $[a, b]$ and if $f(a)$ and $f(b)$ have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval (a, b).



$f(c) = 0$.

OR



$f(c) = 0$.

$\boxed{\text{Proof:}}$ Suppose $f(a) > 0$. Then $f(b) < 0$. (since opposite signs).

Now, f is continuous on $[a, b]$.

$f(b) < 0 < f(a)$. Thus, by IVT, there exists at least one c in $[a, b]$ such that $f(c) = 0$. Now, $c \neq a, b$ as $f(a) > 0$, $f(b) < 0$.

Thus, c is in (a, b).

Π

In case of multiple roots, we divide the interval conveniently.

$\boxed{\text{e.g.}}$ Show that there exists 2 roots in $[0, 4]$ of

$f(x) = x^2 - 4x + 3$.

$\boxed{\text{Soln.}}$ $f(0) = 3 > 0$

$f(2) < 0$

f is a polynomial. So, it is continuous on $[0, 2]$ and we saw

$f(0) > 0$
$f(2) < 0$.

So, ~~there~~ by IVT, there is a root in $(0, 2)$.

$\boxed{\text{Next}}$. $f(2) < 0$

$f(4) = 3 > 0$

Again f is continuous on $[2, 4]$ and we saw $f(2) < 0$, $f(4) > 0$.

Thus, by IVT, there is a zero of f is $(2, 4)$.

Thus, we have found two roots of f is $[0, 4]$.

$\square$

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We will deal with applications:

Two runners begin running from different points on a street; their respective positions at any time t, $0 \leq t \leq 1$ are given by $f(t) = t^5 + 2t - 1$ and $g(t) = 2t - t^2$.

Does either runner catch the other during this time? Carefully justify your answer.

$\boxed{\text{Soln.}}$ [We simply have to ~~show~~ ~~that~~ check whether $f(t) = g(t)$ for some t in $[0, 1]$. Because, if they match, then one has caught up with the other, o.w. $f(t) > g(t)$ forever or $g(t) > f(t)$ forever, i.e. noone catches the other].

[So, we use IVT!!]

Define $h(t) = f(t) - g(t)$. Note h is a polynomial, so it is continuous on $[0, 1]$. We need to figure out if $h(t) = 0$ has a solution in $[0, 1]$.

(As usual, we check signs at end points).

$h(0) = f(0) - g(0) = -1 - 0 < 0$.

$h(1) = f(1) - g(1) = 1 + 2 - 1 - (2 - 1) = 1 > 0$.

Hence, by IVT, there exists some t in $(0, 1)$, such that $h(t) = 0$. i.e. one runner catches up with the other.

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