Quick Review

1. $\int x^3 (x^2+8)^5 \, dx$.

Soln: Let $u = x^2 + 8$. $\frac{du}{dx} = 2x \Rightarrow \frac{du}{2x} = dx$.

$$\int x^3 (x^2+8)^5 dx = \int x^3 \, u^5 \, \frac{du}{2x}$$

$$= \frac{1}{2}\int x^2 u^5 \, du$$

$$= \frac{1}{2}\int (u-8)\, u^5 \, du$$

$$= \frac{1}{2}\left[\int u^6 \, du - 8\int u^5 \, du\right]$$

$$= \frac{1}{2}\left[\frac{u^7}{7} - 8\,\frac{u^6}{6}\right] + C$$

$$= \frac{1}{2}\left[\frac{(x^2+8)^7}{7} - \frac{8}{6}(x^2+8)^6\right] + C.$$

Ans

(2) $\int\left(e^{-3x} - \frac{e^{2x}}{4-e^{2x}}\right)dx$

Soln. $I = \int e^{-3x}\,dx - \int \frac{e^{2x}}{4-e^{2x}}\,dx$.

Let $u = -3x$.

$\frac{du}{-3} = dx$.

Let $v = 4-e^{2x}$

$\frac{dv}{dx} = -2e^{2x}$

$\Rightarrow \frac{dv}{-2e^{2x}} = dx$.

Thus, $I = \int e^{u}\frac{du}{-3} - \int \frac{e^{2x}}{v}\frac{dv}{(-2e^{2x})}$

$= -\frac{1}{3}\int e^{u}\,du + \frac{1}{2}\int \frac{dv}{v}$

$= -\frac{1}{3}e^{u} + \frac{1}{2}\ln(|v|) + C$

$= -\frac{1}{3}e^{-3x} + \frac{1}{2}\ln(|4-e^{2x}|) + C$.

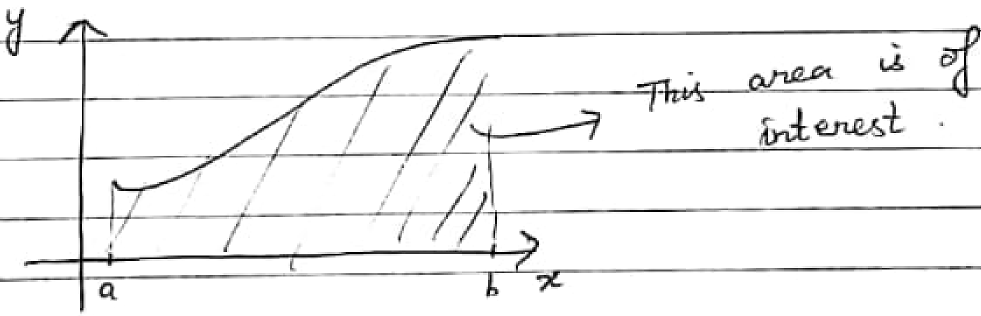
Ans

6.3

Definite Integral.

Given a continuous function $f(x)$ on an interval $[a, b]$, one is often interested in determining the area bounded by the graph of $f(x)$, the x-axis, and the lines $x = a$ and $x = b$.

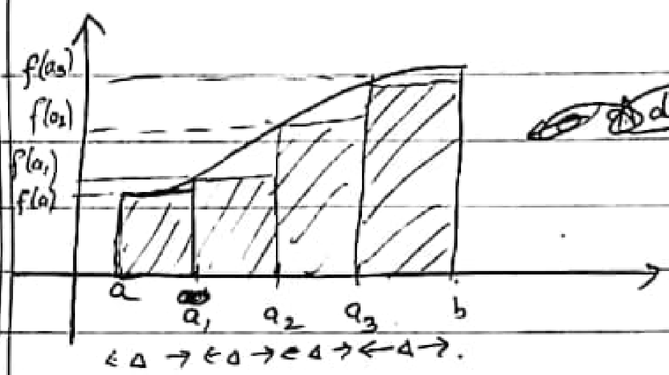
One can begin to approximate this area using the so-called Riemann Sums..



The idea is to break down the interval $[a, b]$ into subintervals of equal length h; choosing some point from the subinterval and forming rectangles whose areas we add up to approximate the above area under $f(x)$.

Scanned by CamScanner

e.g.



Add area

We divide the domain $[a,b]$ into the subintervals $[a,a_1]$, $[a_1,a_2]$, $[a_2,a_3]$, $[a_3,b]$. This is done by first looking at $(b-a)$ and dividing it by the no. of subintervals we want. (in this case, 4).

Let $\Delta = \frac{b-a}{4}$. Then choose $a_1 = a+\Delta$

$a_2 = a+2\Delta$

$a_3 = a+3\Delta$

$b = a+4\Delta$ (automatic).

Here, in this example, I chose the left-end-points of each sub-interval as my representative point to form the rectangles.

Now the sum of the areas of the rectangles is

$$L(f) = f(a).\Delta + f(a_1)\Delta + f(a_2)\Delta + f(a_3)\Delta$$

$$= \left(f(a) + f(a_1) + f(a_2) + f(a_3)\right)\Delta.$$

This is called the left Riemann sum. (left as we chose the left-end points) with four subintervals of equal length.

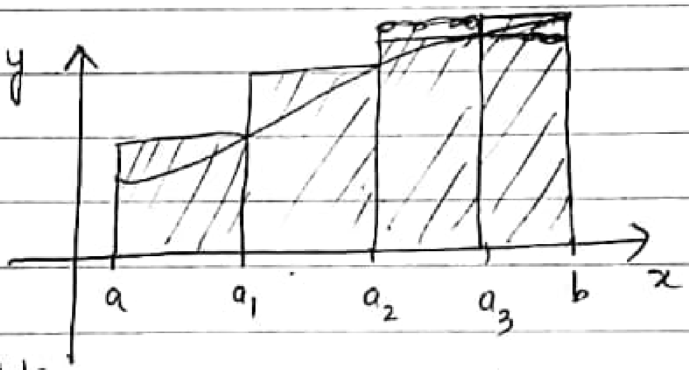
- In general, if I have n sub-intervals, then $\Delta_n = \frac{b-a}{n}$.

And the ~~left~~ Riemann sum is.

$$L(f) = \left(f(a) + f(a_1) + \cdots + f(a_{n-1})\right)\Delta_n.$$

Similarly choosing the right-end points of each subinterval, we get ~~Upper~~ Right Riemann Sum $(R(f))$.

e.g.

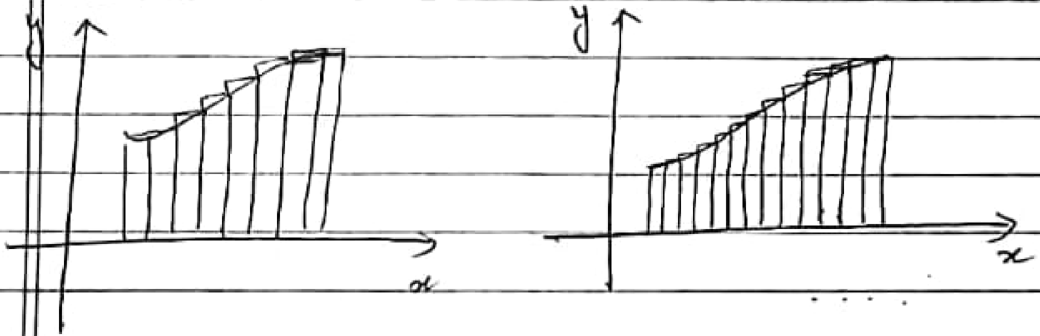


. ~~Upper~~ Right Riemann Sum $= \left(f(a_1) + f(a_2) + f(a_3) + f(b)\right).\Delta_n$

. In general, with n-subintervals.

$$R(f) = \left(f(a_1) + \cdots + f(a_{n-1}) + f(b)\right).\Delta_n$$

- Notice, that the Right Riemann Sum in this case gives me an area greater than the required area whereas left Riemann sum is less than the required area. It might change depending on whether f is increasing or decreasing.
- Also, notice that if we increase the number of intervals n, then we can get a better approximation of our required area.



Thus, the best possible scenario is if we let

$n \to \infty$.

and look at

$$\lim_{n \to \infty} \left(\left[f(x_1) + f(x_2) + \cdots + f(x_n) \right] \Delta_n \right) \quad \text{where } \Delta_n = \frac{b-a}{n}$$

and $x_1, x_2, \ldots, x_n$ are any arbitrary points in the respective subintervals. (- for right Riemann Sum - choose right end points

- for left Riemann Sum - choose left end points)

(Definition).

Let $f : [a, b] \to \mathbb{R}$ be a function.

If $\lim_{n \to \infty} \left([f(x_1) + \cdots + f(x_n)] \Delta_n \right)$ exists

when – Upper Riemann Sum is considered

– Lower Riemann Sum is considered

and in both cases, the limit is the same, then we call this limit the definite integral of f from a to b, denoted by

$\int_a^b f(x)\,dx$. Thus, $\int_a^b f(x)\,dx = \lim_{n \to \infty} \left[f(x_1)\Delta_n + f(x_2)\Delta_n + \cdots + f(x_n)\Delta_n \right]$

where $x_1, \ldots, x_n$ can be (i) the left-end points OR (ii) the right-end points of the subintervals.

Remark: Actually this works even for arbitrarily chosen points from each sub interval.

a – is called lower limit of integration

b – is called upper limit of integration.

- We say f is integrable on $[a, b]$ if the above limit exists.

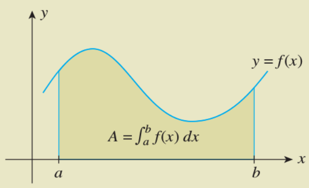
Theorem: If f is continuous, then f is integrable i.e. $\displaystyle\int_a^b f(x)\,dx$ exists.

Geometric Interpretation of $\int_a^b f(x)\,dx$ for $f(x) \geq 0$ on $[a, b]$

If f is nonnegative and continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx$$

is equal to the area of the region under the graph of f on $[a, b]$



Thus, in this case, the definite integral gives the Area under the Curve.

Geometric Interpretation of $\int_a^b f(x)\,dx$ on $[a, b]$

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx$$

is equal to the area of the region above $[a, b]$ minus the area of the region below $[a, b]$ (Figure 15).

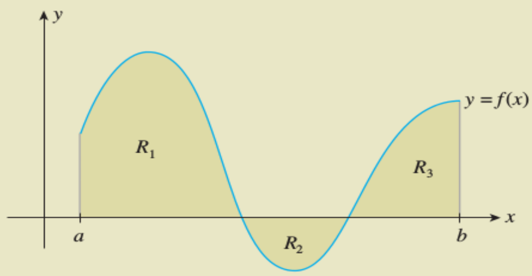


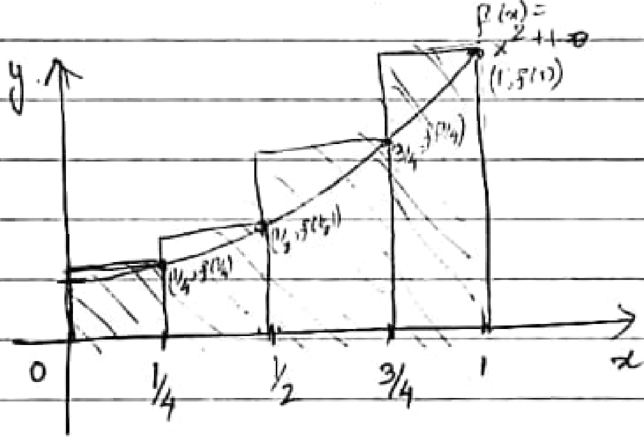
FIGURE 15
$\int_a^b f(x)\,dx = \text{Area of } R_1 - \text{Area of } R_2 + \text{Area of } R_3$

Thus, here it is not exactly the area under the curve. We have to make suitable sign changes if we were to compute the definite integral geometrically.

Remark: Area must always be positive. Thus, if we were to compute an area under the $x-axis$ using integration say e.g. R_2 in the above figure, then since the $f(x)$-values are negative, the integral will come out to be negative. So, area will be negative of the integration. However, in the above scenario, we are not computing area; Thus it says minus of the area; so we computed the area R_2 and then associated a minus sign to it.

Example: Use a Riemann Sum with four subintervals $(n=4)$ to approximate the area under the curve $f(x) = x^2+1$ over the interval $[0,1]$. Choose the representative points to be the right-end points of the subintervals.

Soln:



$n = 4$, $b=1, a=0$

$\Delta = \frac{b-a}{n}$

$= \frac{1-0}{4} = \frac{1}{4}$.

$x_1 = 0 + 1/4$

$x_2 = 0 + 2 \cdot \frac{1}{4} = \frac{1}{2}$

$x_3 = 0 + 3 \cdot 1/4 = 3/4$

The approximate area is

$\frac{1}{4} \cdot f(1/4) + f(1/2) \cdot 1/4 + f(3/4) \cdot 1/4 + f(1) \cdot 1/4$

$= \frac{1}{4}\left(\frac{1}{16}+1\right) + \frac{1}{4}\left(\frac{1}{4}+1\right) + \frac{1}{4}\left(\frac{9}{16}+1\right) + \frac{1}{4}(1+1) \approx 1.4$

Scanned by CamScanner

6.4 The Fundamental Theorem of Calculus (FTC)

Let f be continuous on $[a,b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any anti-derivative of f; i.e. $F'(x) = f(x)$.

Notation If $F'(x) = f(x)$, then we usually write

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

e.g. $F(x)\Big|_3^4$ means $F(4) - F(3)$.

Thus, in order to find

$\int_a^b f(x)\,dx$, we can • first find

$\int f(x)\,dx$ which gives a family of functions $F(x) + C$.

• then find

$$(F(b)+C) - (F(a)+C) = \underbrace{F(b) - F(a)}$$

(thus, can get rid of C and only compute this)

Thus, this theorem enables us to compute the area under a curve!

$\boxed{\text{e.g.}}$ Find the area under $f(x) = x^2 + 1$ over $[0, 1]$

Soln: Need to find $\int_0^1 f(x)\,dx$

$$\int f(x)\,dx = \int (x^2+1)\,dx = \int x^2\,dx + \int dx = \frac{x^3}{3} + x + C.$$

$$\int_0^1 f(x)\,dx = \left(\frac{x^3}{3} + x + C\right)\Bigg|_0^1 = \left(\frac{1}{3}+1\right) - (0+0) = \frac{4}{3}. \qquad \square$$

(Note that C cancels out anyway, so we can remove it)

e.g. Calculate $\int_0^2 e^{3x}\,dx$

Soln. $\int e^{3x}\,dx = \dfrac{e^{3x}}{3} + C$ (using substitution

Thus, $\int_0^2 e^{3x}\,dx = \dfrac{e^{3x}}{3}\Bigg|_0^2$ ~~(Note that C anyway cancels out)~~

$= \frac{1}{3}(e^6 - 1). \qquad \square$

A typical Problem Asked in the Exam

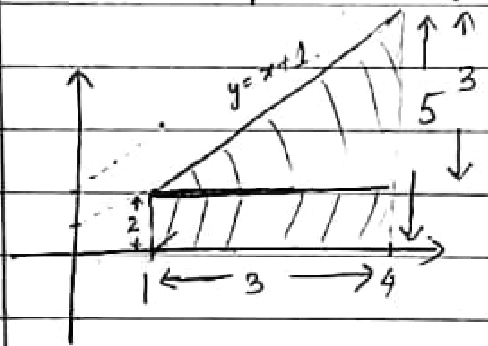
Let A be the area in the xy-plane bounded by the x-axis and the lines $y = x+1$, $x = 1$, $x = 4$.

Determine A by

(i) using geometry

(ii) with a definite integral.

Soln:



This is the graph

(i) Area of triangle $= \frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2}$.

Area of rectangle $= 2 \cdot 3 = 6$.

Total area $A = \frac{9}{2} + 6 = \frac{21}{2} = 10.5$

(ii) $$\int_1^4 (x+1)\,dx = \int_1^4 x\,dx + \int_1^4 dx$$

$$= \frac{x^2}{2}\Big|_1^4 + x\Big|_1^4$$

$$= \left(\frac{16}{2} - \frac{1}{2}\right) + (4-1) = 8 - \frac{1}{2} + 3 = 11 - \frac{1}{2}$$

$$= 10.5.$$

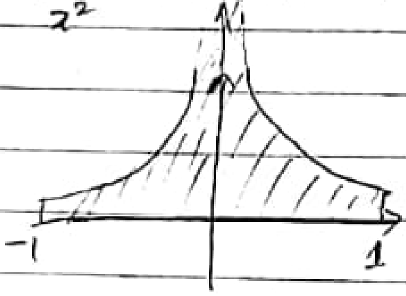
Thus, they match!

Warning

To apply FTC, it is crucial that f is continuous.

e.g. $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\, dx$

$\frac{1}{x^2}$ is not continuous at 0.



Area is extremely big.

However, if we apply FTC, then

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = \left(-\frac{1}{x}\right)\Big|_{-1}^{1} = (-1) - (+1) = -1 - 1 = -2$$

Negative Area! Absurd.

State. [T/F] Let $F(x) = -\frac{1}{x-2}$, and $f(x) = \frac{1}{(x-2)^2}$. Since $F'(x) = f(x)$, by FTC we have $\displaystyle\int_{1}^{3} f(x)\, dx = F(3) - F(1) = -2$.

Ans: False. f is not continuous at $x = 2$.
(Also, negative area!)

Remark: • $\int f(x)\,dx$ is a class of functions $F(x)+C$.

• $\int_a^b f(x)\,dx$ is a real number ← huge difference between the two !!

Net Change Formula

If f' is continuous on $[a,b]$, then

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

i.e. the net change is obtained by integrating the rate of change over the interval under consideration.

e.g. A concert just ended. People are leaving through the gate @ $100t + 300$ people/min. (for $0 \leq t \leq 4$)
How many people left in the first 4 mins?

Soln: Let $f(t)$ be the no. of people walking out at t minutes.
Thus, we need $f(4) - f(0)$.

$$f(4) - f(0) = \int_0^4 (100t + 300)\,dt = \left(50t^2 + 300t\right)\Big|_0^4$$

$$= 50 \times 16 + 300 \times 4 = 2000$$

Ans