STUDENT NAME: ______________________________

INSTRUCTOR: ______________________________

Please sign the pledge:
On my honor as a student, I have neither given nor received aid on this exam.

Directions

Answer each question in the space provided. Please write clearly and legibly. ***Show all of your work in order to receive full credit, and clearly identify your final answer. No books, notes or calculators are allowed.***

For instructor use only

Page	Points	Score
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1. [10 pts] For each of the following two functions, find the domain. ***Record your answer in interval notation.***

   (a) $F(x) = \dfrac{\sqrt{1-x}}{\sqrt{x}}$

   Solution. $\sqrt{1-x}$ is defined provided that $1-x \geq 0$, or $x \leq 1$; also, $\sqrt{x}$ is defined provided that $x \geq 0$, but $\sqrt{x}$ appears in the denominator, so that we must exclude the number $x = 0$. Thus F is defined when x satisfies both $x \leq 1$ and $x > 0$. The domain is $(0, 1]$

   (b) $R(x) = (g \circ f)(x) + h(x)$, where $f(x) = x + 1, g(x) = \frac{1}{3x+5}, h(x) = x^3$

   Solution.
   $R(x) = (g \circ f)(x) + h(x) = \dfrac{1}{3f(x)+5} + x^3 = \dfrac{1}{3(x+1)+5} + x^3 = \dfrac{1}{3x+8} + x^3.$
   Because the denominator of $\frac{1}{3x+8}$ vanishes at $-8/3$, we see that R has domain $\left\{x : x \neq -\dfrac{8}{3}\right\}$ or $(-\infty, -\dfrac{8}{3}) \cup (-\dfrac{8}{3}, \infty)$.

2. [6 pts] Andy is going on a 10-day trip in a few months. He paid for 10 nights at \$100 per night for his hotel room. He is delaying his purchase of a plane ticket, hoping to buy one at a price he will find acceptable. Andy uses a simple "travel quotient" function to figure out which prices are acceptable. The travel quotient $Q(A)$ is given by the airplane ticket price A, divided by S, where S is the sum of the ticket price A and the amount Andy has already spent on lodging.

   (a) Write the rule for the travel quotient $Q(A)$ as a function of A.

   Solution. From the description of the problem,
   $$Q(A) = \frac{A}{1000 + A}$$

   (b) For Andy, an acceptable price for an airplane ticket is any price A so that $Q(A) \leq \frac{1}{9}$. Should Andy buy when the ticket price is \$250?

   Solution. We have $Q(250) = \frac{1}{5}$; so the answer is no, this is not the acceptable price of plane ticket because $Q(250) = \frac{1}{5} > \frac{1}{9}$. Alternate solution:
   $$\begin{aligned} Q(A) \leq \frac{1}{9} \quad & \text{if and only if} \quad \frac{A}{1000+A} \leq \frac{1}{9} \\ & \text{if and only if} \quad 9A \leq 1000 + A \\ & \text{if and only if} \quad 8A \leq 1000 \\ & \text{if and only if} \quad A \leq 125. \end{aligned}$$

   Only prices at \$125 and below are acceptable to Andy.

3. [15 pts] Find the following limits, or explain why the limit does not exist.

(a) $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$

Solution.

$\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9} = \lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}\dfrac{\sqrt{x}+3}{\sqrt{x}+3} = \lim_{x\to 9} \dfrac{x-9}{(x-9)(\sqrt{x}+3)} = \lim_{x\to 9} \dfrac{1}{\sqrt{x}+3} = \dfrac{1}{6}$

(b) $\lim_{x\to\infty} \frac{3x^3+2x+6}{-x^3+8x^2-5x+4}$

Solution.

$\lim_{x\to\infty} \dfrac{3x^3+2x+6}{-x^3+8x^2-5x+4} = \lim_{x\to\infty} \dfrac{3x^3+2x+6}{-x^3+8x^2-5x+4}\dfrac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \dfrac{3+\frac{2}{x^2}+\frac{6}{x^3}}{-1+\frac{8}{x}+\frac{4}{x^3}} = \dfrac{3}{-1} = -3$

(c) $\lim_{x\to 3} \frac{x-3}{|2x-6|}$

Solution.

$\lim_{x\to 3^-} \dfrac{x-3}{|2x-6|} = \lim_{x\to 3^-} \dfrac{x-3}{6-2x} = \lim_{x\to 3^-} \dfrac{x-3}{2(3-x)} = \lim_{x\to 3^-} \dfrac{1}{-2} = -\dfrac{1}{2}$.

$\lim_{x\to 3^+} \dfrac{x-3}{|2x-6|} = \lim_{x\to 3^+} \dfrac{x-3}{2x-6} = \lim_{x\to 3^+} \dfrac{x-3}{2(x-3)} = \lim_{x\to 3^+} \dfrac{1}{2} = \dfrac{1}{2}$

So the limit at 3 does not exist.

4. (a) [4 pts] State precisely what it means for a function $f(x)$ to be continuous at $x = a$.

Solution. The function f is continuous at a number $x = a$ if the following conditions are satisfied

1. $f(a)$ is defined.
2. $\lim_{x\to a} f(x)$ exists.
3. $\lim_{x\to a} f(x) = f(a)$.

(b) [6 pts] Find the values of m and b that make the following function continuous:

$$f(x) = \begin{cases} 5 - x^2 & x \leq -1 \\ mx + b & -1 < x < 1 \\ x^2 + 1 & 1 \leq x \end{cases}$$

Solution. The individual functions are all continuous on their own (being polynomials), so the discontinuities, if any, occur at $x = -1, 1$. So we want $\lim_{x\to -1^-} f(x) = f(-1) = 4 = \lim_{x\to -1^+} f(x) = -m + b$ and $\lim_{x\to 1^-} f(x) = m + b = \lim_{x\to 1^+} f(x) = f(1) = 2$. So the values we want are the solutions to the system.

$$\begin{aligned} -m + b &= 4 \\ m + b &= 2 \end{aligned}$$

We can eliminate the m variable immediately and get $2b = 6$ so that $b = 3$ and $m = -1$.

5. [15 pts] Find the derivatives of the following functions. You **do not** need to simplify your final answer.

   (a) $f(x) = \dfrac{3x+5}{x^2-4x}$

   Solution.

   $$\frac{d}{dx}f(x) = \frac{3(x^2-4x)-(3x+5)(2x-4)}{(x^2-4x)^2}$$

   (b) $g(x) = \sqrt{\sqrt[4]{x} - \frac{1}{x^2}}$

   Solution.

   $$\frac{d}{dx}g(x) = \frac{1}{2}(\sqrt[4]{x} - \tfrac{1}{x^2})^{-\frac{1}{2}}\frac{d}{dx}(\sqrt[4]{x} - \tfrac{1}{x^2}) = \frac{1}{2}(\sqrt[4]{x} - \tfrac{1}{x^2})^{-\frac{1}{2}}(\frac{1}{4}x^{-\frac{3}{4}} - (-2)x^{-3})$$

   (c) $h(t) = (t^2+2t+1)(t^4-t^3-6)$

   Solution.

   $$\frac{d}{dt}h(t) = (t^2+2t+1)(4t^3-3t^2) + (t^4-t^3-6)(2t+2)$$

6. (a) [4 pts] State the limit definition of the derivative of a function $f(x)$.

Solution. The derivative of a function f with respect to x is the function f'

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

(b) [6 pts] Using the limit definition, find the derivative of $f(x) = x^2 + \frac{1}{3}x + \frac{2}{3}$.

Solution.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 + \frac{1}{3}(x+h) + \frac{2}{3} - (x^2 + \frac{1}{3}x + \frac{2}{3})}{h} \\
&= \lim_{h\to 0} \frac{x^2 + h^2 + 2hx + \frac{1}{3}x + \frac{1}{3}h + \frac{2}{3} - x^2 - \frac{1}{3}x - \frac{2}{3}}{h} \\
&= \lim_{h\to 0} \frac{h^2 + 2hx + \frac{1}{3}h}{h} \\
&= \lim_{h\to 0} (h + 2x + \frac{1}{3}) \\
&= 2x + \frac{1}{3}
\end{aligned} \tag{1}$$

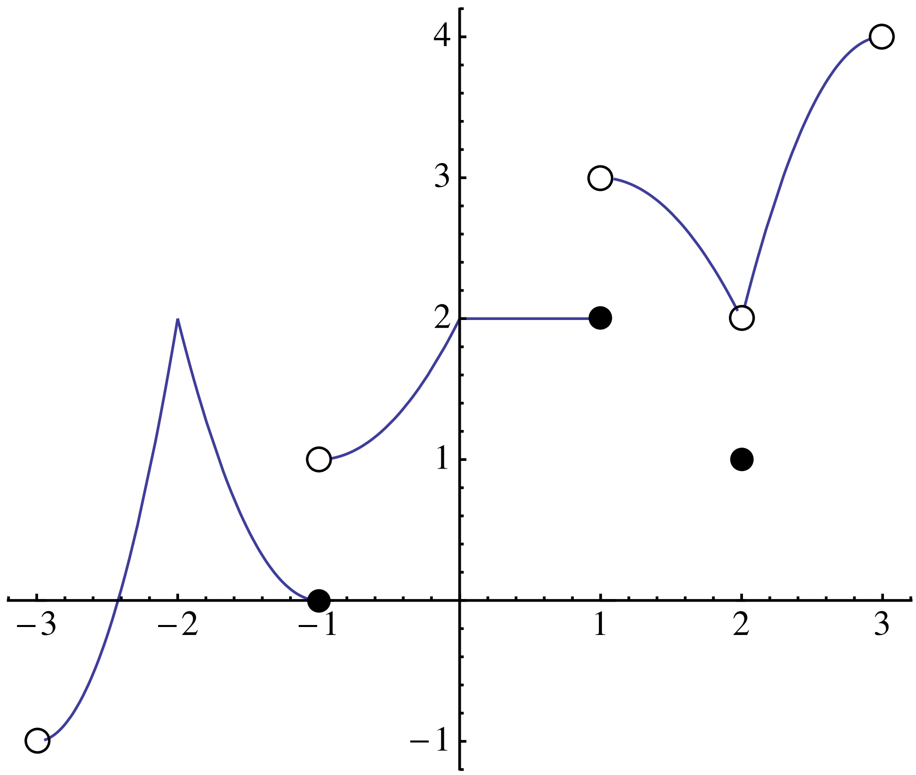
(c) [4 pts] The graph of $f(x) = x^2 + \dfrac{1}{3}x + \dfrac{2}{3}$ has a tangent line L that is parallel to the line $y = \dfrac{7}{3}x + 5$. Find an equation of this tangent line L.

Solution.

The tangent line parallel to $y = \frac{7}{3}x + 5$ must have slope $\frac{7}{3}$. The slope of the tangent line to the graph of the function $f(x)$ at a point (a, b) on the graph of f has slope $f'(a) = 2a + \frac{1}{3}$. So $2a + \frac{1}{3} = \frac{7}{3}$, so $a = 1$, so $b = a^2 + \frac{1}{3}a + \frac{2}{3} = 2$.

Equation of the tangent line $y - 2 = \frac{7}{3}(x-1)$ or $y = \frac{7}{3}x - \frac{1}{3}$

7. [8 pts] Pictured is the graph of the function $f(x)$ for $-3 < x < 3$. Using this graph, answer the following questions about $f(x)$.



(a) For which values of x in $(-3, 3)$ is $f(x)$ not differentiable?
Solution. $x = -2, -1, 0, 1, 2$

(b) For which values of x in $(-3, 3)$ is $f(x)$ not continuous?
Solution. $x = -1, 1, 2$

(c) For which values a in $(-3, 3)$ does $\lim_{x\to a} f(x)$ not exist?
Solution. $x = -1, 1$

(d) What is $\lim_{x\to 2} f(x)$?
Solution. 2

8. [8 pts] Two runners begin running from different points on a street; their respective positions at any time t, $0 \leq t \leq 1$, are given by $f(t) = t^5 + 2t - 1$ and $g(t) = 2t - t^2$.

(a) Why are the functions f and g continuous?

Solution.

Because they are both polynomial functions.

(b) Does either runner catch the other during this time? Carefully justify your answer.

Solution.

One runner catch each other when they are at same positions at the same time or equivalently when $f(t) = g(t)$. Solve the equation:

$t^5 + 2t - 1 = 2t - t^2 \iff t^5 + t^2 - 1 = 0.$

Call $h(t) = t^5 + t^2 - 1$.

Clearly the function h is continuous on $[0, 1]$ since it is a polynomial

Note that $h(0) = -1$, $h(1) = 1$, so that 0 is between $h(0)$ and $h(1)$.

Thus, by Intermediate Value Theorem, there is a point c in $[0, 1]$ such that $h(c) = 0$.

Because h(c) = f(c) - g(c), we conclude, from h(c) = 0, that f(c) = g(c); thus both runners are at the same position at time t = c.

9. [9 pts] Multiple-choice. Circle the correct response.

(a) Let f be a function such that $\lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = 5$. Which of the following must be true?

I) f is continuous at $x = 2$

II) f is differentiable at $x = 2$

III) The derivative of f is continuous at $x = 2$

(a) I only
(b) II only
(c) I and II only
(d) II and III only
(e) I, II, and III

Solution. (c)

(b) If $\lim_{x\to 3} f(x) = 7$, which of the following must be true?

I) f is continuous at $x = 3$

II) f is differentiable at $x = 3$

III) $f(3) = 7$

(a) none
(b) II only
(c) III only
(d) I and III only
(e) I , II, and III

Solution. (a)

(c) Let f and g be differentiable functions such that $f(1) = 2$, $f'(1) = 3$, $f'(2) = -4$, $g(1) = 2$, $g'(1) = -3$, and $g'(2) = 5$. If $h(x) = f(g(x))$, what is $h'(1)$?

(a) -9
(b) -4
(c) 0
(d) 12
(e) 15

Solution. (d)

10. A bacterial colony, originating from a single mother cell placed at the center of a petri dish, spreads outward, maintaining the shape of a disk of radius r, where r is measured in centimeters. See the diagram below.

Growing Bacteria Colony in a Petri Dish

(a) [2 pts] Express the amount of area occupied by this colony as a function of its radius r.

$$A = \pi r^2 \ \ cm^2$$

(b) [3 pts] Find the rate of change of area with respect to radius.

$$\frac{dA}{dr} = 2\pi r \ \ cm^2/cm$$