STUDENT NAME: ____________________

INSTRUCTOR: ____________________

Please sign the pledge:
On my honor as a student, I have neither given nor received aid on this exam.

Directions

Answer each question in the space provided. Please write clearly and legibly. ***Show all of your work—your work must justify your answer, and clearly identify your final answer. No books, notes, or electronic devices of any kind may be used during the exam period. You must simplify results of function evaluations, when it is possible to do so. For example,*** $4^{3/2}$ ***should be evaluated*** ($\boldsymbol{replaced\ by}$ 8)**.**

For instructor use only

Page	Points	Score
2	15	
3	10	
4	11	
5	10	
6	10	
7	12	
8	10	
9	10	
10	6	
11	6	
Total:	100	

1. [15 pts] Find derivatives of the following functions. Do not simplify your answers.

a) $f(x) = (x^3 + 1)^8$

$$f'(x) = 8(x^3 + 1)^7(3x^2)$$

b) $g(x) = \dfrac{e^{5x}}{2x + 5}$

$$g'(x) = \frac{5e^{5x}(2x + 5) - e^{5x}(2)}{(2x + 5)^2}$$

c) $h(x) = x \ln \sqrt{x^2 + 1}$

Notice that by the properties of natural log, $\ln \sqrt{x^2 + 1} = \frac{1}{2} \ln(x^2 + 1)$ so in fact $h(x) = \frac{x}{2} \ln(x^2 + 1)$ and so

$$h'(x) = \frac{1}{2} \ln(x^2 + 1) + \frac{x}{2} \left(\frac{2x}{x^2 + 1} \right)$$

Alternatively,

$$h'(x) = x \frac{1}{\sqrt{x^2 + 1}} \left(\frac{1}{2}(1 + x^2)^{-1/2}(2x) \right) + \ln \sqrt{x^2 + 1}$$

2. [10 pts] Given $x^5 + x^3y = 3y^2 + 1$.

a) Find $\dfrac{dy}{dx}$

We differentiate the equation implicitly with respect to x:

$$\begin{aligned} 5x^4 + 3x^2y + x^3\tfrac{dy}{dx} &= 6y\tfrac{dy}{dx} \\ \implies \quad 5x^4 + 3x^2y &= (6y - x^3)\tfrac{dy}{dx} \\ \implies \quad \tfrac{5x^4+3x^2y}{6y-x^3} &= \tfrac{dy}{dx} \end{aligned}$$

b) Find an equation of the tangent line to this curve at the point $(1, 0)$.

At this point the value of $\frac{dy}{dx}$ (the slope of the tangent line) is $\frac{5+0}{0-1} = -5$, and so the equation of the tangent line is

$$y = -5(x - 1).$$

3. [5 pts] Suppose $g(x)$ is a differentiable function with derivative given by $g'(x) = \dfrac{x}{x^4+1}$. Suppose that the function f is defined for $x \geq 0$ by $f(x) = \sqrt{3x}$. Let $h(x)$ be defined for $x \geq 0$ by $h(x) = g(f(x))$. Find $h'(2)$.

   By the chain rule

$$h'(x) = g'(f(x))f'(x) = g'(\sqrt{3x})\frac{\sqrt{3}}{2\sqrt{x}} = \left(\frac{\sqrt{3x}}{(\sqrt{3x})^4+1}\right)\left(\frac{\sqrt{3}}{2\sqrt{x}}\right) = \frac{3}{2(9x^2+1)}$$

   Therefore $h'(2) = \frac{3}{2(36+1)} = \frac{3}{74}$.

4. [6 pts] a) State the Extreme Value Theorem.

   If f is a continuous function on the closed interval $[a, b]$, then it achieves an absolute maximum and an absolute minimum on that interval

   b) Complete the following definition: the line $y = b$ is a horizontal asymptote of the graph of a function f if:

   either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$ (or both)

5. [10 pts] A steel ball drops through a fluid of varying viscosity so that its *position*, in feet below the surface of the fluid, is given by

$f(t) = t^3 - \frac{1}{4}t^4 \quad \text{for } 0 \le t \le 3,$

where time t is measured in seconds. What is the **greatest velocity** that the ball attains over the time interval $0 \le t \le 3$? Carefully justify your answer.

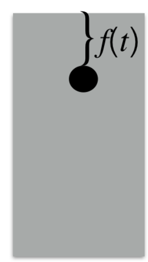
Notice that the velocity if given by

$$v(t) = f'(t) = 3t^2 - t^3.$$

This function is continuous on the interval $[0, 3]$ so by the Extreme Value Theorem it achieves an absolute maximum (that is, a greatest velocity). First we find the critical numbers in $[0, 3]$: since $\frac{dv}{dt} = 6t - 3t^2 = 3t(2 - t)$ we see that $t = 0$ and $t = 2$ correspond to the critical numbers, and now we make a table with those numbers (because one lies in the open interval $(0, 3)$ and the other is an endpoint) as well as the other endpoint to see where $v(t)$ achieves a maximum.

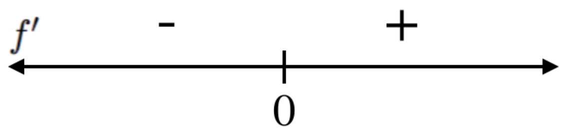
t	$v(t) = 3t^2 - t^3$
0	0
2	$12 - 8 = 4$
3	$27 - 27 = 0$

Therefore, the greatest velocity is achieved is 4 ft/sec (at $t = 2$ seconds).



6. [10 pts] Let $f(x) = \ln(x^2+4)$. You may take for granted that $f'(x) = \dfrac{2x}{x^2+4}$ and $f''(x) = \dfrac{2(4-x^2)}{(x^2+4)^2}$

a) (3 points) Find the interval(s) where f is increasing and where f is decreasing.



or

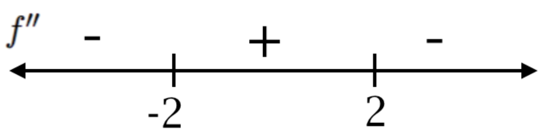
	$(-\infty,0)$	$(0,\infty)$
x	$-$	$+$
x^2+4	$+$	$+$
$f'(x)$	$-$	$-$
behavior $f(x)$	decreasing ↘	↗ increasing

Therefore, f is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$.

b) (2 points) Classify all local maxima and minima.

$x=0$ is the only critical number and by the previous sign-line/table we can see that it yields a relative minimum. That is, $f(0)=\ln(4)$ is a relative minimum value of f and f has no other relative extreme values.

c) (3 points) Find the interval(s) where the graph of f is concave up and those where the graph is concave down.



or

	$(-\infty,-2)$	$(-2,2)$	$(2,\infty)$
$2(4-x^2)$	$-$	$+$	$-$
$(x^2+4)^2$	$+$	$+$	$+$
$f''(x)$	$-$	$+$	$-$
behavior $f(x)$	concave down $\cap$	concave up $\cup$	$\cap$ concave down

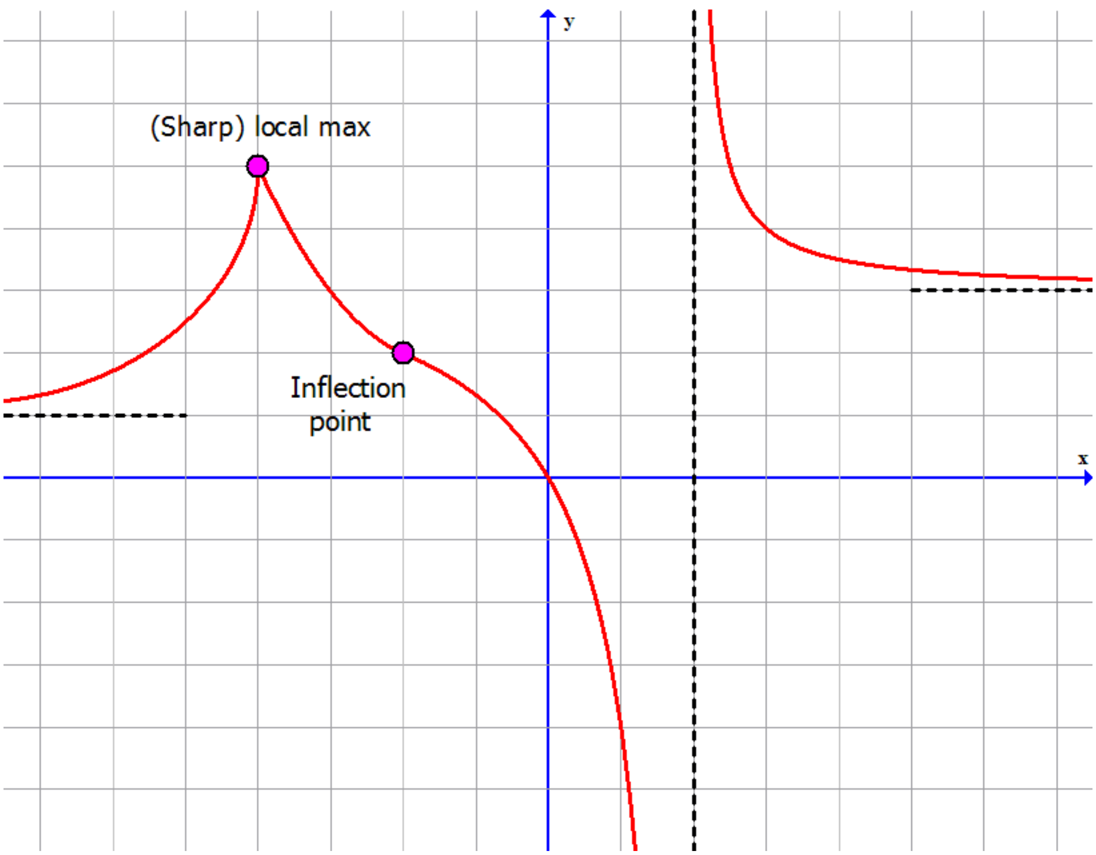
Therefore, the graph of f is concave up on the interval $(-2,2)$ and concave down on the intervals $(-\infty,-2)$ and $(2,\infty)$

d) (2 points) Find all the inflection points on the graph of f (provide both x and y coordinates of all the inflection points).

The concavity of the graph changes at the points $(-2, f(-2))$ and $(2, f(2))$ so these are our only two inflection points.

7. [12 pts] Sketch a graph of $f(x)$, given the following data:

- $f(x)$ is continuous everywhere except at $x = 2$, and it is twice differentiable everywhere except at $x = 2$ and $x = -4$
- $f(x)$ has a vertical asymptote at $x = 2$.
- $\lim_{x\to\infty} f(x) = 3$, $\lim_{x\to-\infty} f(x) = 1$
- $f(x)$ has only one x intercept: $(0, 0)$
- $f'(x)$ is positive on $(-\infty, -4)$, and negative on $(-4, 2)$, $(2, \infty)$
- $f''(x)$ is positive on $(-\infty, -4)$, $(-4, -2)$ and $(2, \infty)$, and negative on $(-2, 2)$
- $f(-4) = 5$, $f(-2) = 2$



8. [10 pts] The number of items produced by a manufacturer is given by

$$p = 100xy^3$$

where x is the amount of capital and y is the amount of labor. At a particular point in time:

(i) the manufacturer has 2 units of capital;

(ii) capital is increasing at a rate of 1 unit per month;

(iii) the manufacturer has 3 units of labor; and

(iv) labor is decreasing at a rate of 0.5 units per month.

Determine the rate of change in the number of items produced at this point in time. Is the number of items produced at this time increasing or decreasing?

Differentiating with respect to time we have

$$\frac{dp}{dt} = 100\left(x3y^2\frac{dy}{dt} + y^3\frac{dx}{dt}\right)$$

At the moment when $x = 2$, $\frac{dx}{dt} = 1$, $y = 3$ and $\frac{dy}{dt} = -0.5$ we have that

$$\frac{dp}{dt} = 100\left(2 \cdot 3 \cdot 3^2 \cdot \left(\frac{-1}{2}\right) + 3^3 \cdot 1\right) = 0.$$

At the instant in time described in the problem, $\frac{dp}{dt} = 0$. Thus, the production level at this instant is neither increasing nor decreasing. (Analogy: suppose that $f'(t_0) = 0$ where $f(t)$ = position of a particle at time t. Then the particle is a rest at time t_0, moving neither to the left nor to the right.)

9. [10 pts] A farmer wishes to fence in a rectangular garden of 100 ft^2. The north-south fences will cost \$1.50 per foot, while the east-west fences will cost \$6 per foot. Find the dimensions of the garden that will minimize the cost. You must fully justify your claim that you have found the dimensions minimizing the cost (using calculus).

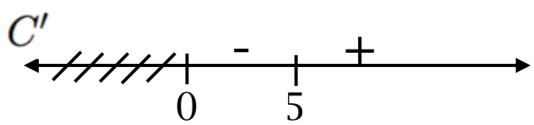
Using x for the length of the east-west fence and y for the length of the north-south fence, as the figure below, we have that

$$\begin{cases} 100 = xy \\ C = 1.5(2y) + 6(2x) = 3y + 12x \end{cases}$$

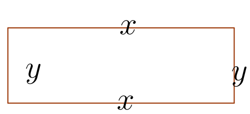
We can use the first equation to write $y = \frac{100}{x}$ and so the cost becomes

$$C(x) = \frac{300}{x} + 12x$$

We minimize C over the interval $(0, \infty)$. Notice that $C'(x) = -\frac{300}{x^2} + 12$ so the only critical number is $x = 5$.



The C' sign chart above shows that C attains its minimum value on $(0, \infty)$ at $x = 5$: the derivative C' is negative over $(0, 5)$; hence, the cost function decreases over this interval, while it must increase over $(5, \infty)$ because on this interval C' is positive. The continuity of C at 5 ensures that cost achieves it absolute minimum over $(0, \infty)$ at $x = 5$. Alternatively, observe that $C''(x) = \frac{600}{x^3}$ and so the graph of the function $C = C(x)$ is always concave up on its domain. Therefore, the critical number must correspond to an absolute minimum. The cost minimizing dimensions are $x = 5$ and $y = 20$.



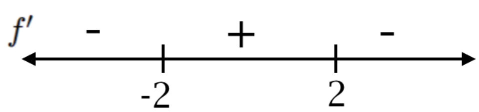
10. [3 pts] True/false question: if the answer is true give a brief explanation. If false, provide an example where the statement fails.

Suppose that $f(x)$ is a differentiable function on $(-\infty, \infty)$ and that $f'(x)+1 > 0$ for all x. Then on the interval $[a, b]$ the function $g(x) = f(x)+x$ has an absolute minimum at a.

True. First of all, $g(x)$ is a continuous, in fact differentiable; thus, by the Extreme Value Theorem, it has an absolute minimum on the interval $[a, b]$. Since $g'(x) = f'(x) + 1$ and $f'(x) + 1 > 0$ for all x we conclude that $g(x)$ is an increasing function. In particular, this implies that the minimum will occur at the left endpoint of the interval $[a, b]$, that is, at $x = a$.

11. [3 pts] Multiple choice questions: circle the correct answer

If g is a differentiable function such that $g(x) < 0$ for all real numbers x and if $f(x)$ is a function such that $f'(x) = (x^2 - 4)g(x)$, which of the following must be true?



	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
$x+2$	$-$	$+$	$+$
$x-2$	$-$	$-$	$+$
$g(x)$	$-$	$-$	$-$
$f'(x)$	$-$	$+$	$-$
behavior $f(x)$	decreasing ↘	↗ increasing	decreasing ↘

We have

Thus,

a) f has a relative maximum at $x = -2$, and a relative minimum at $x = 2$

b) f has a relative minimum at $x = -2$ and a relative maximum at $x = 2$ TRUE

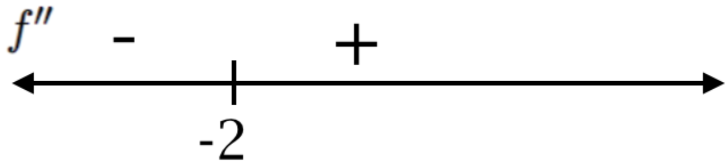
c) f has relative minima at $x = -2$ and at $x = 2$

d) f has relative maxima at $x = -2$ and at $x = 2$

e) it cannot be determined if f has any relative extrema

12. [3 pts] **Consider the function $f(x) = 2xe^x$. For what values of x is the graph of $f(x)$ concave down?**

$f'(x) = 2(xe^x + e^x)$; $f''(x) = 2(xe^x + 2e^x) = 2e^x(x + 2)$. Thus,



and (e) is correct.

a) $x > 2$

b) $x > 1$

c) $x < 2$

d) $x < -1$

e) $x < -2$ TRUE

13. [3 pts] **The graph of a twice-differentiable function f is shown in the figure below. Which of the following appears to be true?**

We see that $f(1) = 0$, that the slope of the line tangent to the graph of f at 1 is positive, so that $f'(1) > 0$. Finally, because the graph of f is concave down near 1 we expect that $f''(1) < 0$. Thus (d) is the correct choice.

a) $f(1) < f'(1) < f''(1)$

b) $f(1) < f''(1) < f'(1)$

c) $f'(1) < f(1) < f''(1)$

d) $f''(1) < f(1) < f'(1)$ TRUE

e) $f''(1) < f'(1) < f(1)$