Official Study Guide Final Exam

Information for Final Exam:

- Tuesday December 12th
- 7:00 pm - 10:00 pm
- Location: Instructors add your exam location
- Topics on the Exam: The Final Exam will be cumulative, at most 30 out of 100 points will be material covered since Exam 2, that is, sections 6.1-6.5. To prepare for the final, you should review topics covered on Exams 1 and 2, assisted by the study guides for those exams. To review material covered since Exam 2, use this guide. The study guides should not be your only material used when studying for the final exam. Reviewing previous WebAssign sets, midterm exams, quizzes, written homework assignments, the textbook, and your notes is all recommended, as is completing posted practice final exams.

  Note: For all limit problems, your work must be well-organized. In particular, for a limit that exists, a string of valid equalities must connect the original limit problem with the value of the limit. For limits at infinity, you won't be required to justify your answer: giving the correct limit will be considered sufficient.

The following list of topics is not exhaustive, it is just meant to highlight the most important aspects of the chapter

Topics to Keep in Mind from Chapter 6:

- If $f(x)$ is a function, all possible antiderivatives of $f(x)$ on an interval of interest are represented by the symbol $\int f(x)dx$. Under this notation, we might say that

  (1) $$\int f(x)dx = \text{ the family of all functions having derivative } f(x);$$

  We might also say,

  (2) $$\int f(x)dx = \text{the general antiderivative of } f(x).$$

  For example, $\int xdx = \frac{x^2}{2} + C$ means that all functions having derivative $f(x) = x$ on say, $(-\infty, \infty)$ are of the form $\frac{x^2}{2} + C$ where C is an arbitrary constant

- Remark: In general, finding antiderivatives is not as straightforward as finding derivatives. There are simple-looking functions such as $f(x) = e^{x^2}$ that do not have "nice antiderivatives".[1] On the final

[1] Roughly speaking, "not nice" here means "can't be expressed as a finite algebraic combination of functions with which you are familiar".

exam, if you are asked to find an antiderivative of a function f or to evaluate an indefinite integral $\int f\,dx$, then you will be dealing with a function f that *does* have a nice antiderivative, which you'll be able to find using the integration/antidifferentiation rules and techniques you have learned in this course.

- You should know the following integration rules.

Rule	Notation
Integral of a constant	$\int kdx = kx + C$
Constants slide through integrals	$\int kf(x)dx = k\int f(x)dx$
The integral of a sum is the sum of the integrals	$\int (f(x)+g(x))\,dx = \int f(x)dx + \int g(x)dx$
Integral of a power of x	$\int x^n dx = \frac{1}{n+1}x^{n+1} + C \qquad n \neq -1$
Integral of $\frac{1}{x}$	$\int \frac{dx}{x} = \ln\lvert x\rvert + C$
Integral of an exponential	$\int e^x dx = e^x + C$

- *Your work on indefinite integrals must be well organized; in particular, you must have a string of valid equalities connecting the original indefinite integral with your final answer.*

- An **Initial Value Problem** (IVP) is a problem of the form

(3)
$$\begin{cases} \frac{dy}{dx} = g(x) \\ y\left(x_0\right) = y_0 \end{cases}$$

That is, you want to find a function $y(x)$ whose derivative is $g(x)$ and whose value at x_0 is y_0. To solve an IVP do the following:

1. Integrate both sides of the differential equation $\frac{dy}{dx} = g(x)$, obtaining
$$y(x) = G(x) + C,$$
where G is an antiderivative of g and C is a constaht
2. We have $y(x) = G(x) + C$, and upon substituting x_0 for x in this equation, we obtain $y_0 = G(x_0) + C$, or $C = y_0 - G(x_0)$. The final solution is $y(x) = G(x) + C$, where $C = y_0 - G(x_0)$.

For example, to solve the IVP

(4)
$$\begin{cases} \frac{dy}{dx} = x^2 - 1 \\ y(1) = -1 \end{cases}$$

We use the first step to write $y(x) = \int \left(x^2 - 1\right) dx = \frac{x^3}{3} - x + C$ and then choose the constant C such that $y(1) = -1$, that is, $\frac{1}{3} - 1 + C = -1$ which gives $C = -\frac{1}{3}$ so $y(x) = \frac{x^3}{3} - x - \frac{1}{3}$ is the solution of the IVP.

As a physical application of initial value problems, we have that the velocity of an object as a function of time $v(t)$ is an antiderivative of its acceleration $a(t)$ and the position of an object as a function of time $s(t)$ is an antiderivative of the velocity $v(t)$, that is,

$$v(t) = \int a(t)dt \quad \text{and} \quad s(t) = \int v(t)dt,$$

where we interpret the preceding equations to specify the forms of v and s, with constants of integration being determined by appropriate initial conditions.

For example if the acceleration as a function of time is $a(t) = t^2 - 7$ then its velocity is $v(t) = \int a(t)dt = \int \left(t^2 - 7\right) dt = \frac{t^3}{3} - 7t + C$ where the constant C is determined once an initial condition is given for v.

- You should be prepared to find indefinite integrals via the method of substitution, often called "u-substitution." See Section 6.2 of the text.
  - For example, to find the integral $\int \frac{\sqrt{\ln x}}{x} dx$ we make the substitution $u = \ln x$. In this case $du = \frac{dx}{x}$ and the integral $\int \frac{\sqrt{\ln x}}{x} dx$ becomes

$$\int \frac{\sqrt{\ln x}}{x} dx = \int \sqrt{u}\, du = \frac{2}{3} u^{\frac{3}{2}} + C = \frac{2}{3} (\ln x)^{\frac{3}{2}} + C \tag{5}$$

- You should know how definite integrals are defined via *Riemann Sums*.

Definition of a Riemann Sum and the Definite Integral: Let f be defined on $[a, b]$. The definite integral of f from a to b, denoted $\int_a^b f(x)dx$ is given by

$$\begin{aligned} \int_a^b f(x) &= \lim_{n\to\infty} [f(x_1)\triangle x + f(x_2)\triangle x + \cdots + f(x_n)\triangle x] \\ &= \lim_{n\to\infty} [f(x_1) + f(x_2) + \cdots + f(x_n)] \left(\frac{b-a}{n}\right) \end{aligned} \tag{6}$$

provided the limit exists and takes the same value independent of how the numbers x_i in $\left[a + (i-1)\frac{(b-a)}{n}, a + i\frac{(b-a)}{n}\right]$ are chosen. Tan calls these numbers $x_1, x_2, \cdots, x_n$ representative points.

- When the limit defining $\int_a^b f(x)dx$ exists we say that f is integrable on $[a, b]$. It is possible to show that if f is continuous on $[a, b]$ then f is integrable on $[a, b]$.
- Because continuous functions are integrable, we can simplify our definition of definite integral if the integrand f is continuous as follows:

1. **Evaluation via left-endpoint approximation:** assume f is continuous on $[a, b]$, then choosing representative points as left-endpoints, we have

$$\int_a^b f(x)dx \equiv \lim_{n\longrightarrow\infty} \left(\frac{b-a}{n}\right) \left[f(a) + f\left(a + \frac{b-a}{n}\right) + \cdots + f\left(a + (n-1)\frac{(b-a)}{n}\right)\right] \tag{7}$$

   For example, Left Endpoint Approximations for $f(x) = x^2$ on $[a, b] = [0, 1]$ yield

$$\int_0^1 x^2 dx = \lim_{n\longrightarrow\infty} \frac{1}{n} \left[(0)^2 + \left(\frac{1}{n}\right)^2 + \cdots + \left(\frac{n-1}{n}\right)^2\right] \tag{8}$$

2. **Evaluation via right-endpoint approximation:** assume f is continuous on $[a, b]$, then choosing representative points as right-endpoints, we have

$$\int_a^b f(x)dx \equiv \lim_{n\longrightarrow\infty} \left(\frac{b-a}{n}\right) \left[f\left(a + \frac{(b-a)}{n}\right) + f\left(a + 2\frac{(b-a)}{n}\right) + \cdots + f(b)\right] \tag{9}$$

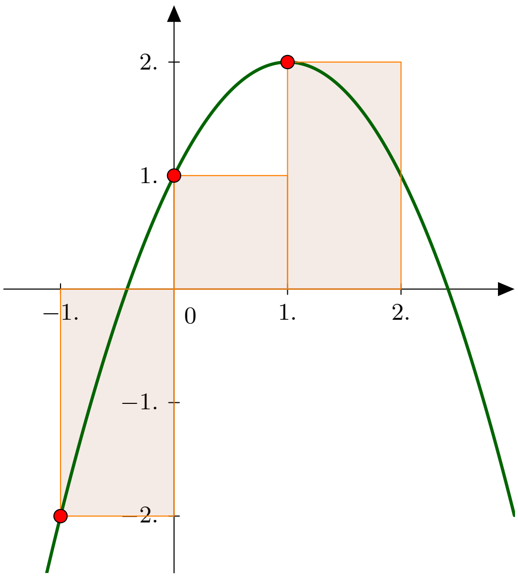
   For example, Right Endpoint Approximations for $f(x) = x^2$ on $[a, b] = [0, 1]$ yield

$$\int_0^1 x^2 dx = \lim_{n\longrightarrow\infty} \frac{1}{n} \left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + 1^2\right] \tag{10}$$

  - **Remark 1): you should be able to compute the Riemann sums for f corresponding to left-endpoint approximations and right-endpoint approximations.** For example, if you are asked to approximate $\int_{-1}^2 (2x - x^2 + 1)dx$ using the left-endpoint approximation using three subdivisions of the interval $[-1, 2]$ you must calculate the following sum: (we are taking

$[a,b] = [-1,2]$, $n = 3$ and $f(x) = 2x - x^2 + 1$ in the formula for the left-endpoint approximation)

$$\left(\frac{2-(-1)}{3}\right)[f(-1) + f(0) + f(1)] = f(-1) + f(0) + f(1) = -2 + 1 + 2 = 1 \tag{11}$$



See also problems 3,4,14 and 15 on pages 428 and 429.

– **Remark 2): you should be able to compare left-endpoint (or right-endpoint) Riemann sums for f on $[a,b]$ to the value of $\int_a^b f(x)dx$ in certain situations.** For instance, when f is positive and increasing on $[a,b]$, you should know that finite left endpoint approximation is an underestimate of $\int_a^b f(x)\,dx$ and a finite right endpoint approximation is an overestimate of $\int_a^b f(x)\,dx$. When f is positive and decreasing on $[a,b]$, then a finite left endpoint approximation is an overestimate of $\int_a^b f(x)\,dx$ and a finite right endpoint approximation is an underestimate of $\int_a^b f(x)\,dx$.

- You should know the following properties of the definite integral.

The Definite Integral and Area: $\int_a^b f(x)dx$ is the **signed or net area** under/over the curve of $y = f(x)$. Area should always be deemed positive. If the curve is above the x axis, then integrating left to right yields the area under the curve and above the x axis, whereas if the curve is below the x axis, the integrating left-to-right yields negative the area above the curve and below the x axis. In general, assuming $a < b$, the integral $\int_a^b f(x)dx$ is a difference of areas: the area under the curve $y = f(x)$ and above the interval $[a,b]$ minus the area above the curve and under the interval $[a,b]$. See the picture on page 427 of the text.

Additional Properties of the definite integral: Suppose that $f(x), g(x)$ are functions defined on $[a,b]$ for which $\int_a^b f(x)dx$, $\int_a^b g(x)dx$ exist. If c is any number between a and b and k any constant then

- $\int_a^a f(x)dx = 0$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- $\int_a^b (f(x) + kg(x))\, dx = \int_a^b f(x)dx + k\int_a^b g(x)dx$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- The last property allows us to split the integral $\int_{c_1}^{c_2} f(x)dx$ as

$$\int_{c_1}^{c_2} f(x)dx = \int_{c_1}^{c_3} f(x)dx + \int_{c_3}^{c_2} f(x)dx \tag{12}$$

regardless of the order of the points c_1, c_2, c_3! For example, $\int_1^{-2} x^2dx = \int_1^0 x^2dx + \int_0^{-2} x^2dx$.

- **You must be able to state the Fundamental Theorem of Calculus**:

Let f be a continuous function on $[a,b]$. Then $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f; that is, $F'(x) = f(x)$.

- When computing definite integrals via the Fundamental Theorem, sometimes you will use the method of substitution as part of the process. There are two natural ways to organize your work (see pages 443 and 444 of the textbook), which we will now illustrate. *Note well: you must organize your work in one of these ways. In particular, the original definite integral must be connected by a string of valid equalities (showing how the Fundamental Theorem of Calculus is being applied) to your final answer*

  For example, suppose that we want to find $\int_0^{\sqrt{2}} xe^{x^2}dx$.

- FIRST METHOD: you compute first the corresponding indefinite integral $\int xe^{x^2}dx$, for which we use the substitution $u = x^2$, $du = 2xdx$ or $\frac{1}{2}\,du = x\,dx$. We have

$$\int xe^{x^2}dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C. \tag{13}$$

  Thus, $F(x) = \frac{1}{2}e^{x^2}$ is an antiderivative of $f(x) = xe^{x^2}$ and the Fundamental Theorem of Calculus allows us to conclude that

$$\int_0^{\sqrt{2}} xe^{x^2}dx = \left(\frac{1}{2}e^{x^2}\right)\Bigg|_0^{\sqrt{2}} = \left(\frac{1}{2}e^2 - \frac{1}{2}e^0\right) = \frac{1}{2}e^2 - \frac{1}{2}. \tag{14}$$

- SECOND METHOD: we choose to work with the definite integral from the beginning, but now we need to keep track of the limits of integration. In order to find $\int_0^{\sqrt{2}} xe^{x^2}dx$ we use again the substitution $u = x^2$, $du = 2xdx$ but now we keep track of the limits of integration. Since

$$\begin{cases} x = 0 \implies u = 0 \\ x = \sqrt{2} \implies u = 2 \end{cases} \tag{15}$$

  the definite integral becomes

$$\int_0^{\sqrt{2}} xe^{x^2}dx = \frac{1}{2}\int_0^2 e^u du = \frac{1}{2}\left(e^u\right)\Big|_0^2 = \frac{1}{2}\left(e^2 - e^0\right) = \frac{1}{2}\left(e^2 - 1\right). \tag{16}$$

The advantage of this method is that after you find the integral in terms of u there is no need to use the substitution again in order to get an expression in terms of x since the new limits of integration take care of that. However, if you use this method be sure to change the limits of integration!

- You should know the Net-Change Formula: The net change in a function f over an interval $[a, b]$ is given by

$$f(b) - f(a) = \int_a^b f'(x)\, dx,$$

provided that f' is continuous in $[a, b]$.

- You should know the formula for the average value. Suppose that $f(x)$ is a function defined on $[a, b]$. Then its **average value over** $[a, b]$ is by definition

$$\frac{1}{b-a} \int_a^b f(x) dx$$