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Let's look at one example and you can try another one on your own later (hopefully).

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Now we have to compute eigen-vectors corresponding to the three distinct eigen values.

Eigen Vector corresponding to $z = 6$

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We need $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ such that

$$\begin{bmatrix} -5 & 4 & 3 \\ 4 & -5 & 0 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Our goal is to bring the matrix $\begin{bmatrix} -5 & 4 & 3 \\ 4 & -5 & 0 \\ 3 & 0 & -5 \end{bmatrix}$ into a simple form of the type $\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$ where $*$ can be any real number and then work with this changed matrix.

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Taking $v_3 = 3$, we have an eigen vector $\begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$.

In a similar way, we can find eigen vectors corresponding to $z = 1, -4$. Follow the same method. Try finding at least one of them.

Algorithm

• Given $A_{3\times 3}$, compute $A - z\mathcal{I}_3$; put $\det(A - z\mathcal{I}_3) = 0$ to get eigen-values a, b, c which may not be distinct. We are only bothered about real roots for the time being.

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• You will see that fixing at most two of the $v_i's$ to be any constants will give you an eigen vector.