MATH 1210–Section 007
Homework 1

Remark: All these problems are from past question papers. Even though you don't need to turn in all of these, make sure you know how to solve them all.

1. Evaluate the following limits or state that they do not exist. You must justify your answer, e.g. using the definition of a limit or a limit property to receive full credit. (Answer any 2, but make sure you know how to do all of them)

a. $\displaystyle\lim_{x\to 4} \frac{\sqrt{x-3}+1}{x^2-5x+4}$

b. $\displaystyle\lim_{x\to 2} \frac{\sqrt{14+t}-4}{t-2}$

c. $\displaystyle\lim_{x\to 2} \frac{x^2+x-2}{x-5}$

d. $\displaystyle\lim_{x\to\infty} \frac{9000x^5-30}{x^5-3000x-15}$

e. $\displaystyle\lim_{x\to 3^-} \frac{3x-x^2}{|3-x|}$

f. $\displaystyle\lim_{x\to -1} \frac{|1+x|}{x}$

g. $\displaystyle\lim_{x\to\infty} \frac{\sqrt{30x}}{x+29}$

2. Write the domains of the following functions in interval notation: (Submit only (b) or (e), but practice all of these)

a. $F(x) = \dfrac{\sqrt{1-x}}{\sqrt{x}}$

b. $R(x) = (g\circ f)(x) + h(x)$ where $f(x) = x+1,\ g(x) = \dfrac{1}{3x+5},\ h(x) = x^3$

c. $f(t) = \dfrac{\sqrt{t-1}}{t^2-2t-3}$

d. $g(x) = \dfrac{x^2+9}{x^2-x-6}$

e. $h(x) = g(f(x))$ where $f(x) = 2x+1,\ g(x) = \dfrac{x+3}{x-1}$

3. Answer any one from the following mainly to practice writing the solution. (My suggestion is do (e), but feel free to choose your own)

a. Is the function $f(x) = \begin{cases} x^2+1, & x \le 3 \\ 12 - \frac{6}{x}, & x > 3 \end{cases}$ continuous at $x=0$? Is it continuous at $x=3$? Justify your answer.

b. Find the value of c that makes $J(x)$ a continuous function on $\mathbb{R}$ (all real numbers)

$$J(x) = \begin{cases} 2x^2+cx-1, & x<1 \\ \sqrt{x+1}, & x \ge 1 \end{cases}$$

c. Let $f(x) = \begin{cases} \frac{10}{x-5} & x<0 \\ x^3+1 & x \ge 0 \end{cases}$. Find the domain of f. Show that f is continuous at $x=2$ but discontinuous at $x=0$.

d. For $f(x) = \begin{cases} 5x+1 & x<1 \\ k & x=1, \\ x^2+5 & x>1 \end{cases}$ determine k that makes the function continuous at $x=1$. Justify your answer.

e. For what value of the constant c is the following function continuous on $(-\infty, \infty)$:

$$f(x) = \begin{cases} cx^2+2x & x<2 \\ x^3-cx & x \geq 2 \end{cases}$$

f. Find where the function f is continuous.

$$f(x) = \begin{cases} -2x+1 & x<1 \\ 0 & x=1 \\ \frac{1}{x-2} & x>1 \end{cases}$$

4. Answer any one of the following.

a. A race track with perimeter 1 mile has two identical semicircles at the ends of a rectangular area. Assuming both semicircles have radius r and the rectangle has length y, both measured in miles. Find a function f in the variable r giving the area enclosed by the race track. Find the domain of f.

b. A rectangular box made of sheet metal is to have a square base and a volume of 100 in^3.

(i) Letting x denote the length of one side of the base, find a function $f(x)$ giving the amount (in square inches) of sheet metal needed to construct the box.

(ii) What if the domain of $f(x)$ you found in (i)?

(iii) How much sheet metal is needed to construct a box with dimensions 5 in $\times$ 5 in $\times$ 5 in?

c. Andy is going on a 10-day trip in a few months. He paid for 10 nights at \$100 per night for his hotel room. He is delaying his purchase of a plane ticket, hoping to buy one at a price he will find acceptable. Andy uses a simple "travel quotient" function to figure out which prices are acceptable. The travel quotient $Q(A)$ is given by the airplane ticket price A, divided by S, where S is the sum of the ticket price A and the amount Andy has already spent on lodging.

(i) Write the rule for the travel quotient $Q(A)$ as a function of A.

(ii) For Andy, an acceptable price for an airplane ticket is any price A so that $Q(A) \leq 9$. Should Andy buy when the ticket price is \$250?

d. The owner of a farm has 3000 yards of fencing with which in enclose a rectangular piece of grazing land along the side of a straight sided river. Fencing is not required next to the river. Let x be the width (perpendicular to the river) and y be the length (parallel to the river) of the enclosed land. Find a function f in terms of x for the area of the grazing land (in square yards). Find the domain of this function given the physical limitations, namely that x and y are positive numbers.

5. Answer any one of these. Make sure you understand all the problems. These are very good problems!

a. Dr. Snowworthy has constructed mathematical models for the amounts of snowfall in Duluth, MN and in Buffalo, NY. According to her model, the amount of snowfall in Duluth, in inches, t months after December, $1 \leq t \leq 4$, is predicted to be $D(t) = 13 + 12t^2 - 2t^3$ and in Buffalo is predicted to be $B(t) = 10 + 2t + 9t^2 - t^3$. According to Dr. Snowworthy's model, is there a time t in the interval $[0, 4]$ such that the amount of snow fallen in Duluth will equal the amount fallen in Buffalo? Carefully justify your answer.

b. Is there a real number that is exactly 1 more than its cube? Explain your answer. (Hint: Observe that -2 is 6 more than its cube and 0 is the cube of itself).

c. Let $f(x) = x^4 + 2x^3 + 5x + 2$. Does f have a root in the interval $(-1, 1)$? Justify.

d. Let $g(x) = x^3 - 4x^2 + x + 6$.

(i) Is g continuous? What is its domain?

(ii) Show that $g(x)$ has at least one real root.

e.. Is the difference of $f(x) = x^5 + 2x^2$ and $g(x) = x^3 + 1$ ever 0 on the interval $[0, 1]$? Explain why or why not.

6. Answer any one:

a. Compute the derivative of $f(x) = 3 - \sqrt{x}$ at the point $x = 2$ from the definition. Find the equation of the tangent line to graph of f at the point $x = 2$.

b. (i) Given a function $f(x)$ such that $f(8) = 7, f'(8) = -2$, find $g'(8)$ where $g(x) = 1 + f(x)\sqrt[3]{x}$.

(ii) Find the equation of the tangent line to the graph of $g(x)$ at the point $(8, g(8))$.

c. Find the equations of all horizontal tangent lines to the graph of $f(x) = x^3 - 12x$.

7. Find derivatives for practice. There is no need to submit these.

a. $g(x) = \dfrac{x-2}{x^2+3x+1} + 5x$

b. $g(s) = \dfrac{-2s^2}{\sqrt{2} + s^3}$

c. $f(v) = (v^{\sqrt{5}} - \dfrac{1}{v^2})(v^3 + 1)$.

d. Suppose $g(x)$ is a differentiable function. Let $f(x) = 3x^2 g(x) - 5x$.

(a) Find $f'(x)$ using the rules of differentiation.

(b) Find $f'(1)$ given that $g(1) = 3$ and $g'(1) = 2$.

(c) What is the equation of the tangent line to the graph of $y = f(x)$ at $x = 1$?

8. Write True/False. If True, provide a brief justification. If False give a counterexample or disprove it. Answer any two

a. If $f'(3)$ exists, then $\lim_{x \to 3} f(x)$ may not always exist. (Hint: If differentiable, what do we know about continuity at that point?)

b. If a function f is differentiable, then the derivative function f' must be continuous.

c. There is a continuous function $f(x)$ defined on $[1, 3]$ such that $f(1) = 0$ and $f(3) = 5$ but $f(x) \neq 2$ for any x between 1 and 3.

d. There is a function $f(x)$ defined on $[1, 3]$ such that $f(1) = 0$ and $f(3) = 5$ but $f(x) \neq 2$ for any x between 1 and 3.

Bonus Problem:*If f is differentiable at $x = a$, then f is continuous at $x = a$*

Step 1: Make a clever substitution for h in the definition of $f'(a)$ to get $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$.

Step 2: Remember that for the derivative $f'(a)$ to be defined, a is required to be in the domain of $f(x)$. So first condition in the definition of continuity of f at a is guaranteed.

Step 3: Using limit laws, show that $\lim_{x \to a} f(x) = f(a)$ is the same as $\lim_{x \to a} \Big(f(x) - f(a)\Big) = 0$.

Step 4: We will show that $\lim_{x \to a} f(x) = f(a)$. Since $f(a)$ exists by Step 2, this will also show that the limit will exist. Thus, we will satisfy the remaining two conditions for continuity.

By step 3, it is enough to show that $\lim_{x \to a} \Big(f(x) - f(a)\Big) = 0$.

Here's how you start:

Let $h(x) = \dfrac{f(x) - f(a)}{x - a}$. By Step 1, $\lim_{x \to a} h(x)$ exists. Now,

$\lim_{x \to a} (f(x) - f(a)) = \lim_{x \to a} \left[\dfrac{f(x) - f(a)}{x - a}(x - a)\right] = \lim_{x \to a} \Big(h(x)(x - a)\Big)$. Use limit laws to complete the proof.