Official Study Guide Exam 1

Information Midterm 1:

- Thursday, September 28th
- 7:00 pm - 8:30 pm
- Location:
- Topics on the Exam: Applied Calculus by Tan, Ninth edition: sections 1.3 to 3.3

The following list of topics is not exhaustive, it is just meant to highlight the most important aspects of each chapter

Topics to Keep in Mind from Chapter 1:

- Practice interval notation since **problem statements are likely to indicate that you must express domains (and potentially ranges) of functions in interval notation**. For example, the domain of $f(x) = \sqrt{x}$ must be written as $[0, \infty)$ instead of $x \geq 0$.
- Some problems from Chapters 2 (and 3) require use of algebra reviewed in sections 1.1 and 1.2; e.g., factorization of polynomials, rationalizing the numerator or denominator of a fraction (in a limit problem), and properties of exponents.
- Solving simple inequalities, for example $2x - 5 \leq 8$.
- Finding the absolute value of numbers (like $|2 - \sqrt{7}|$).
- Properties of lines (slope, intercept, parallel, perpendicular); finding equations of a line (using the point-slope as well as slope intercept formulas).
- You should know formulas like the area/perimeter of circles, rectangles, right triangles and the Pythagorean Theorem.
- You should also know the general quadratic formula. This formula states that if $b^2 - 4ac \geq 0$ then the solutions of the equation $ax^2 + bx + c = 0$ are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Topics to Keep in Mind from Chapter 2:

- Basic facts about functions: recognizing which curves can be the graphs of a function (vertical line test), finding domains and ranges of functions (ranges based on graphs), performing algebraic operations on functions (like adding, subtracting, multiplying, dividing and composing functions)

- Interpret and work with basic mathematical models (e.g, problems 55 and 56 on page 73 and problem 17 on page 85).

- Limit Techniques: multiplying by the conjugate over the conjugate, simplifying an expression before evaluating the limit, calculating one-sided limits (especially useful to study functions that have absolute values or are piecewise defined), limits at infinity. For limits at infinity problems you will be expected to justify your answer with the appropriate algebra. For example, saying that $\lim\limits_{x \longrightarrow \infty} \dfrac{x^2+x+1}{3x^2-3} = \dfrac{1}{3}$ without further justification will be considered incomplete. You can divide the numerator and denominator by x^n, where n is the largest power of x appearing in the denominator,

  $$(1) \qquad \lim_{x \longrightarrow \infty} \frac{x^2+x+1}{3x^2-3} = \lim_{x \longrightarrow \infty} \frac{\left(x^2+x+1\right) \cdot \frac{1}{x^2}}{\left(3x^2-3\right) \cdot \frac{1}{x^2}} = \lim_{x \longrightarrow \infty} \frac{1+\frac{1}{x}+\frac{1}{x^2}}{3-\frac{3}{x^2}} = \frac{1}{3},$$

  or factor out the highest-power term in the numerator and denominator,

  $$(2) \qquad \lim_{x \longrightarrow \infty} \frac{x^2+x+1}{3x^2-3} = \lim_{x \longrightarrow \infty} \frac{x^2\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}{x^2\left(3-\frac{3}{x^2}\right)} = \lim_{x \longrightarrow \infty} \frac{1+\frac{1}{x}+\frac{1}{x^2}}{3-\frac{3}{x^2}} = \frac{1}{3}.$$

- **Your work on limit problems must be well organized; in particular, for a limit that exists, a string of equalities must connect the original limit problem with the value of the limit.**

- **You should know the definition of continuity of a function f at a point a.** That is, if asked "what is the definition of continuity of f at a point a?" you should write "The function f is continuous at a number a provided (1) $f(a)$ is defined, (2) $\lim\limits_{x \to a} f(x)$ exists, and (3) $\lim\limits_{x \to a} f(x) = f(a)$.

- You should know that polynomial functions and rational functions are continuous at every point of their domains and that continuity is preserved under the elementary algebraic operations, for example, that the sum of two continuous functions is a (new) continuous function, the product of continuous functions is a (new) continuous function, etc.

- You should be able to determine at which points a function f is continuous, either from its graph or from its formula (using the theorems discussed in the preceding item or using the definition of continuity at a point).

- **You should be able to state and use the Intermediate Value Theorem.** That is, if asked "write the Intermediate Value Theorem" you should write something like "If f is a continuous function on a closed interval $[a, b]$ and M is any number between $f(a)$ and $f(b)$ then there is at least one number c in $[a, b]$ such that $f(c) = M$". Also, **when using the Intermediate Value Theorem be sure to justify why you can apply it**, in particular, you must justify why your function is continuous. For example, suppose that the problem is

  Show that the equation $x^3 + x - 1 = 0$ has a solution.

  You might respond as follows: "Let $f(x) = x^3 + x - 1$. Observe that $f(0) = -1$ while $f(1) = 1$. Because f is continuous on $[0, 1]$ (it's a polynomial function) and 0 is between $f(0)$ and $f(1)$, the Intermediate Value Theorem tells us that there must be at least one number c between 0 and 1 such that $f(c) = 0$; that is, $c^3+c-1 = 0$. Hence, c is a solution of $x^3 + x - 1 = 0$."

- **You should know the definition of the derivative of a function f.** That is, if asked "write the definition of the derivative of a function f" you should write "the derivative of f at x is given by $f'(x) = \lim\limits_{h \longrightarrow 0} \dfrac{f(x+h) - f(x)}{h}$ provided the limit exists." If asked to find a derivative using the definition, you can't use any of the differentiation rules to find it; you must find $\lim\limits_{h \longrightarrow 0} \dfrac{f(x+h) - f(x)}{h}$ using the limit laws and techniques.

- Geometrical interpretation of derivatives in terms of slopes of tangent lines, namely, $f'(a)$ is the slope of the tangent line to the graph of f at the point $(a, f(a))$.

- Interpretation of Derivatives as rates of change, namely, if $y(x)$ describes a quantity y as a function of a quantity x then $\frac{dy}{dx}$ is the (instantaneous) rate of change of y with respect to x. For example, if $x(t)$ is the position of a particle then $\frac{dx}{dt}$ is the (instantaneous) rate of change of the position x with respect to the time t, that is, $\frac{dx}{dt}$ is the velocity of the particle. (If, e.g., x is expressed in meters and t in seconds, then the units of dx/dt would be meters per second.)

- **Relationship between differentiability and continuity: namely, if $f'(a)$ exists then f is continuous at a.** This means that continuity is a *necessary* condition for the existence of the derivative. However, continuity is not a sufficient condition: a function can be continuous at a without having a derivative at a (think of $f(x) = |x|$ at $a = 0$).

Topics to Keep in Mind from Chapter 3:

Differentiation Rules:

- Rules from Section 3.1:

  (i) $\frac{d}{dx}[c] = 0$ (c, a constant) (ii) $\frac{d}{dx}[x^n] = nx^{n-1}$ for every real number n

(iii) $\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}[f(x)]+\frac{d}{dx}[g(x)]$

(iv) $\frac{d}{dx}\left[f(x)-g(x)\right]=\frac{d}{dx}[f(x)]-\frac{d}{dx}[g(x)]$

(v) $\frac{d}{dx}[cf(x)]=c\frac{d}{dx}[f(x)]$

- Product and Quotient Rules

(vi) Product Rule: $\frac{d}{dx}[f(x)g(x)]=f(x)\frac{d}{dx}[g(x)]+g(x)\frac{d}{dx}[f(x)]$

(vii) Quotient Rule: $\displaystyle \frac{d}{dx}\left[\frac{hi(x)}{lo(x)}\right]=\frac{lo(x)\frac{d}{dx}[hi(x)]-hi(x)\frac{d}{dx}[lo(x)]}{(lo(x))^2}$

- Chain Rule

(viii) General Chain Rule: $\frac{d}{dx}\left[g(f(x))\right]=g'(f(x))f'(x)$ or

$$\frac{d}{dx}\left[g(f(x))\right]=g'(f(x))\frac{d}{dx}[f(x)].$$

(ix) Consequence "General Power Rule" (or "The Power Rule for Functions"): $\frac{d}{dx}\left[\Big(f(x)\Big)^n\right]=n\Big(f(x)\Big)^{n-1}\frac{d}{dx}\left[f(x)\right]$.

Common Pre-Calculus Mistakes

1. **MISTAKE**: $(x+y)^2=x^2+y^2$. Powers don't behave that way. The correct way to expand this expression gives

   (3) $$(x+y)^2=x^2+2xy+y^2$$

2. **MISTAKE**: $\frac{1}{x+y}=\frac{1}{x}+\frac{1}{y}$. The rule for adding fractions gives

   (4) $$\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}$$

3. **MISTAKE**: $\frac{1}{x+y}=\frac{1}{x}+y$. This error comes from carelessness about what's in the denominator

4. **MISTAKE**: $\sqrt{x+y}=\sqrt{x}+\sqrt{y}$. There is no simplified way to write $\sqrt{x+y}$

5. **MISTAKE**: If $x<y$ then $kx<ky$ where k is *any* constant. This is is true only when k is a positive constant. If k is negative you need to reverse the inequality.

6. **MISTAKE**: $\left|\frac{x-2}{x+1}\right|=3$ implies $\frac{x-2}{x+1}=3$: again this is partially correct because you must also work with the equation $\frac{x-2}{x+1}=-3$. The first equation $\frac{x-2}{x+1}=3$ gives $x=-5/2$ and the second equation $\frac{x-2}{x+1}=-3$ gives $x=-1/4$ so the complete solution is $x=-5/2$ or $x=-1/4$

7. **Forgetting to simplify fractions in limits**: It is not correct to say that $\lim_{x\longrightarrow 1}\frac{x^2-1}{x-1}=\frac{0}{0}$ and therefore the limit is undefined. Precisely the point here is that before evaluating a limit you have to do some algebraic manipulation. For example

$$\lim_{x\longrightarrow 1}\frac{x^2-1}{x-1}=\lim_{x\longrightarrow 1}\frac{(x-1)(x+1)}{x-1}=\lim_{x\longrightarrow 1}(x+1)=2 \tag{5}$$

8. **MISTAKE**: $ax=bx$ implies $a=b$. This is fine only if x is not 0. For example $2x=3x$ implies that $x=0$ not that $2=3$.

9. **MISTAKE**: $\frac{d}{dx}\left(2^\pi\right)=\pi 2^{\pi-1}$. 2^π is just a number so its derivative must be 0

$$\frac{d}{dx}\left(2^\pi\right)=0 \tag{6}$$

10. **NOT USING ONE-SIDED LIMITS FOR PROBLEMS WITH CONTINUITY**: For example, if you want to find the value c that makes $J(x)=\begin{cases}2x^2+cx-1 & \text{if } x<1\\ \sqrt{x+3} & \text{if } x\geq 1\end{cases}$ continuous you can't just evaluate both formulas at 1 and say that $2\left(1\right)^2+c\left(1\right)-1=\sqrt{1+3}$. What you need to set equal are the respective one-sided limits and the value of $J(1)$, that is, you must solve

$$\lim_{x\longrightarrow 1^-}J(x)=\lim_{x\longrightarrow 1^+}J(x)=J(1) \tag{7}$$

which ends up giving in the end the original equation you were trying to solve.

11. **NOT USING THE DEFINITION OF THE DERIVATIVE WHEN ASKED TO USE IT**: if a problem says find $f'(x)$ using the definition of the derivative, they mean compute the limit

$$\lim_{h\longrightarrow 0}\frac{f(x+h)-f(x)}{h} \tag{8}$$

You shouldn't find it with the rules of the derivatives.

12. **MISTAKE**: $(f(x)g(x))'=f'(x)g'(x)$. Remember that the correct product rule for derivatives is

$$(f(x)g(x))'=f(x)g'(x)+g(x)f'(x) \tag{9}$$

13. **MISTAKE**: $\left(\frac{f(x)}{g(x)}\right)'=\frac{f'(x)}{g'(x)}$. Remember that the correct quotient rule for derivatives is

$$\left(\frac{f(x)}{g(x)}\right)'=\frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2} \tag{10}$$

14. **The Physical Domain of a Function is Different from its Mathematical Domain**: suppose that we want to find the domain of the function $A(L) = 8L - L^2$. From a mathematical point of view, this function always make sense so we would say that the domain is $\mathbb{R}$. Now, if the function A is *interpreted* as the area of a figure and L is *interpreted* as the length of one of its sides then the domain is no longer $\mathbb{R}$ since we would need $A > 0$ and $L > 0$. In this case we find that the domain for $A(L)$ is $(0, 8)$. It will be also considered correct to say that the domain is $[0, 8]$.

15. **If you use a theorem you should invoke its name and mention why you are allowed to use it**. For example, in problems which are solved with the Intermediate Value Theorem (IVT) you should say explicitly that you will use the IVT and check that the hypothesis of the theorem are satisfied.

16. **MISREADING THE PROBLEM**: it happens a lot. After you finish a problem on the exam, go back and read the question again.

17. **NOT USING COMMON SENSE/NOT CHECKING YOUR ANSWER FOR PLAUSIBILITY:** for example, if the problems asks you to find the area of a region it can't be the case that your final answer is a negative number. Always check that your answer is plausible. For example, it would be strange that if you are asked the volume of a sphere of radius one feet you end up saying that it is 10 trillion cubic feet.

18. **DIFFERENT NOTATIONS FOR THE DERIVATIVES**: you should know the following ways to calculate the derivative of a function $y = f(x)$ (these are all the same, the only difference is their notation)
(11)
$$\begin{cases} f'(x) = \lim_{h \longrightarrow 0} \frac{f(x+h)-f(x)}{h} \\ \\ f'(a) = \lim_{h \longrightarrow 0} \frac{f(a+h)-f(a)}{h} \\ \\ f'(x) = \frac{dy}{dx} = \frac{df}{dx} = \lim_{\triangle x \longrightarrow 0} \frac{\triangle y}{\triangle x} = \lim_{\triangle x \longrightarrow 0} \frac{y(x+\triangle x)-y(x)}{\triangle x} = \lim_{\triangle x \longrightarrow 0} \frac{f(x+\triangle x)-f(x)}{\triangle x} \end{cases}$$

19. **REMARK ABOUT TRUE/FALSE QUESTIONS**: in true/false questions you can't assume that the function has more properties than those which are explicitly mentioned in the question. For example: "True/false: if $f(x)$ is a function such that $f(-3) < 0$ and $f(3) > 0$ then there must exist c between -3 and 3 that satisfies $f(c) = 0$". This is FALSE because you are not told that the function $f(x)$ is continuous. *If* you were told that $f(x)$ is continuous the it would be TRUE since this *would* follow from the Intermediate Value Theorem.