1. Find the general solution to the equation $y'' + 3y' + 2y = (2+t)e^t$.

Solution:

The complementary equation $y'' + 3y' + 2y = 0$ has general solution $y_c(t) = c_1e^{-2t} + c_2e^{-t}$. An annihilating operator for $2e^t$ is $A_1 = (D-1)$. We demand that $L[c_3e^t] = 2e^t$ where $L = D^2+3D+2 = (D+2)(D+1)$ is our principal operator. This gives $c_3 = \frac{1}{3}$ and $y_{p_1} = \frac{1}{3}e^t$ is a particular solution to $y'' + 3y' + 2y = 2e^t$.

An annihilating operator for te^t is $(D-1)^2$ and we demand that $L[c_4e^t + c_5te^t] = te^t$ for our principal operator L. This gives us the requirements $6c_4 + 5c_5 = 0$ and $6c_5 = 1$, so that $c_5 = \frac{1}{6}$ and $c_4 = -\frac{5}{36}$. Thus $y_{p_2} = -\frac{5}{36}e^t + \frac{1}{6}te^t$ is a particular solution to the DE $y'' + 3y' + 2y = te^t$.

The general solution to $y'' + 3y' + 2y = (2+t)e^t$ is $y_c + y_{p_1} + y_{p_2}$, or $c_1e^{-2t} + c_2e^{-t} + \frac{7}{36}e^t + \frac{1}{6}te^t$.

2.

Find the general solution to the following equations using annihilating operators.

(a) $y''' - 4y' = t$.

(b) $y''' - 4y' = 3\cos t$.

(c) $y''' - 4y' = t + 3\cos t$. (Use your work in the previous two parts; no new computations needed!)

Solution:

First note for all three parts of this problem that the equation $y''' - 4y' = 0$ has solution $y_c(t) = c_1 + c_2e^{2t} + c_3e^{-2t}$.

(a) An annihilating operator for t is D^2. The general solution to the DE $D^2(D^3 - 4D)[y] = 0$ is $c_1 + c_2e^{2t} + c_3e^{-2t} + c_4t + c_5t^2$. We demand that $L[c_4t + c_5t^2] = t$ where $L = D^3 - 4D$. This says

$$-4c_4 - 8c_5t = t$$

which forces $c_4 = 0$ and $c_5 = -\frac{1}{8}$. Thus $y_p(t) = -\frac{1}{8}t^2$ and the general solution to $y''' - 4y' = t$ is $y(t) = -\frac{1}{8}t^2 + c_1 + c_2e^{2t} + c_3e^{-2t}$.

(b) An annihilating operator for $3\cos t$ is $D^2 + 1$. The general solution to the DE $(D^2+1)(D^3 - 4D)[y] = 0$ is $c_1 + c_2e^{2t} + c_3e^{-2t} + c_4\cos t + c_5\sin t$. We demand that $L[c_4\cos t + c_5\sin t] = 3\cos t$ where $L = D^3 - 4D$. This says

$$c_4\sin t - c_5\cos t - 4(-c_4\sin t + c_5\cos t) = 3\cos t$$

which forces $c_4 = 0$ and $c_5 = -\frac{3}{5}$. Thus $y_p(t) = -\frac{3}{5}\sin t$ and the general solution to $y''' - 4y' = 3\cos t$ is $y(t) = -\frac{3}{5}\sin t + c_1 + c_2e^{2t} + c_3e^{-2t}$.

(c) A particular solution to $y''' - 4y' = t + 3\cos t$ is $y_p = -\frac{1}{8}t^2 - \frac{3}{5}\sin t$. The general solution is $y(t) = -\frac{1}{8}t^2 - \frac{3}{5}\sin t + c_1 + c_2e^{2t} + c_3e^{-2t}$.

3.

Solve $y'' + 3y' + 2y = \cos t$ by first solving $y'' + 3y' + 2y = e^{it}$ using an annihilating operator with complex coefficients, and then extracting the desired solution from your result.

Solution:

The equation $y'' + 3y' + 2y = 0$ has solution $y_c(t) = c_1 e^{-2t} + c_2 e^{-t}$. An annihilating operator for e^{it} is $D - i$. We demand that $L[c_3 e^{it}] = e^{it}$, where $L = D^2 + 3D + 2$. This gives $-c_3 e^{it} + 3ic_3 e^{it} + 2c_3 e^{it} = e^{it}$ so that

$$c_3 = \frac{1}{1+3i} = \frac{1-3i}{10}.$$

A particular solution to $y'' + 3y' + 2y = \cos t$ is

$$\mathrm{Re}\left(\frac{1-3i}{10}e^{it}\right) = \mathrm{Re}\left(\frac{1-3i}{10}(\cos t + i\sin t)\right) = \frac{1}{10}\cos t + \frac{3}{10}\sin t.$$

The general solution to this DE is $y = c_1 e^{-2t} + c_2 e^{-t} + \frac{1}{10}\cos t + \frac{3}{10}\sin t$.

- $$my'' + ky = F_0 \cos \omega t,$$

Resonanace when

$$\omega = \omega_0.$$

Beats when

$$\omega \neq \omega_0$$

4.

True or False? If m, c and k are positive, then all solutions to

$$my'' + cy' + ky = 2t$$

are unbounded as $t \to \infty$.

Solution:

(c) True. The general solution looks like $y = y_c + y_p$ where y_c is the general solution to $my'' + cy' + ky = 0$ and y_p is a particular solution to $my'' + cy' + ky = 2t$. We know that when m, c, and k are all positive, $y_c(t) \to 0$ as $t \to \infty$. (see Section ATTN). To find y_p we use the demand step $(mD^2 + cD + k)[c_3 + c_4 t] = 2t$.

This will give $c_4 = 2/k$, so that in particular c_4 is not 0. Thus $y_p(t)$ will tend to infinity as $t \to \infty$, and all solutions to $my'' + cy' + ky = 2t$ are unbounded as $t \to \infty$.