## $\boxed{\text{Algorithm}}$

$$\frac{dx}{dt} = f(x,y) \qquad \boxed{\text{e.g}} \quad \frac{dx}{dt} = x(2x+3y-7)$$

$$\frac{dy}{dt} = g(x,y). \qquad \frac{dy}{dt} = y(3x-4y-2).$$

$\boxed{\text{Step 1}}$ Find Equilibrium Points:

$f(x,y) = 0 = g(x,y)$.

- $x(2x+3y-7)=0$
  $y(3x-4y-2)=0.$ ] Solns are ~~$(0,0)$, $(0,\frac{1}{2})$~~

$(0,0)$, $(0,-\frac{1}{2})$, $(\frac{7}{2},0)$, $(2,1)$

$\boxed{\text{Step 2}}$ Compute $J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$

In example,

$$J = \begin{pmatrix} 4x+3y-7 & 3x \\ 3y & 3x-8y-2 \end{pmatrix}$$

$\boxed{\text{Step 3}}$ Compute $J$ at all eq. pts.

$$J_1 = \begin{pmatrix} -7 & 0 \\ 0 & -2 \end{pmatrix}, \quad J_2 = \begin{pmatrix} -\frac{17}{2} & 0 \\ -\frac{3}{2} & 2 \end{pmatrix},$$

$$J_3 = \begin{pmatrix} 7 & \frac{21}{2} \\ 0 & \frac{17}{2} \end{pmatrix}, \quad J_4 = \begin{pmatrix} 4 & 6 \\ 3 & -4 \end{pmatrix}.$$

$\boxed{\text{Step 4}}$ Write the approximating linear system at each equilibrium point

$\boxed{\text{Here}}$ $\boxed{\text{At } (0,0)}$ $X' = J_1 X$, At $\left(\frac{7}{2}, 0\right)$, $X' = J_3 X$

$\boxed{\text{At } (0,-\frac{1}{2})}$ $X' = J_2 X$ $\boxed{\text{At } (2,1)}$ $X' = J_4 X$.

$\boxed{\text{Step 5}}$ For each linear approximating system, study the behaviour of $(0,0)$ for this linear system.

$\boxed{\text{At } (0,0)}$ $X' = J_1 X$; $\Delta > 0$, $\tau < 0$, $\tau^2 - 4\Delta > 0$.
So, $(0,0)$ is a stable node for the linear system.

$\boxed{\text{At } (0,-\frac{1}{2})}$ $X' = J_2 X$; $\Delta < 0$. So, $(0,0)$ is an unstable saddle for this linear system.

$\boxed{\text{At } (\frac{7}{2}, 0)}$ $X' = J_3 X$; $\Delta > 0$, $\tau > 0$, $\tau^2 - 4\Delta > 0$.
So, $(0,0)$ is an unstable node for this linear system

$\boxed{\text{At } (2,1)}$

$X' = J_4 X$ ; $\Delta < 0$. So, $(0,0)$ is an unstable saddle for this linear system.

$\boxed{\text{Step 6}}$ From the behaviour of $(0,0)$ for the linear system classify the corresponding equilibrium points

$\boxed{(0,0)}$. We saw $(0,0)$ for lin. sys. was an unstable node. This is major type. Hence, by Thm 10.2.3,

$(0,0)$ for the non-linear system is an unstable node.

$\boxed{(0,-\frac{1}{2})}$ $(0,0)$ – lin.-sys — unstable saddle. (Major type)

So, $(0,-\frac{1}{2})$ for non-lin system is an unstable saddle

$\boxed{\frac{7}{2},0}$

$(0,0)$ - lin. sys. — unstable node (Major type).

Hence, $\left(\frac{7}{2},0\right)$ is an unstable node.

$\boxed{2,1}$ $(0,0)$ - lin. sys. → unstable saddle (Major type)

So, $(2,1)$ → unstable saddle.

Note that in this example, all equlibria turned out to be “Major Type" – so we used Theorem 10.2.3

**Theorem 10.2.3.** *Suppose we have an autonomous nonlinear system*

$$\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= g(x,y),\end{aligned}\tag{10.24}$$

*where $f(x,y)$ and $g(x,y)$ are continuously differentiable. Suppose $(x_e,y_e)$ is an isolated equilibrium point for (10.24). Consider the related linear system*

$$\mathbf{X}' = \mathbf{J}\mathbf{X}\tag{10.25}$$

*where $\mathbf{J}$ is the Jacobian matrix*

$$\begin{pmatrix} \frac{\partial f}{\partial x}(x_e,y_e) & \frac{\partial f}{\partial y}(x_e,y_e)\\ \frac{\partial g}{\partial x}(x_e,y_e) & \frac{\partial g}{\partial y}(x_e,y_e)\end{pmatrix},$$

*and assume that $(0,0)$ is an isolated equilibrium point for this linear system (this is the same as saying det $\mathbf{J} \neq 0$).*

*If $(0,0)$ is one of the major types (saddle point, spiral point, or nondegenerate node) for the linear system (10.25), then $(x_e,y_e)$ is the same type for the nonlinear system. Moreover, for these major types, the stability of the equilibrium point is the same for the nonlinear system as it was for the linear system.*

In case you come across a problem where the equilibria is “Borderline type” , use Theorem 10.2.4.:

**Theorem 10.2.4.** *Under the same hypotheses as Theorem 10.2.3, we have the following:*

*If the linear system $\mathbf{X}' = \mathbf{J}\mathbf{X}$ has either a degenerate node or a star point at $(0,0)$, then the nonlinear system has either a node, star or spiral at $(x_e,y_e)$, and the stability is the same for both systems.*

*If the linear system $\mathbf{X}' = \mathbf{J}\mathbf{X}$ has a center at $(0,0)$, then the nonlinear system has either a center, a spiral, or a hybrid center/spiral at $(x_e,y_e)$. In this case, we cannot predict the stability of $(x_e,y_e)$ for (10.24).*

Always keep this picture in mind: