3.6 (Part I)

Implicit Differentiation

So far we have ~~represented functions~~ used as

$y = f(x)$ to draw graphs of functions,

or to take derivatives, etc. etc..

This is the case where the dependent variable y is given "explicitly" in terms of x via the function f. However, we ~~might~~ often deal with situations where y can't be expressed explicitly in terms of x. In this case, we will have an equation involving y & x but we may not be able to solve for y.

This y will be given 'implicitly' as some operation on x.

Our goal will be to still make sense ~~of~~ of 'the tangent line to the curve' at some given point on the curve. This, we will achieve through 'implicit differentiation'!

Suppose we are given the following ~~curve~~ equation: ~~whose~~.

$$x^2y + 2y = 3x$$

One can draw the graph of this equation: we get some curve on the plane.

Note that the point $(2,1)$ lies on this curve. $(2^2 \cdot 1 + 2\cdot(1) = 3\cdot(2))$.

We want to ~~define the~~ find out slope of tangent line at $(2,1)$.

$\boxed{\text{Method 1}}$ We are lucky in this case, as we can solve for y explicitly: $y(x^2+2) = 3x$

$$\text{or, } y = \frac{3x}{x^2+2}.$$

Then you use quotient rule to find slope, blublublahblah...

$\boxed{\text{Method 2}}$ However, we can also use implicit differentiation for this. In fact, we are more focused on learning Method 2 as we won't get lucky everytime (eg. $x^2 + xy + y^3 = 7$).

Implicit differentiation: finding $\frac{dy}{dx}$ without solving for y.

Steps

(1) Treat y as a function of x.

(2) Take derivative with respect to x on both sides of the equation (use all rules (product/quotient/chain, whatever required).

(3) Collect the '$\frac{dy}{dx}$' term on one side of the equation and solve for it.

Let us apply these to the problem at hand:

$$x^2y + 2y = 3x.$$

So, $\frac{d}{dx}(x^2y + 2y) = \frac{d}{dx}(3x)$ (using Step 1).

or, $\frac{d}{dx}(x^2y) + \frac{d}{dx}(2y) = 3.$

or, $2xy + x^2\frac{dy}{dx} + 2\frac{dy}{dx} = 3.$ [using product rule to x^2y. $y\frac{d}{dx}(x^2) + x^2\frac{d(y)}{dx} = y(2x) + x^2\frac{dy}{dx}$].

or, $\frac{dy}{dx}\left[x^2+2\right] = 3-2xy.$

or, $\boxed{\frac{dy}{dx} = \frac{3-2xy}{(x^2+2)}}$ Now plug in the point $(2,1)$ to find slope of tangent line at $(2,1)$.

Example

Given $xy^3 + \frac{1}{y} = 1$; $(-2,-1)$ lies on this curve

ⓐ Find the equation of the tangent line of the graph at $(-2,-1)$.

Soln.

$$xy^3 + \frac{1}{y} = 1$$

$$\frac{d}{dx}\left(xy^3 + \frac{1}{y}\right) = \frac{d}{dx}(1).$$

or, $\frac{d}{dx}(xy^3) + \frac{d}{dx}\left(\frac{1}{y}\right) = 0.$

or, $y^3 + x\frac{d}{dx}(y^3) + \frac{d}{dx}(y^{-1}) = 0.$

or, $y^3 + x\,3y^2\frac{dy}{dx} - \frac{1}{y^2}\frac{dy}{dx} = 0.$ (Using chain-rule/Generalized Power Rule $\frac{d}{dx}(y^3) = \frac{d}{dy}(y^3)\cdot\frac{dy}{dx} = 3y^2\frac{dy}{dx}$)

or, $\frac{dy}{dx}\left(3y^2x - \frac{1}{y^2}\right) = -y^3.$

or, $\frac{dy}{dx} = \frac{y^3}{\frac{1}{y^2} - 3y^2x}.$

So, $\left.\frac{dy}{dx}\right|_{(-2,-1)} = \frac{-1}{1 - 3(-2)} = \frac{-1}{7}$

Thus, $y-(-1) = -\frac{1}{7}(x-(-2))$

or, $\boxed{y+1 = -\frac{1}{7}(x+2)}$ is the required answer.

$\boxed{\text{e.g.}}$ $\sqrt{x^2+y^2} - 2y^3 = 3x^2$ Find $\frac{dy}{dx}$

Soln:

$$\frac{d}{dx}\left(\sqrt{x^2+y^2}\right) - 2\frac{d}{dx}(y^3) = \frac{d}{dx}(3x^2).$$

[$\frac{d}{dx}\left(\sqrt{x^2+y^2}\right)$. For the time being, let $u = x^2+y^2$.

Thus, $\frac{d}{dx}(u^{\frac{1}{2}}) = \frac{d}{du}(u^{\frac{1}{2}}) \cdot \frac{du}{dx}$ (Chain-rule)

$$= \frac{1}{2\sqrt{u}} \cdot \left(\frac{d}{dx}(x^2+y^2)\right)$$

$$= \frac{1}{2\sqrt{x^2+y^2}}\left(2x + 2y\frac{dy}{dx}\right)]$$

With practice, these will become very easy & you don't have to write out these steps.

or, $\frac{1}{2\sqrt{x^2+y^2}}\left(2x + 2y\frac{dy}{dx}\right) - 6y^2\frac{dy}{dx} = 6x$.

or, $\frac{dy}{dx}\left(\frac{y}{\sqrt{x^2+y^2}} - 6y^2\right) = 6x - \frac{x}{\sqrt{x^2+y^2}}$.

or, $$\boxed{\frac{dy}{dx} = \frac{6x - \frac{x}{\sqrt{x^2+y^2}}}{\frac{y}{\sqrt{x^2+y^2}} - 6y^2}}$$ Ans

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