Finalized

2.3 Part II: Mathematical Models

pending

2.3.1 Introduction

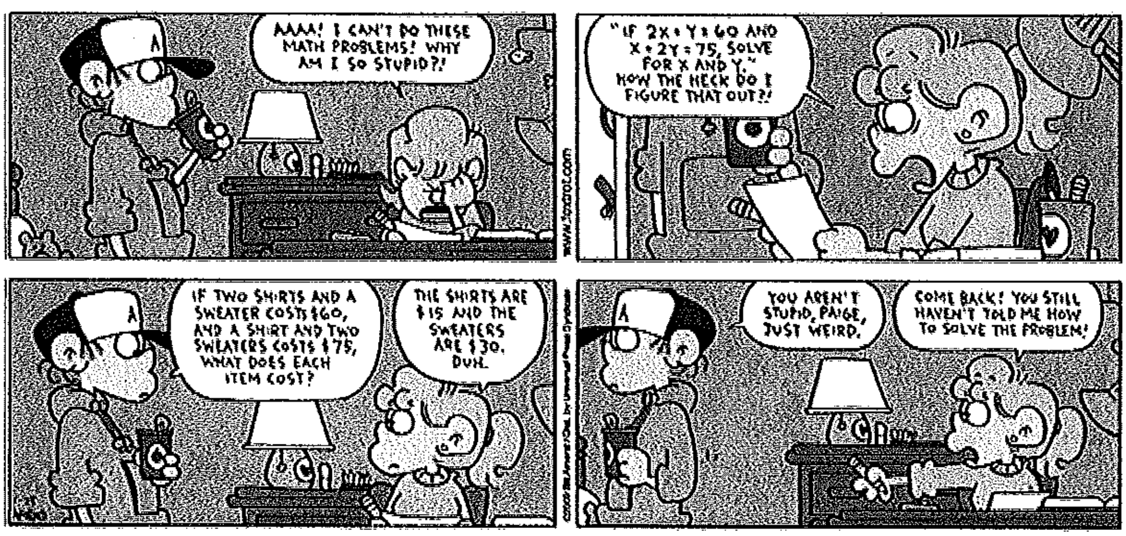
Primary objective: Given a context or scenario with a quantity of interest:

1. Express the quantity of interest as a function of ONE variable.

> General strategy:
> - Draw a picture, if applicable.
> - Identify the quantity of interest (call it Q).
> - Assign as many variables as you need and express Q.
> - If Q has just one variable, then we are good. Otherwise:
>   - Look for unused information from the problem,
>   - Convert the information into a constraint equation,
>   - Use it to eliminate variables in Q.

2. Find the domain of the function.

> General strategy: So say the quantity of interest Q is a function of the variable x. Then the domain of $Q(x)$ is all x such that x is **contextually realistic**. There are two approaches:
> - Set up and solve inequalities using that fact that most physical quantities, like measurements and time, cannot be negative.
> - Draw extreme pictures. In other words, draw pictures that illustrates the smallest possible (and larges possible) x which is still contextually realistic.



Comic by Bill Amend (www.foxtrot.com).

2.3.2 Examples

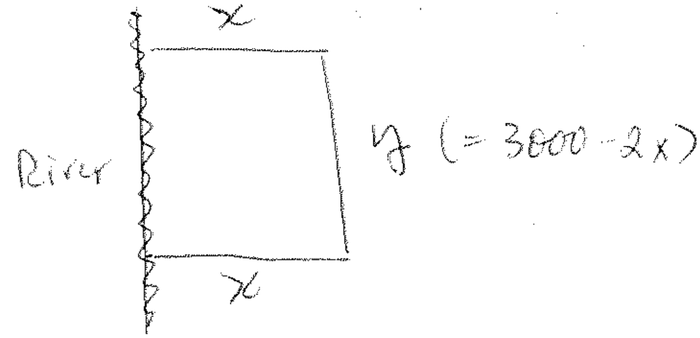
words to use

Example 1. *(Tan 8e, Sect. 2.3, e.g.) The owner of the Rancho Los Feliz has 3000 yards of fencing with which to enclose a rectangular piece of grazing land along the straight portion of a river. Fencing is not required along the river. Denoting each of the portions of the fencing perpendicular to the river by x.*

quantity of interest

(a) *Express the area of the grazing land as a function of x.*

Picture:



Area $A = xy$

But 3000 yards of fencing available

$\Rightarrow \quad 2x + y = 3000 \quad \longleftarrow$ Constraint equation

$\Rightarrow \quad y = 3000 - 2x$.

Thus $A = x(3000 - 2x)$ or $A = 3000x - 2x^2$

(b) Find the domain of the function.

The domain of A (a function of $\underline{x}$)
is All the $\underline{x}$ which yields a realistic picture.

Such x satisfies

$x \geq 0$ and $y \geq 0$.

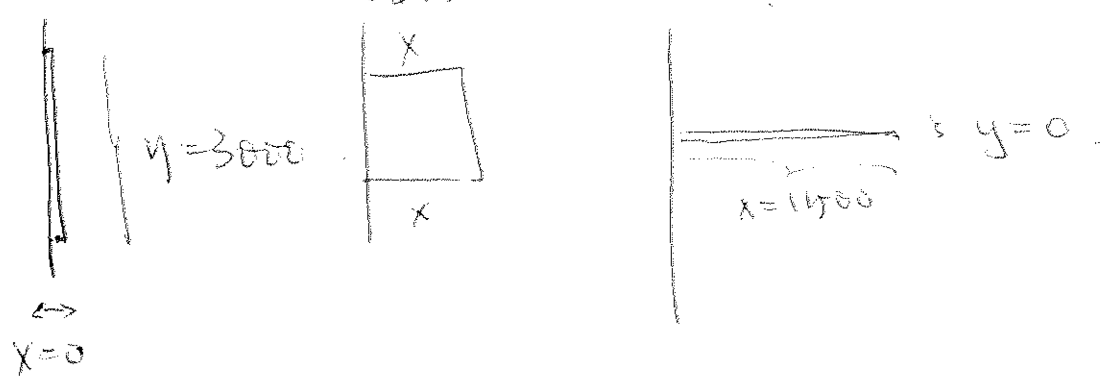
$x \geq 0$ and $3000 - 2x \geq 0$

$x \geq 0$ and $1500 \geq x$

Thus the domain of $A(x)$ is $0 \leq x \leq 1500$,
or $[0, 1500]$.

Extreme pictures.

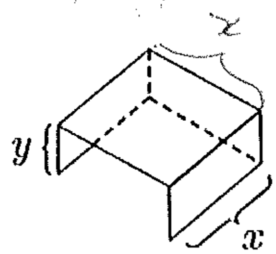
Rmk: If $x = 0$, x between 0 & 1500, If $x = 1500$



Example 2. *David Hasselhoff is designing a garage for KITT, his sentient 1982 Trans-Am. His garage will consist of a square roof with side length x, three sides with height y, and no floor. The garage mush have a volume of 1600 cubic feet.*

(surface area)

(a) *Find a function with variable x giving the number of square feet of material needed to construct the garage.*



(square roof)

Square feet of material needed (call it S)

= Area of roof + 3 x area of a side wall

$= x^2 + 3xy$

But volume of garage is 1600 ft^3

$\Rightarrow x^2 y = 1600 \Rightarrow y = \frac{1600}{x^2}$

constraint equation.

Thus $S = x^2 + 3x\left(\frac{1600}{x^2}\right)$

or $S = x^2 + \frac{4800}{x}$

(b) Find the domain of the function.

The domain of S (a function of x)
is all x such that

$$x > 0 \quad \text{and} \quad y > 0.$$

(Note: x or y cannot be 0 since the volume of garage is 3600.)

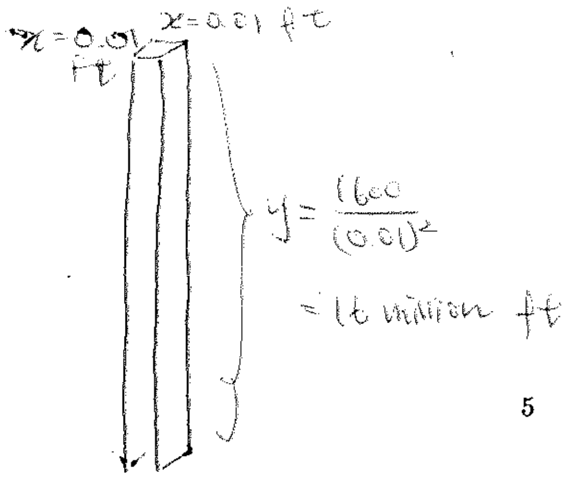
$$x > 0 \quad \text{and} \quad \frac{1600}{x^2} > 0.$$

But as long as $x > 0$,
$\frac{1600}{x^2}$ is positive, so $\frac{1600}{x^2} > 0$
is automatically satisfied.

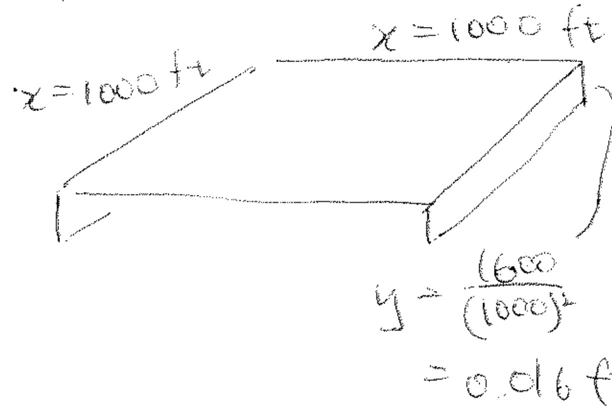
Thus the domain of $S(x)$ is all x
such that $x > 0$, or $(0, \infty)$.

Rmk Extreme picture:

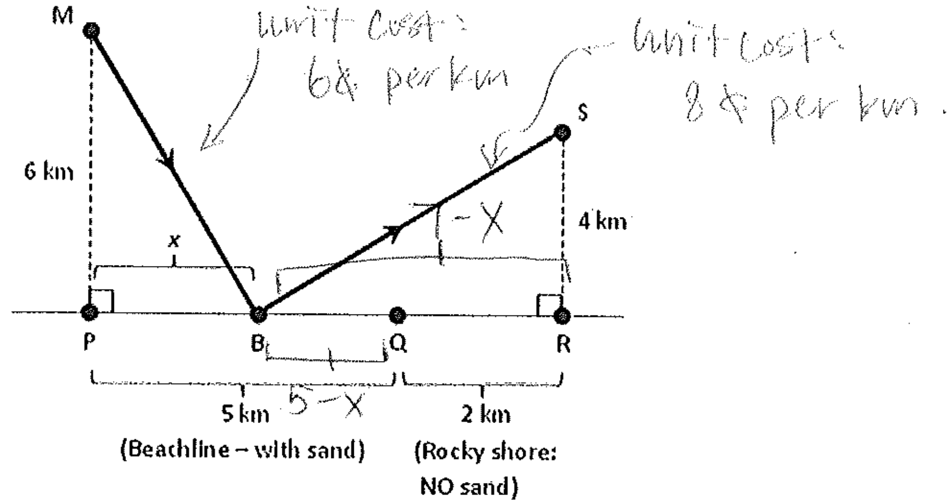
x very close to 0:



x very large positive:



Example 3. *Mary wants to pick up a lot of sand at the beach and transport the sand to her studio for a big art project. Here is a map of her neighbourhood:*



Mary has to drive from her home M to some point B on the beachline PQ (not on QR), pick up the sand, and then drive from B to the studio S. The gas cost for driving from M to B is 6 cents per km, but the gas cost for driving from B to S is higher, namely 8 cents per km, because the car is loaded with lots of sand. Express the (total) gas cost of the whole journey as a function of x and find the domain of the function.

(Total) gas cost for whole journey (call T)

= (Total) cost for portion MB + (Total) cost for portion BS

= Unit cost on MB · (distance from M to B) + Unit cost on BS · (distance from B to S)

= 6 cents per km · ($\sqrt{x^2+6^2}$ km) + 8 cents per km $\sqrt{(7-x)^2+4^2}$ km

$= 6\sqrt{x^2+6^2} + 8\sqrt{(7-x)^2+4^2}$ (cents)

(Extra space)

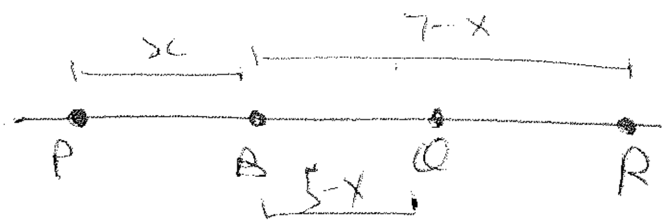
Domain of $T(x)$:

Method 1 B must be between P and Q,

so x must be between 0 and 5

So domain of $T(x)$ is $0 \le x \le 5$, or $[0,5]$

Method 2



B must be between P & Q

∴ Domain of $T(x)$ is all x which simultaneously satisfies

$$x \ge 0, \quad 5 - x \ge 0, \quad (7 - x \ge 0)$$

$$x \ge 0, \quad x \le 5, \quad (x \le 7)$$

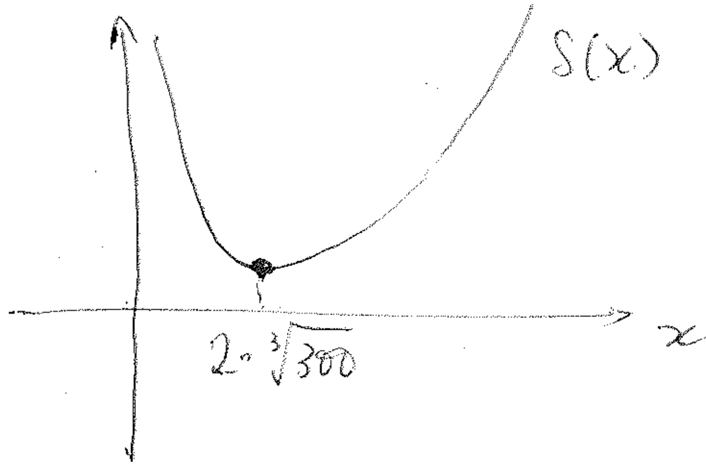
So domain of $T(x)$ is $0 \le x \le 5$

2.3.3 Remark: What are we going to do with these?

Example 2

(Chap 2.4-4.5)

Calculus would tell us that the graph of $S(x) = x^2 + \frac{4800}{x}$ over $(0, \infty)$ looks like



So the garage with $x = 2\cdot\sqrt[3]{300}$

$\left(\text{and } y = \frac{1600}{x^2} = \frac{1600}{2^2(\sqrt[3]{300})^2} = \frac{4}{3}\cdot\sqrt[3]{300}\right)$

would require the least amount of material (and possibly the most effective)