5.6.1 Exponential Growth Model Example

Example 1. *In ideal laboratory setting, a bacteria colony grows exponentially. The experiment started at 4 pm and has 100 bacteria cells in the colony. At 9 pm, the colony expanded to 2000 cells.*

(a) Find the size of the colony at any time t.

(b) How long does it take in total for the size of colony to reach 50,000?

(c) How fast was the colony growing at 5 pm?

Solution: Let $Q(t)$ be the size of the colony at time t. Let $Q_0 = 100$ be the initial population at time $t = 0$ (4 pm). Let k be the growth constant.

(a)

$$Q(t) = 100e^{kt}.$$

We are given $Q(5) = 2000$. Thus,

$$\begin{aligned} 2000 &= 100e^{5k} \\ \implies \ln 20 &= 5k \\ \implies k &= \frac{\ln 20}{5} \end{aligned}$$

Hence,

$$Q(t) = 100e^{\frac{\ln 20}{5}t}.$$

(b)

$$\begin{aligned} 50000 &= 100e^{\frac{\ln 20}{5}t} \\ \implies \ln 500 &= \frac{\ln 20}{5}t \\ \implies t &= 5\frac{\ln 500}{\ln 20} \text{ (hours)} \end{aligned}$$

(c)

$$\begin{aligned} Q(t) &= 100e^{kt} \\ \frac{dQ}{dt} &= 100ke^{kt} \end{aligned}$$

We need this at $t = 1$. Thus, required growth rate is $100 \times \frac{\ln 20}{5} \times e^{\frac{\ln 20}{5}} = 20 \ln 20 e^{\frac{\ln 20}{5}}$ (per hour).

5.6.2 Exponential Decay Model Example

Example 2. *(Carbon Dating) Carbon-14 is a radioactive material that decays exponentially. Skeletal remains of the so-called Pittsburgh Man, unearthed in Pennsylvania, had lost 82% of the Carbon-14 they originally contained. The half-life of Carbon-14 is 5770 years. Determine the approximate age of the bones.*

Solution: Let decay constant be k, let amount of C-14 present after t years be $Q(t)$. Let Q_0 be the initial amount. We have

$$Q(t) = Q_0 e^{kt}.$$

Now we are given that $Q(5770) = \dfrac{Q_0}{2}$,

$$\begin{aligned} \frac{Q_0}{2} &= Q_0 e^{5770k} \\ \implies \frac{1}{2} &= e^{5770k} \\ \implies k &= \frac{\ln \frac{1}{2}}{5770} \end{aligned}$$

We are also given that presently when t years have elapsed, 18% of Q_0 is present. Need to figure out t from this.

$$\begin{aligned} 0.18Q_0 &= Q_0 e^{kt} \\ \implies 0.18 &= e^{kt} \\ \implies t &= \frac{\ln 0.18}{k} \end{aligned}$$

Hence the age of the bones is $\dfrac{\ln 0.18}{\frac{\ln 0.5}{5770}}$ years $= 5770 \times \dfrac{\ln 0.18}{\ln 0.5}$ years.