$\boxed{\text{Quick Review}}$

① $\int_1^2 \left(2x^{2/3} + \frac{x^2}{2}\right) dx$

$= \left(2\,\frac{x^{5/3}}{5/3} + \frac{x^3}{6}\right)\Big|_1^2$

$= \left(\frac{6}{5}\,x^{5/3} + \frac{x^3}{6}\right)\Big|_1^2$

$= \left(\frac{6}{5}\,2^{5/3} + \frac{8}{6}\right) - \left(\frac{6}{5} + \frac{1}{6}\right)$

$= \frac{6}{5}\,2^{5/3} + \frac{7}{6} - \frac{6}{5} = \frac{6}{5}\,2^{5/3} + \frac{35-36}{30}$

$= \frac{6}{5}\,2^{5/3} - \frac{1}{30}$

<u>Ans</u>

② $\int_1^2 \frac{(\ln(3x))^2}{x}\, dx$ Let $u = \ln(3x)$

$\frac{du}{dx} = \frac{3}{3x} = \frac{1}{x} \Rightarrow x\,du = dx$

$$\int \frac{(\ln(3x))^2}{x}\,dx = \int \frac{u^2}{x}\, x\,du = \int u^2 du = \frac{u^3}{3} + C$$
$$= \frac{(\ln(3x))^3}{3} + C.$$

$$\int_1^2 \frac{(\ln(3x))^2}{x}\,dx = \frac{(\ln(3x))^3}{3}\Bigg|_1^2$$
$$= \frac{(\ln 6)^3}{3} - \frac{(\ln 3)^3}{3}.$$

Ans.

[We will learn another way in a few minutes!]

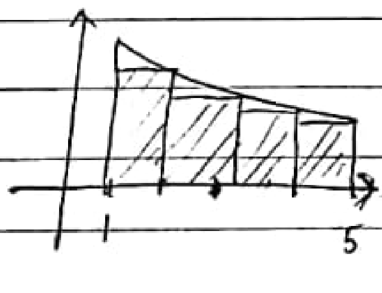
(3) Find the area under the graph of $f(x) = \cancel{}\ \frac{2}{x}$ on $[1,3]$

Ans: $A = \int_1^3 \frac{2}{x}\,dx = 2\int_1^3 \frac{1}{x}\,dx = 2\left(\ln|x|\right)\Big|_1^3 = 2(\ln 3 - \ln 1)$
$= 2\ln 3.$

(4) Suppose we compute the upper Riemann Sum $R(f)$ of a continuous, positive, decreasing function f on $[1,5]$ using subdivision of $[1,5]$ into 4 subintervals. Which is correct?

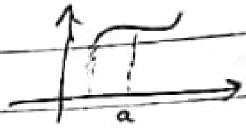
(a) $\int_1^5 f(x)\,dx = R(f)$ (b) $\int_1^5 f(x)\,dx < R(f)$ (c) $\int_1^5 f(x)\,dx > R(f)$.

Ans:



Thus, (c) is the correct answer.

6.5 Properties of Definite Integrals

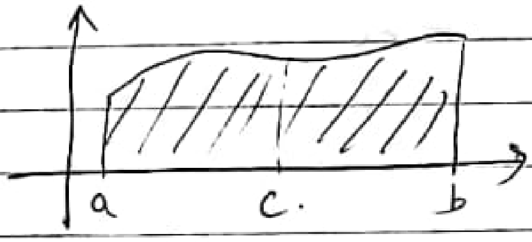
① $\int_a^a f(x)\,dx = 0$ [Area from a to a : 

No Area]

② $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ (for any constants a, b, c).

(usually we apply this for $a<c<b$; but otherwise also it works)

Interpretation:



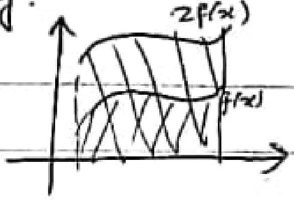
Breaking up the area

③ $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$, c is any constant

④ $\int_a^b \left(f(x) \pm g(x)\right) dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

⑤ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

Interpretation of ③.



Area becomes twice!

U-Substitution for Definite Integrals.

We learn a new method where instead of going to indefinite integration (i.e. without limits), we can keep the limits but suitably change them.

e.g.

$$\int_{x=0}^{x=1} x^2 (x^3+2)^3 \, dx$$

Let $u = x^3 + 2$.

$x=0 \Rightarrow u=2$

$x=1 \Rightarrow u=3$.

$\frac{du}{dx} = 3x^2$.

$$\int_{u=2}^{u=3} x^2 \, u^3 \, \frac{du}{3x^2}$$

$$= \frac{1}{3} \int_{u=2}^{u=3} u^3 \, du = \left. \frac{u^4}{12} \right|_2^3$$

$$= \frac{1}{12}(81 - 16) = \frac{65}{12}$$

Thus, if $u = g(x)$. then $\int_{x=a}^{x=b} \ldots dx$ becomes $\int_{u=g(a)}^{u=g(b)} \ldots du$

Let $\int_0^8 h(s)\,ds = 4$. Find $\int_0^2 6x^2 h(x^3)\,dx$.

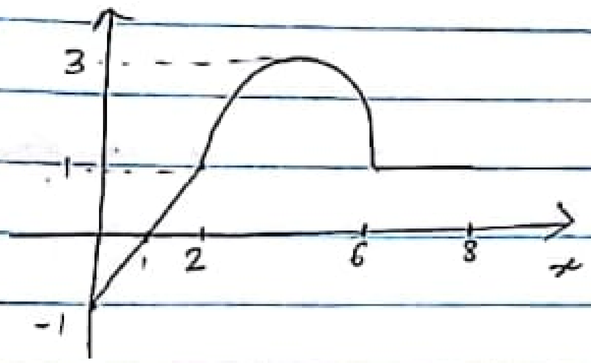
Soln: Let $u = x^3$. Thus $\frac{du}{3x^2} = dx$.

$$\int_0^2 6x^2 h(x^3)\,dx = \int_0^8 6x^2 h(u)\frac{du}{3x^2}$$

$$= \int_0^8 2\,h(u)\,du = 2\int_0^8 h(u)\,du = 2\times 4 = 8.$$

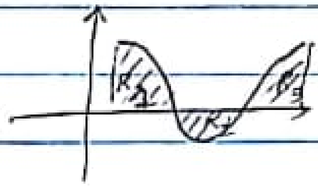
[This exercise shows that variable inside the integral sign is irrelevant! $\int_a^b h(s)\,ds = \int_a^b h(x)\,dx = \int_a^b h(u)\,du \ldots$]

Compute $\int_0^8 f(x)\,dx$ where the graph is given as follows:



The region between 2 to 6 is a semi-circle.

Solⁿ: [Recall $\int_a^b f(x)\,dx = R_1 + R_3 - R_2$



(We are going to apply same technique here; R_2 is the area of the region; so positive)]

$$\int_0^8 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^6 f(x)\,dx + \int_6^8 f(x)\,dx.$$

[Note that $\int_0^1 f(x)\,dx$ has to be negative as graph lies below x-axis]

Now, $\int_0^1 f(x)\,dx = -\frac{1}{2}\times 1 \times 1 = -\frac{1}{2}$

(negative because of the discussion above)

- $\int_1^2 f(x)\,dx = \frac{1}{2}\times 1\times 1 = \frac{1}{2}$.

- $\int_6^8 f(x)\,dx = 1\times 2 = 2$.

- $\int_2^6 f(x)\,dx = \frac{1}{2}\times\pi\times(2)^2 + 1\times 4$.
  $= 2\pi + 4$.

Hence. $\int_0^8 f(x)\,dx = -\frac{1}{2} + \frac{1}{2} + 2\pi + 4 + 2 = 2\pi + 6$. □

OR

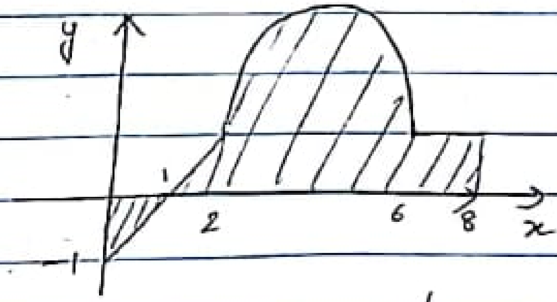
Alternately, you can say $\int_0^8 f(x)\,dx$ = Area above x-axis − Area below x-axis

$= \left(2\pi + 6 + \frac{1}{2}\right) - \frac{1}{2}$

$= 2\pi + 6$.

(Note area below x-axis is $\frac{1}{2}$ as area is always positive. $\int_0^1 f(x)\,dx$ however is negative here as the $f(x)$-values are negative in the region).

Thus, if the question was. compute the shaded area.



then answer is

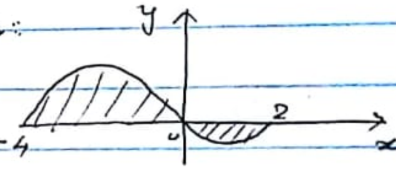
$2\pi + 6 + \frac{1}{2} + \frac{1}{2}$

$= 2\pi + 7$.

(i.e. whenever below x-axis. $-\int_a^b f(x)\,dx$ gives the area; we have to manipulate the sign to get area)

Suppose the graph of a function is:

$f(x)$.



Mark the correct answers:

ⓐ $\int_{-4}^{2} f(x)\,dx = \int_{-4}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx$

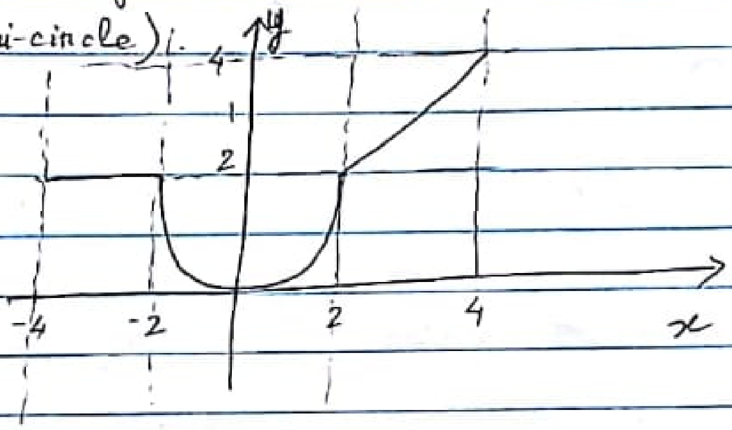
ⓑ Area of the shaded region: $\int_{-4}^{2} f(x)\,dx = \int_{-4}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx$.

ⓒ Area of the shaded region: $\int_{-4}^{0} f(x)\,dx - \int_{0}^{2} f(x)\,dx$.

ⓓ Area of the shaded region: $\left| \int_{-4}^{0} f(x)\,dx \right| + \left| \int_{0}^{2} f(x)\,dx \right|$.

Tricky Example

The graph of a function $f(x)$ on the interval $[-4,4]$ is given as follows: (where between -2 and 2, it's a semi-circle).

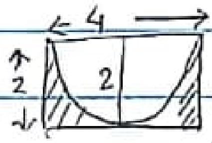


Compute $\int_{-4}^{4} f(x)\,dx$.

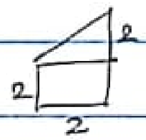
Soln: $\int_{-4}^{4} f(x)\,dx = \int_{-4}^{-2} f(x)\,dx + \int_{-2}^{2} f(x)\,dx + \int_{2}^{4} f(x)\,dx$.

- $\int_{-4}^{-2} f(x)\,dx = 2 \cdot 2 = 4$

- $\int_{-2}^{2} f(x)\,dx = 4 \cdot 2 - \frac{1}{2}\pi(2)^2$
  $= 8 - 2\pi$



- $\int_{2}^{4} f(x)\,dx = \frac{1}{2} \cdot 2 \cdot 2 + 2 \cdot 2 = 6.$



Thus, $\int_{-4}^{4} f(x)\,dx = 4 + 8 - 2\pi + 6 = 18 - 2\pi.$

e.g. Find $\displaystyle\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\,dx$

Soln: Let $u = \ln x$. $\frac{du}{dx} = \frac{1}{x} \Rightarrow x\,du = dx$.

$$\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\,dx = \int_1^4 \frac{1}{x\sqrt{u}}\,x\,du = \int_1^4 \frac{du}{\sqrt{u}}$$

$$= \left.\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right|_1^4$$

$$= 2\sqrt{u}\Big|_1^4$$

$$= 2(2-1) = 2.$$

Average Value Formula.

The average value of a continuous function over a closed interval $[a, b]$ is

$$\boxed{\frac{1}{b-a}\int_a^b f(x)\,dx}$$

e.g.

The internal temperature of a freezer over the three-hour period $0 \leq t \leq 3$ is given by

$f(t) = 3t^2 - t^3$ degree Fahrenheit ($°F$).

(a) ~~…~~ Find the average temperature over $[0, 3]$.

Ans: $\frac{1}{3-0}\int_0^3 f(t)\,dt = \frac{1}{3}\int_0^3 (3t^2 - t^3)\,dt$

$= \frac{1}{3}\left[\int_0^3 3t^2\,dt - \int_0^3 t^3\,dt\right] = \frac{1}{3}\left[\left. t^3\right|_0^3 - \left.\frac{t^4}{4}\right|_0^3\right]$

$= \frac{1}{3}\left[(27-0) - \frac{1}{4}(81-0)\right] = \frac{1}{3}\left(27 - \frac{81}{4}\right)$

$= \frac{9}{4}\ (°F)$