Contd

- Some more important functions:

i) Polynomials $\quad f(x) = a_n x^n + \cdots + a_1 x + a_0$

where n is a non-negative integer

$\boxed{e.g.}$ $\quad f(x) = 2x^2 + 4.$ quadratic polynomial.

(ii) Rational functions $\quad h(x) = \dfrac{f(x)}{g(x)}$ where f, g are both polynomials.

(Remember how to figure out the domain?)

(iii) $\boxed{\text{Power function}}$

$f(x) = x^r$ for some real number r

e.g. $\cdot\, f(x) = x^2$ is a polynomial as well as a power function.

$\cdot\, f(x) = x^{\frac{1}{2}}$ is a power function provided domain is $[0, \infty)$ (right?).

Most functions that we encounter will be combinations of all these functions we talked about.

e.g. If, $f(x) = \sqrt{x^2 - 3x + 4}$.

then $f(x) = \cancel{g(h(x))}\ (g \circ h)(x)$

where $h(x) = x^2 - 3x + 4$
$g(x) = \sqrt{x}$.

Again domains have to be carefully chosen!

e.g. $f(x) = (1+2x)^{\frac{1}{2}} + \dfrac{1}{(x^2+2)^{3/2}}$

~~This is the sum of of~~ 2 ~~power~~ functions.

Let $f_1(x) = 1+2x$. ; $f_3(x) = \sqrt{x}$.

$f_2(x) = (x^2+2)$. ; $f_4(x) = x^{-3/2}$.

Then $f(x) = (f_3 \circ f_1)(x) + (f_4 \circ f_2)(x)$.

One of the major uses of functions is to form mathematical models concerning real life events.

[e.g.] A manufacturer has a monthly fixed cost of \$100,000 and a production cost of \$14 for each unit produced. The product sells for \$20 /unit.

(a) What is the cost function?
(b) What is the revenue function?
(c) What is the profit function?
(d) Compute profit (loss) corresponding to production levels of 12,000 and 20,000 units.

[Soln]. (This is a typical example of mathematical modelling; we'll soon develop more tools which will allow us to delve more into the behaviour (local/global) of the functions we have).

(a) Let $C(x)$ denote cost function to produce x units.
By the data given,

$$C(x) = 100,000 + 14x.$$

[\$$14x$ is the cost to produce x units].

(the unit is \$ for this function).

(b) Let $R(x)$ denote the revenue generated from selling x units.
From data given,

$$R(x) = 20x.$$ (again in \$).

(c) Let $P(x)$ be the profit function. (in \$).

$$P(x) = R(x) - C(x)$$
$$= 20x - 100{,}000 - 14x$$
$$= 6x - 100{,}000.$$

(d) Need to find $P(12000)$ & $P(20{,}000)$.

$$P(12000) = 72000 - 100000 = -28{,}000.$$

So, there's a loss of \$28,000.

$$P(20{,}000) = 120000 - 100000 = 20000$$

There's a profit of \$20000.

$\square$

Other typical examples of modelling involve optimizing cost of fencing/tiling around a given area, etc.

Let us discuss one such example.

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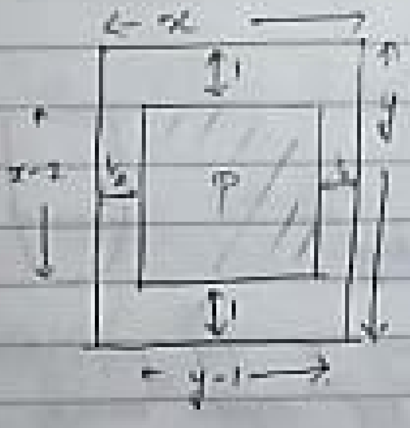
2. A book designer has decided that the pages of a book should have 1-in margins at the top and bottom and $\frac{1}{2}$-in margins on the sides. She further stipulated that each page should have a total area of 50 in^2. Find a function in the variable x, giving the area of the printed part of the page. Find the domain of the function.

$\boxed{\text{Soln}}$ Let x (in) be the length of the page, y (in) be the breadth of the page (this includes the margins as well).



P is the printed area.

The total area (in inch^2) is

$xy = 50$. (from the data)

So, $\boxed{y = \frac{50}{x}}$

$\boxed{\text{The area of P}}$ $P(x) = (x-1)(y-2)$

$= (x-1)\left(\frac{50}{x}-2\right)$.

The domain will be the interval $(1, 25)$.

(we need $P(x) \geq 0$ as it's an area!)

Ans

$x > 1$

$\frac{50}{x} - 2 > 0$

So, $x < 25$

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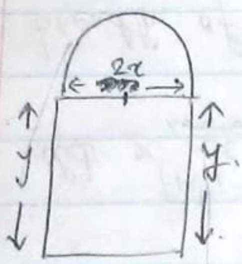
Thus, this was an example, where domain had to be chosen such that it was contextually realistic.

③ A Norman window has the shape of a rectangle surrounded by a semi-circle.

Suppose the Norman window is to have a perimeter of 28 ft. Find a function in one variable giving the area of the window.

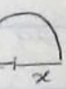
Soln



Let $2x$ be the length & y be the height (in ft)

• Area of the semi-circle.

$$A_1(x) = \frac{1}{2}\pi x^2 \quad (\text{in } ft^2)$$



(since radius is x).

• Area of rectangle: $A_2(x) = 2xy$ (in ft^2).

So, total area (in ft^2):

$$A(x) = \frac{1}{2}\pi x^2 + 2xy$$

Now, from given data, perimeter $= 28$ (in ft).

So, $y + 2x + y + \pi x = 28.$

or, $2y + x(\pi + 2) = 28.$

~~or,~~ $\cancel{x(\pi+2) = 28 - 2y.}$

or, $2y = 28 - x(\pi+2)$

(*) . . . or, $2y = \frac{28 - x(\pi+2)}{\cancel{2}}$ $\cancel{= 14 - x\left(\frac{\pi}{2}+1\right)}$

~~Thus,~~ $\cancel{A(x) = \frac{1}{2}\pi x^2 + 2x\left(14 - x\left(\frac{\pi}{2}+1\right)\right).}$

$$\cancel{= \frac{1}{2}\pi x^2 + 28x - \frac{x^2\pi}{2} - 2x^2}$$

$$\cancel{= -2x^2 - \frac{1}{2}\pi x^2 + 28x.}$$

Thus, $A(x) = \frac{1}{2}\pi x^2 + x\left(28 - x(\pi+2)\right)$

$$= \frac{1}{2}\pi x^2 + 28x - \pi x^2 - 2x^2$$

Note that we need (*) to be positive.

So, $28 - x(\pi+2) > 0.$

or, $x < \frac{28}{\pi+2}.$

Clearly, $x > 0.$ So, domain should be. $\left(0, \frac{28}{\pi+2}\right)$

• Calculus will help us study how to draw these functions; ~~where the~~ at which x-values the function is maximized / minimized, etc.

We will actually maximize, minimize quantities which is of immense importance in real life models.

For example, you ~~get~~ buy I-phones ~~and stuff~~ and they tend to get 'faster' every time a new model comes out. But there is a certain compromise - battery life.

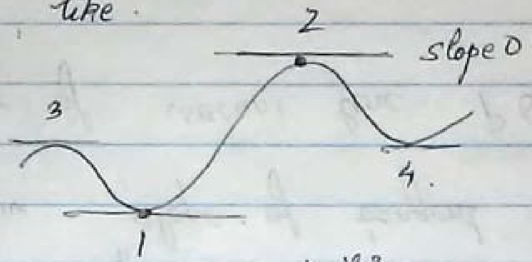
It's about optimizing the two to get the best output.

(This is an example of High Performance Low Power (HPLP) problem; ECE guys work on these things).

The first step towards achieving 'Optimization' zenith is to study 'limits' of functions.

Suppose a profit function is graphed.

Say, it looks like.



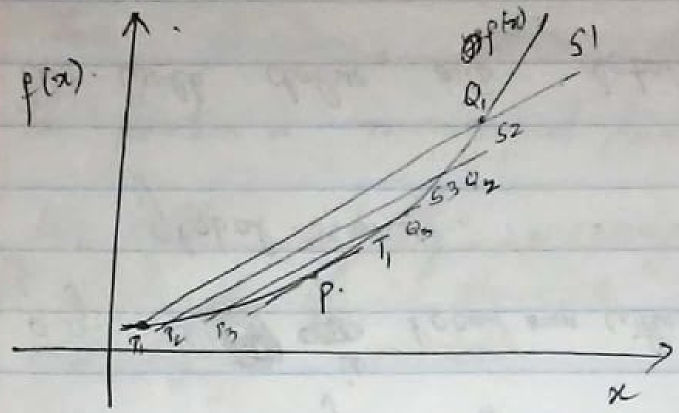
The 2 lines drawn at 1&2 are special cases of 'tangent lines'. ~~These actually~~

Note that the 2 points we chose indicate the maximum / minimum profits; notion of derivatives will tell us that max/min's are indicated by the slope of the tangent lines being 0.

We will study local and global max/min, e.g. 3 is local max whereas 1 is a global min.

We will delve into details soon.

To formulate "slope of tangent line" mathematically:

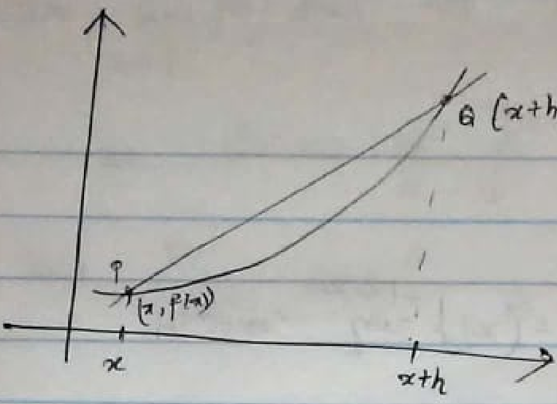


We say S^1, S^2, S^3 are secant lines, and we are 'approaching' the tangent line T_1.

i.e. the points P_i, Q_i are moving closer and closer and eventually we get the tangent line at P.

So, we can define "slope of tangent line of $f(x)$ at x":

= slope of secant line PQ (as Q is pushed arbitrarily close to P, but not at P)



i.e. we are looking at

$$\frac{f(x+h)-f(x)}{(x+h)-x}$$ as h becomes smaller and smaller.

$\left(\begin{array}{c} \parallel \\ \text{slope of } PQ \text{ right?!} \end{array}\right)$

We call this $\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$.

We'll now try to make this rigorous by discussing what "$\lim_{h\to 0}$" means mathematically.

[Remark: "$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$", if it exists, is called the "derivative of f at x"]

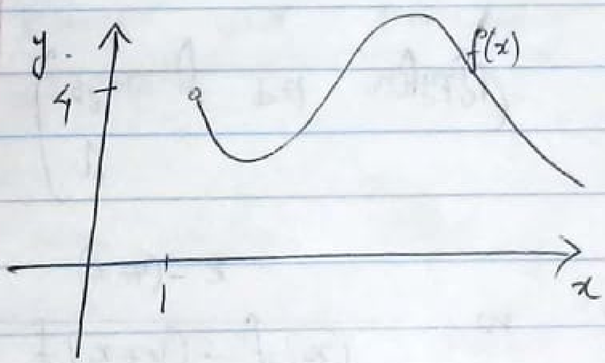
- $\boxed{\text{One-sided limits}}$

We say $\lim_{x \to a^+} f(x) = L$ (resp. $\lim_{x \to a^-} f(x) = L$)

if:
as x gets very close to a from the right (resp., left) but $x \neq a$, all values of $f(x)$ get very close to L. (Note that L can be ∞ or $-\infty$ too).

If there is no such L, we say $\lim_{x \to a^\pm} f(x)$ does not exist (or DNE)

$\boxed{\text{e.g.}}$



f(1) is not defined (we mark it by a circle).

$f : (1, \infty) \to \mathbb{R}$

- However, $\lim_{x \to 1^+} f(x) = 4$ (as the diagram show