MA2040: Probability, Statistics and Stochastic Processes
Problem Set-III

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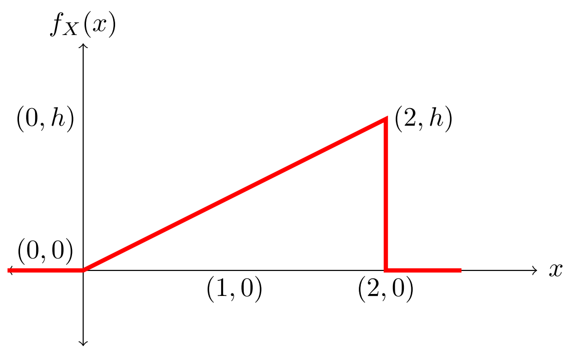
1. If $X_1, X_2, \ldots, X_n$ are independent random variables having the same probability density function $f_X(x)$, what is the probability density function for the random variable $Y = \min\{X_1, X_2, \ldots, X_n\}$?
   Solution:

$$P(Y \geq y) = P(X_1 \geq y, X_2 \geq y, \ldots, X_n \geq y) = P(X_1 \geq y)P(X_2 \geq y) \ldots P(X_n \geq y)$$

   Hence, we obtain that

$$1 - F_Y(y) = (1 - F_X(y))^n$$
$$f_Y(y) = n f_X(y) (1 - F_X(y))^{n-1}$$

2. A random variable X has a probability density function as shown below.



   (a) Determine h
   Solution: Area equals one implies $h = 1$

   (b) Determine the cumulative distribution function

   Solution: $F_X(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2/4 & \text{if } x \in [0, 2] \\ 1 & \text{if } x \geq 2 \end{cases}$

   (c) Compute the mean

   Solution: We have

$$\mathbb{E}(X) = \int_0^2 x f_X(x) dx = \int_0^2 x \cdot \frac{x}{2} dx = 4/3$$

(d) Compute the variance

Solution:

$$\mathbb{E}\left(X^2\right) = \int_0^2 x^2 f_X(x)dx = \int_0^2 x^2 \cdot \frac{x}{2}dx = 2 \implies \mathrm{Var}\left(X\right) = 2/9$$

(e) Determine the probability that $X \in (1,2)$.

Solution:

$$\mathbb{P}\left(X \in (1,2)\right) = \int_1^2 f_X(x)dx = \int_1^2 x/2dx = 3/4$$

3. The median m of a probability density function is defined as the value of m such that

$$\int_{-\infty}^{m} f(x)dx = \int_m^{\infty} f(x)dx = 1/2$$

Essentially, the median splits the distribution into two equal halves. Prove that the median is the best predictor if one wants to minimize the expected value of the absolute error, i.e., $\mathbb{E}\left(|X-c|\right)$ is minimized when c is the median of the underlying distribution.
Solution: We have

$$\mathbb{E}\left(|X-c|\right) = \int_{-\infty}^{c} \left(c-x\right) f_X(x)dx + \int_c^{\infty} \left(x-c\right) f_X(x)dx$$

Differentiating with respect to c, we obtain

$$\int_{-\infty}^{c} f_X(x)dx = \int_c^{\infty} f_X(x)dx$$

We also have that

$$\int_{-\infty}^{c} f_X(x)dx + \int_c^{\infty} f_X(x)dx = 1$$

Hence, this gives us that

$$\int_{-\infty}^{c} f_X(x)dx = \int_c^{\infty} f_X(x)dx = 1/2$$

4. Let X be a random variable, whose pdf is given by

$$f_X(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ xe^{-x^2/2} & \text{if } x > 0 \end{cases}$$

Find the pdf for the random variable $Y = X^2$.
Solution: Since $Y = X^2$ is an increasing function on $[0,\infty)$, we have

$$f_Y(y) = f_X(\sqrt{y})\frac{dx}{dy} = \frac{f_X(\sqrt{y})}{2\sqrt{y}} = 1/2e^{-y/2}$$

5. Let X be a uniform random variable on the interval $[0,1]$. Consider the random variable $Y = g\left(X\right)$, where

$$g(x) = \begin{cases} 1 & \text{if } x \leq 1/3 \\ 2 & \text{else} \end{cases}$$

Find the probability mass function of Y and compute its expected value.

Solution:

$$\mathbb{P}\left(Y = 1\right) = \mathbb{P}\left(X \leq 1/3\right) = 1/3$$
$$\mathbb{P}\left(Y = 2\right) = \mathbb{P}\left(X \geq 1/3\right) = 2/3$$

6. Show the expected value of a random variable X can also be obtained as

$$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > x)\, dx - \int_0^\infty \mathbb{P}(X < -x)\, dx$$

Solution: We have $X = X^+ - X^-$, where $X^+ = \max\{X, 0\}$ and $X^- = \max\{-X, 0\}$. Note that

$$X^+ = \int_0^\infty I(X > t)\, dt$$

and

$$X^- = \int_{-\infty}^0 I(X \leq t)\, dt$$

where I is the indicator function that takes the value 1, when the argument is true and takes the value 0, when the argument is false. We have

$$\mathbb{E}(X) = \mathbb{E}\left(X^+ - X^-\right)$$

which gives us what we want. Note that expectation of the indicator function is nothing but the probability of the argument.

7. A defective coin minting machine produces coins whose probability of heads is a random variable Y with PDF

$$f_Y(y) = \begin{cases} y \exp(y) & \text{if } y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

A coin produced by this machine is selected and tossed repeatedly, with successive tosses assumed independent.

(a) Find the probability that a coin toss results in head.

$$\int_0^1 y f_Y(y) dy$$

(b) Given that a coin toss resulted in heads, find the conditional PDF of Y.

(c) Given that the first coin toss resulted in heads, find the conditional probability of heads on the next toss.

8. Let the random variables X and Y have a joint PDF, which is uniform over the triangles with vertices $(0,0)$, $(0,1)$ and $(1,0)$.

(a) Find the joint PDF of X and Y.

Solution: $f_{XY}(x,y) = 2$ over the triangle.

(b) Find the marginal PDFs.

Solution:

$$f_X(x) = \int_{y=0}^{1-x} 2dy = 2(1-x)$$

$$f_Y(y) = \int_{x=0}^{1-y} 2dx = 2(1-y)$$

(c) Find the conditional PDFs.

Solution:

$$f_{X|Y=y} = \frac{1}{1-y}$$

$$f_{Y|X=x} = \frac{1}{1-x}$$

9. Chennai's temperature is modeled as a normal random variable with a mean temperature of 34°C and a standard deviation of 5°C. What is the probability that the temperature at a randomly chosen time will exceed 45°C?

Solution: $z = \dfrac{45-34}{5} = 2.2$. $\Phi(2.2) = 0.9861$. Hence, desired probability is $1 - 0.9861 = 0.1039$.

10. A surface is ruled with parallel lines, which are at a distance d from each other. Suppose that we throw a needle of length l on the surface at random. What is the probability that the needle with intersect one of the lines? (NOTE: You will need to treat the case $d < l$ and $d > l$ separately.)

Solution: Page 161. Bertsekas.

$$d > l : \frac{2l}{\pi d}$$

$$d < l : \frac{2}{\pi} \arccos\left(\frac{d}{l}\right) + \frac{2}{\pi}\frac{l}{d}\left(1 - \sqrt{1 - \left(\frac{d}{l}\right)^2}\right)$$

11. Consider two continuous random variables Y and Z and a random variable X that is equal to Y with a probability p and equals Z with a probability $1-p$. Obtain the pdf of X interms of the pdf's of Y and Z.

Solution:

$$F_X(x) = \mathbb{P}(X \le x) = p\mathbb{P}(Y \le x) + (1-p)\,\mathbb{P}(Z \le x)$$

Hence,

$$f_X(x) = pf_Y(x) + (1-p)\,f_Z(x)$$