

MA2040: Probability, Statistics and Stochastic Processes

Problem Set-II

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1. Assume that Carlsen meets Anand in the 2020 chess championship. The championship match consists of a sequence of games and each game has three outcomes (i) Anand winning, (ii) Carlsen winning, (iii) A draw. The first player to win a game wins the match. For instance, we could have a sequence of 3 draws followed by a Carlsen victory in the 4th game, which would mean that Carlsen wins the Championship. The probability of a single game ending in

- (a) Carlsen's favour is 0.4
- (b) Anand's favour is 0.2
- (c) draw is 0.4

- i What is the probability of Anand winning the championship?

Solution: Probability of Anand winning the match on the k^{th} game is given by $0.4^{k-1} \times 0.2$, i.e., the first $k - 1$ games have to be drawn and the k^{th} game has to be won by Anand. Hence, the probability of Anand winning the championship is

$$\sum_{k=1}^{\infty} 0.4^{k-1} \times 0.2 = \frac{0.2}{1 - 0.4} = 1/3$$

- ii What is the probability of Carlsen winning the championship?

Solution: Probability of Carlsen winning the match on the k^{th} game is given by $0.4^{k-1} \times 0.4$, i.e., the first $k - 1$ games have to be drawn and the k^{th} game has to be won by Carlsen. Hence, the probability of Carlsen winning the championship is

$$\sum_{k=1}^{\infty} 0.4^{k-1} \times 0.4 = \frac{0.4}{1 - 0.4} = 2/3$$

This can also be obtained as $1 - 1/3 = 2/3$, since the probability of Anand or Carlsen winning the Championship is 1.

- iii What is the Probability Mass Function for the number of games played in the championship?

Solution: For the match to last k games, the first $k - 1$ have to be drawn and the k^{th} game has to be won either by Anand or Carlsen. Hence, $P(k) = 0.4^{k-1} \times (0.4 + 0.2) = 0.6 \times 0.4^{k-1}$.

2. Consider rolling a pair of fair dice. Let X denote the difference between the numbers that show up on the dice, i.e., $X = |D_1 - D_2|$, where D_i is the number that shows up on the i^{th} dice.

- What are the possible values for X ?

Solution: X can take values from 0 to 5.

- What is the probability mass function for X ?

Solution: We have

$$P(0) = \frac{6}{36}, P(1) = \frac{10}{36}, P(2) = \frac{8}{36}, P(3) = \frac{6}{36}, P(4) = \frac{4}{36}, P(5) = \frac{2}{36}$$

- Find the expected value and standard deviation of X .

Solution:

$$\mathbb{E}(X) = \frac{0 \times 6 + 1 \times 10 + 2 \times 8 + 3 \times 6 + 4 \times 4 + 5 \times 2}{36} = \frac{70}{36} = 1.9\bar{4}$$

3. A fair die is rolled repeatedly till an odd prime appears. What is the probability that the number of rolls exceed 5?

Solution: Probability of an odd prime occurring in a single roll is $1/3$ (since only 3 and 5 have to occur). For the number of rolls to exceed 5, we need the first 5 rolls to be not an odd prime. Hence,

the probability of this event is $\left(\frac{2}{3}\right)^5$.

4. Let X be a discrete random variable with mean μ and variance σ^2 . Prove that

$$\mathbb{E}[(X - a)^2] = \sigma^2 + (a - \mu)^2$$

Hence, prove that the mean (or expected value) minimizes $\mathbb{E}[(X - a)^2]$.

Solution: We have

$$\mathbb{E}[(X - a)^2] = \mathbb{E}[(X - \mu + \mu - a)^2] = \mathbb{E}[(X - \mu)^2] + 2\mathbb{E}[(X - \mu)(\mu - a)] + \mathbb{E}[(\mu - a)^2]$$

since the expectation is a linear operator. This immediately gives us that

$$\mathbb{E}[(X - a)^2] = \sigma^2 + (a - \mu)^2$$

since $\mathbb{E}[(X - \mu)(\mu - a)] = (\mu - a)\mathbb{E}[(X - \mu)] = 0$ and $\mathbb{E}[(\mu - a)^2] = (\mu - a)^2$.

5. A production process is partitioned into two independent sub-processes. The probabilities of a defective component in the first and second sub-processes are 0.01 and 0.02, respectively. If 50 units are produced, what is the probability there will be fewer than 3 defective units?

Solution: Probability of a unit being non-defective is $(1 - 0.01) \times (1 - 0.02)$. Hence, the probability of a unit being defective is $1 - (1 - 0.01) \times (1 - 0.02) = 0.0302$. The desired probability is given as

$$\sum_{k=0}^2 \binom{50}{k} (0.0302)^k (0.9698)^{50-k} \approx 0.8082$$

We could also approximate this probability with a Poisson random variable with expected defectives to be $50 \times 0.0302 = 1.51$. Hence, the desired probability is

$$\sum_{k=0}^2 e^{-1.51} \frac{1.51^k}{k!} \approx 0.8063$$

6. Communication channels do not always transmit the correct signal. Suppose that for a particular channel the error rate is 1 in 100, i.e., the probability of incorrect transmission is $1/100$. If 2000 messages are sent in a given week, and it is assumed that their transmissions are independent, what is the probability

that there will be at least 5 errors?

Solution: Desired probability is

$$1 - \sum_{k=0}^4 \binom{2000}{k} (0.01)^k (0.99)^{2000-k} \approx 0.999984$$

Approximating it with Poisson, the expected number of incorrect transmissions is 20, we get the probability as

$$1 - e^{-20} \left(\sum_{k=0}^4 \frac{20^k}{k!} \right) \approx 0.999983$$

7. A casino offers a game of chance for a single player in which a fair coin is tossed at each stage. The initial stake starts at \$1 and is increased by \$1 every time heads appears. The first time tails appears, the game ends and the player wins whatever is in the pot. Thus the player wins \$1 dollar if tails appears on the first toss, \$2 dollars if heads appears on the first toss and tails on the second, \$3 dollars if heads appears on the first two tosses and tails on the third, and so on. Mathematically, the player wins \$ k dollars, when we have the first $k - 1$ tosses to be heads and the k^{th} toss to be a tail. The casino demands a pay of \$3 to enter the game. Will you play the game?

Solution: Probability of winning on the k attempt is given by $\frac{1}{2^k}$, i.e., the first $k - 1$ tosses are heads and the k^{th} toss is a tail. Hence, the expected return is

$$\sum_{k=1}^{\infty} \frac{k}{2^k} = 2$$

Since the casino demands a pay of \$3 to enter the game, which is greater than the expected return, it is not advisable to play the game.

8. Repeat the above if the price money was 2^k instead of k and the casino demands a pay of \$100 to enter the game. Will you still be willing to play the game? (For more details, look up St. Petersburg paradox)

Solution: My expected return in this case is given by

$$\sum_{k=1}^{\infty} \frac{2^k}{2^k}$$

which diverges. Hence, going by the expected value we could play the game. However, I will make a profit only when my return exceeds 100, which happens with a probability of $1/2^{100}$.

9. If X is a discrete random variable, prove that (i) $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$ and (ii) $\text{Var}[aX + b] = a^2\text{Var}(X)$

Solution: Follows immediately from definition.

$$\mathbb{E}[aX + b] = \sum_x (ax + b) p(x) = a \sum_x xp(x) + b \sum_x p(x) = a\mathbb{E}[X] + b$$

10. Data shows that 5% of the individuals reserving tables at a restaurant will not appear. If the restaurant has 50 tables and takes 52 reservations, what is the probability that it will be able to accommodate everyone appearing?

Solution: Probability of failing to accommodate everyone is when 51 individuals or 52 individuals appear. This happens with a probability of

$$\binom{52}{51} (0.95)^{51} \times (0.05) + \binom{52}{52} (0.95)^{52} = 0.95^{51} \times (0.95 + 52 \times 0.05) = 3.55 \times 0.95^{51} \approx 0.2595$$

Hence, desired probability is 0.7405.

11. Electrical power failures in a workplace are modeled as a Poisson experiment with a rate of one every two months.

- (a) What is the probability of having more than 10 failures in a year?

Solution: The mean for a year is 6. Desired probability is $e^{-6} \sum_{k=11}^{\infty} \frac{6^k}{k!} \approx 0.04262$

- (b) What is the probability that the number of failures in a year will differ by more than a standard deviation from the expected number?

Solution: The standard deviation for a year is $\sqrt{6} \approx 2.45$. Hence, the desired probability is

$$1 - e^{-6} \left(\frac{6^4}{4!} + \frac{6^5}{5!} + \frac{6^6}{6!} + \frac{6^7}{7!} + \frac{6^8}{8!} \right) \approx 0.3039$$

12. Table 1 below indicates the joint probabilities per day.

Table 1: Joint probability of weather and power cuts

	Sunny	Rainy
Power cut	0.2	0.15
No power cut	0.6	p

- (a) Find p .

Solution: Sum must be 1. Hence, $p = 0.05$.

- (b) What is the probability that there won't be rain for one week?

Solution: Probability that it will be sunny is $0.2 + 0.6 = 0.8$. Hence, probability it won't rain for one week is 0.8^7 .

- (c) What is the probability that there will be at least one power in the next three days?

Solution: Probability of having no power cuts is 0.65. Hence, probability of having at least one power in the next three days $1 - 0.65^3 = 0.725375$.

- (d) Is there a dependence between weather and power cuts?

Solution: Let X be the event of having a power cut and Y be the event of the day being sunny. We have

$$P(X) = 0.2 + 0.15 = 0.35$$

We have

$$P(X | Y) = \frac{P(X, Y)}{P(Y)} = \frac{0.2}{0.2 + 0.6} = 0.25$$

Hence, we see that there is a dependence between weather and power cuts.

- (e) Find the joint probability, all marginal probabilities, and all conditional probabilities.

Solution: We have $P(X) = 0.35$, $P(X^c) = 0.65$, $P(Y) = 0.8$, $P(Y^c) = 0.2$.

$$P(X | Y) = \frac{P(X, Y)}{P(Y)} = 0.25 \quad P(X^c | Y) = 0.75$$

$$P(X | Y^c) = \frac{P(X, Y^c)}{P(Y^c)} = \frac{0.15}{0.2} = 0.75 \quad P(X^c | Y^c) = 0.25$$

$$P(Y | X) = \frac{P(X, Y)}{P(X)} = 4/7 \quad P(Y^c | X) = 3/7$$

$$P(Y | X^c) = \frac{P(X^c, Y)}{P(X^c)} = 12/13 \quad P(Y^c | X^c) = 1/13$$