MA2040: Probability, Statistics and Stochastic Processes
Problem Set-IV

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1. Let X and Y have joint pdf $f_{X,Y}(x,y) = \begin{cases} 6e^{-(2x+3y)} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

   - Are X and Y independent?

     Solution: We have
     $$f_X(x) = \int_0^\infty f_{X,Y}(x,y)dy = 2e^{-2x}$$
     $$f_Y(y) = \int_0^\infty f_{X,Y}(x,y)dx = 3e^{-3y}$$
     We have $f_{X,Y} = f_X \cdot f_Y$. Hence, X and Y are independent.

   - Find $\mathbb{E}\left(Y \mid X > 2\right)$

     Solution: Since X and Y are independent, conditioning on X doesn't affect Y. Hence,
     $$\mathbb{E}\left(Y \mid X > 2\right) = \mathbb{E}\left(Y\right) = \int_0^\infty 3ye^{-3y}dy = \frac{1}{3}$$

   - Find $\mathbb{P}\left(X > Y\right)$

     Solution: $\mathbb{P}\left(X > Y\right) = \displaystyle\int_0^\infty \mathbb{P}\left(X > y\right) f_Y(y)dy = \int_0^\infty \left(1 - F_X(y)\right) f_Y(y)dy$

2. Let $X \sim \text{Uniform}(1,2)$ and given $X = x$, Y follows an exponential distribution with $Y \sim \text{EXP}\left(\lambda = x\right)$. Find the covariance of X and Y.

   Solution:
   $$f_{XY}\left(x,y\right) = f_{Y=y|X=x}f_X(x) = xe^{-xy}$$
   $$\text{Cov}\left(X,Y\right) = \mathbb{E}\left(XY\right) - \mathbb{E}\left(X\right)\mathbb{E}\left(Y\right)$$
   We have
   $$\mathbb{E}\left(X\right) = \int_{x=1}^2 \int_{y=0}^\infty x^2e^{-xy}dydx = 3/2$$
   $$\mathbb{E}\left(Y\right) = \int_{x=0}^1 \int_{y=0}^\infty xye^{-xy}dydx = \ln(2)$$

$$\mathbb{E}(XY) = \int_{x=1}^{2} \int_{y=0}^{\infty} x^2 y e^{-xy} dy dx = 1$$

3. Let X and Y be independent standard normal random variables. Find covariance of Z and W, where $Z = 1 + X + XY^2$ and $W = 1 + X$.

   Solution: Compute
   $$\mathbb{E}(ZW) - \mathbb{E}(Z)\,\mathbb{E}(W)$$

4. A fair die is rolled n times. Let X be the number of 1's and Y be the number of 2's. Find the correlation coefficient $\rho(X, Y)$.

   Solution:$-1/5$

5. Let 32 people sit around a round table. Each person tosses a fair coin. Anyone whose outcome is different from both his neighbors is taken out from the group. If X is the total number of such persons, find variance of X.

6. The moment generating function of a random variable X is given by
   $$M_X(s) = \frac{2}{2-s}, \ \ \forall s \in (-\infty, 2)$$
   Find the distribution of X.

7. Let $X \sim \text{Binomial}(n, p)$ and $Y \sim \text{Binomial}(m, p)$ be independent random variables. Show that $X+Y \sim \text{Binomial}(m+n, p)$.

8. Show that the function e^{-s} is a moment generating function.

9. Show that if $M(s)$ is a moment generating function, so is $M(cs)$.

10. Show that if $M(s)$ is a moment generating function, $cM(s)$ cannot be a moment generating function for $c \neq 1$.

11. Show that if $M(s)$ is a moment generating function, then $e^{-s}M(s)$ is also a moment generating function.

12. The moment generating function of a random variable X is
    $$M_X(s) = \frac{e^{-2s}}{6} + \frac{e^{-s}}{3} + \frac{e^{s}}{4} + \frac{e^{2s}}{4}$$
    Show that $P(|X| \leq 1) = \frac{7}{12}$.