MA 2040 $\boxed{A}$ **Probability, Statistics and Stochastic Processes** **Max. Marks 20**

Duration 50 minutes **Quiz-I** **February 18, 2019**

Student Name ________________________________ **Student Roll number** ________________

Instructor Name ________________________________ **Classroom** ________________

1. Let X be a random variable on $[1, \infty)$, whose probability density function is given by $p_X(x) = k/x^3$.

   - (1 point) Value of k is $\boxed{2}$
     Solution: We need $\int_1^{\infty} \frac{k}{x^3} dx = 1 \implies k = 2.$
   - (1 point) Cummulative distribution function (CDF) is $\boxed{F_X(x) = \begin{cases} (1 - 1/x^2) & \text{for } x \geq 1 \\ 0 & \text{else} \end{cases}}$
     Solution: **Solution:** We have $F_x(x) = \int_1^x \frac{k}{y^3} dy = \left(1 - \frac{1}{x^2}\right)$ for $x \geq 1$. For $x \leq 1$, $F_X(x) = 0$.
   - (1 point) Expected value is $\boxed{2}$
     Solution: We need $\int_1^{\infty} x \cdot \frac{k}{x^3} dx = k = 2.$
   - (1 point) Variance is $\boxed{\infty}$
     Solution: We have variance to be given by $\mathbb{E}\left(X^2\right) - \mathbb{E}X^2$. We have
     $$\mathbb{E}\left(X^2\right) = \int_1^{\infty} x^2 \cdot \frac{k}{x^3} dx = \infty$$
   - (1 point) Probability that the random variable, X, takes a value less than 2 is $\boxed{3/4}$
     Solution: This is nothing but $F_X(2)$.

2. A box contains three coins: two fair coins and one fake coin. The fake coin has both sides as heads. A coin is picked at random (all three coins are equally likely to be chosen) and tossed.

   - (1 point) Then the probability of getting a head is $\boxed{2/3}$
     Solution:
     $$\mathbb{P}(H) = \mathbb{P}(H \mid \text{fake coin})\,\mathbb{P}(\text{fake coin}) + \mathbb{P}(H \mid \text{fair coin})\,\mathbb{P}(\text{fair coind}) = 1 \times 1/3 + 1/2 \times 2/3 = 2/3$$
   - (1 point) If a head comes up, then the probability that the coin is a fake coin is $\boxed{1/2}$
     Solution:
     $$\mathbb{P}(\text{fake coin} \mid \text{H}) = \frac{\mathbb{P}(\text{H} \mid \text{fake coin})\,\mathbb{P}(\text{fake coin})}{\mathbb{P}(\text{H})} = 1/2$$

3. A coin, displaying 0 and 1 on both sides, and a four sided dice, displaying 1, 2, 3, 4 on its four faces are tied together by a string and tossed/rolled. Let X denote the number displayed on the coin and Y denote the number displayed on the dice. The joint probability mass function $p_{X,Y}(x, y)$ is given below.

$X \backslash Y$	1	2	3	4
0	2/20	3/20	4/20	2/20
1	1/20	2/20	5/20	q

   - (1 point) Value of q is $\boxed{1/20}$
     Solution: Sum of all probabilities must be 1.
   - (1 point) $\mathbb{P}(X = 0)$, i.e., $p_X(0)$ is $\boxed{11/20}$
     Solution: Sum of the first row in the table.

- (1 point) $\mathbb{P}(Y = 3 \mid X = 1)$, i.e., $p_{Y|X=1}(3)$ is $\boxed{5/9}$
  Solution: Restricted sample space is the row corresponding to $X = 1$.
- (1 point) $\mathbb{E}(X \mid Y = 2)$ is $\boxed{2/5}$
  Solution: Restricted sample space is the column corresponding to $Y = 2$.
- (1 point) $\mathrm{Var}(Y)$ is $\boxed{21/25}$
  Solution: We have $P_Y(1) = 3/20$, $P_Y(2) = 5/20$, $P_Y(3) = 9/20$ and $P_Y(4) = 3/20$.

4. A fair coin is tossed thrice. **A** be the event that a head occurs on each of the first two tosses, **B** be the event that a tail occurs on the third toss and **C** be the event that exactly two tails occur in the three tosses.
   Solution: $A = \{HHH, HHT\}$, $B = \{HHT, HTT, THT, TTT\}$, $C = \{HTT, THT, TTH\}$.

   - (1 point) $\mathbb{P}(\mathbf{B} \cap \mathbf{C})$ $\boxed{2/8}$
   - (1 point) $\mathbb{P}(\mathbf{A} \cap \mathbf{B^c})$ $\boxed{1/8}$
   - (1 point) $\mathbb{P}(\mathbf{A^c} \cap \mathbf{C^c})$ $\boxed{3/8}$

5. A candidate can appear for a competitive exam **atmost thrice till he clears it**. Probability of him passing the exam in a given attempt is $p \in (0, 1)$, independent of other attempts. Let X be the number of attempts.

   - (1 point) The probability that he clears the exam is $\boxed{1 - (1 - p)^3}$
     Solution: Probability he doesn't clear is $(1 - p)^3$.
   - (3 points) If $p_X(x)$ denotes the probability mass function of the random variable X, then
     - (1 point) $p_X(1) =$ $\boxed{p}$
     - (1 point) $p_X(2) =$ $\boxed{p(1 - p)}$
     - (1 point) $p_X(3) =$ $\boxed{(1 - p)^2}$
   - (1 point) Expected number of attempts is $\boxed{p^2 - 3p + 3}$

     Solution: $\sum_{k=1}^{3} k p_X(k)$