MA2040: Probability, Statistics and Stochastic Processes
Problem Set-V

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1. Let $\{X_i\}_{i=1}^{64}$ be a random sample from a normal distribution with mean $\mu = 50$ and variance $\sigma^2 = 16$. Find $\mathbb{P}(49 < X_8 < 51)$ and $\mathbb{P}\left(49 < \overline{X} < 51\right)$.
Solution:
$$\mathbb{P}(49 < X_8 < 51) = \mathbb{P}\left(\frac{49-50}{4} < \frac{X_8 - 50}{4} < \frac{51-50}{4}\right) = \Phi(1/4) - \Phi(-1/4) \tag{1}$$
$$= 2\Phi(1/4) - 1 = 2 \times 0.59871 - 1 = 0.19742 \tag{2}$$
$$\mathbb{P}\left(49 < \overline{X} < 51\right) = \mathbb{P}\left(\frac{49-50}{4/\sqrt{64}} < \frac{\overline{X} - 50}{4/\sqrt{64}} < \frac{51-50}{4/\sqrt{64}}\right) = 2\Phi(2) - 1 = 0.9545 \tag{3}$$

2. Let $Y = X_1 + X_2 + \cdots + X_{15}$ be the sum of a random sample of size 15 from the population whose probability density function is given by
$$f_X(x) = \begin{cases} kx^2 & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$
What is the approximate value of $\mathbb{P}(-0.3 < Y < 1.5)$ when one uses Central Limit Theorem?
Solution: We have $k = 3/2$. Mean is 0. Variance is 0.6. From CLT, we have
$$\mathbb{P}(-0.3 < Y < 1.5) = \mathbb{P}\left(-0.3/\sqrt{0.6 \times 15} < Z < 1.5/\sqrt{0.6 \times 15}\right) = \Phi(0.5) - \Phi(-0.1) \tag{4}$$
$$= 0.69146 - 0.46017 \tag{5}$$
$$= 0.23129 \tag{6}$$

3. Light bulbs are installed successively into a socket. If we assume that each bulb has a mean life of 2 months with a standard deviation of 0.25 months, what is the probability that 40 bulbs will last for at least 7 years?
Solution: Let X_i be the life of the i^{th} bulb in months. We want $S_{40} = \displaystyle\sum_{i=1}^{40} X_i \geq 84$. Hence,
$$\mathbb{P}(S_{40} \geq 84) = \mathbb{P}\left(\frac{S_{40} - 40 \times 2}{0.25 \times \sqrt{40}} \geq \frac{84 - 40 \times 2}{0.25 \times \sqrt{40}}\right) = 1 - \Phi(2.529) = 1 - 0.99413 = 0.00587$$

4. A random sample of size 36 is taken from the population whose pdf is given by $f_X(x) = 2e^{-2x}$, for $x \geq 0$. If $\overline{X}$ denotes the sample mean, find $\mathbb{E}\left(\overline{X}\right)$ and $\mathbb{E}\left(\overline{X}^2\right)$. Also find the probability that $\overline{X} \in \left[\frac{1}{4}, \frac{3}{4}\right]$.
Solution: We have the mean of the distribution to be $1/2$ and variance to be $1/4$. Hence, $\mathbb{E}\left(\overline{X}\right) = 1/2$ and $\mathbb{E}\left(\overline{X}^2\right) = 1/4/36 + 1/4 = \dfrac{37}{144}$. The desired probability is again computed by normal approximation, which gives us 0.9974.

5. In order to estimate f, the true fraction of smokers in a large population, Alvin selects n people at random. His estimator M_n is obtained by dividing S_n (the number of smokers in his sample) by n, i.e., $M_n = S_n/n$. Alvin chooses the sample size n to be the smallest possible number for which the Chebyshev inequality yields a guarantee that

$$\mathbb{P}\left(|M_n - f| \geq \epsilon\right) \leq \delta$$

where ϵ and δ are some prespecified tolerances. Determine how the value of n recommended by the Chebyshev inequality changes in the following cases.

(a) The value of ϵ is reduced to half its original value.
Solution: Goes up by a factor of 4.

(b) The value of δ is reduced to half its original value.
Solution: Goes up by a factor of 2.

6. Before starting to play the roulette in a casino, you want to look for biases that you can exploit. You therefore watch 100 rounds that result in a number between 1 and 36, and count the number of rounds for which the result is odd. If the count exceeds 55, you decide that the roulette is not fair. Assuming that the roulette is fair, find an approximation for the probability that you will make the wrong decision. (HINT: Central Limit Theorem)
Solution: Let S be the number of times that the result was odd. This is a binomial random variable, with $n = 100$ and $p = 0.5$. This gives us $\mathbb{E}(S) = 50$ and $\mathrm{Var}(S) = 25$. From normal approximation, we obtain that

$$\mathbb{P}(S > 55) = \mathbb{P}\left(\frac{S-50}{5} > \frac{55-50}{5}\right) = 1 - \Phi(1) = 0.1587$$

7. **Proof of Central Limit Theorem**: Let $\{X_i\}_{i=1}^n$ be a sequence of independent identically distributed random variables with mean zero, with variance, say σ^2, and associated transform $M_X(s)$. We assume that $M_X(s)$ is finite for $s \in (-d, d)$, for some $d \in \mathbb{R}^+$. Let

$$Z_n = \frac{X_1 + X_2 + \cdots + X_n}{\sigma\sqrt{n}}$$

(a) Show that the transform associated with Z_n is given by

$$M_{Z_n}(s) = \left(M_x\left(\frac{s}{\sigma\sqrt{n}}\right)\right)^n$$

Solution:Immediate from definition of mgf.

(b) Write the first two terms and the error term of the Taylor series expansion of M_X around $s = 0$.

(c) From the above conclude that

$$\lim_{n\to\infty} M_{Z_n}(s) = e^{s^2/2}$$

for all s. And note that $e^{s^2/2}$ is the moment generating function of a standard normal random variable.