

MA2040: Probability, Statistics and Stochastic Processes

Problem Set-IV

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1. Let X and Y have joint pdf $f_{X,Y}(x,y) = \begin{cases} 6e^{-(2x+3y)} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

- Are X and Y independent?

Solution: We have

$$f_X(x) = \int_0^\infty f_{X,Y}(x,y)dy = 2e^{-2x}$$

$$f_Y(y) = \int_0^\infty f_{X,Y}(x,y)dx = 3e^{-3y}$$

We have $f_{X,Y} = f_X \cdot f_Y$. Hence, X and Y are independent.

- Find $\mathbb{E}(Y \mid X > 2)$

Solution: Since X and Y are independent, conditioning on X doesn't affect Y . Hence,

$$\mathbb{E}(Y \mid X > 2) = \mathbb{E}(Y) = \int_0^\infty 3ye^{-3y}dy = \frac{1}{3}$$

- Find $\mathbb{P}(X > Y)$

Solution:

$$\mathbb{P}(X > Y) = \int_0^\infty \mathbb{P}(X > y) f_Y(y)dy = \int_0^\infty (1 - F_X(y)) f_Y(y)dy = 1 - 3 \int_0^\infty (1 - e^{-2y}) e^{-3y}dy = 3/5$$

2. Let $X \sim \text{Uniform}(1, 2)$ and given $X = x$, Y follows an exponential distribution with $Y \sim \text{EXP}(\lambda = x)$. Find the covariance of X and Y .

Solution: We have

$$f_{XY}(x,y) = f_{Y|X}f_X = xe^{-xy}$$

This gives us

$$\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 1 - 3/2 \ln(2)$$

3. Let X and Y be independent standard normal random variables. Find covariance of Z and W , where $Z = 1 + X + XY^2$ and $W = 1 + X$.

Solution: $\mathbb{E}(Z) = 1$ and $\mathbb{E}W = 1$. Further, $\mathbb{E}(ZW) = \mathbb{E}(1 + 2X + X^2 + XY^2 + X^2Y^2) = 3$. Hence, covariance is 2.

4. A fair die is rolled n times. Let X be the number of 1's and Y be the number of 2's. Find the correlation coefficient $\rho(X, Y)$.

Solution: Let $Z = X + Y$. X, Y are binomial random variables with $p = 1/6$, while Z is a binomial random variable with $p = 1/3$. Hence, $\text{Var}(X) = \text{Var}(Y) = 5n/36$ and $\text{Var}(Z) = 2n/9$. Hence,

$$\text{Cov}(X, Y) = \frac{1}{2} (2n/9 - 5n/18) = -n/36$$

Hence, $\rho = -1/5$.

5. Let 32 people sit around a round table. Each person tosses a fair coin. Anyone whose outcome is different from both his neighbors is taken out from the group. If X is the total number of such persons, find variance of X .

Solution: Consider the random variable P_i , which takes a value 1, when the i^{th} person's outcome differs from his both neighbors and takes the value 0, when the outcomes of him matches with both his neighbors. We see that $P_i = 1$ with a probability of $1/4$ and $P_i = 0$ with a probability of $3/4$. Now

we have $X = \sum_{i=1}^{32} P_i$, which is a binomial random variable with $n = 32$ and $p = 1/4$. Hence, $\text{Var}(X) = 6$.

6. The moment generating function of a random variable X is given by

$$M_X(s) = \frac{2}{2-s}, \quad \forall s \in (-\infty, 2)$$

Find the distribution of X .

Solution: X is exponential with $\lambda = 2$.

7. Let $X \sim \text{Binomial}(n, p)$ and $Y \sim \text{Binomial}(m, p)$ be independent random variables. Show that $X + Y \sim \text{Binomial}(m + n, p)$.

Solution: Use the MGF to conclude.

8. Show that the function e^{-s} is a moment generating function.

Solution: Consider the random variable which takes only the value -1 , i.e., $\mathbb{P}(X = -1) = 1$. The MGF is e^{-s} .

9. Show that if $M(s)$ is a moment generating function, so is $M(cs)$.

Soln If $M(s)$ is the mgf of X , then consider the random variable $Z = X/c$.

10. Show that if $M(s)$ is a moment generating function, $cM(s)$ cannot be a moment generating function for $c \neq 1$.

Solution: The mgf needs to be 1 at $s = 0$.

11. Show that if $M(s)$ is a moment generating function, then $e^{-s}M(s)$ is also a moment generating function.

Solution: If $M(s)$ is the mgf of X , then consider the random variable $Z = X - 1$.

12. The moment generating function of a random variable X is

$$M_X(s) = \frac{e^{-2s}}{6} + \frac{e^{-s}}{3} + \frac{e^s}{4} + \frac{e^{2s}}{4}$$

Show that $P(|X| \leq 1) = \frac{7}{12}$.

Solution: This is a discrete random variable.

$$\mathbb{P}(X = k) = \begin{cases} 1/6 & \text{if } X = -2 \\ 1/3 & \text{if } X = -1 \\ 1/4 & \text{if } X = 1 \\ 1/4 & \text{if } X = 2 \end{cases}$$