

MA2040: Probability, Statistics and Stochastic Processes

Problem Set-IV

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1. Let X and Y have joint pdf $f_{X,Y}(x,y) = \begin{cases} 6e^{-(2x+3y)} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

- Are X and Y independent?

Solution: We have

$$f_X(x) = \int_0^\infty f_{X,Y}(x,y)dy = 2e^{-2x}$$

$$f_Y(y) = \int_0^\infty f_{X,Y}(x,y)dx = 3e^{-3y}$$

We have $f_{X,Y} = f_X \cdot f_Y$. Hence, X and Y are independent.

- Find $\mathbb{E}(Y \mid X > 2)$

Solution: Since X and Y are independent, conditioning on X doesn't affect Y . Hence,

$$\mathbb{E}(Y \mid X > 2) = \mathbb{E}(Y) = \int_0^\infty 3ye^{-3y}dy = \frac{1}{3}$$

- Find $\mathbb{P}(X > Y)$

2. Let $X \sim \text{Uniform}(1, 2)$ and given $X = x$, Y follows an exponential distribution with $Y \sim \text{EXP}(\lambda = x)$. Find the covariance of X and Y .
3. Let X and Y be independent standard normal random variables. Find covariance of Z and W , where $Z = 1 + X + XY^2$ and $W = 1 + X$.
4. A fair die is rolled n times. Let X be the number of 1's and Y be the number of 2's. Find the correlation coefficient $\rho(X, Y)$.
5. Let 32 people sit around a round table. Each person tosses a fair coin. Anyone whose outcome is different from both his neighbors is taken out from the group. If X is the total number of such persons, find variance of X .
6. The moment generating function of a random variable X is given by

$$M_X(s) = \frac{2}{2-s}, \quad \forall s \in (-\infty, 2)$$

Find the distribution of X .

7. Let $X \sim \text{Binomial}(n, p)$ and $Y \sim \text{Binomial}(m, p)$ be independent random variables. Show that $X + Y \sim \text{Binomial}(m + n, p)$.
8. Show that the function e^{-s} is a moment generating function.
9. Show that if $M(s)$ is a moment generating function, so is $M(cs)$.
10. Show that if $M(s)$ is a moment generating function, $cM(s)$ cannot be a moment generating function for $c \neq 1$.
11. Show that if $M(s)$ is a moment generating function, then $e^{-s}M(s)$ is also a moment generating function.
12. The moment generating function of a random variable X is

$$M_X(s) = \frac{e^{-2s}}{6} + \frac{e^{-s}}{3} + \frac{e^s}{4} + \frac{e^{2s}}{4}$$

Show that $P(|X| \leq 1) = \frac{7}{12}$.