Department of Mathematics, IIT Madras
MA 2040 : Probability, Statistics and Stochastic Processes
~~Solutions to Problem Set - VI~~

1. Note that
   R ∼Binomial(n,p)
   G ∼Binomial(n,1-p)
   Further, $R + G = n$.

   (a)
   $$p_R(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k}, & k = 0, 1, 2, ..., n \\ 0, & \text{otherwise.} \end{cases}$$
   $\mathbb{E}(R) = np$, Var(R)$=np(1-p)$.

   (b) This event is possible in two ways:
   First item goes to red truck and rest $(n-1)$ go to green truck.
   First item goes to green truck and the rest $(n-1)$ go to red truck.
   This implies that the desired probability $= p(1-p)^{n-1} + (1-p)p^{(n-1)}$.

   (c) For $n = 1$, the desired event occurs with probability 1.
   For $n = 2$, the desired probability = P(the first item goes to red truck and second item goes to green truck) + P(the first item goes to green truck and second item goes to red truck)
   $=p(1-p) + (1-p)p = 2p(1-p)$
   For $n \geq 3$, the desired probability is
   $\binom{n}{1} p(1-p)^{n-1} + \binom{n}{1}(1-p)p^n - 1 = np(1-p)^{n-1} + n(1-p)p^{n-1}$

   (d) Recall that $R + G = n$.
   Hence $\mathbb{E}(R - G) = \mathbb{E}(2R - n) = 2\mathbb{E}(R) - n = 2np - n = n(2p - 1)$.
   $Var(R - G) = Var(2R - n) = 4Var(R) = 4np(1-p)$.

   (e) Let A be the event that the first two packages loaded go onto the red truck.
   Note that R can be written as
   $R = X_1 + X_2 + ... + X_n$
   where X_i's are Bernoulli RVs with parameter p. $R \mid A = 2 + X_3 + X_4 ... + X_n$
   $\mathbb{E}(R \mid A) = 2 + (n-2)p$

   $$\begin{aligned} Var(R \mid A) &= Var(2 + X_3 + X_4 ... + X_n) \\ &= Var(X_3) + Var(X_4) + ... + Var(X_n) \\ &= (n-2)p(1-p). \end{aligned}$$

   The possible values of $R \mid A$ are 2,3,4,...,n.

$$p_{R|A}(k) = \begin{cases} \binom{n-2}{k-2} p^{k-2}(1-p)^{n-k}, & k = 2, 3, 4, ..., n \\ 0, & otherwise. \end{cases}$$

2. Probability of passing a quiz =3/4
   Probability of failing a quiz =1/4.

   (a) $\binom{6}{2}(1/4)^2(3/4)^4$

   (b) Let N be the number of quizzes required to get 3 failures.
   Note that N here is the number of trials required to get 3 successes in a Bernoulli with parameter 1/4. Hence $\mathbb{E}(N) = \dfrac{3}{1/4} = 12$ Expected number of quizzes required to get 3 failures = 12.
   Thus expected number of passed quizzes = 12-3= 9. Alternatively, number of quizzes that he will pass before he has failed three times follows Binomial(N,3/4).
   Thus the desired expected value $= \mathbb{E}(N.\frac{3}{4}) = 12\frac{3}{4} = 9$ (Law of iterated expectation).

   (c) Desired probability $= P(Y_2 = 8, Y_3 = 9) = \binom{7}{1}(1/4)^1(3/4)^6(1/4)(1/4)$, where $\binom{7}{1}(1/4)^1(3/4)^6$ is first failure in first 7 quizzes, 1/4 is second failure in 8^{th} quizz, and 1/4 is third failure on 9^{th} quiz.

   (d) Let F denotes the failure and P denote the passed quiz. The desired event happens if and only if on of the following happens:

$$\begin{aligned} &FF... \\ &PFF... \\ &FPFF... \\ &PFPFF... \\ &FPFPFF... \end{aligned}$$

   The desired probability is

$$\begin{aligned} &P(FF) + P(PFF) + P(FPFF) + P(PFPFF) + P(FPFPFF) + ... \\ &= (\frac{1}{4})^2 + \frac{3}{4}(\frac{1}{4})^2 + \frac{3}{4}(\frac{1}{4})^3 + (\frac{3}{4})^2(\frac{1}{4})^3 + (\frac{3}{4})^2(\frac{1}{4})^4 + ... \\ &= [(\frac{1}{4})^2 + \frac{3}{4}(\frac{1}{4})^3 + (\frac{3}{4})^2(\frac{1}{4})^4 + ...] + [\frac{3}{4}(\frac{1}{4})^2 + (\frac{3}{4})^2(\frac{1}{4})^3 + ...] \\ &= x + y(say). \end{aligned}$$

Now

$$\begin{aligned} x &= (\frac{1}{4})^2 + \frac{3}{4}(\frac{1}{4})^3 + (\frac{3}{4})^2(\frac{1}{4})^4 + ... \\ &= (\frac{1}{4})^2[1 + \frac{3}{4}\frac{1}{4} + (\frac{3}{4}\frac{1}{4})^2 + ...] \\ &= (\frac{1}{4})^2\frac{1}{1-3/16} = \frac{1}{16}\frac{16}{13} = \frac{1}{13} \end{aligned}$$

Similarly,

$$\begin{aligned} y &= \frac{3}{4}(\frac{1}{4})^2[1 + \frac{3}{4}\frac{1}{4} + (\frac{3}{4}\frac{1}{4})^2 + ...] \\ &= \frac{3}{4}(\frac{1}{4})^2\frac{16}{13} = \frac{3}{52} \end{aligned}$$

Thus $x + y = \frac{1}{13} + \frac{3}{52}$.

3. Let X be the number of failures before the r^{th} success.

   (a) Let Y be the number of trials to get r successes. Thus, $X + r = Y$

   $$p_Y(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r} \qquad k = r, r+1, \cdots$$

   $$p_X(k) = P_Y(k+r) = \binom{k+r-1}{r-1} p^r (1-p)^k \qquad k = 0, 1, 2, \cdots$$

   (b) $\mathbb{E}(X) = \mathbb{E}(Y - r) = \frac{r}{p} - r = \frac{(1-p)r}{p}$.
   $Var(X) = Var(Y) = \frac{r(1-p)}{p^2}$.

   (c) $P(i^{th}$ failure occurs before the r^{th} success)
   $= P($ there are i or more than i failures before the r^{th} success$) = P(X \geq i)$
   $= \sum_{k=i}^{\infty} \binom{k+r-1}{r-1} p^r (1-p)^k$.

4. Trains arrival $\sim$ Poisson(3)

   (a) Since trains arrival is Poisson, by definition of Poisson process "arrivals in disjoint intervals are independent", hence, P(No trains on days $1, 2$ and $3|1$ train on day $0 = P(N_3 = 0)$.
   We know that N_3 is Poisson(3λ) = Poisson(9). Therefore the required probability $= e^{-9}$.

   (b) We know that inter arrival times are exponential(3) in case of Poisson(3) process. Thus T_2, the time of second arrival, is independent of T_1, the time of first arrival and further $T_2 \sim$ exponential(3)
   Thus $P(T_2 > 3) = \int_3^\infty 3e^{-3x}dx = -\frac{3}{3}e^{-3x}\Big]_3^\infty = e^{-9}$
   Alternately, $P($ next train to arrive takes more than 3 days after the first train on day 0$) = P($ no of trains on days 1,2 and 3$|$ 1 train on day 0$) = e^{-9}$.

(c) $P($ No trains on first two days and 4 trains on 4^{th} day)
$= P($ No trains on day 1$)P($ No trains on day 2$)P($4 trains on day 4$)$
$= e^{-3}e^{-3}\dfrac{e^{-3}3^4}{4!} = \dfrac{e^{-9}3^4}{4!}$.

(d) Note that in Poisson(r) processes, "the time of the k^{th} arrival" has the following density,

$$f_{Y_k}(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{k-1}}{(k-1)!}, \qquad k = 1, 2, \cdots, t > 0$$

Thus time of the 5^{th} arrival has density

$$f_{Y_5}(t) = \frac{3^5}{4!}t^4 e^{-3t}, \qquad t > 0$$

We are interested in,

$$P(X_5 > 2) = \int_2^\infty \frac{3^5}{4!}t^4 e^{-3t}dt$$

$$= \frac{3^5}{24} \times \left[-\frac{1}{81}e^{-3t}(27t^4 + 36t^3 + 36t^2 + 24t + 8)\right]_2^\infty$$

$$\frac{3}{24}e^{-6}(27 \times 16 + 36 \times 8 + 36 \times 4 + 24 \times 2 + 8) = 115e^{-6}.$$

5. (a) A potential customer becomes actual customer with probability p. Hence desired probability $= \binom{5}{3}p^3(1-p)^2$.

(b) P(fifth potential customer to arrive becomes the third actual customer)
=P(any 2 of the first 4 customer becomes real customer).
P(5^{th} customer become 3^{rd} real customer)
$= \binom{4}{2}p^2(1-p)^2.p = \binom{4}{2}p^3(1-p)^2$.

(c) Note that the process arrival of actual customers is Poisson($p\lambda$).
L= arrival of 10^{th} actual customer. $f_L(t) = f_{Y_{10}}(t) = \dfrac{(p\lambda)^{10}t^9e^{-p\lambda t}}{9!}, t \geq 0$.
$\mathbb{E}(L) = \mathbb{E}(Y_{10}) = \dfrac{10}{p\lambda}$.

(d) The required conditional expectation = expected time of arrival of fifth potential customer + expected time of arrival of the seventh actual customer$= \dfrac{5}{\lambda} + \dfrac{7}{\lambda p}$