MA2040: Probability, Statistics and Stochastic Processes
Problem Set-IV

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1. Let X and Y have joint pdf $f_{X,Y}(x,y) = \begin{cases} 6e^{-(2x+3y)} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

   - Are X and Y independent?

     Solution: We have
     $$f_X(x) = \int_0^\infty f_{X,Y}(x,y)dy = 2e^{-2x}$$
     $$f_Y(y) = \int_0^\infty f_{X,Y}(x,y)dx = 3e^{-3y}$$
     We have $f_{X,Y} = f_X \cdot f_Y$. Hence, X and Y are independent.

   - Find $\mathbb{E}(Y \mid X > 2)$

     Solution: Since X and Y are independent, conditioning on X doesn't affect Y. Hence,
     $$\mathbb{E}(Y \mid X > 2) = \mathbb{E}(Y) = \int_0^\infty 3ye^{-3y}dy = \frac{1}{3}$$

   - Find $\mathbb{P}(X > Y)$

     Solution:
     $$\mathbb{P}(X > Y) = \int_0^\infty \mathbb{P}(X > y) f_Y(y)dy = \int_0^\infty (1 - F_X(y)) f_Y(y)dy = 1 - 3\int_0^\infty \left(1 - e^{-2y}\right) e^{-3y}dy = 3/5$$

2. Let $X \sim \text{Uniform}(1,2)$ and given $X = x$, Y follows an exponential distribution with $Y \sim \text{EXP}(\lambda = x)$. Find the covariance of X and Y.

   Solution: We have
   $$f_{XY}(x,y) = f_{Y|X} f_X = xe^{-xy}$$
   This gives us
   $$\text{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 1 - 3/2\ln(2)$$

3. Let X and Y be independent standard normal random variables. Find covariance of Z and W, where $Z = 1 + X + XY^2$ and $W = 1 + X$.

   Solution: $\mathbb{E}(Z) = 1$ and $\mathbb{E}W = 1$. Further, $\mathbb{E}(ZW) = \mathbb{E}\left(1 + 2X + X^2 + XY^2 + X^2Y^2\right) = 3$. Hence, covariance is 2.

4. A fair die is rolled n times. Let X be the number of 1's and Y be the number of 2's. Find the correlation coefficient $\rho(X, Y)$.

   Solution: Let $Z = X + Y$. X, Y are binomial random variables with $p = 1/6$, while Z is a binomial random variable with $p = 1/3$. Hence, $\text{Var}(X) = \text{Var}(Y) = 5n/36$ and $\text{Var}(Z) = 2n/9$. Hence,

   $$\text{Cov}(X, Y) = \frac{1}{2}(2n/9 - 5n/18) = -n/36$$

   Hence, $\rho = -1/5$.

5. Let 32 people sit around a round table. Each person tosses a fair coin. Anyone whose outcome is different from both his neighbors is taken out from the group. If X is the total number of such persons, find variance of X.

   Solution: Consider the random variable P_i, which takes a value 1, when the i^{th} person's outcome differs from his both neighbors and takes the value 0, when the outcomes of him matches with both his neighbors. We see that $P_i = 1$ with a probability of $1/4$ and $P_i = 0$ with a probability of $3/4$. Now we have $X = \displaystyle\sum_{i=1}^{32} P_i$. X is **not a binomial random variable**, since for instance, P_i and P_{i+1} are dependent, (in fact they are also correlated), since

   $$\text{Cov}(P_iP_{i+1}) = \mathbb{E}(P_iP_{i+1}) - \mathbb{E}(P_i)\,\mathbb{E}(P_{i+1}) = 1/8 - 1/4 \times 1/4 = 1/16$$

   Hence, we have that

   $$\text{Var}(X) = \text{Var}\left(\sum_{i=1}^{32} P_i\right) = \sum_{i=1}^{32} \text{Var}(P_i) + \sum_{i \neq j} \text{Cov}(P_iP_j)$$

   Note that if i and j are apart by more than 2, i.e., if $|i - j| \pmod{32} \geq 3$, then P_i and P_j are independent. Hence, $\text{Cov}(P_iP_j) = 0$ for $|i - j| \pmod{32} \geq 3$. The only non-zero covariances involving P_i are along with $P_{i\pm1}$ and $P_{i\pm2}$. (The indices $i \pm 1$ and $i \pm 2$ will be interpreted $\pmod{32}$ always). We have

   $$\text{Cov}(P_iP_{i\pm1}) = \mathbb{E}(P_iP_{i\pm1}) - \mathbb{E}(P_i)\,\mathbb{E}(P_{i\pm1}) = 1/16$$
   $$\text{Cov}(P_iP_{i\pm2}) = \mathbb{E}(P_iP_{i\pm2}) - \mathbb{E}(P_i)\,\mathbb{E}(P_{i\pm2}) = 1/2^4 - 1/4 \times 1/4 = 0$$

   Hence, we obtain

   $$\text{Var}(X) = \sum_{i=1}^{32} \left(\text{Var}(P_i) + \text{Cov}(P_iP_{i+1}) + \text{Cov}(P_iP_{i-1}) + \text{Cov}(P_iP_{i+2}) + \text{Cov}(P_iP_{i-2})\right)$$

   This gives us

   $$\text{Var}(X) = \sum_{i=1}^{32} \left(\frac{1}{4} \times \frac{3}{4} + \frac{1}{16} + \frac{1}{16}\right) = 10$$

6. The moment generating function of a random variable X is given by

$$M_X(s) = \frac{2}{2-s}, \ \ \forall s \in (-\infty, 2)$$

Find the distribution of X.

Solution: X is exponential with $\lambda = 2$.

7. Let $X \sim \text{Binomial}(n,p)$ and $Y \sim \text{Binomial}(m,p)$ be independent random variables. Show that $X+Y \sim \text{Binomial}(m+n,p)$.

Solution:Use the MGF to conclude.

8. Show that the function e^{-s} is a moment generating function.

Solution: Consider the random variable which takes only the value -1, i.e., $\mathbb{P}(X=-1) = 1$. The MGF is e^{-s}.

9. Show that if $M(s)$ is a moment generating function, so is $M(cs)$.

Solution: If $M(s)$ is the mgf of X, then consider the random variable variable $Z = X/c$.

10. Show that if $M(s)$ is a moment generating function, $cM(s)$ cannot be a moment generating function for $c \neq 1$.

Solution: The mgf needs to be 1 at $s = 0$.

11. Show that if $M(s)$ is a moment generating function, then $e^{-s}M(s)$ is also a moment generating function.

Solution: If $M(s)$ is the mgf of X, then consider the random variable variable $Z = X - 1$.

12. The moment generating function of a random variable X is

$$M_X(s) = \frac{e^{-2s}}{6} + \frac{e^{-s}}{3} + \frac{e^{s}}{4} + \frac{e^{2s}}{4}$$

Show that $P(|X| \leq 1) = \dfrac{7}{12}$.

Solution: This is a discrete random variable.

$$\mathbb{P}(X=k) = \begin{cases} 1/6 & \text{if } X = -2 \\ 1/3 & \text{if } X = -1 \\ 1/4 & \text{if } X = 1 \\ 1/4 & \text{if } X = 2 \end{cases}$$