MA2040: Probability, Statistics and Stochastic Processes
Problem Set-VII

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1. A building contractor usually gets his light bulbs from shop A, where he knows that the bulbs last for an expected time of 100 months with a standard deviation of 3 months. A new shop B contacts the building contractor and say that he can supply bulbs which will last more than those from shop A maintaining the same standard deviation. The building contractor has to now decide whether he needs to make an order of 1 lakh light bulbs from shop A or shop B. To do this, he asks 16 customers who purchased from shop B and finds out that that these light bulbs lasted for an average of 103 months. The contractor doesn't like to take risk and decides to use hypothesis testing with a very low significance value of 1% to decide. Will he purchase from shop A or shop B? You may assume that the times are normally distributed.
Solution: Here the null hypothesis is $H_0 : \mu = 100$ months. The alternate hypothesis is to test whether the bulbs from shop B last longer, hence $H_1 : \mu > 100$ months. **This is a right sided hypothesis testing**. We have the sample average to be 103 over 16 samples. We know that $\bar{X}$ is also normally distributed.

   - **Method** 1:

   $$\mathbb{P}\left(\bar{X} \geq 103 \mid H_0 \text{ is true}\right) = \mathbb{P}\left(\frac{\bar{X}-100}{3/\sqrt{16}} \geq \frac{103-100}{3/\sqrt{16}}\right) = 1-\Phi\left(4\right) = 1-0.999968 = 3.2\times 10^{-5} = 0.0032\%$$

   Since the p-value is less than the significance value, the contractor will purchase from the new shop owner.

   - **Method** 2: Another method is to find the critical value above which he will shift. The critical value C satisfies

   $$\mathbb{P}\left(\bar{X} \geq C\right) = 0.01 \implies \mathbb{P}\left(\frac{\bar{X}-100}{3/\sqrt{16}} \geq \frac{C-100}{3/\sqrt{16}}\right) = 0.01$$

   This gives us that
   $$\frac{C-100}{3/\sqrt{16}} = 2.33 \implies C = 100 + \frac{2.33\times 3}{4} = 102$$

   Since the average of 103 exceeds C, the contractor will now purchase from the new shop.

2. A dealer purchases an idli grinder from manufacturer A, where he knows that the expected time to grind 2 kgs of flour to a desired consistency is 15 minutes with a standard deviation of 3 minutes. A new manufacturer B contacts the dealer and proposes that his grinders can grind faster than those from manufacturer A with the same standard deviation of 3 minutes. The dealer samples 10 grinders from B and finds that they take 14 minutes to do the same process. The dealer is willing to take risk and decides to use hypothesis testing with a reasonably high significance value of 10% to decide. Will he purchase from shop A or shop B? You may assume that the times are normally distributed.
Solution: Here the null hypothesis is $H_0 : \mu = 15$ minutes. The alternate hypothesis is to test whether the grinders from manufacturer B grind faster, hence $H_1 : \mu < 15$ minutes. **This is a left sided hypothesis testing**. We have the sample average to be 14 over 10 samples. We know that $\bar{X}$ is also normally distributed.

- **Method** 1:

$$\mathbb{P}\left(\bar{X} \leq 14 \mid H_0 \text{ is true}\right) = \mathbb{P}\left(\frac{\bar{X}-15}{3/\sqrt{10}} \leq \frac{14-15}{3/\sqrt{10}}\right) = \mathbb{P}\left(\frac{\bar{X}-15}{3/\sqrt{10}} \leq -1.054\right) = \Phi\left(-1.054\right) = 0.1468$$

  Since this is greater than the significance value, the contractor will stick with the old manufacturer.

- **Method** 2: Another method is to find the critical value above which he will shift. The critical value C satisfies

$$\mathbb{P}\left(\bar{X} \leq C\right) = 0.1 \implies \mathbb{P}\left(\frac{\bar{X}-15}{3/\sqrt{10}} \leq \frac{C-15}{3/\sqrt{10}}\right) = 0.1$$

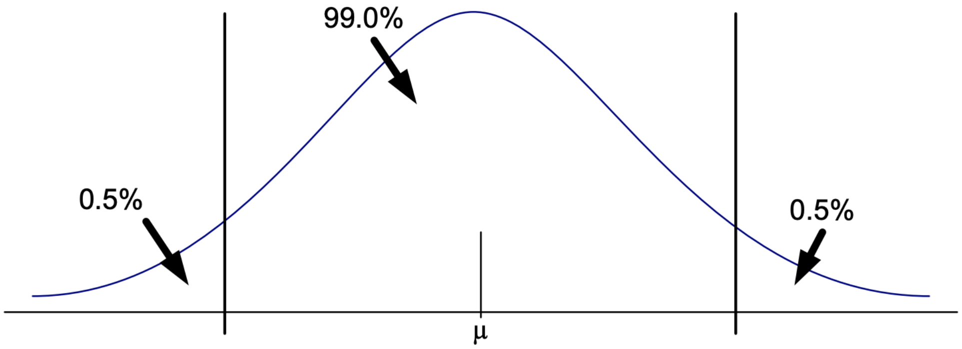
  This gives us that

$$\frac{C-15}{3/\sqrt{10}} = -1.285 \implies C = 15 - \frac{3 \times 1.285}{\sqrt{10}} = 13.78$$

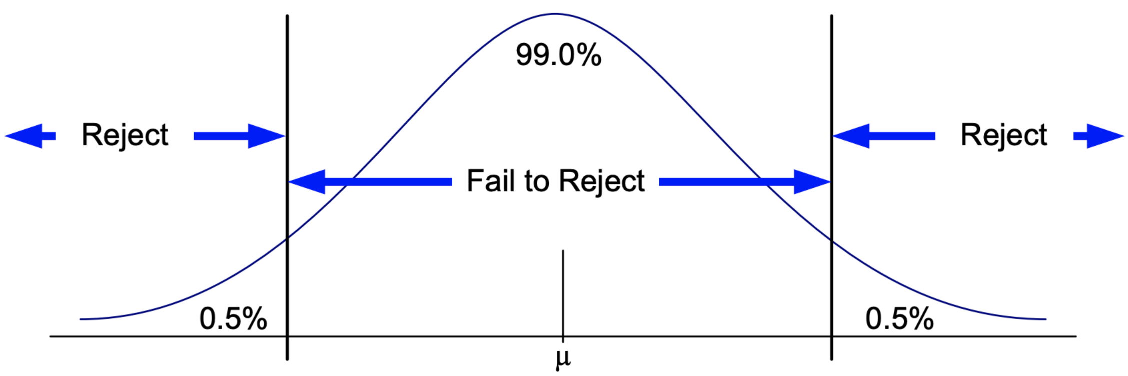
  Since the average of 14 minutes exceeds C, the contractor will stick with the old manufacturer.

3. A premium gold ball production line must produce all of its balls of 45.75 grams in order to get the top rating. Samples are drawn hourly and checked. If the production line gets out os sync with a statistical significance of more than 1%, it must be shut down and repaired. A sample of 18 balls has a mean of 45 grams and a sample standard deviation of 2 grams. Should the production line be repaired?
Solution:

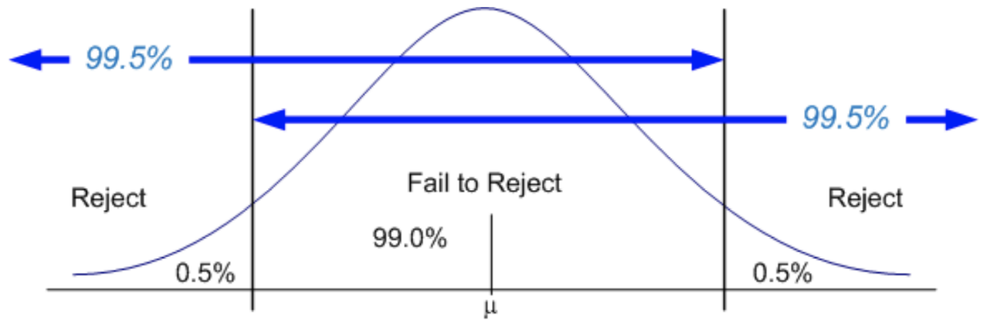
- **Step** 1: Identify the null hypothesis. This is $H_0 : \mu = 45.75$ grams.
- **Step** 2: Identify the alternate hypothesis. This is $H_1 : \mu \neq 45.75$ grams.
- **Step 3:** Identify the distribution to use. Here we are given the sample standard deviation. Hence, we will be using a t-distribution.
- **Step 4:** Identify left or right or two sided test. Here, we are interested in two sided test.
- **Step 5:** Draw the test.



Assign the “Fail to reject” to the appropriate region.



- **Step 6:** Find the critical values. Here $n = 18$ and hence we need to look at t distribution with $n - 1 = 17$ degrees of freedom. We are doing a two sided test, hence the level of significance on both sides must be halved.



We get the t-values as ± 2.898. Hence, the critical values are

$$\frac{C - 45.75}{2/\sqrt{18}} = \pm 2.898 \implies C = 45.75 \pm \frac{5.796}{3\sqrt{2}} = 45.75 \pm 1.366$$

- **Step 7:** Decide. Here we see that the mean falls within the fail to reject region. Hence, stick to the current production line. Do not stop it.

4. In the first problem, the second shop owner (B) says that his bulbs last for an average of 105 months with a standard deviation of 3 months. If the contractor wants his type I error to be 1% and type II error to be 5%, how many samples should he check and what is his critical value for accepting and rejecting the null hypothesis?

   Solution: $H_0 : \mu = 100$ months, $H_1 : \mu = 105$ months. Let the contractor get his information from n previous customers of shop B. We need C such that

   $$\mathbb{P}\left(\bar{X} > C \mid H_0 \text{ is true}\right) = 0.01 \text{ and } \mathbb{P}\left(\bar{X} < C \mid H_1 \text{ is true}\right) = 0.05$$

   This gives us that we need C and n such that

   $$\mathbb{P}\left(\frac{\bar{X}-100}{3/\sqrt{n}} > \frac{C-100}{3/\sqrt{n}}\right) = 0.01 \text{ and } \mathbb{P}\left(\frac{\bar{X}-105}{3/\sqrt{n}} < \frac{C-105}{3/\sqrt{n}}\right) = 0.05$$

   This gives us that
   $$\Phi\left(\frac{C-100}{3/\sqrt{n}}\right) = 0.99 \text{ and } \Phi\left(\frac{C-105}{3/\sqrt{n}}\right) = 0.05$$

   Hence, we obtain that
   $$\frac{C-100}{3/\sqrt{n}} = 2.33 \text{ and } \frac{C-105}{3/\sqrt{n}} = -1.645$$

   Hence,
   $$C - \frac{8}{\sqrt{n}} = 100 \text{ and } C + \frac{4.935}{\sqrt{n}} = 105$$

   Hence, we get
   $$\frac{12.835}{\sqrt{n}} = 5 \implies \sqrt{n} = 2.567 \implies n = 6.58$$

   Hence, $n = 7$. The critical value C is 103.116.

Table of error types		Null hypothesis H_0 in reality is	
		True	False
Decision on null hypothesis H_0 is to	Accept	Correct inference True Negative Probability $= 1 - \alpha$	Type II error False Negative Probability $= \beta$
	Reject	Type I error False Positive Probability $= \alpha$	Correct Inference True Positive Probability $= 1 - \beta$