# Announcement

- Assignment 2 will be posted tonight
    - Due Jan. 23.

# Algorithm Performance

(the Big-*O*)

# Lecture 6

Today:

- Worst-case Behaviour
- Counting Operations
- Performance Considerations
- Time measurements
- Order Notation (the Big-*O*)

# Pessimistic Performance Measure

- Often consider the *worst-case* behaviour as a benchmark.
  - make guarantees about code performance under all circumstances

- Can predict performance by counting the number of “elementary” steps required by algorithm in the worst case
  - derive total steps ($T$) as a function of input size ($N$)

# Analysis of `dup_chk()`

Q. What is $N$?
- The number of elements in the array

```
        int dup_chk(int a[], int length) {
1         int i = length;
N + 1     while (i > 0) {
N             i--;
N             int j = i - 1;
i + 1         while (j >= 0) {
i                 if (a[i] == a[j]) {
                      return 1;
                  }
i                 j--;
              }
          }
1         return 0;
        }
```

Outside of loop: 2 (steps)

Outer loop: $3N + 1$

Inner loop: $3$`i`$ + 1$ for all possible `i` from 0 to $N - 1$.

$= 3/2\ N^2 - 1/2\ N$

1 + 4 + 7 + … + 3(N-1) + 1

Grand total = $3/2\ N^2 + 5/2\ N + 3$

A *quadratic* function!

# Some Math

$$1 + 4 + 7 + \ldots + 3(N-1) + 1$$

# Some Math

$$1 + 4 + 7 + \ldots + 3(N-1) + 1 = (3N-3+1 + 1) * N/2$$

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$$1 + 4 + 7 + \ldots + 3(N-1) + 1 = (3N-3+1 + 1) * N/2$$
$$= 1/2 * (3N-1) * N$$

# Some Math

$$
\begin{aligned}
1 + 4 + 7 + \ldots + 3(N-1) + 1 &= (3N-3+1 + 1) * N/2 \\
&= 1/2 * (3N-1) * N \\
&= 1/2 * (3N^2 - N)
\end{aligned}
$$

# Some Math

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&= 3/2 * N^2 - 1/2 * N
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$$= 1/2 * (3N^2 - N)$$
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Observation 1: The 1/2 *N term doesn’t matter very much

# Some Math

$$1 + 4 + 7 + \ldots + 3(N-1) + 1 = (3N-3+1 + 1) * N/2$$

$$= 1/2 * (3N-1) * N$$

$$= 1/2 * (3N^2 - N)$$

$$= 3/2 * N^2 - 1/2 * N$$

Arithmetic series

Observation 1: The 1/2 *N term doesn’t matter very much

Observation 2: Arithmetic series have N^2 leading terms

1. As N gets bigger, the last number that we add is bigger
2. The number of pairs of numbers is bigger

# Analysis of `dup_chk()`

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- The number of elements in the array

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N          i--;
N          int j = i - 1;
i + 1      while (j >= 0) {
i              if (a[i] == a[j]) {
                   return 1;
               }
i              j--;
           }
       }
1      return 0;
     }
```

Outside of loop: 2 (steps)

Outer loop: $3N + 1$

Inner loop: $3$`i`$ + 1$ for all possible `i` from $0$ to $N - 1$.

$= 3/2\ N^2 - 1/2\ N$

Grand total $= 3/2\ N^2 + 5/2\ N + 3$

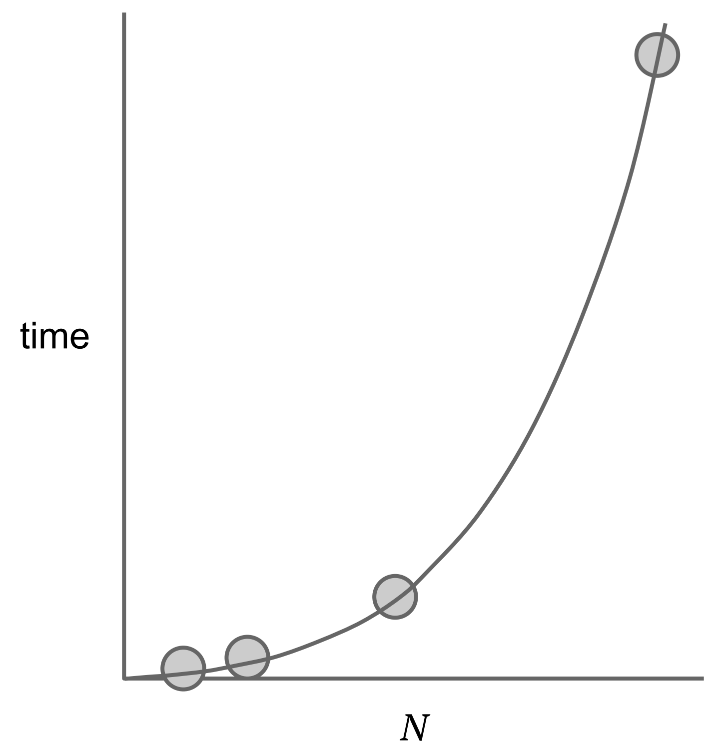
A *quadratic* function!

Observation: The 5/2 * N and 3 terms don't matter very much

# Empirical Measurement

- Another graph - a quadratic this time!
- Confirms predictions: **doubling** (2x) the input size leads to a **quadrupling** (4x) of the running time.

| $N$ | time (in ms) |
|---|---|
| 10000 | 89 |
| 20000 | 365 |
| 40000 | 1424 |
| 100000 | 9011 |



# 2D Maximum Density Problem

Problem: Given a 2-dimensional array ($N$x$N$) of integers, find the 10x10 swatch that yields the largest sum.

Applications:

- Resource management and optimization
- Finding brightest areas of photos

# Algorithm / Code?

- Simple approach: Try all possible positions for the upper left corner
  - $(N-9)$x$(N-9)$ of them
  - use a nested loop
- Total each swatch using a 10x10 nested loop
- A *brute-force* approach!
  - Generate a possible solution [naively]
  - Test it [naively]

# In C

```
int max10by10(int a[N][N]) {
    int best = 0;
    for (int u_row = 0; u_row < N-9; u_row++) {
        for (int u_col = 0; u_col < N-9; u_col++) {
            int total = 0;
            for (int row = u_row; row < u_row+10; row++) {
                for (int col = u_col; col < u_col+10; col++) {
                    total += a[row][col];
                }
            }
            best = max(best, total);
        }
    }
    return best;
}
```

x($N$-9)

x($N$-9)

x10

11

10

10

Approximate Method:

Count the *barometer instructions*, the instructions executed most frequently. Usually, in the innermost loop.

Innermost loop: 11 + 10 + 10 = 31 ops

Total = 31 **x10 x($N$-9) x($N$-9) =310$N^2$**

# Which Performance Measurement?

- Empirical timings
  - run your code on a real machine with various input sizes
  - plot a graph to determine the relationship
- Operation counting
  - assumes all elementary instructions are created equal
- Actual performance can depend on much more than just your algorithm!

# Running Time is Affected By . . .

- CPU speed
- Amount of main memory
- Specialized hardware (e.g., graphics card)
- Operating system
- System configuration (e.g., virtual memory)
- Programming Language
- Algorithm Implementation
- Other Programs
- . . .

# Comparing Algorithm Performance

- There can be many ways to solve a problem, i.e., different algorithms that produce the same result
    - E.g., There are numerous sorting algorithms.
- Compare algorithms by their behaviour for large input sizes, i.e., as $N$ gets large
    - On today's hardware, **most** algorithms perform quickly for small $N$
- Interested in growth rate as a function of $N$
    - E.g., Sum an array: *linear* growth $= O(N)$
    - E.g., Check for duplicates: *quadratic* growth $= O(N^2)$

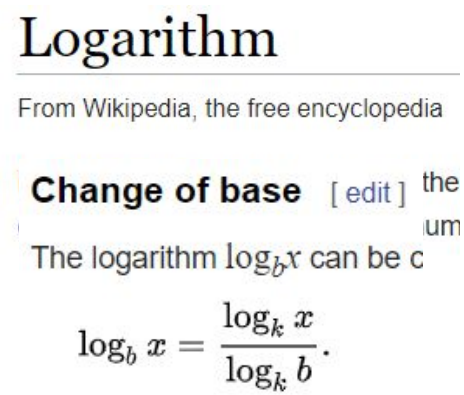
# Order Notation (the Big-$O$)

- Suppose we express the number of operations used in our algorithm as a function of $N$, the size of the problem.
- Intuitively, take the dominant term, remove the leading constant, and put $O( \ . \ . \ . \ )$ around it.

- E.g., $f(N) = 348N^2 - 6956N + 34762 \longrightarrow \boxed{O(N^2)}$

# Formalities of the Big-*O*

- Given a function $T(N)$, we say $T(N) = O(f(N))$ if $T(N)$ is at most a constant times $f(N)$, except perhaps for some small values of $N$.
- Properties:
  - constant factors don't matter
  - low-order terms don't matter



Logarithm

From Wikipedia, the free encyclopedia

**Change of base** [ edit ] the

um

The logarithm $\log_b x$ can be c

$$\log_b x = \frac{\log_k x}{\log_k b}.$$

- Rules:
  - For any $k$ and any function $f(N)$, $k \cdot f(N) = O(f(N))$
    - E.g., $5N = O(N)$
    - E.g., $\log_a N = O(\log_b N)$. Why?
    - Q. Do leading constants really not matter?

# Leading Constants - Experiment

Of course, constant factors affect performance

- E.g., If two different algorithms run in $f_1(N) = 20N^2$ and $f_2(N) = 2N^2$, re… tive… d expect Algorit…
- E.g., Sim… e running Algorithm… e same running time.
- Big-O hi… eading constants - *a hardware independent analysis.*


| **Cray Supercomputer** | vs | **iMac Desktop Personal Computer (2011)** |
|---|---|---|
| $17.6 \times 10^{15}$ instructions per second<br>runs optimized dup_chk( ) code from last time<br>$f(N) = 3/2\ N^2 + 5/2\ N + 3$ | | $40 \times 10^{9}$ instructions per second<br>runs an unoptimized, different dup_chk ( )<br>$f(N) = 30N \log N + 5N + 4$ |

# Experimental Results

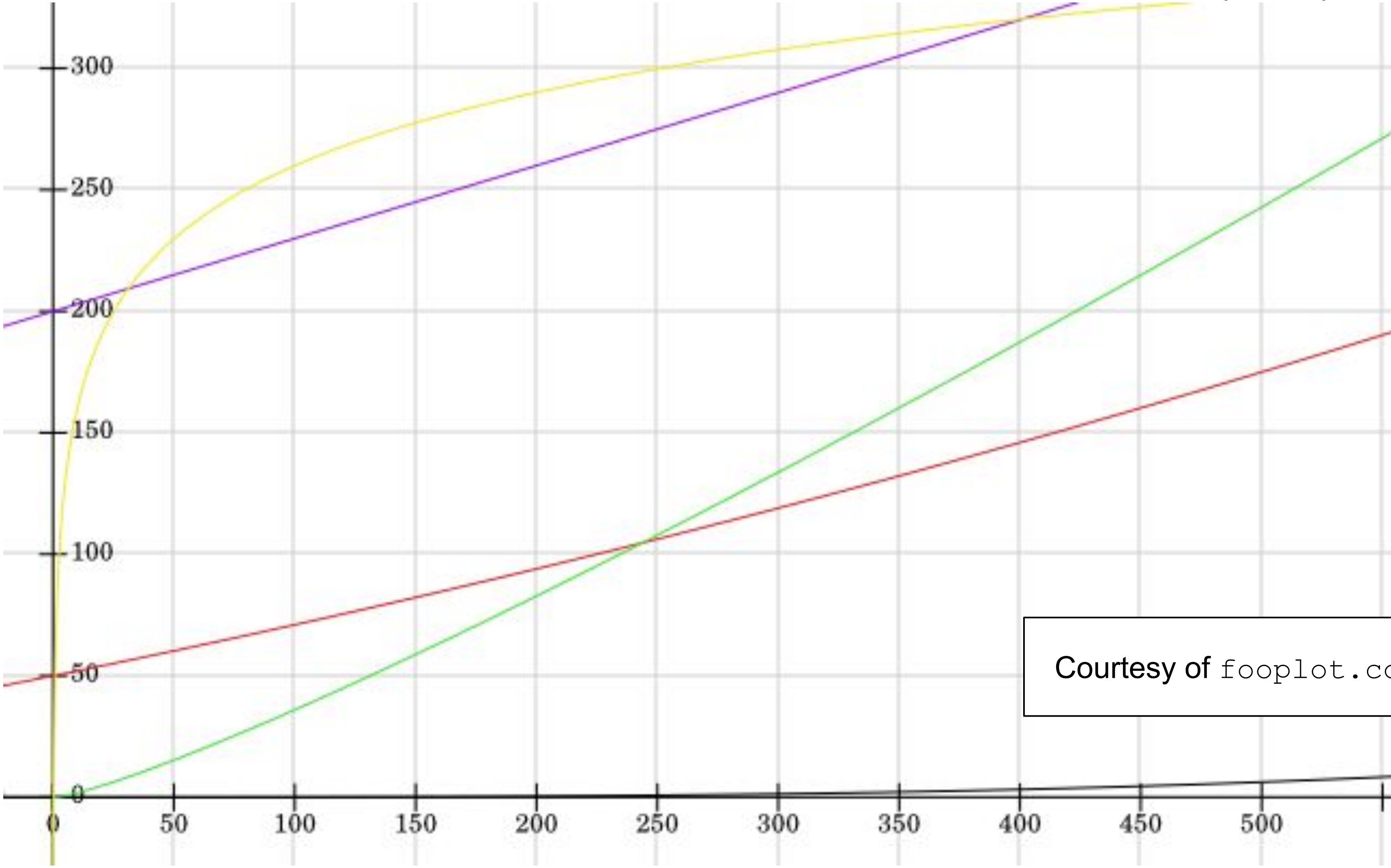
| $N$ | iMac | Cray |
|---|---|---|
| 100,000 | 1.2 ms | 850 ns |
| $10^6$ | 15 ms | 85 μs |
| $10^7$ | 0.2 s | 8.5 ms |
| $10^8$ | 2 s | 0.85 s |
| $10^9$ | 22 s | 1.75 min |
| $10^{10}$ | 4.2 min | 2:22 hr |
| $10^{11}$ | 56 min | 10 days |
| $10^{12}$ | 8:20 hr | 2.7 years |


Conclusions:

- Cray runs $O(N^2)$ algorithm
- iMac runs $O(N \log N)$ algorithm which runs faster than Cray for large $N$ ($10^9$ and beyond)
- Thus slow computer + no opt + $O(N \log N)$ >> fast computer + optimization + $O(N^2)$ algorithm
- **Rule of Thumb: The slower the function grows, the faster the algorithm.**
- For the $O(N^2)$ Cray, a 10x increase in $N$ leads to roughly a 100x increase in running time.
- For the $O(N \log N)$ iMac, a 10x increase in $N$ leads to roughly a 10x increase in running time (for the $N$), plus a little (for the $\log N$).

# Some Plots

100**log(x)**+60 (yellow)
0.3**x**+200 (purple)
0.18**x*log(x)** (green)
0.0001**x^2**+0.2x+50 (red)
0.00000005**x^3** (black)




Courtesy of `fooplot.com`

# Some Plots t

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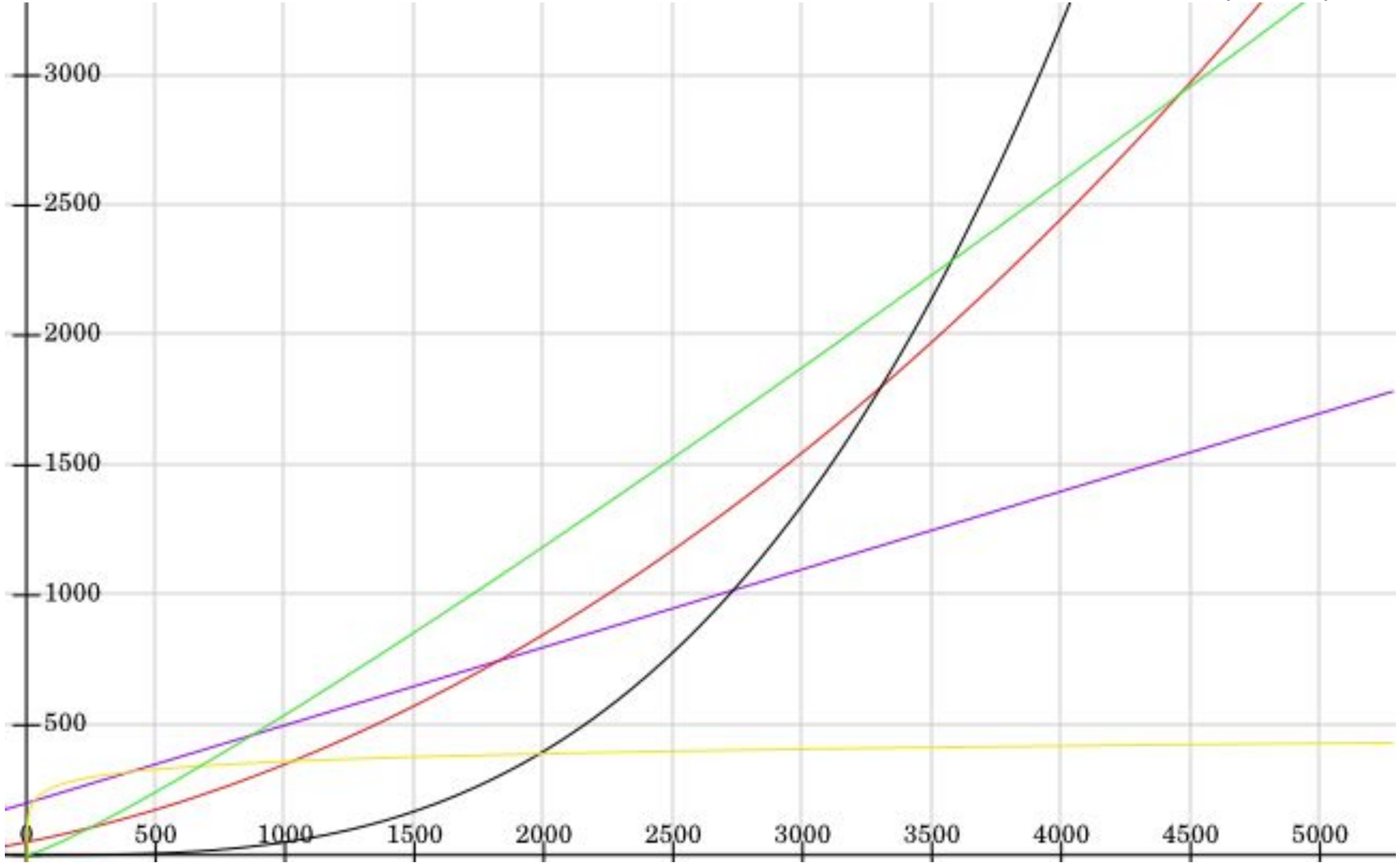

# What Does It All Mean?

- A carefully crafted algorithm can make the difference between a usable and an useless piece of software
- E.g., If it costs one algorithm 0.5s to search 1 billion bank records and another one 0.005s.
- E.g., Or, if $10^9$ isn't "big" how about Google?
- E.g., Real-time systems - where a nearly instant response is required
- "You can't make a racehorse of a pig, but you can make a very fast pig."

# As N Gets Large, The Algorithm is Most Important

“You can’t make a racehorse of a pig, but you can make a very fast pig.”

“When you find yourself trying all sorts of clever implementation tricks to speed up an algorithm, you should step back for a moment and ask if you’re trying to make a racehorse out of a pig. It might be more productive to use your time to look for a racehorse.”

“It’s important to know whether an efficient algorithm is possible. There’s no sense spending time looking for a unicorn. In this situation, the best you can hope for is a fast pig and a short racecourse.”

- Lou Hafer, SFU CS

# Big-*O* and Barometer Instructions

Rule of thumb:

The frequency of the barometer instructions will be proportional to the big-O running time

So, find the most frequent operation(s) and count them!

# Loops ⇒ Multiply

```
int max10by10(int a[N][N]) {
    int best = 0;
    for (int u_row = 0; u_row < N-9; u_row++) {
        for (int u_col = 0; u_col < N-9; u_col++) {
            int total = 0;
            for (int row = u_row; row < u_row+10; row++) {
                for (int col = u_col; col < u_col+10; col++) {
                    total += a[row][col];
                }
            }
            best = max(best, total);
        }
    }
    return best;
}
```

x(N-9)

x(N-0)

x10

x10

barometer instructions

f(N) = 3 **x10 x10 x(N-9) x(N-9)** = $O(N^2)$

# Polynomials

Rule:

The powers of N are ordered according to their exponents, i.e., $N^a = O(N^b)$ if and only if $a \le b$

- E.g., $N^2 = O(N^3)$, but $N^3$ is not $O(N^2)$.

Why are lower-ordered terms not included?

- E.g., If your bank account followed $f(N) = N^2 + N + 1$, you would probably care a lot about the lower-ordered terms for small N, like N=5, as $f(5) = 5^2 + 5 + 1 = \$31$. You'll care about every dollar. But not for larger N, like N=1000, as $f(1000) = 1000^2 + 1000 + 1 = \$1,001,001$. You care most that you have that million bucks, and not much about the \$1000 or the \$1.

# More Rules

3. A logarithm grows more slowly than any other positive power of N.
   - E.g., $\log_2 N = O(N^{1/2})$.
4. If $f(N) = O(g(N))$ and $g(N) = O(h(N))$ then $f(N) = O(h(N))$.
5. If both $f(N)$ and $g(N)$ are $O(h(N))$ then

   $f(N) + g(N) = O(h(N))$
6. If $f_1(N) = O(g_1(N))$ and $f_2(N) = O(g_2(N))$ then $f_1(N) \times f_2(N) = O(g_1(N) \times g_2(N))$

E.g., $(10 + 5N^2)(10\log_2 N + 1) + (5N + \log_2 N)(10N + 2N \log_2 N)$

# Typical Growth Rate Functions

- $O(1)$ – **constant** time
  - The time is independent of **N**, E.g., list look-up
- $O(\log N)$ – **logarithmic** time
  - Usually the log is to the base 2, E.g., binary search
- $O(N)$ – **linear** time, E.g., linear search
- $O(N \log N)$ – E.g., quicksort, mergesort
- $O(N^2)$ – **quadratic** time, e.g. selection sort
- $O(N^k)$ – **polynomial** (where k is a constant)
- $O(2^N)$ – **exponential** time, very slow!