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KU-MATH

FinKont2: Hand-In Exercise #1

Answers (individually composed – but feel free to discuss in groups) must be handed in via Absalon no later than 23:59 CET on Tuesday March 5, 2024. There are 10 equal-weighted (number.letter)-named questions.

The Bachelier Model (30%)

The Bachelier model – named after the French finance (and stochastic processes) pioneer Louis Bachelier who used it in his 1900 dissertation – models the price of a stock, S, by a so-called arithmetic (as oppsosed to geometric) process, which has dynamics of the form

$$dS(t) = \ldots dt + \sigma dW(t),$$

the important point being that the term in front of $dW(t)$ is constant, i.e. it does not have $S(t)$ itself in it.

1.a
Suppose the interest rate is 0 and consider a strike-K expiry-T call-option. Show that its arbitrage-free time-t price is

$$\pi^{\text{call, Bach}}(t) = (S(t) - K)\Phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}\phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right),$$

where Φ and ϕ denote, respectively, the standard normal distribution and density function. Show that the time-t Δ-hedge-ratio of the call-option is

$$\Delta^{\text{call, Bach}}(t) = \Phi\left(\frac{S(t)-K}{\sigma\sqrt{T-t}}\right).$$

Hint: You may without proof use that if $X \sim N(\mu, \sigma^2)$ then

$$\mathrm{E}(X\mathbf{1}_{l\le X\le h}) = \mu\left(\Phi\left(\frac{h-\mu}{\sigma}\right) - \Phi\left(\frac{l-\mu}{\sigma}\right)\right) + \sigma\left(\phi\left(\frac{l-\mu}{\sigma}\right) - \phi\left(\frac{h-\mu}{\sigma}\right)\right),$$

where as usual $\mathbf{1}$ denotes the indicator function.

1.b (where we still assume that the interest rate is 0)
What do the implied volatilities across strikes look like in the Bachelier model? Comment

on the results. ($S(0) = 100$, $T = 1$ and $\sigma = 15$ are reasonable parameters.)

1.c

What does the call-price formula look like in a Bachelier model with a non-zero (but constant) interest rate r? Hint: Careful about time-dependent parameters.

Quanto Hedging and The Kingdom of Denmark Put (40%)

In this exercise we consider an arbitrage-free currency model of Black/Scholes-type. More specifically, we think of and refer to "domestic" as "US" and "foreign" as "Japan". With Björk's Proposition 18.7 (with a sign typo corrected) in mind we write bank-accounts, exchange-rate and Japanese stock dynamics under the US-martingale measure as

$$\begin{aligned} dB_{US}(t) &= r_{US}B_{US}(t)dt \\ dX(t) &= X(t)(r_{US} - r_J)dt + X(t)\sigma_X^\top dW(t) \\ dB_J(t) &= r_J B_J(t)dt \\ dS_J(t) &= S_J(t)(r_J - \sigma_X^\top \sigma_J)dt + S_J(t)\sigma_J^\top dW(t), \end{aligned}$$

where the σ's are 2-dimensional (constant column) vectors. (We don't need a US stock here; we'll get enough fun as it is.)

For numerical experiments, you can/may/should assume that $r_{US} = 0.03$ $r_J = 0.00$, $\sigma_X^\top = (0.1, 0.02)$ a, and $\sigma_J^\top = (0, 0.25)$.

Consider a *quanto put*. This is an option that at time T pays off

$$Y_0(K - S_J(T))^+,$$

where (the constant) Y_0 is some agreed-upon-in-advance exchange-rate; it could be the time-0 exchange rate or a forward exchange rate.

2.a

Show that the arbitrage-free time-t price of the quanto put is $F^{QP}(t, S_J(t))$, where the function F^{QP} is defined by

$$F^{QP}(t,s) = Y_0 e^{-r_{US}(T-t)} \left(K\Phi(-d_2(t,s)) - e^{(r_J - \sigma_X^\top \sigma_J)(T-t)} s\Phi(-d_1(t,s)) \right),$$

Φ is the standard normal distribution function as usual,

$$d_{1/2}(t,s) = \frac{\ln(s/K) + (r_J - \sigma_X^\top \sigma_J \pm ||\sigma_J||^2/2)(T-t)}{\sqrt{T-t}||\sigma_J||},$$

and $||\cdot||$ denotes the Euclidian norm of a vector.
Show that
$$\frac{\partial F^{QP}(t,s)}{\partial s} = Y_0 e^{(r_J - \sigma_X^\top \sigma_J - r_{US})(T-t)} \left(\Phi(d_1(t,s)) - 1\right) =: g(t,s).$$
("=:" and ":=" mean "defined as", with the term closest to ":" being defined.)
Hint: Careful "pattern recognition" brings you back to Black-Scholes, thus requiring no new calculations.

2.b
Consider now the usual set-up for a discrete hedge experiment, i.e. the interval $[0;T]$ is split into n pieces at the equidistant time-points t_i, where we adjust our portfolio.
Illustrate experimentally (say, in the plausible case where $Y_0 = X(0) = 1/100$ and $K = S_J(0) = 30,000$ and $T = 2$) that a strategy that

- at time t_i holds
$$\Delta^{QP}(t_i, S_J(t_i), X(t_i)) := \frac{g(t_i, S_J(t_i))}{X(t_i)}$$
units of the foreign stock, and
- is then made self-financing with the domestic bank-account,

does not replicate the pay-off of the quanto put (even in the limit).

Suppose now that the strategy above in amended by holding (at time t_i)
$$-\Delta^{QP}(t_i, S_J(t_i), X(t_i))S_J(t_i)$$
units of currency, which are instantly deposited in the foreign bank (giving you $-\Delta^{QP} S_J/B_J$ units of it), where they earn interest; it is still kept self-financing via domestic bank-account.
Illustrate experimentally that this strategy ("suitably initiated" and "in the limit") *does* replicate the pay-off of the quanto put.

2.c
Explain (i.e.: prove mathematically) why the strategy from the last part of 2.b works (replication-wise), and why the from the first part does not. (A discrete-time or localized argument will do but a full-fledged, three-holdings continuous-time argument is perhaps easier. But in any case, if your experimental results do not support your conclusions, your credibility is low!)

2.d (A new and unusual type of question)
In short: Write a two-page essay on Emanuel Derman's description of the Kingdom of Denmark Put in his book My Life as a Quant.

First, obtain a copy of the book legally. This can be done via (for instance) the link just given, or (probably, I haven't checked in detail) via the University (or some other) library.
Second, find the part where he writes about the Kingdom of Denmark Put (around pages 209-218 in the 2014 edition) and read it.

By *essay* I mean a self-contained short text (article, report, *fristil*, ...) that contains one or more of the following elements:

- A summary of events. *The 5 W questions: what, where, when, who, and why?*
- An explanation linking Derman's verbal description to the math and calculations from the previous questions. *The H question: how?* For instance: What would be reasonable interest rate and volatility parameter values circa 1990?
- Checking of facts and further sources. For instance: Can you find out how much Goldman Sachs paid the Danish state/government to co-sign, i.e. to guarantee that investors would be paid their promised amounts even in the case of a Goldman Sachs default.

(And when you are done with this, read the whole book!)

A Reflection Theorem (30%)

A relevant reference for this exercise is Poulsen (2006) "Barrier Options and Their Static Hedges: Simple Derivations and Extensions", *Quantitative Finance*, vol. 6(4), pp 327-335. (But your task is provide details; "dot the i's and cross the t's.")

Let us look at the standard, no-dividends Black-Scholes model, put

$$p = 1 - \frac{2r}{\sigma^2},$$

and consider a simple claim with a pay-off at time T specified by a pay-off function g. The arbitrage-free time-t value is

$$\pi^g(t) = e^{-r(T-t)}\mathrm{E}_t^{\mathbb{Q}}(g(S(T))) = e^{-r(T-t)} f(S(t), t),$$

where of course $f(S(t), t) = \mathrm{E}_t^{\mathbb{Q}}(g(S(T)))$. Let $H > 0$ be a constant and define a new function $\widehat{g}$ by

$$\widehat{g}(x) = (x/H)^p \, g\left(H^2/x\right).$$

We call a simple claim with this pay-off function g's *reflected claim* (don't worry why) .

<u>3.a</u>
Define the process Z by

$$Z(t) = \left(\frac{S(t)}{H}\right)^p.$$

Show that $Z(t)/Z(0)$ is a positive, mean-1 $\mathbb{Q}$-martingale. This means that

$$\frac{d\mathbb{Q}^Z}{d\mathbb{Q}} = \frac{Z(T)}{Z(0)}$$

defines a probability measure $\mathbb{Q}^Z \sim \mathbb{Q}$. Show that

$$\pi^{\widehat{g}}(t) = e^{-r(T-t)} \left(\frac{S(t)}{H}\right)^p \mathrm{E}_t^{\mathbb{Q}^Z} \left(g\left(\frac{H^2}{S(T)}\right)\right).$$

<u>3.b</u>
Put $Y(t) = H^2/S(t)$. Show that

$$dY(t) = rY(t)dt + \sigma Y(t) \left(-dW^{\mathbb{Q}^Z}(t)\right),$$

where $W^{\mathbb{Q}^Z}$ is a $\mathbb{Q}^Z$-Brownian motion.

<u>3.c</u>
Explain why this means that the distribution of Y under $\mathbb{Q}^Z$ is the same as the distribution of S under $\mathbb{Q}$. Use this to argue that

$$\pi^{\widehat{g}}(t) = e^{-r(T-t)}(S(t)/H)^p f(H^2/S(t), t).$$

Sketch how this can be used to derive closed-form expressions for barrier option prices.