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Barrier options and their static hedges: simple derivations and extensions

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We use a reflection result to give simple proofs of (well-known) valuation formulas and static hedge portfolio constructions for zero-rebate single-barrier options in the Black–Scholes model. We then illustrate how to extend the ideas to other model types giving (at least) easy-to-program numerical methods and other option types such as options with rebates, and double-barrier and lookback options.

Keywords: Barrier option; Static hedging

1. Introduction

This paper is a survey of valuation and hedging techniques for single-barrier, double-barrier, and lookback options. First (section 2), we use the ideas from Carr *et al*. (1998) to formulate and prove a reflection result related to Geometric Brownian motion. We then (section 3) show how cunning applications of this result establish valuation formulas for all types of zero-rebate single-barrier options in the Black–Scholes model. These formulas have been known since Merton (1973), but the derivation presented here is honestly short (i.e. we do not just leave a lot of tedious calculations to the reader) and self-contained. The key observation is the equivalence of barrier options to simple contingent claims with specifically adjusted pay-off functions. These claims can be written as superpositions of plain vanilla options (puts and calls) and that, in turn (section 4), leads to static hedges, i.e. replicating portfolios that—contrary to usual Δ-hedges—require (close to) no dynamic adjustments. We then demonstrate how these hedge portfolios are obtained as special (but non-trivial) cases of the more general, yet also more intuitive, hedging approach based on partial differential equations (PDEs) from Derman *et al*. (1995); the adjusted pay-off function is simply a numerical by-product. Next, we describe extensions of the approach beyond Black–Scholes model single-barrier option valuation. With general volatility specifications (section 5) these extensions give simple numerical methods, and in the Black–Scholes model they produce closed-form solutions for more complicated option structures with a minimum of extra calculations (section 6). We conclude the paper (section 7) by outlining topics of ongoing research, including a discussion of the practical applicability of the static hedge techniques.

2. A reflection theorem in the Black–Scholes model

We consider the Black–Scholes model with constant short-term interest rate r and a risky asset whose price is a Geometric Brownian motion,

$$\mathrm{d}S(t) = \mu S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W^Q(t),$$

under the equivalent martingale measure Q. Using $\mu = r$ gives the classic model for a non-dividend-paying stock, $\mu = r - d$ models a stock with dividend yield d, $\mu = 0$ models forwards/futures, and in a currency model μ is the difference between the domestic and the foreign interest rate.

Put

$$p = 1 - \frac{2\mu}{\sigma^2},$$

and consider a simple claim with a pay-off at time T specified by a pay-off function g (a 'g-claim' for short). The arbitrage-free time-t value (such values are denoted by π's throughout the paper) is

$$\pi^g(t) = \mathrm{e}^{-r(T-t)}\mathbf{E}_t^Q(g(S(T))) = \mathrm{e}^{-r(T-t)}f(S(t), t),$$

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where, of course, $f(S(t),t) = \mathbf{E}_t^Q(g(S(T)))$, and the Markov property of S ensures that this is non-deceptive notation. Let $H > 0$ be a constant and define a new function $\widehat{g}$ by

$$\widehat{g}(x) = (x/H)^p g(H^2/x).$$

We call a simple claim with this pay-off function g's *reflected claim*. The term reflection is borrowed from physics, and indeed much of what we do can be viewed as illustrating the power of the method of images and the associated Kelvin transform.

The next theorem shows that the g- and $\widehat{g}$-claims are very closely connected and it is the main source of the subsequent results in the paper.

THEOREM 2.1 (*Reflection theorem*). *Let the setup be as above and consider a simple claim with pay-off function $\widehat{g}$. The arbitrage-free time-t value of this $\widehat{g}$-claim is*

$$\pi^{\widehat{g}}(t) = e^{-r(T-t)}(S(t)/H)^p f(H^2/S(t),t). \tag{1}$$

Proof Using the Ito formula (or the well-known form of Geometric Brownian motion) on the process Z defined by

$$Z(t) = \left(\frac{S(t)}{H}\right)^p$$

tells us that

$$\mathrm{d}Z(t) = p\sigma Z(t)\,\mathrm{d}W^Q(t),$$

so $Z(t)/Z(0)$ is a positive, mean-1 Q-martingale. Here the exact form of p is needed. The result would not hold if σ were time-dependent or stochastic. This means that

$$\frac{\mathrm{d}Q^Z}{\mathrm{d}Q} = \frac{Z(T)}{Z(0)}$$

defines a probability measure $Q^Z \sim Q$. Now use the abstract Bayes' formula for conditional means (Karatzas and Shreve 1992, lemma 3.5.3) to write the value of the $\widehat{g}$-claim as

$$\begin{aligned}\pi^{\widehat{g}}(t) &= \mathrm{e}^{-r(T-t)}\mathbf{E}_t^Q\left(\left(\frac{S(T)}{H}\right)^p g\left(\frac{H^2}{S(T)}\right)\right)\\ &= \mathrm{e}^{-r(T-t)}\left(\frac{S(t)}{H}\right)^p \mathbf{E}_t^{Q^Z}\left(g\left(\frac{H^2}{S(T)}\right)\right).\end{aligned}$$

Girsanov's theorem (Karatzas and Shreve 1992, theorem 3.5.1) tells us that

$$\mathrm{d}W^{Q^Z}(t) = \mathrm{d}W^Q(t) - p\sigma\,\mathrm{d}t$$

defines a Q^Z-Brownian motion. Put $Y(t) = H^2/S(t)$. Then the Ito formula and the definition of W^{Q^Z} gives us (again, the particular form of p is needed)

$$\mathrm{d}Y(t) = \mu Y(t)\,\mathrm{d}t + \sigma Y(t)\left(-\mathrm{d}W^{Q^Z}(t)\right),$$

which means that the law of Y under Q^Z is the same as the law of S under Q. Therefore

$$\mathbf{E}_t^{Q^Z}(g(Y(T))) = f(Y(t),t) = f(H^2/S(t),t),$$

and the result follows. □

Reflection principles/results date a long way back in the literature on physics, partial differential equations and stochastic processes, and with Joshi (2003, theorem 10.2) the above formula can be regarded as a finance text-book result. Despite that, it is probably fair to attribute the formula, and certainly the demonstration of its usefulness for option pricing, to Peter Carr (Carr and Chou 1997b, Carr *et al.* 1998). The proof given here differs from Carr's original one by having more abstract probability and less partial integrations. What you prefer is a matter of taste, but in a teaching context, it can make a nice example of how general concepts (Markov, Ito, measure-change, Girsanov) can lead to a concrete formula.

3. Zero-rebate barrier options

Valuation formulas for single-barrier zero-rebate options were given in Merton (1973) (where a footnote covers non-zero rebates), so they are as old as the Black–Scholes formula itself.

Let us show how the reflection theorem makes the pricing easy and has some interesting 'spin-offs'. To do this, consider a zero-rebate knock-out version of a v-claim with barrier B, 'the barrier option' in the following. (We look only at knock-out options; knock-in options are handled by parity.) Such an option has the pay-off

$$v(S(T))\mathbf{1}_{m(T)>B} \text{ (down-and-out case) or}$$
$$v(S(T))\mathbf{1}_{M(T)<B} \text{ (up-and-out case)},$$

where $m(T) = \min_{u\le T} S(u)$ and $M(T) = \max_{u\le T} S(u)$ denote the running minimum and maximum. The two cases are treated completely similarly, so let us focus on down-and-out options. As input to the reflection theorem we use the g-function defined by

$$g(x) = v(x)\mathbf{1}_{x>B}$$

(which is *not* the pay-off of the barrier option) and B in the place of H. With $\widehat{g}$ denoting the g's B-reflected claim as defined in section 2, we can look at the simple claim with pay-off function $h = g - \widehat{g}$ (see figure 1), the 'adjusted pay-off' in the language of Carr and Chou. Note that $h(x) = g(x) = v(x)$ if $x > B$ and equation (1) tells us that the time-t value of the h-claim is

$$\pi^h(t) = \mathrm{e}^{-r(T-t)}\big(f(S(t),t) - (S(t)/B)^p f(B^2/S(t),t)\big). \tag{2}$$

In particular, we see that if $S(t) = B$, then the h-claim has a value of 0. Suppose now that we buy the h-claim, sell it again if the stock price hits the barrier and, if this does not happen, simply hold it until expiry. If the stock price stays above the barrier, we receive $g(S(T))$ at expiry, otherwise we get 0. In other words, exactly the same as the barrier option, so we can read off its value directly from equation (2). This result is, of course, only useful if the g-claim itself has been valued, i.e. we know the f-function. But in the case of calls or puts we easily establish the formulas from (for instance) Björk (2004, sections 18.2-3) . In some cases, one has to be a little careful, because the g-function

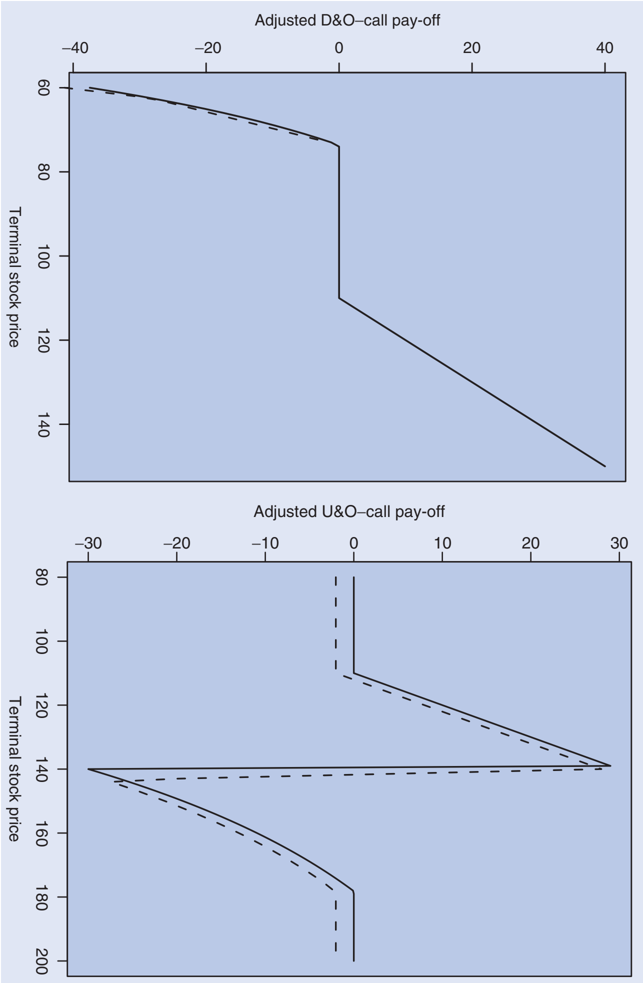

Figure 1. Adjusted pay-off function, h, for the down-and-out call (left) and the up-and-out call (right). The underlying call has a strike of 110, and the barriers levels are, respectively, 90 and 140. The dotted lines show the approximations achieved using, respectively, three and 11 plain vanilla options.

is *not* the regular pay-off function for the v-claim, but rather its truncated-at-B version. But if v is piecewise linear, then so is g, and although the f-function becomes more complicated, we never leave the realm of plain vanilla options. Specifically, for a down-and-out put $g(x) = (K - x)^+ \mathbf{1}_{x>B}$, which is the pay-off from a portfolio containing 1 strike-K put, -1 strike-B puts and $B - K$ strike-B digital puts.

Example 3.1 (*Geometic Brownian motion and its minimum*). As a special case consider the pay-off function $v(x) = \mathbf{1}_{x \geq K}$, put $g(x) = v(x)\mathbf{1}_{x>B}$ for some B, where $0 \leq B \leq \min(K, S(0))$ (so actually, $v = g$). Then clearly

$$f(x,t) = \Phi\left(\frac{\ln(x/K) + (\mu - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right).$$

Let π^h denote the value of the associated h-claim option. On the one hand,

$$\begin{aligned} e^{rT}\pi^h(0) &= \mathbf{E}^Q(\mathbf{1}_{S(T)\geq K}\mathbf{1}_{m(T)>B}) \\ &= Q(S(T) \geq K, m(T) > B), \end{aligned}$$

where $m(T) = \min_{u \leq T} S(u)$. On the other hand, equation (4) tells us that

$$\begin{aligned} e^{rT}\pi^h(0) = \Phi\left(\frac{\ln(S(0)/K) + (\mu - \sigma^2/2)T}{\sigma\sqrt{T}}\right) - \left(\frac{S(0)}{B}\right)^p \\ \times \Phi\left(\frac{\ln(B^2/S(0)K) + (\mu - \sigma^2/2)T}{\sigma\sqrt{T}}\right). \end{aligned}$$

Combining these two equations and viewing the right-hand side as a function of K and B determines the joint distribution of Geometric Brownian motion and its minimum. The joint density, say $\phi_{S,m}$, is obtained by differentiating wrt. K and B. Because the logarithm is monotone, we immediately get the distribution of Brownian motion with drift and its minimum, and by symmetry the distribution of Brownian motion and its maximum; see Jeanblanc *et al.* (2005) for a nice treatment of hitting times and extrema of Brownian motion and related processes and their uses in finance. (Note that we have *not* determined the joint distribution of the three-dimensional variable $(m(T), S(T), M(T))$, but only two of its two-dimensional marginals. The simultaneous distribution is considerably more complicated, but can be found using the double-barrier analysis in section 6.) Putting $K = B$ gives the (complementary) distribution function for the minimum of the Geometric Brownian motion. With $\tau = \inf\{u | S(u) = B\}$ denoting the first hitting time to the level B we have that $Q(\tau \leq x) = 1 - Q(\tau > x) = 1 - Q(m(x) > B)$, and by differentiating (wrt. x) we get the first hitting time density. With the joint and the marginal densities at hand, the density of the minimum conditional on $S(T) = x$ is simply their ratio, $\phi_{S,m}(x,y)/\phi_S(x)$ in obvious notation. This can also be interpreted as the minimum of a Brownian bridge, and can be quite useful for efficient computations (Beaglehole *et al.* 1997).

4. Static hedging

Any simple claim, particularly the h-claim, can be replicated by a portfolio of plain vanilla puts and calls, so we can devise static hedges for the barrier option. We simply buy puts and calls at time 0 such that the pay-off function is matched at expiry. Note, however, that for a general p, the h-function is not piecewise linear, so the static hedge portfolio involves a continuum of options. In fact, h could be discontinuous at the barrier level; for instance, this happens for an up-and-out call, as seen in figure 1. Then matching the pay-off becomes problematic in practice. Further, the hedge is strictly only semi-static, because it must be unwound (the portfolio of puts and calls sold) if the barrier is hit.

If we assume that $\mu = 0$ and consider a call option, $v(x) = (x - K)^+$, then we find that

$$\hat{g}(x) = (x/B)(B^2/x - K)^+ = (K/B)(B^2/K - x)^+,$$

which is the pay-off of K/B puts with strike B^2/K. So the knock-out call can be hedged by buying the plain vanilla

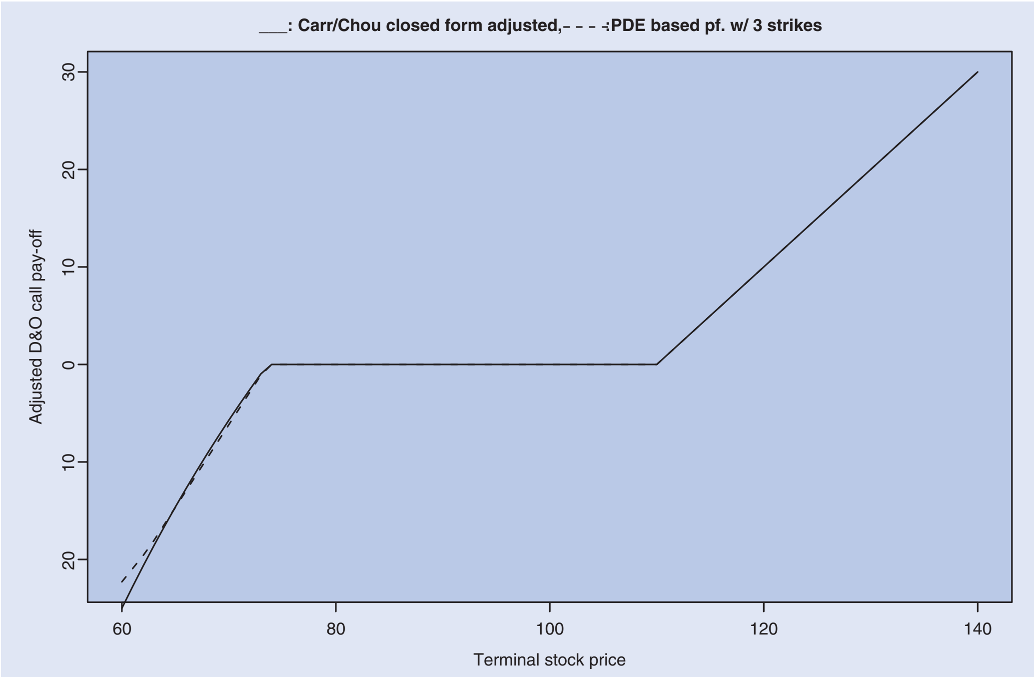

Figure 2. The true adjusted pay-off function for the strike-110 barrier-90 down-and-out call, and (dotted line) the numerical approximation found using expiry-1 options with three hedge strike levels (at 73.64, 64 and 50 and with time match points, t_j's, of 0.25, 0.5 and 0.75) as hedge instruments.

call and shorting K/B strike-B^2/K puts. This result, a version of the put/call-symmetry, was a starting point of static hedging; see Carr *et al.* (1998) and the references therein.

A different approach to static hedging is presented in Derman *et al.* (1995). It takes as outset the partial differential equation (PDE) formulation of the pricing problem: if the barrier option is still alive, its value is of the form $\pi^g(t) = F(S(t), t)$ (so $F(x,t) = \mathrm{e}^{-r(T-t)} f(x,t)$ in earlier notation, but here it is most convenient to work with F), where the function F solves

$$
\begin{aligned}
&7F_t + \frac{1}{2}\sigma^2 x^2 F_{xx} + \mu x F_x = rF && \text{on the } \textit{alive region}, \\
&F(B,t) = R(B,t) \text{ for } t < T && \text{(known rebate at the barrier)}, \\
&F(x,T) = g(x) && \text{(pay-off at expiry)}.
\end{aligned}
$$

Using puts and/or calls, a portfolio that matches the known values at expiry, and along the barrier, can be found as the solution to a linear system of equations. To illustrate, consider a 0-rebate down-and-out call option. First, split $[0; T]$ into n intervals via points $0 = t_0 < t_1 < \cdots < t_n = T$, and let Put(spot, time | strike, expiry) denote put option prices (and likewise for calls). Next, suppose that n (strike, expiry)-pairs (K_j, T_j) have been chosen *in some suitable way*. Now, find the solution α to

$$A\alpha + u = 0,$$

where A is an $n \times n$ matrix with entries $A_{i,j} = \mathrm{Put}(B, t_i | K_j, T_j)$ and u is an n-vector with entries $\mathrm{Call}(B, t_i | K, T)$. A portfolio with the (K, T)-call and α_j in the (K_j, T_j)-put then matches that value of the barrier option along the barrier (or at least at the t_i-points) and above the barrier at expiry (provided all K_j's are $\leq B$). A static hedge in other words.

Derman *et al.* (1995) suggest using $T_j = t_{j-1}$ and $K_j = B$ (calender-spread hedging with strikes along the barrier), which makes the A-matrix triangular so that we can solve for α_j's in one easy-to-explain backward-working pass.

Another choice is $T_j = T$ for all j and K_j's that are different and below the barrier. In this way, the α-portfolio is a simple T-claim, and we can plot its pay-off function, and, as illustrated in figure 2, even for very few hedge strike levels, this (strictly numerical) method gives a very accurate approximation to the adjusted pay-off function that we derived in closed-form earlier by a different technique.

5. Beyond Black–Scholes

As noted in its proof, the reflection theorem depends critically on the Black–Scholes model assumption. With (local) volatility of the form $\sigma(S(t), t)$, even in the simple case of time-dependent volatility or drift, it breaks down. As shown by Andersen *et al.* (2002), the PDE approach to static hedging can be generalized in a conceptually straightforward manner to give hedges and values in local volatility models, stochastic volatility models and even jump diffusions. There are a number of non-trivial implementational/numerical issues to this. These are analysed at length in Nalholm and Poulsen (2006a), but in this paper we confine ourselves to two cases: zero drift and time-(only-)dependent coefficients.

5.1. The drift-free case

In this case Q^Z is in fact the well-known stock-numeraire martingale measure. If $\mu \equiv 0$ and volatility is a stochastic process independent of the Brownian motion driving S, then (as noted by Andreasen (2001)) we can proceed exactly as in the proof of the reflection theorem. The independence means that the distribution of volatility is the same under Q and Q^Z, and we conclude that

$$\begin{aligned}\tilde{f}(S(t), \sigma(t), t) &:= \mathbf{E}_t^Q(g(S(t)) = \pi^g(t) \\ &= \frac{S(t)}{H}\mathbf{E}_t^{Q^Z}\left(g\left(\frac{H^2}{S(t)}\right)\right) = \pi^{\hat{g}}(t).\end{aligned}$$

However, if S and σ are correlated, σ does not have the same distribution under Q and Q^Z, and the result above does not hold. In some cases, the Heston model for instance, the dynamics of σ are only *parametrically* changed, and there are symmetry relations, but they do not appear to provide either closed-form solutions for values, or static hedges.

5.2. Time-dependent drift and volatility

Using time-varying drift and volatility may serve as a first attempt at capturing deviations from the basic Black–Scholes model. One could imagine using the (easily observable) time-0 maturity-t forward rate as $r(t)$, and the volatility proxy $\tilde{\sigma}(t) = \mathbf{E}^Q(\sigma(\cdot, t))$ that can be inferred from prices of plain vanilla options (Derman and Kani 1998).

To the author's knowledge there is no simple closed-form solution for barrier option values in the case of time-dependent drift and volatility; results are given either as integrals that require numerical treatment (see Roberts and Shortland (1997), for instance), or as approximations, which can be quite tight (Lo *et al.* 2003).

It is interesting to note that the PDE-based hedge method in the previous section gives a simple-to-implement way of determining the barrier option value numerically: along the barrier, use the time-varying drift and volatility version of the Black–Scholes formula (where r and σ^2 are replaced by their forward-in-time averages) to determine the A matrix and u vector of the previous section. The α-portfolio still constitutes a hedge, and its weights are found by solving a linear system of equations.

Alternatively, one could try to get back to the drift-free case by working with $X(t) := \mathrm{e}^{-\int_0^t \mu(u)\,\mathrm{d}u} S(t)$. But as barrier hits are defined in terms of S, not the transformed process X, hitting times of curved barriers have to be calculated. Time-change techniques go some way towards that, but the law of the hitting time is known only up to the solution of an integral equation (Jeanblanc *et al.* 2005, proposition 1.24).

6. Other option types

6.1. Barrier options with rebates

Using the distribution functions/densities just found, valuation formulas for barrier options with rebates (as given in Rubinstein and Reiner (1991), for instance) can be derived by direct, but sometimes cumbersome, calculations. However, using the ideas from Carr and Picron (1999) concerning stationary securities together with optional stopping, the formulas can be derived without any need for new calculations. To this end, assume that $r \geq 0$, let a be a constant and look at

$$\begin{aligned}X(t) &= \mathrm{e}^{-rt} S^a(t) \\ &= S^a(0)\exp((a(\mu - \sigma^2/2) - r)t + a\sigma W(t)).\end{aligned}$$

If a solves

$$a(\mu - \sigma^2/2) - r = -a^2\sigma^2/2,$$

then X is a martingale. This quadratic equation has the two roots,

$$a_{+/-} = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} + \frac{2r}{\sigma^2}},$$

where, as earlier, $p = 1 - 2\mu/\sigma^2$. The roots are both real, and a_+ is strictly positive, and a_- negative (strictly so, unless in the trivial case where $r = \mu = 0$).

Let us now look specifically at a down-and-out option that pays the rebate R when the barrier is hit. (The up-and-out case is treated similarly, except some inequalities must be reversed because $a_- < 0$. Cases where the rebate is paid at expiry are easily dealt with.) Remembering that we have already valued the zero-rebate version, and what has to be calculated is $R\mathbf{E}^Q(\mathrm{e}^{-r\tau}\mathbf{1}_{\tau<T})$, where $\tau = \inf\{u | S(u) = B\}$. Optional stopping (Karatzas and Shreve 1992, theorem 1.3.22) with the bounded stopping time $\tau \wedge T$ gives us

$$\begin{aligned}S^{a_+}(0) &= X(0) = \mathbf{E}^Q(X_{\tau\wedge T}) \\ &= \mathbf{E}^Q(\mathrm{e}^{-r\tau}B^{a_+}\mathbf{1}_{\tau<T}) + \mathrm{e}^{-rT}\mathbf{E}^Q\left(S^{a_+}(T)\mathbf{1}_{\inf_{u\leq T} S(u)>B)}\right),\end{aligned}$$

meaning that (because $a_+ > 0$ we can rewrite the indicator function)

$$\begin{aligned}&R\mathbf{E}^Q(\mathrm{e}^{-r\tau}\mathbf{1}_{\tau<T}) \\ &= \frac{R}{B^{a_+}}\left(S^{a_+}(0) - \mathrm{e}^{-rT}\mathbf{E}^Q(S^{a_+}(T)\mathbf{1}_{\inf_{u\leq T} S^{a_+}(u)>B^{a_+}})\right).\end{aligned}$$

The last term looks new but is not. Note first that S^{a_+} is itself a Geometric Brownian motion with drift-rate r (follows immediately because $\mathrm{e}^{-rt}S^{a_+}(t)$ is a martingale) and volatility $a_+\sigma$. Second, note that the last term is (minus) the value of a strike-0 call with transformed barrier B^{a_+} and rebate 0 written on the transformed process S^{a_+}. Therefore, its value can be found from the zero-rebate down-and-out call with

$$\sigma \rightsquigarrow a_+\sigma,\ \mu \rightsquigarrow r,\ S(0) \rightsquigarrow S^{a_+}(0),\ K = 0, \text{ and } B \rightsquigarrow B^{a_+}.$$

So the rebate-R barrier-B call corresponds to a long position in the rebate-0 call, R/B^{a_+} units of the a_+-security, and R/B^{a_+} units short in the strike-0, rebate-0 barrier-B^{a_+} call on the a_+-security.

6.2. Lookback options

Barrier option pay-offs depend on the maximum or minimum of the underlying, but only through indicator functions. Lookback options have pay-offs that depend more explicitly on the extreme value. For instance, a lookback call-option pays $S(T) - m(T)$ at time T, where, as before, m denotes the running minimum. This makes the pricing and hedging more difficult, not least so because we have to be careful when determining the value at a time-point after initiation, where it will depend non-trivially on both the current stock price and the extremum to date. Equipped with the joint distribution of S and the extremum, valuation formulas can be found by what Björk (2004, section 18.5) describes as 'a series of elementary, but extremely tedious, partial integrations' (and that is just for time-0 values).

But as shown in Carr and Chou (1997a, section 8) and Jeanblanc *et al.* (2005), there is an easier way. To keep the analysis simple, we now look at a contract that pays $m(T)$ at time T, and because there is no real loss of generality in this, we refer to it as *the* lookback option.

As a stepping-stone, we need to consider so-called one-touch options. For a given barrier level B, this is a contract with pay-off

$$\mathbf{1}_{m(T) \leq B},$$

and which depends on $m(T)$ in exactly the indicator-function way that allows us to value and statically hedge the contract using previous results. In particular, the one-touch option is equivalent to a simple claim with pay-off function

$$h_{\mathrm{OT}}(x; B) = \begin{cases} 0, & \text{for } x > B, \\ 1 + (x/B)^p, & \text{for } x \leq B. \end{cases}$$

(The one-touch option is an in-option, so we first find the adjusted pay-off of the out-version, and then use in/out-parity, which simply amounts to subtracting from 1.)

Next note that

$$m(T) = m(t) - \int_0^{m(t)} \mathbf{1}_{m(T) \leq B}\, \mathrm{d}B, \tag{3}$$

so when viewed from time t, receiving the pay-off $m(T)$ is equivalent to receiving

$$\begin{aligned} h_m(S(T), m(t)) &= m(t) - \int_0^{m(t)} h_{\mathrm{OT}}(S(T); B)\, \mathrm{d}B \\ &= m(t) - \int_{\min(m(t),\, S(T))}^{m(t)} (1 + (S(T)/B)^p)\, \mathrm{d}B. \end{aligned}$$

The last term is just an integral of a power function (two, if you are pedantic), and when $p \neq 1$ ($p = 1$ leads to options on ordinary Brownian motion, which are easily dealt with, although the resulting formulas appear a little different, involving the normal *density* function) we get

$$\begin{aligned} &h_m(S(T); m(t)) \\ &\quad = \begin{cases} m(t), & \text{for } S(T) > m(t), \\ \dfrac{2-p}{1-p} S(T) & \\ \quad + \dfrac{(m(t))^{1-p}}{p-1} (S(T))^p, & \text{for } S(T) \leq m(t). \end{cases} \end{aligned}$$

Viewed from time t—remember that $m(t)$ is known then (but not before)—this pay-off is that of a portfolio of digital options and two types of gap options, the last of which is written on the transformed stock-price S^p. This is easily valued.

Equation (3) shows that the lookback option can be statically hedged by a portfolio with (a continuum of) one-touch options. Each one-touch digital can be statically hedged by puts and calls. There is a small complication though: the static hedge portfolio of a given one-touch digital must be liquidated when its barrier is hit. This means that liquidation takes place every time the minimum changes. And that happens almost surely on an uncountable set of measure 0. But at least we *have found* a hedge portfolio, the usual (stock, bank-account) hedge portfolio, whose existence is ensured by the martingale representation and is quite tricky to find (Bermin 2000).

6.3. Double-barrier options

A double-barrier (knock-out) version of a v-claim pays

$$v(S(T))\mathbf{1}_{m(T) > L,\, M(T) < U} \text{ at time } T,$$

where, as before, M and m are, respectively, the running maximum and minimum.

Inspired by the single-barrier analysis let us look at

$$g_0(x) = v(x)\mathbf{1}_{L < x < U}.$$

To value the double-barrier option an immediate idea is to apply the reflection theorem twice to g_0; once with the role of H played by L, once with U as H. This will not quite work, though, because the U-reflected claim will make a non-zero contribution (that varies with time to expiry) along the L-barrier, where the g_0 and the L-reflected claims net out. Therefore, the combined portfolio does not have value 0 along the L-barrier (and likewise for the U-barrier). But it is a step in the right direction, because the U-reflected claim only pays off above U, so its value when $S(t) = L(< U)$ is typically small. To finish the job the trick is to do a series of reflections as in Carr and Chou (1997a, section 6) or, more recently, Sbuelz (2005). Specifically, let us split the positive real line into the regions shown in

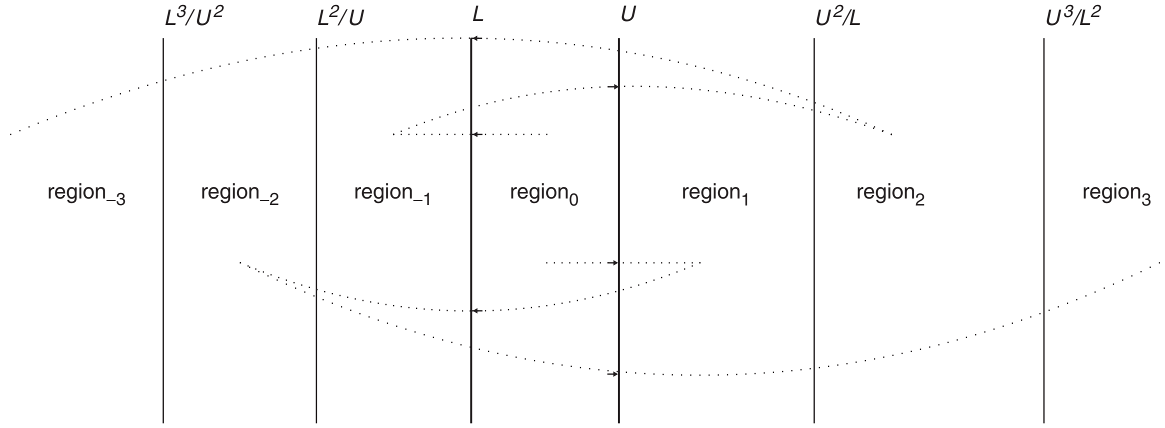

Figure 3. The reflection regions and the connections.

figure 3, i.e.

$$\text{region}_i = \begin{cases} \left[\left(\dfrac{U}{L}\right)^{i-1} U; \left(\dfrac{U}{L}\right)^{i} U\right], & \text{for } i > 0, \\ [L; U], & \text{for } i = 0, \\ \left[\left(\dfrac{U}{L}\right)^{i} L; \left(\dfrac{U}{L}\right)^{i+1} L\right], & \text{for } i < 0. \end{cases}$$

Reflection of g_0 through L gives a claim with pay-off function

$$\begin{aligned}\widehat{g}_0(x) &= \left(\frac{x}{L}\right)^p g_0\left(\frac{L^2}{x}\right) = \left(\frac{x}{L}\right)^p v\left(\frac{L^2}{x}\right)\mathbf{1}_{L<L^2/x<U} \\ &= \left(\frac{x}{L}\right)^p v\left(\frac{L^2}{x}\right)\mathbf{1}_{\text{region}_{-1}}(x) =: -g_{-1}(x).\end{aligned}$$

By the reflection theorem

$$\begin{aligned}\pi^{g_{-1}}(t) &= -\mathrm{e}^{-r(T-t)}\left(\frac{S(t)}{L}\right)^p f_0\left(\frac{L^2}{S(t)}, t\right) \\ &=: -\mathrm{e}^{-r(T-t)} f_{-1}(S(t), t),\end{aligned}$$

so, as we want, the g_0- and the g_{-1}-claim net out when $S(t) = L$. To remove the g_{-1}-contribution along the U-barrier, we reflect its pay-off through U, i.e. look at

$$\begin{aligned}\widehat{g}_{-1}(x) &= \left(\frac{x}{U}\right)^p g_{-1}\left(\frac{U^2}{x}\right) \\ &= \left(\frac{x}{U}\right)^p \left(\frac{U^2/x}{L}\right)^p v\left(\frac{L^2}{U^2/x}\right)\mathbf{1}_{L^2/U<U^2/x<L} \\ &= \left(\frac{U}{L}\right)^p v\left(x\frac{L^2}{U^2}\right)\mathbf{1}_{\text{region}_2}(x) =: -g_2(x).\end{aligned}$$

The reflection theorem tells us that

$$\begin{aligned}\pi^{g_2}(t) &= -\mathrm{e}^{-r(T-t)}\left(\frac{S(t)}{U}\right)^p f_{-1}\left(\frac{U^2}{S(t)}, t\right) \\ &= \mathrm{e}^{-r(T-t)}\left(\frac{U}{L}\right)^p f_0\left(S(t)\frac{L^2}{U^2}, t\right),\end{aligned}$$

which exactly equals $-\pi^{g_{-1}}$ when $S(t) = U$. We now reflect g_2 through L ($\rightsquigarrow g_{-3}$), g_{-3} through U, and so on. A pattern quickly emerges:

$$\pi^{g_i}(t) = \begin{cases} \left(\dfrac{U}{L}\right)^{jp} f_0\left(S(t)\left(\dfrac{L}{U}\right)^{2j}, t\right), \\ \qquad \text{for } j = 0, 2, 4, \ldots, \\ -\left(\dfrac{S(t)}{U}\right)^p \left(\dfrac{L}{U}\right)^{jp} f_0\left(\left(\dfrac{U^2}{S(t)}\right)\left(\dfrac{U}{L}\right)^{2j}, t\right), \\ \qquad \text{for } i = -1, -3, -5, \ldots. \end{cases}$$

We can repeat the procedure starting with a reflection of g_0 through U. This analysis is similar up to some changes of signs on indices and barriers.

This leads to a valuation formula in terms of an infinite sum ($\sum_{-\infty}^{\infty}$) that only involves f_0 evaluated at appropriate points. This is the best we can realistically hope for given the results in the literature (Kunitomo and Ikeda 1992). The formula is most easily conveyed in pseudo-code form. The algorithm for a double knock-out call is given below (in the syntax of the R language), but to change to a different option type (put, digital), all that has to be altered is the definition of `f0`.

```
DoubleBarrier=function(S, T, L, U, T, r, mu,
sigma,Nterms){
  p<-1-2*mu/sigma^2
  f0<-function(y){
    f0<-BSCall(y,K,T,r,mu,sigma)-BSCall
    (y,U,[same])-(U-K)*BSDigital
    (y,U,[same])
  }
  Value<-0
  for (j in -Nterms:Nterms){
    EvenTerm<-(U/L)^(j*p)*f0((L/U)^(2*j)*S)
    OddTerm<(S/U)^p*(L/U)^(j*p)*f0((U/L)^
    (2*j)*(U^2/S))
    Value<-Value+EvenTerm+OddTerm
  }
  DoubleBarrier<-Value
}
```

Table 1. Values of double-barrier knock-out calls. The first values are calculated using the algorithm from the Kunitomo–Ikeda paper (with eight terms), and the Carr–Chou values use the algorithm described in this paper. The parameters (besides those indicated above) are $S(t) = K = 1000$, $T - t = \frac{1}{2}$, $r = \mu = 0.05$, and $\sigma = 0.2$.

			Carr–Chou Nterms			
L	U	Kunitomo–Ikeda	0	1	3	3
500	1500	66.1289	66.1289	66.1289	66.1289	66.1289
800	1200	22.0820	22.1128	22.0820	22.0820	22.0820
900	1100	1.78676	3.01120	1.78693	1.78676	1.78676
950	1050	0.00057	0.24683	0.07458	0.00207	0.00057

(For very small ($L \approx 0$) and/or very large ($U \approx \infty$) barriers, i.e. to value single-barrier or plain vanilla options, it is not advisable to use the algorithm above verbatim; some numerical care must be taken to avoid adding or multiplying numbers of vastly different sizes. We thank Morten Nalholm for this observation.)

Table 1 shows numerical results for the implementation of the algorithm. Only a few terms are needed in the $-\infty$ to $+\infty$ summation to obtain a very precise result. The closer L and U are, the closer the option is to expiry, the closer $S(t)$ is to a boundary, and the more terms are needed.

Several extensions follow easily. Exponentially curved boundaries are treated by viewing the options as written on processes with transformed drifts. (The two series of reflections are independent, so the curvatures do not have to be the same.) Transformation also establishes results for regular Brownian motion, for instance its probability of staying within a strip. Non-zero rebates at hit, say R_L and R_U, can be handled using a portfolio (b_{a_+}, b_{a_-}) of the two stationary securities from the rebate analysis chosen such that

$$b_{a_+} U^{a_+} + b_{a_-} U^{a_-} = R_U \text{ and } b_{a_+} L^{a_+} + b_{a_-} L^{a_-} = R_L.$$

7. Conclusion and current research

Relying on a reflection theorem, we have given self-contained, simple and short derivations of values of exotic options up to and including double barriers in a Black–Scholes model. We have also shown how static hedges can be found (in more ways than one) and that gives rise to easy-to-implement numerical methods.

Few truly closed-form option valuation formulas are known outside the Black–Scholes model. The reflection/symmetry techniques are a promising tool for remedying this, and although each step forward seems to require a 'consider the following odd construction'-trick, we can surely expect more from that source in the future; Carr and Lee (2005) offer some recent contributions.

It can be argued that the applicability of static hedging has been 'oversold'. Nice mathematics (reflection) or simple numerics (PDE-based methods)—the focus of this paper—do not translate directly into practical usefulness. First, the static hedges require a continuum of plain vanilla options to be traded frictionlessly. Clearly, that is not the case in practice; in fact, one would expect transaction costs for option trading to be markedly larger than for trading in the underlying, which would level the playing field. Second, static hedging does not alleviate the exploding Greeks problem of discontinuous barrier options, such as (see the right-hand graph of figure 1) the up-and-out call. Nalholm and Poulsen (2006a) demonstrate that careful choice of hedge instruments as well as regularization techniques is necessary to achieve practically feasible results. A third issue is model risk. Imagine that we base our static hedge on the Black–Scholes model (or some adjusted version thereof, as traders often do), how bad are we off? Intuitively, there is mixed evidence. On the one hand, we use more natural hedge instruments (and instruments that allow us to 'hedge other Greeks than Δ' in trader language). On the other hand, the hedges involve options that are (deep) out-of-the-money when we buy them (their strikes have to be at least at the barrier) and at-the-money when we need to sell them. Such positions will be very sensitive to the existence of a volatility smile or skew and its dynamic behaviour. Recent papers analyse this problem from different angles. Engelmann *et al.* (2006) undertake a comprehensive empirical investigation of German option data, and find that the use of plain vanilla options to hedge barrier contracts (in a way that is not exactly similar but comparable to the static hedging approach of this paper) offers considerable improvements, but that details matter. Nalholm and Poulsen (2006a) run a simulation horse-race between hedging strategies in various non-Black–Scholes models. The results show that while quite sensitive to model risk, non-naive static hedge strategies are more robust than usual dynamic hedges to exactly which model generates the smile. A third approach is taken by Maruhn and Sachs (2005), who minimize (numerically) the sensitivity of hedge portfolio values to uncertain Black–Scholes parameters. Again, the findings indicate that static hedging can be made useful, but that it does not happen automatically.

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