

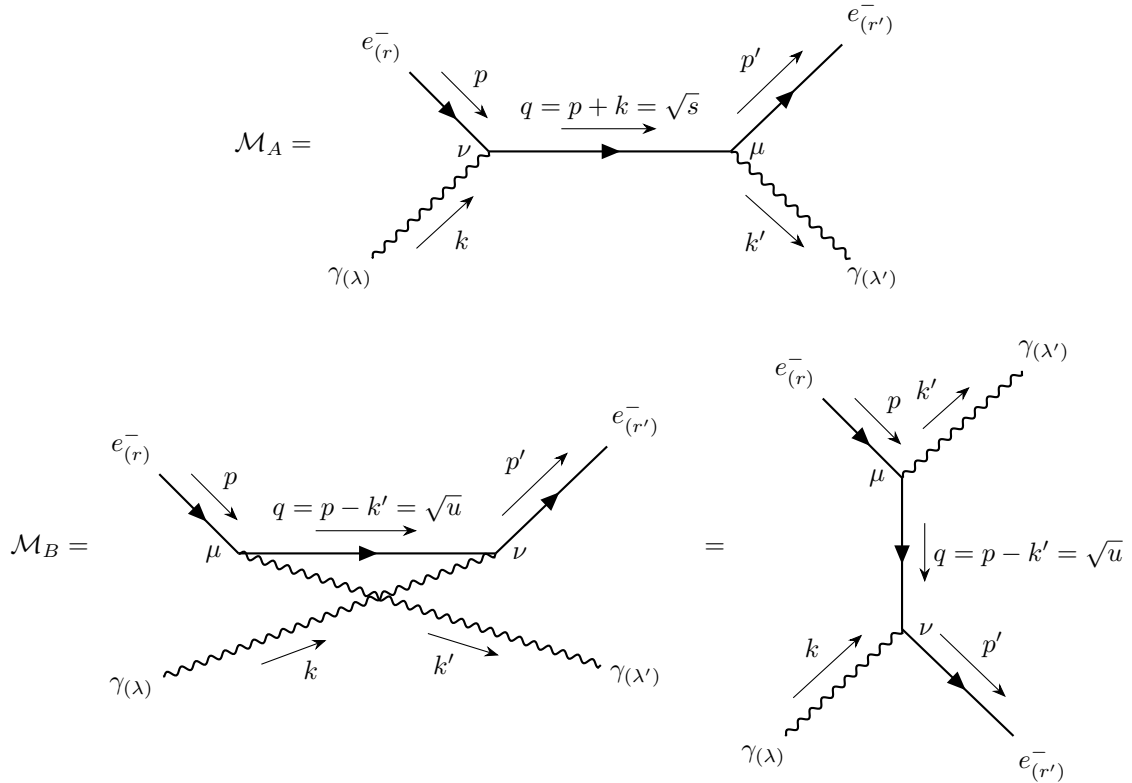
# Chapter 1

## QED Processes at Lowest order

### 1.1 Fogli di Emma <3

### 1.2 $e^- \gamma \rightarrow e^- \gamma$ (Compton)

Let's examine a process with external bosons: *Compton scattering*, or  $e^- \gamma \rightarrow e^- \gamma$ . This process is described by two independent diagrams, since they are topologically different:



We wrote the diagram of  $\mathcal{M}_B$  in two topologically equivalent forms: in the first one is clear the topological relation with diagram of  $\mathcal{B}$  (this is useful to find the relative sign between diagrams  $A$  and  $B$ ), while in the second one is clear that it describes a  $u$ -channel.

### 1.3 Sezioni di esempio

Consider the case in which the initial state is a single particle and the final state is given by  $n$  particles. We are therefore considering a decay process. Assume for the moment that particles are indistinguishable.



The rules of quantum mechanics tell us that the probability for this process is obtained by taking the squared modulus of the amplitude and summing over all possible final states

$$\begin{aligned}
 |S_{fi}^{CN}|^2 &= |(2\pi)^4 \delta^4(p - p') M_{fi}^{CN}|^2 \\
 &= (2\pi)^4 \delta^4(p - p') (VT) |M_{fi}^{CN}|^2 \\
 &= (2\pi)^4 \delta^4(p - p') (VT) \frac{1}{2\omega_i n V} \prod_{l=1}^{n_f} \left( \frac{1}{2\omega_l V} \right) |\mathcal{M}_{fi}|^2
 \end{aligned}$$

**Note:**  $(\delta^4(p - p'))^2 = \delta^4(p - p') \delta^4(p - p') = \delta^4(p - p') \delta^4(0) = \delta^4(p - p') \frac{VT}{(2\pi)^4}$   
 We use the final space and time in order to remove divergent terms during calculation

We must now sum this expression over all final states. Since we are working in a finite volume  $V$ , this is the sum over the possible discrete values of the momenta of the final particles .  
 Since  $p_i = (2\pi/L)n_i$ , we have  $dn_i = (L/2\pi)dp_i$  and  $d^3n_i = (V/(2\pi)^3)d^3p$  where  $d^3n_i$  is the infinitesimal phase space related to a final state in which the  $i$ -th particle has momentum between  $p_i$  and  $p_i + dp_i$   
 Let  $d\omega$  be the probability for a decay in which in the final state the  $i$ -th particle has momentum between  $p_i$  and  $p_i dp_i$

$$d\omega = |S_{fi}^{SN}|^2 \prod_{l=1}^{n_f} \left( \frac{V d^3 p_l}{(2\pi)^3} \right)$$

This is the probability that the decay takes place in any time between  $-T/2$  and  $T/2$ . We are more interested in the differential decay rate  $d\Gamma_{fi}$ , which is the decay probability per unit of time:

$$d\Gamma_{fi} = \frac{d\omega}{T} = (2\pi)^4 \delta^4(p_i - p_f) \frac{|\mathcal{M}_{fi}|^2}{2\omega_{p_{in}}} \prod_{l=1}^{n_f} \frac{d^3 p_l}{(2\pi)^3 2\omega_l}$$

**Notes:**

- (i)  $d\Gamma_{fi}$  = differential decay rate
- (ii)  $p_f$  = sum over final momenta
- (iii)  $\omega_{p_{in}}$  = initial energy
- (iv)  $|\mathcal{M}_{fi}|^2$  = Feynman amplitude of the process (depends on final momenta  $p_i$ )

It is useful to define the **(differential) n-body phase space** as

$$d\Phi_{(n_f)} = (2\pi)^4 \delta^4(p_i - p_f) \prod_{l=1}^{n_f} \frac{d^3 p_l}{(2\pi)^3 2\omega_l}$$

Therefore the differential decay rate can be written as

$$d\Gamma_{fi} = \frac{1}{2\omega_{p_{in}}} |\mathcal{M}_{fi}|^2 d\Phi_{(n_f)}$$

The decay rate is defined as

$$\Gamma_{fi} = \int d\Gamma_{fi} \rightarrow \text{integration over all possible final momenta}$$

and its meaning is  $\Gamma \equiv \text{trans. probability} \times \text{unit of time} \times \text{init. particle}$

Notice that if  $n$  of the final particles are identical, configurations that differ by a permutation are not distinct and therefore the phase space is reduced by a factor  $1/n!$

If we have a system of  $N(0)$  particles, the time evolution of the number of particles  $N(t)$  is

$$\frac{dN}{dt} = -\Gamma N \quad \Rightarrow \quad N(t) = N(0)e^{-\Gamma t}$$

Notice that decay rate is not invariant

$$[\Gamma] = [E] = \frac{1}{T} \quad (\text{in natural units})$$

If we define the **lifetime** as  $\tau = 1/\Gamma \Rightarrow N(t) = N(0)\exp(-t/\tau)$  this changes under Lorentz tfm. If we consider two reference frames  $o$  and  $o'$

$$\Gamma' = \frac{\Gamma}{\gamma} < \Gamma \quad \tau' = \gamma\tau > \tau$$

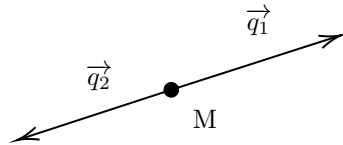
$\gamma = (1 - v)^{-1/2}$ , where  $v$  is the speed of  $o'$  in  $o$  in natural units,  $\gamma > 1$ . Therefore a particle in a moving frame has a longer lifetime than in the rest frame

### Example 1: muon lifetime

For a muon in the rest frame  $\tau_\mu^{RF} = 2.2 \times 10^{-6} s$ , but if we observe it in the lab frame  $\tau_\mu^{LAB} = \gamma\tau_\mu^{RF} \simeq 2 \times 10^{-5} s$  since  $E_\mu = 1 \text{ GeV}$ ,  $m_\mu = 0.1 \text{ GeV}$   
 $\Rightarrow \gamma = E_\mu/m_\mu \simeq 10$

### Example 2: $1 \rightarrow 2$ decay

Consider the decay of a particle of a mass  $M$  into two particles of masses  $m_1, m_2$ . Since the phase space is Lorentz invariant, we can compute it in the frame that we prefer. We use the rest frame for the initial particle.



We don't impose a priori conservation of momentum since it's imposed by the delta function.

$$p = (M, 0) \quad q_1 = (\omega_1, \mathbf{q}_1) \quad q_2 = (\omega_2, \mathbf{q}_2)$$

$$d\Phi_{(2)} = (2\pi)^4 \delta^4(\underbrace{P_i - P_f}_{=p-q_1-q_2}) \frac{d^3 q_1}{(2\pi)^3 2\omega_1} \frac{d^3 q_2}{(2\pi)^3 2\omega_2}$$

I have 6 integration parameters, 4 constraint given by  $\delta^4$ , so I have 2 independent variables. Integrating over  $d^3 q_2$

$$d\Phi'_{(2)} = \int d\Phi_{(2)} = \frac{1}{(2\pi)^2} \delta(M - \omega_1 - \omega_2) \frac{1}{4\omega_1\omega_2} d^3 q_1$$

in this way, the condition  $\mathbf{q}_2 = -\mathbf{q}_1$  vanishes. We have to impose it again when we calculate  $d\Gamma$  (we omit this detail)

Usually the 4-th non independent parameter is eliminated by integration over modulus of  $q_1$ , leaving free 2 parameters for the angles.  $d^3 q_1 \rightarrow \mathbf{q}_1^2 d|\mathbf{q}_2| d\Omega_1$ .

Notice that  $M - \omega_1 - \omega_2 = M - \sqrt{\mathbf{q}_1^2 + m_1^2} - \sqrt{\mathbf{q}_2^2 + m_2^2}$  and then the  $\delta$  implies

$$\hat{q}_1^2 = \frac{1}{2M} \left( M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2 \right)^{1/2}$$

$|\mathbf{q}_1|$  is the only zero of  $f(|\hat{q}_1|) = M - \omega_1 - \omega_2$ .

We also have

$$|f'(|\hat{q}_1|)| = \frac{\partial \omega_1}{\partial |\mathbf{q}_1|} + \frac{\partial \omega_2}{\partial |\mathbf{q}_1|} = |\hat{q}_1| \left( \frac{\omega_1 + \omega_2}{\omega_1 \omega_2} \right)$$

Using

$$\delta(f(x)) = \sum_{x_0 = \text{zero of } f(x)} \frac{\delta(x - x_0)}{|f'(x_0)|}$$

and performing integration over  $d|\mathbf{q}_1|$  we obtain

$$d\Phi_{(2)}'' = \int d\Phi_{(2)}' = \frac{1}{16\pi^2} \frac{|\hat{q}_1|}{M} d\Omega_1$$

Using this result we obtain the  $1 \rightarrow 2$  decay rate in function of the solid angle (in the rest frame)

$$\left( \frac{d\Gamma_{RF}}{d\Omega} \right) = \frac{1}{64\pi^4 M^3} [M^4 - 2M^2(m_1^2 + m_2^2) + (m_1 - m_2)^2]^{1/2} |\mathcal{M}_{RF}|^2$$

In a general frame we can easily obtain an analogous formula, just consider  $d\Gamma = 1/(2\omega_i n) |\mathcal{M}_{fi}|^2 d\Phi_{(nf)}$  in a general frame. remember that  $dI_{(nf)}$  is invariant

We have 2 important limit cases:

(A) If  $m_1 = m_2 = m$  (for example  $Z \rightarrow e^+ e^-$ )

$$|\hat{q}_1| = \frac{M}{2} \left( 1 - \frac{4m^2}{M^2} \right)^{1/2}$$

$$\left( \frac{d\Gamma_{RF}}{d\Omega} \right) = \frac{1}{64\pi^2 M} \left( 1 - \frac{4m^2}{M^2} \right)^{1/2} |\mathcal{M}_{fi}|^2$$

(B) If  $m_1 = m, m_2 = 0$  (for example  $W^\pm \rightarrow e^\pm \bar{\nu}$ )

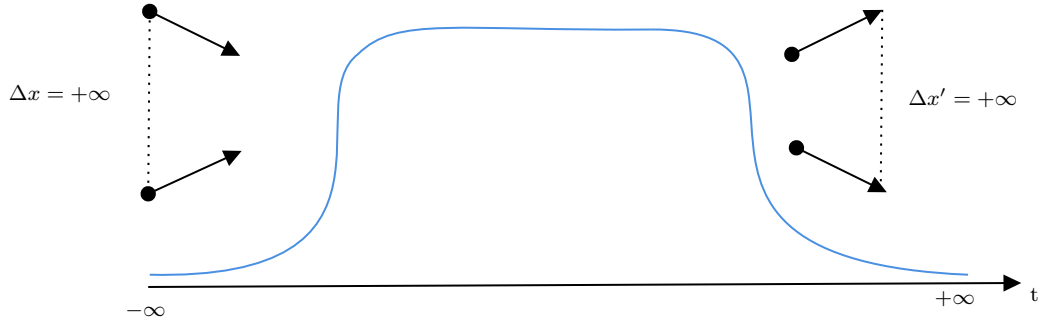
$$|\hat{q}_1| = \frac{M}{2} \left( 1 - \frac{4m^2}{M^2} \right)^{1/2}$$

$$\left( \frac{d\Gamma_{RF}}{d\Omega} \right) = \frac{1}{64\pi^2 M} \left( 1 - \frac{m^2}{M^2} \right)^{1/2} |\mathcal{M}_{fi}|^2$$

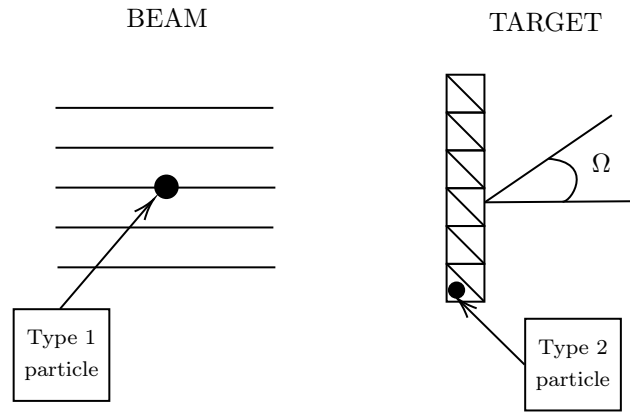
**Notes:** If we have two identical particles in the final state, the calculation of the phase is different

$$d\Phi_{(2)}^{\text{identical}} = \frac{1}{2} d\Phi_{(2)}^{\text{distinguishable}}$$

## 1.4 Cross section ( $\leftrightarrow$ scattering process)



### Scattering in the lab frame



Consider a beam of particles with mass  $m_1$ , number (assuming a uniform distribution) density  $n_1^{(0)}$  (subscript 0 is meant to stress that these are number densities in a specific frame, that with particle 2 at rest) and velocity  $v_1$  impinging on a target made of particles with mass  $m_2$  and number density  $n_2^{(0)}$  at rest.

Let  $N_s$  be the number of scattering events that place per unit volume and per unit time

$$\frac{N_t}{T} \varphi_1 N_2 \sigma = (n_1^{(0)} v_1) (n_2^{(0)} V) \sigma$$

More formally we have

$$dN_s = \sigma v_1 n_1^{(0)} n_2^{(0)} dV dV$$

with:

- (i) T: unit of time
- (ii)  $\varphi_1$ : flux of the beam  $\varphi_1 = n_1^{(0)} v_1$
- (iii)  $N_2$ : particles per unit volume in the detector  $N_2 = n_2^{(0)} V$
- (iv)  $\sigma$ : proportionality constant

Dimensional analysis shows  $[\sigma] = [L]^2$  and then  $\sigma$ , called cross section, can be interpreted as an “effective area”.