Contents

Chapter 1

Introduction

1.1 Free Fields Theories

1.1.1 Free Fields

Here we recall the expressions of quantum free fields.
Notice that real (i.e. physical) fields are never free because we have interactions, but using interaction picture we reconduct the problem in a simper one, where fields are described by free fields. This can be able with a proper choice.

$$\phi_I(x) \equiv \phi_{\text{free}}(x) \qquad \phi_I = \text{field in interacting picture}$$

Complex Scalar Field

Euler-Lagrange equation for the scalar field:

$$(\Box + m^2)\varphi = 0$$

Fourier expantion:

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} \big(e^{-ikx}a(k) + e^{ikx}b^\dagger(k)\big)_{k_0=\omega_k=\sqrt{m^2+\mathbf{k}^2}}$$

In the real case

$$\varphi^\dagger(x) = \varphi(x) \Rightarrow a(k) = b(k)$$

Dirac Spinorial Field

Euler-Lagrange equation for the Dirac field:

$$(i\partial\!\!\!/ - m)\psi = 0$$

Fourier expantion:

$$\psi(x) = \frac{1}{(2\pi)^{2/3}} \int \frac{d^3k}{\sqrt{2\omega_k}} \sum_{r=1,2} \big(e^{-ikx}u_r(k)c_r(k) + e^{ikx}v_r(k)d_r^\dagger(k)\big)_{k_0=\omega_k}$$

where $u_r(k)/v_r(k)$ are the $\varepsilon > 0/\varepsilon < 0$ spinors with helicity indicated by r.
Spinors are normalized according to

$$\begin{cases} \bar{u_r}(k)u_s(k) = 2m\delta_{rs} & \bar{u_r}(k)v_s(k) = 0 \\ \bar{v_r}(k)v_r(k) = -2m\delta_{rs} & \bar{v_r}(k)u_s(k) = 0 \end{cases}$$

Real E-M Vector Field

Euler-Lagrange equation for the real E-M field:

$$\partial_\mu F^{\mu\nu} = \Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Fourier expantion:

$$A^\mu(x) = \frac{1}{(2\pi)^{2/3}} \int \frac{d^3k}{\sqrt{2\omega_k}} \sum_{\lambda=1,2} \left(e^{-ikx}\varepsilon^\mu_{(\lambda)} a_\lambda(k) + e^{ikx}\varepsilon^{\mu\dagger}_{(\lambda)}(k) a^\dagger_\lambda(k)\right)_{k_0=\omega_k=|\mathbf{k}|}$$

where polarization vectors can be chosen as follows:

$$\varepsilon^\mu_{(1)} = (0,1,0,0) \qquad \varepsilon^\mu_{(2)} = (0,0,1,0)$$

I can complexify the field substituting $a^\dagger_\lambda$ with another operator b_λ (analogously to the scalar field).

1.1.2 Fock Space of Free Fields

See Maggiore. See 6.1

We impose the existence of vacuum state $|0\rangle$, with $\langle 0|0\rangle = 1$, then using creation operators we obtain other states $(a^\dagger)^n |0\rangle$, which are n-particles states.
In QFT we normalized states in a covariant way, instead of QM normalization $\int \psi^*\psi = 1$:

$$|1(p)\rangle \equiv (2\pi)^{3/2}\sqrt{2\omega_p}\; o^\dagger(p)\,|0\rangle \tag{1.1a}$$

$$\langle 1(p)|1(p')\rangle = (2\pi)^3(2\omega_p)\delta^3(p-p') \tag{1.1b}$$

Where the term $(2\omega_p)\delta^3(p-p')$ is covariant under Lorentz transformations.[I] Dimostrare che è covariante

1.1.3 Contraction of Fields with States

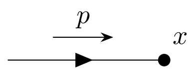
If we have a state $|e^-_s(p)\rangle$ that describes an electron with momentum p and Dirac index s, then

$$\left|e^-_s(p)\right\rangle = (2\pi)^{2/3}\sqrt{2\omega_p}\; c^\dagger_s(p)\,|0\rangle$$

Given a field ψ that describes a particle annihilation (or antiparticle creation) in the coordinate x we have

$$\begin{aligned}
\langle 0|\,\psi(x)\left|e^-_s(p)\right\rangle &= \langle 0|\,(\psi_+(x)+\psi_-(x))\left|e^-_s(p)\right\rangle \\
&= \frac{(2\pi)^{3/2}}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}} e^{-ikx}\sqrt{2\omega_p} \sum_r \langle 0|\, c_r(k) c^\dagger_s(p)\,|0\rangle\, u_r(k) \\
&= \int \mathrm{d}^3k \left(\frac{2\omega_p}{2\omega_k}\right)^{1/2} \sum_r \delta_{rs}\delta^{(3)}(\bar{p}-\bar{k}) u_r(k)\,\langle 0|0\rangle\, 0 e^{-ikx} \\
&= e^{-ikx} u_s(p)
\end{aligned}$$

where we used $\langle 0|\, c_r(k)c^\dagger_s(p)\,|0\rangle = \langle 0|\,\{c_r(k), c^\dagger_s(p)\}\,|0\rangle = \delta_{rs}\;\delta^3(\bar{p}-\bar{k})$.
The factor e^{-ikx} is required for the $\delta^{(4)}$ conservation. We see that the relativistic normalization leads to Feyman rules without normalization factors:

$$e^{-ipx}u_s(p) =$$ 

[I]We indicated with $o^\dagger$ a generic creation operator.

1.2 S-Matrix

1.2.1 Interaction Picture

In the interaction picture, with $H = H_0 + H_{\text{int}}$, where H_{int} is the interaction hamiltonian

(i) Fields ϕ_I evolves like in the free theory (respect to H_0)

(ii) State evolves with the following evolution operator

$$U_I(t,t_0) \equiv e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t_0}$$
$$|\alpha, t\rangle = U_I(t-t_0)\,|\alpha, t_0\rangle \qquad i\partial_t U_I(t-t_0) = H_I^{\text{int}}(t) U_I(t,t_0)$$

(iii) Operators in interaction picture are (let $O_H(t)$ be a generic operator in Heisenberg picture and $O_I(t)$ the same operator in Interaction picture)

$$O_I(t) = e^{iH_0 t} e^{-iHt}\; O_H(t) e^{iHt} e^{-iH_0 t}$$

Notice that, in general

$$[H_I(t), H^0] \neq 0 \neq [H_I^{\text{int}}, H^0]$$

and if $t \neq t'$ we also have

$$[H_I^{\text{int}}(t), H_I^{\text{int}}(t')] \neq 0$$

1.2.2 S-matrix and States Evolution

The S**-matrix** is a well defined operator defined as

$$S = \lim_{\substack{t_0 \to -\infty \\ t \to +\infty}} U_I(t,t_0)$$

We compute S by perturbation obtaining

$$\begin{aligned} S &= T\left(\exp\left(-i\int \mathrm{d}^4x \mathcal{H}_I^{\text{int}}(x)\right)\right) \\ &= \sum_{n=0}^{+\infty} \frac{(-i)^n}{n!} \int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n\, T\left[\mathcal{H}_I^{\text{int}}(x_1) \ldots \mathcal{H}_I^{\text{int}}(x_n)\right] \end{aligned}$$

S matrix has some relevant properties

(i) Unitariety (since Hamiltonian is hermitian)

(ii) Behaves as a scalar under Lorentz transformations, and then is an invariant quantity (notice that in general $\mathcal{H}_I^{\text{int}}$ is not invariant).
In the case of $\mathcal{H}_I^{\text{int}}$ invariant (for example if, as in many theories, $\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$, for example in QED) is easy to prove that S is invariant, since all n-th derivatives of $\exp\left(\int \mathcal{H}\right)$ are invariant, and so also S is invariant

1.2.3 S-matrix and transition probabilities

Assume that there's no interaction for $x, t \to \pm\infty$.
Consider canonically normalized (CN) states $|\psi| = 1$:

$$|\psi_i\rangle_{CN} \equiv |\psi(-\infty)\rangle_{CN} \qquad |\psi(+\infty)\rangle_{CN} \equiv S\,|\psi_i\rangle_{CN}$$

(both are free particle states). Elements of S are in the form

$$S_{fi}^{CN} = {}_{CN}\langle\psi_f| S\,|\psi_i\rangle_{CN}$$

This leads to a probabilistic interpretation of S-matrix elements.

$$\boxed{\textit{The squared amplitude } |S_{fi}^{CN}|^2 \textit{ is the transition probability of } |\psi_i\rangle_{CN} \textit{ into } |\psi_f\rangle_{CN}.}$$

Notice that the requirement $\sum_f |S_{fi}^{CN}| = 1$ is satisfied automatically. In the case of covariant normalization

$$\langle 1(p)|1(p')\rangle = (2\pi)^3(2\omega_p)\delta^3(\mathbf{p}-\mathbf{p}')$$

we have the following relation between matrix elements

$$S_{fi}^{CN} = {}_{CN}\langle\psi_f|S\,|\psi_i\rangle_{CN} = \frac{\langle\psi_f|\,S\,|\psi_i\rangle}{\|\psi_i\|\,\|\psi_f\|} = \frac{S_{fi}}{\|\psi_i\|\,\|\psi_f\|}$$

We can define the **Feynman Amplitude** $\mathcal{M}_{fi}$ as

$$S_{fi} = (2\pi)^4\delta^4(p_i - p_f)\mathcal{M}_{fi}$$

and it can be obtained directly starting from Feynman rules (used with the covariant normalization described in Sec. 1.1.3)

1.3 Discrete space normalization

Usually, in order to make arguments cleaner, or to avoid problems with divergent terms in calculations, we first consider a system in a cubic box with spatial valume $V = L^3$. At the end of computations V will be sent to infinity. Sometimes we will do something similar also for time.
For a discrete space we must use a different normalization.

In a box, momentum of a particle is quantized

$$p_i = \left(\frac{2\pi}{L}\right)n_i \qquad n_i \in \mathbb{Z}$$

and we must adopt the following rule for integrals

$$\int \mathrm{d}^3p\, f(\mathbf{p}) \quad \to \quad \sum_{\mathbf{n}}\left(\frac{2\pi}{L}\right)f_{\mathbf{n}} \qquad \mathbf{n} = (n_1, n_2, n_3)$$

We must adopt also the following

$$\delta^3(\mathbf{p}-\mathbf{p}') \quad \to \quad \left(\frac{L}{2\pi}\right)^3\delta_{\mathbf{n}\mathbf{n}'}$$

in this way

$$\int \mathrm{d}^3p\,\delta^3(\mathbf{p}-\mathbf{p}') = 1 \quad \to \quad \sum_{\mathbf{n}}\left(\frac{2\pi}{L}\right)^3\left(\frac{L}{2\pi}\right)^3\delta_{\mathbf{n}\mathbf{n}'} = 1$$

In particular

$$\delta^3(0) \to \left(\frac{L}{2\pi}\right)^3$$

When we consider also a finite amount of time we have

$$\delta^4(0) \to \left(\frac{L}{2\pi}\right)\left(\frac{T}{2\pi}\right)$$

Using (1.1b), we have

$$\langle 1(p)|1(p)\rangle = (2\pi)^3 2\omega_p\delta^3(0) = 2\omega_p V$$

so normalization of states (1.1a) becomes ($o^\dagger$ is a general creation operator)

$$|1(p)\rangle = \sqrt{2\omega_p V}\,|1(p)\rangle_{CN}$$

1.3.1 S-Matrix in discrete space

Using the latter equation, S_{fi}^{CN} reads

$$
\begin{aligned}
S_{fi}^{CN} &= \prod_{j=1}^{n_i}\left(\frac{1}{2\omega_j V}\right)^{1/2}\prod_{l=1}^{n_f}\left(\frac{1}{2\omega_l V}\right)^{1/2} S_{fi} \\
&= (2\pi)^4\delta^4(p_i - p_f)\left\{\prod_{j=1}^{n_i}\left(\frac{1}{2\omega_j V}\right)^{1/2}\prod_{l=1}^{n_f}\left(\frac{1}{2\omega_l V}\right)^{1/2}\mathcal{M}_{fi}\right\} \\
&= (2\pi)^4\delta^4(p_i - p_f)M_{fi}^{CN}
\end{aligned}
$$

In the second passage we use the definition of $\mathcal{M}_{fi}$, omitting the quantization of δ^4. M_{fi}^{CN} is the **canonically normalized Feynman amplitude**.

Chapter 2

The S-matrix and physical observables

2.1 Decay Rate

Consider the case in which the initial state is a single particle and the final state is given by n particles. We are therefore considering a decay process. Assume for the moment that particles are indistinguishable.

The rules of quantum mechanics tell us that the probability for this process is obtained by taking the squared modulus of the amplitude and summing over all possible final states

$$\begin{aligned}
\left|S_{fi}^{CN}\right|^2 &= \left|(2\pi)^4\delta^4(p-p')M_{fi}^{CN}\right|^2 \\
&= (2\pi)^4\delta^4(p-p')(VT)\left|M_{fi}^{CN}\right|^2 \\
&= (2\pi)^4\delta^4(p-p')(VT)\frac{1}{2\omega_i nV}\prod_{l=1}^{n_f}\left(\frac{1}{2\omega_l V}\right)|\mathcal{M}_{fi}|^2
\end{aligned}$$

Note: $\left(\delta^4(p-p')\right)^2 = \delta^4(p-p')\delta^4(p-p') = \delta^4(p-p')\delta^4(0) = \delta^4(p-p')\frac{VT}{(2\pi)^4}$
We use the final space and time in order to remove divergent terms during calculation

We must now sum this expression over all final states. Since we are working in a finite volume V, this is the sum over the possible discrete values of the momenta of the final particles .
Since $p_i = (2\pi/L)n_i$, we have $dn_i = (L/2\pi)\mathrm{d}p_i$ and $\mathrm{d}^3n_i = (V/(2\pi)^3)\mathrm{d}^3p$ where d^3n_i is the infinitesimal phase space related to a final state in which the i-th particle has momentum between p_i and $p_i + \mathrm{d}p_i$
Let $\mathrm{d}\omega$ be the probability for a decay in which in the final state the i-th particle has momentum between p_i and $p_i\mathrm{d}p_i$

$$\mathrm{d}\omega = \left|S_{fi}^{SN}\right|^2\prod_{l=1}^{n_f}\left(\frac{V\mathrm{d}^3p_l}{(2\pi)^3}\right)$$

This is the probability that the decay takes place in any time between $-T/2$ and $T/2$. We are more interested in the differential decay rate $\mathrm{d}\Gamma_{fi}$, which is the decay probability per unit of time:

$$\mathrm{d}\Gamma_{fi} = \frac{\mathrm{d}\omega}{T} = (2\pi)^4\delta^4(p_i - p_f)\frac{|\mathcal{M}_{fi}|^2}{2\omega_{p_{in}}}\prod_{l=1}^{n_f}\frac{\mathrm{d}^3p_l}{(2\pi)^3 2\omega_l}$$

Notes:

- $\mathrm{d}\Gamma_{fi}$ = differential decay rate
- p_f = sum over final momenta
- $\omega_{p_{in}}$ = initial energy
- $|\mathcal{M}_{fi}|^2$ = Feynman amplitude of the process (depends on final momenta p_i)

It is useful to define the **(differential) n-body phase space** as

$$\mathrm{d}\Phi_{(n_f)} = (2\pi)^4\delta^4(p_i - p_f)\prod_{l=1}^{n_f}\frac{\mathrm{d}^3 p_l}{(2\pi)^3 2\omega_l}$$

Therefore the differential decay rate can be written as

$$\mathrm{d}\Gamma_{fi} = \frac{1}{2\omega_{p_{in}}}|\mathcal{M}_{fi}|^2\mathrm{d}\Phi_{(n_f)}$$

The decay rate is defined as

$$\Gamma_{fi} = \int \mathrm{d}\Gamma_{fi} \qquad \rightarrow \text{integration over all possible final momenta}$$

and its meaning is $\Gamma \equiv$ trans. probability $\times$ unit of time $\times$ init. particle
Notice that if n of the final particles are identical, configurations that differ by a permutation are not distinct and therefore the phase space is reduced by a factor $1/h!$
If we have a system of N(0) particles, the time evolution of the number of particles N(t) is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\Gamma N \quad \Rightarrow \quad N(t) = N(0)e^{-\Gamma t}$$

Notice that decay rate is not invariant

$$[\Gamma] = [E] = \frac{1}{T} \qquad \text{(in natural units)}$$

If we define the **lifetime** as $\tau = 1/\Gamma \Rightarrow N(t) = N(0)\exp(-t/\tau)$ this changes under Lorentz tfm. If we consider two reference frames o and o'

$$\Gamma' = \frac{\Gamma}{\gamma} < \Gamma \qquad \tau' = \gamma\tau > \tau$$

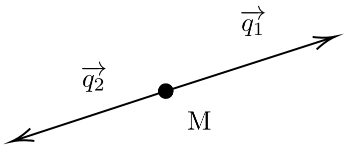
$\gamma = (1-v)^{-1/2}$, where v is the speed of o' in o in natural units, $\gamma > 1$. Therefore a article in a moving frame has a longer lifetime then in the rest frame

Example 1: muon lifetime

For a muon in the rest frame $\tau_\mu^{RF} = 2.2 \times 10^{-6}s$, but if we observe it in the lab frame
$\tau_\mu^{LAB} = \gamma\tau_\mu^{RF} \simeq 2 \times 10^{-5}s$ since $E_\mu = 1$ GeV, $m_\mu = 0.1$ GeV
$\Rightarrow \gamma = E_\mu/m_\mu \simeq 10$

Example 2: $1 \rightarrow 2$ decay

Consider the decay of a particle of a mass M into two particles of masses m_1, m_2. Since the phase space in Lorentz invariant, we can compute it in the frame that we prefer. We use the rest frame for the initial particle.



We don't impose a priori conservation of momentum since it's imposed by the delta function.

$$p = (M, 0) \qquad q_1 = (\omega_1, \mathbf{q_1}) \qquad q_2 = (\omega_2, q_2)$$
$$\mathrm{d}\Phi_{(2)} = (2\pi)^4 \delta^4 \underbrace{(P_i - P_f)}_{=p-q_1-q_2} \frac{\mathrm{d}^3 q_1}{(2\pi)^3 2\omega_1} \frac{\mathrm{d}^3 q_2}{(2\pi)^3 2\omega_2}$$

I have 6 integration parameters, 4 constraint given by δ^4, so I have 2 independent variables. Integrating over $\mathrm{d}^3 q_2$

$$\mathrm{d}\Phi'_{(2)} = \int \mathrm{d}\Phi_{(2)} = \frac{1}{(2\pi)^2} \delta(M - \omega_1 - \omega_2) \frac{1}{4\omega_1\omega_2} \mathrm{d}^3 q_1$$

in this way, the condition $\mathbf{q_2} = \mathbf{q_1}$ vanish. We have to impose it again when we calculate $\mathrm{d}\Gamma$ (we omit this detail)
Usually the 4-th non independent parameter is eliminated by integration over modulus of q_1, leaving free 2 parameters for the angles. $\mathrm{d}^3 q_1 \to \mathbf{q_1}^2 \mathrm{d}|\mathbf{q_2}| \mathrm{d}\Omega_1$.

Notice that $M - \omega_1 - \omega_2 = M - \sqrt{\mathbf{q_1}^2 + m_1^2} - \sqrt{\mathbf{q_2} + m_2^2}$ and then the δ implies

$$\hat{q_1}^2 = \frac{1}{2M} \left(M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2 \right)^{1/2}$$

$|\mathbf{q_1}|$ is the only zero of $f(|\hat{q_1}|) = M - \omega_1 - \omega_2$.
We also have

$$|f'(|\hat{q_1}|)| = \frac{\partial\omega_1}{\partial|\mathbf{q_1}|} + \frac{\partial\omega_2}{\partial|\mathbf{q_1}|} = |\hat{q_1}| \left(\frac{\omega_1 + \omega_2}{\omega_1\omega_2} \right)$$

Using

$$\delta\big(f(x)\big) = \sum_{x_0 = \text{zero of f(x)}} \frac{\delta(x - x_0)}{|f'(x_0)|}$$

and performing integration over $\mathrm{d}|\mathbf{q_1}|$ we obtain

$$\mathrm{d}\Phi''_{(2)} = \int \mathrm{d}\Phi'_{(2)} = \frac{1}{16\pi^2} \frac{|\hat{q_1}|}{M} \mathrm{d}\Omega_1$$

Using this result we obtain the $1 \to 2$ decay rate in function of the solid angle (in the rest frame)

$$\left(\frac{\mathrm{d}\Gamma_{RF}}{\mathrm{d}\Omega} \right) = \frac{1}{64\pi^4 M^3} \left[M^4 - 2M^2(m_1^2 + m_2^2) + (m_1 - m_2^2)^2 \right]^{1/2} |\mathcal{M}_{RF}|^2$$

In a general frame we can easily obtain an analogous formula, just consider $\mathrm{d}\Gamma = 1/(2\omega_i n)|\mathcal{M}_{fi}|^2 \mathrm{d}\Phi_{(nf)}$ in a general frame. remember that $\mathrm{d}I_{(nf)}$ is invariant

We have 2 important limit cases:

(A) If $m_1 = m_2 = m$ (for example $Z \to e^+e^-$)

$$|\hat{q_1}| = \frac{M}{2} \left(1 - \frac{4m^2}{M^2} \right)^{1/2}$$
$$\left(\frac{\mathrm{d}\Gamma_{RF}}{\mathrm{d}\Omega} \right) = \frac{1}{64\pi^2 M} \left(1 - \frac{4m^2}{M^2} \right)^{1/2} |\mathcal{M}_{fi}|^2$$

(B) If $m_1 = m, m_2 = 0$ (for example $W^\pm \to e^\pm \overset{(-)}{\nu}$)

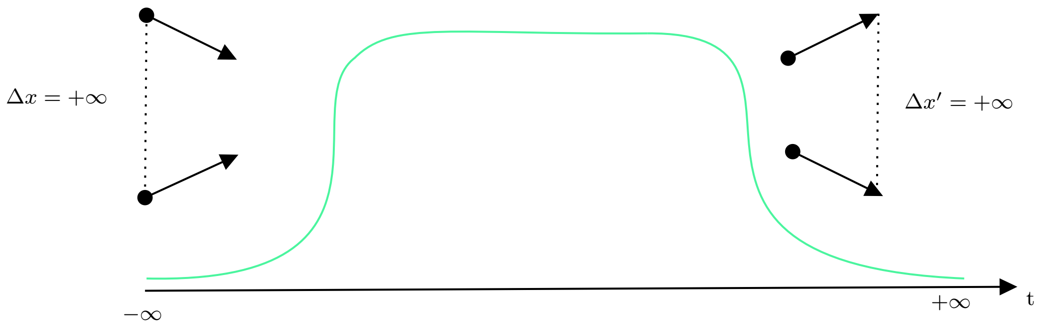
$$|\hat{q}_1| = \frac{M}{2}\left(1 - \frac{4m^2}{M^2}\right)^{1/2}$$

$$\left(\frac{\mathrm{d}\Gamma_{RF}}{\mathrm{d}\Omega}\right) = \frac{1}{64\pi^2 M}\left(1 - \frac{m^2}{M^2}\right)^{1/2} |\mathcal{M}_{fi}|^2$$

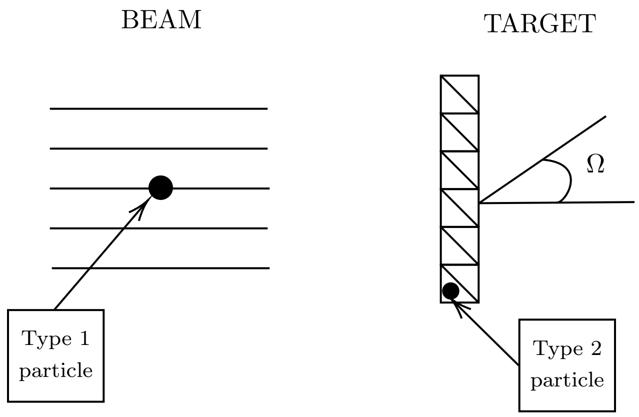
Notes: If we have two identical particles in the final state, the calculation of the phase is different

$$\mathrm{d}\Phi_{(2)}^{\text{identical}} = \frac{1}{2}\mathrm{d}\Phi_{(2)}^{\text{distinguishable}}$$

2.2 Cross section ($\leftrightarrow$ scattering process)



Scattering in the lab frame



Consider a beam of particles with mass m_1, number (assuming a uniform distribution) density $n_1^{(0)}$ (subscript 0 is meant to stress that these are number densities in a specific frame, that with particle 2 at rest) and velocity v_1 inpining on a target made of particles with mass m_2 and number density $n_2^{(0)}$ at rest.
Let N_s be the number of scattering events that place per unit volume and per unit time

$$\frac{N_t}{T}\varphi_1 N_2 \sigma = \left(n_1^{(0)} v_1\right)\left(n_2^{(0)} V\right)\sigma$$

More formally we have

$$\mathrm{d}N_s = \sigma v_1 n_1^{(0)} n_2^{(0)}\ \mathrm{d}V\mathrm{d}V$$

with:

- T: unit of time

- φ_1: flux of the beam $\varphi_1 = n_1^{(0)} v_1$
- N_2: particles per unit volume in the detector $N_2 = n_2^{(0)} V$
- σ: proportionality constant

Dimensional analysis shows $[\sigma] = [L]^2$ and then σ, called cross section, can be interpreted as an "effective area"
Consider the case where just one particle collides with another particle (2 particles scattering). We can obtain this condition imposing $n_{1,2}^{(0)} = 1/V$, in this way the scattering per unit of volume is the same as two particles scattering
Notice that this situation is the more physical. The probability of 3 particles collision in the same d^3x and $\mathrm{d}t$ is almost 0

If $\mathrm{d}\omega$ is, again, the probability for a process in which in the final state the i-th particle has momentum between p_i and $p_i + \mathrm{d}p_i$

$$\mathrm{d}\omega = (2\pi)^4 \delta^4(p_i - p_f) VT \left(\frac{1}{2\omega_1 V}\right)\left(\frac{1}{2\omega_2 V}\right) \prod_{l=1}^{n_f} \left(\frac{\mathrm{d}^3 pl}{(2\pi)^3 2\omega_l}\right) |\mathcal{M}_{fi}|^2$$

we can define the **differential cross section** (in the lab frame) $\mathbf{d}\boldsymbol{\sigma}$ as

$$\begin{aligned}
(\mathrm{d}\sigma)_{\mathrm{LAB}} &= \frac{\mathrm{d}\omega}{n_1^{(0)} n_2^{(0)} v_1 VT} \\
&= \frac{V}{Tv_1}\mathrm{d}\omega \\
&= (2\pi)^4 \delta^4(p_i - p_f) \prod_{l_1}^{n_f} \left(\frac{\mathrm{d}^3 p_l}{(2pi)^3 2\omega_l}\right) \frac{|\mathcal{M}_{fi}|^2}{4\omega_1\omega_2 v_1} \qquad \text{with } \omega_2 = m_2
\end{aligned}$$

The result is then

$$(\mathrm{d}\sigma)_{\mathrm{LAB}} = \frac{1}{4\omega_1 v_1 m_2} |\mathcal{M}_{fi}|^2 \ \mathrm{d}\Phi_{(nf)}$$

All quantities refers to the rest frame for particle 2 ($\omega_1 = \omega_1^{(0)}, v_1 = v_1^{(0)}$)
In order to obtain a covariant relation for $\mathrm{d}\sigma$, we notice that the only non-covariant factor in the latter relation is $(I_{12})_{\mathrm{LAB}} = \omega_1^{(0)} m_2$. This factor can be substituted with a covariant one

$$I_{12} = \left[(p_1 p_2)^2 - m_1^2 m_2^2\right]^{1/2}$$

called **covariant flux factor**. This factor is obviously covariant, we just have to prove that in the lab frame it coincides with $(I_{12})_{\mathrm{LAB}}$. In the lab frame we have $p_1 = (\omega_1, \mathbf{p_1}^{(0)})$ and $p_2 = (m_2, \mathbf{0})$, so the previous formula becomes

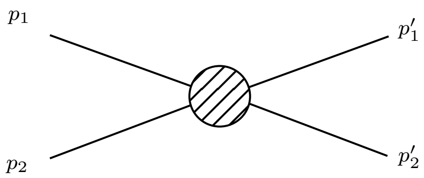
$$\left[m_2^2(\omega_1^{(0)})^2 - m_1 m_2\right]^{1/2} = m_2\left((\omega_1^{(0)})^2 - p_1^2\right)^{1/2} = m_2\left|\mathbf{p_1^{(0)}}\right| = m_2\omega_1^{(0)} v_1^{(0)}$$

So the final result is

$$\mathrm{d}\sigma = \frac{\left|\mathcal{M}_{fi}^2\right|}{4I_{12}} \mathrm{d}\ \Phi_{(nf)}$$

Example 3: $2 \to 2$ scattering

Consider a scattering process $2 \to 2$. We consider an initial state with two particles with masses m_1, m_2 and four momenta p_1, p_2 and a final state with masses m'_1, m'_2 and four momenta p'_1, p'_2



with

$$p_1 = (\omega_1, \mathbf{p_1}) \qquad p_2 = (\omega_2, \mathbf{p_2}) \qquad p'_1 = (\omega'_1, \mathbf{p'_1}) \qquad p'_2 = (\omega'_2, \mathbf{p'_2})$$

With a procedure identical to $1 \to 1$ decay (imposing also $\mathbf{p'_2} = \mathbf{p_1} + \mathbf{p_2} - \mathbf{p'_1}$ when we calculate $\mathrm{d}\sigma$), we obtain

$$\mathrm{d}\Phi'_{(2)} = \frac{1}{(2\pi)^2} \frac{1}{4\omega'_1\omega'_2} \delta(\omega_1 + \omega_2 - \omega'_1 - \omega'_2)\ \mathrm{d}^3 p'_1$$

In order to integrate over $\mathrm{d}|\mathbf{p'_1}|$ it is useful to introduce Mandelstam variables s, t and u

$$s = (p1 + p2)^2 \qquad t = (p1 - p'_1)^2 \qquad u = (p_1 - p'_2)^2$$

These variables are clearly Lorentz invariants, and satisfy (using $p_1 + p_2 = p'_1 + p'_2$) the relation

$$s + t + u = m_1^2 + m_2^2 + (m'_1)^2 + (m'_2)^2$$

It is useful to work in the center of mass frame, where the incoming particles have $p_1 = (\omega_1, \mathbf{p})$ and $p_2 = (\omega_2, -\mathbf{p})$. Computing s in the CM frame we obtain $s = (\omega_1 + \omega_2)^2 = (\omega'_1 + \omega'_2)^2$ and then

$$p_1 + p_2 = (\sqrt{s}, 0)$$
$$\delta(\omega_1 + \omega_2 - \omega'_1 - \omega'_2) = \delta(\sqrt{s} - \omega'_1 - \omega'_2)$$

With a procedure identical to the one used for $1 \to 2$ decay ($M \leftrightarrow \sqrt{s}$) we obtain

$$\left(\mathrm{d}\Phi''_{(2)}\right)_{CM} = \frac{1}{16\pi^2} \frac{\left|\widehat{p'_1}\right|_{CM}}{\sqrt{s}}\ \mathrm{d}\Omega'_1$$
$$\left|\widehat{p'_1}\right|_{CM} = \frac{1}{2\sqrt{s}} \left[s^2 + (m_1'^2 + m_2'^2)^2 - 2s(m_1'^2 + m_2'^2)\right]^{1/2}$$

The covariant flux factor in CM frame reads

$$(I_{12})_{CM} = \left[\mathbf{p}^2(\omega_1 + \omega_2)^2\right]^{1/2} = |\mathbf{p}|\sqrt{s}$$

So we obtain the final result

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega'_1}\right)_{CM} = \frac{1}{64\pi^2 s} \frac{\left|\widehat{p'_1}\right|_{CM}}{|\mathbf{p}|}\ |\mathcal{M}_{fi}|^2_{CM}$$
$$= \frac{1}{128\pi^2 s^{3/2}} \frac{1}{|\mathbf{p}|} \left[s^2 + (m_1'^2 - m_2'^2)^2 - 2s(m_1'^2 + m_2'^2)\right]^{1/2} |\mathcal{M}_{fi}|^2_{CM}$$

We have two important limit cases

(A) If $m_1 = m_1', m_2 = m_2'$, i. e. elastic scattering (for example $e^-\mu^\dagger \to e^-\mu^\dagger$)

$$\left.\begin{array}{l} |\mathbf{p_1}| = |\mathbf{p_1'}|, \omega_1 = \omega_1' \\ |\mathbf{p_2}| = |\mathbf{p_2'}|, \omega_2 = \omega_2' \end{array}\right\} \quad \left|\hat{p_1'}\right|_{CM} = |\mathbf{p}|$$

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_1'}\right)_{CM} = \frac{1}{64\pi^2}\frac{|\mathcal{M}_{fi}|^2}{s}$$

(B) If $m_1 = m_2 \simeq 0, m_1' = m_2' = M$ (for example $e^-e^+ \to \mu^-\mu^+$)

$$|\mathbf{p_1}| = |\mathbf{p}|, \omega_1 = \frac{\sqrt{s}}{2}$$

$$\left|\hat{p_1'}\right| = \frac{\sqrt{s}}{2}\left(1 - \frac{1}{4M^2}s\right)^{1/2} \tag{2.1}$$

$4M^2/s$ processes with $s < 4M^2$ are unphysical

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_1'}\right)_{CM} = \frac{1}{64\pi^2}\left(1 - \frac{4M^2}{s}\right)\frac{|\mathcal{M}_{fi}|^2}{s}$$

The previous formulas are valid also for particles with spin, if the initial and final spin states are known; in this case the initial state has the form $|i\rangle = |\mathbf{p_1}, \mathbf{s_1}; \ldots; \mathbf{p_n}, \mathbf{s_n}\rangle$, and similarly for the final state.
However, experimentally is more common that we do not know the initial spin configuration and we accept in the detector all final spin configurations; in this case, to compare with experiment, we must

(i) Considering all (equally present) polarization of initial state, i.e. *average* over the initial spin configuration

(ii) Considering all final polarizations, i.e. *sum* over all possible final configurations

Defining the **unpolarized Feynman amplitude** $\overline{|\mathcal{M}_{fi}|}$ as

$$\overline{|\mathcal{M}_{fi}|^2} = \frac{1}{\text{n initial polarizations}} = \sum_{\substack{\text{initial}\\\text{spins}}}\sum_{\substack{\text{final}\\\text{spins}}} |\mathcal{M}_{fi}|^2$$

we just hace to substitute $|\mathcal{M}_{fi}|^2 \to \overline{|\mathcal{M}_{fi}|^2}$ in all previous formulas
For example, in $2 \to 2$ scattering

$$\frac{1}{\text{n init pol}} = \frac{1}{(2s_a+1)(2s_b+1)} \qquad \text{where } s_a = \text{spin of particle 1; and } s_b = \text{spin of particle 2}$$

Chapter 3

QED processes at lowest order

3.1 The QED Lagrangian and its Symmetries

Mandl, sec 11.1 - Maggiore, sec 7.1

Quantum electrodynamic (QED) describes the interactions between (or any other charged spin 1/2 particle) and photons. QED is described by the lagrangian

$$\mathcal{L}_{\text{QED}} = \underbrace{\overline{\psi}(i\partial\!\!\!/ - m)\psi}_{\mathcal{L}_D^{(0)}} \underbrace{- \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\mathcal{L}_{EM}} \underbrace{- qA_n\overline{\psi}\gamma^\mu\psi}_{\mathcal{L}_{int}} \underbrace{- \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2}_{\mathcal{L}_{GF}}$$

(i) $\mathcal{L}_D^{(0)}$ is the langrangian for the free Dirac field

(ii) $\mathcal{L}_{EM}$ is the lagrangian for the free EM field. In order to quatize the E-n field we have to add the term $\mathcal{L}_{GF}$ (gauge fixing). For other purposes this term can be omitted. Usually the choice $\xi = 1$, called Feynman gauge, is the simplest choice for quantization

(iii) $\mathcal{L}_{int}$ describes the interaction between Dirac field and EM-field. Notice that the term $\mathcal{L}_D = \mathcal{L}_D^{(0)} + \mathcal{L}_{int}$ can be obtained from $\mathcal{L}_D^{(0)}$ with the "minimal substitution" $\partial_\mu \to \partial_\mu + iqA_\mu = D_\mu$, i.e. using covariant derivative D_μ instead of ∂_μ in the dirac lagrangian.
Notice that $\mathcal{L}_D$ exhibits local symmetry, while $\mathcal{L}_D^{(0)}$ doesn't

Besides Lorentz invariance, the QED exhibits following symmetries:

- **Global U(1) symmetry**

$$\begin{cases} \psi(x) \to \psi'(x) = e^{i\alpha}\psi(x) & \alpha \in \mathbb{R} \\ A^\mu(x) \to A'^\mu(x) = A^\mu(x) \end{cases}$$

There is therefore an associated conserved Noether current

$$j^\mu = q\overline{\psi}\gamma^\mu\psi \quad \to \quad \partial_\mu j^\mu = 0$$

and a U(1) charge which is conserved by the EM interaction

$$Q = q\int \mathrm{d}^3x\ \psi^\dagger\psi \qquad \frac{\mathrm{d}Q}{\mathrm{d}t} = 0$$

- **Local U(1) symmetry** (gauge symmetry)

$$\begin{cases} \psi(x) \to \psi'(x) = e^{iq\alpha(x)}\psi(x) \\ A^\mu \to A'^\mu(x) = A^\mu(x) - iq\partial^\mu\alpha(x) \end{cases}$$

notice that the global U(1) symmetry is a sequence of the local U(1) symmetry, taking $\alpha(x)$ constant)

The covariant derivative of ψ $D_\mu\psi$ behaves as a spinor (remember that D_μ transforms as a vector):

$$\begin{aligned} D_\mu\psi \to (D_\mu\psi)' = D'_\mu\psi' &= (D_\mu - iq\partial_\mu\alpha)(e^{iq\alpha(x)}\psi(x)) \\ &= (\partial_\mu + iqA_\mu - iq\partial_\mu\alpha)(e^{iq\alpha}\psi) \\ &= e^{iq\alpha}(\partial_\mu + iqA_\mu)\psi \\ &= e^{iq\alpha(x)}(D_\mu\psi) \end{aligned}$$

This implies that $\mathcal{L}_D$ is invariant. Since $\mathcal{L}_{EM}$ is invariant, the full lagrangian is invariant

3.2 Flavors in QED and the SU(3) Flavor Global Symmetry

QED describes interactions of the photon field with several kind of leptons, not only electron and positrons. Particles that differs only by their mass are called **flavours**. The next table describes leptons

in QED. There are two families of leptons that differs by their charge. We indicate with (-) (minus) particles with negative charge, and with (+) (plus) particles with positive charge (antiparticles)
Flavours in QED

Leptons[I]	e	μ	τ	ν_e	ν_μ	ν_τ
q	-1	-1	-1	0	0	0
$m[MeV]$	0.5	105	1777	$\simeq 0$	$\simeq 0$	$\simeq 0$

The dirac lagrangian $\mathcal{L}_D$ can be modified in order to consider all possible leptons

$$\mathcal{L}_D = \sum_{i=1}^{n} \bar{\psi}_i(i\not{D} - m_i)\psi_i \simeq \sum_{i=1}^{n_l} \bar{\psi}_i(i\not{D} - m_i)\psi_i + \sum_{j=1}^{n_n} \bar{\psi}_i(i\not{\partial})\psi_i$$

with n: number of leptons, n_l: number of electrically charged particles, n_n: number of neutrinos.
In the last term the interaction term vanishes because of q=0
If we adopt a matrix notation

$$\Psi_C = \begin{pmatrix} \psi_e \\ \psi_\mu \\ \psi_\tau \end{pmatrix} \qquad \Psi_N = \begin{pmatrix} \psi_{\nu_e} \\ \psi_{\nu_\mu} \\ \psi_{\nu_\tau} \end{pmatrix}$$

$$\bar{\Psi}_C = (\bar{\psi}_e, \bar{\psi}_\mu, \bar{\psi}_\tau) \qquad \bar{\Psi}_N = (\bar{\psi}_{\nu_e}, \bar{\psi}_{\nu_\mu}, \bar{\psi}_{\nu_\tau})$$

The subscript C stands for “charged”, and N for “neutral”
We can define following matrices

$$M_C = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \qquad Q_C = (-1)\mathbb{1}_{3\times3} \qquad M_N \simeq Q_N = \emptyset_{3\times3}$$

and the generalization of the covariant derivative $D_\mu = \partial_\mu \cdot \mathbb{1}_{3\times3} + iA_\mu Q_C$ we obtain

$$\mathcal{L}_D = \underbrace{\bar{\Psi}_C(i\not{D} - M_C)\Psi_C}_{\mathcal{L}_C} + \underbrace{\bar{\Psi}_N(i\not{\partial}\mathbb{1}_{3\times3})\Psi_N}_{\mathcal{L}_N}$$

The term $\mathcal{L}_C$ and $\mathcal{L}_N$ are respectively the **Charged Sector** and the **Neutral Sector** of $\mathcal{L}_D$
The basis defined by Ψ_C and Ψ_N is called **physical basis**, since physical particles are identified by their mass.

[I] Neutrinos admit anly global U(l) symmetry
Neutrino masses are in the order $m_\nu \approx 10^{-6}me \leq 1eV$

3.2.1 Global Symmetry of Neutral and Charged Sector

Let's consider a U(3) transformation

$$\Psi(x) \to \Psi'(x) = U\Psi(x) \qquad U^\dagger U = \mathbb{1}_{3\times 3}$$

Spinors and vector are left invariant
The neutral sector is left invariant under U(3) tfm

$$\mathcal{L}_N \to \mathcal{L}'_N = \bar{\Psi}'_N(i\not{\partial}\mathbb{1}_{3\times 3})\Psi'_N = \bar{\Psi}_N U^\dagger(i\not{\partial}\mathbb{1}_{3\times 3})U\Psi_N = \mathcal{L}_N$$

The charged sector is not invariant because of the mass term

$$\mathcal{L} \to \mathcal{L}'_C = \bar{\Psi}_C(i\not{D} - U^\dagger M_C U)\Psi_C \neq \mathcal{L}_C \quad \text{in general}$$

In order to obtain global symmetries of the flavor QED lagrangian, we search the subgroup of U(3) made by matrices $U-g$ that satisfies

$$U_g^\dagger M_C U_g = M_c \qquad U_g \in U(3) \tag{3.1}$$

We can prove that $U(3) \overset{isomorphism}{\simeq} U(1)\times SU(3)$, with

(i) $|\det(\mathrm{U}(3)| = 1$

(ii) $\det(\mathrm{U}(1)) = e^{i\theta}$ [II]

(iii) $\det(\mathrm{SU}(3)) = 1$ [III]

Since generators of SU(3) are Gell-Mann matrices λ_a (for a $= 1,\ldots,8$), generators of U(3) are

$$\mathbb{1}\times\{\lambda_1,\ldots,\lambda_8\}$$

we can also prove that equation 3.1 is satisfied only by diagonal matrix
Diagonal generators of SU(3) are

$$\lambda_0 = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \qquad \lambda_3 = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix} \qquad \lambda_8 = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$$

Up to a phase, matrices U_g are in the form

$$U_g = e^{i\alpha_0\lambda_0}\; e^{i\alpha_3\lambda_3}\; e^{i\alpha_8\lambda_8}$$

For i = 0, 3 ,8 we have $[\lambda_i, M_c] = 0$ and then the equation 3.1 is satisfied: if we take $\alpha_i \ll 1$

$$(1 - i\alpha_i\lambda_i)M_C(1 + i\alpha_i\lambda_i) = (M_C - i\alpha_i[\lambda_i, M_C] + o(\alpha_i^2)) \simeq M_C$$

We then obtained that the global group of symmetry is generated by the algebra

$$\mathcal{G} = \{\lambda_0, \lambda_3, \lambda_8\} = U(1)^3 \subset U(3)$$

I define the following basic of $\mathcal{G}$:

$$\lambda_e = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} \qquad \lambda_\mu = \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix} \qquad \lambda_\tau = \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$$

These matrices generates phase tfm for each kind of leptons (λ_e generates phase tfm for e, ecc ...)
Conserved quantities of this group are 3, and corresponding to the number of particles of each type (Antiparticles are counted with negative sign)

[II] Indicando con $e^{i\theta}$ il determinante delle matrici $U_g \in U(3)$
[III] SU(3) è l'insieme di livello dato da $(\arg\circ\det)^{-1}(0)$

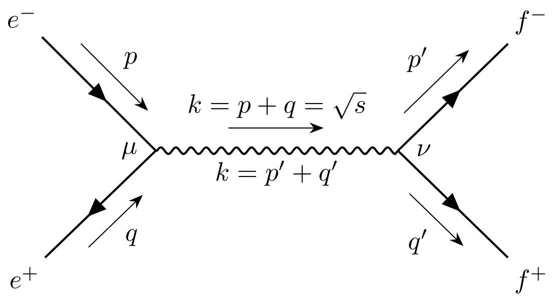
Example 4

$\mu^- \to e^-\gamma$ is forbidden in QED. This is an example of conserved charges due to unitary symmetry that have nothing to do with electric charge.
Flavours changing in neutral sector are forbidden too

3.3 QED Feynman Rules $\to$ fogli stampati (23-26)

3.4 $e^+e^- \to f^+f^-$

This diagram is called s-channel



With $k = p' + q'$ we impose the 4 momentum conservation
We have

$$S_{fi} = (2\pi)^4\delta^4(p + q - p' - q')\mathcal{M}_{fi}$$

with Feynman amplitude

$$\begin{aligned}
\mathcal{M}_{fi} &= (-iq)^2\left[\bar{u}_{r'}(p')\gamma^{\nu}\ v_{s'}(q')\right]^{\mathrm{IV}}\left[\bar{v}_s(q)\gamma^{\mu}\ u_r(p)\right]^{\mathrm{V}} D^F_{\mu\nu}(k) \\
&= (-iq)^2\frac{-ig_{\mu\nu}}{k^2 + i\varepsilon}\ \left[\bar{u}_{r'}(p')\gamma^{\nu}\ v_{s'}(q')\right]\left[\bar{v}_s(q)\gamma^{\mu}\ u_r(p)\right] \\
&= \frac{iq^2}{s + i\varepsilon}\ \left[\bar{u}_{r'}(p')\gamma_{\mu}\ v_{s'}(q')\right]\left[\bar{v}_s(q)\gamma^{\mu}\ u_r(p)\right]
\end{aligned}$$

In the second passage $\xi = 1$. We can prove that this choice has no importance, see Maggiore pg 187

Using the identity $(\bar{u}\gamma^{\mu}v)^* = \bar{v}\gamma^{\mu}u$ (that can be proved by direct calculation using $(\gamma^{\mu})^{\dagger} = \gamma^0\gamma^{\mu}\gamma^0$) we obtain (we omit polarization indexes) dimmi se ti va bene così

$$\begin{aligned}
|\mathcal{M}_{fi}|^2 &= \mathcal{M}_{fi}\mathcal{M}^*_{fi} = \frac{q^4}{s^2}\big(\bar{u}(p')\gamma_{\mu}v(q')\big) \times \big(\bar{v}(q')\gamma_{\nu}\,u(p')\big) \qquad \longrightarrow \text{f-current} \\
|\mathcal{M}_{fi}|^2 &= \mathcal{M}_{fi}\mathcal{M}^*_{fi} = \frac{q^4}{s^2}\big(\bar{v}(q)\gamma^{\mu}u(p)\big) \times \big(\bar{u}(p)\gamma^{\nu}\,v(q)\big) \qquad \longrightarrow \text{e-current}
\end{aligned}$$

At this point, we are still free to specify any particular spinors $u_r(p)$, $\bar{v}_{s'}(p')$ and so on, corresponding to any desired spin states of the fermions.

3.4.1 Sum Over Fermion Spins. Squared Averaged Feynman Amplitude

The Feynman amplitude simplifies considerably when we throw away the spin information. We want to compute

$$|\overline{\mathcal{M}_{fi}}|^2 = \underbrace{\frac{1}{2}\sum_s \frac{1}{2}\sum_r}_{\substack{\text{average over}\\ \text{the initial states}}} \underbrace{\sum_{s'}\sum_{r'}}_{\substack{\text{sum over}\\ \text{final states}}} |\mathcal{M}(r, s \to r', s')|^2$$

[V] indica il percorso $e^- \to \mu \to e^+$ nel diagramma
[V] indica il percorso $f^- \to \nu \to f^+$ nel diagramma

This sum can be performed using completeness relations for dirac spinors

$$\sum_r u_r(p)\bar{u}_r(p) = \not{p} + m \qquad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m$$

Writing spinors indexes explicitly

$$\begin{aligned}\sum_{rs} \text{e-current} &= \sum_{rs} \bar{v}^s_a(q)(\gamma^\mu)_{ab}\, u^r_b(p)\, \bar{u}^r_c(p)(\gamma^\nu)_{cd}\, v^s_d(q) \\ &= (\not{q} - m)_{da}\, \gamma^\mu_{ab}\, (\not{p} + m)_{bc}\, \gamma^\nu_{cd} \\ &= \mathrm{Tr}\big[(\not{q} - m)\gamma^\mu(\not{p} + m)\gamma^\nu\big]\end{aligned}$$

and similarly

$$\sum_{r's'} \text{f-current} = \mathrm{Tr}\big[(\not{p}' + m)\gamma_\mu(\not{q}' - m)\gamma_\nu\big]$$

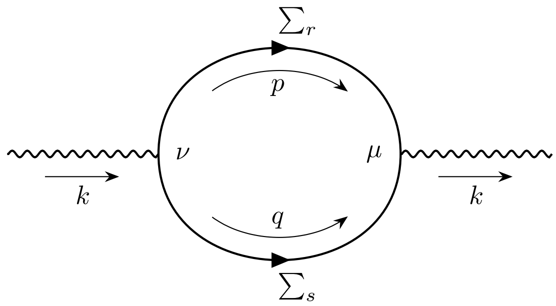
So we obtain

$$\overline{|\mathcal{M}_{fi}|}^2 = \frac{q^4}{4s^2}\,\mathrm{Tr}\big[(\not{q} - m)\gamma^\mu(\not{p} + m)\gamma^\nu\big]\,\mathrm{Tr}\big[(\not{p}' + m)\gamma_\mu(\not{q}' - m)\gamma_\nu\big]$$

The spinors u and v have disappeared, leaving us with a much cleaner expression in terms of γ matrices. This trick is very genera: any QED amplitude involving external fermions, when squared and summed or averaged over spins, can be converted in this way to traces of products of Dirac matrices

There is a trick to obtain previous formula using only Feynman rules. If we set $\mathcal{M}_{fi} = \mathcal{M}_e\mathcal{M}_f$ (we divide $\mathcal{M}_{fi}$ in 2 factors related to e and f), then $|\mathcal{M}_{fi}|^2 = \mathcal{M}_e^*\mathcal{M}_e\,\mathcal{M}_f^*\mathcal{M}_f$. We have

$$\sum_{rs} \mathcal{M}_e^*\,\mathcal{M}_e =$$



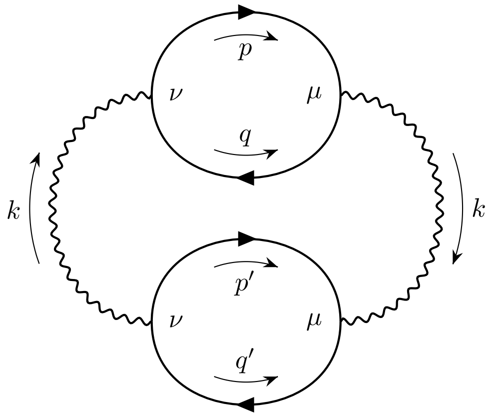
The closed fermion line contributes in the final amplitude with a factor

$$\frac{q^2}{2s}\,\mathrm{Tr}\big[\underbrace{(\not{p} + m)}_{\text{Fermion}}\gamma^\nu\ \underbrace{(\not{q} - m)}_{\text{Antifermion}}\ \gamma^\mu\big]$$

The factor $-q^2/(2s)$ is obtained using full diagram (s is given by photon propagator)

Nei tuoi schemi in mezzo alle 4 linee continue ci sono delle sommatorie. Non ho idea di come metterle, sorry

$$|\overline{\mathcal{M}_{fi}}|^2 =$$



Going back to the expression of $|\mathcal{M}_{fi}|^2$ that we obtain, now we need to calculate explicitly γ matrices traces

$$\mathrm{Tr}\big[(\not{p}+m)\gamma^\nu(\not{q}-m)\gamma^\mu\big] = \mathrm{Tr}\big[\not{p}\,\gamma^\nu\not{q}\,\gamma^\mu\big] - m^2\,\mathrm{Tr}[\gamma^\nu\;\gamma^\mu] + m\big(\mathrm{Tr}\big[\gamma^\nu\not{q}\,\gamma^\mu\big] - \mathrm{Tr}\big[\not{p}\,\gamma^\nu\gamma^\mu\big]\big)$$

Now we prove some properties of $\gamma-$matrices traces (other properties in Peskin pg 133-135)

(I)

$$\begin{aligned}
\mathrm{Tr}[\gamma^\mu] &= \mathrm{Tr}\big[\gamma^5\,\gamma^5\,\gamma^\mu\big] &&\to (\gamma^5)^2 = \mathbb{1}\\
&= -\,\mathrm{Tr}\big[\gamma^5\,\gamma^\mu\,\gamma^5\big] &&\to \{\gamma^5,\gamma^\mu\} = 0\\
&= -\,\mathrm{Tr}\big[\gamma^5\,\gamma^5\,\gamma^\mu\big] &&\to \text{cyclicity}\\
&= -\,\mathrm{Tr}[\gamma^\mu]\\
&\Rightarrow \mathrm{Tr}[\gamma^\mu] = 0
\end{aligned}$$

(II)

$$\begin{aligned}
\mathrm{Tr}[\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma] &= \mathrm{Tr}[2g^{\mu\nu}\,\gamma^\rho\,\gamma^\sigma - \gamma^\nu\,\gamma^\mu\,\gamma^\rho\,\gamma^\sigma]\\
&= 2g^{\mu\nu}\,\mathrm{Tr}[\gamma^\rho\,\gamma^\sigma] - \mathrm{Tr}[\gamma^\nu 2g^{\mu\rho}\gamma^\sigma - \gamma^\nu\,\gamma^\rho\,\gamma^\mu\,\gamma^\sigma]\\
&= 8g^{\mu\nu}g^{\rho\sigma} - 8g^{\mu\rho}g^{\nu\sigma} + \mathrm{Tr}[\gamma^\nu\,\gamma^\rho\,2g^{\mu\sigma} - \gamma^\nu\,\gamma^\rho\,\gamma^\sigma\,\gamma^\mu]\\
&= 8\big(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\big) - \mathrm{Tr}[\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma]\\
&\Rightarrow \mathrm{Tr}[\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma] = 4\big(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\big)
\end{aligned}$$

From previous calculation, using induction, can be easily proved that $\mathrm{Tr}\,[\underbrace{\gamma^\mu\ldots\gamma^\sigma}_{\substack{\text{odd \# of}\\ \gamma\text{ matrices}}}] = 0$

With these relations we obtain

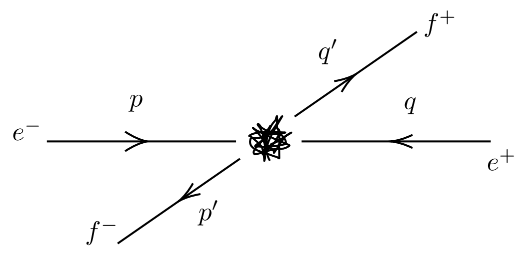
- $$\begin{aligned}
\mathrm{Tr}\big[(\not{p}+m)\gamma^\nu(\not{q}-m)\gamma^\sigma\big] &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})p_\mu q_\rho - m^2 4g^{\nu\sigma}\\
&= 4(p^\nu q^\sigma - g^{\nu\sigma}p\cdot q + p^\sigma q^\nu - g^{\nu\sigma}m^2)\\
&= 4\big(p^\nu q^\sigma + p^\sigma q^\nu - g^{\nu\sigma}(p\cdot q + m^2)\big)\\
&\text{with m} = \text{mass of } e^I
\end{aligned}$$

- $$\mathrm{Tr}\big[(\not{p}'+m)\gamma_\mu(\not{q}'-m)\gamma_\nu\big] = 4\big(p'_\mu\,q'_\nu + p'_\nu\,q'_\mu\,g_{\mu\nu}(p'\cdot q' + M^2)\big)\qquad \text{with M} = \text{mass of } f^I$$

- $$\begin{aligned}
\overline{|\mathcal{M}_{fi}|^2} &= \frac{4q^4}{s^2}\,\big(p^\mu q^\nu + p^\nu q^\mu - g^{\mu\nu}(p\cdot q + m^2)\big)\big(p'_\mu\,q'_\nu + p'_\nu\,q_\mu - g_{\mu\nu}(p'\cdot q' + M^2)\big)\\
&\approx \frac{8q^4}{s^2}\big((p\cdot p')(q\cdot q') + (p\cdot q')(p'\cdot q) + M^2(p\cdot q)\big)\qquad \text{if } m \ll M
\end{aligned}$$

To obtain a more explicit formula we must specialize to a particular frame of reference and express the vectors p, q, p', q', k in terms of the basis kinematic variables (energies and angles) in that frame. In practice, the choice of frame will be dictated by the experimental conditions.
We want to calculate cross section in the center of mass frame.

$$\begin{aligned}
p\; &= (\omega_p,\; \boldsymbol{p})\\
q\; &= (\omega_q,\; \boldsymbol{q})\\
\boldsymbol{p}+\boldsymbol{q} &= 0
\end{aligned}$$



$$\begin{aligned}
p'\; &= (\omega_{p'},\; \boldsymbol{p})\\
q'\; &= (\omega_{q'},\; \boldsymbol{q})\\
\boldsymbol{p}'+\boldsymbol{q}' &= 0
\end{aligned}$$

In the CM frame we have following kinematics relations

$$\left.\begin{array}{r}(p+q)=(\omega_p+\omega_q,\mathbf{p}+\mathbf{q})=(\omega_p+\omega_q,0)\\ s=(p+q)^2=(\omega_p+\omega_q)^2\end{array}\right\}=(p+q)=(\sqrt{s},0)$$

and

$$\omega_p=(p+q)^2=(\omega_p+\omega_q)^2$$

Same for final momenta

$$(p'+q')=(\sqrt{s},0)$$
$$\omega_{p'}=\omega_{q'}=\frac{\sqrt{s}}{2}$$

For product momenta

$$\begin{aligned}
(p\cdot q)&=\omega^2-\mathbf{pq}=\omega^2|\mathbf{p}|^2=2\omega^2-m^2\overset{m=0}{\simeq}2\omega^2\\
(p'\cdot q')&=\omega^2-\mathbf{pq}=\omega^2|\mathbf{p}|^2=2\omega^2-M^2\\
(p\cdot p')&=\omega^2-\mathbf{pp'}=\omega^2-|\mathbf{p}||\mathbf{p}'|\cos\theta\overset{m=0}{\simeq}\omega(\omega-|\mathbf{p}'|\cos\theta)\overset{m=0}{\simeq}(q\cdot q')\\
(p\cdot q')&=\omega^2+\mathbf{p}\,\mathbf{p}'=\omega^2+|\mathbf{p}||\mathbf{p}'|\cos\theta\simeq\omega(\omega+|\mathbf{p}'|\cos\theta)\simeq(q\cdot p')\\
|\mathbf{p}'|^2&=\omega^2-M^2=\omega^2\left(1-\frac{M^2}{\omega^2}\right)=\frac{s}{4}\left(1-\frac{4M^2}{s}\right)\\
|\mathbf{p}|^2&=\omega^2+m^2\overset{m=0}{\simeq}\omega^2=\frac{s}{4}
\end{aligned}$$

Now we can rewrite $\overline{|\mathcal{M}_{fi}|^2}$ in terms of ω and θ. If $m=0$ is valid (e.g. $m_e/m_\mu=1/200$)

$$\begin{aligned}
\overline{|\mathcal{M}_{fi}|^2}&\simeq\frac{8q^4}{s^2}\left[(\omega^4-2\omega^3|p'|\cos\theta+\omega^2|p'|^2\cos^2\theta)+(\omega^4+2\omega^3|p'|\cos\theta+\omega^2|p'|^2\cos^2\theta)+2M^2\omega^2\right]\\
&=\frac{8q^4}{s^2}\left[\frac{2s}{4}\left(\frac{s}{4}+\frac{s}{4}\left(1-\frac{4M^2}{s}\right)\cos^2\theta\right)+2M^2\frac{s}{4}\right]\\
&=q^4\left[\left(1+\frac{4M^2}{s}\right)+\left(1-\frac{4M^2}{s}\right)\cos^2\theta\right]
\end{aligned}$$

Cross section formula in the CM frame reads

$$\begin{aligned}
\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)_{CM}&=\frac{1}{64\pi^4}\;\frac{1}{s}\;\frac{|p'|}{|q'|}\;|\overline{\mathcal{M}}|^2_{CM}\\
&=\frac{q^4}{64\pi^2}\;\frac{1}{s}\;\left(1-\frac{4M^2}{s}\right)^{1/2}\left[\left(1+\frac{4M^2}{s}\right)+\left(1-\frac{4M^2}{s}\right)\cos^2\theta\right]\\
&=\frac{\alpha^2}{4s}\left(1-\frac{4M^2}{s}\right)^{1/2}\left[\left(1+\frac{4M^2}{s}\right)+\left(1-\frac{4M^2}{s}\right)\cos^2\theta\right]\quad\text{with }\alpha=\frac{e^2}{4\pi}\approx\frac{1}{137}
\end{aligned}\tag{3.2}$$

Integrating over $\mathrm{d}\Omega$ we find the total unpolarized cross section

$$(\bar{\sigma})_{TOT}=\int\mathrm{d}\Omega\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)_{CM}=\frac{4\pi\alpha^2}{3s}\left(1-\frac{4M^2}{s}\right)^{1/2}\left(1+\frac{2M^2}{s}\right)+\sigma(\alpha^3)\tag{3.3}$$

Notes: $\left(1-(4M^2)/s\right)$ in the equation 3.2 impose a physical constraint for the scattering process: $s=4\omega^2>4M^2$. Energy of initial particles must be grater than the mass of final particles $\omega>M$ (we changed the case $m\approx0$)
In equation 3.3, *the same term*, shows that for $m\approx M$ the cross section vanish and (see previous formulas) there is almost no dependence on the angle, moreover the term $\sigma(\alpha^3)$ is trabscured since we considering only first order of perturbative expansion.

We used approximation $m \approx 0$ because usually in experiments $\omega \approx 1GeV$ and $m_\mu \approx 200me$ $m_\mu = 105MeV$. For very high experiments, $\omega \approx TeV$, we can omit also f mass, approximating $M \approx 0$. This is called **ultra relativistic regime** (only energies are taken into account)

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)^{UR}_{CM} = \frac{\alpha^2}{4s} \underbrace{(1+\cos\theta)}_{\substack{\text{scattering amplitude}\\ \text{is higher at small angles}}} \qquad (\bar{\sigma})^{UR}_{TOT} = \frac{4\pi\alpha^2}{3s} + o\left(\frac{M^2}{s}\right)$$

Let's summarize how we obtained these results. The method extends in a straightforward way to the calculation of unpolarized cross section for other QED processes at lowest order. The general procedure is as follows:

(1) Draw the diagram(s) for the desired process

(2) Use Feynman rules to write down the amplitude $\mathcal{M}_{fi}$

(3) Square the amplitude and average or sum over spins, using completeness relations (for processing involving photons in the final state there is an analogous completeness relation, we will derive it in Compton scattering)

(4) Evaluate traces using γ-matrices proprieties; collect terms and simplify the answer as much as possible

(5) Specialize to a particular frame of reference, and draw a picture of the kinematic variable in that frame. Express all 4-momentum vectors in terms of suitably chosen set of variables such as E and θ

(6) Plug the resulting expression for $\left|\overline{\mathcal{M}_{fi}}\right|^2$ into the cross-section formula and integrate over phase space variables that are not measured to obtain a differential cross section in the desired form

Exercise 1

Calculate cross sections of following precesses ($f \neq e$) (unpolarized)

- $e^-\, f^- \rightarrow e^-\, f^-$ (t-channel)
- $e^-\, f^+ \rightarrow e^-\, f^+$ (u-channel)

3.4.2 Polarized Scattering Relations between helicity and chiarality.

Perkin, sec 5.2 - Schwartz, sec 13.3 - Mandl, sec 8.4 **The ultra relativistic limit and helicity amplitudes**

In the previous calculation we obtained the amplitude polarized

$$\mathcal{M}_{fi} = \frac{iq^2}{s}\left(\bar{u}_{r'}(p')\gamma^\mu v_{s'}(q')\right)\left(\bar{v}_s\gamma_\mu u_r(p)\right)$$

Calculation of the polarized cross section allows us to understand better the unpolarized cross section, for example show us where the factor $(1+\cos^2\theta)$ comes from.

We must choose a basis of polarization states. The best choice is to quantize each spin along the direction of particle's motion, that is, to use states of definite helicity.

In general, helicity projectors are hard to be used for lower energies, so we work in the ultra relativistic limit. Recall that in the massless limit, the left- and right-handed helicity states of a Dirac particle live in different representations of the Lorentz group. We might expect them to behave independently, and in fact they do.

We would like to use the spin sum identities to write the squared amplitude in term of traces as before, ever though we now want to consider only one set of polarizations at a time. We note that in the ultra-relativistic limit, helicity is related to chiarality, so we can use chirality projectors, that are much simpler.

In particular, let $\Lambda_\pm$ be energy projectors, $\Pi_\pm$ be helicity projectors, and $P_{L,R}$ be chirality projectors, then following relations holds:

$$\Pi_\pm \Lambda_+ = P_{R,L}\Lambda_+ \qquad \Pi_\pm \Lambda_- = P_{L,R}\Lambda_-$$

where

$$P_R = \frac{1+\gamma_5}{2} \qquad P_L = \frac{1-\gamma_5}{2}$$

And for spinors in UR limit:

$$\begin{aligned} u_+ &= \Pi_+\, u = u_1 = u_R && \rightarrow \text{right handed spinor} && h = \frac{1}{2} \\ u_- &= \Pi_-\, u = u_2 = u_L && \rightarrow \text{left handed spinor} && h = -\frac{1}{2} \end{aligned}$$

with h = helicity eigenvalue

$$\begin{aligned} v_+ &= \Pi_+\, v = v_2 = v_L && \rightarrow \text{left handed spinor} && h = \frac{1}{2} \\ v_- &= \Pi_-\, v = v_1 = v_R && \rightarrow \text{right handed spinor} && h = -\frac{1}{2} \end{aligned}$$

We notice that for antiparticles the relations between chirality and helicity is inverted. This can be easily interpreted using Dirac's Holes Theory, for example an antiparticle with positive helicity is the hole left by a particle with positive helicity (since both momenta and spin change sign), and so particles and antiparticles with same helicity must have inverse chirality.

Using chirality projectors properties we can write Dirac currents as

$$\begin{aligned} \bar{v}\gamma^\mu u &= s\bar{v}(P_L + P_R)\gamma^\mu\,(P_L + P_R)u \\ &= \bar{v}P_L\,\gamma^\mu\,P_L\,u + \bar{v}P_R\,\gamma^\mu\,P_R\,u + \bar{v}P_L\,\gamma^\mu\,P_R\,u + \bar{v}P_R\,\gamma^\mu\,P_L\,u \end{aligned}$$

Using γ matrices properties

$$P_{R,L}\,\gamma^\mu = \frac{\gamma^\mu \pm \gamma^5\,\gamma^\mu}{2} = \frac{\gamma^\mu \pm \overbrace{\{\gamma^5, \gamma^\mu\}}^{=0} \mp \gamma^\mu\,\gamma^5}{2} = \gamma^\mu\,\frac{1 \mp \gamma^5}{2} = \gamma^\mu\,P_{L,R}$$

we obtain

$$\begin{aligned} \bar{v}\gamma^\mu\,u &= \bar{v}\gamma^\mu\,P_R\,P_L\,u + \bar{v}\gamma^\mu\,P_L\,P_R\,u + \bar{v}\gamma^\mu\,P_R\,P_R\,u + \bar{v}\gamma^\mu\,P_L\,P_L\,u \\ &= \bar{v}\gamma^\mu\,P_R\,u + \bar{v}\gamma^\mu\,P_L\,u = \bar{v}\gamma^\mu_R\,u + \bar{v}\gamma^\mu_L\,u = J^\mu_R + J^\mu_L \end{aligned}$$

where we are defined $\gamma^\mu_{R,L} = \gamma^\mu\, P_{RL}$ and the left- and right-currents:

$$J^\mu_L = \bar{v}\,\gamma^\mu_L\, u \qquad J^\mu_R = \bar{v}\,\gamma^\mu_R\, u$$

We define right- and left-handed spinors as follows:

$$\begin{aligned} u_L = P_L\, u \quad u_R = P_R\, u \\ v_L = P_L\, v \quad v_R = P_R\, u \end{aligned}$$

and them conjugated as (let ψ be a generic spinors)

$$\begin{aligned} \psi_L = P_L\,\psi \quad &\rightarrow \quad \bar{\psi}_L = \bar{\psi}\, P_R \\ \psi_R = P_R\,\psi \quad &\rightarrow \quad \bar{\psi}_R = \bar{\psi}\, P_L \end{aligned}$$

With this notation I can rewrite left- and right-handed currents as

$$\begin{aligned} \bar{v}\,\gamma^\mu\, u = \bar{v}\,\gamma^\mu_R\; u + \bar{v}\,\gamma^\mu_L\, u = \bar{v}_R\gamma^\mu\, u_R + \bar{v}_L\gamma^\mu\, u_L \\ \bar{u}\,\gamma^\mu\, v = \bar{u}\,\gamma^\mu_R\; v + \bar{u}\,\gamma^\mu\, v = \bar{u}_R\,\gamma^\mu\, v_R + \bar{u}_L\,\gamma^\mu\, v_L \end{aligned}$$

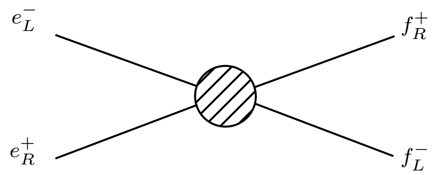
Here is evident that a deric current can be divided in its left handed and right handed components

Going back to the Feynman amplitude

$$\begin{aligned} \mathcal{M}_{fi} = \frac{iq^2}{s} &\big(\bar{u}(p')\,\gamma^\mu_L\, v(q')\big)\big(\bar{v}(q)\,\gamma^L_\mu\, u(p)\big)\times \quad \rightarrow \mathcal{M}_{LL} \\ &\times \big(\bar{u}(p')\,\gamma^\mu_L\, v(q')\big)\big(\bar{v}(q)\,\gamma^R_\mu\, u(p)\big)\times \quad \rightarrow \mathcal{M}_{LR} \\ &\times \big(\bar{u}(p')\,\gamma^\mu_R\, v(q')\big)\big(\bar{v}(q)\,\gamma^L_\mu\, u(p)\big)\times \quad \rightarrow \mathcal{M}_{RL} \\ &\times \big(\bar{u}(p')\,\gamma^\mu_R\, v(q')\big)\big(\bar{v}(q)\,\gamma^R_\mu\, u(p)\big) \quad \rightarrow \mathcal{M}_{RR} \end{aligned}$$

If I did not used the UK limit, I would have obtained 16 independent terms, instead of 4.
Each factor in the latter, corresponds to a different Feynman diagram with different left- and right-handed, initial and final particles, for example

$\mathcal{M}_{LL} =$



With

$$\mathcal{M}_{LL} = UR\left(e^-\left(-\frac{1}{2}\right) + e^+\left(\frac{1}{2}\right) \rightarrow f^-\left(-\frac{1}{2}\right) + f^+\left(\frac{1}{2}\right)\right)$$

Having inserted these projection operators, we are now free to sum over the electron and positron spins in the squared amplitude. As a cosequence of projector operators, we saw that the amplitude vanishes (in UR limit) unless the eletron and positron have opposite helicity, or equivalently, unless their spinors have the same helicity.
Let's calculate squared Feynman amplitude of $\mathcal{M}_{LL}$, summing over spins

$$|\mathcal{M}_{LL}|^2 = \sum_{\text{spins}} \underbrace{\left|\bar{u}_r(p')\,\gamma^\mu_L\, v_{s'}(q')\right|^2}_{\mathbf{B}} \underbrace{\left|\bar{v}_s(q)\,\gamma^L_\mu\, u_r(p)\right|^2}_{\mathbf{A}}$$

with

$$\begin{aligned} \mathbf{A} &= \sum_{rs} \big(\bar{v}_s(q)\,\gamma^L_\mu\, u_r(p)\big)\big(\bar{u}_r(p)\,\gamma^l_\nu\, v_s(q)\big) \\ &= \mathrm{Tr}\big[\not{q}\,\gamma^L_\mu\,\not{p}\,\gamma^L_\nu\big] = \mathrm{Tr}\big[q^\rho\,\gamma_\rho\,\gamma_\mu\, P_L\, p^\sigma\,\gamma_\sigma\,\gamma_\nu\, P_L\big] \\ &= \mathrm{Tr}\big[q^\rho\,\gamma_\rho\,\gamma_\mu\, p^\sigma\,\gamma_\sigma\, P_R\gamma_\nu\, P_L\big] = \mathrm{Tr}\left[\not{q}\,\gamma_\mu\,\not{p}\,\gamma_\nu\left(\frac{1-\gamma^5}{2}\right)\right] \\ &= \frac{1}{2}\,\mathrm{Tr}\big[\not{q}\,\gamma_\mu\,\not{p}\,\gamma_\nu\big] - \frac{1}{2}\,\mathrm{Tr}\big[\not{q}\,\gamma_\mu\,\not{p}\,\gamma_\nu\,\gamma_5\big] \end{aligned}$$

Recall:

$$\mathrm{Tr}\big[\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma\big] = 4\big(g^{\mu\nu}\,g^{\rho\sigma} - g^{\mu\rho}\,g^{\nu\sigma} + g^{\mu\sigma}\,g^{\nu\rho}\big)$$
$$\mathrm{Tr}\big[\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma\,\gamma^5\big] = -4i\varepsilon^{\mu\nu\rho\sigma}$$

$$\mathbf{A} = 2\big(q_\mu p_\nu - q\cdot p g_{\mu\nu} + q_\nu p_\mu - i\varepsilon_{\alpha\mu\beta\nu}\,q^\alpha p^\beta\big)$$
$$\mathbf{B} = 2\big(p'^\mu q'^\nu - q'\cdot p' g^{\mu\nu} + p'^\nu q'^\mu - i\varepsilon^{\rho\mu\sigma\nu}p'_\rho q'_\sigma\big)$$

And we finally obtain

$$\begin{aligned}
|\mathcal{M}_{LL}|^2 &= \frac{4q^2}{s^2}\big[2(p\cdot p')(q\cdot q') + 2(p\cdot q')(p'\cdot q) - \varepsilon^{\rho\mu\sigma\nu}\,\varepsilon_{\alpha\mu\beta\nu}\,q^\alpha p^\beta p'_\rho q'_\sigma\big] \\
&= \frac{8q^2}{s^2}\big[(p\cdot p')(q\cdot q') + (p\cdot q')(p'\cdot q) - (p\cdot p')(q\cdot q') + (p\cdot q')(q\cdot p')\big] \\
&= \frac{16q^2}{s^2}(p\cdot q')(p'\cdot q)
\end{aligned}$$

In general frame

$$u = (p-q')^2 = (q-p')^2 = 2m^2 - 2pq' \simeq -2pq' = -2qp'$$

and in the center of mass frame

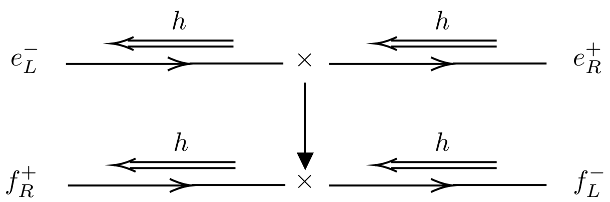
$$(p\cdot q') = (q\cdot p') = \omega^2(1+\cos\theta)$$

so the squared amplitude in the center of mass frame reads

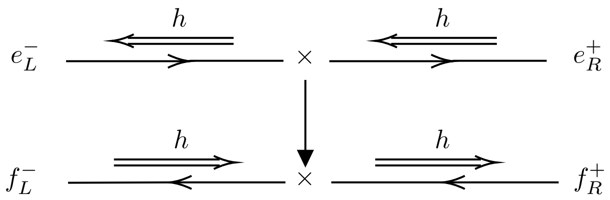
$$|\mathcal{M}_{LL}|^2 = \frac{16q^4}{s^2}\omega^4(1+\cos\theta)^2 = q^4(1+\cos\theta)^2 = q^4\frac{u^2}{s^2}$$

Notice that this amplitude vanished for $\theta = \pm\pi$
In particular

(i) $|\mathcal{M}_{LL}|^2$ is max when $\theta = 0$



(ii) $|\mathcal{M}_{LL}|^2$ is zero when $\theta = \pi$



Amplitude vanished when helicity states are orthogonal. This is what we would expect, since for $\theta = \pi$ the total angular momentum of the final state is opposite to that of the initial state

Exercise 2

Compute $|\mathcal{M}_{RR}|^2, |\mathcal{M}_{LR}|^2, |\mathcal{M}_{RL}|^2$
Solution:

$$|\mathcal{M}_{RR}|^2 = q^4(1+\cos\theta)^2 = q^4\frac{u^2}{s^2}$$
$$|\mathcal{M}_{LR}|^2 = |\mathcal{M}_{RL}|^2 = q^4(1-\cos\theta)^2 = q^4\frac{t^2}{s^2}$$

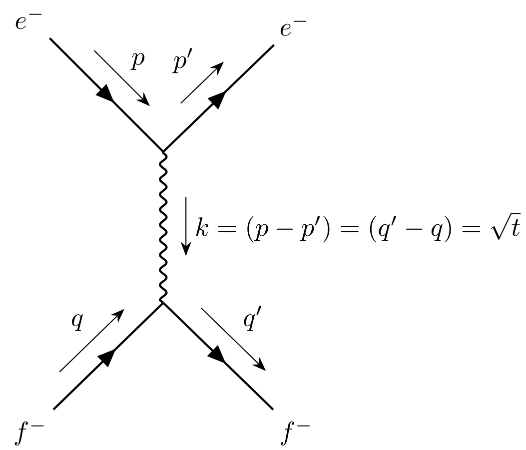
Actually $|\mathcal{M}_{LL}|^2 = |\mathcal{M}_{RR}|^2$ and $|\mathcal{M}_{LR}|^2 = |\mathcal{M}_{RL}|^2$ are consequences of parity invariance

Adding up all four contributions, and dividing by 4 to average over the electron and positron states, we recover the unpolarized cross section dirived previously

$$\begin{aligned}|\mathcal{M}|^2 &= \frac{1}{4}\left(|\mathcal{M}_{LL}|^2 + |\mathcal{M}_{RR}|^2 + |\mathcal{M}_{RL}|^2 + |\mathcal{M}_{LR}|^2\right)\\ &= q^4(1+\cos^2\theta)\end{aligned}$$

3.4.3 Mandelstam variables and crossing symmetries

Peskin, sec 5.4
Let's consider a different but closely related QED process: $e^- f^- \to e^- f^-$



The Feynman amplitude of this process (in Feynmann gauge) reads

$$\mathcal{M}_{fi} = \frac{iq^2}{t}\big(\bar{u}(p')\,\gamma^\mu\,u(p)\big)\big(\bar{u}(q')\,\gamma_\mu\,u(q)\big)$$

The relation between this process and $e^+e^- \to f^+f^-$ becomes clear when we compute the squared amplitude, average and summed over spins:

$$\overline{|\mathcal{M}_{fi}|}^2 = \frac{q^4}{4t^2}\,\mathrm{Tr}\big[(\not{p}+m)\gamma^\nu(\not{p}'+m)\gamma^\mu\big]\,\mathrm{Tr}\big[(\not{q}+m)\gamma_\nu(\not{q}'+M)\gamma_\mu\big]$$

Recall that for $e^-e^+ \to f^-f^+$

$$\overline{|\mathcal{M}_{fi}|}^2_{e^-e^+} = \frac{q^4}{4s^2}\,\mathrm{Tr}\big[(\not{p}+m)\gamma^\nu(\not{q}-m)\big]\,\mathrm{Tr}\big[(\not{q}'-m)\gamma_\nu(\not{p}'+m)\gamma_\mu\big]$$

Notice that following symmetry relations hold

$$\begin{aligned} e^+e^- &\leftrightarrow e^-f^- \\ p &\leftrightarrow p \\ q &\leftrightarrow -p' \\ p' &\leftrightarrow q' \\ q' &\leftrightarrow -q \end{aligned} \qquad \xrightarrow{\text{in Mandelstam variables}} \qquad \begin{aligned} e^+e^- &\leftrightarrow e^-f^- \\ s=(p+q)^2 &\leftrightarrow t=(p-p')^2 \\ t=(p-p')^2 &\leftrightarrow u=(p-q')^2 \\ u=(p-q')^2 &\leftrightarrow s=(p+q)^2 \end{aligned}$$

So instead of evaluating traces from scratch, we can just make the same replacement in our previous result. Setting $m=0$ we find

$$\begin{aligned} \left|\overline{\mathcal{M}_{fi}}\right|^2 &= \frac{8q^4}{t^2}\left((p\cdot q')(p'\cdot q)+(p\cdot q)(p'\cdot q')-M^2(p\cdot p')\right) \\ &\underset{\substack{\text{UR limit}\\ n=0}}{\approx} 2q^4\left(\frac{t^2+u^2}{s^2}\right) \end{aligned}$$

Where we used following relation for the UR limit

$$\begin{aligned} s &= 2pq = 2p'q' \\ t &= -2pp' = -2qq' \\ u &= -2pq' = -2p'q \end{aligned}$$

In the center of mass frame, in the UR limit, we have

$$\begin{aligned} \left|\overline{\mathcal{M}_{fi}}\right|^2_{CM} &= 2e^4\left[\frac{4+(1+\cos\theta)^2}{(1-\cos\theta)^2}\right] \\ \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{CM} &= \frac{\alpha^2}{2s}\left[\frac{4+(1+\cos\theta)^2}{(1-\cos\theta)^2}\right] \end{aligned}$$

Where θ is the angle between f^-_{init} and e^-_{fin}

Diagramma foglio 41

3.4.4 Crossing symmetry

The trick we made use of here, namely the relation between the two processes $e^+e^- \to f^+f^-$ and $e^-f^- \to e^-f^-$, is our first example of a type of relation known as **crossing symmetry**:

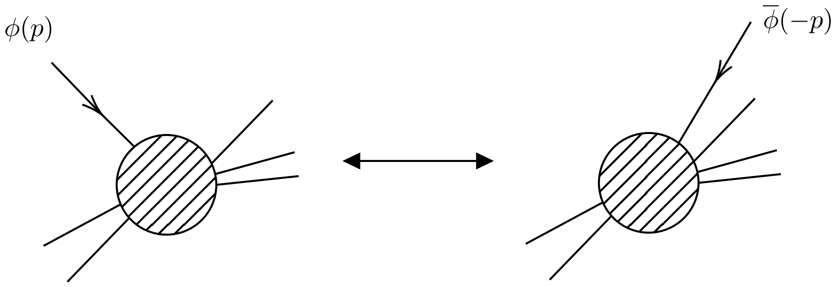
> *The S-matrix for any process involving a particle with momentum p in the initial state is equal to the S-matrix for an otherwise identical process but with an antiparticle of momentum $k=-p$ in the final state*
>
> $$\mathcal{M}\big(\varphi(p)+\dots\to\dots\big)=\pm\mathcal{M}\big(\dots\to\dots+\bar{\varphi}(k)\big)$$

Note that there is no value of p for which p and k are both physically allowed, since the particle must have $p^0>0$ and the antiparticle $k^0>0$. So technically, we should say that either amplitude can be obtained from the other by analytic continuation

previous relation follows directly from Feynman rules. Relative sign between amplitude is not predicted by Feynman rules, but it can be obtained from commuting/anticommuting proprieties of field. For example, for spinors fields

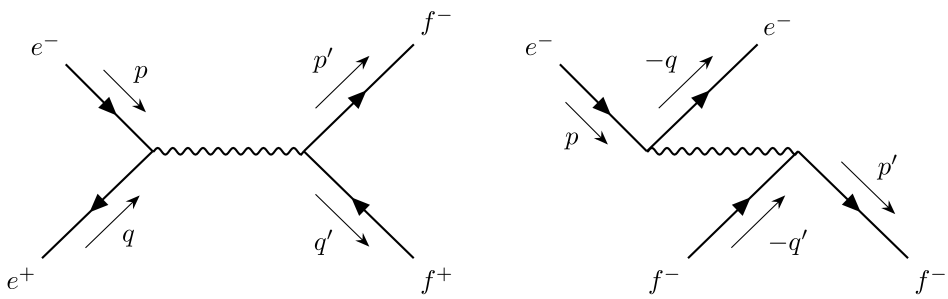
$$\begin{cases} \text{odd permutations} & \to \quad \text{minus sign} \\ \text{ever permutations} & \to \quad \text{plus sign} \end{cases}$$

For squared amplitudes the relative sign is irrelevant but it's important when we have to sum different diagrams contributions
The diagrams that contribute to the two amplitudes fall into a natural one-to-one correspondence, where corresponding diagrams differ inly by changing the incoming φ into the outgoing $\bar{\varphi}$



For example, in the $e^-e^+ \to f^+f^-$ and $e^-f^- \to e^-f^-$ correspondence we have

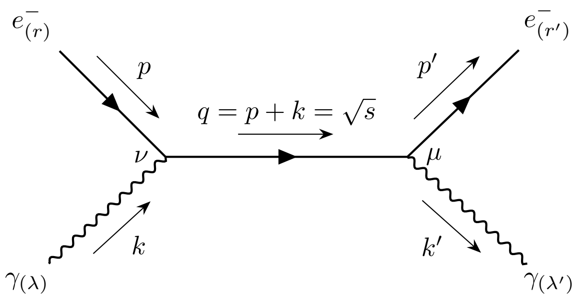
Controlla se sono giusti

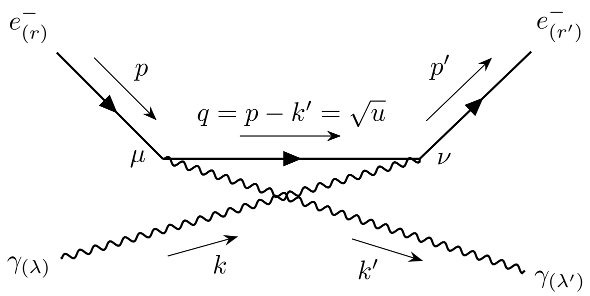
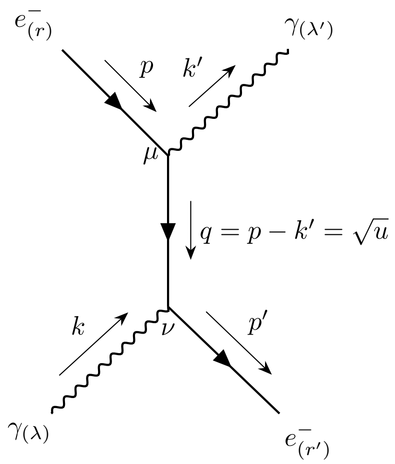


3.5 $e^-\gamma \to e^-\gamma$ (Compton)

See Peskin, sec 5.5
Let's examine a process with external bosons: *Compton scattering*, or $e^-\gamma \to e^-\gamma$. This process is described by two independent diagrams, since they are topologically different:

$\mathcal{M}_A =$



$\mathcal{M}_B =$  $=$ 

We wrote the diagram of $\mathcal{M}_B$ in two topologically equivalent forms: in the first one is clear the topological relation with diagram of $\mathcal{M}_A$ (this is useful to find the relative sign between diagrams A and B: it's clear that diagrams differs for the permutation of two bosons), while in the second one is clear that it describes a u-channel.

Amplitudes reads, using Feynman rules

$$\begin{aligned}\mathcal{M}_A &= \bar{u}_{r'}(p')(-iq\gamma^\mu)\varepsilon_\mu^{\lambda' *}(k')\tilde{S}_F(p+k)(-iq\gamma^\nu)\varepsilon_\nu^\lambda(k)u_r(p) \\ &= -q^2\varepsilon_\mu^{\lambda' *}(k')\varepsilon_\nu^\lambda(k)\left[\bar{u}_{r'}(p')\gamma^\mu\tilde{S}_F(p+k)\gamma^\nu u_r(p)\right]\end{aligned}$$

$$\mathcal{M}_B = -q^2\varepsilon_\mu^{\lambda' *}(k')\varepsilon_\nu^\lambda(k)\left[\bar{u}_{r'}(p')\gamma^\nu\tilde{S}_F(p-k')\gamma^\mu u_r(p)\right]$$

(Recall that $\tilde{S}_F$ is a matrix, so elements in the squared bracket must be written in this order)
Because of anticommuting relations for bosons, these amplitudes must be summed up in the total amplitude. The explicit form of Feynman propagator for the Dirac field reads

$$\tilde{S}_F(p) = \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon} = \frac{i}{\not{p}-m+i\varepsilon}$$

so total amplitude is

$$\mathcal{M} = -iq^2\varepsilon_\mu^{\lambda' *}(k')\varepsilon_\nu^\lambda(k)\bar{u}_{r'}(p')\left[\frac{\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu}{(p+k)^2-m^2} + \frac{\gamma^\nu(\not{p}-\not{k}'+m)\gamma^\mu}{(p-k')^2-m^2}\right]u_r(p)$$

We make some simplifications before squaring this expression. Since $p^2=m^2$ and $k^2=0$:

$$(p+k)^2-m^2 = 2p\cdot k \qquad (p-k')^2-m^2 = -2p\cdot k'$$

To simplify numerators, I can use Dirac algebra:

$$\begin{aligned}(\not{p}+m)\gamma^\nu u(p) &= (p_\mu\gamma^\mu\gamma^\nu + m\gamma^\nu)u(p) = (2g^{\mu\nu}p_\mu - p_\mu\gamma^\nu\gamma^\mu + m\gamma^\nu)u(p) \\ &= 2p^\nu u(p) - \gamma^\nu\underbrace{(\not{p}-m)}_{2m\,\Lambda_-(p)}u(p) = 2p^\nu u(p)\end{aligned}$$

Using these tricks we obtain

$$\mathcal{M} = -iq^2\varepsilon_\mu^{\lambda' *}(k')\varepsilon_\nu^\lambda(k)\bar{u}_{r'}(p')\left[\frac{\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{-\gamma^\nu\not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{-2p\cdot k'}\right]u_r(p)$$

3.5.1 The Ward Identities and sum over the photon polarizations

See Mandl, sec 8.3

The next step in the calculation will be to square this expression for $\mathcal{M}$ and sum or average over electron and photon polarization states. The sum over electron polarizations can be performed as before, using $\sum u(p)\bar{u}(p) = \not{p} + m$. Fortunately, there is a similar trick for summing over photons polarization vectors. Gauge invariance of the theory implies the gauge invariance of the matrix elements, i.e. of the Feynman amplitudes. It is, of course, only the matrix element itself, corresponding to the sum of all possible Feynman graphs in a given order of perturbation theory, which must be gauge invariant. For example, for the Compton scattering, the individual amplitudes $\mathcal{A}$ and $\mathcal{B}$ are not gauge invariants, but their sum $\mathcal{M}$ is.

For any process involving external photons, the Feynman amplitude $\mathcal{M}$ is of the form

$$\mathcal{M} = \varepsilon_\alpha^{\lambda_1}(k_1)\varepsilon_\beta^{\lambda_2}(k_2)\dots L^{\alpha\beta\cdots}(k_1, k_2, \dots) \tag{3.4}$$

with one polarization vector $\varepsilon^{\lambda_i}(k_i)$ for each external photon, and the tensor amplitude $L^{\alpha\beta\cdots}(k_1, k_2, \dots)$ independent of these polarization vectors.

The polarization vectors are of course gauge dependent. For example, for a free photon described in the Lorentz gauge by the plane wave

$$A^\mu(x) = \text{const} \cdot \varepsilon_\lambda^\mu(k) e^{\pm ikx}$$

the gauge transformation

$$A^\mu \to A'^\mu(x) = A^\mu(x) + \partial^\mu \alpha(x) \qquad \text{with} \quad \alpha(x) = \tilde{\alpha}(k)e^{\pm ikx}$$

implies

$$\varepsilon_\lambda^\mu(k) \to \varepsilon'^\mu_\lambda(k) = \varepsilon_\lambda^\mu(k) \pm ik^\mu \tilde{\alpha}(k)$$

Invariance of the amplitude Eq.(3.4) under this transformation requires

$$k_1^\alpha L_{\alpha,\beta,\dots}(k_1, k_2, \dots) = k_1^\beta L_{\alpha,\beta,\dots}(k_1, k_2, \dots) = \dots = 0$$

i.e. when any external photon polarization vector is replaced by the corresponding four momentum, the amplitude must vanish. This is the statement of the *Ward Identity*:

> *If $\mathcal{M}(k) = \varepsilon_\mu(k)L^\mu(k)$ is the amplitude for some QED process involving an external photon with momentum k, then this amplitude vanishes if we replace ε_μ with k_μ:*
>
> $$k_\mu L^\mu(k) = 0$$

Exercise 3

Verify explicitly the Ward Identity for the Feynman amplitude of Compton scattering

See Peskin, sec 5.5

Returning to our derivation of the polarization sum formula for squared scattering amplitude. Writing in general

$$\mathcal{M} = \varepsilon_\mu^{(\lambda)}(k)L^\mu(k)$$

then the sum over polarizations of the photon with momentum k reads

$$\sum_{\lambda=1,2} |\mathcal{M}|^2 = \sum_{\lambda=1,2} \varepsilon_\mu^{(\lambda)}(k)\varepsilon_\nu^{(\lambda)*}(k)L^\mu(k)L^{\nu\dagger}(k)$$

Because of the covariance of the theory we can do the calculation in a specific frame. In order to simplify the analysis we choose the frame where the photon moves along the $\hat{z}$ axis:

$$k^\mu = (|k|, 0, 0, |k|)$$

In this case the Ward Identity reads

$$0 = k_\mu L^\mu = |k| \left(L^0 - L^3\right) \qquad \longrightarrow \qquad L^0 = L^3$$

Recall that in this frame

$$\varepsilon_\mu^{(1)}(k) = (0,1,0,0) \qquad \varepsilon_\mu^{(2)}(k) = (0,0,1,0)$$

So we have

$$\sum_{\lambda=1,2} \varepsilon_\mu^{(\lambda)}(k)\varepsilon_\nu^{(\lambda)*}(k)L^\mu(k)L^{\nu\dagger}(k) = \left|L^1\right|^2 + \left|L^2\right|^2 = \left|L^1\right|^2 + \left|L^2\right|^2 + \left|L^3\right|^2 - \left|L^0\right|^2 = -g_{\mu\nu}L^\mu L^\nu$$

So we obtained the general rule to simplify photons polarization sum[VI]

$$\boxed{\sum_{\lambda=1,2} \varepsilon_\mu^{(\lambda)}(k)\varepsilon_\nu^{(\lambda)*}(k)L^\mu(k)L^{\nu\dagger}(k) \quad \longrightarrow \quad -g_{\mu\nu}}$$

3.5.2 The Klein-Nishima formula and the Thomson scattering

See Peskin, sec. 5.5

To obtain the unpolarized cross section for Compton scattering, we use the covariant method described in the previous section. Writing

$$\mathcal{M} = \varepsilon_\mu^{\lambda' *}(k')\varepsilon_\nu^\lambda(k)\left(L^{\mu\nu}(k,k')\right)_{r,r'}$$

with

$$\left(L^{\mu\nu}(k,k')\right)_{r,r'} = -iq^2\bar{u}_{r'}(p')\left[\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{-\gamma^\nu \not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{-2p\cdot k'}\right]u_r(p)$$

we obtain

$$\begin{aligned}
|\bar{\mathcal{M}}|^2 &= \frac{1}{4}\left(\sum_{\lambda'}\varepsilon_\mu^{(\lambda')*}(k')\varepsilon_\rho^{(\lambda')}(k')\right)\left(\sum_\lambda \varepsilon_\nu^{(\lambda)*}(k)\varepsilon_\sigma^{(\lambda)}(k')\right)\sum_{r,r'}\left(L^{\mu\nu}\right)_{r,r'}\left(L^{\rho\sigma}\right)^\dagger_{r,r'} \\
&= \frac{1}{4}g_{\mu\rho}g_{\nu\sigma}\sum_{r,r'}\left(L^{\mu\nu}\right)_{r,r'}\left(L^{\rho\sigma}\right)^\dagger_{r,r'} = \frac{1}{4}\left(L^{\mu\nu}\right)_{r,r'}\left(L_{\mu\nu}\right)^\dagger_{r,r'} \\
&= \frac{q^4}{4}\operatorname{Tr}\left[(\not{p}'+m)\left(\frac{\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{\gamma^\nu\not{k}'\gamma^\mu - 2\gamma^\nu p^\mu}{2p\cdot k'}\right)\times\right. \\
&\qquad \left.\times\,(\not{p}+m)\left(\frac{\gamma_\nu\not{k}\gamma_\mu + 2\gamma_\mu p_\nu}{2p\cdot k} + \frac{\gamma_\mu\not{k}'\gamma_\nu - 2\gamma_\nu p_\mu}{2p\cdot k'}\right)\right] \\
&= \frac{q^4}{4}\left\{\frac{T_{AA}}{(2p\cdot k)^2} + \frac{T_{BB}}{(2p\cdot k')^2} + \frac{T_{AB}+T_{BA}}{(2p\cdot k)(2p\cdot k')}\right\}
\end{aligned}$$

where

$$\begin{aligned}
T_{AA} &= \operatorname{Tr}\left[(\not{p}'+m)(\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu)(\not{p}+m)(\gamma_\nu\not{k}\gamma_\mu + 2\gamma_\mu p_\nu)\right] \\
T_{BB} &= \operatorname{Tr}\left[(\not{p}'+m)(\gamma^\nu\not{k}'\gamma^\mu - 2\gamma^\nu p^\mu)(\not{p}+m)(\gamma_\mu\not{k}'\gamma_\nu - 2\gamma_\nu p_\mu)\right] \\
T_{AB} &= \operatorname{Tr}\left[(\not{p}'+m)(\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu)(\not{p}+m)(\gamma_\mu\not{k}'\gamma_\nu - 2\gamma_\nu p_\mu)\right] \\
T_{BA} &= \operatorname{Tr}\left[(\not{p}'+m)(\gamma^\nu\not{k}'\gamma^\mu - 2\gamma^\nu p^\mu)(\not{p}+m)(\gamma_\nu\not{k}\gamma_\mu + 2\gamma_\mu p_\nu)\right]
\end{aligned}$$

[VI]Notice that we could prove (see Peskin) that even if we took $\lambda = 0,1,2,3$, we could have obtained that the unphysical time-like and longitudinal photons can be consistently omitted from QED calculations, since in any event the squared amplitudes for producing these states cancel to give zero total probability.

Notice that $T_{BB} = T_{AA}(k \leftrightarrow -k')$ and $T_{BA} = T_{AB}(k \leftrightarrow -k')$, we need therefore only calculate T_{AA} and T_{AB}.
Considering T_{AA}, there are 16 terms inside the trace, but half contains an odd number of γ matrices and therefore vanishes. Other terms are

$$
\begin{aligned}
1 &= \mathrm{Tr}\big[\not{p}'\gamma^\mu \not{k}\gamma^\nu \not{p}\gamma_\nu \not{k}\gamma_\mu\big] \\
2 &= 2\,\mathrm{Tr}\big[\not{p}'\gamma^\mu \not{k}\gamma^\nu \not{p}\gamma_\mu p_\nu\big] = 2\,\mathrm{Tr}\big[\not{p}'\gamma^\mu \not{k}\not{p}\not{p}\gamma_\mu\big] \\
3 &= 2\,\mathrm{Tr}\big[\not{p}'\gamma^\mu p^\nu \not{p}\gamma_\nu \not{k}\gamma_\mu\big] = 2\,\mathrm{Tr}\big[\not{p}'\gamma^\mu \not{p}\not{p}\not{k}\gamma_\mu\big] \\
4 &= 4\,\mathrm{Tr}\big[\not{p}'\gamma^\mu p^\nu \not{p}\gamma_\mu p_\nu\big] = 4p^2\,\mathrm{Tr}\big[\not{p}'\gamma^\mu \not{p}\gamma_\mu\big] \\
5 &= m^2\,\mathrm{Tr}[\gamma^\mu \not{k}\gamma^\nu \gamma_\nu \not{k}\gamma_\mu] \\
6 &= 2m^2\,\mathrm{Tr}[\gamma^\mu \not{k}\gamma^\nu \gamma_\mu p_\nu] = 2m^2\,\mathrm{Tr}\big[\gamma^\mu \not{k}\not{p}\gamma_\mu\big] \\
7 &= 2m^2\,\mathrm{Tr}[\gamma^\mu p^\nu \gamma_\nu \not{k}\gamma_\mu] = 2m^2\,\mathrm{Tr}\big[\gamma^\mu \not{p}\not{k}\gamma_\mu\big] \\
8 &= 4m^2\,\mathrm{Tr}[\gamma^\mu p^\nu \gamma_\mu p_\nu] = 4m^2p^2\,\mathrm{Tr}[\gamma^\mu\gamma_\mu]
\end{aligned}
$$

In order to simplify above formulas we recall the proprieties of contractions of γ matrices, i.e. products in the form $\gamma^\mu A \gamma^\mu$ where A is a matrix:

(i) $\gamma^\mu\gamma_\mu = 4\mathbb{1}$

(ii) $\gamma^\mu \not{p}\gamma_\mu = -2\not{p}$

(iii) $\gamma^\mu \not{p}\not{q}\gamma_\mu = 4p\cdot q$

(iv) $\gamma^\mu \not{p}\not{q}\not{k}\gamma_\mu = -2\not{k}\not{q}\not{p}$

Using these proprieties, cyclicity of the trace and anticommuting proprieties of gamma matrices[VII], we obtain (remember that $p^2 = m^2$ and $k^2 = 0$):

$$
\begin{aligned}
1 &= \mathrm{Tr}\big[\not{p}'\gamma^\mu \not{k}\gamma^\nu \not{p}\gamma_\nu \not{k}\gamma_\mu\big] = -2\,\mathrm{Tr}\big[\not{p}'\gamma^\mu \not{k}\not{p}\not{k}\gamma_\mu\big] = 4\,\mathrm{Tr}\big[\not{p}'\not{k}\not{p}\not{k}\big] = -4\,\mathrm{Tr}\Big[\not{p}'\not{k}^2\not{p}\Big] + 8(p\cdot k)\,\mathrm{Tr}\big[\not{p}'\not{k}\big] = 32(p\cdot k)(p'\cdot k) \\
2 &= 2\,\mathrm{Tr}\big[\not{p}'\gamma^\mu \not{k}\not{p}\not{p}\gamma_\mu\big] = -4\,\mathrm{Tr}\big[\not{p}'\not{p}\not{p}\not{k}\big] = -4m^2\,\mathrm{Tr}\big[\not{p}'\not{k}\big] = -16m^2(p'\cdot k) \\
3 &= 2\,\mathrm{Tr}\big[\not{p}'\gamma^\mu \not{p}\not{p}\not{k}\gamma_\mu\big] = 2m^2\,\mathrm{Tr}\big[\not{p}'\gamma^\mu \not{k}\gamma_\mu\big] = -4m^2\,\mathrm{Tr}\big[\not{p}'\not{k}\big] = -16m^2(p'\cdot k) \\
4 &= 4p^2\,\mathrm{Tr}\big[\not{p}'\gamma^\mu \not{p}\gamma_\mu\big] = -8m^2\,\mathrm{Tr}\big[\not{p}'\not{p}\big] = -32m^2(p'\cdot p) \\
5 &= m^2\,\mathrm{Tr}[\gamma^\mu \not{k}\gamma^\nu \gamma_\nu \not{k}\gamma_\mu] = 4m^2\,\mathrm{Tr}[\gamma^\mu \not{k}\not{k}\gamma_\mu] = 0 \\
6 &= 2m^2\,\mathrm{Tr}\big[\gamma^\mu \not{k}\not{p}\gamma_\mu\big] = 8m^2(k\cdot p)\,\mathrm{Tr}[\mathbb{1}] = 32m^2(k\cdot p) \\
7 &= 2m^2\,\mathrm{Tr}\big[\gamma^\mu \not{p}\not{k}\gamma_\mu\big] = 8m^2(p\cdot k)\,\mathrm{Tr}[\mathbb{1}] = 32m^2(p\cdot k) \\
8 &= 4m^2p^2\,\mathrm{Tr}[\gamma^\mu\gamma_\mu] = 16m^4\,\mathrm{Tr}[\mathbb{1}] = 64m^4
\end{aligned}
$$

At the end we find

$$
\begin{aligned}
T_{AA} &= 16\left(4m^4 - 2m^2 p\cdot p' + 4m^2 p\cdot k - 2m^2 p'\cdot k + 2(p\cdot k)(p'\cdot k)\right) \\
&= 16\left(2m^4 + m^2(s-m^2) - \frac{1}{2}(s-m^2)(u-m^2)\right)
\end{aligned}
$$

[VII] $\not{A}\not{B} = A_\mu B_\nu \gamma^\mu\gamma^\nu = A_\mu B_\nu(2g^{\mu\nu}\mathbb{1} - \gamma^\nu\gamma^\mu) = 2(A\cdot B)\,\mathbb{1} - \not{B}\not{A} \quad \rightarrow \quad \not{A}\not{A} = A^2\,\mathbb{1}$
$\mathrm{Tr}\big[\not{A}\not{B}\big] = 2(A\cdot B)\,\mathrm{Tr}[\mathbb{1}] - \mathrm{Tr}\big[\not{B}\not{A}\big] = 8(A\cdot B) - \mathrm{Tr}\big[\not{A}\not{B}\big] \quad \rightarrow \quad \mathrm{Tr}\big[\not{A}\not{B}\big] = 4(A\cdot B)\mathbb{1}$

where we introduced Mandelstam variables:

$$
\begin{aligned}
s &= (p+k)^2 = 2p\cdot k + m^2 = 2p'\cdot k' + m^2 \\
t &= (p'-p)^2 = -2p\cdot p' + 2m^2 = -2k\cdot k' \\
u &= (k'-p)^2 = -2k'\cdot p + m^2 = -2k\cdot p' + m^2
\end{aligned}
$$

Sending $k \leftrightarrow -k'$ $(s \leftrightarrow u)$ we can immediately write

$$
\begin{aligned}
T_{BB} &= 16\left(4m^4 - 2m^2 p\cdot p' - 4m^2 p\cdot k' + 2m^2 p'\cdot k' + 2(p\cdot k')(p'\cdot k')\right) \\
&= 16\left(2m^4 + m^2(u-m^2) - \frac{1}{2}(u-m^2)(s-m^2)\right)
\end{aligned}
$$

Exercise 4

Compute the elements T_{AB} and T_{BA}

Evaluating the traces in T_{AB} and T_{BA} requires about the same amount of work as we have just done. The answer is

$$
\begin{aligned}
T_{AB} = T_{BA} &= -16\left(4m^4 + m^2(p\cdot k - p\cdot k')\right) \\
&= -16\left(2m^4 + \frac{m^2}{2}((s-m^2)-(u-m^2))\right)
\end{aligned}
$$

Putting together the pieces of the unpolarized Feynman amplitude for Compton scattering we obtain

$$
\begin{aligned}
|\bar{\mathcal{M}}|^2 &= 2q^4\left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right] \\
&= 2q^4\left[-\left(\frac{u-m^2}{s-m^2} + \frac{s-m^2}{u-m^2}\right) + 4m^2\left(\frac{1}{s-m^2} + \frac{1}{u-m^2}\right) + 4m^4\left(\frac{1}{s-m^2} + \frac{1}{u-m^2}\right)^2\right]
\end{aligned}
\tag{3.5}
$$

To turn this expression into a cross section we must decide a frame of reference and draw a piture of the kinematics. We will analyze two different frames

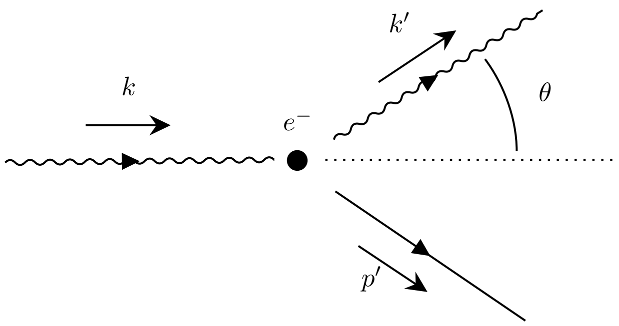
(i) *"Lab" frame*, in which the electron is initially at rest, this frame is useful for low energy incoming photons: $\omega_\gamma \ll m_e$;

(ii) *c.o.m. frame*, in which the center of mass is at rest, this frame is useful for high energy incoming photons: $\omega_\gamma \gg m_e$, where we can set $m_e = 0$

3.5.3 Lab frame - Low energy photon

See also Mandl sec. 8.6

In the low energy case, I can verify if QED prediction agrees with Thomason law for low energies scattering.



$$
\begin{aligned}
p &= (m_e, 0) \\
k &= (\omega_k, \mathbf{k}) \\
p' &= (E', \mathbf{p}') \\
k' &= (\omega_{k'}, \mathbf{k}')
\end{aligned}
$$

We will express the cross section in terms of ω and θ. We can find ω', the energy of the final photon, using the following trick:

$$\begin{aligned} m^2 = (p')^2 = (p+k-k')^2 &= p^2 + 2p\cdot(k-k') - 2k\cdot k' \\ &= m^2 + 2m(\omega_k - \omega_{k'}) - 2\omega_k\omega_{k'}(1-\cos\theta) \end{aligned}$$

hence, we obtain Compton's formula for the shift in the photon wavelength:

$$\Delta\lambda = \left(\frac{1}{\omega_{k'}} - \frac{1}{\omega_k}\right) = \frac{1-\cos\theta}{m}$$

For our purposes, however, is more useful to solve for $\omega_{k'}$:

$$\omega_{k'} = \frac{\omega_k}{1+\frac{\omega_k}{m}(1-\cos\theta)} \tag{3.6}$$

The unpolarized amplitude in the Lab frame is

$$\begin{aligned} |\bar{\mathcal{M}}|^2_{\mathrm{LAB}} &= 2q^4\left[\left(\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}}\right) + 2m\left(\frac{1}{\omega_k} - \frac{1}{\omega_{k'}}\right) + m^2\left(\frac{1}{\omega_k} - \frac{1}{\omega_{k'}}\right)^2\right] \\ &= 2q^4\left[\left(\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}}\right) - \sin^2\theta\right] \end{aligned}$$

The covariant flux factor reads

$$I_{\mathrm{LAB}} = [(p\cdot k)^2 - m_e^2 m_\gamma^2]^{1/2} = |p\cdot k| = m_e\omega_k$$

The 2-body phase space

$$\begin{aligned} \int \mathrm{d}\Phi_{(2)} &= \int \frac{\mathrm{d}^3k'}{(2\pi)^3 2\omega_{k'}}\frac{\mathrm{d}^3p'}{(2\pi)^3 2E'}(2\pi)^4\delta^4(k'+p'-k-p) = \int \frac{\omega_{k'}^2\mathrm{d}\omega_{k'}\mathrm{d}\Omega}{(2\pi)^2}\frac{1}{4\omega_{k'}E'}\delta(\omega_{k'}+E'-\omega_k-m) \\ &= \int \frac{\omega_{k'}^2\mathrm{d}\omega_{k'}\mathrm{d}\Omega}{(2\pi)^2}\frac{1}{4\omega_{k'}E'}\frac{\delta(\omega_{k'}-|\mathbf{k}'|)}{\left|\frac{\partial(\omega_{k'}+E'-\omega_k-m)}{\partial|k'|}\right|_{\omega_{k'}=|\mathbf{k}'|}} = \int \mathrm{d}\Omega\frac{|\mathbf{k}'|^2}{16\pi^2\omega_{k'}E'}\left|\frac{\partial(\omega_{k'}+E')}{\partial|k'|}\right|^{-1}_{\omega_{k'}=|\mathbf{k}'|} \end{aligned}$$

where

$$E' = \left(m^2 + (\mathbf{k}-\mathbf{k}')^2\right)^{1/2} = \left[m^2 + \omega_k^2 + \omega_{k'}^2 - 2\omega_{k'}\omega_k\cos\theta\right]^{1/2}$$

$$\frac{\partial E'}{\partial|k'|} = \frac{\omega_{k'} - \omega_k\cos\theta}{E'}$$

and

Controlla l'ultimo passaggio

$$\left|\frac{\partial(\omega_{k'}+E')}{\partial|k'|}\right|_{\omega_{k'}=|\mathbf{k}'|} = \left|1 + \frac{\omega_{k'}-\omega_k\cos\theta}{E'}\right| = \frac{m\omega_k}{E'\omega_{k'}}$$

So the unpolarized cross section is

$$\begin{aligned} \left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)_{\mathrm{LAB}} &= \frac{\overline{|\mathcal{M}|}^2_{\mathrm{LAB}}}{4I_{\mathrm{LAB}}}\frac{\mathrm{d}\Phi_{(2)}}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\frac{|\mathbf{k}'|^2}{I_{\mathrm{LAB}}\omega_{k'}E'}\left|\frac{\partial(\omega_{k'}+E')}{\partial|k'|}\right|^{-1}\overline{|\mathcal{M}|}^2_{\mathrm{LAB}} \\ &= \frac{q^4}{32\pi^2}\frac{1}{m^2}\left(\frac{\omega_{k'}}{\omega_k}\right)^2\left(\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}} - \sin^2\theta\right) \\ &= \frac{\alpha^2}{2}\frac{1}{m^2}\left(\frac{\omega_{k'}}{\omega_k}\right)^2\left(\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}} - \sin^2\theta\right) \end{aligned}$$

where $\omega_{k'}/\omega_k$ is given by (3.6) and in the last step we used $\alpha = e^2/(4\pi)$. Writing $\mathrm{d}\Omega = (2\pi)\mathrm{d}\cos\theta$ we obtain

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\cos\theta}\right)_{\mathrm{LAB}} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega_{k'}}{\omega_k}\right)^2\left(\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}} - \sin^2\theta\right) \tag{3.7}$$

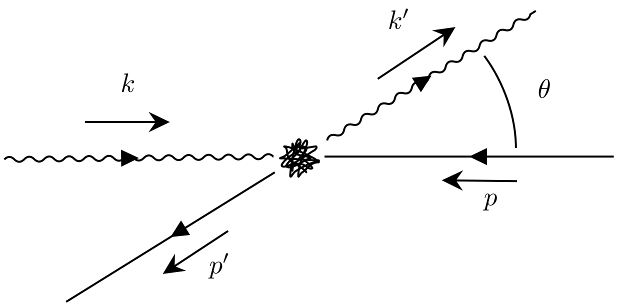
This is the (spin-averaged) *Klein-Nishina formula*. In the low energy limit $\omega_k \ll m$, from (3.6) we have $\omega_{k'} \approx \omega_k$, i.e. the kinetic energy of the recoil electron is negligible, and Eq.(3.7) reduces to the familiar Thomson cross-section for scattering of classical electromagnetic radiation by a free electron:

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\cos\theta}\right)_{\mathrm{LAB}} \overset{\omega_k \ll m}{=} \frac{\pi\alpha^2}{m^2}\left(1+\cos^2\theta\right) \quad \rightarrow \quad (\bar{\sigma})_{\mathrm{LAB}} = \frac{8\pi\alpha^2}{3m^2} \equiv \frac{8}{3}\pi r_e^2$$

We have calculated the full relativistic corrections for the Thomson formula.

3.5.4 C.o.M. frame - High energy photon

See Peskin sec. 5.5, and Schwartz sec. 13.5.4 To analyze the high-energy behaviour of the Compton scattering cross section, it is easiest to work in the center-of-mass frame.



$$\begin{aligned} p &= (E, -\mathbf{k}) \\ k &= (\omega_k, \mathbf{k}) \\ p' &= (E', -\mathbf{k}') \\ k' &= (\omega_k', \mathbf{k}') \end{aligned}$$

The kinematics of the reaction in the high energy limit ($m \approx 0$) looks like

$$\begin{aligned} E &= \sqrt{\mathbf{k}^2 + m^2} \overset{m=0}{\approx} |\mathbf{k}| = \omega_k \\ E' &= \sqrt{\mathbf{k}'^2 + m^2} \overset{m=0}{\approx} |\mathbf{k}'| = \omega_{k'} \end{aligned}$$

$$\begin{aligned} p \cdot k &= E\omega_k + |\mathbf{k}|^2 = \omega_k(E + \omega_k) \overset{m=0}{\approx} 2\omega_k^2 \\ p \cdot p' &= E\,E' - \mathbf{k}\mathbf{k}' = E\,E' - |\mathbf{k}||\mathbf{k}'|\cos\theta = E\,E' - \omega_k\omega_{k'}\cos\theta \overset{m=0}{\approx} \omega_k\omega_{k'}(1-\cos\theta) \\ p \cdot k' &= E\,\omega_{k'} + \mathbf{k}\mathbf{k}' = E\,\omega_{k'} + |\mathbf{k}||\mathbf{k}'|\cos\theta = \omega_{k'}(E + \omega_k\cos\theta) \overset{m=0}{\approx} \omega_k\omega_{k'}(1+\cos\theta) \end{aligned}$$

We also have

$$\begin{aligned} s &= (p+k)^2 = m^2 + 2p\cdot k = m^2 + 2\omega_k(E+\omega_k) \overset{m=0}{\approx} 4\omega_k^2 \quad \rightarrow \quad \omega_k \overset{m=0}{\approx} \frac{\sqrt{s}}{2} \\ s &= (p'+k')^2 = m^2 + 2p'\cdot k' = m^2 + 2\omega_{k'}(E'+\omega_k') \overset{m=0}{\approx} 4\omega_{k'}^2 \quad \rightarrow \quad \omega_{k'} \overset{m=0}{\approx} \frac{\sqrt{s}}{2} \end{aligned}$$

Plugging these values into Eq.(3.5)

$$\overline{|\mathcal{M}|}^2 = 2q^4\left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right]$$

for c.o.m. frame with $E \gg m$ we have

$$\overline{|\mathcal{M}|}^2_{\mathrm{CM}} \approx 2q^4\left(\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'}\right) \approx 2q^4\left(\frac{1+\cos\theta}{2} + \frac{2}{1+\cos\theta}\right)$$

we notice that the term $p\cdot k/p\cdot k'$ becomes divergent when the electron is emitted in the backward direction ($\theta \approx \pi$), while other terms are all of $\mathcal{O}(1)$ or smaller.

Notice that two initial diagrams $\mathcal{M}_A$, s-channel, and $\mathcal{M}_B$, u-channel, give contributions to the total amplitude proportional to[VIII]

$$\mathcal{M}_A \quad \rightarrow \quad \frac{1}{2p\cdot k} = \frac{1}{s-m^2} \qquad\qquad \mathcal{M}_B \quad \rightarrow \quad \frac{1}{2p\cdot k'} = \frac{1}{u-m^2}$$

Here is clear the relation between the momentum of the channel and the contribution to the total Feynman amplitude. The divergent contribution is due to the square of the u-channel diagram, we can see that for $\theta=\pi$ we have $u=(p-k')^2=m^2-2p\cdot k'\approx m^2-2\omega_k^2(1+\cos\theta)\approx m^2$ i.e. the divergent contribution is related to the situation where the initial electron emits a photon with all its kinetic energy and then absorbs all the energy of the initial photon. The amplitude is large at $\theta\approx\pi$ because the denominator of the propagator is then small ($\sim m^2$) compared to s. This kind of divergence is called *Infra-Red divergence*. We can correct the divergent term (unphysical) considering higher terms in the Taylor expantion of E in m:

This is a Infra-Red divergence, related to Sudakov logs. Add some comment. See Mandl 8.9

$$E=\sqrt{\mathbf{k}^2+m^2}=|\mathbf{k}|+\frac{m^2}{2|\mathbf{k}|}+o(m^3)\overset{m\approx 0}{\approx}\omega_k+\frac{m^2}{2\omega_k}$$

$$p\cdot k'=\omega_{k'}(E+\omega_k\cos\theta)\overset{m=0}{\approx}\omega_{k'}\left(\omega_k+\frac{m^2}{2\omega_k}+\omega_k\cos\theta\right)=\omega_k^2\left(1+\cos\theta+\frac{m^2}{2\omega_k^2}\right)$$

$$\overline{|\mathcal{M}|}^2_{\text{CM}}\approx 2q^4\left(\frac{p\cdot k'}{p\cdot k}+\frac{p\cdot k}{p\cdot k'}\right)\approx 2q^4\left(\frac{1+\cos\theta}{2}+\frac{2}{1+\cos\theta+\frac{m^2}{2\omega_k^2}}\right)$$

Now we can compute the cross section in the CM frame (we can use the formula for elastic scattering):

$$\begin{aligned}\left(\frac{\mathrm{d}\bar\sigma}{\mathrm{d}\Omega}\right)_{\text{CM}}&=\frac{1}{64\pi^2}\frac{\overline{|\mathcal{M}|}^2_{\text{CM}}}{s}\approx\frac{q^4}{32\pi^2 s}\left(\frac{1+\cos\theta}{2}+\frac{2}{1+\cos\theta+\frac{m^2}{2\omega_k^2}}\right)\\&\approx\frac{\alpha^2}{2s}\left(\frac{1+\cos\theta}{2}+\frac{2}{1+\cos\theta+\frac{m^2}{2\omega_k^2}}\right)\end{aligned}$$

or

$$\left(\frac{\mathrm{d}\bar\sigma}{\mathrm{d}(\cos\theta)}\right)_{\text{CM}}\approx\frac{\pi\alpha^2}{s}\left(\frac{1+\cos\theta}{2}+\frac{2}{1+\cos\theta+\frac{m^2}{2\omega_k^2}}\right)$$

Recall that the electron mass m can be neglected completely in this formula if it were not necessary to cutoff the singularity for $\theta=0$.
The total Compton scattering cross section reads:

$$\begin{aligned}\bar\sigma_{\text{total}}=\int_{-1}^{1}\mathrm{d}(\cos\theta)\left(\frac{\mathrm{d}\bar\sigma}{\mathrm{d}(\cos\theta)}\right)_{\text{CM}}&\approx\frac{\pi\alpha^2}{s}\int_{-1}^{1}\mathrm{d}(\cos\theta)\left(\frac{1+\cos\theta}{2}+\frac{2}{1+\cos\theta+\frac{m^2}{2\omega_k^2}}\right)\\&=\frac{\pi\alpha^2}{s}+\frac{2\pi\alpha^2}{s}\log\left(\frac{s}{m^2}\right)\end{aligned}$$

The main dependence α^2/s follows from dimensional analysis. But the singularity associated to backward scattering of photons leads to an enhancement by an extra logarithm of the energy, called *Sudakov logarithm*.

Inserire diagrammi di Feynman pag 168 Peskin (ruotati appropriatamente)

Exercise 5: Pair Annihilation into Photons

Find the total cross section of the annihilation process $e^+e^-\rightarrow 2\gamma$

[VIII] You can easily verify this statement looking at the calculation on the beginning of this section.

3.6 Scattering by an external E.M. field and the Rutherford formula

3.6.1 $e^-p \to e^-p$ - Rutherford scattering

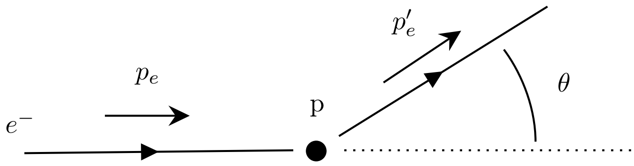
See Schwartz sec. 13.4

Now let us go back to the problem of scattering of an electron by a Coulomb potential. Recall the classical Rutherford scattering formula,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m_e^2 e^4}{4|\mathbf{p}_i|^4 \sin^4 \frac{\theta}{2}}$$

where $|\mathbf{p}_i| = |\mathbf{p}_f|$ is the magnitude of the incoming electron momentum, which. is the same as the magnitude of the outgoing electron momentum for elastic scattering. Rutherford calculated this using classical mechanics to describe how an electron would get deflected in a central potential, as from atomic nucleus.

We study the process in the CM frame for the proton.



$$\begin{aligned} p_e &= (\omega_e, \mathbf{p_e}) \\ p_p &= (M_p, 0) \\ p'_e &= (\omega'_e, \mathbf{p_e}') \\ p'_p &= (\omega'_p, \mathbf{p_p}') \end{aligned}$$

We neglect the recoil of the proton because of its huge mass ($\mathbf{p_p}' \approx 0$) (we are not considering high energy case).

$$(\omega'_p, \mathbf{p_p}') = \left(M + \frac{|\mathbf{p_p}'|^2}{2M} + o\left(|\mathbf{p_p}'|^3\right), \mathbf{p_p}' \right) \approx (M, 0)$$

If we can neglect the electron mass ($m_e \approx 0$) we also have

$$\omega_e = \sqrt{m^2 + \mathbf{p}_e^2} = |\mathbf{p}_e| + \frac{m^2}{2|\mathbf{p}_e|} + o(m^3) \approx |\mathbf{p}_e|$$

$$\omega'_e = \sqrt{m^2 + \mathbf{p}_e'^2} = |\mathbf{p}'_e| + \frac{m^2}{2|\mathbf{p}'_e|} + o(m^3) \approx |\mathbf{p}'_e|$$

Then energy conservation reads

$$\omega_e + M_p = \omega'_e + \omega'_p \quad \to \quad \omega_e = \omega'_e + \frac{|\mathbf{p_p}'|^2}{2M_p} + o\left(|\mathbf{p_p}'|^3\right) \quad \to \quad \omega_e \approx \omega'_e$$

The only quantity that shows a remarkable variation is the angle θ (variation can be $\mathcal{O}(1)$):

$$\begin{aligned} (p_e - p'_e)^2 &= 2m^2 - 2p_e \cdot p'_e = 2m^2 - 2\omega_e\omega'_e + 2|\mathbf{p}_e||\mathbf{p}'_e| \cos\theta \approx -2|\mathbf{p}_e|^2(1 - \cos\theta) \\ (p_p - p'_p)^2 &= 2M^2 - 2p_p \cdot p'_p = 2M^2 - 2M\omega'_p \approx -|\mathbf{p}'_p|^2 \end{aligned}$$

and because of momentum conservation $p_e + p_p = p'_e + p'_p$ we have

$$-2|\mathbf{p}_e|^2(1 - \cos\theta) \approx -|\mathbf{p}'_p|^2 \quad \to \quad \cos\theta \approx 1 - \frac{1}{2}\frac{|\mathbf{p}'_p|^2}{|\mathbf{p}_e|^2}$$

In order to give a description of this process using QED, we modify the QED lagrangian, so that we introduce also protons in our theory. We consider a low energy process, where the proton can be consider as a fundamental particle, described as a spin 1/2 fermion. Then we can do the same trick we used for QED flavours. The modified Lagrangian reads:[IX]

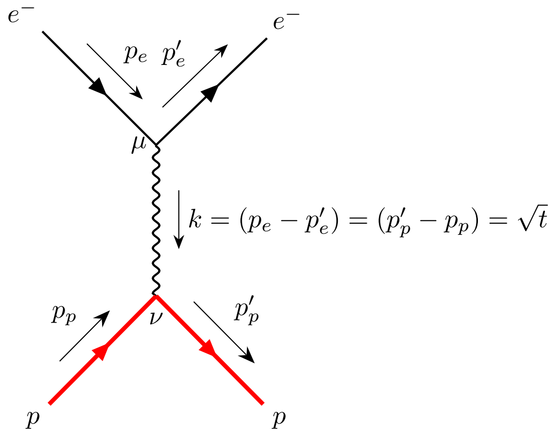
$$\mathcal{L} = \bar{\psi}_e(i\partial\!\!\!/ - m_e)\psi_e + \bar{\psi}_p(i\partial\!\!\!/ - M_p)\psi_p + \underbrace{q_e\bar{\psi}_e A\!\!\!/\psi_e + q_p\bar{\psi}_p A\!\!\!/\psi_p}_{\mathcal{L}_{\text{int}}}$$

Notice that $\mathcal{L}_{\text{int}}$ in order to obtain a description of Rutherford scattering we need to consider at least 2-nd order processes, i.e. 2 vertex. The two-vertex S matrix element becomes

$$\begin{aligned} S_{(2)} &= \frac{(-i)^2}{2!}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{T}\left\{\mathrm{N}\left[q_e\bar{\psi}_e A\!\!\!/\psi_e + q_p\bar{\psi}_p A\!\!\!/\psi_p\right]_x \mathrm{N}\left[q_e\bar{\psi}_e A\!\!\!/\psi_e + q_p\bar{\psi}_p A\!\!\!/\psi_p\right]_y\right\} \\ &= \frac{1}{2}(-iq_e)(-iq_p)\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{T}\left\{\mathrm{N}\left[\bar{\psi}_e A\!\!\!/\psi_e\right]_x \mathrm{N}\left[\bar{\psi}_p A\!\!\!/\psi_p\right]_y + \mathrm{N}\left[\bar{\psi}_e A\!\!\!/\psi_e\right]_y \mathrm{N}\left[\bar{\psi}_p A\!\!\!/\psi_p\right]_x\right\} \\ &= (-iq_e)(-iq_p)\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{T}\left\{\mathrm{N}\left[\bar{\psi}_e A\!\!\!/\psi_e\right]_x \mathrm{N}\left[\bar{\psi}_p A\!\!\!/\psi_p\right]_y\right\} \end{aligned}$$

where in the first step we omitted products in the integral that does not describe interaction between electron and proton, and in the second step we change integral variables in the second term of the time product.

The $e^-p \to e^-p$ process is similar to the process $e^-f^- \to e^-f^-$. Both processes are described by a t-channel



Feynman amplitude for this process is (we consider the gauge fixing term for EM field with $\zeta = 1$)

$$\mathcal{M} = i\frac{q_eq_p}{t}\left(\bar{u}_{r'}(p'_e)\gamma^\mu u_r(p_e)\right)_e \left(\bar{u}_{s'}(p'_p)\gamma_\mu u_s(p_p)\right)_p$$

where lower indices means that spinors are related respectively to electron and proton fields. Since proton is at rest its spinors takes a simple form

$$u_s(p_e = (M,0)) = \sqrt{2M}\begin{pmatrix}\xi_s \\ 0\end{pmatrix} \qquad \bar{u}_s(p'_e = (M,0)) = \sqrt{2M}\begin{pmatrix}\xi_s^\dagger & 0\end{pmatrix}$$

and the proton current reads[X]

$$\left(\bar{u}_{s'}(p'_p)\gamma_\mu u_s(p_p)\right)_p = 2M\begin{pmatrix}\xi'^\dagger_s & 0\end{pmatrix}\gamma_\mu\begin{pmatrix}\xi_s \\ 0\end{pmatrix} = 2Mg_{\mu0}\delta_{ss'}$$

so we obtain

$$\mathcal{M}_{\text{LAB}} = i\frac{q_eq_p}{t}\left(\bar{u}_{r'}(p'_e)g_{\mu0}\gamma^\mu u_r(p_e)\right)_e 2M\delta_{ss'} = i\frac{q_eq_p}{t}\left(\bar{u}_{r'}(p'_e)\gamma^0 u_r(p_e)\right)_e 2M\delta_{ss'}$$

[IX] See Schwartz sec. 5.2

[X] The last identity can be easly proved using an explicit representation of gamma matrices.

The unpolarized squared amplitude in the lab frame reads

$$\overline{|\mathcal{M}|}^2_{\text{LAB}} = \left(\frac{q_e q_p}{t}\right)^2 \left(\sum_{s,s'} 4M^2 \delta_{ss'}\right) \frac{1}{2} \sum_{rr'} \left(\bar{u}_{r'}(p'_e)\gamma^0 u_r(p_e)\right)_e \left(\bar{u}_r(p_e)\gamma_0 u_{r'}(p'_e)\right)_e$$
$$= \left(\frac{q_e q_p}{t}\right)^2 2M^2 \operatorname{Tr}\left[(\not{p}_e + m)\gamma^0(\not{p}'_e + m)\gamma_0\right] = \cdots = 16\left(\frac{q_e q_p}{t}\right)^2 M^2 \omega_e^2 \left(1 - v^2 \sin^2\left(\frac{\theta}{2}\right)\right)$$

where we define the magnitude of the speed of the electron

Verify last step, probably is wrong. See both Schwartz and Mandl. Also next calculation can be wrong

$$v = \frac{|\mathbf{p}_e|}{m}$$

We also have

$$t = -4\omega_e^2 v^2 \sin^2\left(\frac{\theta}{2}\right)$$

Putting pieces together we obtain

$$\overline{|\mathcal{M}|}^2_{\text{LAB}} = (q_e q_p)^2 \frac{\omega_e^2 M^2}{|\mathbf{p}_e|^4} \left(\frac{1 - v^2 \sin^2\left(\frac{\theta}{2}\right)}{\sin^4\left(\frac{\theta}{2}\right)}\right)$$

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\omega}\right)_{\text{LAB}} = \frac{\alpha^2\left(1 - v^2 \sin^2\left(\frac{\theta}{2}\right)\right)}{4\omega_e^2 v^4 \sin^4\left(\frac{\theta}{2}\right)} \qquad (q_p = -q_e = e)$$

The latter is known as *Mott formula*. In the non-relativistic limit we can use $v \ll 1$ and $p_e \ll \omega_e \sim m_e$, thus

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\omega}\right)^{\text{non-rel.}}_{\text{LAB}} = \frac{\alpha^2}{4\omega_e^2 v^4 \sin^2\left(\frac{\theta}{2}\right)}$$

which is the Rutherford formula. We can consider a generic nucleous with atomic number Z, then

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\omega}\right)^{\text{non-rel.}}_{\text{LAB}} = \frac{\alpha^2 Z^2}{4\omega_e^2 v^4 \sin^2\left(\frac{\theta}{2}\right)}$$

3.6.2 Generic external E.M. field

See Mandl sec. 8.7

We can use a more general approach, where instead of considering the proton as a fundamental particle, we consider it as a source of E.M. field, considered as an external field in our theory.

So far, the electromagnetic field has been described by a quantized field, involving photon creation and annihilation operators. In some problem, where the quantum fluctuations are unimportant, it may be adequate to describe the field as a purely classical function of the space-time coordinates. In cases such as Rutherford scattering, we consider an applied external electromagnetic field $A^{\text{ext}}(x)$, such as the Coulomb field of a heavy nucleus. More generally, one may have to consider both types of field, replacing A by the sum of the quantized and the classical fields, $A(x) + A^{\text{ext}}(x)$.

In the last section, we have seen that matrix element of the scattering amplitude reads

$$S_{fi} = \langle f|\, S_{(2)}\, |i\rangle = \left\langle e^-_{r'}(p'_e) p_{(s')}(M)\right|\, S_{(2)}\, \left|e^-_r(p_e) p_{(s)}(M)\right\rangle$$
$$= \int \mathrm{d}^4x \left\langle e^-_{r'}(p'_e)\right| (-iq_e)\left[\bar{\psi}_e\gamma^\mu\psi_e\right]_x \left|e^-_r(p_e)\right\rangle \int \mathrm{d}^4y \left\langle p_{(s')}(M)\right| (-iq_p)\left[\bar{\psi}_p\gamma^\nu\psi_p\right]_y \left|p_{(s)}(M)\right\rangle iD^F_{\mu\nu}(x-y)$$
$$= -iq_e \int \mathrm{d}^4x \left\langle e^-_{r'}(p'_e)\right| \left[\bar{\psi}_e\gamma^\mu\psi_e\right]_x \left|e^-_r(p_e)\right\rangle \int \mathrm{d}^4y D^F_{\mu\nu}(x-y) \underbrace{q_p \left\langle p_{(s')}(M)\right| \left[\bar{\psi}_p\gamma^\nu\psi_p\right]_y \left|p_{(s)}(M)\right\rangle}_{J^\nu_p(y)}$$

where $J^\nu_p(y)$ is a 4-current density, associated to the proton field. We can therefore define the classic field (i.e. a function)

$$A^{\text{ext}}_\mu(x) = \int \mathrm{d}^4y D^F_{\mu\nu}(x-y) J^\nu_p(y)$$

which Maxwell equation reads[XI]

Metti qualche commento sulle funzioni di green

$$\Box A^\mu_{\text{ext}}(x) = \int \mathrm{d}^4 y \left(\Box D^F_{\mu\nu}(x-y)\right) J^\nu_p(y) = J^\nu_p(x)$$

i.e. $A^\mu_{\text{ext}}(x)$ can be interpreted as a classic Maxwell field with external source $J^\nu_p(x)$. With this notation we obtain

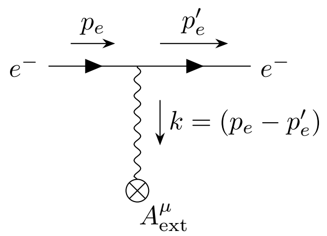
$$S_{fi} = -iq_e \int \mathrm{d}^4 x \left\langle e^-_{r'}(p'_e)\right| \left[\bar{\psi}_e \not{A}_{\text{ext}} \psi_e\right]_x \left|e^-_r(p_e)\right\rangle$$

We obtained the same result if we would have defined the QED interaction lagrangian as

$$\mathcal{L}_{\text{int}} = -q_e \bar{\psi}\gamma^\mu \psi (A_\mu + A^{\text{ext}}_\mu)$$

and we considered its expantion at the first order (the field A_μ vanishes in the matrix element because of the sandwich with initial and final states). Notice that A_μ is a quantum field (i.e. an operator) while A^{ext}_μ is a classic field (i.e. a function).

From a "graphical" point of view we introduce a new symbol, marking external source by a crossed circle



As a simple example, let's assume that A^μ_{ext} is a static potential.[XII] In this case we obtain same result as classic Rutherford scattering. First we express the external field in momentum space[XIII]

$$A^\mu_{\text{ext}}(x) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3 k \, e^{i\mathbf{k}\cdot\mathbf{x}} \varepsilon^\mu_{\text{ext}}(\mathbf{k})$$

Then the S matrix element at first order reads

$$\begin{aligned}
S_{fi} &= -iq_e \int \mathrm{d}^4 x \left\langle e^-_{r'}(p'_e)\right| \bar{\psi}_e \not{A}_{\text{ext}} \psi_e \left|e^-_r(p_e)\right\rangle \\
&= -iq_e \int \mathrm{d}^4 x \left\langle e^-_{r'}(p'_e)\right| \bar{\psi}_e \gamma^\mu \psi_e \left|e^-_r(p_e)\right\rangle \frac{1}{(2\pi)^3} \int \mathrm{d}^3 k \, e^{i\mathbf{k}\cdot\mathbf{x}} \varepsilon^{\text{ext}}_\mu(\mathbf{k}) \\
&= -iq_e \frac{1}{(2\pi)^3} \int \mathrm{d}^4 x \, \mathrm{d}^3 k \, \bar{u}(p'_e)\gamma^\mu u(p_e) \, \varepsilon^{\text{ext}}_\mu(\mathbf{k}) \, e^{-ip\cdot x + ip'\cdot x + i\mathbf{k}\cdot\mathbf{x}} \\
&= -iq_e \int \mathrm{d}x \, \mathrm{d}^3 k \, \delta^3(\mathbf{k} + \mathbf{p} - \mathbf{p}')\bar{u}(p'_e)\gamma^\mu u(p_e) \, \varepsilon^{\text{ext}}_\mu(\mathbf{k}) \, e^{-i(\omega_e - \omega'_e)x} \\
&= -iq_e (2\pi)\delta(\omega_e - \omega'_e)\bar{u}(p'_e)\gamma^\mu u(p_e) \, \varepsilon^{\text{ext}}_\mu(\mathbf{p}' - \mathbf{p}) \\
&= (2\pi)\delta(\omega_e - \omega'_e)\mathcal{M}^{\text{ext}}_{fi}
\end{aligned}$$

where

$$\mathcal{M}^{\text{ext}}_{fi} = -iq_e \bar{u}(p'_e)\gamma^\mu u(p_e) \, \varepsilon^{\text{ext}}_\mu(k)$$

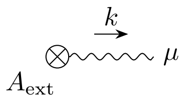
Analysing $\mathcal{M}^{\text{ext}}_{fi}$, we notice that the Feynman rules for QED with external field are the usual ones, using factors $\tilde{A}^\mu_{\text{ext}}(k)$ instead of polarization vector for the Maxwell field. Set again $\varepsilon^{\text{ext}}_\mu(k) = \tilde{A}^{\text{ext}}_\mu(k)$ we have the following rules

[XI] Here we used proprieties of Green functions

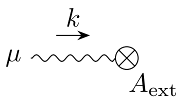
[XII] This is not a restrictive assumption.

[XIII] Notice that $\varepsilon^\mu_{\text{ext}}(\mathbf{k}) = \tilde{A}^\mu_{\text{ext}}(\mathbf{k})$, but we use this notiation in analogy with the quantum Maxwell field expantion.

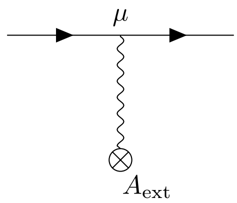
$\varepsilon_\mu^{\text{ext}}(k) =$



$\varepsilon_\mu^{\text{ext}\,*}(k) =$



$-iq_e\gamma^\mu =$



Other rules are unchanged.

Let's consider the cross section for the incoming electron[XIV]

$$\begin{aligned}
\mathrm{d}\omega_{fi} &= |S_{fi}^{CN}|^2 \frac{V\mathrm{d}^3p'_e}{(2\pi)^3} = (2\pi)^2\delta(\omega_e - \omega'_e)\frac{T}{2\pi}\left(\frac{V\mathrm{d}^3p'_e}{(2\pi)^3}\right)|M_{fi}^{\text{ext}}|^2 \\
&= (2\pi)\delta(\omega_e - \omega'_e)T\frac{1}{2\omega_e V}\frac{1}{2\omega'_e V}\left(\frac{V\mathrm{d}^3p'_e}{(2\pi)^3}\right)|\mathcal{M}_{fi}^{\text{ext}}|^2 \\
&= (2\pi)\delta(\omega_e - \omega'_e)\frac{T}{2\omega_e V}\left(\frac{\mathrm{d}^3p'_e}{(2\pi)^3\,2\omega'_e}\right)|\mathcal{M}_{fi}^{\text{ext}}|^2
\end{aligned}$$

and (in lab frame, with number densities $n_i = 1/V$)

$$\mathrm{d}\sigma = \frac{\mathrm{d}\omega_{fi}}{n_e n_p v_e VT} = \frac{V}{Tv_e}\mathrm{d}\omega_{fi} = \frac{1}{16\pi^2}\delta(\omega_e - \omega'_e)\frac{|\mathcal{M}_{fi}^{\text{ext}}|^2}{\omega_e\omega'_e v_e}\mathrm{d}^3p'_e$$

Writing $\mathrm{d}^3p'_e = |p'_e|^2\,\mathrm{d}|p'_e|\,\mathrm{d}\Omega \approx \omega_e'^2\,\mathrm{d}\omega'_e\,\mathrm{d}\Omega$, integrating over the magnitude of final momentum $|p'_e|$, and averaging over polarizations, we obtain ($|p'_e| \approx \omega'_e$)

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)_{\text{LAB}} = \frac{1}{16\pi^2}\frac{|p'_e|^2}{\omega_e v_e}\overline{|\mathcal{M}_{fi}^{\text{ext}}|}^2_{\text{LAB}}$$

The squared unpolarized amplitude reads

$$\begin{aligned}
\overline{|\mathcal{M}_{fi}^{\text{ext}}|}^2_{\text{LAB}} &= \frac{1}{2}\sum_{rr'} q_e^2\varepsilon_\mu^{\text{ext}}(k)\varepsilon_\nu^{\text{ext}\,*}(k)\bar{u}_{r'}(p'_e)\gamma^\mu u_r(p_e)\bar{u}_r(p_e)\gamma^\nu u_{r'}(p'_e) \\
&= \frac{1}{2}q_e^2\varepsilon_\mu^{\text{ext}}(k)\varepsilon_\nu^{\text{ext}\,*}(k)\,\mathrm{Tr}\Big[(\not{p}'_e + m)\gamma^\mu(\not{p}_e + m)\gamma^\nu\Big]
\end{aligned}$$

For example, in the Rutherford scattering, the external field is a Coulomb field, i.e.

$$A^\mu_{\text{ext}}(x) = \left(\frac{Zq_p}{4\pi|\mathbf{x}|}, 0, 0, 0\right)$$

Taking its Fourier transform

$$\varepsilon^\mu_{\text{ext}}(x) = \left(\frac{Zq_p}{|\mathbf{k}|^2}, 0, 0, 0\right)$$

we obtain

Complete calculations, see also previous section. Notice that mandl use wrong normalization

[XIV] Respect to the calculations done in chapter 2, we must make the replace $(2\pi)^4\delta^4(P_f - P_i) \to (2\pi)\delta(\omega'_e - \omega_e)$. Notice that in the first line the amplitude is canonically normalized.

$$\overline{|\mathcal{M}_{fi}^{\text{ext}}|}^2_{\text{LAB}} = \frac{q_e^2 q_p^2 Z^2}{2|\mathbf{k}|^4} \operatorname{Tr}\Big[(\not{p}'_e + m)\gamma^0(\not{p}_e + m)\gamma^0\Big] = \dots$$

Introducing scattering angle we obtain again the Mott formula

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)_{\text{LAB}} = \dots = \frac{(\alpha Z)^2}{4\omega_e^2 v_e^4}\left(\frac{1 - v_e^2 \sin^2(\theta/2)}{\sin^4(\theta/2)}\right)$$

for scattering of relativistic electrons by a Coulomb field. In the non-relativistic limit ($v_e \ll 1$) this reduces to the Rutherford scattering formula

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)_{\text{LAB}} = \frac{(\alpha Z)^2}{4m^2 v_e^4 \sin^4(\theta/2)}$$

We have here only considered the nucleus as a point charge. We only mention that the treatment is easily modified to the realistic case of a nucleus whose charge is distributed over a finite volume. For high energy electrons, this leads to an important method of investigating nuclear charge distribution.

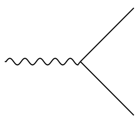
Chapter 4

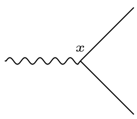
QED processes at higher order

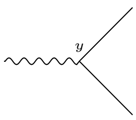
4.1 Beyond the tree-level

Up to now we have calculated QED processes in the lowset order of perturbation theory. On taking higher order into account we expect correction contributions from both real and virtual radiation. However, these corrections drag with them divergences in the corresponding integrals which make the results unphysical. In order to remove these inherent effects in the higher order perturbative approach, we can consider adopting the tools provided by regularization and renormalization theory based on the concept of modifying quantities to keep the finite and well-defined feature of the physical theory.

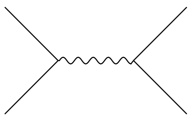
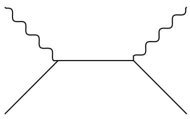
Feynman diagrams representing higher order corrections contain additional vertices, compared with those describing the process in the lowest order of perturbation theory. In the following we write down some of these diagrams:

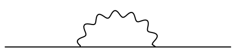
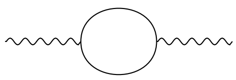
$S_{(1)}$ $\Rightarrow$  Not Physical

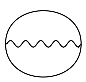
$S_{(2)}$ $\Rightarrow$ 

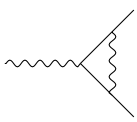
$\times$ 

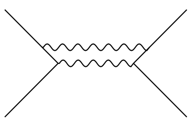
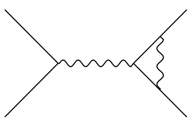
0 propagators

 ,  1 propagator

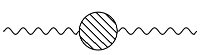
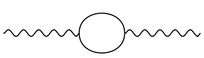
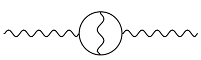
 ,  2 propagators

 (Vacuum) 3 propagators

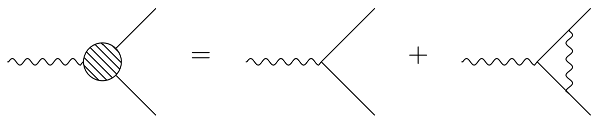
$S_{(3)}$ $\Rightarrow$ 

$S_{(4)}$ $\Rightarrow$  , 

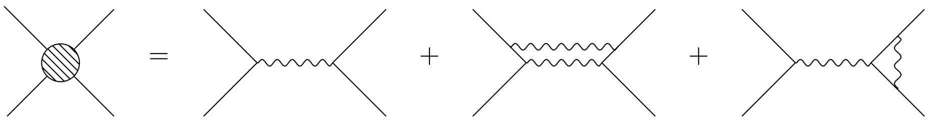
The 1 loop diagrams with 2 external particles are higher order corrections of the free propagator:

 $=$ $+$  $+$ 

Diagrams at third order loops modifies tree level interactions



And at fourth order loops affects the scattering amplitude



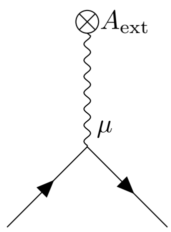
I call processes without loops (*tree*) α_{EM}**-order processes**, then processes with n loops are $\alpha_{EM} \circ (\alpha_{EM}^n)$ **order processes**.

We have so far shown how to calculate processes in the order $\alpha_{EM}(1+o(\alpha_{EM})+\dots)$. Since QED is a perturbative theory, theoretical result we obtained so far (considering only tree-level diagrams) coincides with experimental results only up to the first order in α_{EM}. In order to obtain better results we must consider higher order contibutions. See next example:

Example 5: Magnetic Dipole Momentum of e^-

Mandl, sec 9.6.1

The magnetic moment of a particle shows up through the scattering of the particle by a magnetic field. For this reason we shall one more study the elastic scattering of an electron by a static potential. We considered this process in lowest order in sec.3.6.2, whose diagram at lowest order is:



We can rewrite the lowest order scattering amplitude as follows (using *Gordon Identity*)[a]

$$iq_e\bar{u}(p')\gamma^\mu u(p) = i\left(\frac{q_e}{2m_e}\right)\bar{u}(p')\left\{(p'+p)^\mu + 2i\Sigma^{\mu\nu}(p'-p)_\nu\right\}u(p) \tag{4.1}$$

Physically this means that γ^μ (related to QED interaction) using Dirac e.o.m. can be rewritten in terms of

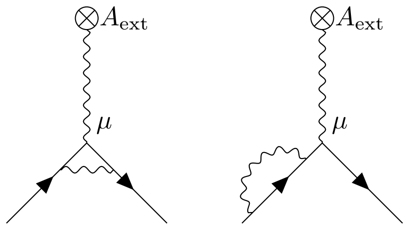
(i) $(p'+p)^\mu$ (which is related to e^- coupling)

(ii) $g_e\Sigma^{\mu\nu}$ (where $\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ is the Lorentz generator and $g_e = 2$ is the gyromagnetic ratio)

(iii) $(p'-p)_\nu$ (which is related to spin coupling)

The the non-relativistic limit of slowing moving particles and static magnetic field, the second term in (4.1) is just the amplitude for the scattering of a spin 1/2 particle with magnetic moment $(-e/2m)$, i.e. with gyromagnetic ratio $g_e = 2$. In QM the factor $g_e = 2$ is given by experimental results, while QFT predict this value at lowest order.

At higher orders, QED predicts $o(\alpha_{EM})$ corrections to g_e. Let's consider higher order diagrams, i.e. diagrams with one loop, for example[b]



All possible diagrams with one loop gives a $o(\alpha_{EM})$ correction. In 1948 Schwinger did the calculation and obtained another term in the amplitude of the process, namely

$$i\left(\frac{q_e}{2m}\right)\bar{u}(p')\left\{\left(\frac{\alpha}{2\pi}\right)2i\Sigma^{\mu\nu}(p'-p)_\nu\right\}u(p)$$

Then the QED prediction at $o(\alpha_{EM})$ is

$$g_e = 2\left(1+\frac{\alpha}{2\pi}\right)+o(\alpha^2) \quad \to \quad \left(\frac{g_e-2}{2}\right)_{TH} = \frac{\alpha}{2\pi} = 0.00116$$

Kusch and Foley (respectively in 1947 and 1948) done experiments and found

$$\left(\frac{g_e-2}{2}\right)_{EXP} = \frac{\alpha}{2\pi} = 0.00119 \pm 0.00005$$

This was the first prove that we need QED to describe EM interactions with high precision.

Subsequently, both theory and experiment have been greatly refined. Theoretically, the high order corrections of order α^2, α^3 and α^4 have been calculated. The result of these heavy calculations is

$$10^{12}\left(\frac{g_e-2}{2}\right)_{TH} = 1159652183 \pm 8$$

while the experimental value is

$$10^{12}\left(\frac{g_e-2}{2}\right)_{EXP} = 1159652182 \pm 7$$

The agreement can only be described as remarkable.

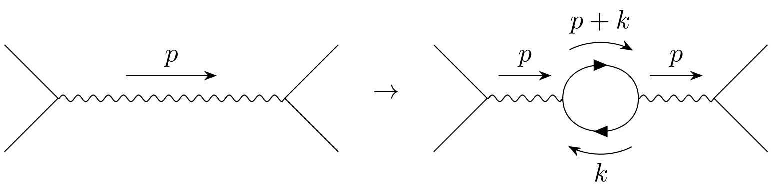
[a] See Mandl for explicit derivation.
[b] All possible diagrams with one loop are shown in Mandl.

When calculating diagrams with loops often one finds divergences. Consider for example the photon self-energy

Example 6: Photon Self-Energy

Mandl sec. 9.2

Consider the effect of the photon self-energy insertion in the photon propagator



In the Feynman amplitudes, the insertion of the photon propagator corresponds to the replacement

$$D_{F\alpha\beta}(p) \quad \to \quad D_{F\alpha\mu}(p)\left(-iq_e)^2\Pi^{\mu\nu}(p,m)\right)D_{F\nu\beta}(p)$$

where

$$(-iq_e)^2\Pi^{\mu\nu}(p,m) = (-1)(-iq_e)^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[((\not{p}+\not{k})+m)\gamma^\mu(\not{k}+m)\gamma^\nu\right]}{[(p+k)^2-m^2][k^2-m^2]} \tag{4.2}$$

This formula express the possibility of creating virtual electrons with any momentum k. This leads to some problem since the integral is divergent. Take $\Lambda > 0$. Then:

$$\Pi^{\mu\nu}(p,m) = \int_0^\Lambda(\dots) + \lim_{\Lambda_0\to\infty}\int_\Lambda^{\Lambda_0}(\dots) \equiv \Pi^{\mu\nu}_{Fin}(p,m) + \Pi^{\mu\nu}_{Div}(p,m)$$

Suppose $\Lambda \gg p, m$, then

$$\begin{aligned}\Pi^{\mu\nu}_{Div}(p,m) &= \lim_{\Lambda_0\to\infty}\int_\Lambda^{\Lambda_0}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}[\not{k}\gamma^\mu\not{k}\gamma^\nu]}{k^4}\\ &\propto \lim_{\Lambda_0\to\infty}\int_\Lambda^{\Lambda_0}\mathrm{d}^4k\frac{k^\mu k^\nu}{k^4}\\ &\simeq \lim_{\Lambda_0\to\infty}\Lambda_0^2\end{aligned}$$

i.e. the integral is quadratic divergent.

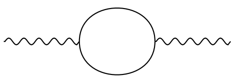
4.2 Superficial degree of divergence and renormalizability condition on the coupling constant

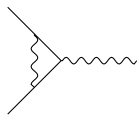
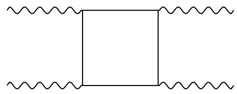
Peskin sec. 10.1

The divergence shown in the previous example (for $k\to\infty$) is called **ultraviolet divergence**. We can also say that the **superficial degree of divergence** D of this diagram is $D=2$.

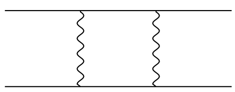
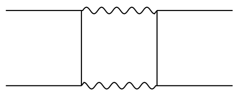
Exercise 6

Verify that following diagrams are superficially divergent ($D\geq 0$):



Verify that following diagrams are not superficially divergent ($D<0$):

There is a simple formula to determinate the superficial degree of divergence of a diagram. First we introduce some notation for diagrams:

(i) D is the superficial degree of divergence

(ii) L is the number of loops (i.e. the number of independent momentum in Feynman graph)

(iii) $E_{B/F}$ is the number of external bosons/fermions

(iv) $n_{B/F}$ is the number of bosons/fermions attached to each vertex

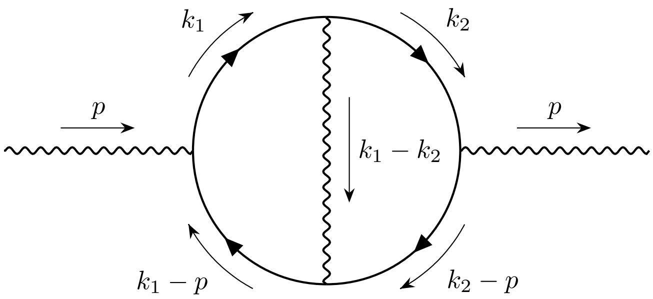
(v) $P_{B/F}$ is the number of bosonic/fermionic propagators

(vi) V is the number of vertex in the diagram

First, notice that the following topological propriety holds:

$$L = P_B + P_F - V + 1$$

Example 7



$$\Rightarrow \begin{cases} 2 \text{ loops } (k_1, k_2) \\ 5 \text{ props} \\ 4 \text{ vertex} \end{cases}$$

Since the number of bosons and the number of fermions attached to each vertex is constant,[I] then the number of vertices can also be express in terms of external particles:

$$V = \frac{2P_B + E_B}{n_B} = \frac{2P_F + E_F}{n_f}$$

Notice that each boson propagator decrease the degree of divergence by 2 since it's related to a factor $\frac{1}{k^2}$, while for fermionic propagators I have a factor $\frac{1}{\not{k}} = \frac{\not{k}}{k}$ so it decrease the degree of divergence by 1. Then I obtain

$$D = 4L - 2P_B - P_F$$

Putting all these relations together I obtain:

$$D = 4 - \left(4 - n_B - \frac{3}{2}n_F\right) V - E_B - \frac{3}{2}E_F$$

Notice that this formula holds for any theory involving bosons and fermions, since we didn't made any restrictive assumption on the theory. The dimension of the lagrangian is 4, while dimensions of bosonic and fermionic fields are respectively 1 and 3/2, therefore let $[g]$ be the dimension of the coupling constant g of my theory, I have

$$[g] = 4 - n_B - \frac{3}{2}n_F$$

and then

$$\boxed{D = 4 - [g]V - E_B - \frac{3}{2}E_F} \tag{4.3}$$

Example 8

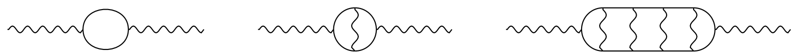
In QED $n_B = 1$ and $n_F = 2$, so $[g] = 4 - n_B - \frac{3}{2}n_F = 0$ as we expected since interaction lagrangian is $\mathcal{L}_{int} = q\bar{\psi}\not{A}\psi$ and $[q] = 0$.

Notice that from (4.3) follows that in theories with $[g] = 0$ (therefore QED is included) the degree divergence is independent by the number of vertices in diagrams.

Example 9

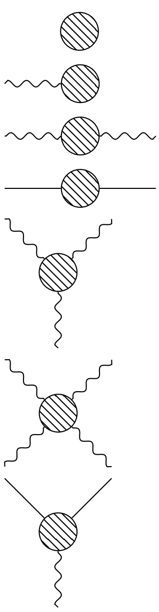
[I] For any theory, n_B and n_F are fixed by the interaction lagrangian. For example, in QED $n_B = 1$ and $n_F = 2$

Following diagrams have all $D = 2$:



A necessary condition for renormalizability is that $[g] \leq 0$, otherwise if $[g] < 0$ then D increases with the number of vertex and, therefore, with the order of the perturbative expansion, making perturbative expansion useless.

In spite of the superficial degree of divergente, doing explicit calculations we find that actually the degree of divergence of diagrams may be smaller that the one we obtained through the topological analysis, for example in QED we have these seven amplitudes whose superficial degree of divergence is ≥ 0:



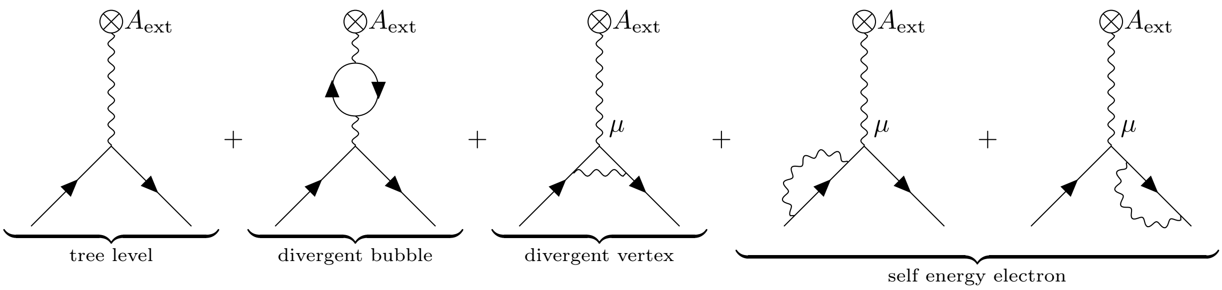
= Vacuum	$D = 4$	Irrelevant
= Tadpole	$D = 3$	Vanishes
= Photon Self-Energy	$D = 2$	Log divergence
= Electron Self-Energy	$D = 1$	Log divergence
$= 3\gamma$	$D = 1$	Vanishes
$= 4\gamma$	$D = 0$	Finite
= Vertex	$D = 0$	Log Divergence

This is due to the additional degrees of freedom of QED. When a theory has additional symmetries (as gauge symmetry for QED $\to$ Ward Identity) the divergence can be smaller than superficial divergence D.

4.3 Basic idea behind the renormalization procedure

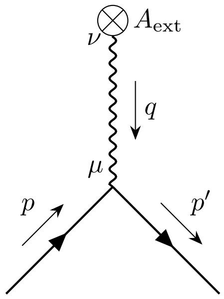
Halzen sec. 7.2; Mandl sec. 9.1, 9.2

Let's consider again the scattering by an external potential problem. Now we consider higher corrections in the total amplitude (in the previous analysis we considered only the first diagram):



We already know Feynman amplitude for the first diagram

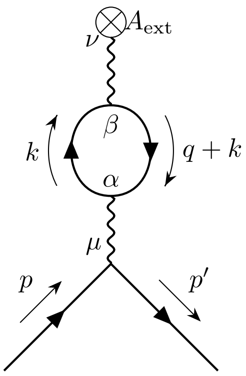
$$\mathcal{M}_0 =$$



$$= (-ie)\bar{u}(p')\gamma^\mu u(p) g_{\mu\nu}\varepsilon^\nu_{\text{ext}}(q)$$

For the diagram with the photon self-energy we can use previous computation obtaining

$$\mathcal{M}_1 =$$



$$= (-ie)\bar{u}(p')\gamma^\mu u(p)\left(-i\frac{g_{\mu\alpha}}{q^2}\right)(-ie)^2\Pi^{\alpha\beta}(q,m)\left(-i\frac{g_{\beta\nu}}{q^2}\right)\varepsilon^\nu_{\text{ext}}(q)$$

where $\Pi^{\alpha\beta}(q,m)$ was defined in (4.2):

$$\Pi^{\alpha\beta}(q,m) = (-1)\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[(\not{q}+\not{k}+m)\gamma^\alpha(\not{k}+m)\gamma^\beta\right]}{[(q+k)^2-m^2][k^2-m^2]}$$

This integral is quadratically divergent for large k. In order to handle it, we must regularize it, that is, we most modify it so that it becomes a well-defined finite integral, that is, we must modify it so that it becomes a well-defined finite integral. For example, this could be archived by multiplying int integrand in the previous equation by the convergence factor

$$\left(\frac{-\Lambda^2_\infty}{k^2-\Lambda^2_\infty}\right)^2$$

Here, Λ_∞ is a **cut-off** parameter. For large, but finite, values of Λ_∞, the integral now behaves like $\int \mathrm{d}^4k/k^6$ for large k, and is well- defined and convergent. For $\Lambda_\infty \to \infty$, the factor tends to unity, and the original theory is restored. One can think of such convergence factor either as a mathematical device, introduced to overcome a very unsatisfactory feature of QED, or as a genuine modification of QED at very high energies, i.e. at very small distances, which should show up in experiments at sufficently high energies.

Let assume that the theory has been already regularized in this way, so that all expression are well-defined, finite and gauge invariant.

After regularization $\Pi^{\alpha\beta}(q^2)$ expression can be simplified. It follows from Lorentz invariance that $\Pi^{\alpha\beta}(q^2)$ must be of the form

$$e^2\Pi^{\alpha\beta}(q^2) = -ig^{\alpha\beta}I(q^2) + iq^\alpha q^\beta K(q^2)$$

since this is the most general second-rank tensor which can be formed using only the four-vector q^μ. From the Ward Identity (i.e. gauge invariance) follows that the second term proportional to the photon momentum q give vanishing contributions, hence, it can be omitted. $I(q^2)$ takes the form

$$I(q^2) = \frac{\alpha}{3\pi}\int_{m^2}^{\Lambda_\infty}\frac{\mathrm{d}k^2}{|k|^2} - \frac{2\alpha}{\pi}\int_0^1 \mathrm{d}z(1-z)\log\left(1-\frac{q^2z(1-z)}{m^2}\right)$$

where m is the mass of the electron and Λ_∞ is the cutoff parameter that we introduced in order to regularize the integral. Here we notice that the divergence is due to the first integral, and has logarithmic behaviour. After we will have solved the integral we have to take the limit $\Lambda \to \infty$. With this procedure we avoid the calculation of indeterminated divergent integrals.

In the approximation $q^2 \ll m^2$ we obtain following formula for $I(q^2)$:

$$I(q^2) \simeq \frac{\alpha}{3\pi} \log\left(\frac{\Lambda_\infty^2}{m^2}\right) + \frac{\alpha}{15\pi} \frac{q^2}{m^2} + o\left(\frac{q^2}{m^2}\right)$$

Unless we can dispose of the infinite part of $I(q^2)$ the result will not be physically meaningful.

The way to proceed is best explained by returning to Rutherford scattering. Recall $\varepsilon^\nu_{\text{ext}}(q) = (Ze/q^2, 0, 0, 0)$, including the loop contribution to the tree-order amplitude, we obtain for the small q^2 limit the amplitude

$$\mathcal{M} = -i\frac{Ze^2}{q^2}\left(1 - \frac{e^2}{12\pi^2}\log\left(\frac{\Lambda_\infty^2}{m^2}\right) - \frac{e^2}{60\pi^2}\frac{q^2}{m^2} + o\left(\alpha^2, \frac{q^4}{m^4}\right)\right)$$

Now, notice that the parameter e is just a theoretical parameter, called **bare parameter** $e_B = e$, that cannot be measured. Then I can introduce a new parameter called **renormalized parameter**

$$e_R \equiv e_B\left(1 - \frac{e^2}{12\pi^2}\log\frac{\Lambda_\infty^2}{m^2}\right)^{1/2}$$

With this new variable Feynman amplitude is

$$\mathcal{M} = -i\frac{Ze_R^2}{q^2}\left(1 - \frac{e_R^2}{60\pi^2}\frac{q^2}{m^2} + o\left(\alpha^2, \frac{q^4}{m^4}\right)\right)$$

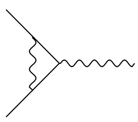
In this way we obtained a copmletely physical amplitude. Since the measurable quantities are scattering amplitudes, I solved the problem of divergent contributions in my theory due to photons self-energy (in the first order).

4.3.1 Renormalizable theories

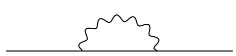
Up to now we saw that a sufficent condition for a theory to be renormalizable is that all the divergences can be absorbed into redefinitions of physical parameters.

In QED I have only two kind of particles: photons and electrons (and positrons). Therefore only parameters are the charge and the mass of the electron, which are the only parameter I can use to renormalize the theory.

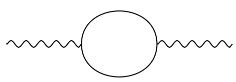
I have three kind of divergences:



$$= -ieZ_1\gamma^\mu$$



$$= \frac{+iZ_2}{p^2 - m^2}$$



$$= -i\frac{g^{\mu\nu}}{p^2}Z_3$$

i.e. I have 3 divergent terms to be removed: Z_1, Z_2, Z_3. Since I have only 2 parameters I can use to remove these divergent terms, it seems that QED is not renormalizable. Luckily, doing explicit calculations, I obtain that thanks to gauge invariance (Ward Identities) I obtain $Z_1 = Z_2$ and therefore QED is renormalizable using opportune definitions of free parameters:

$$e_R = e_B\left(1 - \frac{\delta e}{e}\right) \qquad\qquad m_R = m_B\left(1 - \frac{\delta m}{m}\right)$$

where $\delta e/e$ and $\delta m/m$ are divergent terms. We obtained that QED is renormalizable at any order.

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