

Chapter 1

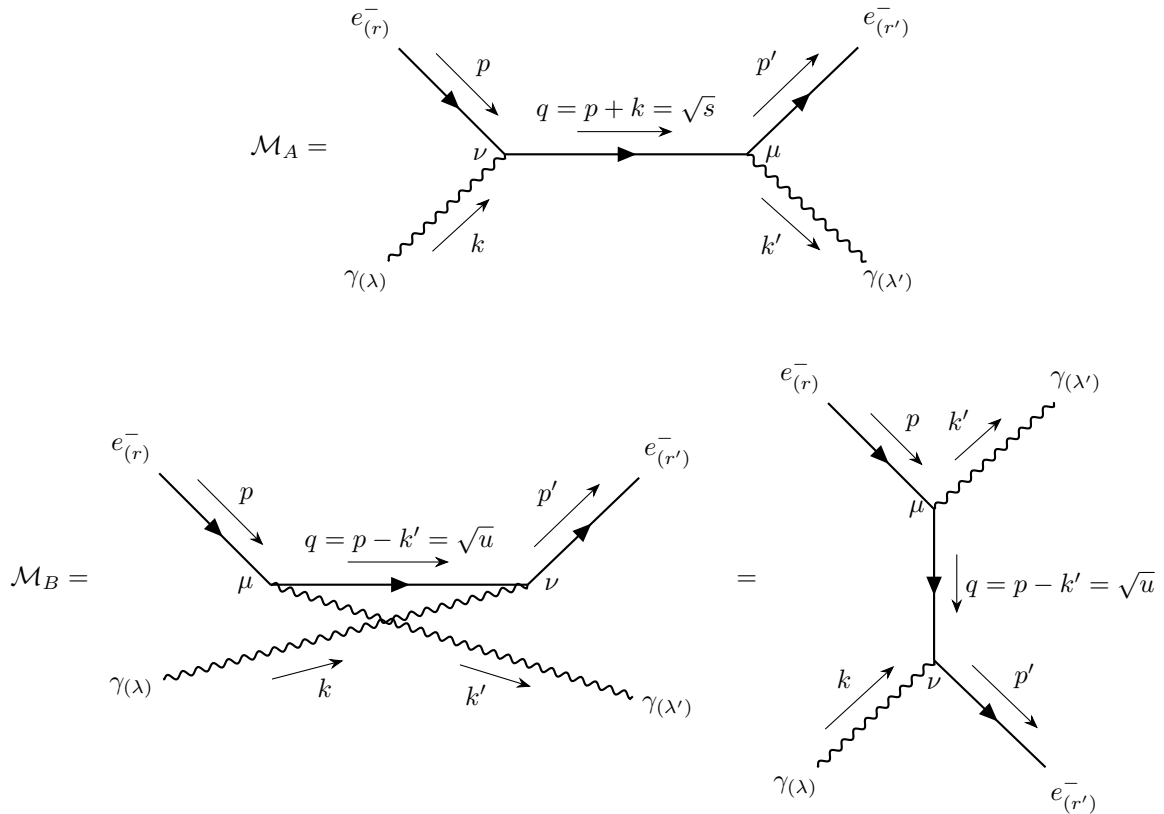
QED Processes at Lowest order

1.1 Fogli di Emma <3

1.2 $e^- \gamma \rightarrow e^- \gamma$ (Compton)

See Peskin, sec 5.5

Let's examine a process with external bosons: *Compton scattering*, or $e^- \gamma \rightarrow e^- \gamma$. This process is described by two independent diagrams, since they are topologically different:



We wrote the diagram of \mathcal{M}_B in two topologically equivalent forms: in the first one is clear the topological relation with diagram of \mathcal{M}_A (this is useful to find the relative sign between diagrams A and B : it's clear that diagrams differs for the permutation of two bosons), while in the second one is clear that it describes a u -channel.

Amplitudes reads, using Feynman rules

$$\begin{aligned}
\mathcal{M}_A &= \bar{u}_{r'}(p')(-iq\gamma^\mu)\varepsilon_\mu^{\lambda'*}(k')\tilde{S}_F(p+k)(-iq\gamma^\nu)\varepsilon_\nu^\lambda(k)u_r(p) \\
&= -q^2\varepsilon_\mu^{\lambda'*}(k')\varepsilon_\nu^\lambda(k)\left[\bar{u}_{r'}(p')\gamma^\mu\tilde{S}_F(p+k)\gamma^\nu u_r(p)\right] \\
\mathcal{M}_B &= -q^2\varepsilon_\mu^{\lambda'*}(k')\varepsilon_\nu^\lambda(k)\left[\bar{u}_{r'}(p')\gamma^\nu\tilde{S}_F(p-k')\gamma^\mu u_r(p)\right]
\end{aligned}$$

(Recall that \tilde{S}_F is a matrix, so elements in the squared bracket must be written in this order)
Because of anticommuting relations for bosons, these amplitudes must be summed up in the total amplitude. The explicit form of Feynman propagator for the Dirac field reads

$$\tilde{S}_F(p) = \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon} = \frac{i}{\not{p} - m + i\varepsilon}$$

so total amplitude is

$$\mathcal{M} = -iq^2\varepsilon_\mu^{\lambda'*}(k')\varepsilon_\nu^\lambda(k)\bar{u}_{r'}(p')\left[\frac{\gamma^\mu(\not{p} + \not{k} + m)\gamma^\nu}{(p+k)^2 - m^2} + \frac{\gamma^\nu(\not{p} - \not{k}' + m)\gamma^\mu}{(p-k')^2 - m^2}\right]u_r(p)$$

We make some simplifications before squaring this expression. Since $p^2 = m^2$ and $k^2 = 0$:

$$(p+k)^2 - m^2 = 2p \cdot k \quad (p-k')^2 - m^2 = -2p \cdot k'$$

To simplify numerators, I can use Dirac algebra:

$$\begin{aligned}
(\not{p} + m)\gamma^\nu u(p) &= (p_\mu\gamma^\mu\gamma^\nu + m\gamma^\nu)u(p) = (2g^{\mu\nu}p_\mu - p_\mu\gamma^\nu\gamma^\mu + m\gamma^\nu)u(p) \\
&= 2p^\nu u(p) - \underbrace{\gamma^\nu(\not{p} - m)}_{2m\Lambda_-(p)}u(p) = 2p^\nu u(p)
\end{aligned}$$

Using these tricks we obtain

$$\mathcal{M} = -iq^2\varepsilon_\mu^{\lambda'*}(k')\varepsilon_\nu^\lambda(k)\bar{u}_{r'}(p')\left[\frac{\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p \cdot k} + \frac{-\gamma^\nu\not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{-2p \cdot k'}\right]u_r(p)$$

1.2.1 The Ward Identities and sum over the photon polarizations

See Mandl, sec 8.3

The next step in the calculation will be to square this expression for \mathcal{M} and sum or average over electron and photon polarization states. The sum over electron polarizations can be performed as before, using $\sum u(p)\bar{u}(p) = \not{p} + m$. Fortunately, there is a similar trick for summing over photons polarization vectors. Gauge invariance of the theory implies the gauge invariance of the matrix elements, i.e. of the Feynman amplitudes. It is, of course, only the matrix element itself, corresponding to the sum of all possible Feynman graphs in a given order of perturbation theory, which must be gauge invariant. For example, for the Compton scattering, the individual amplitudes \mathcal{A} and \mathcal{B} are not gauge invariants, but their sum \mathcal{M} is.

For any process involving external photons, the Feynman amplitude \mathcal{M} is of the form

$$\mathcal{M} = \varepsilon_{\alpha}^{\lambda_1}(k_1)\varepsilon_{\beta}^{\lambda_2}(k_2)\dots L^{\alpha\beta\dots}(k_1, k_2, \dots) \quad (1.1)$$

with one polarization vector $\varepsilon^{\lambda_i}(k_i)$ for each external photon, and the tensor amplitude $L^{\alpha\beta\dots}(k_1, k_2, \dots)$ independent of these polarization vectors.

The polarization vectors are of course gauge dependent. For example, for a free photon described in the Lorentz gauge by the plane wave

$$A^\mu(x) = \text{const} \cdot \varepsilon_\lambda^\mu(k)e^{\pm ikx}$$

the gauge transformation

$$A^\mu \rightarrow A'^\mu(x) = A^\mu(x) + \partial^\mu \alpha(x) \quad \text{with} \quad \alpha(x) = \tilde{\alpha}(k) e^{\pm i k x}$$

implies

$$\varepsilon_\lambda^\mu(k) \rightarrow \varepsilon'^\mu_\lambda(k) = \varepsilon_\lambda^\mu(k) \pm i k^\mu \tilde{\alpha}(k)$$

Invariance of the amplitude Eq.(1.1) under this transformation requires

$$k_1^\alpha L_{\alpha,\beta,\dots}(k_1, k_2, \dots) = k_1^\beta L_{\alpha,\beta,\dots}(k_1, k_2, \dots) = \dots = 0$$

i.e. when any external photon polarization vector is replaced by the corresponding four momentum, the amplitude must vanish. This is the statement of the *Ward Identity*:

If $\mathcal{M}(k) = \varepsilon_\mu(k) L^\mu(k)$ is the amplitude for some QED process involving an external photon with momentum k , then this amplitude vanishes if we replace ε_μ with k_μ :

$$k_\mu L^\mu(k) = 0$$

Sistemare grafica

Creare ambiente esercizio, questo è un esercizio

Example 1

Verify explicitly the Ward Identity for the Feynman amplitude of Compton scattering

See Peskin, sec 5.5

Returning to our derivation of the polarization sum formula for squared scattering amplitude. Writing in general

$$\mathcal{M} = \varepsilon_\mu^{(\lambda)}(k) L^\mu(k)$$

then the sum over polarizations of the photon with momentum k reads

$$\sum_{\lambda=1,2} |\mathcal{M}|^2 = \sum_{\lambda=1,2} \varepsilon_\mu^{(\lambda)}(k) \varepsilon_\nu^{(\lambda)*}(k) L^\mu(k) L^{\nu\dagger}(k)$$

Because of the covariance of the theory we can do the calculation in a specific frame. In order to simplify the analysis we choose the frame where the photon moves along the \hat{z} axis:

$$k^\mu = (|k|, 0, 0, |k|)$$

In this case the Ward Identity reads

$$0 = k_\mu L^\mu = |k| (L^0 - L^3) \quad \longrightarrow \quad L^0 = L^3$$

Recall that in this frame

$$\varepsilon_\mu^{(1)}(k) = (0, 1, 0, 0) \quad \varepsilon_\mu^{(2)}(k) = (0, 0, 1, 0)$$

So we have

$$\sum_{\lambda=1,2} \varepsilon_\mu^{(\lambda)}(k) \varepsilon_\nu^{(\lambda)*}(k) L^\mu(k) L^{\nu\dagger}(k) = |L^1|^2 + |L^2|^2 = |L^1|^2 + |L^2|^2 + |L^3|^2 - |L^0|^2 = -g_{\mu\nu} L^\mu L^\nu$$

So we obtained the general rule to simplify photons polarization sum¹

$$\sum_{\lambda=1,2} \varepsilon_\mu^{(\lambda)}(k) \varepsilon_\nu^{(\lambda)*}(k) L^\mu(k) L^{\nu\dagger}(k) \quad \longrightarrow \quad -g_{\mu\nu}$$

Creare ambiente per riquadrare le formule

¹Notice that we could prove that even if we took $\lambda = 0, 1, 2, 3$, we could have obtained that the unphysical time-like and longitudinal photons can be consistently omitted from QED calculations, since in any event the squared amplitudes for producing these states cancel to give zero total probability.

1.2.2 The Klein-Nishima formula and the Thomson scattering

See (?)

1.3 Sezioni di esempio

Consider the case in which the initial state is a single particle and the final state is given by n particles. We are therefore considering a decay process. Assume for the moment that particles are indistinguishable.



The rules of quantum mechanics tell us that the probability for this process is obtained by taking the squared modulus of the amplitude and summing over all possible final states

$$\begin{aligned} |S_{fi}^{CN}|^2 &= |(2\pi)^4 \delta^4(p - p') M_{fi}^{CN}|^2 \\ &= (2\pi)^4 \delta^4(p - p') (VT) |M_{fi}^{CN}|^2 \\ &= (2\pi)^4 \delta^4(p - p') (VT) \frac{1}{2\omega_i n V} \prod_{l=1}^{n_f} \left(\frac{1}{2\omega_l V} \right) |\mathcal{M}_{fi}|^2 \end{aligned}$$

Note: $(\delta^4(p - p'))^2 = \delta^4(p - p') \delta^4(p - p') = \delta^4(p - p') \delta^4(0) = \delta^4(p - p') \frac{VT}{(2\pi)^4}$

We use the final space and time in order to remove divergent terms during calculation

We must now sum this expression over all final states. Since we are working in a finite volume V , this is the sum over the possible discrete values of the momenta of the final particles .

Since $p_i = (2\pi/L)n_i$, we have $dn_i = (L/2\pi)dp_i$ and $d^3n_i = (V/(2\pi)^3)d^3p$ where d^3n_i is the infinitesimal phase space related to a final state in which the i -th particle has momentum between p_i and $p_i + dp_i$

Let $d\omega$ be the probability for a decay in which in the final state the i -th particle has momentum between p_i and $p_i dp_i$

$$d\omega = |S_{fi}^{SN}|^2 \prod_{l=1}^{n_f} \left(\frac{V d^3p_l}{(2\pi)^3} \right)$$

This is the probability that the decay takes place in any time between $-T/2$ and $T/2$. We are more interested in the differential decay rate $d\Gamma_{fi}$, which is the decay probability per unit of time:

$$d\Gamma_{fi} = \frac{d\omega}{T} = (2\pi)^4 \delta^4(p_i - p_f) \frac{|\mathcal{M}_{fi}|^2}{2\omega_{p_{in}}} \prod_{l=1}^{n_f} \frac{d^3p_l}{(2\pi)^3 2\omega_l}$$

Notes:

- (i) $d\Gamma_{fi}$ = differential decay rate
- (ii) p_f = sum over final momenta
- (iii) $\omega_{p_{in}}$ = initial energy
- (iv) $|\mathcal{M}_{fi}|^2$ = Feynman amplitude of the process (depends on final momenta p_i)

It is useful to define the **(differential) n-body phase space** as

$$d\Phi_{(n_f)} = (2\pi)^4 \delta^4(p_i - p_f) \prod_{l=1}^{n_f} \frac{d^3 p_l}{(2\pi)^3 2\omega_l}$$

Therefore the differential decay rate can be written as

$$d\Gamma_{fi} = \frac{1}{2\omega_{p_{in}}} |\mathcal{M}_{fi}|^2 d\Phi_{(n_f)}$$

The decay rate is defined as

$$\Gamma_{fi} = \int d\Gamma_{fi} \rightarrow \text{integration over all possible final momenta}$$

and its meaning is $\Gamma \equiv \text{trans. probability} \times \text{unit of time} \times \text{init. particle}$

Notice that if n of the final particles are identical, configurations that differ by a permutation are not distinct and therefore the phase space is reduced by a factor $1/n!$

If we have a system of $N(0)$ particles, the time evolution of the number of particles $N(t)$ is

$$\frac{dN}{dt} = -\Gamma N \Rightarrow N(t) = N(0)e^{-\Gamma t}$$

Notice that decay rate is not invariant

$$[\Gamma] = [E] = \frac{1}{T} \quad (\text{in natural units})$$

If we define the **lifetime** as $\tau = 1/\Gamma \Rightarrow N(t) = N(0)\exp(-t/\tau)$ this changes under Lorentz tfm. If we consider two reference frames o and o'

$$\Gamma' = \frac{\Gamma}{\gamma} < \Gamma \quad \tau' = \gamma\tau > \tau$$

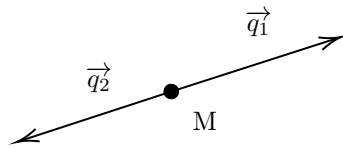
$\gamma = (1 - v)^{-1/2}$, where v is the speed of o' in o in natural units, $\gamma > 1$. Therefore a particle in a moving frame has a longer lifetime than in the rest frame

Example 2: muon lifetime

For a muon in the rest frame $\tau_\mu^{RF} = 2.2 \times 10^{-6} s$, but if we observe it in the lab frame $\tau_\mu^{LAB} = \gamma \tau_\mu^{RF} \simeq 2 \times 10^{-5} s$ since $E_\mu = 1 \text{ GeV}$, $m_\mu = 0.1 \text{ GeV}$
 $\Rightarrow \gamma = E_\mu/m_\mu \simeq 10$

Example 3: $1 \rightarrow 2$ decay

Consider the decay of a particle of a mass M into two particles of masses m_1, m_2 . Since the phase space is Lorentz invariant, we can compute it in the frame that we prefer. We use the rest frame for the initial particle.



We don't impose a priori conservation of momentum since it's imposed by the delta function.

$$p = (M, 0) \quad q_1 = (\omega_1, \mathbf{q}_1) \quad q_2 = (\omega_2, \mathbf{q}_2)$$

$$d\Phi_{(2)} = (2\pi)^4 \delta^4 \underbrace{(P_i - P_f)}_{=p-q_1-q_2} \frac{d^3 q_1}{(2\pi)^3 2\omega_1} \frac{d^3 q_2}{(2\pi)^3 2\omega_2}$$

I have 6 integration parameters, 4 constraint given by δ^4 , so I have 2 independent variables. Integrating over $d^3 q_2$

$$d\Phi'_{(2)} = \int d\Phi_{(2)} = \frac{1}{(2\pi)^2} \delta(M - \omega_1 - \omega_2) \frac{1}{4\omega_1 \omega_2} d^3 q_1$$

in this way, the condition $\mathbf{q}_2 = \mathbf{q}_1$ vanish. We have to impose it again when we calculate $d\Gamma$ (we omit this detail)

Usually the 4-th non independent parameter is eliminated by integration over modulus of q_1 , leaving free 2 parameters for the angles. $d^3 q_1 \rightarrow \mathbf{q}_1^2 d|\mathbf{q}_2| d\Omega_1$.

Notice that $M - \omega_1 - \omega_2 = M - \sqrt{\mathbf{q}_1^2 + m_1^2} - \sqrt{\mathbf{q}_2^2 + m_2^2}$ and then the δ implies

$$\hat{q}_1^2 = \frac{1}{2M} \left(M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2 \right)^{1/2}$$

$|\mathbf{q}_1|$ is the only zero of $f(|\hat{q}_1|) = M - \omega_1 - \omega_2$.

We also have

$$|f'(|\hat{q}_1|)| = \frac{\partial \omega_1}{\partial |\mathbf{q}_1|} + \frac{\partial \omega_2}{\partial |\mathbf{q}_1|} = |\hat{q}_1| \left(\frac{\omega_1 + \omega_2}{\omega_1 \omega_2} \right)$$

Using

$$\delta(f(x)) = \sum_{x_0 = \text{zero of } f(x)} \frac{\delta(x - x_0)}{|f'(x_0)|}$$

and performing integration over $d|\mathbf{q}_1|$ we obtain

$$d\Phi''_{(2)} = \int d\Phi'_{(2)} = \frac{1}{16\pi^2} \frac{|\hat{q}_1|}{M} d\Omega_1$$

Using this result we obtain the $1 \rightarrow 2$ decay rate in function of the solid angle (in the rest frame)

$$\left(\frac{d\Gamma_{RF}}{d\Omega} \right) = \frac{1}{64\pi^4 M^3} [M^4 - 2M^2(m_1^2 + m_2^2) + (m_1 - m_2)^2]^{1/2} |\mathcal{M}_{RF}|^2$$

In a general frame we can easily obtain an analogous formula, just consider $d\Gamma = 1/(2\omega_i n) |\mathcal{M}_{fi}|^2 d\Phi_{(nf)}$ in a general frame. remember that $dI_{(nf)}$ is invariant

We have 2 important limit cases:

(A) If $m_1 = m_2 = m$ (for example $Z \rightarrow e^+ e^-$)

$$|\hat{q}_1| = \frac{M}{2} \left(1 - \frac{4m^2}{M^2} \right)^{1/2}$$

$$\left(\frac{d\Gamma_{RF}}{d\Omega} \right) = \frac{1}{64\pi^2 M} \left(1 - \frac{4m^2}{M^2} \right)^{1/2} |\mathcal{M}_{fi}|^2$$

(B) If $m_1 = m, m_2 = 0$ (for example $W^\pm \rightarrow e^\pm \bar{\nu}$)

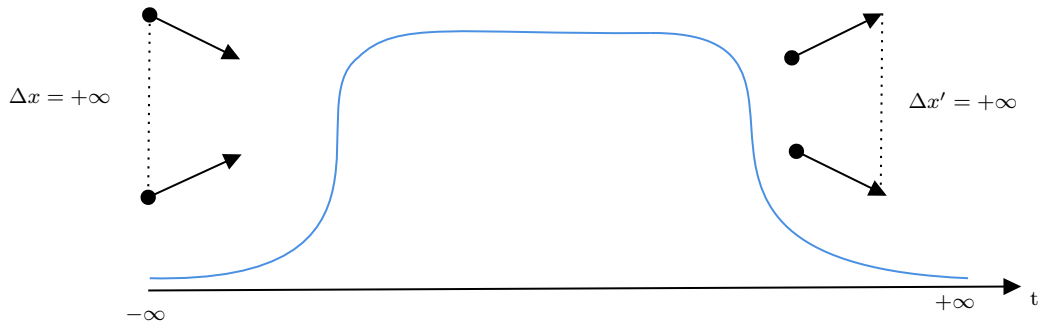
$$|\hat{q}_1| = \frac{M}{2} \left(1 - \frac{4m^2}{M^2} \right)^{1/2}$$

$$\left(\frac{d\Gamma_{RF}}{d\Omega} \right) = \frac{1}{64\pi^2 M} \left(1 - \frac{m^2}{M^2} \right)^{1/2} |\mathcal{M}_{fi}|^2$$

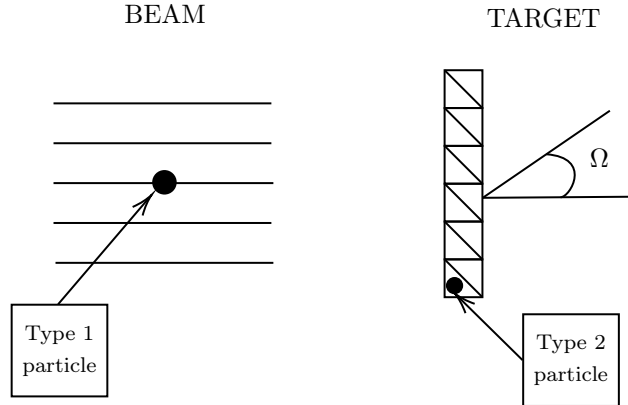
Notes: If we have two identical particles in the final state, the calculation of the phase is different

$$d\Phi_{(2)}^{\text{identical}} = \frac{1}{2} d\Phi_{(2)}^{\text{distinguishable}}$$

1.4 Cross section (\leftrightarrow scattering process)



Scattering in the lab frame



Consider a beam of particles with mass m_1 , number (assuming a uniform distribution) density $n_1^{(0)}$ (subscript 0 is meant to stress that these are number densities in a specific frame, that with particle 2 at rest) and velocity v_1 impinging on a target made of particles with mass m_2 and number density $n_2^{(0)}$ at rest.

Let N_s be the number of scattering events that place per unit volume and per unit time

$$\frac{N_t}{T} \varphi_1 N_2 \sigma = (n_1^{(0)} v_1) (n_2^{(0)} V) \sigma$$

More formally we have

$$dN_s = \sigma v_1 n_1^{(0)} n_2^{(0)} dV dV$$

with:

(i) T: unit of time

(ii) φ_1 : flux of the beam $\varphi_1 = n_1^{(0)} v_1$

(iii) N_2 : particles per unit volume in the detector $N_2 = n_2^{(0)} V$

(iv) σ : proportionality constant

Dimensional analysis shows $[\sigma] = [L]^2$ and then σ , called cross section, can be interpreted as an “effective area”.