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# Chapter 1

## Introduction

### 1.1 Free Theories Lagrangians

#### 1.1.1 Complex Scalar Field

$$(\square + m^2)\varphi = 0$$

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} (e^{-ikx} a(k) + e^{ikx} b^\dagger(k))_{k_0=\omega_k}$$
$$\omega_k = \sqrt{m^2 + \mathbf{k}^2}$$

In the real case  $\varphi^\dagger(x) = \varphi(x) \Rightarrow a(k) = b(k)$

#### 1.1.2 Dirac Spinorial Field

$$(i\not{\partial} - m)\psi = 0$$

$$\psi(x) = \frac{1}{(2\pi)^{2/3}} \int \frac{d^3k}{\sqrt{2\omega_k}} \sum_{r=1,2} (e^{-ikx} u_r(k) c_r(k) + e^{ikx} v_r(k) d_r^\dagger(k))$$

where  $u_r(k)/v_r(k)$  are the  $\varepsilon > 0/\varepsilon < 0$  spinors

Spinors are normalized according to

$$\begin{cases} \bar{u}_r(k) u_s(k) = 2m \delta_{rs} & \bar{u}_r(k) v_s(k) = 0 \\ \bar{v}_r(k) v_s(k) = -2m \delta_{rs} & \bar{v}_r(k) u_s(k) = 0 \end{cases}$$

#### 1.1.3 E-M Vector Field

$$\partial_\mu F^{\mu\nu} = \square A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0, \text{ where } F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$A^\mu(x) = \frac{1}{(2\pi)^{2/3}} \int \frac{d^3k}{\sqrt{2\omega_k}} \sum_\lambda (e^{-ikx} \varepsilon_{(\lambda)}^\mu a_\lambda(k) + e^{ikx} \varepsilon_{(\lambda)}^{\mu\dagger}(k) a_\lambda^\dagger(k))_{k_0=\omega_k=|\mathbf{k}|}$$
$$\varepsilon_{(1)}^\mu = (0, 1, 0, 0) \quad \varepsilon_{(2)}^\mu = (0, 0, 1, 0)$$

I can complexify the field substituting  $a_\lambda^\dagger$  with another operator  $b_\lambda$  (analogously to the scalar field)

Notice that real fields are never free because we have interactions, but using interaction picture we reconstruct the problem in a simpler one, where fields are described by free fields. This can be done with a proper choice.

$$\Phi_I(x) \equiv \Phi_{\text{free}}(x) \quad \Phi_I = \text{interacting}$$

## 1.2 Fock Space of Free Fields

See Maggiore. See 6.1

We impose the existence of vacuum state  $|0\rangle$ , and using creation operators we obtain other states  $(a^\dagger)^n |0\rangle$ , which are n-particles states.

In QFT we normalized states in a covariant way, instead of QM normalization  $\int \psi^* \psi = 1$

$$\begin{aligned} |1(p)\rangle &\equiv (2\pi)^{3/2} \sqrt{2\omega_k} a^\dagger(p) |0\rangle \\ \langle 1(p) | 1(p') \rangle &= (2\pi)^3 (2\omega_p) \delta^3(p - p')^1 \end{aligned}$$

$(2\omega_p) \delta^3(p - p')$ : Covariant under Lorentz tfm

## 1.3 Contraction of Fields with States

If we have a state  $|e_s^-(p)\rangle$  that describes an electron with momentum p and Dirac index s, then

$$|e_s^-(p)\rangle = (2\pi)^{2/3} \sqrt{2\omega_p} c_s^\dagger(p) |0\rangle$$

given a field  $\psi$  that describes a particle annihilation (or antiparticle creation) in x we have

$$\begin{aligned} \langle 0 | \psi(x) | e_s^-(p) \rangle &= \langle 0 | (\psi_+(x) + \psi_-(x)) | e_s^-(p) \rangle \\ &= \frac{(2\pi)^{3/2}}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} e^{-ikx} \sqrt{2\omega_p} \sum_r \langle 0 | c_r(k) c_s^\dagger(p) | 0 \rangle u_r(k) \\ &= \int d^3k \left( \frac{2\omega_p}{2\omega_k} \right)^{1/2} \sum_r \delta_{rs} \delta^{(3)}(\bar{p} - \bar{k}) u_r(k) \langle 0 | 0 \rangle e^{-ikx} \\ &= e^{-ikx} u_s(p) \end{aligned}$$

$$c_r(k) c_s^\dagger(p) = \{c_r(k), c_s^\dagger(p)\} = \delta_{rs} \delta^3(\bar{p} - \bar{k})$$

The factor  $e^{-ikx}$  is required for the  $\delta^{(4)}$  conservation, and we see that the relativistic normalization leads to the relation ( $\rightarrow$  Feynman rule)

$$\begin{array}{c} \longrightarrow \\ \xrightarrow{p} \end{array} = e^{-ipx} u_s(p)$$

In this case there is no normalization factors in the Feynman rule

## 1.4 S-matrix and State Evolution

In the interaction picture, with  $H = H_0 + H_{\text{int}}$ , with  $H_{\text{int}}$  = interaction hamiltonian

- (i) Fields  $\Phi_I$  evolves like in the free theory (respect to  $H_0$ )
- (ii) State evolves with the following evolution operator

$$\begin{aligned} U_I(t, t_0) &\equiv e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t_0} \\ |\alpha, t\rangle &= U_I(t - t_0) |\alpha, t_0\rangle \quad i\partial_t U_I(t - t_0) = H_I^{\text{int}}(t) U_I(t, t_0) \end{aligned}$$

Notice that, in general

$$[H_I(t), H_0] \neq 0 \neq [H_I^{\text{int}}, H_0]$$

and if  $t \neq t'$  we also have

$$[H_I^{\text{int}}(t), H_I^{\text{int}}(t')] \neq 0 \quad \text{with } O_I(t) = e^{iH_0 t} e^{-iHt} O_H e^{iHt} e^{-iH_0 t}$$

The S-matrix is a well defined operator defined as

$$S = \lim_{\substack{t_0 \rightarrow -\infty \\ t \rightarrow +\infty}} U_I(t, t_0)$$

We compute S by perturbation obtaining

$$\begin{aligned} S &= T \left( \exp \left( -i \int d^4x \mathcal{H}_I^{\text{int}}(x) \right) \right) \\ &= \sum_{n=0}^{+\infty} \frac{(-i)^n}{n!} \int d^4x_1 \dots d^4x_n T(\mathcal{H}_I^{\text{int}}(x_1) \dots \mathcal{H}_I^{\text{int}}(x_n)) \end{aligned}$$

S has some relevant properties

- (i) Unitary (since hamiltonian is hermitian)
- (ii) Behaves as a scalar under Lorentz tms, and then is an invariant quantity (notice that in general  $\mathcal{H}_I^{\text{int}}$  is not invariant  
In the case of  $\mathcal{H}_I^{\text{int}}$  is invariant (for example if  $\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$ , as in many theories, one of them id QED) is easy to prove that S is invariant, since all n-th derivatives of  $\exp(\int \mathcal{H})$  are invariant, and so also S is invariant

## 1.5 S-matrix and transition probabilities

We suppose that there's no interactions for  $x, t \rightarrow +\infty$

## 1.6 S-matrix and transition probabilities

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Consider a canonically normalized (CN) state  $|\psi\rangle = 1$ :

$$|\psi_i\rangle_{CN} \equiv |\psi(-\infty)\rangle_{CN} \quad |\psi(+\infty)\rangle_{CN} \equiv S |\psi_i\rangle_{CN} \quad \rightarrow \text{both are free particle states}$$

Elements of S are in the form

$$S_{fi}^{CN} = \langle \psi_f | S | \psi_i \rangle_{CN}$$

This leads to a probabilistic interpretation of S-matrix elements.

$|S_{fi}^{CN}|^2$  = propability of evolution of  $|\psi_i\rangle_{CN}$  into  $|\psi_f\rangle_{CN}$ , since the condition  $\sum_f |S_{fi}^{CN}| = 1$  is satisfied automatically

In the case of covariant normalization

$$\langle 1(p) | 1(p') \rangle = (2\pi)^3 (2\omega p) \delta^3(\mathbf{p} - \mathbf{p}')$$

we have the following relation between matrix elements

$$S_{fi}^{CN} = \langle \psi_f | S | \psi_i \rangle_{CN} = \frac{\langle \psi_f | S | \psi_i \rangle}{\|\psi_i\| \|\psi_f\|} = \frac{S_{fi}}{\|\psi_i\| \|\psi_f\|}$$

We can define the **Feynman Amplitude**  $\mathcal{M}_{fi}$  as

$$S_{fi} = (2\pi)^4 \delta^4(p_i - p_f) \mathcal{M}_{fi}$$

and it can be obtained directly starting from Feynman rules (calculated with the covariant normalization)

## 1.7 Discrete space normalization

Usually, in order to make arguments cleaner, or to avoid problems with divergent terms, we first consider a system in a cubic box with spatial volume  $V = L^3$ .

At the end of computations  $V$  will be sent to infinity. Sometimes we will do something similar also for time.

For a discrete space we must use a different normalization.

In a box, momentum of a particle is quantized (1-dim case)

$$p_i = \left(\frac{2\pi}{L}\right)n_i \quad n_i \in \mathbb{Z}$$

and we must adopt the following rule for integrals

$$\int d^3p f(\mathbf{p}) \rightarrow \sum_{\mathbf{n}} \left(\frac{2\pi}{L}\right)^3 f_{\mathbf{n}} \quad \mathbf{n} = (n_1, n_2, n_3)$$

We must adopt also the following

$$\delta^3(\mathbf{p} - \mathbf{p}') \rightarrow \left(\frac{L}{2\pi}\right)^3 \delta_{\mathbf{nn}'}$$

in this way

$$\int d^3p \delta^3(\mathbf{p} - \mathbf{p}') = 1 \rightarrow \sum_{\mathbf{n}} \left(\frac{2\pi}{L}\right)^3 \left(\frac{L}{2\pi}\right)^3 \delta_{\mathbf{nn}'} = 1$$

Some useful relations are

$$\delta^3(0) \rightarrow \left(\frac{L}{2\pi}\right)^3$$

$$\delta^4(0) \rightarrow \left(\frac{L}{2\pi}\right) \left(\frac{T}{2\pi}\right) \rightarrow \text{Only if we consider a finite amount of time}$$

Normalization of state becomes

$$|1(p)\rangle = (2\pi)^{2/3} \sqrt{2\omega_p V} o^\dagger(p) |0\rangle = (2\pi)^{2/3} \sqrt{2\omega_p V} |1(p)\rangle_{CN}$$

$$\langle 1(p) | 1(p) \rangle = (2\pi)^3 2\omega_p V \delta^3(0) = 2\omega_p V$$

Using the latter equation,  $S_{fi}^{CN}$  reads

$$S_{fi}^{CN} = \prod_{j=1}^{n_i} \left(\frac{1}{2\omega_j V}\right)^{1/2} \prod_{l=1}^{n_f} \left(\frac{1}{2\omega_l V}\right)^{1/2} S_{fi}$$

$$= (2\pi)^4 \delta^4(p_i - p_f) \left\{ \prod_{j=1}^{n_i} \left(\frac{1}{2\omega_j V}\right)^{1/2} \prod_{l=1}^{n_f} \left(\frac{1}{2\omega_l V}\right)^{1/2} \mathcal{M}_{fi} \right\}$$

$$= (2\pi)^4 \delta^4(p_i - p_f) M_{fi}^{CN}$$

In the first passage  $(2\pi)^{2/3}$  factors vanish because of  $\delta^3$  factors inside  $S_{fi}$  related to the sandwich  $\langle \psi_f | S | \psi_i \rangle$

In the second passage we use  $df_n$  of  $\mathcal{M}_{fi}$ , omitting the quantization of  $\delta^4$

$M_{fi}^{CN}$  is the **canonically normalized Feynman amplitude**

## Chapter 2

# The S-matrix and physical observables

### 2.1 Decay Rate

Consider the case in which the initial state is a single particle and the final state is given by  $n$  particles. We are therefore considering a decay process. Assume for the moment that particles are indistinguishable.



The rules of quantum mechanics tell us that the probability for this process is obtained by taking the squared modulus of the amplitude and summing over all possible final states

$$\begin{aligned} |S_{fi}^{CN}|^2 &= |(2\pi)^4 \delta^4(p - p') M_{fi}^{CN}|^2 \\ &= (2\pi)^4 \delta^4(p - p') (VT) |M_{fi}^{CN}|^2 \\ &= (2\pi)^4 \delta^4(p - p') (VT) \frac{1}{2\omega_i n V} \prod_{l=1}^{n_f} \left( \frac{1}{2\omega_l V} \right) |\mathcal{M}_{fi}|^2 \end{aligned}$$

**Note:**  $(\delta^4(p - p'))^2 = \delta^4(p - p') \delta^4(p - p') = \delta^4(p - p') \delta^4(0) = \delta^4(p - p') \frac{VT}{(2\pi)^4}$

We use the final space and time in order to remove divergent terms during calculation

We must now sum this expression over all final states. Since we are working in a finite volume  $V$ , this is the sum over the possible discrete values of the momenta of the final particles .

Since  $p_i = (2\pi/L)n_i$ , we have  $dn_i = (L/2\pi)dp_i$  and  $d^3n_i = (V/(2\pi)^3)d^3p$  where  $d^3n_i$  is the infinitesimal phase space related to a final state in which the  $i$ -th particle has momentum between  $p_i$  and  $p_i + dp_i$ . Let  $d\omega$  be the probability for a decay in which in the final state the  $i$ -th particle has momentum between  $p_i$  and  $p_i dp_i$

$$d\omega = |S_{fi}^{SN}|^2 \prod_{l=1}^{n_f} \left( \frac{V d^3 p_l}{(2\pi)^3} \right)$$

This is the probability that the decay takes place in any time between  $-T/2$  and  $T/2$ . We are more interested in the differential decay rate  $d\Gamma_{fi}$ , which is the decay probability per unit of time:

$$d\Gamma_{fi} = \frac{d\omega}{T} = (2\pi)^4 \delta^4(p_i - p_f) \frac{|\mathcal{M}_{fi}|^2}{2\omega_{p_{in}}} \prod_{l=1}^{n_f} \frac{d^3 p_l}{(2\pi)^3 2\omega_l}$$

**Notes:**

- (i)  $d\Gamma_{fi}$  = differential decay rate
- (ii)  $p_f$  = sum over final momenta
- (iii)  $\omega_{p_{in}}$  = initial energy
- (iv)  $|\mathcal{M}_{fi}|^2$  = Feynman amplitude of the process (depends on final momenta  $p_i$ )

It is useful to define the **(differential) n-body phase space** as

$$d\Phi_{(n_f)} = (2\pi)^4 \delta^4(p_i - p_f) \prod_{l=1}^{n_f} \frac{d^3 p_l}{(2\pi)^3 2\omega_l}$$

Therefore the differential decay rate can be written as

$$d\Gamma_{fi} = \frac{1}{2\omega_{p_{in}}} |\mathcal{M}_{fi}|^2 d\Phi_{(n_f)}$$

The decay rate is defined as

$$\Gamma_{fi} = \int d\Gamma_{fi} \rightarrow \text{integration over all possible final momenta}$$

and its meaning is  $\Gamma \equiv \text{trans. probability} \times \text{unit of time} \times \text{init. particle}$

Notice that if n of the final particles are identical, configurations that differ by a permutation are not distinct and therefore the phase space is reduced by a factor  $1/n!$

If we have a system of  $N(0)$  particles, the time evolution of the number of particles  $N(t)$  is

$$\frac{dN}{dt} = -\Gamma N \Rightarrow N(t) = N(0)e^{-\Gamma t}$$

Notice that decay rate is not invariant

$$[\Gamma] = [E] = \frac{1}{T} \quad (\text{in natural units})$$

If we define the **lifetime** as  $\tau = 1/\Gamma \Rightarrow N(t) = N(0)\exp(-t/\tau)$  this changes under Lorentz tfm. If we consider two reference frames o and o'

$$\Gamma' = \frac{\Gamma}{\gamma} < \Gamma \quad \tau' = \gamma\tau > \tau$$

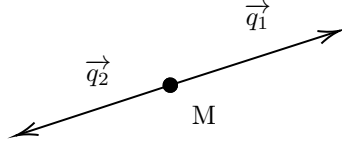
$\gamma = (1 - v)^{-1/2}$ , where v is the speed of o' in o in natural units,  $\gamma > 1$ . Therefore a particle in a moving frame has a longer lifetime than in the rest frame

### Example 1: muon lifetime

For a muon in the rest frame  $\tau_\mu^{RF} = 2.2 \times 10^{-6} s$ , but if we observe it in the lab frame  $\tau_\mu^{LAB} = \gamma \tau_\mu^{RF} \simeq 2 \times 10^{-5} s$  since  $E_\mu = 1 \text{ GeV}$ ,  $m_\mu = 0.1 \text{ GeV}$   
 $\Rightarrow \gamma = E_\mu/m_\mu \simeq 10$

### Example 2: $1 \rightarrow 2$ decay

Consider the decay of a particle of a mass M into two particles of masses  $m_1, m_2$ . Since the phase space is Lorentz invariant, we can compute it in the frame that we prefer. We use the rest frame for the initial particle.



We don't impose a priori conservation of momentum since it's imposed by the delta function.

$$p = (M, 0) \quad q_1 = (\omega_1, \mathbf{q}_1) \quad q_2 = (\omega_2, \mathbf{q}_2)$$

$$d\Phi_{(2)} = (2\pi)^4 \delta^4 \underbrace{(P_i - P_f)}_{=p-q_1-q_2} \frac{d^3 q_1}{(2\pi)^3 2\omega_1} \frac{d^3 q_2}{(2\pi)^3 2\omega_2}$$

I have 6 integration parameters, 4 constraint given by  $\delta^4$ , so I have 2 independent variables. Integrating over  $d^3 q_2$

$$d\Phi'_{(2)} = \int d\Phi_{(2)} = \frac{1}{(2\pi)^2} \delta(M - \omega_1 - \omega_2) \frac{1}{4\omega_1 \omega_2} d^3 q_1$$

in this way, the condition  $\mathbf{q}_2 = \mathbf{q}_1$  vanish. We have to impose it again when we calculate  $d\Gamma$  (we omit this detail)

Usually the 4-th non independent parameter is eliminated by integration over modulus of  $q_1$ , leaving free 2 parameters for the angles.  $d^3 q_1 \rightarrow \mathbf{q}_1^2 d|\mathbf{q}_2| d\Omega_1$ .

Notice that  $M - \omega_1 - \omega_2 = M - \sqrt{\mathbf{q}_1^2 + m_1^2} - \sqrt{\mathbf{q}_2^2 + m_2^2}$  and then the  $\delta$  implies

$$q_1^2 = \frac{1}{2M} \left( M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2 \right)^{1/2}$$

$|\mathbf{q}_1|$  is the only zero of  $f(|\hat{q}_1|) = M - \omega_1 - \omega_2$ .

We also have

$$|f'(|\hat{q}_1|)| = \frac{\partial \omega_1}{\partial |\mathbf{q}_1|} + \frac{\partial \omega_2}{\partial |\mathbf{q}_1|} = |\hat{q}_1| \left( \frac{\omega_1 + \omega_2}{\omega_1 \omega_2} \right)$$

Using

$$\delta(f(x)) = \sum_{x_0 = \text{zero of } f(x)} \frac{\delta(x - x_0)}{|f'(x_0)|}$$

and performing integration over  $d|\mathbf{q}_1|$  we obtain

$$d\Phi''_{(2)} = \int d\Phi'_{(2)} = \frac{1}{16\pi^2} \frac{|\hat{q}_1|}{M} d\Omega_1$$

Using this result we obtain the  $1 \rightarrow 2$  decay rate in function of the solid angle (in the rest frame)

$$\left( \frac{d\Gamma_{RF}}{d\Omega} \right) = \frac{1}{64\pi^4 M^3} [M^4 - 2M^2(m_1^2 + m_2^2) + (m_1 - m_2)^2]^{1/2} |\mathcal{M}_{RF}|^2$$

In a general frame we can easily obtain an analogous formula, just consider  $d\Gamma = 1/(2\omega_i n) |\mathcal{M}_{fi}|^2 d\Phi_{(nf)}$  in a general frame. remember that  $dI_{(nf)}$  is invariant

We have 2 important limit cases:

(A) If  $m_1 = m_2 = m$  (for example  $Z \rightarrow e^+ e^-$ )

$$|\hat{q}_1| = \frac{M}{2} \left( 1 - \frac{4m^2}{M^2} \right)^{1/2}$$

$$\left( \frac{d\Gamma_{RF}}{d\Omega} \right) = \frac{1}{64\pi^2 M} \left( 1 - \frac{4m^2}{M^2} \right)^{1/2} |\mathcal{M}_{fi}|^2$$



(B) If  $m_1 = m, m_2 = 0$  (for example  $W^\pm \rightarrow e^\pm \bar{\nu}$ )

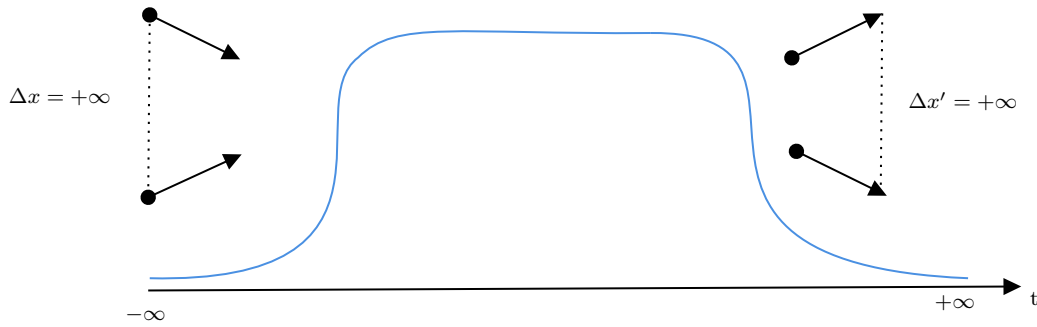
$$|\hat{q}_1| = \frac{M}{2} \left( 1 - \frac{4m^2}{M^2} \right)^{1/2}$$

$$\left( \frac{d\Gamma_{RF}}{d\Omega} \right) = \frac{1}{64\pi^2 M} \left( 1 - \frac{m^2}{M^2} \right)^{1/2} |\mathcal{M}_{fi}|^2$$

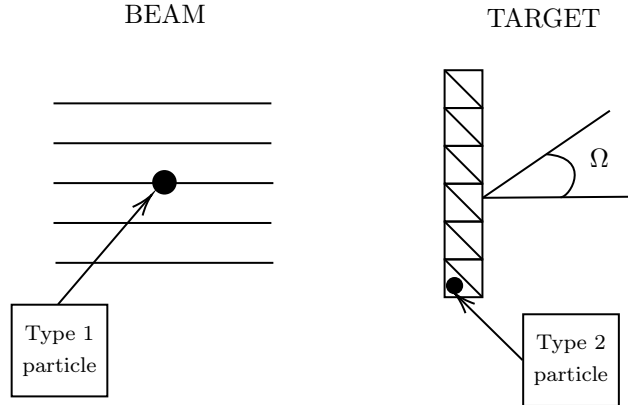
**Notes:** If we have two identical particles in the final state, the calculation of the phase is different

$$d\Phi_{(2)}^{\text{identical}} = \frac{1}{2} d\Phi_{(2)}^{\text{distinguishable}}$$

## 2.2 Cross section ( $\leftrightarrow$ scattering process)



Scattering in the lab frame



Consider a beam of particles with mass  $m_1$ , number (assuming a uniform distribution) density  $n_1^{(0)}$  (subscript 0 is meant to stress that these are number densities in a specific frame, that with particle 2 at rest) and velocity  $v_1$  impinging on a target made of particles with mass  $m_2$  and number density  $n_2^{(0)}$  at rest.

Let  $N_s$  be the number of scattering events that place per unit volume and per unit time

$$\frac{N_t}{T} \varphi_1 N_2 \sigma = (n_1^{(0)} v_1) (n_2^{(0)} V) \sigma$$

More formally we have

$$dN_s = \sigma v_1 n_1^{(0)} n_2^{(0)} dV dt$$

with:

(i) T: unit of time

(ii)  $\varphi_1$ : flux of the beam  $\varphi_1 = n_1^{(0)} v_1$

(iii)  $N_2$ : particles per unit volume in the detector  $N_2 = n_2^{(0)} V$

(iv)  $\sigma$ : proportionality constant

Dimensional analysis shows  $[\sigma] = [L]^2$  and then  $\sigma$ , called cross section, can be interpreted as an “effective area”

Consider the case where just one particle collides with another particle (2 particles scattering). We can obtain this condition imposing  $n_{1,2}^{(0)} = 1/V$ , in this way the scattering per unit of volume is the same as two particles scattering

Notice that this situation is the more physical. The probability of 3 particles collision in the same  $d^3x$  and  $dt$  is almost 0

If  $d\omega$  is, again, the probability for a process in which in the final state the  $i$ -th particle has momentum between  $p_i$  and  $p_i + dp_i$

$$d\omega = (2\pi)^4 \delta^4(p_i - p_f) VT \left( \frac{1}{2\omega_1 V} \right) \left( \frac{1}{2\omega_2 V} \right) \prod_{l=1}^{n_f} \left( \frac{d^3 p_l}{(2\pi)^3 2\omega_l} \right) |\mathcal{M}_{fi}|^2$$

we can define the **differential cross section** (in the lab frame)  $d\sigma$  as

$$\begin{aligned} (d\sigma)_{\text{LAB}} &= \frac{d\omega}{n_1^{(0)} n_2^{(0)} v_1 VT} \\ &= \frac{V}{T v_1} d\omega \\ &= (2\pi)^4 \delta^4(p_i - p_f) \prod_{l=1}^{n_f} \left( \frac{d^3 p_l}{(2\pi)^3 2\omega_l} \right) \frac{|\mathcal{M}_{fi}|^2}{4\omega_1 \omega_2 v_1} \quad \text{with } \omega_2 = m_2 \end{aligned}$$

The result is then

$$(d\sigma)_{\text{LAB}} = \frac{1}{4\omega_1 v_1 m_2} |\mathcal{M}_{fi}|^2 d\Phi_{(nf)}$$

All quantities refers to the rest frame for particle 2 ( $\omega_1 = \omega_1^{(0)}$ ,  $v_1 = v_1^{(0)}$ )

In order to obtain a covariant relation for  $d\sigma$ , we notice that the only non-covariant factor in the latter relation is  $(I_{12})_{\text{LAB}} = \omega_1^{(0)} m_2$ . This factor can be substituted with a covariant one

$$I_{12} = [(p_1 p_2)^2 - m_1^2 m_2^2]^{1/2}$$

called **covariant flux factor**. This factor is obviously covariant, we just have to prove that in the lab frame it coincides with  $(I_{12})_{\text{LAB}}$ . In the lab frame we have  $p_1 = (\omega_1, \mathbf{p}_1^{(0)})$  and  $p_2 = (m_2, \mathbf{0})$ , so the previous formula becomes

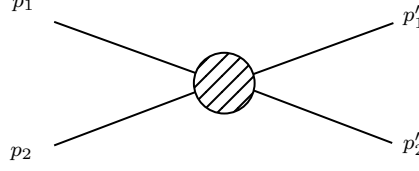
$$[m_2^2 (\omega_1^{(0)})^2 - m_1 m_2]^{1/2} = m_2 ((\omega_1^{(0)})^2 - p_1^2)^{1/2} = m_2 |\mathbf{p}_1^{(0)}| = m_2 \omega_1^{(0)} v_1^{(0)}$$

So the final result is

$$d\sigma = \frac{|\mathcal{M}_{fi}^2|}{4I_{12}} d\Phi_{(nf)}$$

### Example 3: $2 \rightarrow 2$ scattering

Consider a scattering process  $2 \rightarrow 2$ . We consider an initial state with two particles with masses  $m_1, m_2$  and four momenta  $p_1, p_2$  and a final state with masses  $m'_1, m'_2$  and four momenta  $p'_1, p'_2$



With

$$p_1 = (\omega_1, \mathbf{p}_1) \quad p_2 = (\omega_2, \mathbf{p}_2) \quad p'_1 = (\omega'_1, \mathbf{p}'_1) \quad p'_2 = (\omega'_2, \mathbf{p}'_2)$$

With a procedure identical to  $1 \rightarrow 1$  decay (imposing also  $\mathbf{p}'_2 = \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1$  when we calculate  $d\sigma$ ), we obtain

$$d\Phi'_{(2)} = \frac{1}{(2\pi)^2} \frac{1}{4\omega'_1\omega'_2} \delta(\omega_1 + \omega_2 - \omega'_1 - \omega'_2) d^3p'_1$$

In order to integrate over  $d|\mathbf{p}'_1|$  it is useful to introduce Mandelstam variables  $s, t$  and  $u$

$$s = (p_1 + p_2)^2 \quad t = (p_1 - p'_1)^2 \quad u = (p_1 - p'_2)^2$$

These variables are clearly Lorentz invariants, and satisfy (using  $p_1 + p_2 = p'_1 + p'_2$ ) the relation

$$s + t + u = m_1^2 + m_2^2 + (m'_1)^2 + (m'_2)^2$$

It is useful to work in the center of mass frame, where the incoming particles have  $p_1 = (\omega_1, \mathbf{p})$  and  $p_2 = (\omega_2, -\mathbf{p})$ . Computing  $s$  in the CM frame we obtain  $s = (\omega_1 + \omega_2)^2 = (\omega'_1 + \omega'_2)^2$  and then

$$p_1 + p_2 = (\sqrt{s}, 0) \\ \delta(\omega_1 + \omega_2 - \omega'_1 - \omega'_2) = \delta(\sqrt{s} - \omega'_1 - \omega'_2)$$

With a procedure identical to the one used for  $1 \rightarrow 2$  decay ( $M \leftrightarrow \sqrt{s}$ ) we obtain

$$(d\Phi''_{(2)})_{CM} = \frac{1}{16\pi^2} \frac{|\hat{p}'_1|_{CM}}{\sqrt{s}} d\Omega'_1 \\ |\hat{p}'_1|_{CM} = \frac{1}{2\sqrt{s}} \left[ s^2 + (m_1'^2 + m_2'^2)^2 - 2s(m_1'^2 + m_2'^2) \right]^{1/2}$$

The covariant flux factor in CM frame reads

$$(I_{12})_{CM} = [\mathbf{p}^2(\omega_1 + \omega_2)^2]^{1/2} = |\mathbf{p}| \sqrt{s}$$

So we obtain the final result

$$\left( \frac{d\sigma}{d\Omega'_1} \right)_{CM} = \frac{1}{64\pi^2 s} \frac{|\hat{p}'_1|_{CM}}{|\mathbf{p}|} |\mathcal{M}_{fi}|_{CM}^2 \\ = \frac{1}{128\pi^2 s^{3/2}} \frac{1}{|\mathbf{p}|} [s^2 + (m_1'^2 - m_2'^2)^2 - 2s(m_1'^2 + m_2'^2)]^{1/2} |\mathcal{M}_{fi}|_{CM}^2$$

We have two important limit cases

(A) If  $m_1 = m'_1, m_2 = m'_2$ , i. e. elastic scattering (for example  $e^- \mu^\dagger \rightarrow e^- \mu^\dagger$ )

$$\left. \begin{aligned} |\mathbf{p}_1| &= |\mathbf{p}'_1|, \omega_1 = \omega'_1 \\ |\mathbf{p}_2| &= |\mathbf{p}'_2|, \omega_2 = \omega'_2 \end{aligned} \right\} \quad |\hat{p}_1|_{CM} = |\mathbf{p}|$$

$$\left( \frac{d\sigma}{d\Omega'_1} \right)_{CM} = \frac{1}{64\pi^2} \frac{|\mathcal{M}_{fi}|^2}{s}$$

(B) If  $m_1 = m_2 \simeq 0, m'_1 = m'_2 = M$  (for example  $e^- e^+ \rightarrow \mu^- \mu^+$ )

$$\begin{aligned} |\mathbf{p}_1| &= |\mathbf{p}|, \omega_1 = \frac{\sqrt{s}}{2} \\ |\hat{p}_1| &= \frac{\sqrt{s}}{2} \left( 1 - \frac{1}{4M^2} s \right)^{1/2} \end{aligned} \quad (2.1)$$

$4M^2/s$  processes with  $s < 4M^2$  are unphysical

$$\left( \frac{d\sigma}{d\Omega'_1} \right)_{CM} = \frac{1}{64\pi^2} \left( 1 - \frac{4M^2}{s} \right) \frac{|\mathcal{M}_{fi}|^2}{s}$$

The previous formulas are valid also for particles with spin, if the initial and final spin states are known; in this case the initial state has the form  $|i\rangle = |\mathbf{p}_1, \mathbf{s}_1; \dots; \mathbf{p}_n, \mathbf{s}_n\rangle$ , and similarly for the final state.

However, experimentally is more common that we do not know the initial spin configuration and we accept in the detector all final spin configurations; in this case, to compare with experiment, we must

- (i) Considering all (equally present) polarization of initial state, i.e. *average* over the initial spin configuration
- (ii) Considering all final polarizations, i.e. *sum* over all possible final configurations

Defining the **unpolarized Feynman amplitude**  $|\overline{\mathcal{M}_{fi}}|$  as

$$|\overline{\mathcal{M}_{fi}}|^2 = \frac{1}{\text{n initial polarizations}} = \sum_{\text{initial spins}} \sum_{\text{final spins}} |\mathcal{M}_{fi}|^2$$

we just have to substitute  $|\mathcal{M}_{fi}|^2 \rightarrow |\overline{\mathcal{M}_{fi}}|^2$  in all previous formulas

For example, in  $2 \rightarrow 2$  scattering

$$\frac{1}{\text{n init pol}} = \frac{1}{(2s_a + 1)(2s_b + 1)} \quad \text{where } s_a = \text{spin of particle 1; and } s_b = \text{spin of particle 2}$$