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Chapter 1

Introduction

1.1 Free Theories Lagrangians

1.1.1 Complex Scalar Field

$(\Box + m^2)\varphi = 0$

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} \big(e^{-ikx} a(k) + e^{ikx} b^\dagger(k)\big)_{k_0=\omega_k}$$

$$\omega_k = \sqrt{m^2 + \mathbf{k}^2}$$

In the real case $\varphi^\dagger(x) = \varphi(x) \Rightarrow a(k) = b(k)$

1.1.2 Dirac Spinorial Field

$(i\partial\!\!\!/ - m)\psi = 0$

$$\psi(x) = \frac{1}{(2\pi)^{2/3}} \int \frac{d^3k}{\sqrt{2\omega_k}} \sum_{r=1,2} \big(e^{-ikx} u_r(k) c_r(k) + e^{ikx} v_r(k) d_r^\dagger(k)\big)$$

where $u_r(k)/v_r(k)$ are the $\varepsilon > 0/\varepsilon < 0$ spinors
Spinors are normalized according to

$$\begin{cases} \bar{u_r}(k) u_s(k) = 2m\delta_{rs} & \bar{u_r}(k) v_s(k) = 0 \\ \bar{v_r}(k) v_r(k) = -2m\delta_{rs} & \bar{v_r}(k) u_s(k) = 0 \end{cases}$$

1.1.3 E-M Vector Field

$\partial_\mu F^{\mu\nu} = \Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0$, where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$A^\mu(x) = \frac{1}{(2\pi)^{2/3}} \int \frac{d^3k}{\sqrt{2\omega_k}} \sum_\lambda \big(e^{-ikx} \varepsilon^\mu_{(\lambda)} a_\lambda(k) + e^{ikx} \varepsilon^{\mu\dagger}_{(\lambda)}(k) a^\dagger_\lambda(k)\big)_{k_0=\omega_k=|\mathbf{k}|}$$

$$\varepsilon^\mu_{(1)} = (0,1,0,0) \qquad \varepsilon^\mu_{(2)} = (0,0,1,0)$$

I can complexify the field substituting $a^\dagger_\lambda$ with another operator b_λ (analogously to the scalar field)
Notice that real fields are never free because we have interactions, but using interaction picture e reconduct the problem in a simper one, where filed are described by free fields. This can be able with a proper choice.

$$\Phi_I(x) \equiv \Phi_{\text{free}}(x) \qquad \Phi_I = \text{interacting}$$

1.2 Fock Space of Free Fields

See Maggiore. See 6.1

We impose the existence of vacuum state $|0\rangle$, and using creation operators we obtain other states $(a^\dagger)^n\,|0\rangle$, which are n-particles states.

In QFT we normalized states in a covariant way, instead of QM normalization $\int \psi^*\psi = 1$

$$|1(p)\rangle \equiv (2\pi)^{3/2}\sqrt{2\omega_k}\; o^\dagger(p)\,|0\rangle$$
$$\langle 1(p)\,|\,1(p')\rangle = (2\pi)^3(2\omega p)\delta^3(p-p')^1$$

$(2\omega p)\delta^3(p-p')$: Covariant under Lorentz tfm

1.3 Contraction of Fields with States

If we have a state $|e_s^-(p)\rangle$ that describes an electron with momentum p and Dirac index s, then

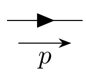
$$\left|e_s^-(p)\right\rangle = (2\pi)^{2/3}\sqrt{2\omega p}\; c_s^\dagger(p)\,|0\rangle$$

given a field ψ that describes a particle annihilation (or antiparticle creation) in x we have

$$\begin{aligned}
\left\langle 0\,\middle|\,\psi(x)\,\middle|\,e_s^-(p)\right\rangle &= \left\langle 0\,\middle|\,(\psi_+(x)+\psi_-(x))\,\middle|\,e_s^-(p)\right\rangle \\
&= \frac{(2\pi)^{3/2}}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}}e^{-ikx}\sqrt{2\omega_p}\;\sum_r \left\langle 0\,\middle|\,c_r(k)c_s^\dagger(p)\,\middle|\,0\right\rangle u_r(k) \\
&= \int \mathrm{d}^3k\Big(\frac{2\omega_p}{2\omega_k}\Big)^{1/2}\;\sum_r \delta_{rs}\delta^{(3)}(\bar{p}-\bar{k})u_r(k)\,\langle 0\,|\,0\rangle\, e^{-ikx} \\
&= e^{-ikx}u_s(p)
\end{aligned}$$

$c_r(k)c_s^\dagger(p) = \{c_r(k), c_s^\dagger(p)\} = \delta_{rs}\;\delta^3(\bar{p}-\bar{k})$

The factor e^{-ikx} is required for the $\delta^{(4)}$ conservation, and we see that the relativistic normalization leads to the relation ($\to$ Feyman rule)



$$= e^{-ipx}u_s(p)$$

In this case there is no normalization factors in the Feynman rule

1.4 S-matrix and State Evolution

In the interaction picture, with $H = H_0 + H_{\text{int}}$, with H_{int}= interaction hamiltonian

(i) Fields Φ_I evolves like in the free theory (respect to H_0)

(ii) State evolves with the following evolution operator

$$U_I(t,t_0) \equiv e^{iH_0t}e^{-iH(t-t_0)}e^{-iH_0t_0}$$
$$|\alpha,t\rangle = U_I(t-t_0)\,|\alpha,t_0\rangle \qquad i\partial_tU_I(t-t_0) = H_I^{\text{int}}(t)U_I(t,t_0)$$

Notice that, in general

$$[H_I(t),H_0] \neq 0 \neq [H_I^{\text{int}},H_0]$$

and if $t\neq t'$ we also have

$$[H_I^{\text{int}}(t),H_I^{\text{int}}(t')\neq 0 \qquad \text{with } O_I(t) = e^{iH_0t}e^{-iHt}\;O_He^{iHt}e^{-iH_0t}$$

The S-matrix is a well defined operator defined as

$$S = \lim_{\substack{t_0 \to -\infty \\ t \to +\infty}} U_I(t, t_0)$$

We compute S by perturbation obtaining

$$\begin{aligned} S &= T\left(\exp\left(-i\int \mathrm{d}^4 x \mathcal{H}_I^{\text{int}}(x)\right)\right) \\ &= \sum_{n=0}^{+\infty} \frac{(-i)^n}{n!} \int \mathrm{d}^4 x_1 \dots \mathrm{d}^4 x_n \, T(\mathcal{H}_I^{\text{int}}(x_1) \dots \mathcal{H}_I^{\text{int}}(x_n)) \end{aligned}$$

S has some relevant properties

(i) Unitary (since hamiltonian is hermitian)

(ii) Behaves as a scalar under Lorentz tfms, and then is an invariant quantity (notice that in general $\mathcal{H}_I^{\text{int}}$ is not invariant
In the case of $\mathcal{H}_I^{\text{int}}$ is invariant (for example if $\mathcal{H}_{\text{int}} = -\mathscr{L}_{\text{int}}$, as in many theories, one of them id QED) is easy to prove that S in invariant, since all n-th derivatives of $\exp\left(\int \mathcal{H}\right)$ are invariant, and so also S is invariant

1.5 S-matrix and transition probabilities

We suppose that there's no interactions for $x, t \to +\infty$

1.6 S-matrix and transition probabilities

we suppose that there's no interaction for $x, t \to +\infty$
Consider a canonically normalized (CN) state $|\psi| = 1$:

$$|\psi_i\rangle_{CN} \equiv |\psi(-\infty)\rangle_{CN} \qquad |\psi(+\infty)\rangle_{CN} \equiv S\,|\psi_i\rangle_{CN} \quad \to \text{both are free particle states}$$

Elements of S are in the form

$$S_{fi}^{CN} = \langle \psi_f \,|\, S \,|\, \psi_i \rangle_{CN}$$

This leads to a probabilistic interpretation of S-matrix elements.
$\left|S_{fi}^{CN}\right|^2$ = propability of evolution of $|\psi_i\rangle_{CN}$ into $|\psi_f\rangle_{CN}$, since the condition $\sum_f \left|S_{fi}^{CN}\right| = 1$ is satisfied automatically
In the case of covariant normalization

$$\langle 1(p) \,|\, 1(p') \rangle = (2\pi)^3 (2\omega p) \delta^3(\mathbf{p} - \mathbf{p}')$$

we have the following relation between matrix elements

$$S_{fi}^{CN} = \langle \psi_f \,|\, S \,|\, \psi_i \rangle_{CN} = \frac{\langle \psi_f \,|\, S \,|\, \psi_i \rangle}{\|\psi_i\| \, \|\psi_f\|} = \frac{S_{fi}}{\|\psi_i\| \, \|\psi_f\|}$$

We can define the **Feynman Amplitude** $\mathcal{M}_{fi}$ as

$$S_{fi} = (2\pi)^4 \delta^4(p_i - p_f) \mathcal{M}_{fi}$$

and it can be obtained directly starting from Feynman rules (calculated with the covariant normalization)

1.7 Discrete space normalization

Usually, in order to make arguments cleaner, or to avoid problems with divergent terms, we first consider a system in a cubic box with spatial valume $V = L^3$.
At the end of computations V will be sent to infinity. Sometimes we will do something similar also for time.
For a discrete space we must use a different normalization.
In a box, momentum of a particle is quantized (1-dim case)

$$p_i = \left(\frac{2\pi}{L}\right)n_i \qquad n_i \in \mathbb{Z}$$

and we must adopt the following rule for integrals

$$\int \mathrm{d}^3p\, f(\mathbf{p}) \quad \to \quad \sum_{\mathbf{n}}\left(\frac{2\pi}{L}\right)f_{\mathbf{n}} \qquad \mathbf{n} = (n_1, n_2, n_3)$$

We must adopt also the following

$$\delta^3(\mathbf{p} - \mathbf{p}') \quad \to \quad \left(\frac{L}{2\pi}\right)^3 \delta_{\mathbf{nn}'}$$

in this way

$$\int \mathrm{d}^3p\, \delta^3(\mathbf{p} - \mathbf{p}') = 1 \quad \to \quad \sum_{\mathbf{n}}\left(\frac{2\pi}{L}\right)^3\left(\frac{L}{2\pi}\right)^3 \delta_{\mathbf{nn}'} = 1$$

Some useful relations are

$$\delta^3(0) \to \left(\frac{L}{2\pi}\right)^3$$

$$\delta^4(0) \to \left(\frac{L}{2\pi}\right)\left(\frac{T}{2\pi}\right) \to \text{Only if we consider a finite amount of time}$$

Normalization of state becomes

$$|1(p)\rangle = (2\pi)^{2/3}\sqrt{2\omega_p V} o^\dagger(p)\,|0\rangle = (2\pi)^{2/3}\sqrt{2\omega_p V}\,|1(p)\rangle_{CN}$$
$$\langle 1(p)\,|\,1(p)\rangle = (2\pi)^3 2\omega_p V \delta^3(0) = 2\omega_p V$$

Using the latter equation, S_{fi}^{CN} reads

$$\begin{aligned} S_{fi}^{CN} &= \prod_{j=1}^{n_i}\left(\frac{1}{2\omega_j V}\right)^{1/2}\prod_{l=1}^{n_f}\left(\frac{1}{2\omega_l V}\right)^{1/2} S_{fi} \\ &= (2\pi)^4\delta^4(p_i - p_f)\left\{\prod_{j=1}^{n_i}\left(\frac{1}{2\omega_j V}\right)^{1/2}\prod_{l=1}^{n_f}\left(\frac{1}{2\omega_l V}\right)^{1/2}\mathcal{M}_{fi}\right\} \\ &= (2\pi)^4\delta^4(p_i - p_f)M_{fi}^{CN} \end{aligned}$$

In the first passage $(2\pi)^{2/3}$ factors vanish because of δ^3 factors inside S_{fi} related to the sandwich $\langle\psi_f\,|\,S\,|\,\psi_i\rangle$
In the second passage we use df_n of $\mathcal{M}_{fi}$, omitting the quantization of δ^4
M_{fi}^{CN} is the **canonically normalized Feynman amplitude**

Chapter 2

The S-matrix and physical observables

2.1 Decay Rate

Consider the case in which the initial state is a single particle and the final state is given by n particles. We are therefore considering a decay process. Assume for the moment that particles are indistinguishable.

The rules of quantum mechanics tell us that the probability for this process is obtained by taking the squared modulus of the amplitude and summing over all possible final states

$$
\begin{aligned}
\left|S^{CN}_{fi}\right|^2 &= \left|(2\pi)^4\delta^4(p-p')M^{CN}_{fi}\right|^2 \\
&= (2\pi)^4\delta^4(p-p')(VT)\left|M^{CN}_{fi}\right|^2 \\
&= (2\pi)^4\delta^4(p-p')(VT)\frac{1}{2\omega_i nV}\prod_{l=1}^{n_f}\left(\frac{1}{2\omega_l V}\right)|\mathcal{M}_{fi}|^2
\end{aligned}
$$

Note: $\left(\delta^4(p-p')\right)^2 = \delta^4(p-p')\delta^4(p-p') = \delta^4(p-p')\delta^4(0) = \delta^4(p-p')\frac{VT}{(2\pi)^4}$
We use the final space and time in order to remove divergent terms during calculation

We must now sum this expression over all final states. Since we are working in a finite volume V, this is the sum over the possible discrete values of the momenta of the final particles .
Since $p_i = (2\pi/L)n_i$, we have $dn_i = (L/2\pi)\mathrm{d}p_i$ and $\mathrm{d}^3n_i = (V/(2\pi)^3)\mathrm{d}^3p$ where d^3n_i is the infinitesimal phase space related to a final state in which the i-th particle has momentum between p_i and $p_i + \mathrm{d}p_i$
Let $\mathrm{d}\omega$ be the probability for a decay in which in the final state the i-th particle has momentum between p_i and $p_i\mathrm{d}p_i$

$$
\mathrm{d}\omega = \left|S^{SN}_{fi}\right|^2\prod_{l=1}^{n_f}\left(\frac{V\mathrm{d}^3p_l}{(2\pi)^3}\right)
$$

This is the probability that the decay takes place in any time between $-T/2$ and $T/2$. We are more interested in the differential decay rate $\mathrm{d}\Gamma_{fi}$, which is the decay probability per unit of time:

$$
\mathrm{d}\Gamma_{fi} = \frac{\mathrm{d}\omega}{T} = (2\pi)^4\delta^4(p_i-p_f)\frac{|\mathcal{M}_{fi}|^2}{2\omega_{p_{in}}}\prod_{l=1}^{n_f}\frac{\mathrm{d}^3p_l}{(2\pi)^3 2\omega_l}
$$

Notes:

(i) $\mathrm{d}\Gamma_{fi}$ = differential decay rate

(ii) p_f = sum over final momenta

(iii) $\omega_{p_{in}}$ = initial energy

(iv) $|\mathcal{M}_{fi}|^2$ = Feynman amplitude of the process (depends on final momenta p_i)

It is useful to define the **(differential) n-body phase space** as

$$\mathrm{d}\Phi_{(n_f)} = (2\pi)^4 \delta^4(p_i - p_f) \prod_{l=1}^{n_f} \frac{\mathrm{d}^3 p_l}{(2\pi)^3 2\omega_l}$$

Therefore the differential decay rate can be written as

$$\mathrm{d}\Gamma_{fi} = \frac{1}{2\omega_{p_{in}}} |\mathcal{M}_{fi}|^2 \, \mathrm{d}\Phi_{(n_f)}$$

The decay rate is defined as

$$\Gamma_{fi} = \int \mathrm{d}\Gamma_{fi} \qquad \rightarrow \text{integration over all possible final momenta}$$

and its meaning is $\Gamma \equiv$ trans. probability $\times$ unit of time $\times$ init. particle
Notice that if n of the final particles are identical, configurations that differ by a permutation are not distinct and therefore the phase space is reduced by a factor $1/h$!
If we have a system of N(0) particles, the time evolution of the number of particles N(t) is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\Gamma N \quad \Rightarrow \quad N(t) = N(0)e^{-\Gamma t}$$

Notice that decay rate is not invariant

$$[\Gamma] = [E] = \frac{1}{T} \qquad \text{(in natural units)}$$

If we define the **lifetime** as $\tau = 1/\Gamma \Rightarrow N(t) = N(0)\exp(-t/\tau)$ this changes under Lorentz tfm. If we consider two reference frames o and o'

$$\Gamma' = \frac{\Gamma}{\gamma} < \Gamma \qquad \tau' = \gamma\tau > \tau$$

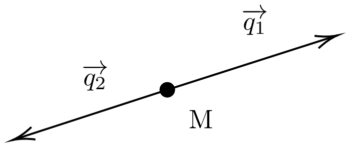
$\gamma = (1-v)^{-1/2}$, where v is the speed of o' in o in natural units, $\gamma > 1$. Therefore a article in a moving frame has a longer lifetime then in the rest frame

Example 1: muon lifetime

For a muon in the rest frame $\tau_\mu^{RF} = 2.2 \times 10^{-6} s$, but if we observe it in the lab frame
$\tau_\mu^{LAB} = \gamma\tau_\mu^{RF} \simeq 2 \times 10^{-5} s$ since $E_\mu = 1$ GeV, $m_\mu = 0.1$ GeV
$\Rightarrow \gamma = E_\mu/m_\mu \simeq 10$

Example 2: 1 → 2 decay

Consider the decay of a particle of a mass M into two particles of masses m_1, m_2. Since the phase space in Lorentz invariant, we can compute it in the frame that we prefer. We use the rest frame for the initial particle.



We don't impose a priori conservation of momentum since it's imposed by the delta function.

$$p = (M, 0) \qquad q_1 = (\omega_1, \mathbf{q_1}) \qquad q_2 = (\omega_2, q_2)$$

$$\mathrm{d}\Phi_{(2)} = (2\pi)^4 \delta^4 \underbrace{(P_i - P_f)}_{=p-q_1-q_2} \frac{\mathrm{d}^3 q_1}{(2\pi)^3 2\omega_1} \frac{\mathrm{d}^3 q_2}{(2\pi)^3 2\omega_2}$$

I have 6 integration parameters, 4 constraint given by δ^4, so I have 2 independent variables. Integrating over $\mathrm{d}^3 q_2$

$$\mathrm{d}\Phi'_{(2)} = \int \mathrm{d}\Phi_{(2)} = \frac{1}{(2\pi)^2} \delta(M - \omega_1 - \omega_2) \frac{1}{4\omega_1\omega_2} \mathrm{d}^3 q_1$$

in this way, the condition $\mathbf{q_2} = \mathbf{q_1}$ vanish. We have to impose it again when we calculate $\mathrm{d}\Gamma$ (we omit this detail)
Usually the 4-th non independent parameter is eliminated by integration over modulus of q_1, leaving free 2 parameters for the angles. $\mathrm{d}^3 q_1 \to \mathbf{q_1}^2 \mathrm{d}\left|\mathbf{q_2}\right| \mathrm{d}\Omega_1$.

Notice that $M - \omega_1 - \omega_2 = M - \sqrt{\mathbf{q_1}^2 + m_1^2} - \sqrt{\mathbf{q_2} + m_2^2}$ and then the δ implies

$$\hat{q_1}^2 = \frac{1}{2M} \left(M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2 \right)^{1/2}$$

$|\mathbf{q_1}|$ is the only zero of $f(|\hat{q_1}|) = M - \omega_1 - \omega_2$.
We also have

$$|f'(|\hat{q_1}|)| = \frac{\partial \omega_1}{\partial \left|\mathbf{q_1}\right|} + \frac{\partial \omega_2}{\partial \left|\mathbf{q_1}\right|} = |\hat{q_1}| \left(\frac{\omega_1 + \omega_2}{\omega_1 \omega_2} \right)$$

Using

$$\delta\big(f(x)\big) = \sum_{x_0 = \text{zero of f(x)}} \frac{\delta(x - x_0)}{|f'(x_0)|}$$

and performing integration over $\mathrm{d}\left|\mathbf{q_1}\right|$ we obtain

$$\mathrm{d}\Phi''_{(2)} = \int \mathrm{d}\Phi'_{(2)} = \frac{1}{16\pi^2} \frac{|\hat{q_1}|}{M} \mathrm{d}\Omega_1$$

Using this result we obtain the $1 \to 2$ decay rate in function of the solid angle (in the rest frame)

$$\left(\frac{\mathrm{d}\Gamma_{RF}}{\mathrm{d}\Omega} \right) = \frac{1}{64\pi^4 M^3} \left[M^4 - 2M^2(m_1^2 + m_2^2) + (m_1 - m_2^2)^2 \right]^{1/2} \left|\mathcal{M}_{RF}\right|^2$$

In a general frame we can easily obtain an analogous formula, just consider $\mathrm{d}\Gamma = 1/(2\omega_i n) \left|\mathcal{M}_{fi}\right|^2 \mathrm{d}\Phi_{(nf)}$ in a general frame. remember that $\mathrm{d}I_{(nf)}$ is invariant

We have 2 important limit cases:

(A) If $m_1 = m_2 = m$ (for example $Z \to e^+ e^-$)

$$|\hat{q_1}| = \frac{M}{2} \left(1 - \frac{4m^2}{M^2} \right)^{1/2}$$

$$\left(\frac{\mathrm{d}\Gamma_{RF}}{\mathrm{d}\Omega} \right) = \frac{1}{64\pi^2 M} \left(1 - \frac{4m^2}{M^2} \right)^{1/2} \left|\mathcal{M}_{fi}\right|^2$$

(B) If $m_1 = m, m_2 = 0$ (for example $W^{\pm} \to e^{\pm} \overset{(-)}{\nu}$)

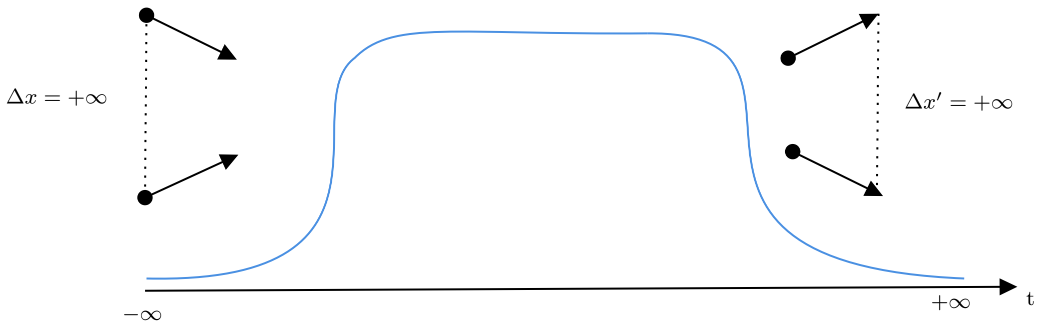
$$|\hat{q}_1| = \frac{M}{2}\left(1 - \frac{4m^2}{M^2}\right)^{1/2}$$

$$\left(\frac{\mathrm{d}\Gamma_{RF}}{\mathrm{d}\Omega}\right) = \frac{1}{64\pi^2 M}\left(1 - \frac{m^2}{M^2}\right)^{1/2} |\mathcal{M}_{fi}|^2$$

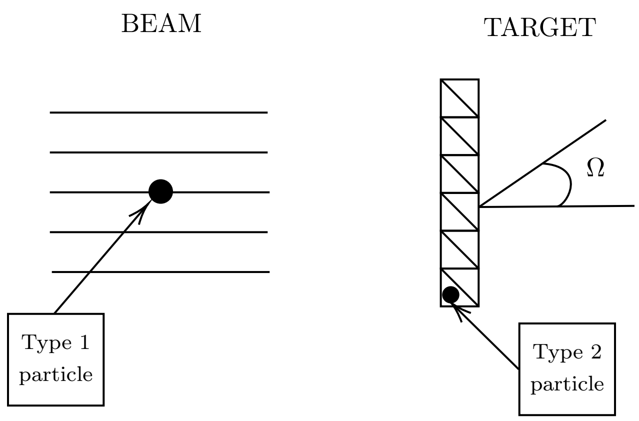
Notes: If we have two identical particles in the final state, the calculation of the phase is different

$$\mathrm{d}\Phi_{(2)}^{\text{identical}} = \frac{1}{2}\mathrm{d}\Phi_{(2)}^{\text{distinguishable}}$$

2.2 Cross section ($\leftrightarrow$ scattering process)



Scattering in the lab frame



Consider a beam of particles with mass m_1, number (assuming a uniform distribution) density $n_1^{(0)}$ (subscript 0 is meant to stress that these are number densities in a specific frame, that with particle 2 at rest) and velocity v_1 inpining on a target made of particles with mass m_2 and number density $n_2^{(0)}$ at rest.
Let N_s be the number of scattering events that place per unit volume and per unit time

$$\frac{N_t}{T}\varphi_1 N_2 \sigma = \left(n_1^{(0)} v_1\right)\left(n_2^{(0)} V\right)\sigma$$

More formally we have

$$\mathrm{d}N_s = \sigma v_1 n_1^{(0)} n_2^{(0)}\ \mathrm{d}V \mathrm{d}V$$

with:

(i) T: unit of time

(ii) φ_1: flux of the beam $\varphi_1 = n_1^{(0)} v_1$

(iii) N_2: particles per unit volume in the detector $N_2 = n_2^{(0)} V$

(iv) σ: proportionality constant

Dimensional analysis shows $[\sigma] = [L]^2$ and then σ, called cross section, can be interpreted as an "effective area".

ciao