Chapter 1

QED Processes at Lowest order

1.1 The QED Lagrangian and its Symmetries

Mandl, sec 11.1 - Maggiore, sec 7.1

Quantum electrodynamic (QED) describes the interactions between (or any other charged spin 1/2 particle) and photons. QED is described by the lagrangian

$$\mathcal{L}_{\text{QED}} = \underbrace{\overline{\psi}(i\not{\partial} - m)\psi}_{\mathcal{L}_D^{(0)}} \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\mathcal{L}_{EM}} - \underbrace{qA_n\overline{\psi}\gamma^\mu\psi}_{\mathcal{L}_{int}} - \underbrace{\frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2}_{\mathcal{L}_{GF}}$$

(i) $\mathcal{L}_D^{(0)}$ is the langrangian for the free Dirac field

(ii) $\mathcal{L}_{EM}$ is the lagrangian for the free EM field. In order to quatize the E-n field we have to add the term $\mathcal{L}_{GF}$ (gauge fixing). For other purposes this term can be omitted. Usually the choice $\xi = 1$, called Feynman gauge, is the simplest choice for quantization

(iii) $\mathcal{L}_{int}$ describes the interaction between Dirac field and EM-field. Notice that the term $\mathcal{L}_D = \mathcal{L}_D^{(0)} + \mathcal{L}_{int}$ can be obtained from $\mathcal{L}_D^{(0)}$ with the "minimal substitution" $\partial_\mu \to \partial_\mu + iqA_\mu = D_\mu$, i.e. using covariant derivative D_μ instead of ∂_μ in the dirac lagrangian.
Notice that $\mathcal{L}_D$ exhibits local symmetry, while $\mathcal{L}_D^{(0)}$ doesn't

Besides Lorentz invariance, the QED exhibits following symmetries:

- **Global U(1) symmetry**

$$\begin{cases} \psi(x) \to \psi'(x) = e^{i\alpha}\psi(x) & \alpha \in \mathbb{R} \\ A^\mu(x) \to A'^\mu(x) = A^\mu(x) \end{cases}$$

  There is therefore an associated conserved Noether current

$$j^\mu = q\overline{\psi}\gamma^\mu\psi \quad \to \quad \partial_\mu j^\mu = 0$$

  and a U(1) charge which is conserved by the EM interaction

$$Q = q\int \mathrm{d}^3x\ \psi^\dagger\psi \qquad \frac{\mathrm{d}Q}{\mathrm{d}t} = 0$$

- **Local U(1) symmetry** (gauge symmetry)

$$\begin{cases} \psi(x) \to \psi'(x) = e^{iq\alpha(x)}\psi(x) \\ A^\mu \to A'^\mu(x) = A^\mu(x) - iq\partial^\mu\alpha(x) \end{cases}$$

notice that the global U(1) symmetry is a sequence of the local U(1) symmetry, taking $\alpha(x)$ constant)

The covariant derivative of ψ $D_\mu\psi$ behaves as a spinor (remember that D_μ transforms as a vector):

$$\begin{aligned} D_\mu\psi \to (D_\mu\psi)' = D'_\mu\psi' &= (D_\mu - iq\partial_\mu\alpha)(e^{iq\alpha(x)}\psi(x)) \\ &= (\partial_\mu + iqA_\mu - iq\partial_\mu\alpha)(e^{iq\alpha}\psi) \\ &= e^{iq\alpha}(\partial_\mu + iqA_\mu)\psi \\ &= e^{iq\alpha(x)}(D_\mu\psi) \end{aligned}$$

This implies that $\mathcal{L}_D$ is invariant. Since $\mathcal{L}_{EM}$ is invariant, the full lagrangian is invariant

1.2 Flavors in QED and the SU(3) Flavor Global Symmetry

QED describes interactions of the photon field with several kind of leptons, not only electron and positrons. Particles that differs only by their mass are called **flavours**. The next table describes leptons

in QED. There are two families of leptons that differs by their charge. We indicate with (-) (minus) particles with negative charge, and with (+) (plus) particles with positive charge (antiparticles)
Flavours in QED

Leptons[I]	e	μ	τ	ν_e	ν_μ	ν_τ
q	-1	-1	-1	0	0	0
$m[MeV]$	0.5	105	1777	$\simeq 0$	$\simeq 0$	$\simeq 0$

The dirac lagrangian $\mathcal{L}_D$ can be modified in order to consider all possible leptons

$$\mathcal{L}_D = \sum_{i=1}^{n} \bar{\psi}_i(i\not{D} - m_i)\psi_i \simeq \sum_{i=1}^{n_l} \bar{\psi}_i(i\not{D} - m_i)\psi_i + \sum_{j=1}^{n_n} \bar{\psi}_i(i\not{\partial})\psi_i$$

with n: number of leptons, n_l: number of electrically charged particles, n_n: number of neutrinos.
In the last term the interaction term vanishes because of q=0
If we adopt a matrix notation

$$\Psi_C = \begin{pmatrix} \psi_e \\ \psi_\mu \\ \psi_\tau \end{pmatrix} \qquad \Psi_N = \begin{pmatrix} \psi_{\nu_e} \\ \psi_{\nu_\mu} \\ \psi_{\nu_\tau} \end{pmatrix}$$

$$\bar{\Psi}_C = (\bar{\psi}_e, \bar{\psi}_\mu, \bar{\psi}_\tau) \qquad \bar{\Psi}_N = (\bar{\psi}_{\nu_e}, \bar{\psi}_{\nu_\mu}, \bar{\psi}_{\nu_\tau})$$

The subscript C stands for "charged", and N for "neutral"
We can define following matrices

$$M_C = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \qquad Q_C = (-1)\mathbb{1}_{3\times3} \qquad M_N \simeq Q_N = \emptyset_{3\times3}$$

and the generalization of the covariant derivative $D_\mu = \partial_\mu \cdot \mathbb{1}_{3\times3} + iA_\mu Q_C$ we obtain

$$\mathcal{L}_D = \underbrace{\bar{\Psi}_C(i\not{D} - M_C)\Psi_C}_{\mathcal{L}_C} + \underbrace{\bar{\Psi}_N(i\not{\partial}\mathbb{1}_{3\times3})\Psi_N}_{\mathcal{L}_N}$$

The term $\mathcal{L}_C$ and $\mathcal{L}_N$ are respectively the **Charged Sector** and the **Neutral Sector** of $\mathcal{L}_D$
The basis defined by Ψ_C and Ψ_N is called **physical basis**, since physical particles are identified by their mass.

[I]Neutrinos admit anly global U(l) symmetry
Neutrino masses are in the order $m_\nu \approx 10^{-6}me \leq 1eV$

1.2.1 Global Symmetry of Neutral and Charged Sector

Let's consider a U(3) transformation

$$\Psi(x) \to \Psi'(x) = U\Psi(x) \qquad U^\dagger U = \mathbb{1}_{3\times 3}$$

Spinors and vector are left invariant
The neutral sector is left invariant under U(3) tfm

$$\mathcal{L}_N \to \mathcal{L}'_N = \bar{\Psi}'_N(i\partial\!\!\!/\mathbb{1}_{3\times 3})\Psi'_N = \bar{\Psi}_N U^\dagger (i\partial\!\!\!/\mathbb{1}_{3\times 3})U\Psi_N = \mathcal{L}_N$$

The charged sector is not invariant because of the mass term

$$\mathcal{L} \to \mathcal{L}'_C = \bar{\Psi}_C(iD\!\!\!\!/ - U^\dagger M_C U)\Psi_C \neq \mathcal{L}_C \quad \text{in general}$$

In order to obtain global symmetries of the flavor QED lagrangian, we search the subgroup of U(3) made by matrices $U-g$ that satisfies

$$U_g^\dagger M_C U_g = M_c \qquad U_g \in U(3) \tag{1.1}$$

We can prove that $U(3) \overset{isomorphism}{\simeq} U(1) \times SU(3)$, with

(i) $|\det(\mathrm{U}(3)| = 1$

(ii) $\det(\mathrm{U}(1)) = e^{i\theta}$ [II]

(iii) $\det(\mathrm{SU}(3)) = 1$ [III]

Since generators of SU(3) are Gell-Mann matrices λ_a (for a $= 1, \ldots, 8$), generators of U(3) are

$$\mathbb{1} \times \{\lambda_1, \ldots, \lambda_8\}$$

we can also prove that equation **??** is satisfied only by diagonal matrix
Diagonal generators of SU(3) are

$$\lambda_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Up to a phase, matrices U_g are in the form

$$U_g = e^{i\alpha_0\lambda_0}\ e^{i\alpha_3\lambda_3}\ e^{i\alpha_8\lambda_8}$$

For i $=$ 0, 3 ,8 we have $[\lambda_i, M_c] = 0$ and then the equation **??** is satisfied: if we take $\alpha_i \ll 1$

$$(1 - i\alpha_i\lambda_i)M_C(1 + i\alpha_i\lambda_i) = (M_C - i\alpha_i[\lambda_i, M_C] + o(\alpha_i^2)) \simeq M_C$$

We then obtained that the global group of symmetry is generated by the algebra

$$\mathcal{G} = \{\lambda_0, \lambda_3, \lambda_8\} = U(1)^3 \subset U(3)$$

I define the following basic of $\mathcal{G}$:

$$\lambda_e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_\mu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_\tau = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These matrices generates phase tfm for each kind of leptons (λ_e generates phase tfm for e, ecc ...)
Conserved quantities of this group are 3, and corresponding to the number of particles of each type (Antiparticles are counted with negative sign)

[II] Indicando con $e^{i\theta}$ il determinante delle matrici $U_g \in U(3)$
[III] SU(3) è l'insieme di livello dato da $(\arg \circ \det)^{-1}(0)$

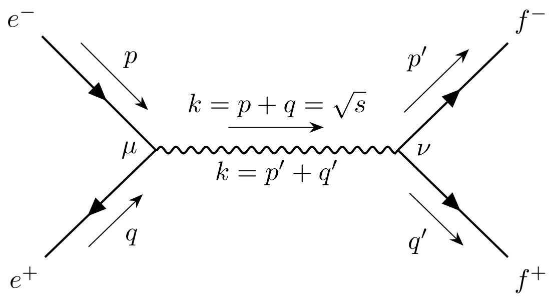
Example 1

$\mu^- \to e^- \gamma$ is forbidden in QED. This is an example of conserved charges due to unitary symmetry that have nothing to do with electric charge.
Flavours changing in neutral sector are forbidden too

1.3 QED Feynman Rules $\to$ fogli stampati (23-26)

1.4 $e^+e^- \to f^+f^-$

This diagram is called s-channel



With $k = p' + q'$ we impose the 4 momentum conservation
We have

$$S_{fi} = (2\pi)^4 \delta^4 (p + q - p' - q') \mathcal{M}_{fi}$$

with Feynman amplitude

$$\begin{aligned}
\mathcal{M}_{fi} &= (-iq)^2 \left[\bar{u}_{r'}(p')\gamma^\nu \; v_{s'}(q')\right]^{\mathrm{IV}} \left[\bar{v}_s(q)\gamma^\mu \; u_r(p)\right]^{\mathrm{V}} D^F_{\mu\nu}(k) \\
&= (-iq)^2 \frac{-ig_{\mu\nu}}{k^2 + i\varepsilon} \left[\bar{u}_{r'}(p')\gamma^\nu \; v_{s'}(q')\right] \left[\bar{v}_s(q)\gamma^\mu \; u_r(p)\right] \\
&= \frac{iq^2}{s + i\varepsilon} \left[\bar{u}_{r'}(p')\gamma_\mu \; v_{s'}(q')\right] \left[\bar{v}_s(q)\gamma^\mu \; u_r(p)\right]
\end{aligned}$$

In the second passage $\xi = 1$. We can prove that this choice has no importance, see Maggiore pg 187

Using the identity $(\bar{u}\gamma^\mu v)^* = \bar{v}\gamma^\mu u$ (that can be proved by direct calculation using $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$) we obtain (we omit polarization indexes) gurda foglio 27

$$|\mathcal{M}_{fi}|^2 = \mathcal{M}_{fi}\mathcal{M}^*_{fi} = \frac{q^4}{s^2} \underbrace{\left(\bar{u}(p')\gamma_\mu v(q')\right) \left(\bar{v}(q)\gamma^\mu u(p)\right)}_{=??}$$

At this point, we are still free to specify any particular spinors $u_r(p)$, $\bar{v}_{s'}(p')$ and so on, corresponding to any desired spin states of the fermions.

1.4.1 Sum Over Fermion Spins. Squared Averaged Feynman Amplitude

The Feynman amplitude simplifies considerably when we throw away the spin information. We want to compute

$$|\overline{\mathcal{M}_{fi}}|^2 = \underbrace{\frac{1}{2}\sum_s \frac{1}{2}\sum_r}_{\substack{\text{average over}\\ \text{the initial states}}} \underbrace{\sum_{s'} \sum_{r'}}_{\substack{\text{sum over}\\ \text{final states}}} |\mathcal{M}(r, s \to r', s')|^2$$

[V]indica il percorso $e^- \to \mu \to e^+$ nel diagramma
[V]indica il percorso $f^- \to \nu \to f^+$ nel diagramma

This sum can be performed using completeness relations for dirac spinors

$$\sum_r u_s(p)\bar{u}_s(p) = \not{p} + m \qquad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m$$

Writing spinors indexes explicitly

$$\begin{aligned}\sum_{rs} \text{e-current} &= \sum_{rs} \bar{v}^s_a(q)(\gamma^\mu)_{ab}\, u^r_b(p)\, \bar{u}^r_c(p)(\gamma^\nu)_{cd}\, v^s_d(q) \\ &= (\not{q} - m)_{da}\, \gamma^\mu_{ab}\, (\not{p} + m)_{bc}\, \gamma^\nu_{cd} \\ &= \mathrm{Tr}\big[(\not{q} - m)\gamma^\mu(\not{p} + m)\gamma^\nu\big]\end{aligned}$$

and similarly

$$\sum_{r's'} \text{f-current} = \mathrm{Tr}\big[(\not{p}' + m)\gamma_\mu(\not{q}' - m)\gamma_\nu\big]$$

So we obtain

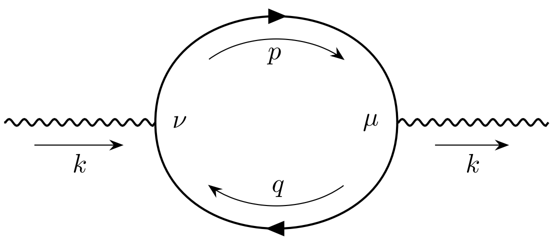
$$\overline{|\mathcal{M}_{fi}|^2} = \frac{q^4}{4s^2}\mathrm{Tr}\big[(\not{q} - m)\gamma^\mu(\not{p} + m)\gamma^\nu\big]\mathrm{Tr}\big[(\not{p}' + m)\gamma_\mu(\not{q}' - m)\gamma_\nu\big]$$

The spinors u and v have disappeared, leaving us with a much cleaner expression in terms of γ matrices. This trick is very genera: any QED amplitude involving external fermions, when squared and summed or averaged over spins, can be converted in this way to traces of products of Dirac matrices

There is a trick to obtain previous formula using only Feynman rules. If we set $\mathcal{M}_{fi} = \mathcal{M}_e \mathcal{M}_f$ (we divide $\mathcal{M}_{fi}$ in 2 factors related to e and f), then $|\mathcal{M}_{fi}|^2 = \mathcal{M}_e^* \mathcal{M}_e\, \mathcal{M}_f^* \mathcal{M}_f$. We have

Da completare

$$\sum_{rs} \mathcal{M}_e^*\, \mathcal{M}_e =$$



The closed fermion line contributes in the final amplitude with a factor

$$\frac{q^2}{2s}\mathrm{Tr}\big[\underbrace{(\not{p} + m)}_{\text{Fermion}}\gamma^\nu\ \underbrace{(\not{q} - m)}_{\text{Antifermion}}\ \gamma^\mu\big]$$

The factor $-q^2/(2s)$ is obtained using full diagram (s is given by photon propagator)

Fare diagramma di Feynman

Going back to the expression of $|\mathcal{M}_{fi}|^2$ that we obtain, now we need to calculate explicitly γ matrices traces

$$\mathrm{Tr}\big[(\not{p} + m)\gamma^\nu(\not{q} - m)\gamma^\mu\big] = \mathrm{Tr}\big[\not{p}\ \gamma^\nu \not{q}\ \gamma^\mu\big] - m^2\mathrm{Tr}[\gamma^\nu\ \gamma^\mu] + m\big(\mathrm{Tr}[\gamma^\nu \not{q}\ \gamma^\mu] - \mathrm{Tr}[\not{p}\ \gamma^\nu\gamma^\mu]\big)$$

Now we prove some properties of $\gamma-$matrices traces (other properties in Peskin pg 133-135)

(I)

$$\begin{aligned}\mathrm{Tr}[\gamma^\mu] &= \mathrm{Tr}[\gamma^5\,\gamma^5\,\gamma^\mu] && \to (\gamma^5)^2 = \mathbb{1} \\ &= -\mathrm{Tr}[\gamma^5\,\gamma^\mu\,\gamma^5] && \to \{\gamma^5, \gamma^\mu\} = 0 \\ &= -\mathrm{Tr}[\gamma^5\,\gamma^5\,\gamma^\mu] && \to \text{cyclicity} \\ &= -\mathrm{Tr}[\gamma^\mu] \\ &\Rightarrow \mathrm{Tr}[\gamma^\mu] = 0\end{aligned}$$

(II)

$$
\begin{aligned}
\mathrm{Tr}[\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma] &= \mathrm{Tr}[2g^{\mu\nu}\,\gamma^\rho\,\gamma^\sigma - \gamma^\nu\,\gamma^\mu\,\gamma^\rho\,\gamma^\sigma] \\
&= 2g^{\mu\nu}\mathrm{Tr}[\gamma^\rho\,\gamma^\sigma] - \mathrm{Tr}[\gamma^\nu 2g^{\mu\rho}\gamma^\sigma - \gamma^\nu\,\gamma^\rho\,\gamma^\mu\,\gamma^\sigma] \\
&= 8g^{\mu\nu}g^{\rho\sigma} - 8g^{\mu\rho}g^{\nu\sigma} + \mathrm{Tr}[\gamma^\nu\,\gamma^\rho\,2g^{\mu\sigma} - \gamma^\nu\,\gamma^\rho\,\gamma^\sigma\,\gamma^\mu] \\
&= 8\big(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\big) - \mathrm{Tr}[\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma] \\
&\Rightarrow \mathrm{Tr}[\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma] = 4\big(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\big)
\end{aligned}
$$

From previous calculation, using induction, can be easily proved that $\mathrm{Tr}[\underbrace{\gamma^\mu \ldots \gamma^\sigma}_{\substack{\text{odd } \# \text{ of}\\ \gamma\text{ matrices}}}] = 0$

With these relations we obtain

•

$$
\begin{aligned}
\mathrm{Tr}[(\not{p}+m)\gamma^\nu(\not{q}-m)\gamma^\sigma] &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})p_\mu q_\rho - m^2 4g^{\nu\sigma} \\
&= 4(p^\nu q^\sigma - g^{\nu\sigma}p\cdot q + p^\sigma q^\nu - g^{\nu\sigma}m^2) \\
&= 4\big(p^\nu q^\sigma + p^\sigma q^\nu - g^{\nu\sigma}(p\cdot q + m^2)\big) \\
&\text{with m} = \text{mass of } e^I
\end{aligned}
$$

•

$$
\mathrm{Tr}[(\not{p}'+m)\gamma_\mu(\not{q}'-m)\gamma_\nu] = 4\big(p'_\mu\,q'_\nu + p'_\nu\,q'_\mu\,g_{\mu\nu}(p'\cdot q' + M^2)\big) \qquad \text{with M} = \text{mass of } f^I
$$

•

$$
\begin{aligned}
\overline{|\mathcal{M}_{fi}|^2} &= \frac{4q^4}{s^2}\,\big(p^\mu q^\nu + p^\nu q^\mu - g^{\mu\nu}(p\cdot q + m^2)\big)\big(p'_\mu\,q'_\nu + p'_\nu\,q_\mu - g_{\mu\nu}(p'\cdot q' + M^2)\big) \\
&\approx \frac{8q^4}{s^2}\big((p\cdot p')(q\cdot q') + (p\cdot q')(p'\cdot q) + M^2(p\cdot q)\big) \qquad \text{if } m \ll M
\end{aligned}
$$

To obtain a more explicit formula we must specialize to a particular frame of reference and express the vectors p, q, p', q', k in terms of the basis kinematic variables (energies and angles) in that frame. In practice, the choice of frame will be dictated by the experimental conditions.
We want to calculate cross section in the center of mass frame.

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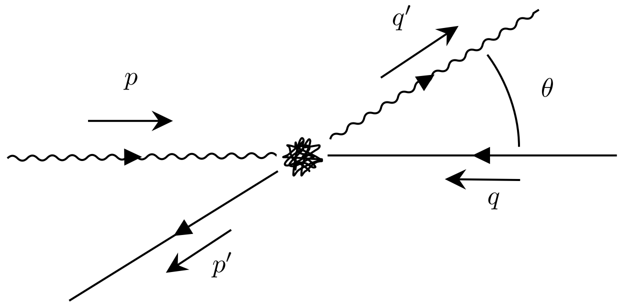
Da sistemare



$$
\begin{aligned}
p &= (\omega_p, \mathbf{p}) \\
q &= (\omega_q, \mathbf{q}) \\
\mathbf{p}+\mathbf{q} &= 0 \\
p' &= (\omega_{p'}, \mathbf{p}') \\
q' &= (\omega_{q'}, \mathbf{q}') \\
\mathbf{p}+\mathbf{q} &= 0
\end{aligned}
$$

In the CM frame we have following kinematics relations

$$
\left.\begin{aligned}
(p+q) = (\omega_p + \omega_q, \mathbf{p}+\mathbf{q}) = (\omega_p + \omega_q, 0) \\
s = (p+q)^2 = (\omega_p + \omega_q)^2
\end{aligned}\right\} = (p+q) = (\sqrt{s}, 0)
$$

and

$$
\omega_p = (p+q)^2 = (\omega_p + \omega_q)^2
$$

Same for final momenta

$$
(p'+q') = (\sqrt{s}, 0)
$$

$$
\omega_{p'} = \omega_{q'} = \frac{\sqrt{s}}{2}
$$

For product momenta

$$
\begin{aligned}
(p \cdot q) &= \omega^2 - \mathbf{pq} = \omega^2 \left|\mathbf{p}\right|^2 = 2\omega^2 - m^2 \overset{m=0}{\simeq} 2\omega^2 \\
(p' \cdot q') &= \omega^2 - \mathbf{pq} = \omega^2 \left|\mathbf{p}\right|^2 = 2\omega^2 - M^2 \\
(p \cdot p') &= \omega^2 - \mathbf{pp'} = \omega^2 - |\mathbf{p}|\,|\mathbf{p}'|\cos\theta \overset{m=0}{\simeq} \omega(\omega - |\mathbf{p}'|\cos\theta) \overset{m=0}{\simeq} (q \cdot q') \\
(p \cdot q') &= \omega^2 + \mathbf{p}\,\mathbf{p}' = \omega^2 + |\mathbf{p}|\,|\mathbf{p}'|\cos\theta \simeq \omega(\omega + |\mathbf{p}'|\cos\theta) \simeq (q \cdot p') \\
\left|\mathbf{p}'\right|^2 &= \omega^2 - M^2 = \omega^2\left(1 - \frac{M^2}{\omega^2}\right) = \frac{s}{4}\left(1 - \frac{4M^2}{s}\right) \\
\left|\mathbf{p}\right|^2 &= \omega^2 + m^2 \overset{m=0}{\simeq} \omega^2 = \frac{s}{4}
\end{aligned}
$$

Now we can rewrite $\overline{\left|\mathcal{M}_{fi}\right|^2}$ in terms of ω and θ. If $m = 0$ is valid (e.g. $m_e/m_\mu = 1/200$)

$$
\begin{aligned}
\overline{\left|\mathcal{M}_{fi}\right|^2} &\simeq \frac{8q^4}{s^2}\left[(\omega^4 - 2\omega^3\left|p'\right|\cos\theta + \omega^2\left|p'\right|^2\cos^2\theta) + (\omega^4 + 2\omega^3\left|p'\right|\cos\theta + \omega^2\left|p'\right|^2\cos^2\theta) + 2M^2\omega^2\right] \\
&= \frac{8q^4}{s^2}\left[\frac{2s}{4}\left(\frac{s}{4} + \frac{s}{4}\left(1 - \frac{4M^2}{s}\right)\cos^2\theta\right) + 2M^2\frac{s}{4}\right] \\
&= q^4\left[\left(1 + \frac{4M^2}{s}\right) + \left(1 - \frac{4M^2}{s}\right)\cos^2\theta\right]
\end{aligned}
$$

Cross section formula in the CM frame reads

$$
\begin{aligned}
\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)_{CM} &= \frac{1}{64\pi^4}\,\frac{1}{s}\,\frac{|p'|}{|q'|}\,\left|\overline{\mathcal{M}}\right|^2_{CM} \\
&= \frac{q^4}{64\pi^2}\,\frac{1}{s}\left(1 - \frac{4M^2}{s}\right)^{1/2}\left[\left(1 + \frac{4M^2}{s}\right) + \left(1 - \frac{4M^2}{s}\right)\cos^2\theta\right] \\
&= \frac{\alpha^2}{4s}\left(1 - \frac{4M^2}{s}\right)^{1/2}\left[\left(1 + \frac{4M^2}{s}\right) + \left(1 - \frac{4M^2}{s}\right)\cos^2\theta\right] \quad \text{with } \alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}
\end{aligned}
\tag{1.2}
$$

Integrating over $\mathrm{d}\Omega$ we find the total unpolarized cross section

$$
(\bar{\sigma})_{TOT} = \int \mathrm{d}\Omega\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)_{CM} = \frac{4\pi\alpha^2}{3s}\left(1 - \frac{4M^2}{s}\right)^{1/2}\left(1 + \frac{2M^2}{s}\right) + \sigma(\alpha^3) \tag{1.3}
$$

Notes: $\left(1 - (4M^2)/s\right)$ in the equation **??** impose a physical constraint for the scattering process: $s = 4\omega^2 > 4M^2$. Energy of initial particles must be grater than the mass of final particles $\omega > M$ (we changed the case $m \approx 0$)
In equation **??**, *the same term*, shows that for $m \approx M$ the cross section vanish and (see previous formulas) there is almost no dependence on the angle, moreover the term $\sigma(\alpha^3)$ is trabscured since we considering only first order of perturbative expansion.

We used approximation $m \approx 0$ because usually in experiments $\omega \approx 1GeV$ and $m_\mu \approx 200me$ $m_\mu = 105MeV$. For very high experiments, $\omega \approx TeV$, we can omit also f mass, approximating $M \approx 0$. This is called **ultra relativistic regime** (only energies are taken into account)

$$
\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)^{UR}_{CM} = \frac{\alpha^2}{4s}\underbrace{(1 + \cos\theta)}_{\substack{\text{scattering amplitude} \\ \text{is higher at small angles}}} \qquad (\bar{\sigma})^{UR}_{TOT} = \frac{4\pi\alpha^2}{3s} + o\left(\frac{M^2}{s}\right)
$$

Let's summarize how we obtained these results. The method extends in a straightforward way to the calculation of unpolarized cross section for other QED processes at lowest order. The general procedure is as follows:

(1) Draw the diagram(s) for the desired process

(2) Use Feynman rules to write down the amplitude $\mathcal{M}_{fi}$

(3) Square the amplitude and average or sum over spins, using completeness relations (for processing involving photons in the final state there is an analogous completeness relation, we will derive it in Compton scattering)

(4) Evaluate traces using γ-matrices proprieties; collect terms and simplify the answer as much as possible

(5) Specialize to a particular frame of reference, and draw a picture of the kinematic variable in that frame. Express all 4-momentum vectors in terms of suitably chosen set of variables such as E and θ

(6) Plug the resulting expression for $|\overline{\mathcal{M}_{fi}}|^2$ into the cross-section formula and integrate over phase space variables that are not measured to obtain a differential cross section in the desired form

Exercise 1

Calculate cross sections of following precesses ($f \neq e$) (unpolarized)

- $e^- f^- \to e^- f^-$ (t-channel)
- $e^- f^+ \to e^- f^+$ (u-channel)

1.4.2 Polarized Scattering Relations between helicity and chiarality.

Perkin, sec 5.2 - Schwartz, sec 13.3 - Mandl, sec 8.4 **The ultra relativistic limit and helicity amplitudes**

In the previous calculation we obtained the amplitude polarized

$$\mathcal{M}_{fi} = \frac{iq^2}{s}\big(\bar{u}_{r'}(p')\gamma^\mu v_{s'}(q')\big)\big(\bar{v}_s\gamma_\mu u_r(p)\big)$$

Calculation of the polarized cross section allows us to understand better the unpolarized cross section, for example show us where the factor $(1+\cos^2\theta)$ comes from.
We must choose a basis of polarization states. The best choice is to quantize each spin along the direction of particle's motion, that is, to use states of definite helicity.
In general, helicity projectors are hard to be used for lower energies, so we work in the ultra relativistic limit. Recall that in the massless limit, the left- and right-handed helicity states of a Dirac particle live in different representations of the Lorentz group. We might expect them to behave independently, and in fact they do.
We would like to use the spin sum identities to write the squared amplitude in term of traces as before, ever though we now want to consider only one set of polarizations at a time. We note that in the ultra-relativistic limit, helicity is related to chiarality, so we can use chirality projectors, that are much simpler.
In particular, let $\Lambda_\pm$ be energy projectors, $\Pi_\pm$ be helicity projectors, and $P_{L,R}$ be chirality projectors, then following relations holds:

$$\Pi_\pm\Lambda_+ = P_{R,L}\Lambda_+ \qquad \Pi_\pm\Lambda_- = P_{L,R}\Lambda_-$$

where

$$P_R = \frac{1+\gamma_5}{2} \qquad P_L = \frac{1-\gamma_5}{2}$$

And for spinors in UR limit:

$$u_+ = \Pi_+\,u = u_1 = u_R \qquad \to \text{right handed spinor} \qquad h = \frac{1}{2}$$

$$u_- = \Pi_-\,u = u_2 = u_L \qquad \to \text{left handed spinor} \qquad h = -\frac{1}{2}$$

with h = helicity eigenvalue

disegni foglio 35

$$
\begin{aligned}
v_+ &= \Pi_+ \, v = v_2 = v_L && \to \text{left handed spinor} && h = \frac{1}{2} \\
v_- &= \Pi_- \, v = v_1 = v_R && \to \text{right handed spinor} && h = -\frac{1}{2}
\end{aligned}
$$

We notice that for antiparticles the relations between chirality and helicity is inverted. This can be easily interpretated using Dirac's Holes Theory, for example an antiparticle with positive helicity is the hole left by a particle with posotive helicity (since both momenta and spin change sign), and so particles and antiparticles with same helicity must have inverse chirality. disegni foglio 35

Using chirality projectors properties we can write Dirac currents as

$$
\begin{aligned}
\bar{v}\gamma^\mu u &= \bar{v}(P_L + P_R)\gamma^\mu \, (P_L + P_R)u \\
&= \bar{v}P_L \, \gamma^\mu \, P_L \, u + \bar{v}P_R \, \gamma^\mu \, P_R \, u + \bar{v}P_L \, \gamma^\mu \, P_R \, u + \bar{v}P_R \, \gamma^\mu \, P_L \, u
\end{aligned}
$$

Using γ matrices properties

$$
P_{R,L} \, \gamma^\mu = \frac{\gamma^\mu \pm \gamma^5 \, \gamma^\mu}{2} = \frac{\gamma^\mu \pm \overbrace{\{\gamma^5, \gamma^\mu\}}^{=0} \mp \gamma^\mu \, \gamma^5}{2} = \gamma^\mu \, \frac{1 \mp \gamma^5}{2} = \gamma^\mu \, P_{L,R}
$$

we obtain

$$
\begin{aligned}
\bar{v}\gamma^\mu \, u &= \bar{v}\gamma^\mu \, P_R \, P_L \, u + \bar{v}\gamma^\mu \, P_L \, P_R \, u + \bar{v}\gamma^\mu \, P_R \, P_R \, u + \bar{v}\gamma^\mu \, P_L \, P_L \, u \\
&= \bar{v}\gamma^\mu \, P_R \, u + \bar{v}\gamma^\mu \, P_L \, u = \bar{v}\gamma^\mu_R \, u + \bar{v}\gamma^\mu_L \, u = J^\mu_R + J^\mu_L
\end{aligned}
$$

where we are defined $\gamma^\mu_{R,L} = \gamma^\mu \, P_{RL}$ and the left- and right-currents:

$$
J^\mu_L = \bar{v} \, \gamma^\mu_L \, u \qquad J^\mu_R = \bar{v} \, \gamma^\mu_R \, u
$$

We define right- and left-handed spinors as follows:

$$
\begin{aligned}
u_L &= P_L \, u \quad u_R = P_R \, u \\
v_L &= P_L \, v \quad v_R = P_R \, u
\end{aligned}
$$

and them conjugated as (let ψ be a generic spinors)

$$
\begin{aligned}
\psi_L &= P_L \, \psi \quad \to \quad \bar{\psi}_L = \bar{\psi} \, P_R \\
\psi_R &= P_R \, \psi \quad \to \quad \bar{\psi}_R = \bar{\psi} \, P_L
\end{aligned}
$$

With this notation I can rewrite left- and right-handed currents as

$$
\begin{aligned}
\bar{v} \, \gamma^\mu \, u &= \bar{v} \, \gamma^\mu_R \, u + \bar{v} \, \gamma^\mu_L \, u = \bar{v}_R \gamma^\mu \, u_R + \bar{v}_L \gamma^\mu \, u_L \\
\bar{u} \, \gamma^\mu \, v &= \bar{u} \, \gamma^\mu_R \, v + \bar{u} \, \gamma^\mu \, v = \bar{u}_R \, \gamma^\mu \, v_R + \bar{u}_L \, \gamma^\mu \, v_L
\end{aligned}
$$

Here is evident that a deric current can be divided in its left handed and right handed components

Going back to the Feynman amplitude

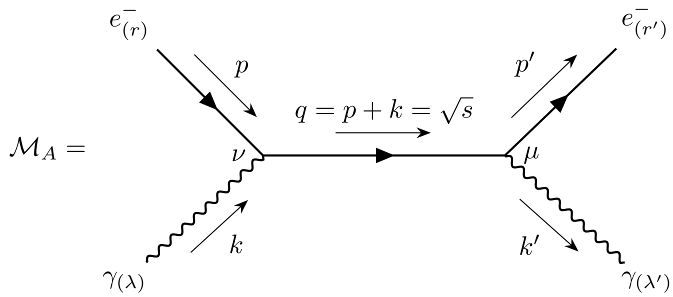
$$
\begin{aligned}
\mathcal{M}_{fi} = \frac{iq^2}{s} &\big(\bar{u}(p') \, \gamma^\mu_L \, v(q')\big)\big(\bar{v}(q) \, \gamma^L_\mu \, u(p)\big) \times \quad \to \mathcal{M}_{LL} \\
&\times \big(\bar{u}(p') \, \gamma^\mu_L \, v(q')\big)\big(\bar{v}(q) \, \gamma^R_\mu \, u(p)\big) \times \quad \to \mathcal{M}_{LR} \\
&\times \big(\bar{u}(p') \, \gamma^\mu_R \, v(q')\big)\big(\bar{v}(q) \, \gamma^L_\mu \, u(p)\big) \times \quad \to \mathcal{M}_{RL} \\
&\times \big(\bar{u}(p') \, \gamma^\mu_R \, v(q')\big)\big(\bar{v}(q) \, \gamma^R_\mu \, u(p)\big) \quad \to \mathcal{M}_{RR}
\end{aligned}
$$

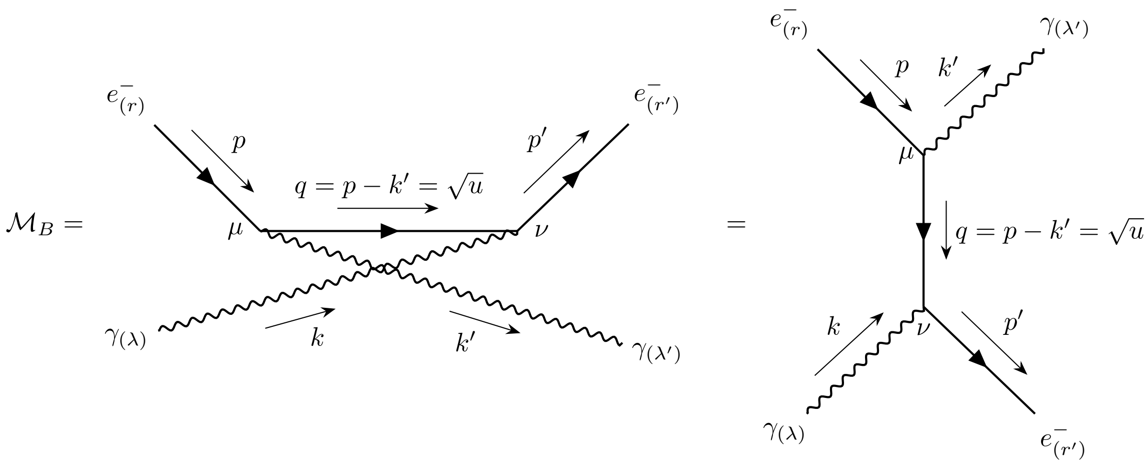
If I did not used the UK limit, I would have obtained 16 independent terms, instead of 4.
Each factor in the latter, corresponds to a different Feynman diagram with different left- and right-handed, initial and final particles, for example FOGLIO 37 diagramma di Feynman

1.5 $e^-\gamma \to e^-\gamma$ (Compton)

See Peskin, sec 5.5

Let's examine a process with external bosons: *Compton scattering*, or $e^-\gamma \to e^-\gamma$. This process is described by two independent diagrams, since they are topologically different:





We wrote the diagram of $\mathcal{M}_B$ in two topologically equivalent forms: in the first one is clear the topological relation with diagram of $\mathcal{M}_A$ (this is useful to find the relative sign between diagrams A and B: it's clear that diagrams differs for the permutation of two bosons), while in the second one is clear that it describes a u-channel.

Amplitudes reads, using Feynman rules

$$\begin{aligned}
\mathcal{M}_A &= \bar{u}_{r'}(p')(-iq\gamma^\mu)\varepsilon_\mu^{\lambda' *}(k')\tilde{S}_F(p+k)(-iq\gamma^\nu)\varepsilon_\nu^\lambda(k)u_r(p) \\
&= -q^2\varepsilon_\mu^{\lambda' *}(k')\varepsilon_\nu^\lambda(k)\left[\bar{u}_{r'}(p')\gamma^\mu\tilde{S}_F(p+k)\gamma^\nu u_r(p)\right]
\end{aligned}$$

$$\mathcal{M}_B = -q^2\varepsilon_\mu^{\lambda' *}(k')\varepsilon_\nu^\lambda(k)\left[\bar{u}_{r'}(p')\gamma^\nu\tilde{S}_F(p-k')\gamma^\mu u_r(p)\right]$$

(Recall that $\tilde{S}_F$ is a matrix, so elements in the squared bracket must be written in this order)

Because of anticommuting relations for bosons, these amplitudes must be summed up in the total amplitude. The explicit form of Feynman propagator for the Dirac field reads

$$\tilde{S}_F(p) = \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon} = \frac{i}{\not{p}-m+i\varepsilon}$$

so total amplitude is

$$\mathcal{M} = -iq^2\varepsilon_\mu^{\lambda' *}(k')\varepsilon_\nu^\lambda(k)\bar{u}_{r'}(p')\left[\frac{\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu}{(p+k)^2-m^2} + \frac{\gamma^\nu(\not{p}-\not{k}'+m)\gamma^\mu}{(p-k')^2-m^2}\right]u_r(p)$$

We make some simplifications before squaring this expression. Since $p^2 = m^2$ and $k^2 = 0$:

$$(p+k)^2 - m^2 = 2p \cdot k \qquad (p-k')^2 - m^2 = -2p \cdot k'$$

To simplify numerators, I can use Dirac algebra:

$$\begin{aligned}(\not{p} + m)\gamma^\nu u(p) &= (p_\mu \gamma^\mu \gamma^\nu + m\gamma^\nu)u(p) = (2g^{\mu\nu}p_\mu - p_\mu \gamma^\nu \gamma^\mu + m\gamma^\nu)u(p) \\ &= 2p^\nu u(p) - \gamma^\nu \underbrace{(\not{p} - m)}_{2m\,\Lambda_-(p)} u(p) = 2p^\nu u(p)\end{aligned}$$

Using these tricks we obtain

$$\mathcal{M} = -iq^2 \varepsilon_\mu^{\lambda' *}(k')\varepsilon_\nu^\lambda(k)\bar{u}_{r'}(p') \left[\frac{\gamma^\mu \not{k} \gamma^\nu + 2\gamma^\mu p^\nu}{2p \cdot k} + \frac{-\gamma^\nu \not{k}' \gamma^\mu + 2\gamma^\nu p^\mu}{-2p \cdot k'} \right] u_r(p)$$

1.5.1 The Ward Identities and sum over the photon polarizations

See Mandl, sec 8.3

The next step in the calculation will be to square this expression for $\mathcal{M}$ and sum or average over electron and photon polarization states. The sum over electron polarizations can be performed as before, using $\sum u(p)\bar{u}(p) = \not{p} + m$. Fortunately, there is a similar trick for summing over photons polarization vectors. Gauge invariance of the theory implies the gauge invariance of the matrix elements, i.e. of the Feynman amplitudes. It is, of course, only the matrix element itself, corresponding to the sum of all possible Feynman graphs in a given order of perturbation theory, which must be gauge invariant. For example, for the Compton scattering, the individual amplitudes $\mathcal{A}$ and $\mathcal{B}$ are not gauge invariants, but their sum $\mathcal{M}$ is.

For any process involving external photons, the Feynman amplitude $\mathcal{M}$ is of the form

$$\mathcal{M} = \varepsilon_\alpha^{\lambda_1}(k_1)\varepsilon_\beta^{\lambda_2}(k_2)\dots L^{\alpha\beta\cdots}(k_1, k_2, \dots) \tag{1.4}$$

with one polarization vector $\varepsilon^{\lambda_i}(k_i)$ for each external photon, and the tensor amplitude $L^{\alpha\beta\cdots}(k_1, k_2, \dots)$ independent of these polarization vectors.

The polarization vectors are of course gauge dependent. For example, for a free photon described in the Lorentz gauge by the plane wave

$$A^\mu(x) = \text{const} \cdot \varepsilon_\lambda^\mu(k)e^{\pm ikx}$$

the gauge transformation

$$A^\mu \to A'^\mu(x) = A^\mu(x) + \partial^\mu \alpha(x) \qquad \text{with} \quad \alpha(x) = \tilde{\alpha}(k)e^{\pm ikx}$$

implies

$$\varepsilon_\lambda^\mu(k) \to \varepsilon'^\mu_\lambda(k) = \varepsilon_\lambda^\mu(k) \pm ik^\mu \tilde{\alpha}(k)$$

Invariance of the amplitude Eq.(**??**) under this transformation requires

$$k_1^\alpha L_{\alpha,\beta,\dots}(k_1, k_2, \dots) = k_1^\beta L_{\alpha,\beta,\dots}(k_1, k_2, \dots) = \cdots = 0$$

i.e. when any external photon polarization vector is replaced by the corresponding four momentum, the amplitude must vanish. This is the statement of the *Ward Identity*: Sistemare grafica

> If $\mathcal{M}(k) = \varepsilon_\mu(k)L^\mu(k)$ is the amplitude for some QED process involving an external photon with momentum k, then this amplitude vanishes if we replace ε_μ with k_μ:
>
> $$k_\mu L^\mu(k) = 0$$

Exercise 2

Verify explicitly the Ward Identity for the Feynman amplitude of Compton scattering

See Peskin, sec 5.5

Returning to our derivation of the polarization sum formula for squared scattering amplitude. Writing in general

$$\mathcal{M} = \varepsilon_\mu^{(\lambda)}(k) L^\mu(k)$$

then the sum over polarizations of the photon with momentum k reads

$$\sum_{\lambda=1,2} |\mathcal{M}|^2 = \sum_{\lambda=1,2} \varepsilon_\mu^{(\lambda)}(k) \varepsilon_\nu^{(\lambda)*}(k) L^\mu(k) L^{\nu\dagger}(k)$$

Because of the covariance of the theory we can do the calculation in a specific frame. In order to simplify the analysis we choose the frame where the photon moves along the $\hat{z}$ axis:

$$k^\mu = (|k|, 0, 0, |k|)$$

In this case the Ward Identity reads

$$0 = k_\mu L^\mu = |k| \left(L^0 - L^3\right) \qquad \longrightarrow \qquad L^0 = L^3$$

Recall that in this frame

$$\varepsilon_\mu^{(1)}(k) = (0,1,0,0) \qquad \varepsilon_\mu^{(2)}(k) = (0,0,1,0)$$

So we have

$$\sum_{\lambda=1,2} \varepsilon_\mu^{(\lambda)}(k) \varepsilon_\nu^{(\lambda)*}(k) L^\mu(k) L^{\nu\dagger}(k) = \left|L^1\right|^2 + \left|L^2\right|^2 = \left|L^1\right|^2 + \left|L^2\right|^2 + \left|L^3\right|^2 - \left|L^0\right|^2 = -g_{\mu\nu} L^\mu L^\nu$$

So we obtained the general rule to simplify photons polarization sum[VI]

Creare ambiente per riquadrare le formule

$$\boxed{\sum_{\lambda=1,2} \varepsilon_\mu^{(\lambda)}(k) \varepsilon_\nu^{(\lambda)*}(k) L^\mu(k) L^{\nu\dagger}(k) \quad \longrightarrow \quad -g_{\mu\nu}}$$

1.5.2 The Klein-Nishima formula and the Thomson scattering

See Peskin, sec. 5.5

To obtain the unpolarized cross section for Compton scattering, we use the covariant method described in the previous section. Writing

$$\mathcal{M} = \varepsilon_\mu^{\lambda' *}(k') \varepsilon_\nu^{\lambda}(k) \left(L^{\mu\nu}(k,k')\right)_{r,r'}$$

with

$$\left(L^{\mu\nu}(k,k')\right)_{r,r'} = -iq^2 \bar{u}_{r'}(p') \left[\frac{\gamma^\mu \not{k} \gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{-\gamma^\nu \not{k}' \gamma^\mu + 2\gamma^\nu p^\mu}{-2p\cdot k'}\right] u_r(p)$$

we obtain

$$\begin{aligned}
|\bar{\mathcal{M}}|^2 &= \frac{1}{4}\left(\sum_{\lambda'} \varepsilon_\mu^{(\lambda')*}(k') \varepsilon_\rho^{(\lambda')}(k')\right)\left(\sum_{\lambda} \varepsilon_\nu^{(\lambda)*}(k) \varepsilon_\sigma^{(\lambda)}(k')\right) \sum_{r,r'} \left(L^{\mu\nu}\right)_{r,r'} \left(L^{\rho\sigma}\right)^\dagger_{r,r'} \\
&= \frac{1}{4} g_{\mu\rho} g_{\nu\sigma} \sum_{r,r'} \left(L^{\mu\nu}\right)_{r,r'} \left(L^{\rho\sigma}\right)^\dagger_{r,r'} = \frac{1}{4} \left(L^{\mu\nu}\right)_{r,r'} \left(L_{\mu\nu}\right)^\dagger_{r,r'} \\
&= \frac{q^4}{4} \mathrm{Tr}\Bigg[(\not{p}' + m)\left(\frac{\gamma^\mu \not{k} \gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{\gamma^\nu \not{k}' \gamma^\mu - 2\gamma^\nu p^\mu}{2p\cdot k'}\right) \times \\
&\qquad \times (\not{p} + m)\left(\frac{\gamma_\nu \not{k} \gamma_\mu + 2\gamma_\mu p_\nu}{2p\cdot k} + \frac{\gamma_\mu \not{k}' \gamma_\nu - 2\gamma_\nu p_\mu}{2p\cdot k'}\right)\Bigg] \\
&= \frac{q^4}{4}\left\{\frac{T_{AA}}{(2p\cdot k)^2} + \frac{T_{BB}}{(2p\cdot k')^2} + \frac{T_{AB} + T_{BA}}{(2p\cdot k)(2p\cdot k')}\right\}
\end{aligned}$$

[VI] Notice that we could prove (see Peskin) that even if we took $\lambda = 0,1,2,3$, we could have obtained that the unphysical time-like and longitudinal photons can be consistently omitted from QED calculations, since in any event the squared amplitudes for producing these states cancel to give zero total probability.

where

$$
\begin{aligned}
T_{AA} &= \mathrm{Tr}\left[(\not{p}' + m)(\gamma^\mu \not{k} \gamma^\nu + 2\gamma^\mu p^\nu)(\not{p} + m)(\gamma_\nu \not{k} \gamma_\mu + 2\gamma_\mu p_\nu)\right] \\
T_{BB} &= \mathrm{Tr}\left[(\not{p}' + m)(\gamma^\nu \not{k}' \gamma^\mu - 2\gamma^\nu p^\mu)(\not{p} + m)(\gamma_\mu \not{k}' \gamma_\nu - 2\gamma_\nu p_\mu)\right] \\
T_{AB} &= \mathrm{Tr}\left[(\not{p}' + m)(\gamma^\mu \not{k} \gamma^\nu + 2\gamma^\mu p^\nu)(\not{p} + m)(\gamma_\mu \not{k}' \gamma_\nu - 2\gamma_\nu p_\mu)\right] \\
T_{BA} &= \mathrm{Tr}\left[(\not{p}' + m)(\gamma^\nu \not{k}' \gamma^\mu - 2\gamma^\nu p^\mu)(\not{p} + m)(\gamma_\nu \not{k} \gamma_\mu + 2\gamma_\mu p_\nu)\right]
\end{aligned}
$$

Notice that $T_{BB} = T_{AA}(k \leftrightarrow -k')$ and $T_{BA} = T_{AB}(k \leftrightarrow -k')$, we need therefore only calculate T_{AA} and T_{AB}.
Considering T_{AA}, there are 16 terms inside the trace, but half contains an odd number of γ matrices and therefore vanishes. Other terms are

$$
\begin{aligned}
(1) &= \mathrm{Tr}[\not{p}' \gamma^\mu \not{k} \gamma^\nu \not{p} \gamma_\nu \not{k} \gamma_\mu] \\
(2) &= 2\mathrm{Tr}[\not{p}' \gamma^\mu \not{k} \gamma^\nu \not{p} \gamma_\mu p_\nu] = 2\mathrm{Tr}[\not{p}' \gamma^\mu \not{k} \not{p} \not{p} \gamma_\mu] \\
(3) &= 2\mathrm{Tr}[\not{p}' \gamma^\mu p^\nu \not{p} \gamma_\nu \not{k} \gamma_\mu] = 2\mathrm{Tr}[\not{p}' \gamma^\mu \not{p} \not{p} \not{k} \gamma_\mu] \\
(4) &= 4\mathrm{Tr}[\not{p}' \gamma^\mu p^\nu \not{p} \gamma_\mu p_\nu] = 4p^2 \mathrm{Tr}[\not{p}' \gamma^\mu \not{p} \gamma_\mu] \\
(5) &= m^2 \mathrm{Tr}[\gamma^\mu \not{k} \gamma^\nu \gamma_\nu \not{k} \gamma_\mu] \\
(6) &= 2m^2 \mathrm{Tr}[\gamma^\mu \not{k} \gamma^\nu \gamma_\mu p_\nu] = 2m^2 \mathrm{Tr}[\gamma^\mu \not{k} \not{p} \gamma_\mu] \\
(7) &= 2m^2 \mathrm{Tr}[\gamma^\mu p^\nu \gamma_\nu \not{k} \gamma_\mu] = 2m^2 \mathrm{Tr}[\gamma^\mu \not{p} \not{k} \gamma_\mu] \\
(8) &= 4m^2 \mathrm{Tr}[\gamma^\mu p^\nu \gamma_\mu p_\nu] = 4m^2 p^2 \mathrm{Tr}[\gamma^\mu \gamma_\mu]
\end{aligned}
$$

In order to simplify above formulas we recall the proprieties of contractions of γ matrices, i.e. products in the form $\gamma^\mu A \gamma^\mu$ where A is a matrix:

(i) $\gamma^\mu \gamma_\mu = 4\mathbb{1}$

(ii) $\gamma^\mu \not{p} \gamma_\mu = -2\not{p}$

(iii) $\gamma^\mu \not{p} \not{q} \gamma_\mu = 4p \cdot q$

(iv) $\gamma^\mu \not{p} \not{q} \not{k} \gamma_\mu = -2\not{k} \not{q} \not{p}$

Using these proprieties, cyclicity of the trace and anticommuting proprieties of gamma matrices[VII], we obtain (remember that $p^2 = m^2$ and $k^2 = 0$):

$$
\begin{aligned}
(1) &= \mathrm{Tr}[\not{p}' \gamma^\mu \not{k} \gamma^\nu \not{p} \gamma_\nu \not{k} \gamma_\mu] = -2\mathrm{Tr}[\not{p}' \gamma^\mu \not{k} \not{p} \not{k} \gamma_\mu] = 4\mathrm{Tr}[\not{p}' \not{k} \not{p} \not{k}] = -4\mathrm{Tr}[\not{p}' \not{k}^2 \not{p}] + 8(p \cdot k)\mathrm{Tr}[\not{p}' \not{k}] = 32(p \cdot k)(p' \cdot k) \\
(2) &= 2\mathrm{Tr}[\not{p}' \gamma^\mu \not{k} \not{p} \not{p} \gamma_\mu] = -4\mathrm{Tr}[\not{p}' \not{p} \not{p} \not{k}] = -4m^2 \mathrm{Tr}[\not{p}' \not{k}] = -16m^2 (p' \cdot k) \\
(3) &= 2\mathrm{Tr}[\not{p}' \gamma^\mu \not{p} \not{p} \not{k} \gamma_\mu] = 2m^2 \mathrm{Tr}[\not{p}' \gamma^\mu \not{k} \gamma_\mu] = -4m^2 \mathrm{Tr}[\not{p}' \not{k}] = -16m^2 (p' \cdot k) \\
(4) &= 4p^2 \mathrm{Tr}[\not{p}' \gamma^\mu \not{p} \gamma_\mu] = -8m^2 \mathrm{Tr}[\not{p}' \not{p}] = -32m^2 (p' \cdot p) \\
(5) &= m^2 \mathrm{Tr}[\gamma^\mu \not{k} \gamma^\nu \gamma_\nu \not{k} \gamma_\mu] = 4m^2 \mathrm{Tr}[\gamma^\mu \not{k} \not{k} \gamma_\mu] = 0 \\
(6) &= 2m^2 \mathrm{Tr}[\gamma^\mu \not{k} \not{p} \gamma_\mu] = 8m^2 (k \cdot p)\mathrm{Tr}[\mathbb{1}] = 32m^2 (k \cdot p) \\
(7) &= 2m^2 \mathrm{Tr}[\gamma^\mu \not{p} \not{k} \gamma_\mu] = 8m^2 (p \cdot k)\mathrm{Tr}[\mathbb{1}] = 32m^2 (p \cdot k) \\
(8) &= 4m^2 p^2 \mathrm{Tr}[\gamma^\mu \gamma_\mu] = 16m^4 \mathrm{Tr}[\mathbb{1}] = 64m^4
\end{aligned}
$$

[VII] $\not{A}\not{B} = A_\mu B_\nu \gamma^\mu \gamma^\nu = A_\mu B_\nu (2g^{\mu\nu}\mathbb{1} - \gamma^\nu \gamma^\mu) = 2(A \cdot B)\,\mathbb{1} - \not{B}\not{A} \quad \rightarrow \quad \not{A}\not{A} = A^2\,\mathbb{1}$
$\mathrm{Tr}[\not{A}\not{B}] = 2(A \cdot B)\,\mathrm{Tr}[\mathbb{1}] - \mathrm{Tr}[\not{B}\not{A}] = 8(A \cdot B) - \mathrm{Tr}[\not{A}\not{B}] \quad \rightarrow \quad \mathrm{Tr}[\not{A}\not{B}] = 4(A \cdot B)\mathbb{1}$

At the end we find

$$\begin{aligned}T_{AA} &= 16\left(4m^4 - 2m^2 p\cdot p' + 4m^2 p\cdot k - 2m^2 p'\cdot k + 2(p\cdot k)(p'\cdot k)\right)\\ &= 16\left(2m^4 + m^2(s-m^2) - \frac{1}{2}(s-m^2)(u-m^2)\right)\end{aligned}$$

where we introduced Mandelstam variables:

$$\begin{aligned}s &= (p+k)^2 = 2p\cdot k + m^2 = 2p'\cdot k' + m^2\\ t &= (p'-p)^2 = -2p\cdot p' + 2m^2 = -2k\cdot k'\\ u &= (k'-p)^2 = -2k'\cdot p + m^2 = -2k\cdot p' + m^2\end{aligned}$$

Sending $k \leftrightarrow -k'$ $(s \leftrightarrow u)$ we can immediately write

$$\begin{aligned}T_{BB} &= 16\left(4m^4 - 2m^2 p\cdot p' - 4m^2 p\cdot k' + 2m^2 p'\cdot k' + 2(p\cdot k')(p'\cdot k')\right)\\ &= 16\left(2m^4 + m^2(u-m^2) - \frac{1}{2}(u-m^2)(s-m^2)\right)\end{aligned}$$

Exercise 3

Compute the elements T_{AB} and T_{BA}

Evaluating the traces in T_{AB} and T_{BA} requires about the same amount of work as we have just done. The answer is

$$\begin{aligned}T_{AB} = T_{BA} &= -16\left(4m^4 + m^2(p\cdot k - p\cdot k')\right)\\ &= -16\left(2m^4 + \frac{m^2}{2}((s-m^2) - (u-m^2))\right)\end{aligned}$$

Putting together the pieces of the unpolarized Feyanmn amplitude for Compton scattering we obtain

$$\begin{aligned}|\bar{\mathcal{M}}|^2 &= 2q^4\left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right]\\ &= 2q^4\left[-\left(\frac{u-m^2}{s-m^2} + \frac{s-m^2}{u-m^2}\right) + 4m^2\left(\frac{1}{s-m^2} + \frac{1}{u-m^2}\right) + 4m^4\left(\frac{1}{s-m^2} + \frac{1}{u-m^2}\right)^2\right]\end{aligned} \tag{1.5}$$

To turn this expression into a cross section we must decide a frame of reference and draw a piture of the kinematics. We will analyze two different frames

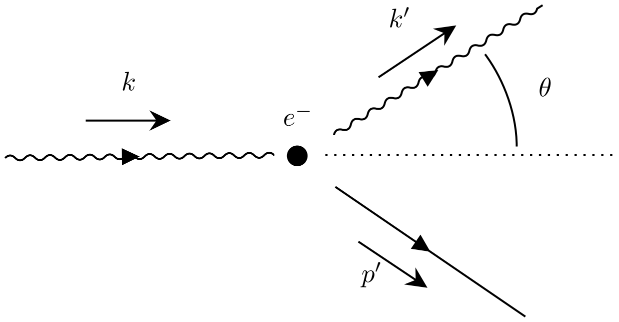
(i) *"Lab" frame*, in which the electron is initially at rest, this frame is useful for low energy incoming photons: $\omega_\gamma \ll m_e$;

(ii) *c.o.m. frame*, in which the center of mass is at rest, this frame is useful for high energy incoming photons: $\omega_\gamma \gg m_e$, where we can set $m_e = 0$

1.5.3 Lab frame - Low energy photon

See also Mandl sec. 8.6

In the low energy case, I can verify if QED prediction agrees with Thomason law for low energies scattering.



$$
\begin{aligned}
p &= (m_e, 0)\\
k &= (\omega_k, \mathbf{k})\\
p' &= (E', \mathbf{p}')\\
k' &= (\omega_{k'}, \mathbf{k}')
\end{aligned}
$$

We will express the cross section in terms of ω and θ. We can find ω', the energy of the final photon, using the following trick:

$$
\begin{aligned}
m^2 = (p')^2 &= (p+k-k')^2 = p^2 + 2p\cdot(k-k') - 2k\cdot k'\\
&= m^2 + 2m(\omega_k - \omega_{k'}) - 2\omega_k\omega_{k'}(1-\cos\theta)
\end{aligned}
$$

hence, we obtain Compton's formula for the shift in the photon wavelength:

$$
\Delta\lambda = \left(\frac{1}{\omega_{k'}} - \frac{1}{\omega_k}\right) = \frac{1-\cos\theta}{m}
$$

For our purposes, however, is more useful to solve for $\omega_{k'}$:

$$
\omega_{k'} = \frac{\omega_k}{1 + \frac{\omega_k}{m}(1-\cos\theta)} \tag{1.6}
$$

The unpolarized amplitude in the Lab frame is

$$
\begin{aligned}
|\bar{\mathcal{M}}|^2_{\text{LAB}} &= 2q^4\left[\left(\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}}\right) + 2m\left(\frac{1}{\omega_k} - \frac{1}{\omega_{k'}}\right) + m^2\left(\frac{1}{\omega_k} - \frac{1}{\omega_{k'}}\right)^2\right]\\
&= 2q^4\left[\left(\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}}\right) - \sin^2\theta\right]
\end{aligned}
$$

The covariant flux factor reads

$$
I_{\text{LAB}} = [(p\cdot k)^2 - m_e^2 m_\gamma^2]^{1/2} = |p\cdot k| = m_e\omega_k
$$

The 2-body phase space

$$
\begin{aligned}
\int \mathrm{d}\Phi_{(2)} &= \int \frac{\mathrm{d}^3k'}{(2\pi)^3 2\omega_{k'}}\frac{\mathrm{d}^3p'}{(2\pi)^3 2E'}(2\pi)^4\delta^4(k'+p'-k-p) = \int \frac{\omega_{k'}^2\mathrm{d}\omega_{k'}\mathrm{d}\Omega}{(2\pi)^2}\frac{1}{4\omega_{k'}E'}\delta(\omega_{k'}+E'-\omega_k-m)\\
&= \int \frac{\omega_{k'}^2\mathrm{d}\omega_{k'}\mathrm{d}\Omega}{(2\pi)^2}\frac{1}{4\omega_{k'}E'}\frac{\delta(\omega_{k'}-|\mathbf{k}'|)}{\left|\frac{\partial(\omega_{k'}+E'-\omega_k-m)}{\partial|k'|}\right|_{\omega_{k'}=|\mathbf{k}'|}} = \int \mathrm{d}\Omega\frac{|\mathbf{k}'|^2}{16\pi^2\omega_{k'}E'}\left|\frac{\partial(\omega_{k'}+E')}{\partial|k'|}\right|^{-1}_{\omega_{k'}=|\mathbf{k}'|}
\end{aligned}
$$

where

$$
E' = \left(m^2 + (\mathbf{k}-\mathbf{k}')^2\right)^{1/2} = \left[m^2 + \omega_k^2 + \omega_{k'}^2 - 2\omega_{k'}\omega_k\cos\theta\right]^{1/2}
$$

$$
\frac{\partial E'}{\partial|k'|} = \frac{\omega_{k'} - \omega_k\cos\theta}{E'}
$$

and

Controlla l'ultimo passaggio

$$
\left|\frac{\partial(\omega_{k'}+E')}{\partial|k'|}\right|_{\omega_{k'}=|\mathbf{k}'|} = \left|1 + \frac{\omega_{k'} - \omega_k\cos\theta}{E'}\right| = \frac{m\omega_k}{E'\omega_{k'}}
$$

So the unpolarized cross section is

$$
\begin{aligned}
\left(\frac{\mathrm{d}\bar\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{LAB}} &= \frac{\overline{|\mathcal{M}|}^2_{\mathrm{LAB}}}{4I_{\mathrm{LAB}}}\frac{\mathrm{d}\Phi_{(2)}}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\frac{|\mathbf{k}'|^2}{I_{\mathrm{LAB}}\omega_{k'}E'}\left|\frac{\partial(\omega_{k'}+E')}{\partial|k'|}\right|^{-1}\overline{|\mathcal{M}|}^2_{\mathrm{LAB}} \\
&= \frac{q^4}{32\pi^2}\frac{1}{m^2}\left(\frac{\omega_{k'}}{\omega_k}\right)^2\left(\frac{\omega_{k'}}{\omega_k}+\frac{\omega_k}{\omega_{k'}}-\sin^2\theta\right) \\
&= \frac{\alpha^2}{2}\frac{1}{m^2}\left(\frac{\omega_{k'}}{\omega_k}\right)^2\left(\frac{\omega_{k'}}{\omega_k}+\frac{\omega_k}{\omega_{k'}}-\sin^2\theta\right)
\end{aligned}
$$

where $\omega_{k'}/\omega_k$ is given by (**??**) and in the last step we used $\alpha = e^2/(4\pi)$. Writing $\mathrm{d}\Omega = (2\pi)\mathrm{d}\cos\theta$ we obtain

$$
\left(\frac{\mathrm{d}\bar\sigma}{\mathrm{d}\cos\theta}\right)_{\mathrm{LAB}} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega_{k'}}{\omega_k}\right)^2\left(\frac{\omega_{k'}}{\omega_k}+\frac{\omega_k}{\omega_{k'}}-\sin^2\theta\right) \tag{1.7}
$$

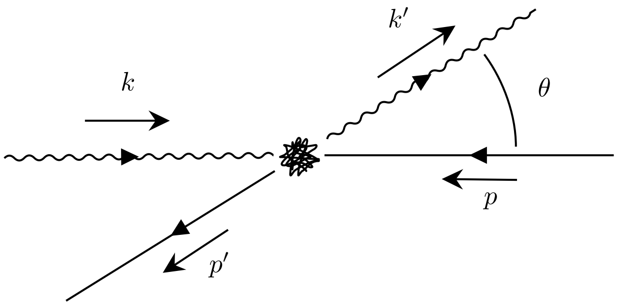
This is the (spin-averaged) *Klein-Nishina formula*. In the low energy limit $\omega_k \ll m$, from (**??**) we have $\omega_{k'} \approx \omega_k$, i.e. the kinetic energy of the recoil electron is negligible, and Eq.(**??**) reduces to the familiar Thomson cross-section for scattering of classical electromagnetic radiation by a free electron:

$$
\left(\frac{\mathrm{d}\bar\sigma}{\mathrm{d}\cos\theta}\right)_{\mathrm{LAB}} \overset{\omega_k \ll m}{=} \frac{\pi\alpha^2}{m^2}\left(1+\cos^2\theta\right) \quad \to \quad (\bar\sigma)_{\mathrm{LAB}} = \frac{8\pi\alpha^2}{3m^2} \equiv \frac{8}{3}\pi r_e^2
$$

We have calculated the full relativistic corrections for the Thomson formula.

1.5.4 C.o.M. frame - High energy photon

See Peskin sec. 5.5, and Schwartz sec. 13.5.4 To analyze the high-energy behaviour of the Compton scattering cross section, it is easiest to work in the center-of-mass frame.



$p = (E, -\mathbf{k})$
$k = (\omega_k, \mathbf{k})$
$p' = (E', -\mathbf{k}')$
$k' = (\omega'_k, \mathbf{k}')$

The kinematics of the reaction in the high energy limit ($m \approx 0$) looks like

$$
\begin{aligned}
E &= \sqrt{\mathbf{k}^2+m^2} \overset{m=0}{\approx} |\mathbf{k}| = \omega_k \\
E' &= \sqrt{\mathbf{k}'^2+m^2} \overset{m=0}{\approx} |\mathbf{k}'| = \omega_{k'}
\end{aligned}
$$

$$
\begin{aligned}
p\cdot k &= E\omega_k + |\mathbf{k}|^2 = \omega_k(E+\omega_k) \overset{m=0}{\approx} 2\omega_k^2 \\
p\cdot p' &= E\,E' - \mathbf{k}\mathbf{k}' = E\,E' - |\mathbf{k}||\mathbf{k}'|\cos\theta = E\,E' - \omega_k\omega_{k'}\cos\theta \overset{m=0}{\approx} \omega_k\omega_{k'}(1-\cos\theta) \\
p\cdot k' &= E\,\omega_{k'} + \mathbf{k}\mathbf{k}' = E\,\omega_{k'} + |\mathbf{k}||\mathbf{k}'|\cos\theta = \omega_{k'}(E+\omega_k\cos\theta) \overset{m=0}{\approx} \omega_k\omega_{k'}(1+\cos\theta)
\end{aligned}
$$

We also have

$$
\begin{aligned}
s &= (p+k)^2 = m^2 + 2p\cdot k = m^2 + 2\omega_k(E+\omega_k) \overset{m=0}{\approx} 4\omega_k^2 \quad \to \quad \omega_k \overset{m=0}{\approx} \frac{\sqrt{s}}{2} \\
s &= (p'+k')^2 = m^2 + 2p'\cdot k' = m^2 + 2\omega_{k'}(E'+\omega'_k) \overset{m=0}{\approx} 4\omega_{k'}^2 \quad \to \quad \omega_{k'} \overset{m=0}{\approx} \frac{\sqrt{s}}{2}
\end{aligned}
$$

Plugging these values into Eq.(**??**)

$$\overline{|\mathcal{M}|}^2 = 2q^4 \left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2 \left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'} \right) + m^4 \left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'} \right)^2 \right]$$

for c.o.m. frame with $E \gg m$ we have

$$\overline{|\mathcal{M}|}^2_{\mathrm{CM}} \approx 2q^4 \left(\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} \right) \approx 2q^4 \left(\frac{1+\cos\theta}{2} + \frac{2}{1+\cos\theta} \right)$$

we notice that the term $p\cdot k/p\cdot k'$ becomes divergent when the electron is emitted in the backward direction ($\theta \approx \pi$), while other terms are all of $\mathcal{O}(1)$ or smaller.

Notice that two initial diagrams $\mathcal{M}_A$, s-channel, and $\mathcal{M}_B$, u-channel, give contributions to the total amplitude proportional to[VIII]

$$\mathcal{M}_A \quad \rightarrow \quad \frac{1}{2p\cdot k} = \frac{1}{s-m^2} \qquad\qquad \mathcal{M}_B \quad \rightarrow \quad \frac{1}{2p\cdot k'} = \frac{1}{u-m^2}$$

Here is clear the relation between the momentum of the channel and the contribution to the total Feynman amplitude. The divergent contribution is due to the square of the u-channel diagram, we can see that for $\theta = \pi$ we have $u = (p-k')^2 = m^2 - 2p\cdot k' \approx m^2 - 2\omega_k^2(1+\cos\theta) \approx m^2$ i.e. the divergent contribution is related to the situation where the initial electron emits a photon with all its kinetic energy and then absorbs all the energy of the initial photon. The amplitude is large at $\theta \approx \pi$ because the denominator of the propagator is then small ($\sim m^2$) compared to s. This kind of divergence is called *Infra-Red divergence*.

We can correct the divergent term (unphysical) considering higher terms in the Taylor expantion of E in m:

This is a Infra-Red divergence, related to Sudakov logs. Add some comment. See Mandl 8.9

$$E = \sqrt{\mathbf{k}^2 + m^2} = |\mathbf{k}| + \frac{m^2}{2|\mathbf{k}|} + o(m^3) \overset{m\approx 0}{\approx} \omega_k + \frac{m^2}{2\omega_k}$$

$$p\cdot k' = \omega_{k'}(E + \omega_k\cos\theta) \overset{m=0}{\approx} \omega_{k'}\left(\omega_k + \frac{m^2}{2\omega_k} + \omega_k\cos\theta\right) = \omega_k^2\left(1+\cos\theta+\frac{m^2}{2\omega_k^2}\right)$$

$$\overline{|\mathcal{M}|}^2_{\mathrm{CM}} \approx 2q^4 \left(\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} \right) \approx 2q^4 \left(\frac{1+\cos\theta}{2} + \frac{2}{1+\cos\theta+\frac{m^2}{2\omega_k^2}} \right)$$

Now we can compute the cross section in the CM frame (we can use the formula for elastic scattering):

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} = \frac{1}{64\pi^2}\frac{\overline{|\mathcal{M}|}^2_{\mathrm{CM}}}{s} \approx \frac{q^4}{32\pi^2 s}\left(\frac{1+\cos\theta}{2} + \frac{2}{1+\cos\theta+\frac{m^2}{2\omega_k^2}} \right)$$
$$\approx \frac{\alpha^2}{2s}\left(\frac{1+\cos\theta}{2} + \frac{2}{1+\cos\theta+\frac{m^2}{2\omega_k^2}} \right)$$

or

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}(\cos\theta)}\right)_{\mathrm{CM}} \approx \frac{\pi\alpha^2}{s}\left(\frac{1+\cos\theta}{2} + \frac{2}{1+\cos\theta+\frac{m^2}{2\omega_k^2}} \right)$$

Recall that the electron mass m can be neglected completely in this formula if it were not necessary to cutoff the singularity for $\theta = 0$.
The total Compton scattering cross section reads:

$$\bar{\sigma}_{\mathrm{total}} = \int_{-1}^{1} \mathrm{d}(\cos\theta)\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}(\cos\theta)}\right)_{\mathrm{CM}} \approx \frac{\pi\alpha^2}{s}\int_{-1}^{1}\mathrm{d}(\cos\theta)\left(\frac{1+\cos\theta}{2} + \frac{2}{1+\cos\theta+\frac{m^2}{2\omega_k^2}} \right)$$
$$= \frac{\pi\alpha^2}{s} + \frac{2\pi\alpha^2}{s}\log\left(\frac{s}{m^2}\right)$$

[VIII] You can easily verify this statement looking at the calculation on the beginning of this section.

The main dependence α^2/s follows from dimensional analysis. But the singularity associated to backward scattering of photons leads to an enhancement by an extra logarithm of the energy, called *Sudakov logarithm*.

Inserire diagrammi di Feynman pag 168 Peskin (ruotati appropriatamente)

Exercise 4: Pair Annihilation into Photons

Find the total cross section of the annihilation process $e^+e^- \to 2\gamma$

1.6 Scattering by an external E.M. field and the Rutherford formula

1.6.1 $e^-p \to e^-p$ - Rutherford scattering

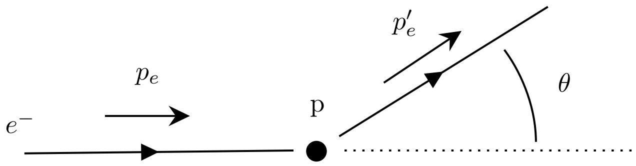
See Schwartz sec. 13.4

Now let us go back to the problem of scattering of an electron by a Coulomb potential. Recall the classical Rutherford scattering formula,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m_e^2 e^4}{4|\mathbf{p}_i|^4 \sin^4 \frac{\theta}{2}}$$

where $|\mathbf{p}_i| = |\mathbf{p}_f|$ is the magnitude of the incoming electron momentum, which. is the same as the magnitude of the outgoing electron momentum for elastic scattering. Rutherford calculated this using classical mechanics to describe how an electron would get deflected in a central potential, as from atomic nucleus.

We study the process in the CM frame for the proton.



$$\begin{aligned} p_e &= (\omega_e, \mathbf{p_e}) \\ p_p &= (M_p, 0) \\ p'_e &= (\omega'_e, \mathbf{p_e}') \\ p'_p &= \left(\omega'_p, \mathbf{p_p}'\right) \end{aligned}$$

We neglect the recoil of the proton because of its huge mass ($\mathbf{p_p}' \approx 0$) (we are not considering high energy case).

$$(\omega'_p, \mathbf{p_p}') = \left(M + \frac{|\mathbf{p_p}'|^2}{2M} + o\left(|\mathbf{p_p}'|^3\right), \mathbf{p_p}'\right) \approx (M, 0)$$

If we can neglect the electron mass ($m_e \approx 0$) we also have

$$\omega_e = \sqrt{m^2 + \mathbf{p}_e^2} = |\mathbf{p}_e| + \frac{m^2}{2|\mathbf{p}_e|} + o(m^3) \approx |\mathbf{p}_e|$$

$$\omega'_e = \sqrt{m^2 + \mathbf{p}_e'^2} = |\mathbf{p}'_e| + \frac{m^2}{2|\mathbf{p}'_e|} + o(m^3) \approx |\mathbf{p}'_e|$$

Then energy conservation reads

$$\omega_e + M_p = \omega'_e + \omega'_p \quad \to \quad \omega_e = \omega'_e + \frac{|\mathbf{p_p}'|^2}{2M_p} + o\left(|\mathbf{p_p}'|^3\right) \quad \to \quad \omega_e \approx \omega'_e$$

The only quantity that shows a remarkable variation is the angle θ (variation can be $\mathcal{O}(1)$):

$$\begin{aligned} (p_e - p'_e)^2 &= 2m^2 - 2p_e \cdot p'_e = 2m^2 - 2\omega_e\omega'_e + 2|\mathbf{p}_e||\mathbf{p}'_e| \cos\theta \approx -2|\mathbf{p}_e|^2(1 - \cos\theta) \\ (p_p - p'_p)^2 &= 2M^2 - 2p_p \cdot p'_p = 2M^2 - 2M\omega'_p \approx -|\mathbf{p}'_p|^2 \end{aligned}$$

and because of momentum conservation $p_e + p_p = p'_e + p'_p$ we have

$$-2|\mathbf{p}_e|^2(1-\cos\theta) \approx -|\mathbf{p}'_p|^2 \quad \to \quad \cos\theta \approx 1 - \frac{1}{2}\frac{|\mathbf{p}'_p|^2}{|\mathbf{p}_e|^2}$$

In order to give a description of this process using QED, we modify the QED lagrangian, so that we introduce also protons in our theory. We consider a low energy process, where the proton can be consider as a fundamental particle, described as a spin 1/2 fermion. Then we can do the same trick we used for QED flavours. The modified Lagrangian reads:[IX]

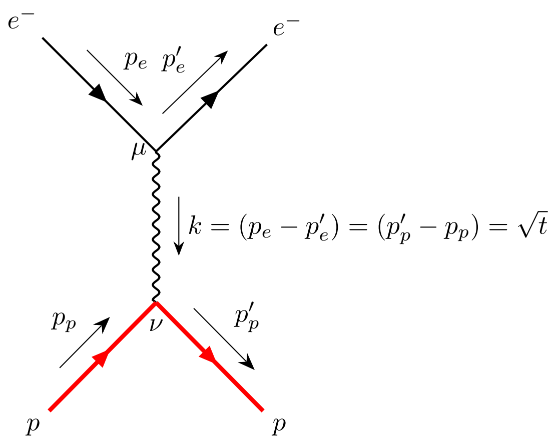
$$\mathcal{L} = \bar{\psi}_e(i\partial\!\!\!/ - m_e)\psi_e + \bar{\psi}_p(i\partial\!\!\!/ - M_p)\psi_p + \underbrace{q_e\bar{\psi}_e A\!\!\!/\psi_e + q_p\bar{\psi}_p A\!\!\!/\psi_p}_{\mathcal{L}_{\text{int}}}$$

Notice that $\mathcal{L}_{\text{int}}$ in order to obtain a description of Rutherford scattering we need to consider at least 2-nd order processes, i.e. 2 vertex. The two-vertex S matrix element becomes

$$\begin{aligned}
S_{(2)} &= \frac{(-i)^2}{2!}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{T}\left\{\mathrm{N}\left[q_e\bar{\psi}_e A\!\!\!/\psi_e + q_p\bar{\psi}_p A\!\!\!/\psi_p\right]_x \mathrm{N}\left[q_e\bar{\psi}_e A\!\!\!/\psi_e + q_p\bar{\psi}_p A\!\!\!/\psi_p\right]_y\right\} \\
&= \frac{1}{2}(-iq_e)(-iq_p)\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{T}\left\{\mathrm{N}\left[\bar{\psi}_e A\!\!\!/\psi_e\right]_x \mathrm{N}\left[\bar{\psi}_p A\!\!\!/\psi_p\right]_y + \mathrm{N}\left[\bar{\psi}_e A\!\!\!/\psi_e\right]_y \mathrm{N}\left[\bar{\psi}_p A\!\!\!/\psi_p\right]_x\right\} \\
&= (-iq_e)(-iq_p)\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{T}\left\{\mathrm{N}\left[\bar{\psi}_e A\!\!\!/\psi_e\right]_x \mathrm{N}\left[\bar{\psi}_p A\!\!\!/\psi_p\right]_y\right\}
\end{aligned}$$

where in the first step we omitted products in the integral that does not describe interaction between electron and proton, and in the second step we change integral variables in the second term of the time product.

The $e^-p \to e^-p$ process is similar to the process $e^-f^- \to e^-f^-$. Both processes are described by a t-channel



Feynman amplitude for this process is (we consider the gauge fixing term for EM field with $\zeta = 1$)

$$\mathcal{M} = i\frac{q_e q_p}{t}\left(\bar{u}_{r'}(p'_e)\gamma^\mu u_r(p_e)\right)_e \left(\bar{u}_{s'}(p'_p)\gamma_\mu u_s(p_p)\right)_p$$

where lower indices means that spinors are related respectively to electron and proton fields. Since proton is at rest its spinors takes a simple form

$$u_s(p_e = (M,0)) = \sqrt{2M}\begin{pmatrix}\xi_s \\ 0\end{pmatrix} \qquad \bar{u}_s(p'_e = (M,0)) = \sqrt{2M}\begin{pmatrix}\xi_s^\dagger & 0\end{pmatrix}$$

[IX] See Schwartz sec. 5.2

and the proton current reads[X]

$$\left(\bar{u}_{s'}(p'_p)\gamma_\mu u_s(p_p)\right)_p = 2M\begin{pmatrix}\xi'^\dagger_s & 0\end{pmatrix}\gamma_\mu\begin{pmatrix}\xi_s\\0\end{pmatrix} = 2Mg_{\mu0}\delta_{ss'}$$

so we obtain

$$\mathcal{M}_{\mathrm{LAB}} = i\frac{q_eq_p}{t}\left(\bar{u}_{r'}(p'_e)g_{\mu0}\gamma^\mu u_r(p_e)\right)_e 2M\delta_{ss'} = i\frac{q_eq_p}{t}\left(\bar{u}_{r'}(p'_e)\gamma^0 u_r(p_e)\right)_e 2M\delta_{ss'}$$

The unpolarized squared amplitude in the lab frame reads

$$\begin{aligned}\overline{|\mathcal{M}|}^2_{\mathrm{LAB}} &= \left(\frac{q_eq_p}{t}\right)^2\left(\sum_{s,s'}4M^2\delta_{ss'}\right)\frac{1}{2}\sum_{rr'}\left(\bar{u}_{r'}(p'_e)\gamma^0u_r(p_e)\right)_e\left(\bar{u}_r(p_e)\gamma_0u_{r'}(p'_e)\right)_e\\ &= \left(\frac{q_eq_p}{t}\right)^2 2M^2\mathrm{Tr}\left[(\not{p}_e+m)\gamma^0(\not{p}'_e+m)\gamma_0\right] = \cdots = 16\left(\frac{q_eq_p}{t}\right)^2M^2\omega_e^2\left(1-v^2\sin^2\left(\frac{\theta}{2}\right)\right)\end{aligned}$$

Verify last step, probably is wrong. See both Schwartz and Mandl. Also next calculation can be wrong

where we define the magnitude of the speed of the electron

$$v = \frac{|\mathbf{p}_e|}{m}$$

We also have

$$t = -4\omega_e^2v^2\sin^2\left(\frac{\theta}{2}\right)$$

Putting pieces together we obtain

$$\overline{|\mathcal{M}|}^2_{\mathrm{LAB}} = (q_eq_p)^2\frac{\omega_e^2M^2}{|\mathbf{p}_e|^4}\left(\frac{1-v^2\sin^2\left(\frac{\theta}{2}\right)}{\sin^4\left(\frac{\theta}{2}\right)}\right)$$

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\omega}\right)_{\mathrm{LAB}} = \frac{\alpha^2\left(1-v^2\sin^2\left(\frac{\theta}{2}\right)\right)}{4\omega_e^2v^4\sin^4\left(\frac{\theta}{2}\right)} \qquad (q_p=-q_e=e)$$

The latter is known as *Mott formula.* In the non-relativistic limit we can use $v \ll 1$ and $p_e \ll \omega_e \sim m_e$, thus

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\omega}\right)^{\text{non-rel.}}_{\mathrm{LAB}} = \frac{\alpha^2}{4\omega_e^2v^4\sin^2\left(\frac{\theta}{2}\right)}$$

which is the Rutherford formula. We can consider a generic nucleous with atomic number Z, then

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\omega}\right)^{\text{non-rel.}}_{\mathrm{LAB}} = \frac{\alpha^2Z^2}{4\omega_e^2v^4\sin^2\left(\frac{\theta}{2}\right)}$$

1.6.2 Generic external E.M. field

See Mandl sec. 8.7

We can use a more general approach, where instead of considering the proton as a fundamental particle, we consider it as a source of E.M. field, considered as an external field in our theory.

So far, the electromagnetic field has been described by a quantized field, involving photon creation and annihilation operators. In some problem, where the quantum fluctuations are unimportant, it may be adequate to describe the field as a purely classical function of the space-time coordinates. In cases such as Rutherford scattering, we consider an applied external electromagnetic field $A^{\mathrm{ext}}(x)$, such as the Coulomb field of a heavy nucleus. More generally, one may have to consider both types of field, replacing A by the sum of the quantized and the classical fields, $A(x) + A^{\mathrm{ext}}(x)$.

[X] The last identity can be easly proved using an explicit representation of gamma matrices.

Without the external EM field, the matrix element of the scattering amplitude reads

$$
\begin{aligned}
S_{fi} &= \langle f | \, S_{(2)} \, | i \rangle = \langle e^-_{r'}(p'_e) p_{(s')}(M) | \, S_{(2)} \, | e^-_r(p_e) p_{(s)}(M) \rangle \\
&= \int \mathrm{d}^4x \, \langle e^-_{r'}(p'_e) | \, (-iq_e) \left[\bar\psi_e \gamma^\mu \psi_e \right]_x | e^-_r(p_e) \rangle \int \mathrm{d}^4y \, \langle p_{(s')}(M) | \, (-iq_p) \left[\bar\psi_p \gamma^\nu \psi_p \right]_y | p_{(s)}(M) \rangle \, i D^F_{\mu\nu}(x-y) \\
&= -iq_e \int \mathrm{d}^4x \, \langle e^-_{r'}(p'_e) | \left[\bar\psi_e \gamma^\mu \psi_e \right]_x | e^-_r(p_e) \rangle \int \mathrm{d}^4y D^F_{\mu\nu}(x-y) \underbrace{q_p \, \langle p_{(s')}(M) | \left[\bar\psi_p \gamma^\mu \psi_p \right]_y | p_{(s)}(M) \rangle}_{J^\nu_p(y)}
\end{aligned}
$$

where $J^\nu_p(y)$ is the 4-current density associated to the proton field. We can therefore define the field

$$A^\mu_{\text{ext}}(x) = \int \mathrm{d}^4y D^F_{\mu\nu}(x-y) J^\nu_p(y)$$

which Maxwell equation reads[XI]

Metti qualche commento sulle funzioni di green

$$\Box A^\mu_{\text{ext}}(x) = \int \mathrm{d}^4y \left(\Box D^F_{\mu\nu}(x-y) \right) J^\nu_p(y) = J^\nu_p(x)$$

i.e. $A^\mu_{\text{ext}}(x)$ is a Maxwell (classic) field with external source $J^\nu_p(x)$. With this notation we obtain

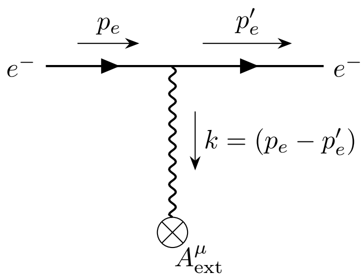
$$S_{fi} = -iq_e \int \mathrm{d}^4x \, \langle e^-_{r'}(p'_e) | \left[\bar\psi_e \not{A}_{\text{ext}} \psi_e \right]_x | e^-_r(p_e) \rangle$$

We obtained the same result if we would have defined the QED interaction lagrangian as

$$\mathcal{L}_{\text{int}} = -q_e \bar\psi \gamma^\mu \psi (A_\mu + A^{\text{ext}}_\mu)$$

and we considered its expantion at the first order (the field A_μ vanishes in the matrix element because of the form of initial and final states). Notice that A_μ is a quantum field (i.e. an operator) while A^{ext}_μ is a classic field (i.e. a function).

The Feynman diagram for this process is



Let's assume that A^μ_{ext} is a static potential. In this case we obtain same result as classic Rutherford scattering. First we express the external field in momentum space[XII]

$$A^\mu_{\text{ext}}(x) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3k \, e^{i\mathbf{k}\cdot\mathbf{x}} \varepsilon^\mu_{\text{ext}}(\mathbf{k})$$

Then the S matrix element at first order reads

$$
\begin{aligned}
S_{fi} &= -iq_e \int \mathrm{d}^4x \, \langle e^-_{r'}(p'_e) | \, \bar\psi_e \not{A}_{\text{ext}} \psi_e \, | e^-_r(p_e) \rangle \\
&= -iq_e \int \mathrm{d}^4x \, \langle e^-_{r'}(p'_e) | \, \bar\psi_e \gamma^\mu \psi_e \, | e^-_r(p_e) \rangle \frac{1}{(2\pi)^3} \int \mathrm{d}^3k \, e^{i\mathbf{k}\cdot\mathbf{x}} \varepsilon^{\text{ext}}_\mu(\mathbf{k}) \\
&= -iq_e \frac{1}{(2\pi)^3} \int \mathrm{d}^4x \, \mathrm{d}^3k \, \bar u(p'_e) \gamma^\mu u(p_e) \, \varepsilon^{\text{ext}}_\mu(\mathbf{k}) \, e^{-ip\cdot x + ip'\cdot x + i\mathbf{k}\cdot\mathbf{x}} \\
&= -iq_e \int \mathrm{d}x \, \mathrm{d}^3k \, \delta^3(\mathbf{k} + \mathbf{p} - \mathbf{p}') \bar u(p'_e) \gamma^\mu u(p_e) \, \varepsilon^{\text{ext}}_\mu(\mathbf{k}) \, e^{-i(\omega_e - \omega'_e)x} \\
&= -iq_e (2\pi) \delta(\omega_e - \omega'_e) \bar u(p'_e) \gamma^\mu u(p_e) \, \varepsilon^{\text{ext}}_\mu(\mathbf{p}' - \mathbf{p})
\end{aligned}
$$

[XI] Here we used proprieties of Green functions

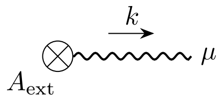
[XII] Notice that $\varepsilon^\mu_{\text{ext}}(\mathbf{k}) = \tilde{A}^\mu_{\text{ext}}(\mathbf{k})$, but we use this notiation in analogy with the quantum Maxwell field expantion.

and

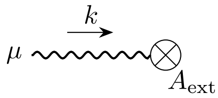
$$\mathcal{M}_{fi}^{\text{ext}} = -iq_e \bar{u}(p_e')\gamma^\mu u(p_e)\, \varepsilon_\mu^{\text{ext}}(k)$$

Notice that the Feynman rules for QED with external field are the same as in the free case, using terms $\tilde{A}^\mu_{\text{ext}}(k)$ instead of polarization vector for the Maxwell field. Set $\varepsilon_\mu^{\text{ext}}(k) = \tilde{A}_\mu^{\text{ext}}(k)$ we have

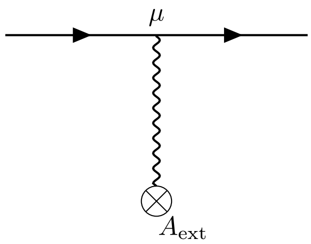
$\varepsilon_\mu^{\text{ext}}(k) =$



$\varepsilon_\mu^{\text{ext}\,*}(k) =$



$-iq_e\gamma^\mu =$



Other rules are unchanged.