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Chapter 1

Introduction

1.1 Free Theories Lagrangians

1.1.1 Complex Scalar Field

$(\Box + m^2)\varphi = 0$

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} \big(e^{-ikx}a(k) + e^{ikx}b^\dagger(k)\big)_{k_0=\omega_k}$$
$$\omega_k = \sqrt{m^2 + \mathbf{k}^2}$$

In the real case $\varphi^\dagger(x) = \varphi(x) \Rightarrow a(k) = b(k)$

1.1.2 Dirac Spinorial Field

$(i\partial\!\!\!/ - m)\psi = 0$

$$\psi(x) = \frac{1}{(2\pi)^{2/3}} \int \frac{d^3k}{\sqrt{2\omega_k}} \sum_{r=1,2} \big(e^{-ikx}u_r(k)c_r(k) + e^{ikx}v_r(k)d_r^\dagger(k)\big)$$

where $u_r(k)/v_r(k)$ are the $\varepsilon > 0/\varepsilon < 0$ spinors
Spinors are normalized according to

$$\begin{cases} \bar{u_r}(k)u_s(k) = 2m\delta_{rs} & \bar{u_r}(k)v_s(k) = 0 \\ \bar{v_r}(k)v_r(k) = -2m\delta_{rs} & \bar{v_r}(k)u_s(k) = 0 \end{cases}$$

1.1.3 E-M Vector Field

$\partial_\mu F^{\mu\nu} = \Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0$, where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$A^\mu(x) = \frac{1}{(2\pi)^{2/3}} \int \frac{d^3k}{\sqrt{2\omega_k}} \sum_\lambda \big(e^{-ikx}\varepsilon^\mu_{(\lambda)}a_\lambda(k) + e^{ikx}\varepsilon^{\mu\dagger}_{(\lambda)}(k)a_\lambda^\dagger(k)\big)_{k_0=\omega_k=|\mathbf{k}|}$$
$$\varepsilon^\mu_{(1)} = (0,1,0,0) \qquad \varepsilon^\mu_{(2)} = (0,0,1,0)$$

I can complexify the field substituting $a_\lambda^\dagger$ with another operator b_λ (analogously to the scalar field)
Notice that real fields are never free because we have interactions, but using interaction picture e reconduct the problem in a simper one, where filed are described by free fields. This can be able with a proper choice.

$$\Phi_I(x) \equiv \Phi_{\text{free}}(x) \qquad \Phi_I = \text{interacting}$$

1.2 Fock Space of Free Fields

See Maggiore. See 6.1
We impose the existence of vacuum state $|0\rangle$, and using creation operators we obtain other states $(a^\dagger)^n\,|0\rangle$, which are n-particles states.
In QFT we normalized states in a covariant way, instead of QM normalization $\int \psi^*\psi = 1$

$$
\begin{aligned}
|1(p)\rangle &\equiv (2\pi)^{3/2}\sqrt{2\omega_k}\; o^\dagger(p)\,|0\rangle \\
\langle 1(p)\,|\,1(p')\rangle &= (2\pi)^3(2\omega p)\delta^3(p-p')^{\mathrm{I}}
\end{aligned}
$$

$(2\omega p)\delta^3(p-p')$: Covariant under Lorentz tfm

1.3 Contraction of Fields with States

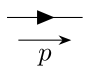
If we have a state $|e_s^-(p)\rangle$ that describes an electron with momentum p and Dirac index s, then

$$
\left|e_s^-(p)\right\rangle = (2\pi)^{2/3}\sqrt{2\omega p}\; c_s^\dagger(p)\,|0\rangle
$$

given a field ψ that describes a particle annihilation (or antiparticle creation) in x we have

$$
\begin{aligned}
\left\langle 0\,\middle|\,\psi(x)\,\middle|\,e_s^-(p)\right\rangle &= \left\langle 0\,\middle|\,(\psi_+(x)+\psi_-(x))\,\middle|\,e_s^-(p)\right\rangle \\
&= \frac{(2\pi)^{3/2}}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}}e^{-ikx}\sqrt{2\omega_p}\;\sum_r \left\langle 0\,\middle|\,c_r(k)c_s^\dagger(p)\,\middle|\,0\right\rangle u_r(k) \\
&= \int \mathrm{d}^3k\Big(\frac{2\omega_p}{2\omega_k}\Big)^{1/2}\;\sum_r \delta_{rs}\delta^{(3)}(\bar{p}-\bar{k})u_r(k)\,\langle 0\,|\,0\rangle\, e^{-ikx} \\
&= e^{-ikx}u_s(p)
\end{aligned}
$$

$c_r(k)c_s^\dagger(p) = \{c_r(k), c_s^\dagger(p)\} = \delta_{rs}\;\delta^3(\bar{p}-\bar{k})$
The factor e^{-ikx} is required for the $\delta^{(4)}$ conservation, and we see that the relativistic normalization leads to the relation ($\rightarrow$ Feyman rule)


$= e^{-ipx}u_s(p)$

In this case there is no normalization factors in the Feynman rule

1.4 S-matrix and State Evolution

In the interaction picture, with $H = H_0 + H_{\text{int}}$, with H_{int}= interaction hamiltonian

(i) Fields Φ_I evolves like in the free theory (respect to H_0)

(ii) State evolves with the following evolution operator

$$
\begin{gathered}
U_I(t,t_0) \equiv e^{iH_0t}e^{-iH(t-t_0)}e^{-iH_0t_0} \\
|\alpha, t\rangle = U_I(t-t_0)\,|\alpha, t_0\rangle \qquad i\partial_t U_I(t-t_0) = H_I^{\text{int}}(t)U_I(t,t_0)
\end{gathered}
$$

Notice that, in general

$$
[H_I(t), H_0] \neq 0 \neq [H_I^{\text{int}}, H_0]
$$

and if $t \neq t'$ we also have

$$
[H_I^{\text{int}}(t), H_I^{\text{int}}(t') \neq 0 \qquad \text{with } O_I(t) = e^{iH_0t}e^{-iHt}\;O_He^{iHt}e^{-iH_0t}
$$

The S-matrix is a well defined operator defined as

$$S = \lim_{\substack{t_0 \to -\infty \\ t \to +\infty}} U_I(t, t_0)$$

We compute S by perturbation obtaining

$$\begin{aligned} S &= T\left(\exp\left(-i\int \mathrm{d}^4x \mathcal{H}_I^{\text{int}}(x)\right)\right) \\ &= \sum_{n=0}^{+\infty} \frac{(-i)^n}{n!} \int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n \, T(\mathcal{H}_I^{\text{int}}(x_1) \ldots \mathcal{H}_I^{\text{int}}(x_n)) \end{aligned}$$

S has some relevant properties

(i) Unitary (since hamiltonian is hermitian)

(ii) Behaves as a scalar under Lorentz tfms, and then is an invariant quantity (notice that in general $\mathcal{H}_I^{\text{int}}$ is not invariant
In the case of $\mathcal{H}_I^{\text{int}}$ is invariant (for example if $\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$, as in many theories, one of them id QED) is easy to prove that S in invariant, since all n-th derivatives of $\exp\left(\int \mathcal{H}\right)$ are invariant, and so also S is invariant

1.5 S-matrix and transition probabilities

we suppose that there's no interaction for $x, t \to +\infty$
Consider a canonically normalized (CN) state $|\psi| = 1$:

$$|\psi_i\rangle_{CN} \equiv |\psi(-\infty)\rangle_{CN} \qquad |\psi(+\infty)\rangle_{CN} \equiv S\,|\psi_i\rangle_{CN} \quad \to \text{both are free particle states}$$

Elements of S are in the form

$$S_{fi}^{CN} = \langle \psi_f \,|\, S \,|\, \psi_i \rangle_{CN}$$

This leads to a probabilistic interpretation of S-matrix elements.
$\left|S_{fi}^{CN}\right|^2$ = propability of evolution of $|\psi_i\rangle_{CN}$ into $|\psi_f\rangle_{CN}$, since the condition $\sum_f \left|S_{fi}^{CN}\right| = 1$ is satisfied automatically
In the case of covariant normalization

$$\langle 1(p) \,|\, 1(p') \rangle = (2\pi)^3 (2\omega p) \delta^3(\mathbf{p} - \mathbf{p}')$$

we have the following relation between matrix elements

$$S_{fi}^{CN} = \langle \psi_f \,|\, S \,|\, \psi_i \rangle_{CN} = \frac{\langle \psi_f \,|\, S \,|\, \psi_i \rangle}{\|\psi_i\| \, \|\psi_f\|} = \frac{S_{fi}}{\|\psi_i\| \, \|\psi_f\|}$$

We can define the **Feynman Amplitude** $\mathcal{M}_{fi}$ as

$$S_{fi} = (2\pi)^4 \delta^4(p_i - p_f) \mathcal{M}_{fi}$$

and it can be obtained directly starting from Feynman rules (calculated with the covariant normalization)

1.6 Discrete space normalization

Usually, in order to make arguments cleaner, or to avoid problems with divergent terms, we first consider a system in a cubic box with spatial valume $V = L^3$.
At the end of computations V will be sent to infinity. Sometimes we will do something similar also for time.

For a discrete space we must use a different normalization.
In a box, momentum of a particle is quantized (1-dim case)

$$p_i = \left(\frac{2\pi}{L}\right) n_i \qquad n_i \in \mathbb{Z}$$

and we must adopt the following rule for integrals

$$\int \mathrm{d}^3 p \, f(\mathbf{p}) \quad \to \quad \sum_{\mathbf{n}} \left(\frac{2\pi}{L}\right) f_{\mathbf{n}} \qquad \mathbf{n} = (n_1, n_2, n_3)$$

We must adopt also the following

$$\delta^3(\mathbf{p} - \mathbf{p}') \quad \to \quad \left(\frac{L}{2\pi}\right)^3 \delta_{\mathbf{n}\mathbf{n}'}$$

in this way

$$\int \mathrm{d}^3 p \, \delta^3(\mathbf{p} - \mathbf{p}') = 1 \quad \to \quad \sum_{\mathbf{n}} \left(\frac{2\pi}{L}\right)^3 \left(\frac{L}{2\pi}\right)^3 \delta_{\mathbf{n}\mathbf{n}'} = 1$$

Some useful relations are

$$\delta^3(0) \to \left(\frac{L}{2\pi}\right)^3$$

$$\delta^4(0) \to \left(\frac{L}{2\pi}\right)\left(\frac{T}{2\pi}\right) \to \text{Only if we consider a finite amount of time}$$

Normalization of state becomes

$$|1(p)\rangle = (2\pi)^{2/3} \sqrt{2\omega_p V} o^\dagger(p) \, |0\rangle = (2\pi)^{2/3} \sqrt{2\omega_p V} \, |1(p)\rangle_{CN}$$
$$\langle 1(p) \, | \, 1(p) \rangle = (2\pi)^3 2\omega_p V \delta^3(0) = 2\omega_p V$$

Using the latter equation, S_{fi}^{CN} reads

$$\begin{aligned} S_{fi}^{CN} &= \prod_{j=1}^{n_i} \left(\frac{1}{2\omega_j V}\right)^{1/2} \prod_{l=1}^{n_f} \left(\frac{1}{2\omega_l V}\right)^{1/2} S_{fi} \\ &= (2\pi)^4 \delta^4(p_i - p_f) \left\{ \prod_{j=1}^{n_i} \left(\frac{1}{2\omega_j V}\right)^{1/2} \prod_{l=1}^{n_f} \left(\frac{1}{2\omega_l V}\right)^{1/2} \mathcal{M}_{fi} \right\} \\ &= (2\pi)^4 \delta^4(p_i - p_f) M_{fi}^{CN} \end{aligned}$$

In the first passage $(2\pi)^{2/3}$ factors vanish because of δ^3 factors inside S_{fi} related to the sandwich $\langle \psi_f \, | \, S \, | \, \psi_i \rangle$
In the second passage we use df_n of $\mathcal{M}_{fi}$, omitting the quantization of δ^4
M_{fi}^{CN} is the **canonically normalized Feynman amplitude**

Chapter 2

The S-matrix and physical observables

2.1 Decay Rate

Consider the case in which the initial state is a single particle and the final state is given by n particles. We are therefore considering a decay process. Assume for the moment that particles are indistinguishable.

The rules of quantum mechanics tell us that the probability for this process is obtained by taking the squared modulus of the amplitude and summing over all possible final states

$$
\begin{aligned}
\left|S_{fi}^{CN}\right|^2 &= \left|(2\pi)^4\delta^4(p-p')M_{fi}^{CN}\right|^2 \\
&= (2\pi)^4\delta^4(p-p')(VT)\left|M_{fi}^{CN}\right|^2 \\
&= (2\pi)^4\delta^4(p-p')(VT)\frac{1}{2\omega_i nV}\prod_{l=1}^{n_f}\left(\frac{1}{2\omega_l V}\right)|\mathcal{M}_{fi}|^2
\end{aligned}
$$

Note: $\left(\delta^4(p-p')\right)^2 = \delta^4(p-p')\delta^4(p-p') = \delta^4(p-p')\delta^4(0) = \delta^4(p-p')\frac{VT}{(2\pi)^4}$
We use the final space and time in order to remove divergent terms during calculation

We must now sum this expression over all final states. Since we are working in a finite volume V, this is the sum over the possible discrete values of the momenta of the final particles .
Since $p_i = (2\pi/L)n_i$, we have $dn_i = (L/2\pi)\mathrm{d}p_i$ and $\mathrm{d}^3n_i = (V/(2\pi)^3)\mathrm{d}^3p$ where d^3n_i is the infinitesimal phase space related to a final state in which the i-th particle has momentum between p_i and $p_i + \mathrm{d}p_i$
Let $\mathrm{d}\omega$ be the probability for a decay in which in the final state the i-th particle has momentum between p_i and $p_i\mathrm{d}p_i$

$$
\mathrm{d}\omega = \left|S_{fi}^{SN}\right|^2\prod_{l=1}^{n_f}\left(\frac{V\mathrm{d}^3p_l}{(2\pi)^3}\right)
$$

This is the probability that the decay takes place in any time between $-T/2$ and $T/2$. We are more interested in the differential decay rate $\mathrm{d}\Gamma_{fi}$, which is the decay probability per unit of time:

$$
\mathrm{d}\Gamma_{fi} = \frac{\mathrm{d}\omega}{T} = (2\pi)^4\delta^4(p_i-p_f)\frac{|\mathcal{M}_{fi}|^2}{2\omega_{p_{in}}}\prod_{l=1}^{n_f}\frac{\mathrm{d}^3p_l}{(2\pi)^3 2\omega_l}
$$

Notes:

(i) $\mathrm{d}\Gamma_{fi}$ = differential decay rate

(ii) p_f = sum over final momenta

(iii) $\omega_{p_{in}}$ = initial energy

(iv) $|\mathcal{M}_{fi}|^2$ = Feynman amplitude of the process (depends on final momenta p_i)

It is useful to define the **(differential) n-body phase space** as

$$\mathrm{d}\Phi_{(n_f)} = (2\pi)^4\delta^4(p_i - p_f)\prod_{l=1}^{n_f}\frac{\mathrm{d}^3 p_l}{(2\pi)^3 2\omega_l}$$

Therefore the differential decay rate can be written as

$$\mathrm{d}\Gamma_{fi} = \frac{1}{2\omega_{p_{in}}}|\mathcal{M}_{fi}|^2\,\mathrm{d}\Phi_{(n_f)}$$

The decay rate is defined as

$$\Gamma_{fi} = \int \mathrm{d}\Gamma_{fi} \qquad \rightarrow \text{integration over all possible final momenta}$$

and its meaning is $\Gamma \equiv$ trans. probability $\times$ unit of time $\times$ init. particle
Notice that if n of the final particles are identical, configurations that differ by a permutation are not distinct and therefore the phase space is reduced by a factor $1/h$!
If we have a system of N(0) particles, the time evolution of the number of particles N(t) is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\Gamma N \quad \Rightarrow \quad N(t) = N(0)e^{-\Gamma t}$$

Notice that decay rate is not invariant

$$[\Gamma] = [E] = \frac{1}{T} \qquad \text{(in natural units)}$$

If we define the **lifetime** as $\tau = 1/\Gamma \Rightarrow N(t) = N(0)\exp(-t/\tau)$ this changes under Lorentz tfm. If we consider two reference frames o and o'

$$\Gamma' = \frac{\Gamma}{\gamma} < \Gamma \qquad \tau' = \gamma\tau > \tau$$

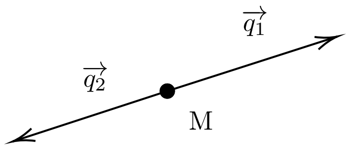
$\gamma = (1-v)^{-1/2}$, where v is the speed of o' in o in natural units, $\gamma > 1$. Therefore a article in a moving frame has a longer lifetime then in the rest frame

Example 1: muon lifetime

For a muon in the rest frame $\tau_\mu^{RF} = 2.2 \times 10^{-6}s$, but if we observe it in the lab frame
$\tau_\mu^{LAB} = \gamma\tau_\mu^{RF} \simeq 2 \times 10^{-5}s$ since $E_\mu = 1$ GeV, $m_\mu = 0.1$ GeV
$\Rightarrow \gamma = E_\mu/m_\mu \simeq 10$

Example 2: 1 → 2 decay

Consider the decay of a particle of a mass M into two particles of masses m_1, m_2. Since the phase space in Lorentz invariant, we can compute it in the frame that we prefer. We use the rest frame for the initial particle.



We don't impose a priori conservation of momentum since it's imposed by the delta function.

$$p = (M, 0) \qquad q_1 = (\omega_1, \mathbf{q_1}) \qquad q_2 = (\omega_2, q_2)$$
$$\mathrm{d}\Phi_{(2)} = (2\pi)^4 \delta^4 \underbrace{(P_i - P_f)}_{=p-q_1-q_2} \frac{\mathrm{d}^3 q_1}{(2\pi)^3 2\omega_1} \frac{\mathrm{d}^3 q_2}{(2\pi)^3 2\omega_2}$$

I have 6 integration parameters, 4 constraint given by δ^4, so I have 2 independent variables. Integrating over $\mathrm{d}^3 q_2$

$$\mathrm{d}\Phi'_{(2)} = \int \mathrm{d}\Phi_{(2)} = \frac{1}{(2\pi)^2} \delta(M - \omega_1 - \omega_2) \frac{1}{4\omega_1\omega_2} \mathrm{d}^3 q_1$$

in this way, the condition $\mathbf{q_2} = \mathbf{q_1}$ vanish. We have to impose it again when we calculate $\mathrm{d}\Gamma$ (we omit this detail)
Usually the 4-th non independent parameter is eliminated by integration over modulus of q_1, leaving free 2 parameters for the angles. $\mathrm{d}^3 q_1 \to \mathbf{q_1}^2 \mathrm{d}\,|\mathbf{q_2}|\,\mathrm{d}\Omega_1$.

Notice that $M - \omega_1 - \omega_2 = M - \sqrt{\mathbf{q_1}^2 + m_1^2} - \sqrt{\mathbf{q_2} + m_2^2}$ and then the δ implies

$$\hat{q_1}^2 = \frac{1}{2M} \left(M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2 \right)^{1/2}$$

$|\mathbf{q_1}|$ is the only zero of $f(|\hat{q_1}|) = M - \omega_1 - \omega_2$.
We also have

$$|f'(|\hat{q_1}|)| = \frac{\partial \omega_1}{\partial\,|\mathbf{q_1}|} + \frac{\partial \omega_2}{\partial\,|\mathbf{q_1}|} = |\hat{q_1}| \left(\frac{\omega_1 + \omega_2}{\omega_1\omega_2} \right)$$

Using

$$\delta\big(f(x)\big) = \sum_{x_0 = \text{zero of f(x)}} \frac{\delta(x - x_0)}{|f'(x_0)|}$$

and performing integration over $\mathrm{d}\,|\mathbf{q_1}|$ we obtain

$$\mathrm{d}\Phi''_{(2)} = \int \mathrm{d}\Phi'_{(2)} = \frac{1}{16\pi^2} \frac{|\hat{q_1}|}{M} \mathrm{d}\Omega_1$$

Using this result we obtain the $1 \to 2$ decay rate in function of the solid angle (in the rest frame)

$$\left(\frac{\mathrm{d}\Gamma_{RF}}{\mathrm{d}\Omega} \right) = \frac{1}{64\pi^4 M^3} \left[M^4 - 2M^2(m_1^2 + m_2^2) + (m_1 - m_2^2)^2 \right]^{1/2} |\mathcal{M}_{RF}|^2$$

In a general frame we can easily obtain an analogous formula, just consider $\mathrm{d}\Gamma = 1/(2\omega_i n)\,|\mathcal{M}_{fi}|^2\,\mathrm{d}\Phi_{(nf)}$ in a general frame. remember that $\mathrm{d}I_{(nf)}$ is invariant

We have 2 important limit cases:

(A) If $m_1 = m_2 = m$ (for example $Z \to e^+e^-$)

$$|\hat{q_1}| = \frac{M}{2} \left(1 - \frac{4m^2}{M^2} \right)^{1/2}$$
$$\left(\frac{\mathrm{d}\Gamma_{RF}}{\mathrm{d}\Omega} \right) = \frac{1}{64\pi^2 M} \left(1 - \frac{4m^2}{M^2} \right)^{1/2} |\mathcal{M}_{fi}|^2$$

(B) If $m_1 = m, m_2 = 0$ (for example $W^{\pm} \to e^{\pm} \overset{(-)}{\nu}$)

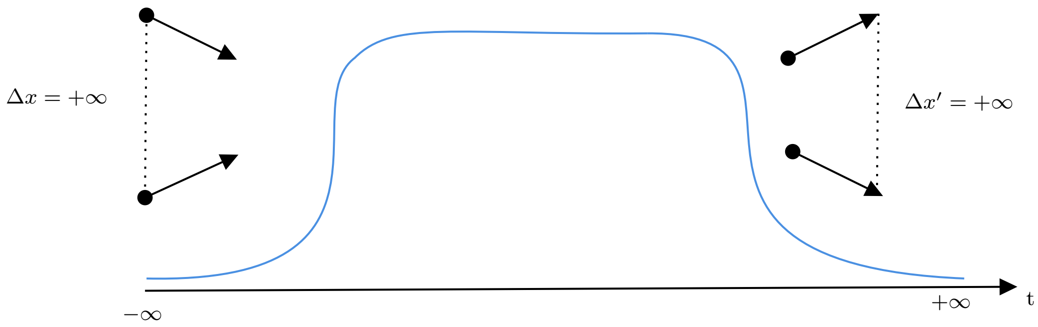
$$|\hat{q}_1| = \frac{M}{2}\left(1 - \frac{4m^2}{M^2}\right)^{1/2}$$

$$\left(\frac{\mathrm{d}\Gamma_{RF}}{\mathrm{d}\Omega}\right) = \frac{1}{64\pi^2 M}\left(1 - \frac{m^2}{M^2}\right)^{1/2} |\mathcal{M}_{fi}|^2$$

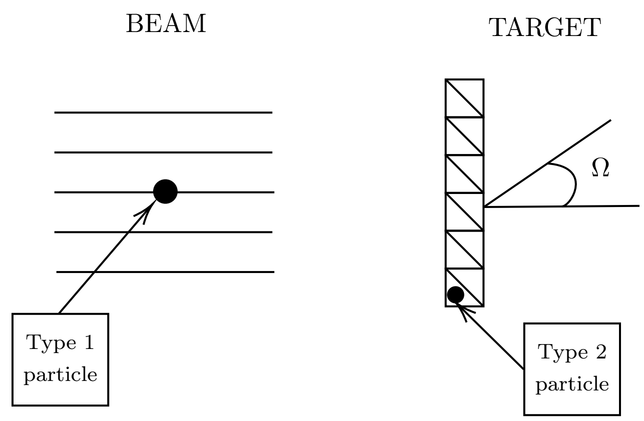
Notes: If we have two identical particles in the final state, the calculation of the phase is different

$$\mathrm{d}\Phi_{(2)}^{\text{identical}} = \frac{1}{2}\mathrm{d}\Phi_{(2)}^{\text{distinguishable}}$$

2.2 Cross section ($\leftrightarrow$ scattering process)



Scattering in the lab frame



Consider a beam of particles with mass m_1, number (assuming a uniform distribution) density $n_1^{(0)}$ (subscript 0 is meant to stress that these are number densities in a specific frame, that with particle 2 at rest) and velocity v_1 inpining on a target made of particles with mass m_2 and number density $n_2^{(0)}$ at rest.
Let N_s be the number of scattering events that place per unit volume and per unit time

$$\frac{N_t}{T}\varphi_1 N_2 \sigma = \left(n_1^{(0)} v_1\right)\left(n_2^{(0)} V\right)\sigma$$

More formally we have

$$\mathrm{d}N_s = \sigma v_1 n_1^{(0)} n_2^{(0)} \, \mathrm{d}V \mathrm{d}V$$

with:

(i) T: unit of time

(ii) φ_1: flux of the beam $\varphi_1 = n_1^{(0)} v_1$

(iii) N_2: particles per unit volume in the detector $N_2 = n_2^{(0)} V$

(iv) σ: proportionality constant

Dimensional analysis shows $[\sigma] = [L]^2$ and then σ, called cross section, can be interpreted as an "effective area"
Consider the case where just one particle collides with another particle (2 particles scattering). We can obtain this condition imposing $n_{1,2}^{(0)} = 1/V$, in this way the scattering per unit of volume is the same as two particles scattering
Notice that this situation is the more physical. The probability of 3 particles collision in the same d^3x and $\mathrm{d}t$ is almost 0

If $\mathrm{d}\omega$ is, again, the probability for a process in which in the final state the i-th particle has momentum between p_i and $p_i + \mathrm{d}p_i$

$$\mathrm{d}\omega = (2\pi)^4 \delta^4(p_i - p_f) VT \left(\frac{1}{2\omega_1 V}\right)\left(\frac{1}{2\omega_2 V}\right) \prod_{l=1}^{n_f} \left(\frac{\mathrm{d}^3 pl}{(2\pi)^3 2\omega_l}\right) |\mathcal{M}_{fi}|^2$$

we can define the **differential cross section** (in the lab frame) $\mathbf{d}\sigma$ as

$$\begin{aligned}
(\mathrm{d}\sigma)_{\mathrm{LAB}} &= \frac{\mathrm{d}\omega}{n_1^{(0)} n_2^{(0)} v_1 VT} \\
&= \frac{V}{Tv_1}\mathrm{d}\omega \\
&= (2\pi)^4 \delta^4(p_i - p_f) \prod_{l_1}^{n_f} \left(\frac{\mathrm{d}^3 p_l}{(2pi)^3 2\omega_l}\right) \frac{|\mathcal{M}_{fi}|^2}{4\omega_1\omega_2 v_1} \qquad \text{with } \omega_2 = m_2
\end{aligned}$$

The result is then

$$(\mathrm{d}\sigma)_{\mathrm{LAB}} = \frac{1}{4\omega_1 v_1 m_2} |\mathcal{M}_{fi}|^2 \ \mathrm{d}\Phi_{(nf)}$$

All quantities refers to the rest frame for particle 2 ($\omega_1 = \omega_1^{(0)}, v_1 = v_1^{(0)}$)
In order to obtain a covariant relation for $\mathrm{d}\sigma$, we notice that the only non-covariant factor in the latter relation is $(I_{12})_{\mathrm{LAB}} = \omega_1^{(0)} m_2$. This factor can be substituted with a covariant one

$$I_{12} = \left[(p_1 p_2)^2 - m_1^2 m_2^2\right]^{1/2}$$

called **covariant flux factor**. This factor is obviously covariant, we just have to prove that in the lab frame it coincides with $(I_{12})_{\mathrm{LAB}}$. In the lab frame we have $p_1 = (\omega_1, \mathbf{p_1}^{(0)})$ and $p_2 = (m_2, \mathbf{0})$, so the previous formula becomes

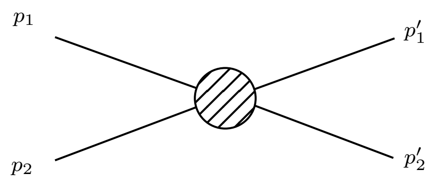
$$\left[m_2^2(\omega_1^{(0)})^2 - m_1 m_2\right]^{1/2} = m_2\left((\omega_1^{(0)})^2 - p_1^2\right)^{1/2} = m_2 \left|\mathbf{p_1^{(0)}}\right| = m_2 \omega_1^{(0)} v_1^{(0)}$$

So the final result is

$$\mathrm{d}\sigma = \frac{\left|\mathcal{M}_{fi}^2\right|}{4I_{12}} \mathrm{d}\ \Phi_{(nf)}$$

Example 3: $2 \to 2$ scattering

Consider a scattering process $2 \to 2$. We consider an initial state with two particles with masses m_1, m_2 and four momenta p_1, p_2 and a final state with masses m_1', m_2' and four momenta p_1', p_2'



With

$$p_1 = (\omega_1, \mathbf{p_1}) \qquad p_2 = (\omega_2, \mathbf{p_2}) \qquad p_1' = (\omega_1', \mathbf{p_1'}) \qquad p_2' = (\omega_2', \mathbf{p_2'})$$

With a procedure identical to $1 \to 1$ decay (imposing also $\mathbf{p_2'} = \mathbf{p_1} + \mathbf{p_2} - \mathbf{p_1'}$ when we calculate $\mathrm{d}\sigma$), we obtain

$$\mathrm{d}\Phi'_{(2)} = \frac{1}{(2\pi)^2} \frac{1}{4\omega_1'\omega_2'} \delta(\omega_1 + \omega_2 - \omega_1' - \omega_2')\ \mathrm{d}^3 p_1'$$

In order to integrate over $\mathrm{d}\,|\mathbf{p_1'}|$ it is useful to introduce Mandelstam variables s, t and u

$$s = (p1 + p2)^2 \qquad t = (p1 - p_1')^2 \qquad u = (p_1 - p_2')^2$$

These variables are clearly Lorentz invariants, and satisfy (using $p_1 + p_2 = p_1' + p_2'$) the relation

$$s + t + u = m_1^2 + m_2^2 + (m_1')^2 + (m_2')^2$$

It is useful to work in the center of mass frame, where the incoming particles have $p_1 = (\omega_1, \mathbf{p})$ and $p_2 = (\omega_2, -\mathbf{p})$. Computing s in the CM frame we obtain $s = (\omega_1 + \omega_2)^2 = (\omega_1' + \omega_2')^2$ and then

$$p_1 + p_2 = (\sqrt{s}, 0)$$
$$\delta(\omega_1 + \omega_2 - \omega_1' - \omega_2') = \delta(\sqrt{s} - \omega_1' - \omega_2')$$

With a procedure identical to the one used for $1 \to 2$ decay ($M \leftrightarrow \sqrt{s}$) we obtain

$$\left(\mathrm{d}\Phi''_{(2)}\right)_{CM} = \frac{1}{16\pi^2} \frac{\left|\widehat{p_1'}\right|_{CM}}{\sqrt{s}}\ \mathrm{d}\Omega_1'$$
$$\left|\widehat{p_1'}\right|_{CM} = \frac{1}{2\sqrt{s}} \left[s^2 + (m_1'^2 + m_2'^2)^2 - 2s(m_1'^2 + m_2'^2)\right]^{1/2}$$

The covariant flux factor in CM frame reads

$$(I_{12})_{CM} = \left[\mathbf{p}^2(\omega_1 + \omega_2)^2\right]^{1/2} = |\mathbf{p}|\sqrt{s}$$

So we obtain the final result

$$\begin{aligned} \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_1'}\right)_{CM} &= \frac{1}{64\pi^2 s} \frac{\left|\widehat{p_1'}\right|_{CM}}{|\mathbf{p}|}\ |\mathcal{M}_{fi}|^2_{CM} \\ &= \frac{1}{128\pi^2 s^{3/2}} \frac{1}{|\mathbf{p}|} \left[s^2 + (m_1'^2 - m_2'^2)^2 - 2s(m_1'^2 + m_2'^2)\right]^{1/2} |\mathcal{M}_{fi}|^2_{CM} \end{aligned}$$

We have two important limit cases

(A) If $m_1 = m'_1, m_2 = m'_2$, i. e. elastic scattering (for example $e^-\mu^\dagger \to e^-\mu^\dagger$)

$$\left.\begin{array}{l} |\mathbf{p_1}| = |\mathbf{p'_1}|\,, \omega_1 = \omega'_1 \\ |\mathbf{p_2}| = |\mathbf{p'_2}|\,, \omega_2 = \omega'_2 \end{array}\right\} \quad \left|\hat{p'_1}\right|_{CM} = |\mathbf{p}|$$

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega'_1}\right)_{CM} = \frac{1}{64\pi^2}\frac{|\mathcal{M}_{fi}|^2}{s}$$

(B) If $m_1 = m_2 \simeq 0, m'_1 = m'_2 = M$ (for example $e^-e^+ \to \mu^-\mu^+$)

$$|\mathbf{p_1}| = |\mathbf{p}|\,, \omega_1 = \frac{\sqrt{s}}{2}$$

$$\left|\hat{p'_1}\right| = \frac{\sqrt{s}}{2}\left(1 - \frac{1}{4M^2}s\right)^{1/2} \tag{2.1}$$

$4M^2/s$ processes with $s < 4M^2$ are unphysical

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega'_1}\right)_{CM} = \frac{1}{64\pi^2}\left(1 - \frac{4M^2}{s}\right)\frac{|\mathcal{M}_{fi}|^2}{s}$$

The previous formulas are valid also for particles with spin, if the initial and final spin states are known; in this case the initial state has the form $|i\rangle = |\mathbf{p_1}, \mathbf{s_1}; \ldots; \mathbf{p_n}, \mathbf{s_n}\rangle$, and similarly for the final state.
However, experimentally is more common that we do not know the initial spin configuration and we accept in the detector all final spin configurations; in this case, to compare with experiment, we must

(i) Considering all (equally present) polarization of initial state, i.e. *average* over the initial spin configuration

(ii) Considering all final polarizations, i.e. *sum* over all possible final configurations

Defining the **unpolarized Feynman amplitude** $\overline{|\mathcal{M}_{fi}|}$ as

$$\overline{|\mathcal{M}_{fi}|^2} = \frac{1}{\text{n initial polarizations}} = \sum_{\substack{\text{initial}\\\text{spins}}}\sum_{\substack{\text{final}\\\text{spins}}} |\mathcal{M}_{fi}|^2$$

we just hace to substitute $|\mathcal{M}_{fi}|^2 \to |\mathcal{M}_{fi}|^2$ in all previous formulas
For example, in $2 \to 2$ scattering

$$\frac{1}{\text{n init pol}} = \frac{1}{(2s_a+1)(2s_b+1)} \qquad \text{where } s_a = \text{spin of particle 1; and } s_b = \text{spin of particle 2}$$

Chapter 3

QED Processes at Lowest order

3.1 The QED Lagrangian and its Symmetries

Mandl, sec 11.1 - Maggiore, sec 7.1

Quantum electrodynamic (QED) describes the interactions between (or any other charged spin 1/2 particle) and photons. QED is described by the lagrangian

$$\mathcal{L}_{\text{QED}} = \underbrace{\overline{\psi}(i\partial\!\!\!/ - m)\psi}_{\mathcal{L}_D^{(0)}} \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\mathcal{L}_{EM}} - \underbrace{qA_n\overline{\psi}\gamma^\mu\psi}_{\mathcal{L}_{int}} - \underbrace{\frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2}_{\mathcal{L}_{GF}}$$

(i) $\mathcal{L}_D^{(0)}$ is the langrangian for the free Dirac field

(ii) $\mathcal{L}_{EM}$ is the lagrangian for the free EM field. In order to quatize the E-n field we have to add the term $\mathcal{L}_{GF}$ (gauge fixing). For other purposes this term can be omitted. Usually the choice $\xi = 1$, called Feynman gauge, is the simplest choice for quantization

(iii) $\mathcal{L}_{int}$ describes the interaction between Dirac field and EM-field. Notice that the term $\mathcal{L}_D = \mathcal{L}_D^{(0)} + \mathcal{L}_{int}$ can be obtained from $\mathcal{L}_D^{(0)}$ with the "minimal substitution" $\partial_\mu \to \partial_\mu + iqA_\mu = D_\mu$, i.e. using covariant derivative D_μ instead of ∂_μ in the dirac lagrangian.
Notice that $\mathcal{L}_D$ exhibits local symmetry, while $\mathcal{L}_D^{(0)}$ doesn't

Besides Lorentz invariance, the QED exhibits following symmetries:

(I) **Global U(1) symmetry**

$$\begin{cases} \psi(x) \to \psi'(x) = e^{i\alpha}\psi(x) & \alpha \in \mathbb{R} \\ A^\mu(x) \to A'^\mu(x) = A^\mu(x) \end{cases}$$

There is therefore an associated conserved Noether current

$$j^\mu = q\overline{\psi}\gamma^\mu\psi \quad \to \quad \partial_\mu j^\mu = 0$$

and a U(1) charge which is conserved by the EM interaction

$$Q = q\int \mathrm{d}^3x \ \psi^\dagger\psi \qquad \frac{\mathrm{d}Q}{\mathrm{d}t} = 0$$

(II) **Local U(1) symmetry** (gauge symmetry)

$$\begin{cases} \psi(x) \to \psi'(x) = e^{iq\alpha(x)}\psi(x) \\ A^\mu \to A'^\mu(x) = A^\mu(x) - iq\partial^\mu\alpha(x) \end{cases}$$

notice that the global U(1) symmetry is a sequence of the local U(1) symmetry, taking $\alpha(x)$ constant)

The covariant derivative of ψ $D_\mu\psi$ behaves as a spinor (remember that D_μ transforms as a vector):

$$\begin{aligned} D_\mu\psi \to (D_\mu\psi)' = D'_\mu\psi' &= (D_\mu - iq\partial_\mu\alpha)(e^{iq\alpha(x)}\psi(x)) \\ &= (\partial_\mu + iqA_\mu - iq\partial_\mu\alpha)(e^{iq\alpha}\psi) \\ &= e^{iq\alpha}(\partial_\mu + iqA_\mu)\psi \\ &= e^{iq\alpha(x)}(D_\mu\psi) \end{aligned}$$

This implies that $\mathcal{L}_D$ is invariant. Since $\mathcal{L}_{EM}$ is invariant, the full lagrangian is invariant

3.2 Flavors in QED and the SU(3) Flavor Global Symmetry

QED describes interactions of the photon field with several kind of leptons, not only electron and positrons. Particles that differs only by their mass are called **flavours**. The next table describes leptons

in QED. There are two families of leptons that differs by their charge. We indicate with (-) (minus) particles with negative charge, and with (+) (plus) particles with positive charge (antiparticles)
Flavours in QED

Leptons[I]	e	μ	τ	ν_e	ν_μ	ν_τ
q	-1	-1	-1	0	0	0
$m[MeV]$	0.5	105	1777	$\simeq 0$	$\simeq 0$	$\simeq 0$

The dirac lagrangian $\mathcal{L}_D$ can be modified in order to consider all possible leptons

$$\mathcal{L}_D = \sum_{i=1}^{n} \bar{\psi}_i(i\not{D} - m_i)\psi_i \simeq \sum_{i=1}^{n_l} \bar{\psi}_i(i\not{D} - m_i)\psi_i + \sum_{j=1}^{n_n} \bar{\psi}_i(i\not{\partial})\psi_i$$

with n: number of leptons, n_l: number of electrically charged particles, n_n: number of neutrinos.
In the last term the interaction term vanishes because of q=0
If we adopt a matrix notation

$$\Psi_C = \begin{pmatrix} \psi_e \\ \psi_\mu \\ \psi_\tau \end{pmatrix} \qquad \Psi_N = \begin{pmatrix} \psi_{\nu_e} \\ \psi_{\nu_\mu} \\ \psi_{\nu_\tau} \end{pmatrix}$$

$$\bar{\Psi}_C = (\bar{\psi}_e, \bar{\psi}_\mu, \bar{\psi}_\tau) \qquad \bar{\Psi}_N = (\bar{\psi}_{\nu_e}, \bar{\psi}_{\nu_\mu}, \bar{\psi}_{\nu_\tau})$$

The subscript C stands for "charged", and N for "neutral"
We can define following matrices

$$M_C = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \qquad Q_C = (-1)\mathbb{1}_{3\times3} \qquad M_N \simeq Q_N = \emptyset_{3\times3}$$

and the generalization of the covariant derivative $D_\mu = \partial_\mu \cdot \mathbb{1}_{3\times3} + iA_\mu Q_C$ we obtain

$$\mathcal{L}_D = \underbrace{\bar{\Psi}_C(i\not{D} - M_C)\Psi_C}_{\mathcal{L}_C} + \underbrace{\bar{\Psi}_N(i\not{\partial}\mathbb{1}_{3\times3})\Psi_N}_{\mathcal{L}_N}$$

The term $\mathcal{L}_C$ and $\mathcal{L}_N$ are respectively the **Charged Sector** and the **Neutral Sector** of $\mathcal{L}_D$
The basis defined by Ψ_C and Ψ_N is called **physical basis**, since physical particles are identified by their mass.

[I]Neutrinos admit anly global U(l) symmetry
Neutrino masses are in the order $m_\nu \approx 10^{-6}me \leq 1eV$

3.2.1 Global Symmetry of Neutral and Charged Sector

Let's consider a U(3) transformation

$$\Psi(x) \to \Psi'(x) = U\Psi(x) \qquad U^\dagger U = \mathbb{1}_{3\times 3}$$

Spinors and vector are left invariant
The neutral sector is left invariant under U(3) tfm

$$\mathcal{L}_N \to \mathcal{L}'_N = \bar{\Psi}'_N (i\not{\partial}\mathbb{1}_{3\times 3})\Psi'_N = \bar{\Psi}_N U^\dagger (i\not{\partial}\mathbb{1}_{3\times 3}) U\Psi_N = \mathcal{L}_N$$

The charged sector is not invariant because of the mass term

$$\mathcal{L} \to \mathcal{L}'_C = \bar{\Psi}_C (i\not{D} - U^\dagger M_C U)\Psi_C \neq \mathcal{L}_C \quad \text{in general}$$

In order to obtain global symmetries of the flavor QED lagrangian, we search the subgroup of U(3) made by matrices $U-g$ that satisfies

$$U_g^\dagger M_C U_g = M_c \qquad U_g \in U(3) \tag{3.1}$$

We can prove that $U(3) \overset{isomorphism}{\simeq} U(1) \times SU(3)$, with

(i) $|\det(\text{U}(3)| = 1$

(ii) $\det(\text{U}(1)) = e^{i\theta}$ [II]

(iii) $\det(\text{SU}(3)) = 1$ [III]

Since generators of SU(3) are Gell-Mann matrices λ_a (for a $= 1, \dots, 8$), generators of U(3) are

$$\mathbb{1} \times \{\lambda_1, \dots, \lambda_8\}$$

we can also prove that equation 3.1 is satisfied only by diagonal matrix
Diagonal generators of SU(3) are

$$\lambda_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Up to a phase, matrices U_g are in the form

$$U_g = e^{i\alpha_0\lambda_0}\ e^{i\alpha_3\lambda_3}\ e^{i\alpha_8\lambda_8}$$

For i = 0, 3 ,8 we have $[\lambda_i, M_c] = 0$ and then the equation 3.1 is satisfied: if we take $\alpha_i \ll 1$

$$(1 - i\alpha_i\lambda_i)M_C(1 + i\alpha_i\lambda_i) = (M_C - i\alpha_i[\lambda_i, M_C] + o(\alpha_i^2)) \simeq M_C$$

We then obtained that the global group of symmetry is generated by the algebra

$$\mathcal{G} = \{\lambda_0, \lambda_3, \lambda_8\} = U(1)^3 \subset U(3)$$

I define the following basic of $\mathcal{G}$:

$$\lambda_e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_\mu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_\tau = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These matrices generates phase tfm for each kind of leptons (λ_e generates phase tfm for e, ecc ...)
Conserved quantities of this group are 3, and corresponding to the number of particles of each type (Antiparticles are counted with negative sign)

[II] Indicando con $e^{i\theta}$ il determinante delle matrici $U_g \in U(3)$
[III] SU(3) è l'insieme di livello dato da $(\arg \circ \det)^{-1}(0)$

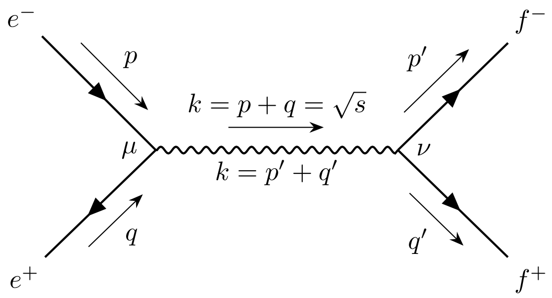
Example 4

$\mu^- \to e^-\gamma$ is forbidden in QED. This is an example of conserved charges due to unitary symmetry that have nothing to do with electric charge.
Flavours changing in neutral sector are forbidden too

3.3 QED Feynman Rules → fogli stampati (23-26)

3.4 $e^+e^- \to f^+f^-$

This diagram is called s-channel



With $k = p' + q'$ we impose the 4 momentum conservation
We have

$$S_{fi} = (2\pi)^4\delta^4(p + q - p' - q')\mathcal{M}_{fi}$$

with Feynman amplitude

$$\begin{aligned}\mathcal{M}_{fi} &= (-iq)^2\left[\bar{u}_{r'}(p')\gamma^\nu\ v_{s'}(q')\right]^{\mathrm{IV}}\left[\bar{v}_s(q)\gamma^\mu\ u_r(p)\right]^{\mathrm{V}}D^F_{\mu\nu}(k)\\ &= (-iq)^2\frac{-ig_{\mu\nu}}{k^2+i\varepsilon}\left[\bar{u}_{r'}(p')\gamma^\nu\ v_{s'}(q')\right]\left[\bar{v}_s(q)\gamma^\mu\ u_r(p)\right]\\ &= \frac{iq^2}{s+i\varepsilon}\left[\bar{u}_{r'}(p')\gamma_\mu\ v_{s'}(q')\right]\left[\bar{v}_s(q)\gamma^\mu\ u_r(p)\right]\end{aligned}$$

In the second passage $\xi = 1$. We can prove that this choice has no importance, see Maggiore pg 187

Using the identity $(\bar{u}\gamma^\mu v)^* = \bar{v}\gamma^\mu u$ (that can be proved by direct calculation using $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$) we obtain (we omit polarization indexes)

$$|\mathcal{M}_{fi}|^2 = \mathcal{M}_{fi}\mathcal{M}*_{fi} = \frac{q^4}{s^2}\big(\bar{u}(p')\gamma_\mu v(q')\big)\ \big(\bar{v}(q)\gamma^\mu u(p)\big)$$
$$=??$$

At this point, we are still free to specify any particular spinors $u_r(p)$, $\bar{v}_{s'}(p')$ and so on, corresponding to any desired spin states of the fermions.

3.4.1 Sum Over Fermion Spins. Squared Averaged Feynman Amplitude

The Feynman amplitude simplifies considerably when we throw away the spin information. We want to compute

$$\overline{|\mathcal{M}_{fi}|^2} = \underbrace{\frac{1}{2}\sum_s\frac{1}{2}\sum_r}_{\substack{\text{average over}\\ \text{the initial states}}}\ \underbrace{\sum_{s'}\sum_{r'}}_{\substack{\text{sum over}\\ \text{final states}}}|\mathcal{M}(r,s\to r',s')|^2$$

[V] indica il percorso $e^- \to \mu \to e^+$ nel diagramma
[V] indica il percorso $f^- \to \nu \to f^+$ nel diagramma

This sum can be performed using completeness relations for dirac spinors

$$\sum_r u_s(p)\bar{u}_s(p) = \not{p} + m \qquad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m$$

Writing spinors indexes explicitly

$$\begin{aligned}\sum_{rs} \text{e-current} &= \sum_{rs} \bar{v}^s_a(q)(\gamma^\mu)_{ab}\, u^r_b(p)\, \bar{u}^r_c(p)(\gamma^\nu)_{cd}\, v^s_d(q) \\ &= (\not{q} - m)_{da}\, \gamma^\mu_{ab}\, (\not{p} + m)_{bc}\, \gamma^\nu_{cd} \\ &= \mathrm{Tr}\big[(\not{q} - m)\gamma^\mu(\not{p} + m)\gamma^\nu\big]\end{aligned}$$

and similarly

$$\sum_{r's'} \text{f-current} = \mathrm{Tr}\big[(\not{p}' + m)\gamma_\mu(\not{q}' - m)\gamma_\nu\big]$$

So we obtain

$$\overline{|\mathcal{M}_{fi}|^2} = \frac{q^4}{4s^2}\mathrm{Tr}\big[(\not{q} - m)\gamma^\mu(\not{p} + m)\gamma^\nu\big]\mathrm{Tr}\big[(\not{p}' + m)\gamma_\mu(\not{q}' - m)\gamma_\nu\big]$$

The spinors u and v have disappeared, leaving us with a much cleaner expression in terms of γ matrices. This trick is very genera: any QED amplitude involving external fermions, when squared and summed or averaged over spins, can be converted in this way to traces of products of Dirac matrices

There is a trick to obtain previous formula using only Feynman rules. If we set $\mathcal{M}_{fi} = \mathcal{M}_e \mathcal{M}_f$ (we divide $\mathcal{M}_{fi}$ in 2 factors related to e and f), then $|\mathcal{M}_{fi}|^2 = \mathcal{M}_e^* \mathcal{M}_e\, \mathcal{M}_f^* \mathcal{M}_f$. We have

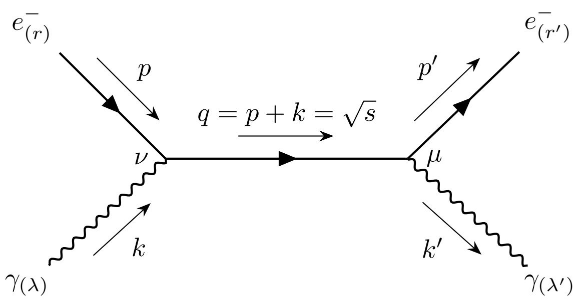
Feynman diagram

3.5 $e^-\gamma \to e^-\gamma$ (Compton)

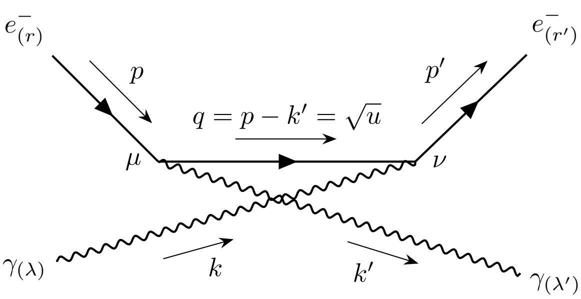
See Peskin, sec 5.5

Let's examinate a process with external bosons: *Compton scattering*, or $e^-\gamma \to e^-\gamma$. This process is described by two independent diagrams, since they are topologically different:

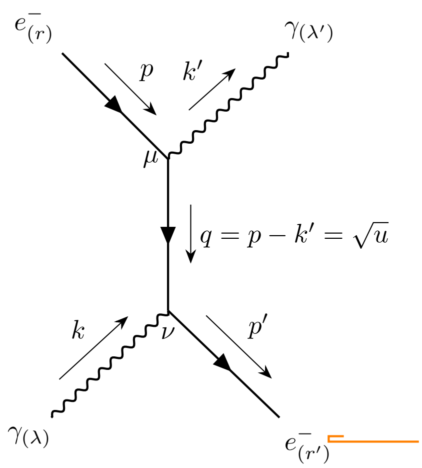
$\mathcal{M}_A =$



$\mathcal{M}_B =$



$=$



We wrote the diagram of $\mathcal{M}_B$ in two topologically equivalent forms: in the first one is clear the topological relation with diagram of $\mathcal{M}_A$ (this is useful to find the relative sign between diagrams A and B: it's clear that diagrams differs for the permutation of two bosons), while in the second one is clear that it describes a u-channel.

Amplitudes reads, using Feynman rules

$$\begin{aligned}\mathcal{M}_A &= \bar{u}_{r'}(p')(-iq\gamma^\mu)\varepsilon_\mu^{\lambda'*}(k')\tilde{S}_F(p+k)(-iq\gamma^\nu)\varepsilon_\nu^\lambda(k)u_r(p) \\ &= -q^2\varepsilon_\mu^{\lambda'*}(k')\varepsilon_\nu^\lambda(k)\left[\bar{u}_{r'}(p')\gamma^\mu\tilde{S}_F(p+k)\gamma^\nu u_r(p)\right]\end{aligned}$$

$$\mathcal{M}_B = -q^2\varepsilon_\mu^{\lambda'*}(k')\varepsilon_\nu^\lambda(k)\left[\bar{u}_{r'}(p')\gamma^\nu\tilde{S}_F(p-k')\gamma^\mu u_r(p)\right]$$

(Recall that $\tilde{S}_F$ is a matrix, so elements in the squared braket must be written in this order)
Because of anticommuting relations for bosons, these amplitudes must be summed up in the total amplitude. The explicit form of Feynman propagator for the Dirac field reads

$$\tilde{S}_F(p) = \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon} = \frac{i}{\not{p}-m+i\varepsilon}$$

so total amplitude is

$$\mathcal{M} = -iq^2\varepsilon_\mu^{\lambda'*}(k')\varepsilon_\nu^\lambda(k)\bar{u}_{r'}(p')\left[\frac{\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu}{(p+k)^2-m^2} + \frac{\gamma^\nu(\not{p}-\not{k}'+m)\gamma^\mu}{(p-k')^2-m^2}\right]u_r(p)$$

We make some simplifications before squaring this expression. Since $p^2 = m^2$ and $k^2 = 0$:

$$(p+k)^2 - m^2 = 2p\cdot k \qquad (p-k')^2 - m^2 = -2p\cdot k'$$

To simplify numerators, I can use Dirac algebra:

$$\begin{aligned}(\not{p}+m)\gamma^\nu u(p) &= (p_\mu\gamma^\mu\gamma^\nu + m\gamma^\nu)u(p) = (2g^{\mu\nu}p_\mu - p_\mu\gamma^\nu\gamma^\mu + m\gamma^\nu)u(p) \\ &= 2p^\nu u(p) - \gamma^\nu\underbrace{(\not{p}-m)}_{2m\,\Lambda_-(p)}u(p) = 2p^\nu u(p)\end{aligned}$$

Using these tricks we obtain

$$\mathcal{M} = -iq^2\varepsilon_\mu^{\lambda'*}(k')\varepsilon_\nu^\lambda(k)\bar{u}_{r'}(p')\left[\frac{\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{-\gamma^\nu\not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{-2p\cdot k'}\right]u_r(p)$$

3.5.1 The Ward Identities and sum over the photon polarizations

See Mandl, sec 8.3
The next step in the calculation will be to square this expression for $\mathcal{M}$ and sum or average over electron and photon polarization states. The sum over electron polarizations can be performed as before, using $\sum u(p)\bar{u}(p) = \not{p}+m$. Fortunately, there is a similar trick for summing over photons polarization vectors. Gauge invariance of the theory implies the gauge invariance of the matrix elements, i.e. of the Feynman amplitudes. It is, of course, only the matrix element itself, corresponding to the sum of all possible Feynman graphs in a given order of perturbation theory, which must be gauge invariant. For example, for the Compton scattering, the individual amplitudes $\mathcal{A}$ and $\mathcal{B}$ are not gauge invariants, but their sum $\mathcal{M}$ is.
For any process involving external photons, the Feynman amplitude $\mathcal{M}$ is of the form

$$\mathcal{M} = \varepsilon_\alpha^{\lambda_1}(k_1)\varepsilon_\beta^{\lambda_2}(k_2)\ldots L^{\alpha\beta\cdots}(k_1,k_2,\ldots) \tag{3.2}$$

with one polarization vector $\varepsilon^{\lambda_i}(k_i)$ for each external photon, and the tensor amplitude $L^{\alpha\beta\cdots}(k_1,k_2,\ldots)$ independent of these polarization vectors.

The polarization vectors are of course gauge dependent. For example, for a free photon described in the Lorentz gauge by the plane wave

$$A^\mu(x) = \text{const} \cdot \varepsilon^\mu_\lambda(k) e^{\pm ikx}$$

the gauge transformation

$$A^\mu \to A'^\mu(x) = A^\mu(x) + \partial^\mu \alpha(x) \qquad \text{with} \quad \alpha(x) = \tilde{\alpha}(k) e^{\pm ikx}$$

implies

$$\varepsilon^\mu_\lambda(k) \to \varepsilon'^\mu_\lambda(k) = \varepsilon^\mu_\lambda(k) \pm i k^\mu \tilde{\alpha}(k)$$

Invariance of the amplitude Eq.(3.2) under this transformation requires

$$k_1^\alpha L_{\alpha,\beta,\dots}(k_1, k_2, \dots) = k_1^\beta L_{\alpha,\beta,\dots}(k_1, k_2, \dots) = \dots = 0$$

i.e. when any external photon polarization vector is replaced by the corresponding four momentum, the amplitude must vanish. This is the statement of the *Ward Identity*:

Sistemare grafica

> If $\mathcal{M}(k) = \varepsilon_\mu(k) L^\mu(k)$ is the amplitude for some QED process involving an external photon with momentum k, then this amplitude vanishes if we replace ε_μ with k_μ:
>
> $$k_\mu L^\mu(k) = 0$$

Creare ambiente exercise, questo è un esercizio

Example 5
Verify explicitly the Ward Identity for the Feynman amplitude of Compton scattering

See Peskin, sec 5.5

Returning to our derivation of the polarization sum formula for squared scattering amplitude. Writing in general

$$\mathcal{M} = \varepsilon^{(\lambda)}_\mu(k) L^\mu(k)$$

then the sum over polarizations of the photon with momentum k reads

$$\sum_{\lambda=1,2} |\mathcal{M}|^2 = \sum_{\lambda=1,2} \varepsilon^{(\lambda)}_\mu(k) \varepsilon^{(\lambda)*}_\nu(k) L^\mu(k) L^{\nu\dagger}(k)$$

Because of the covariance of the theory we can do the calculation in a specific frame. In order to simplify the analysis we choose the frame where the photon moves along the $\hat{z}$ axis:

$$k^\mu = (|k|, 0, 0, |k|)$$

In this case the Ward Identity reads

$$0 = k_\mu L^\mu = |k| \left(L^0 - L^3\right) \qquad \longrightarrow \qquad L^0 = L^3$$

Recall that in this frame

$$\varepsilon^{(1)}_\mu(k) = (0, 1, 0, 0) \qquad \varepsilon^{(2)}_\mu(k) = (0, 0, 1, 0)$$

So we have

$$\sum_{\lambda=1,2} \varepsilon^{(\lambda)}_\mu(k) \varepsilon^{(\lambda)*}_\nu(k) L^\mu(k) L^{\nu\dagger}(k) = \left|L^1\right|^2 + \left|L^2\right|^2 = \left|L^1\right|^2 + \left|L^2\right|^2 + \left|L^3\right|^2 - \left|L^0\right|^2 = -g_{\mu\nu} L^\mu L^\nu$$

So we obtained the general rule to simplify photons polarization sum[VI]

Creare ambiente per riquadrare le formule

$$\sum_{\lambda=1,2} \varepsilon^{(\lambda)}_\mu(k) \varepsilon^{(\lambda)*}_\nu(k) L^\mu(k) L^{\nu\dagger}(k) \quad \longrightarrow \quad -g_{\mu\nu}$$

[VI] Notice that we could prove (see Peskin) that even if we took $\lambda = 0, 1, 2, 3$, we could have obtained that the unphysical time-like and longitudinal photons can be consistently omitted from QED calculations, since in any event the squared amplitudes for producing these states cancel to give zero total probability.

3.5.2 The Klein-Nishima formula and the Thomson scattering

See Peskin, sec. 5.5

To obtain the unpolarized cross section for Compton scattering, we use the covariant method described in the previous section. Writing

$$\mathcal{M} = \varepsilon_{\mu}^{\lambda' *}(k')\varepsilon_{\nu}^{\lambda}(k) \left(L^{\mu\nu}(k,k')\right)_{r,r'}$$

with

$$\left(L^{\mu\nu}(k,k')\right)_{r,r'} = -iq^2 \bar{u}_{r'}(p') \left[\frac{\gamma^{\mu}\not{k}\gamma^{\nu} + 2\gamma^{\mu}p^{\nu}}{2p\cdot k} + \frac{-\gamma^{\nu}\not{k}'\gamma^{\mu} + 2\gamma^{\nu}p^{\mu}}{-2p\cdot k'} \right] u_r(p)$$

we obtain

$$\begin{aligned}
|\bar{\mathcal{M}}|^2 &= \frac{1}{4}\left(\sum_{\lambda'} \varepsilon_{\mu}^{(\lambda')*}(k')\varepsilon_{\rho}^{(\lambda')}(k')\right)\left(\sum_{\lambda} \varepsilon_{\nu}^{(\lambda)*}(k)\varepsilon_{\sigma}^{(\lambda)}(k')\right)\sum_{r,r'} \left(L^{\mu\nu}\right)_{r,r'} \left(L^{\rho\sigma}\right)^{\dagger}_{r,r'} \\
&= \frac{1}{4} g_{\mu\rho}g_{\nu\sigma} \sum_{r,r'} \left(L^{\mu\nu}\right)_{r,r'} \left(L^{\rho\sigma}\right)^{\dagger}_{r,r'} = \frac{1}{4}\left(L^{\mu\nu}\right)_{r,r'} \left(L_{\mu\nu}\right)^{\dagger}_{r,r'} \\
&= \frac{q^4}{4}\mathrm{Tr}\Bigg[(\not{p}'+m)\left(\frac{\gamma^{\mu}\not{k}\gamma^{\nu} + 2\gamma^{\mu}p^{\nu}}{2p\cdot k} + \frac{\gamma^{\nu}\not{k}'\gamma^{\mu} - 2\gamma^{\nu}p^{\mu}}{2p\cdot k'}\right) \times \\
&\qquad \times (\not{p}+m)\left(\frac{\gamma_{\nu}\not{k}\gamma_{\mu} + 2\gamma_{\mu}p_{\nu}}{2p\cdot k} + \frac{\gamma_{\mu}\not{k}'\gamma_{\nu} - 2\gamma_{\nu}p_{\mu}}{2p\cdot k'}\right)\Bigg] \\
&= \frac{q^4}{4}\left\{\frac{T_{AA}}{(2p\cdot k)^2} + \frac{T_{BB}}{(2p\cdot k')^2} + \frac{T_{AB}+T_{BA}}{(2p\cdot k)(2p\cdot k')}\right\}
\end{aligned}$$

where

$$\begin{aligned}
T_{AA} &= \mathrm{Tr}\left[(\not{p}'+m)(\gamma^{\mu}\not{k}\gamma^{\nu} + 2\gamma^{\mu}p^{\nu})(\not{p}+m)(\gamma_{\nu}\not{k}\gamma_{\mu} + 2\gamma_{\mu}p_{\nu})\right] \\
T_{BB} &= \mathrm{Tr}\left[(\not{p}'+m)(\gamma^{\nu}\not{k}'\gamma^{\mu} - 2\gamma^{\nu}p^{\mu})(\not{p}+m)(\gamma_{\mu}\not{k}'\gamma_{\nu} - 2\gamma_{\nu}p_{\mu})\right] \\
T_{AB} &= \mathrm{Tr}\left[(\not{p}'+m)(\gamma^{\mu}\not{k}\gamma^{\nu} + 2\gamma^{\mu}p^{\nu})(\not{p}+m)(\gamma_{\mu}\not{k}'\gamma_{\nu} - 2\gamma_{\nu}p_{\mu})\right] \\
T_{BA} &= \mathrm{Tr}\left[(\not{p}'+m)(\gamma^{\nu}\not{k}'\gamma^{\mu} - 2\gamma^{\nu}p^{\mu})(\not{p}+m)(\gamma_{\nu}\not{k}\gamma_{\mu} + 2\gamma_{\mu}p_{\nu})\right]
\end{aligned}$$

Notice that $T_{BB} = T_{AA}(k \leftrightarrow -k')$ and $T_{BA} = T_{AB}(k \leftrightarrow -k')$, we need therefore only calculate T_{AA} and T_{AB}.
Considering T_{AA}, there are 16 terms inside the trace, but half contains an odd number of γ matrices and therefore vanishes. Other terms are

$$\begin{aligned}
1 &= \mathrm{Tr}[\not{p}'\gamma^{\mu}\not{k}\gamma^{\nu}\not{p}\gamma_{\nu}\not{k}\gamma_{\mu}] \\
2 &= 2\mathrm{Tr}[\not{p}'\gamma^{\mu}\not{k}\gamma^{\nu}\not{p}\gamma_{\mu}p_{\nu}] = 2\mathrm{Tr}[\not{p}'\gamma^{\mu}\not{k}\not{p}\not{p}\gamma_{\mu}] \\
3 &= 2\mathrm{Tr}[\not{p}'\gamma^{\mu}p^{\nu}\not{p}\gamma_{\nu}\not{k}\gamma_{\mu}] = 2\mathrm{Tr}[\not{p}'\gamma^{\mu}\not{p}\not{p}\not{k}\gamma_{\mu}] \\
4 &= 4\mathrm{Tr}[\not{p}'\gamma^{\mu}p^{\nu}\not{p}\gamma_{\mu}p_{\nu}] = 4p^2\mathrm{Tr}[\not{p}'\gamma^{\mu}\not{p}\gamma_{\mu}] \\
5 &= m^2\mathrm{Tr}[\gamma^{\mu}\not{k}\gamma^{\nu}\gamma_{\nu}\not{k}\gamma_{\mu}] \\
6 &= 2m^2\mathrm{Tr}[\gamma^{\mu}\not{k}\gamma^{\nu}\gamma_{\mu}p_{\nu}] = 2m^2\mathrm{Tr}[\gamma^{\mu}\not{k}\not{p}\gamma_{\mu}] \\
7 &= 2m^2\mathrm{Tr}[\gamma^{\mu}p^{\nu}\gamma_{\nu}\not{k}\gamma_{\mu}] = 2m^2\mathrm{Tr}[\gamma^{\mu}\not{p}\not{k}\gamma_{\mu}] \\
8 &= 4m^2\mathrm{Tr}[\gamma^{\mu}p^{\nu}\gamma_{\mu}p_{\nu}] = 4m^2p^2\mathrm{Tr}[\gamma^{\mu}\gamma_{\mu}]
\end{aligned}$$

In order to simplify above formulas we recall the proprieties of contractions of γ matrices, i.e. products in the form $\gamma^{\mu}A\gamma^{\mu}$ where A is a matrix:

(i) $\gamma^\mu \gamma_\mu = 4\mathbb{1}$

(ii) $\gamma^\mu \not{p} \gamma_\mu = -2\not{p}$

(iii) $\gamma^\mu \not{p}\not{q} \gamma_\mu = 4p \cdot q$

(iv) $\gamma^\mu \not{p}\not{q}\not{k} \gamma_\mu = -2\not{k}\not{q}\not{p}$

Using these proprieties, cyclicity of the trace and anticommuting proprieties of gamma matrices[VII], we obtain (remember that $p^2 = m^2$ and $k^2 = 0$):

$$
\begin{aligned}
&1 = \mathrm{Tr}[\not{p}'\gamma^\mu \not{k}\gamma^\nu \not{p}\gamma_\nu \not{k}\gamma_\mu] = -2\mathrm{Tr}[\not{p}'\gamma^\mu \not{k}\not{p}\not{k}\gamma_\mu] = 4\mathrm{Tr}[\not{p}'\not{k}\not{p}\not{k}] = -4\mathrm{Tr}[\not{p}'\not{k}^2\not{p}] + 8(p\cdot k)\mathrm{Tr}[\not{p}'\not{k}] = 32(p\cdot k)(p'\cdot k)\\
&2 = 2\mathrm{Tr}[\not{p}'\gamma^\mu \not{k}\not{p}\not{p}\gamma_\mu] = -4\mathrm{Tr}[\not{p}'\not{p}\not{p}\not{k}] = -4m^2\mathrm{Tr}[\not{p}'\not{k}] = -16m^2(p'\cdot k)\\
&3 = 2\mathrm{Tr}[\not{p}'\gamma^\mu \not{p}\not{p}\not{k}\gamma_\mu] = 2m^2\mathrm{Tr}[\not{p}'\gamma^\mu \not{k}\gamma_\mu] = -4m^2\mathrm{Tr}[\not{p}'\not{k}] = -16m^2(p'\cdot k)\\
&4 = 4p^2\mathrm{Tr}[\not{p}'\gamma^\mu \not{p}\gamma_\mu] = -8m^2\mathrm{Tr}[\not{p}'\not{p}] = -32m^2(p'\cdot p)\\
&5 = m^2\mathrm{Tr}[\gamma^\mu \not{k}\gamma^\nu \gamma_\nu \not{k}\gamma_\mu] = 4m^2\mathrm{Tr}[\gamma^\mu \not{k}\not{k}\gamma_\mu] = 0\\
&6 = 2m^2\mathrm{Tr}[\gamma^\mu \not{k}\not{p}\gamma_\mu] = 8m^2(k\cdot p)\mathrm{Tr}[\mathbb{1}] = 32m^2(k\cdot p)\\
&7 = 2m^2\mathrm{Tr}[\gamma^\mu \not{p}\not{k}\gamma_\mu] = 8m^2(p\cdot k)\mathrm{Tr}[\mathbb{1}] = 32m^2(p\cdot k)\\
&8 = 4m^2p^2\mathrm{Tr}[\gamma^\mu \gamma_\mu] = 16m^4\mathrm{Tr}[\mathbb{1}] = 64m^4
\end{aligned}
$$

At the end we find

$$
\begin{aligned}
T_{AA} &= 16\left(4m^4 - 2m^2 p\cdot p' + 4m^2 p\cdot k - 2m^2 p'\cdot k + 2(p\cdot k)(p'\cdot k)\right)\\
&= 16\left(2m^4 + m^2(s-m^2) - \frac{1}{2}(s-m^2)(u-m^2)\right)
\end{aligned}
$$

where we introduced Mandelstam variables:

$$
\begin{aligned}
s &= (p+k)^2 = 2p\cdot k + m^2 = 2p'\cdot k' + m^2\\
t &= (p'-p)^2 = -2p\cdot p' + 2m^2 = -2k\cdot k'\\
u &= (k'-p)^2 = -2k'\cdot p + m^2 = -2k\cdot p' + m^2
\end{aligned}
$$

Sending $k \leftrightarrow -k'$ ($s \leftrightarrow u$) we can immediately write

$$
\begin{aligned}
T_{BB} &= 16\left(4m^4 - 2m^2 p\cdot p' - 4m^2 p\cdot k' + 2m^2 p'\cdot k' + 2(p\cdot k')(p'\cdot k')\right)\\
&= 16\left(2m^4 + m^2(u-m^2) - \frac{1}{2}(u-m^2)(s-m^2)\right)
\end{aligned}
$$

[VII] $\not{A}\not{B} = A_\mu B_\nu \gamma^\mu\gamma^\nu = A_\mu B_\nu(2g^{\mu\nu}\mathbb{1} - \gamma^\nu\gamma^\mu) = 2(A\cdot B)\,\mathbb{1} - \not{B}\not{A} \quad \rightarrow \quad \not{A}\not{A} = A^2\,\mathbb{1}$
$\mathrm{Tr}[\not{A}\not{B}] = 2(A\cdot B)\,\mathrm{Tr}[\mathbb{1}] - \mathrm{Tr}[\not{B}\not{A}] = 8(A\cdot B) - \mathrm{Tr}[\not{A}\not{B}] \quad \rightarrow \quad \mathrm{Tr}[\not{A}\not{B}] = 4(A\cdot B)\mathbb{1}$