Chapter 1

QED Processes at Lowest order

1.1 The QED Lagrangian and its Symmetries

Mandl, sec 11.1 - Maggiore, sec 7.1
Quantum electrodynamic (QED) describes the interactions between (or any other charged spin 1/2 particle) and photons. QED is described by the lagrangian

$$\mathcal{L}_{\text{QED}} = \underbrace{\overline{\psi}(i\partial\!\!\!/ - m)\psi}_{\mathcal{L}_D^{(0)}} \underbrace{- \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\mathcal{L}_{EM}} \underbrace{- qA_n\overline{\psi}\gamma^\mu\psi}_{\mathcal{L}_{int}} \underbrace{- \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2}_{\mathcal{L}_{GF}}$$

(i) $\mathcal{L}_D^{(0)}$ is the langrangian for the free Dirac field

(ii) $\mathcal{L}_{EM}$ is the lagrangian for the free EM field. In order to quatize the E-n field we have to add the term $\mathcal{L}_{GF}$ (gauge fixing). For other purposes this term can be omitted. Usually the choice $\xi = 1$, called Feynman gauge, is the simplest choice for quantization

(iii) $\mathcal{L}_{int}$ describes the interaction between Dirac field and EM-field. Notice that the term $\mathcal{L}_D = \mathcal{L}_D^{(0)} + \mathcal{L}_{int}$ can be obtained from $\mathcal{L}_D^{(0)}$ with the "minimal substitution" $\partial_\mu \to \partial_\mu + iqA_\mu = D_\mu$, i.e. using covariant derivative D_μ instead of ∂_μ in the dirac lagrangian.
Notice that $\mathcal{L}_D$ exhibits local symmetry, while $\mathcal{L}_D^{(0)}$ doesn't

Besides Lorentz invariance, the QED exhibits following symmetries:

(I) **Global U(1) symmetry**

$$\begin{cases} \psi(x) \to \psi'(x) = e^{i\alpha}\psi(x) & \alpha \in \mathbb{R} \\ A^\mu(x) \to A'^\mu(x) = A^\mu(x) \end{cases}$$

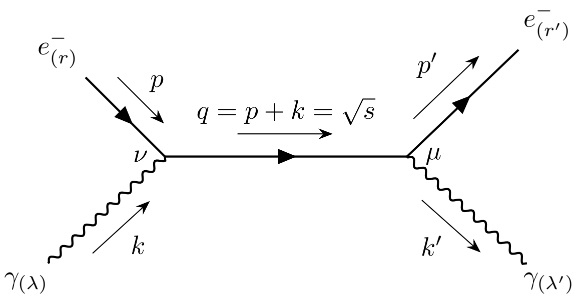
There is therefore an associated conserved Noether current

$$j^\mu = q\bar{\psi}\gamma^\mu\psi \quad \to \quad \partial_\mu j^\mu = 0$$

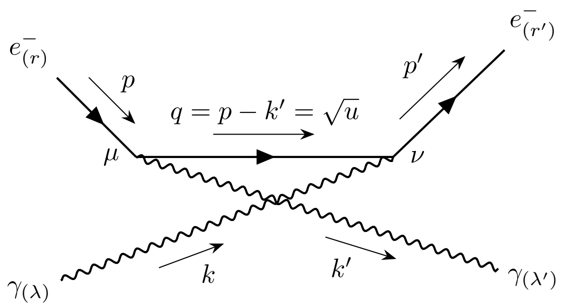
1.2 $e^-\gamma \to e^-\gamma$ (Compton)

See Peskin, sec 5.5
Let's examine a process with external bosons: *Compton scattering*, or $e^-\gamma \to e^-\gamma$. This process is described by two independent diagrams, since they are topologically different:

$\mathcal{M}_A =$



$\mathcal{M}_B =$



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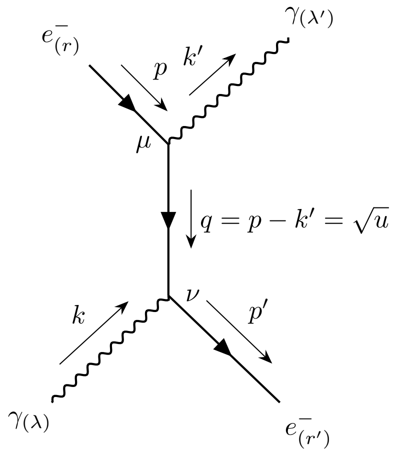


We wrote the diagram of $\mathcal{M}_B$ in two topologically equivalent forms: in the first one is clear the topological relation with diagram of $\mathcal{M}_A$ (this is useful to find the relative sign between diagrams A and B: it's clear that diagrams differs for the permutation of two bosons), while in the second one is clear that it describes a u-channel.

Amplitudes reads, using Feynman rules

$$\begin{aligned}
\mathcal{M}_A &= \bar{u}_{r'}(p')(-iq\gamma^\mu)\varepsilon_\mu^{\lambda'*}(k')\tilde{S}_F(p+k)(-iq\gamma^\nu)\varepsilon_\nu^\lambda(k)u_r(p) \\
&= -q^2\varepsilon_\mu^{\lambda'*}(k')\varepsilon_\nu^\lambda(k)\left[\bar{u}_{r'}(p')\gamma^\mu\tilde{S}_F(p+k)\gamma^\nu u_r(p)\right]
\end{aligned}$$

$$\mathcal{M}_B = -q^2\varepsilon_\mu^{\lambda'*}(k')\varepsilon_\nu^\lambda(k)\left[\bar{u}_{r'}(p')\gamma^\nu\tilde{S}_F(p-k')\gamma^\mu u_r(p)\right]$$

(Recall that $\tilde{S}_F$ is a matrix, so elements in the squared braket must be written in this order)
Because of anticommuting relations for bosons, these amplitudes must be summed up in the total amplitude. The explicit form of Feynman propagator for the Dirac field reads

$$\tilde{S}_F(p) = \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon} = \frac{i}{\not{p}-m+i\varepsilon}$$

so total amplitude is

$$\mathcal{M} = -iq^2\varepsilon_\mu^{\lambda'*}(k')\varepsilon_\nu^\lambda(k)\bar{u}_{r'}(p')\left[\frac{\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu}{(p+k)^2-m^2} + \frac{\gamma^\nu(\not{p}-\not{k}'+m)\gamma^\mu}{(p-k')^2-m^2}\right]u_r(p)$$

1.3 Sezioni di esempio

Consider the case in which the initial state is a single particle and the final state is given by n particles. We are therefore considering a decay process. Assume for the moment that particles are indistinguishable.

The rules of quantum mechanics tell us that the probability for this process is obtained by taking the squared modulus of the amplitude and summing over all possible final states

$$\begin{aligned} \left|S_{fi}^{CN}\right|^2 &= \left|(2\pi)^4\delta^4(p-p')M_{fi}^{CN}\right|^2 \\ &= (2\pi)^4\delta^4(p-p')(VT)\left|M_{fi}^{CN}\right|^2 \\ &= (2\pi)^4\delta^4(p-p')(VT)\frac{1}{2\omega_i nV}\prod_{l=1}^{n_f}\left(\frac{1}{2\omega_l V}\right)|\mathcal{M}_{fi}|^2 \end{aligned}$$

Note: $\left(\delta^4(p-p')\right)^2 = \delta^4(p-p')\delta^4(p-p') = \delta^4(p-p')\delta^4(0) = \delta^4(p-p')\frac{VT}{(2\pi)^4}$
We use the final space and time in order to remove divergent terms during calculation

We must now sum this expression over all final states. Since we are working in a finite volume V, this is the sum over the possible discrete values of the momenta of the final particles .
Since $p_i = (2\pi/L)n_i$, we have $dn_i = (L/2\pi)\mathrm{d}p_i$ and $\mathrm{d}^3n_i = (V/(2\pi)^3)\mathrm{d}^3p$ where d^3n_i is the infinitesimal phase space related to a final state in which the i-th particle has momentum between p_i and $p_i + \mathrm{d}p_i$
Let $\mathrm{d}\omega$ be the probability for a decay in which in the final state the i-th particle has momentum between p_i and $p_i\mathrm{d}p_i$

$$\mathrm{d}\omega = \left|S_{fi}^{SN}\right|^2\prod_{l=1}^{n_f}\left(\frac{V\mathrm{d}^3p_l}{(2\pi)^3}\right)$$

This is the probability that the decay takes place in any time between $-T/2$ and $T/2$. We are more interested in the differential decay rate $\mathrm{d}\Gamma_{fi}$, which is the decay probability per unit of time:

$$\mathrm{d}\Gamma_{fi} = \frac{\mathrm{d}\omega}{T} = (2\pi)^4\delta^4(p_i-p_f)\frac{|\mathcal{M}_{fi}|^2}{2\omega_{p_{in}}}\prod_{l=1}^{n_f}\frac{\mathrm{d}^3p_l}{(2\pi)^3 2\omega_l}$$

Notes:

(i) $\mathrm{d}\Gamma_{fi}$ = differential decay rate

(ii) p_f = sum over final momenta

(iii) $\omega_{p_{in}}$ = initial energy

(iv) $|\mathcal{M}_{fi}|^2$ = Feynman amplitude of the process (depends on final momenta p_i)

It is useful to define the **(differential) n-body phase space** as

$$\mathrm{d}\Phi_{(n_f)} = (2\pi)^4\delta^4(p_i-p_f)\prod_{l=1}^{n_f}\frac{\mathrm{d}^3p_l}{(2\pi)^3 2\omega_l}$$

Therefore the differential decay rate can be written as

$$\mathrm{d}\Gamma_{fi} = \frac{1}{2\omega_{p_{in}}} |\mathcal{M}_{fi}|^2 \, \mathrm{d}\Phi_{(n_f)}$$

The decay rate is defined as

$$\Gamma_{fi} = \int \mathrm{d}\Gamma_{fi} \quad \rightarrow \text{integration over all possible final momenta}$$

and its meaning is $\Gamma \equiv$ trans. probability $\times$ unit of time $\times$ init. particle
Notice that if n of the final particles are identical, configurations that differ by a permutation are not distinct and therefore the phase space is reduced by a factor $1/h!$
If we have a system of N(0) particles, the time evolution of the number of particles N(t) is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\Gamma N \quad \Rightarrow \quad N(t) = N(0)e^{-\Gamma t}$$

Notice that decay rate is not invariant

$$[\Gamma] = [E] = \frac{1}{T} \qquad \text{(in natural units)}$$

If we define the **lifetime** as $\tau = 1/\Gamma \Rightarrow N(t) = N(0)\exp(-t/\tau)$ this changes under Lorentz tfm. If we consider two reference frames o and o'

$$\Gamma' = \frac{\Gamma}{\gamma} < \Gamma \qquad \tau' = \gamma\tau > \tau$$

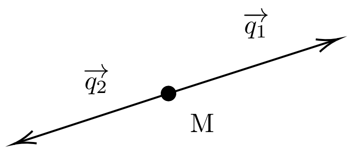
$\gamma = (1-v)^{-1/2}$, where v is the speed of o' in o in natural units, $\gamma > 1$. Therefore a article in a moving frame has a longer lifetime then in the rest frame

Example 1: muon lifetime

For a muon in the rest frame $\tau_\mu^{RF} = 2.2 \times 10^{-6}s$, but if we observe it in the lab frame $\tau_\mu^{LAB} = \gamma\tau_\mu^{RF} \simeq 2 \times 10^{-5}s$ since $E_\mu = 1$ GeV, $m_\mu = 0.1$ GeV
$\Rightarrow \gamma = E_\mu/m_\mu \simeq 10$

Example 2: 1 $\rightarrow$ 2 decay

Consider the decay of a particle of a mass M into two particles of masses m_1, m_2. Since the phase space in Lorentz invariant, we can compute it in the frame that we prefer. We use the rest frame for the initial particle.



We don't impose a priori conservation of momentum since it's imposed by the delta function.

$$p = (M, 0) \qquad q_1 = (\omega_1, \mathbf{q_1}) \qquad q_2 = (\omega_2, q_2)$$
$$\mathrm{d}\Phi_{(2)} = (2\pi)^4\delta^4 \underbrace{(P_i - P_f)}_{=p-q_1-q_2} \frac{\mathrm{d}^3 q_1}{(2\pi)^3 2\omega_1} \frac{\mathrm{d}^3 q_2}{(2\pi)^3 2\omega_2}$$

I have 6 integration parameters, 4 constraint given by δ^4, so I have 2 independent variables.

Integrating over $\mathrm{d}^3 q_2$

$$\mathrm{d}\Phi'_{(2)} = \int \mathrm{d}\Phi_{(2)} = \frac{1}{(2\pi)^2}\delta(M-\omega_1-\omega_2)\frac{1}{4\omega_1\omega_2}\mathrm{d}^3 q_1$$

in this way, the condition $\mathbf{q_2} = \mathbf{q_1}$ vanish. We have to impose it again when we calculate $\mathrm{d}\Gamma$ (we omit this detail)
Usually the 4-th non independent parameter is eliminated by integration over modulus of q_1, leaving free 2 parameters for the angles. $\mathrm{d}^3 q_1 \to \mathbf{q_1}^2 \mathrm{d}\,|\mathbf{q_2}|\,\mathrm{d}\Omega_1$.

Notice that $M-\omega_1-\omega_2 = M - \sqrt{\mathbf{q_1}^2+m_1^2} - \sqrt{\mathbf{q_2}+m_2^2}$ and then the δ implies

$$\hat{q_1}^2 = \frac{1}{2M}\left(M^4 - 2M^2(m_1^2+m_2^2) + (m_1^2-m_2^2)^2\right)^{1/2}$$

$|\mathbf{q_1}|$ is the only zero of $f(|\hat{q_1}|) = M-\omega_1-\omega_2$.
We also have

$$|f'(|\hat{q_1}|)| = \frac{\partial\omega_1}{\partial\,|\mathbf{q_1}|} + \frac{\partial\omega_2}{\partial\,|\mathbf{q_1}|} = |\hat{q_1}|\left(\frac{\omega_1+\omega_2}{\omega_1\omega_2}\right)$$

Using

$$\delta\big(f(x)\big) = \sum_{x_0=\text{zero of f(x)}} \frac{\delta(x-x_0)}{|f'(x_0)|}$$

and performing integration over $\mathrm{d}\,|\mathbf{q_1}|$ we obtain

$$\mathrm{d}\Phi''_{(2)} = \int \mathrm{d}\Phi'_{(2)} = \frac{1}{16\pi^2}\frac{|\hat{q_1}|}{M}\mathrm{d}\Omega_1$$

Using this result we obtain the $1 \to 2$ decay rate in function of the solid angle (in the rest frame)

$$\left(\frac{\mathrm{d}\Gamma_{RF}}{\mathrm{d}\Omega}\right) = \frac{1}{64\pi^4 M^3}\left[M^4 - 2M^2(m_1^2+m_2^2) + (m_1-m_2^2)^2\right]^{1/2}|\mathcal{M}_{RF}|^2$$

In a general frame we can easily obtain an analogous formula, just consider $\mathrm{d}\Gamma = 1/(2\omega_i n)\,|\mathcal{M}_{fi}|^2\,\mathrm{d}\Phi_{(nf)}$ in a general frame. remember that $\mathrm{d}I_{(nf)}$ is invariant

We have 2 important limit cases:

(A) If $m_1 = m_2 = m$ (for example $Z \to e^+e^-$)

$$|\hat{q_1}| = \frac{M}{2}\left(1-\frac{4m^2}{M^2}\right)^{1/2}$$
$$\left(\frac{\mathrm{d}\Gamma_{RF}}{\mathrm{d}\Omega}\right) = \frac{1}{64\pi^2 M}\left(1-\frac{4m^2}{M^2}\right)^{1/2}|\mathcal{M}_{fi}|^2$$

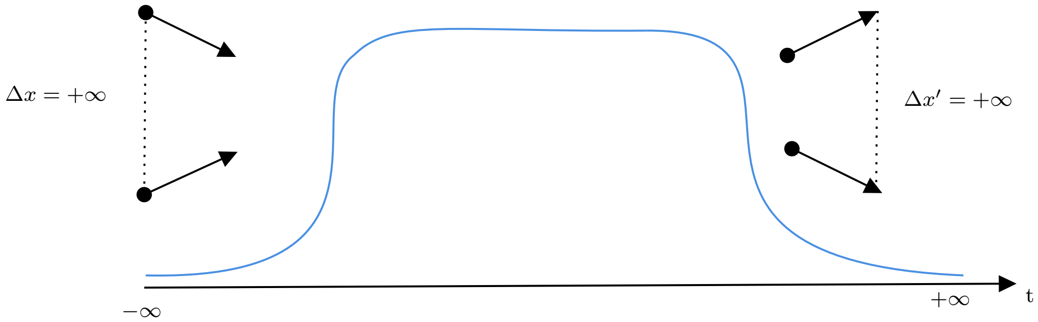
(B) If $m_1 = m, m_2 = 0$ (for example $W^\pm \to e^\pm \overset{(-)}{\nu}$)

$$|\hat{q_1}| = \frac{M}{2}\left(1-\frac{4m^2}{M^2}\right)^{1/2}$$
$$\left(\frac{\mathrm{d}\Gamma_{RF}}{\mathrm{d}\Omega}\right) = \frac{1}{64\pi^2 M}\left(1-\frac{m^2}{M^2}\right)^{1/2}|\mathcal{M}_{fi}|^2$$

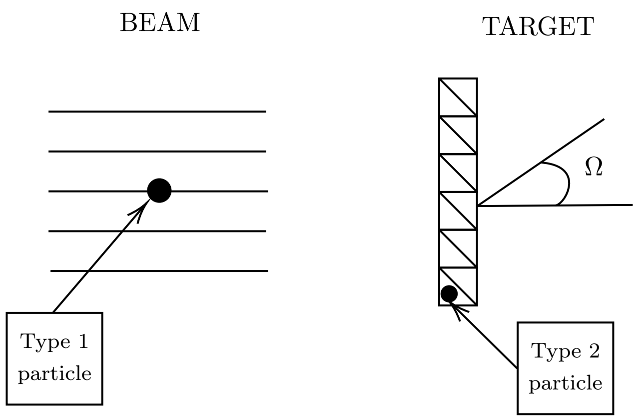
Notes: If we have two identical particles in the final state, the calculation of the phase is different

$$\mathrm{d}\Phi_{(2)}^{\text{identical}} = \frac{1}{2}\mathrm{d}\Phi_{(2)}^{\text{distinguishable}}$$

1.4 Cross section ($\leftrightarrow$ scattering process)



Scattering in the lab frame



Consider a beam of particles with mass m_1, number (assuming a uniform distribution) density $n_1^{(0)}$ (subscript 0 is meant to stress that these are number densities in a specific frame, that with particle 2 at rest) and velocity v_1 inpining on a target made of particles with mass m_2 and number density $n_2^{(0)}$ at rest.
Let N_s be the number of scattering events that place per unit volume and per unit time

$$\frac{N_t}{T}\varphi_1 N_2\sigma = \big(n_1^{(0)}v_1\big)\big(n_2^{(0)}V\big)\sigma$$

More formally we have

$$\mathrm{d}N_s = \sigma v_1 n_1^{(0)} n_2^{(0)}\ \mathrm{d}V\mathrm{d}V$$

with:

(i) T: unit of time

(ii) φ_1: flux of the beam $\varphi_1 = n_1^{(0)}v_1$

(iii) N_2: particles per unit volume in the detector $N_2 = n_2^{(0)}V$

(iv) σ: proportionality constant

Dimensional analysis shows $[\sigma] = [L]^2$ and then σ, called cross section, can be interpreted as an "effective area".