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Chapter 1

Introduction

1.1 Free Fields Theories

1.1.1 Free Fields

Here we recall the expressions of quantum free fields. Notice that real (i.e. physical) fields are never free because we have interactions, but using interaction picture we reconduct the problem in a simper one, where fields are described by free fields. This can be possible with a proper choice

$$\varphi_I(x) \equiv \varphi_{\text{free}}(x) \qquad \varphi_I = \text{field in interacting picture}$$

Complex Scalar Field

Euler-Lagrange equation for the scalar field:

$$(\Box + m^2)\phi = 0$$

Fourier expantion:

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} \left(e^{-ikx} a(k) + e^{ikx} b^\dagger(k)\right)_{k_0=\omega_k=\sqrt{m^2+\boldsymbol{k}^2}}$$

In the real case

$$\phi^\dagger(x) = \phi(x) \Rightarrow a(k) = b(k)$$

Dirac Spinorial Field

Euler-Lagrange equation for the Dirac field:

$$(i\partial\!\!\!/ - m)\psi = 0$$

Fourier expantion:

$$\psi(x) = \frac{1}{(2\pi)^{2/3}} \int \frac{d^3k}{\sqrt{2\omega_k}} \sum_{r=1,2} \left(e^{-ikx} u_r(k) c_r(k) + e^{ikx} v_r(k) d_r^\dagger(k)\right)_{k_0=\omega_k}$$

where $u_r(k)$ and $v_r(k)$ are respectively the $\varepsilon > 0$ and $\varepsilon < 0$ spinors with helicity indicated by r. Spinors are normalized according to

$$\begin{cases} \bar{u_r}(k) u_s(k) = 2m\delta_{rs} & \bar{u_r}(k) v_s(k) = 0 \\ \bar{v_r}(k) v_r(k) = -2m\delta_{rs} & \bar{v_r}(k) u_s(k) = 0 \end{cases}$$

Real E-M Vector Field

Euler-Lagrange equation for the real E-M field:

$$\partial_\mu F^{\mu\nu} = \Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Fourier expantion:

$$A^\mu(x) = \frac{1}{(2\pi)^{2/3}} \int \frac{d^3k}{\sqrt{2\omega_k}} \sum_{\lambda=1,2} \left(e^{-ikx}\varepsilon^\mu_{(\lambda)} a_\lambda(k) + e^{ikx}\varepsilon^{\mu\dagger}_{(\lambda)}(k) a^\dagger_\lambda(k)\right)_{k_0=\omega_k=|\boldsymbol{k}|}$$

where polarization vectors can be chosen as follows:

$$\varepsilon^\mu_{(1)} = (0,1,0,0) \qquad \varepsilon^\mu_{(2)} = (0,0,1,0)$$

I can complexify the field substituting $a^\dagger_\lambda$ with another operator b_λ (analogously to the scalar field).

1.1.2 Fock Space of Free Fields

See Maggiore. See 6.1

We impose the existence of vacuum state $|0\rangle$, with $\langle 0|0\rangle = 1$, then using creation operators we obtain other states $(a^\dagger)^n |0\rangle$, which are n-particles states.
In QFT we normalized states in a covariant way, instead of QM normalization $\int \psi^*\psi = 1$:

$$|1(p)\rangle \equiv (2\pi)^{3/2}\sqrt{2\omega_p}\; o^\dagger(p)\,|0\rangle \tag{1.1a}$$

$$\langle 1(p)|1(p')\rangle = (2\pi)^3(2\omega_p)\delta^3(p-p') \tag{1.1b}$$

where the term $(2\omega_p)\delta^3(p-p')$ is covariant under Lorentz transformations and $o^\dagger$ denotes a generic creation operator.

1.1.3 Contraction of Fields with States

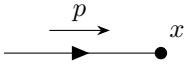
If we have a state $|e^-_s(p)\rangle$ that describes an electron with momentum p and Dirac index s, then

$$\left|e^-_s(p)\right\rangle = (2\pi)^{2/3}\sqrt{2\omega_p}\; c^\dagger_s(p)\,|0\rangle$$

Given a field ψ that describes a particle annihilation (or antiparticle creation) in the coordinate x we have

$$\begin{aligned}
\langle 0|\,\psi(x)\left|e^-_s(p)\right\rangle &= \langle 0|\,(\psi_+(x) + \psi_-(x))\left|e^-_s(p)\right\rangle \\
&= \frac{(2\pi)^{3/2}}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}} e^{-ikx}\sqrt{2\omega_p}\; \sum_r \langle 0|\,c_r(k)c^\dagger_s(p)\,|0\rangle\, u_r(k) \\
&= \int \mathrm{d}^3k \Big(\frac{2\omega_p}{2\omega_k}\Big)^{1/2}\; \sum_r \delta_{rs}\delta^{(3)}(\bar{p}-\bar{k}) u_r(k)\,\langle 0|0\rangle\, 0e^{-ikx} \\
&= e^{-ikx}u_s(p)
\end{aligned}$$

where we used $\langle 0|\,c_r(k)c^\dagger_s(p)\,|0\rangle = \langle 0|\,\{c_r(k), c^\dagger_s(p)\}\,|0\rangle = \delta_{rs}\;\delta^3(\bar{p}-\bar{k})$. The factor e^{-ikx} is required for the $\delta^{(4)}$ conservation. We see that the relativistic normalization leads to Feyman rules without normalization factors:

$e^{-ipx}u_s(p) =$ 

1.2 S-Matrix

1.2.1 Interaction Picture

In the interaction picture, with $H = H_0 + H_{\text{int}}$, where H_{int} is the interaction Hamiltonian

(i) fields φ_I evolves like in the free theory (respect to H_0)

(ii) states evolve with the following evolution operator

$$U_I(t,t_0) \equiv e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t_0}$$
$$|\alpha,t\rangle = U_I(t-t_0)\,|\alpha,t_0\rangle \qquad i\partial_t U_I(t-t_0) = H_I^{\text{int}}(t) U_I(t,t_0)$$

(iii) operators in interaction picture are (let $O_H(t)$ be a generic operator in Heisenberg picture and $O_I(t)$ the same operator in Interaction picture)

$$O_I(t) = e^{iH_0 t} e^{-iHt}\; O_H(t) e^{iHt} e^{-iH_0 t}$$

Notice that, in general

$$[H_I(t), H_0] \neq 0 \neq [H_I^{\text{int}}, H_0]$$

and if $t \neq t'$ we also have

$$[H_I^{\text{int}}(t), H_I^{\text{int}}(t')] \neq 0$$

1.2.2 S-matrix and States Evolution

The **S-matrix** is a well defined operator defined as

$$S = \lim_{\substack{t_0 \to -\infty \\ t \to +\infty}} U_I(t,t_0)$$

We compute it by perturbation obtaining

$$\begin{aligned} S &= T\left(\exp\left(-i\int \mathrm{d}^4x\, \mathcal{H}_I^{\text{int}}(x)\right)\right) \\ &= \sum_{n=0}^{+\infty} \frac{(-i)^n}{n!} \int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n\, T\big[\mathcal{H}_I^{\text{int}}(x_1)\ldots\mathcal{H}_I^{\text{int}}(x_n)\big] \end{aligned}$$

S-matrix has some relevant properties:

(i) unitarity (since Hamiltonian is hermitian)

(ii) behaves as a scalar under Lorentz transformations, and then is an invariant quantity (notice that in general $\mathcal{H}_I^{\text{int}}$ is not invariant). In the case of $\mathcal{H}_I^{\text{int}}$ invariant (for example if, as in many theories, $\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$, for example in QED) is easy to prove that S is invariant, since all n-th derivatives of $\exp\big(\int \mathcal{H}\big)$ are invariant, and so also S is invariant.

1.2.3 S-matrix and transition probabilities

Assume that there's no interaction for $x, t \to \pm\infty$. Consider canonically normalized (CN) states $|\psi| = 1$:

$$|\psi_i\rangle_{CN} \equiv |\psi(-\infty)\rangle_{CN} \qquad |\psi(+\infty)\rangle_{CN} \equiv S\,|\psi_i\rangle_{CN}$$

(both are free particle states). Elements of S are in the form

$$S_{fi}^{CN} = {}_{CN}\langle\psi_f|S\,|\psi_i\rangle_{CN}$$

This leads to a probabilistic interpretation of S-matrix elements.

The squared amplitude $|S_{fi}^{CN}|^2$ is the transition probability of $|\psi_i\rangle_{CN}$ into $|\psi_f\rangle_{CN}$.

Notice that the requirement $\sum_f |S_{fi}^{CN}| = 1$ is satisfied automatically. Moreover the canonical normalization is required in order to interpret such amplitude as a transition amplitude. In the case of covariant normalization

$$\langle 1(p)|1(p')\rangle = (2\pi)^3(2\omega_p)\delta^3(\boldsymbol{p}-\boldsymbol{p}')$$

we have the following relation between matrix elements

$$S_{fi}^{CN} = {}_{CN}\langle\psi_f|S\,|\psi_i\rangle_{CN} = \frac{\langle\psi_f|\,S\,|\psi_i\rangle}{\|\psi_i\|\,\|\psi_f\|} = \frac{S_{fi}}{\|\psi_i\|\,\|\psi_f\|}$$

We can define the **Feynman Amplitude** $\mathcal{M}_{fi}$ as

$$S_{fi} = (2\pi)^4\delta^4(p_i - p_f)\mathcal{M}_{fi}$$

and it can be obtained directly starting from Feynman rules (used with the covariant normalization described in Sec. 1.1.3)

1.3 Discrete space normalization

Usually, in order to make arguments clearer, or to avoid problems with divergent terms in calculations, we first consider a system in a cubic box with spatial volume $V = L^3$. At the end of computations V will be sent to infinity. Sometimes we will do something similar also for time. For a discrete space we must use a different normalization.

In a box, the momentum of a particle is quantized

$$p_i = \left(\frac{2\pi}{L}\right)n_i \qquad n_i \in \mathbb{Z}$$

and we must adopt the following rule for integrals

$$\int \mathrm{d}^3p\, f(\boldsymbol{p}) \quad \to \quad \sum_{\boldsymbol{n}}\left(\frac{2\pi}{L}\right)f_{\boldsymbol{n}} \qquad \boldsymbol{n} = (n_1, n_2, n_3)$$

We must adopt also the following

$$\delta^3(\boldsymbol{p}-\boldsymbol{p}') \quad \to \quad \left(\frac{L}{2\pi}\right)^3\delta_{\boldsymbol{n}\boldsymbol{n}'}$$

in this way the defining relation for the delta function works also in the discrete version

$$\int \mathrm{d}^3p\,\delta^3(\boldsymbol{p}-\boldsymbol{p}') = 1 \quad \to \quad \sum_{\boldsymbol{n}}\left(\frac{2\pi}{L}\right)^3\left(\frac{L}{2\pi}\right)^3\delta_{\boldsymbol{n}\boldsymbol{n}'} = 1$$

In particular

$$\delta^3(0) \to \left(\frac{L}{2\pi}\right)^3$$

When we consider also a finite amount of time we have

$$\delta^4(0) \to \left(\frac{L}{2\pi}\right)\left(\frac{T}{2\pi}\right)$$

Using (1.1b), we have

$$\langle 1(p)|1(p)\rangle = (2\pi)^3 2\omega_p\delta^3(0) = 2\omega_p V$$

so normalization of states (1.1a) becomes ($o^\dagger$ is a general creation operator)

$$|1(p)\rangle = \sqrt{2\omega_p V}\,|1(p)\rangle_{CN}$$

1.3.1 S-Matrix in discrete space

Using the latter equation, S_{fi}^{CN} reads

$$\begin{aligned} S_{fi}^{CN} &= \prod_{j=1}^{n_i}\left(\frac{1}{2\omega_j V}\right)^{1/2}\prod_{l=1}^{n_f}\left(\frac{1}{2\omega_l V}\right)^{1/2} S_{fi} \\ &= (2\pi)^4\delta^4(p_i-p_f)\left\{\prod_{j=1}^{n_i}\left(\frac{1}{2\omega_j V}\right)^{1/2}\prod_{l=1}^{n_f}\left(\frac{1}{2\omega_l V}\right)^{1/2}\mathcal{M}_{fi}\right\} \\ &= (2\pi)^4\delta^4(p_i-p_f)\mathcal{M}_{fi}^{CN} \end{aligned}$$

In the second passage we use the definition of $\mathcal{M}_{fi}$, omitting the quantization of δ^4. $\mathcal{M}_{fi}^{CN}$ is the **canonically normalized Feynman amplitude**:

$$S_{fi} = (2\pi)^4\delta^4(p_i-p_f)\mathcal{M}_{fi} \quad \leftrightarrow \quad S_{fi}^{CN} = (2\pi)^4\delta^4(p_i-p_f)\mathcal{M}_{fi}^{CN}$$

with

$$\mathcal{M}_{fi}^{CN} = \prod_{j=1}^{n_i}\left(\frac{1}{2\omega_j V}\right)^{1/2}\prod_{l=1}^{n_f}\left(\frac{1}{2\omega_l V}\right)^{1/2}\mathcal{M}_{fi}$$

Chapter 2

The S-matrix and physical observables

Maggiore sec. 6.2-6.4; Schwartz chap. 5; Mandl sec. 8.1

2.1 Decay Rate

Consider the case in which the initial state is a single particle and the final state is given by n particles. We are therefore considering a decay process. Assume for the moment that particles are indistinguishable.

The rules of quantum mechanics tell us that the probability for this process is obtained by taking the squared modulus of the amplitude and summing over all possible final states

$$\begin{aligned}\left|S_{fi}^{CN}\right|^2 &= \left|(2\pi)^4\delta^4(p-p')M_{fi}^{CN}\right|^2 \\ &= (2\pi)^4\delta^4(p-p')(VT)\left|M_{fi}^{CN}\right|^2 \\ &= (2\pi)^4\delta^4(p-p')(VT)\frac{1}{2\omega_{in}V}\prod_{l=1}^{n_f}\left(\frac{1}{2\omega_l V}\right)|\mathcal{M}_{fi}|^2\end{aligned}$$

where we used

$$\left(\delta^4(p-p')\right)^2 = \delta^4(p-p')\delta^4(p-p') = \delta^4(p-p')\delta^4(0) = \delta^4(p-p')\frac{VT}{(2\pi)^4}$$

Now we must sum this expression over all final states. Since we are working in a finite volume V, this is the sum over the possible discrete values of the momenta of the final particles . Recalling $p_i = (2\pi/L)n_i$, we have $dn_i = (L/2\pi)\mathrm{d}p_i$ and $\mathrm{d}^3n_i = (V/(2\pi)^3)\mathrm{d}^3p$ where d^3n_i is the infinitesimal phase space related to a final state in which the i-th particle has momentum between p_i and $p_i + \mathrm{d}p_i$.

Let $\mathrm{d}\omega$ be the probability for a decay in which in the final state the i-th particle has momentum between p_i and $p_i + \mathrm{d}p_i$:

$$\mathrm{d}\omega = \left|S_{fi}^{SN}\right|^2\prod_{l=1}^{n_f}\left(\frac{V\mathrm{d}^3p_l}{(2\pi)^3}\right) = (2\pi)^4\delta^4(p-p')T\frac{|\mathcal{M}_{fi}|^2}{2\omega_{in}}\prod_{l=1}^{n_f}\frac{\mathrm{d}^3p_l}{(2\pi)^3 2\omega_l}$$

This is the probability that the decay takes place in any time between $-T/2$ and $T/2$. We are more interested in the differential decay rate $\mathrm{d}\Gamma_{fi}$, which is the decay probability per unit of time:

$$\mathrm{d}\Gamma_{fi} = \frac{\mathrm{d}\omega}{T} = (2\pi)^4\delta^4(p_i - p_f)\frac{|\mathcal{M}_{fi}|^2}{2\omega_{in}}\prod_{l=1}^{n_f}\frac{\mathrm{d}^3p_l}{(2\pi)^3 2\omega_l}$$

Recall the meaning of the quantities in the previous relation:

- $\mathrm{d}\Gamma_{fi}$ = differential decay rate
- p_f = sum over final momenta
- ω_{in} = initial energy
- $|\mathcal{M}_{fi}|^2$ = Feynman amplitude of the process (depends on final momenta p_i)

It is useful to define the **(differential) n-body phase space** as

$$\mathrm{d}\Phi_{(n_f)} = (2\pi)^4 \delta^4(p_i - p_f) \prod_{l=1}^{n_f} \frac{\mathrm{d}^3 p_l}{(2\pi)^3 2\omega_l}$$

Notice that if n of the final particles are identical, configurations that differ by a permutation are not distinct and therefore the phase space is reduced by a factor $1/n!$.

Therefore the differential decay rate can be written as

$$\mathrm{d}\Gamma_{fi} = \frac{1}{2\omega_{in}} |\mathcal{M}_{fi}|^2 \mathrm{d}\Phi_{(n_f)}$$

The decay rate is defined as

$$\Gamma_{fi} = \int \mathrm{d}\Gamma_{fi} \quad \rightarrow \text{integration over all possible final momenta}$$

and its meaning is $\Gamma \equiv$ trans. probability $\times$ unit of time $\times$ init. particle.

If we have a system of $N(0)$ particles, the time evolution of the number of particles $N(t)$ is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\Gamma N \quad \Rightarrow \quad N(t) = N(0) e^{-\Gamma t}$$

Notice that decay rate is not invariant

$$\big[\Gamma\big] = \big[E\big] = \frac{1}{T}$$

hence defining the **lifetime** as $\tau = 1/\Gamma \Rightarrow N(t) = N(0)\exp(-t/\tau)$ this changes under Lorentz transformations. If we consider two reference frames o and o'

$$\Gamma' = \frac{\Gamma}{\gamma} < \Gamma \qquad \tau' = \gamma\tau > \tau$$

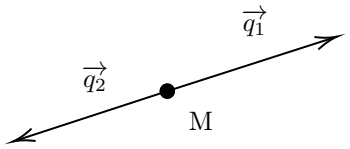
with $\gamma = (1-v)^{-1/2}$, where v is the speed of o' in o in natural units, $\gamma > 1$. Therefore a particle in a moving frame has a longer lifetime then in the rest frame.

Example 1: muon lifetime

For a muon in the rest frame $\tau_\mu^{RF} = 2.2 \times 10^{-6} s$, but if we observe it in the lab frame $\tau_\mu^{LAB} = \gamma \tau_\mu^{RF} \simeq 2 \times 10^{-5} s$ since $E_\mu = 1$ GeV, $m_\mu = 0.1$ GeV imply $\gamma = E_\mu / m_\mu \simeq 10$.

Example 2: $1 \rightarrow 2$ decay

Consider the decay of a particle of a mass M into two particles of masses m_1, m_2. Since the phase space in Lorentz invariant, we can compute it in the frame that we prefer. We use the rest frame for the initial particle.



We don't impose a priori conservation of momentum since it's imposed by the delta function.

$$p = (M, 0) \qquad q_1 = (\omega_1, \boldsymbol{q_1}) \qquad q_2 = (\omega_2, q_2)$$
$$\mathrm{d}\Phi_{(2)} = (2\pi)^4 \delta^4 \underbrace{(P_i - P_f)}_{=p-q_1-q_2} \frac{\mathrm{d}^3 q_1}{(2\pi)^3 2\omega_1} \frac{\mathrm{d}^3 q_2}{(2\pi)^3 2\omega_2}$$

I have 6 integration parameters, 4 constraints given by δ^4, so I have 2 independent variables. Integrating over $\mathrm{d}^3 q_2$

$$\mathrm{d}\Phi'_{(2)} = \int \mathrm{d}\Phi_{(2)} = \frac{1}{(2\pi)^2} \delta(M - \omega_1 - \omega_2) \frac{1}{4\omega_1 \omega_2} \mathrm{d}^3 q_1$$

in this way, the condition $\boldsymbol{q_2} = \boldsymbol{q_1}$ vanish. We have to impose it again when we calculate $\mathrm{d}\Gamma$. Usually the 4-th non independent parameter is eliminated by integration over modulus of q_1, leaving free 2 parameters for the angles, $\mathrm{d}^3 q_1 \to \boldsymbol{q_1}^2 \mathrm{d}|\boldsymbol{q_1}|\mathrm{d}\Omega_1$. Notice that $M - \omega_1 - \omega_2 = M - \sqrt{\boldsymbol{q_1}^2 + m_1^2} - \sqrt{\boldsymbol{q_1}^2 + m_2^2}$ and then the δ fixes $\boldsymbol{q_1}^2$ to the value

$$\hat{\boldsymbol{q_1}}^2 = \frac{1}{2M} \left(M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2 \right)^{1/2}$$

Writing $f(|\boldsymbol{q_1}|) = M - \omega_1 - \omega_2$, such that $f(|\hat{\boldsymbol{q_1}}|) = 0$, we have

$$|f'(|\hat{\boldsymbol{q_1}}|)| = \frac{\partial \omega_1}{\partial |\boldsymbol{q_1}|} + \frac{\partial \omega_2}{\partial |\boldsymbol{q_1}|} = |\hat{\boldsymbol{q_1}}| \left(\frac{\omega_1 + \omega_2}{\omega_1 \omega_2} \right)$$

and then using

$$\delta\big(f(x)\big) = \sum_{x_0 = \text{zero of f(x)}} \frac{\delta(x - x_0)}{|f'(x_0)|}$$

we can perform the integration over $\mathrm{d}|\boldsymbol{q_1}|$ obtaining

$$\mathrm{d}\Phi''_{(2)} = \int \mathrm{d}\Phi'_{(2)} = \frac{1}{16\pi^2} \frac{|\hat{\boldsymbol{q_1}}|}{M} \mathrm{d}\Omega_1$$

Using this result we obtain the $1 \to 2$ decay rate in function of the solid angle (in the rest frame of the initial particle, $\omega_{in} = M$)

$$\left(\frac{\mathrm{d}\Gamma_{RF}}{\mathrm{d}\Omega} \right) = \frac{1}{64\pi^4 M^3} \left[M^4 - 2M^2(m_1^2 + m_2^2) + (m_1 - m_2^2)^2 \right]^{1/2} |\mathcal{M}_{RF}|^2$$

In a general frame we can easily obtain an analogous formula, just consider the generic relation $\mathrm{d}\Gamma = 1/(2\omega_{in}) |\mathcal{M}_{fi}|^2 \mathrm{d}\Phi_{(n_f)}$. Recall that $\mathrm{d}\Phi_{(n_f)}$ is invariant.

We have 2 important limit cases:

(a) If $m_1 = m_2 = m$ (for example $Z \to e^+ e^-$)

$$|\hat{\boldsymbol{q_1}}| = \frac{M}{2} \left(1 - \frac{4m^2}{M^2} \right)^{1/2}$$
$$\left(\frac{\mathrm{d}\Gamma_{RF}}{\mathrm{d}\Omega} \right) = \frac{1}{64\pi^2 M} \left(1 - \frac{4m^2}{M^2} \right)^{1/2} |\mathcal{M}_{fi}|^2$$

(b) If $m_1 = m$, $m_2 = 0$ (for example $W^{\pm} \to e^{\pm} \overset{(-)}{\nu}$)

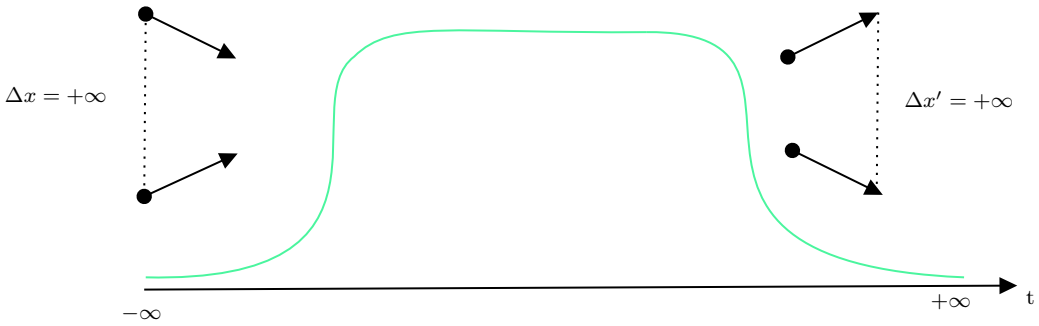
$$|\hat{\boldsymbol{q}}_{\mathbf{1}}| = \frac{M}{2}\left(1 - \frac{4m^2}{M^2}\right)^{1/2}$$

$$\left(\frac{\mathrm{d}\Gamma_{RF}}{\mathrm{d}\Omega}\right) = \frac{1}{64\pi^2 M}\left(1 - \frac{m^2}{M^2}\right)^{1/2} |\mathcal{M}_{fi}|^2$$

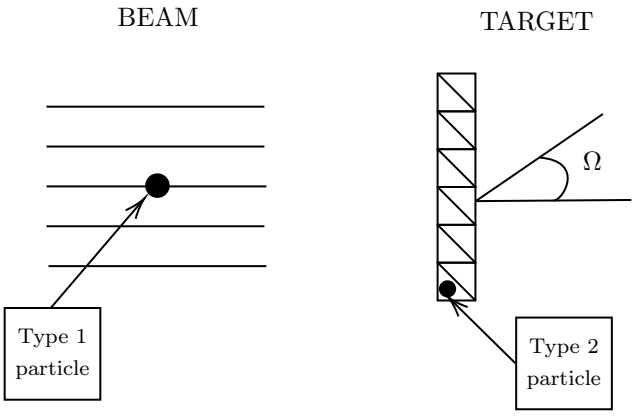
Notice that if we have two identical particles in the final state, the calculation of the phase space is different

$$\mathrm{d}\Phi_{(2)}^{\text{identical}} = \frac{1}{2}\mathrm{d}\Phi_{(2)}^{\text{distinguishable}}$$

2.2 Cross section (scattering process)



Scattering in the lab frame



Consider a beam of particles with mass m_1 with number density (assuming a uniform distribution) $n_1^{(0)}$ (subscript 0 is meant to stress that these are number densities in a specific frame, with particle 2 at rest) and velocity v_1 colliding on a target made of particles with mass m_2 and number density $n_2^{(0)}$ at rest. Let N_s be the number of scattering events that place per unit volume and per unit time

$$\frac{N_s}{T}\phi_1 N_2 \sigma = \left(n_1^{(0)} v_1\right)\left(n_2^{(0)} V\right)\sigma$$

More formally we have

$$\mathrm{d}N_s = \sigma v_1 n_1^{(0)} n_2^{(0)}\ \mathrm{d}V\mathrm{d}t$$

with:

- T: unit of time;

- ϕ_1: flux of the beam $\phi_1 = n_1^{(0)} v_1$;
- N_2: particles per unit volume in the detector $N_2 = n_2^{(0)} V$;
- σ: proportionality constant, called **cross section**.

Dimensional analysis shows $[\sigma] = [L]^2$ and then σ, called cross section, can be interpreted as an "effective area".

Consider the case where just one particle collides with another particle (2 particles scattering). We can obtain this condition imposing $n_{1,2}^{(0)} = 1/V$, in this way the scattering per unit of volume is the same as two particles scattering. Notice that this situation is the more physical one. The probability of 3 particles collision in the same d^3x and $\mathrm{d}t$ is almost 0. Let $\mathrm{d}\omega$ be, again, the probability for a process in which in the final state the i-th particle has momentum between p_i and $p_i + \mathrm{d}p_i$:

$$\mathrm{d}\omega = (2\pi)^4 \delta^4(p_i - p_f) VT \left(\frac{1}{2\omega_1 V}\right)\left(\frac{1}{2\omega_2 V}\right) \prod_{l=1}^{n_f}\left(\frac{\mathrm{d}^3 p_l}{(2\pi)^3 2\omega_l}\right) |\mathcal{M}_{fi}|^2$$

We can now define the **differential cross section** (in the lab frame) $\mathrm{d}\sigma$ as

$$(\mathrm{d}\sigma)_{\mathrm{LAB}} = \frac{\mathrm{d}\omega}{n_1^{(0)} n_2^{(0)} v_1 VT} = \frac{V}{Tv_1}\mathrm{d}\omega = (2\pi)^4 \delta^4(p_i - p_f) \frac{|\mathcal{M}_{fi}|^2}{4\omega_1 \omega_2 v_1} \prod_{l_1}^{n_f}\left(\frac{\mathrm{d}^3 p_l}{(2\pi)^3 2\omega_l}\right)$$

with $\omega_2 = m_2$. The result is then (all quantities are considered in the LAB frame)

$$(\mathrm{d}\sigma)_{\mathrm{LAB}} = \frac{1}{4\omega_1 v_1 m_2} |\mathcal{M}_{fi}|^2 \, \mathrm{d}\Phi_{(n_f)}$$

In order to obtain a covariant relation for $\mathrm{d}\sigma$, we notice that the only non-covariant factor in the latter relation is $(I_{12})_{\mathrm{LAB}} = \omega_1^{(0)} m_2$. This factor can be substituted with a covariant one

$$I_{12} = \left[(p_1 p_2)^2 - m_1^2 m_2^2\right]^{1/2}$$

called **covariant flux factor**. This factor is obviously covariant, we just have to prove that in the lab frame it coincides with $(I_{12})_{\mathrm{LAB}}$. In the lab frame we have $p_1 = (\omega_1, \boldsymbol{p_1}^{(0)})$ and $p_2 = (m_2, \mathbf{0})$, so the previous formula becomes

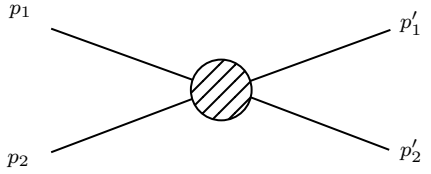
$$\left[m_2^2 (\omega_1^{(0)})^2 - m_1 m_2\right]^{1/2} = m_2 \left((\omega_1^{(0)})^2 - p_1^2\right)^{1/2} = m_2 \left|\boldsymbol{p_1^{(0)}}\right| = m_2 \omega_1^{(0)} v_1^{(0)}$$

So the final result is

$$\mathrm{d}\sigma = \frac{\left|\mathcal{M}_{fi}^2\right|}{4I_{12}} \mathrm{d}\Phi_{(nf)}$$

Example 3: $2 \to 2$ **scattering**

Consider a scattering process $2 \to 2$. We consider an initial state with two particles with masses m_1, m_2 and four momenta p_1, p_2 and a final state with masses m_1', m_2' and four momenta p_1', p_2'



with

$$p_1 = (\omega_1, \boldsymbol{p_1}) \qquad p_2 = (\omega_2, \boldsymbol{p_2}) \qquad p_1' = (\omega_1', \boldsymbol{p_1'}) \qquad p_2' = (\omega_2', \boldsymbol{p_2'})$$

With a procedure identical to $1 \to 1$ decay (imposing also $\boldsymbol{p_2'} = \boldsymbol{p_1} + \boldsymbol{p_2} - \boldsymbol{p_1'}$ when we calculate

$\mathrm{d}\sigma$), we obtain

$$\mathrm{d}\Phi'_{(2)} = \frac{1}{(2\pi)^2}\frac{1}{4\omega'_1\omega'_2}\delta(\omega_1+\omega_2-\omega'_1-\omega'_2)\ \mathrm{d}^3p'_1$$

In order to integrate over $\mathrm{d}\left|\boldsymbol{p'_1}\right|$ it is useful to introduce Mandelstam variables s, t and u

$$s=(p_1+p_2)^2 \qquad t=(p_1-p'_1)^2 \qquad u=(p_1-p'_2)^2$$

These variables are clearly Lorentz invariants, and satisfy (using $p_1+p_2=p'_1+p'_2$) the relation

$$s+t+u=m_1^2+m_2^2+(m'_1)^2+(m'_2)^2$$

It is useful to work in the center of mass frame, where the incoming particles have $p_1=(\omega_1,\boldsymbol{p})$ and $p_2=(\omega_2,-\boldsymbol{p})$. Computing s in the CM frame we obtain $s=(\omega_1+\omega_2)^2=(\omega'_1+\omega'_2)^2$ and then

$$p_1+p_2=(\sqrt{s},0)$$
$$\delta(\omega_1+\omega_2-\omega'_1-\omega'_2)=\delta(\sqrt{s}-\omega'_1-\omega'_2)$$

With a procedure identical to the one used for $1\to 2$ decay ($M\leftrightarrow\sqrt{s}$) we obtain

$$\left(\mathrm{d}\Phi''_{(2)}\right)_{CM}=\frac{1}{16\pi^2}\frac{\left|\hat{\boldsymbol{p'_1}}\right|_{CM}}{\sqrt{s}}\ \mathrm{d}\Omega'_1$$
$$\left|\hat{\boldsymbol{p'_1}}\right|_{CM}=\frac{1}{2\sqrt{s}}\left[s^2+(m'^2_1+m'^2_2)^2-2s(m'^2_1+m'^2_2)\right]^{1/2}$$

The covariant flux factor in CM frame reads

$$(I_{12})_{CM}=\left[\boldsymbol{p}^2(\omega_1+\omega_2)^2\right]^{1/2}=|\boldsymbol{p}|\sqrt{s}$$

So we obtain the final result

$$\begin{aligned}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega'_1}\right)_{CM}&=\frac{1}{64\pi^2 s}\frac{\left|\hat{\boldsymbol{p'_1}}\right|_{CM}}{|\boldsymbol{p}|}\ |\mathcal{M}_{fi}|^2_{CM}\\&=\frac{1}{128\pi^2 s^{3/2}}\frac{1}{|\boldsymbol{p}|}\left[s^2+(m'^2_1-m'^2_2)^2-2s(m'^2_1+m'^2_2)\right]^{1/2}|\mathcal{M}_{fi}|^2_{CM}\end{aligned}$$

We have two important limit cases

(A) If $m_1=m'_1, m_2=m'_2$, i. e. elastic scattering (for example $e^-\mu^+\to e^-\mu^+$)

$$\left.\begin{aligned}|\boldsymbol{p_1}|&=\left|\boldsymbol{p'_1}\right|,\ \omega_1=\omega'_1\\|\boldsymbol{p_2}|&=\left|\boldsymbol{p'_2}\right|,\ \omega_2=\omega'_2\end{aligned}\right\}\quad\to\quad\left|\hat{\boldsymbol{p'_1}}\right|_{CM}=|\boldsymbol{p}|$$

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega'_1}\right)_{CM}=\frac{1}{64\pi^2}\frac{|\mathcal{M}_{fi}|^2}{s}$$

(B) If $m_1=m_2\simeq 0, m'_1=m'_2=M$ (for example $e^-e^+\to\mu^-\mu^+$, $m_e=1\mathrm{MeV}$, $m_\mu=100\mathrm{MeV}$)

$$|\boldsymbol{p_1}|=|\boldsymbol{p}|,\ \omega_1=\frac{\sqrt{s}}{2}$$
$$\left|\hat{\boldsymbol{p'_1}}\right|=\frac{\sqrt{s}}{2}\left(1-\frac{1}{4M^2}s\right)^{1/2} \tag{2.1}$$

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_1'}\right)_{CM} = \frac{1}{64\pi^2}\left(1-\frac{4M^2}{s}\right)\frac{\left|\mathcal{M}_{fi}\right|^2}{s}$$

Notice that in this case we obtained that processes with $4M^2 > s$ are unphysical.

The previous formulas are valid also for particles with spin, if the initial and final spin states are known; in this case the initial state has the form $|i\rangle = |\boldsymbol{p_1}, \boldsymbol{s_1}; \ldots; \boldsymbol{p_n}, \boldsymbol{s_n}\rangle$, and similarly for the final state. However, experimentally is more common that we do not know the initial spin configuration and we accept in the detector all final spin configurations; in this case, to compare with experiment, we must

(i) consider all (equally present) polarizations of initial state, i.e. *average* over the initial spin configuration

(ii) consider all final polarizations, i.e. *sum* over all possible final configurations

Defining the **unpolarized Feynman amplitude** $\overline{|\mathcal{M}_{fi}|}$ as

$$\overline{\left|\mathcal{M}_{fi}\right|^2} = \frac{1}{\text{n° initial polarizations}} \sum_{\substack{\text{initial}\\ \text{spins}}} \sum_{\substack{\text{final}\\ \text{spins}}} \left|\mathcal{M}_{fi}\right|^2$$

we just have to substitute $\left|\mathcal{M}_{fi}\right|^2 \to \overline{\left|\mathcal{M}_{fi}\right|}^2$ in all previous formulas. For example, in $2 \to 2$ scattering

$$\frac{1}{\text{n° initial polarizations}} = \frac{1}{(2s_a+1)(2s_b+1)}$$

where s_a and s_b are the spin of the first and the second particle respectively.

Chapter 3

QED processes at lowest order

Halzen chap. 6; Peskin chap. 5; Mandl chap. 8; Schwartz chap. 13

3.1 The QED Lagrangian and its Symmetries

Mandl sec. 11.1; Maggiore sec. 7.1

Quantum electrodynamic (QED) describes the interactions between electrons (or any other charged spin 1/2 particle) and photons. QED is described by the Lagrangian

$$\mathcal{L}_{\text{QED}} = \underbrace{\overline{\psi}(i\partial\!\!\!/ - m)\psi}_{\mathcal{L}_D^{(0)}} \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\mathcal{L}_{EM}} \underbrace{-qA_\mu\overline{\psi}\gamma^\mu\psi}_{\mathcal{L}_{int}}$$

(i) $\mathcal{L}_D^{(0)}$ is the Langrangian for the free Dirac field;

(ii) $\mathcal{L}_{EM}$ is the Lagrangian for the free EM field. In order to quatize the EM field we have to add the gauge fixing term

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2$$

For other purposes this term can be omitted. Usually the choice $\xi = 1$, called Feynman gauge, is the simplest choice for quantization;

(iii) $\mathcal{L}_{int}$ describes the interaction between Dirac field and EM-field. The term $\mathcal{L}_D = \mathcal{L}_D^{(0)} + \mathcal{L}_{int}$ can be obtained from $\mathcal{L}_D^{(0)}$ with the "minimal substitution" $\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$, i.e. using the covariant derivative D_μ instead of ∂_μ in the Dirac Lagrangian. Notice that $\mathcal{L}_D$ exhibits local symmetry, while $\mathcal{L}_D^{(0)}$ doesn't.

Beside Lorentz invariance, the QED exhibits following symmetries:

- **Global $U(1)$ symmetry**

$$\begin{cases} \psi(x) \to \psi'(x) = e^{i\alpha}\psi(x) & \alpha \in \mathbb{R} \\ A^\mu(x) \to A'^\mu(x) = A^\mu(x) \end{cases}$$

There is therefore an associated conserved Noether current

$$j^\mu = q\overline{\psi}\gamma^\mu\psi$$

such that $\partial_\mu j^\mu = 0$ and a $U(1)$ charge which is conserved by the EM interaction

$$Q = q\int \mathrm{d}^3x\ \psi^\dagger\psi \qquad \frac{\mathrm{d}Q}{\mathrm{d}t} = 0$$

- **Local $U(1)$ symmetry** (gauge symmetry):

$$\begin{cases} \psi(x) \to \psi'(x) = e^{iq\alpha(x)}\psi(x) \\ A^\mu \to A'^\mu(x) = A^\mu(x) - iq\partial^\mu\alpha(x) \end{cases}$$

Notice that the global $U(1)$ symmetry is a consequence of the local $U(1)$ symmetry, taking $\alpha(x)$ constant).

The covariant derivative of ψ, $D_\mu\psi$, behaves as a spinor under gauge transformations (remember that D_μ transforms as a vector), indeed:

$$\begin{aligned} D_\mu\psi \to (D_\mu\psi)' = D'_\mu\psi' &= (D_\mu - iq\partial_\mu\alpha)(e^{iq\alpha(x)}\psi(x)) \\ &= (\partial_\mu + iqA_\mu - iq\partial_\mu\alpha)(e^{iq\alpha}\psi) = e^{iq\alpha}(\partial_\mu + iqA_\mu)\psi \\ &= e^{iq\alpha(x)}(D_\mu\psi) \end{aligned}$$

This implies that $\mathcal{L}_D$ is invariant under local $U(1)$ transformations. Since $\mathcal{L}_{EM}$ is invariant ($F^{\mu\nu}$ is itself invariant under gauge transformations), the full Lagrangian (without $\mathcal{L}_{GF}$) is invariant.

The term $\mathcal{L}_{GF}$ breaks the gauge symmetry, and is introduced for this reason in the quantization of the fields. In this way only physical states appears in the Fock space.

3.2 Flavors in QED and the SU(3) flavor global symmetry

QED describes interactions of the photon field with several kind of leptons, not only electron and positrons. Particles that differs only by their mass are called **flavours**. The next table describes leptons in QED. There are two families of leptons that differs by their charge. We label with $(-)$ particles with negative charge, and with $(+)$ particles with positive charge (antiparticles).

Leptons[I]	e	μ	τ	ν_e	ν_μ	ν_τ
q	-1	-1	-1	0	0	0
m [MeV]	0.5	105	1777	$\simeq 0$	$\simeq 0$	$\simeq 0$

The Dirac Lagrangian $\mathcal{L}_D$ can be modified in order to consider all possible leptons

$$\mathcal{L}_D = \sum_{i=1}^{n} \bar\psi_i(i\not{D} - m_i)\psi_i \simeq \sum_{i=1}^{n_l} \bar\psi_i(i\not{D} - m_i)\psi_i + \sum_{j=1}^{n_n} \bar\psi_i(i\not{\partial})\psi_i$$

with n: number of leptons, n_l: number of electrically charged particles, n_n: number of neutrinos.
In the last term the interaction term vanishes because of q=0 and the masses of neutrinos are neglected.

If we adopt a matrix notation for the fields

$$\Psi_C = \begin{pmatrix} \psi_e \\ \psi_\mu \\ \psi_\tau \end{pmatrix} \qquad \Psi_N = \begin{pmatrix} \psi_{\nu_e} \\ \psi_{\nu_\mu} \\ \psi_{\nu_\tau} \end{pmatrix}$$
$$\bar\Psi_C = (\bar\psi_e, \bar\psi_\mu, \bar\psi_\tau) \qquad \bar\Psi_N = (\bar\psi_{\nu_e}, \bar\psi_{\nu_\mu}, \bar\psi_{\nu_\tau})$$

(the subscript C stands for "charged", and N for "neutral") and we define following matrices

$$M_C = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \qquad Q_C = (-1)\mathbb{1}_{3\times3} \qquad M_N \simeq Q_N = \emptyset_{3\times3}$$

and the generalization of the covariant derivative $D_\mu = \partial_\mu \cdot \mathbb{1}_{3\times3} + iA_\mu Q_C$ then we obtain

$$\mathcal{L}_D = \underbrace{\bar\Psi_C(i\not{D} - M_C)\Psi_C}_{\mathcal{L}_C} + \underbrace{\bar\Psi_N(i\not{\partial}\,\mathbb{1}_{3\times3})\Psi_N}_{\mathcal{L}_N}$$

The term $\mathcal{L}_C$ and $\mathcal{L}_N$ are respectively the **charged sector** and the **neutral sector** of $\mathcal{L}_D$. The basis defined by Ψ_C and Ψ_N is called **physical basis**, since physical particles are identified by their mass.

[I] Neutrinos admit only global $U(1)$ symmetry, and their masses are of the order $m_\nu \approx 10^{-6}\, m_e \leq 1\text{eV}$.

3.2.1 Global symmetry of neutral and charged sector

Let's consider a U(3) transformation

$$\Psi(x) \to \Psi'(x) = U\Psi(x) \qquad U^\dagger U = \mathbb{1}_{3\times 3}$$

(spinors and vectors are left invariant). The neutral sector is left invariant under U(3), indeed

$$\mathcal{L}_N \to \mathcal{L}'_N = \bar{\Psi}'_N(i\partial\!\!\!/\,\mathbb{1}_{3\times 3})\Psi'_N = \bar{\Psi}_N U^\dagger(i\partial\!\!\!/\,\mathbb{1}_{3\times 3})U\Psi_N = \mathcal{L}_N$$

conversely, the charged sector is not invariant because of the mass term, which in general do not commute with U and $U^\dagger$:

$$\mathcal{L} \to \mathcal{L}'_C = \bar{\Psi}_C(i\not{D} - U^\dagger M_C U)\Psi_C \neq \mathcal{L}_C$$

In order to obtain global symmetries of $\mathcal{L}_D$, we search the subgroup of U(3) described by matrices U_g that satisfies

$$U_g^\dagger M_C U_g = M_C \qquad U_g \in U(3) \tag{3.1}$$

One can prove that $U(3) \underset{\text{iso}}{\simeq} U(1) \times SU(3)$, meaning that their Lie algebras are isomorphic, $\mathfrak{u}(3) \underset{\text{iso}}{\simeq} \mathfrak{u}(1) \times \mathfrak{su}(3)$. Since generators of $SU(3)$ are the eight Gell-Mann matrices λ_a (for a $= 1, \dots, 8$) and the only generator of $U(1)$ is $\mathbb{1}$ then generators of U(3) are of the form

$$\mathbb{1} \times \{\lambda_1, \dots, \lambda_8\}$$

One can also prove that equation 3.1 is satisfied only by diagonal matrix, and the only diagonal Gell-Mann matrices are

$$\lambda_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

For i = 0, 3 ,8 we have $[\lambda_i, M_c] = 0$ and then the equation 3.1 is satisfied: if we take $\alpha_i \ll 1$

$$(1 - i\alpha_i\lambda_i)M_C(1 + i\alpha_i\lambda_i) = (M_C - i\alpha_i[\lambda_i, M_C] + o(\alpha_i^2)) \simeq M_C$$

We then obtained that the global group of symmetries is generated by the algebra

$$\mathcal{G} = \{\lambda_0, \lambda_3, \lambda_8\} = U(1)^3 \subset U(3)$$

and elements of the generated subgroup of symmetries are of the form

$$U_g = e^{i\alpha} e^{i\alpha_0\lambda_0}\ e^{i\alpha_3\lambda_3}\ e^{i\alpha_8\lambda_8}$$

I define the following basis of $\mathcal{G}$:

$$\lambda_e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_\mu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_\tau = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These matrices generates phase tfm for each kind of leptons (λ_e generates phase transformations for e, etc.). Conserved quantities of this group are 3, and correspond to the number of particles of each type (antiparticles are counted with negative sign). This is an example of conserved charges due to unitary symmetry that have nothing to do with electric charge.

For instance, the process $\mu^- \to e^-\gamma$ is forbidden in QED, since the number of electrons and muons are not conserved.

Flavour changes in neutral sector are forbidden too, since $\mathcal{L}_N$ is invariant under such group of global symmetries too.

3.3 QED Feynman Rules

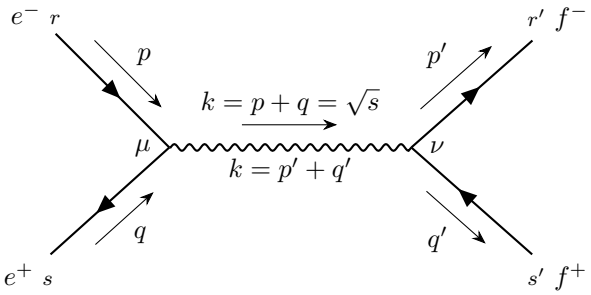
See Schwartz sec. 13.Intro, 13.1

3.4 Process $e^+e^- \to f^+f^-$

Peskin sec. 5.1

We evaluate the cross-section for the process $e^+e^- \to f^+f^-$, where f is a generic fermion couples with the photon (e.g. another electron of a muon or a tauon). We take $f \neq e$ in order to simplify the problem.

At the tree level there is only one Feynman diagram



which is called s-channel since the internal line describe a photon with momentum $\sqrt{s}$. The label r, s, r', s' denote the spin of the spinors.

Taking into account the momentum conservation we have

$$S_{fi} = (2\pi)^4 \delta^4(p + q - p' - q')\mathcal{M}_{fi}$$

with Feynman amplitude[II]

$$\begin{aligned}
\mathcal{M}_{fi} &= (-iq)^2 \big[\bar{u}_{r'}(p')\gamma^\nu \; v_{s'}(q')\big] \big[\bar{v}_s(q)\gamma^\mu \; u_r(p)\big] D^F_{\mu\nu}(k) \\
&= (-iq)^2 \frac{-ig_{\mu\nu}}{k^2 + i\varepsilon} \; \big[\bar{u}_{r'}(p')\gamma^\nu \; v_{s'}(q')\big] \big[\bar{v}_s(q)\gamma^\mu \; u_r(p)\big] \\
&= \frac{iq^2}{s + i\varepsilon} \; \big[\bar{u}_{r'}(p')\gamma_\mu \; v_{s'}(q')\big] \big[\bar{v}_s(q)\gamma^\mu \; u_r(p)\big]
\end{aligned}$$

In the second step we chosen the propagator with parameter $\xi = 1$. One can prove that the amplitude do not depend on the choice of ξ, see Maggiore pag 187.

Using the identity $(\bar{u}\gamma^\mu v)^* = \bar{v}\gamma^\mu u$ (which can be proved by direct calculation using $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$) we obtain (we omit polarization indices)

$$|\mathcal{M}_{fi}|^2 = \mathcal{M}_{fi}\mathcal{M}^*_{fi} = \frac{q^4}{s^2} \underbrace{\big[\bar{u}(p')\gamma_\mu v(q')\big] \big[\bar{v}(q')\gamma_\nu \, u(p')\big]}_{f \text{ current}} \underbrace{\big[\bar{v}(q)\gamma^\mu u(p)\big] \big[\bar{u}(p)\gamma^\nu \, v(q)\big]}_{e \text{ current}}$$

At this point, we are still free to specify any particular spinors $u_r(p)$, $\bar{v}_{s'}(p')$ and so on, corresponding to any desired spin states of the fermions.

3.4.1 Sum Over Fermion Spins. Squared Averaged Feynman Amplitude

Peskin sec. 5.1

The Feynman amplitude simplifies considerably when we throw away the spin information. We want to compute

$$\big|\overline{\mathcal{M}_{fi}}\big|^2 = \underbrace{\frac{1}{2}\sum_s \frac{1}{2}\sum_r}_{\substack{\text{average over} \\ \text{initial states}}} \underbrace{\sum_{s'} \sum_{r'}}_{\substack{\text{sum over} \\ \text{final states}}} |\mathcal{M}(r, s, r', s')|^2$$

[II] Notice that the first square braked correspond to the fermionic path on the right, flowing in the direction $f^- \to \nu \to f^+$. The second square bracket instead correspond to the fermionic path on the left, $e^- \to \mu \to e^+$.

This sum can be performed using completeness relations for Dirac spinors

$$\sum_r u_r(p)\bar{u}_r(p) = \not{p} + m \qquad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m$$

Writing spinors indexes explicitly

$$\begin{aligned}\sum_{rs} e\text{-current} &= \sum_{rs} \bar{v}_a^s(q)(\gamma^\mu)_{ab}\, u_b^r(p)\, \bar{u}_c^r(p)(\gamma^\nu)_{cd}\, v_d^s(q) \\ &= (\not{q} - m)_{da}\, \gamma^\mu_{ab}\, (\not{p} + m)_{bc}\, \gamma^\nu_{cd} \\ &= \mathrm{Tr}\big[(\not{q} - m)\gamma^\mu(\not{p} + m)\gamma^\nu\big]\end{aligned}$$

and similarly

$$\sum_{r's'} f\text{-current} = \mathrm{Tr}\big[(\not{p}' + M)\gamma_\mu(\not{q}' - M)\gamma_\nu\big]$$

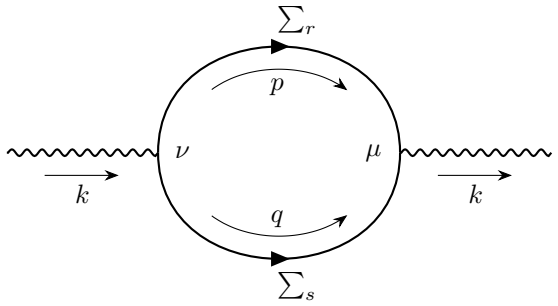
where m is the mass of the electron while M is the mass of the fermion f. So we obtain

$$\big|\overline{\mathcal{M}_{fi}}\big|^2 = \frac{q^4}{4s^2}\,\mathrm{Tr}\big[(\not{q} - m)\gamma^\mu(\not{p} + m)\gamma^\nu\big]\,\mathrm{Tr}\big[(\not{p}' + M)\gamma_\mu(\not{q}' - M)\gamma_\nu\big]$$

The spinors u and v have disappeared, leaving us with a much cleaner expression in terms of γ matrices. This trick is very general: any QED amplitude involving external fermions, when squared and summed or averaged over spins, can be converted in this way to traces of products of Dirac matrices.

There is a trick to obtain previous formula using only Feynman rules. If we set $\mathcal{M}_{fi} = \mathcal{M}_e\mathcal{M}_f$ (we divide $\mathcal{M}_{fi}$ in e and f currents), then $|\mathcal{M}_{fi}|^2 = \mathcal{M}_e^*\mathcal{M}_e\,\mathcal{M}_f^*\mathcal{M}_f$. We have

$\sum_{rs} \mathcal{M}_e^*\,\mathcal{M}_e =$

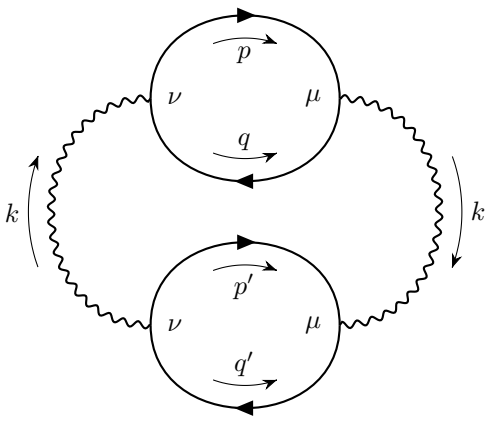


and then the closed fermion line contributes in the final amplitude with a factor

$$\frac{q^2}{2s}\,\mathrm{Tr}\big[(\not{p} + m)\gamma^\nu(\not{q} - m)\gamma^\mu\big]$$

Then the complete unpolarized squared Feynman amplitude is given by (summations are omitted)

$\big|\overline{\mathcal{M}_{fi}}\big|^2 =$



Going back to the expression of $|\mathcal{M}_{fi}|^2$ that we obtained, now we need to calculate explicitly the traces of γ matrices

$$\mathrm{Tr}\big[(\not{p}+m)\gamma^\nu(\not{q}-m)\gamma^\mu\big] = \mathrm{Tr}\big[\not{p}\,\gamma^\nu\not{q}\,\gamma^\mu\big] - m^2\,\mathrm{Tr}[\gamma^\nu\,\gamma^\mu] + m\big(\mathrm{Tr}\big[\gamma^\nu\not{q}\,\gamma^\mu\big] - \mathrm{Tr}\big[\not{p}\,\gamma^\nu\gamma^\mu\big]\big)$$

Now we prove some properties of γ matrices traces (other useful properties can be found in Peskin, page 133-135):

(I)

$$\begin{aligned}
\mathrm{Tr}[\gamma^\mu] &= \mathrm{Tr}\big[\gamma^5\,\gamma^5\,\gamma^\mu\big] && \leftarrow (\gamma^5)^2 = \mathbb{1}\\
&= -\,\mathrm{Tr}\big[\gamma^5\,\gamma^\mu\,\gamma^5\big] && \leftarrow \{\gamma^5,\gamma^\mu\} = 0\\
&= -\,\mathrm{Tr}\big[\gamma^5\,\gamma^5\,\gamma^\mu\big] && \leftarrow \text{cyclicity of the trace}\\
&= -\,\mathrm{Tr}[\gamma^\mu]\\
&= 0
\end{aligned}$$

(II)

$$\begin{aligned}
\mathrm{Tr}[\gamma^\mu\gamma^\nu] &= 2g^{\mu\nu}\,\mathrm{Tr}[\mathbb{1}_{4\times4}] - \mathrm{Tr}[\gamma^\nu\gamma^\mu] && \leftarrow \{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\\
&= 8g^{\mu\nu} - \mathrm{Tr}[\gamma^\mu\gamma^\nu] && \leftarrow \text{cyclicity of the trace}\\
&= 4g^{\mu\nu}
\end{aligned}$$

(III)

$$\begin{aligned}
\mathrm{Tr}[\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma] &= 2g^{\mu\nu}\,\mathrm{Tr}[\gamma^\rho\,\gamma^\sigma] - \mathrm{Tr}[\gamma^\nu\,\gamma^\mu\,\gamma^\rho\,\gamma^\sigma]\\
&= 2g^{\mu\nu}\,\mathrm{Tr}[\gamma^\rho\,\gamma^\sigma] - 2g^{\mu\rho}\,\mathrm{Tr}[\gamma^\nu\gamma^\sigma] + \mathrm{Tr}[\gamma^\nu\,\gamma^\rho\,\gamma^\mu\,\gamma^\sigma]\\
&= 2g^{\mu\nu}\,\mathrm{Tr}[\gamma^\rho\,\gamma^\sigma] - 2g^{\mu\rho}\,\mathrm{Tr}[\gamma^\nu\gamma^\sigma] + 2g^{\mu\sigma}\,\mathrm{Tr}[\gamma^\nu\gamma^\rho] - \mathrm{Tr}[\gamma^\nu\,\gamma^\rho\,\gamma^\sigma\,\gamma^\mu]\\
&= 8\big(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\big) - \mathrm{Tr}[\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma]\\
&= 4\big(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\big)
\end{aligned}$$

(IV) From previous calculation, using induction, can be easily proved that $\mathrm{Tr}\big[\underbrace{\gamma^\mu\ldots\gamma^\sigma}_{\substack{\text{odd \# of}\\ \gamma\text{ matrices}}}\big] = 0$.

With these relations we obtain

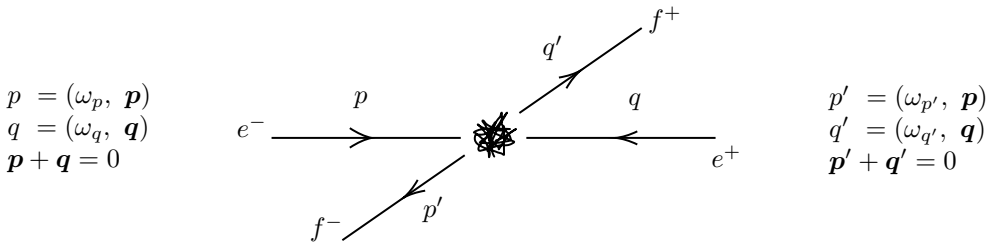
$$\begin{aligned}
\mathrm{Tr}\big[(\not{p}+m)\gamma^\nu(\not{q}-m)\gamma^\sigma\big] &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})p_\mu q_\rho - m^2 4g^{\nu\sigma}\\
&= 4(p^\nu q^\sigma - g^{\nu\sigma}p\cdot q + p^\sigma q^\nu - g^{\nu\sigma}m^2)\\
&= 4\big(p^\nu q^\sigma + p^\sigma q^\nu - g^{\nu\sigma}(p\cdot q + m^2)\big)
\end{aligned}$$

therefore

$$\begin{aligned}
\big|\overline{\mathcal{M}_{fi}}\big|^2 &= \frac{4q^4}{s^2}\left(p^\mu q^\nu + p^\nu q^\mu - g^{\mu\nu}(p\cdot q + m^2)\right)\left(p'_\mu\,q'_\nu + p'_\nu\,q'_\mu - g_{\mu\nu}(p'\cdot q' + M^2)\right)\\
&\approx \frac{8q^4}{s^2}\big((p\cdot p')(q\cdot q') + (p\cdot q')(p'\cdot q) + M^2(p\cdot q)\big) \qquad \text{if } m \ll M
\end{aligned}$$

To obtain a more explicit formula we must specialize to a particular frame of reference and express the vectors p, q, p', q', k in terms of the basis kinematic variables (energies and angles) in that frame. In practice, the choice of frame will be dictated by the experimental conditions.

We want to calculate cross section in the center of mass frame (in the next picture the arrow indicate the direction of the momenta, do not care about particle / antiparticle convention, moreover the angle between $\boldsymbol{p}$ and $\boldsymbol{p'}$ is called θ:



In the CM frame we have following kinematics relations

$$\left.\begin{array}{c}(p+q)=(\omega_p+\omega_q,\boldsymbol{p}+\boldsymbol{q})=(\omega_p+\omega_q,0)\\ s=(p+q)^2=(\omega_p+\omega_q)^2\end{array}\right\}\quad\Rightarrow\quad (p+q)=(\sqrt{s},0)$$

and

$$\omega_p=\sqrt{|\boldsymbol{p}|^2+m^2}=\sqrt{|\boldsymbol{q}|^2+m^2}=\omega_q\quad\Rightarrow\quad \omega_{p'}=\omega_{q'}=\frac{\sqrt{s}}{2}$$

Equivalent relations holds also for final momenta, therefore we have

$$\begin{gathered}(p+q)=(p'+q')=(\sqrt{s},0)\\ \omega_p=\omega_q=\omega_{p'}=\omega_{q'}=\frac{\sqrt{s}}{2}\equiv\omega\\ \boldsymbol{p}+\boldsymbol{q}=\boldsymbol{p'}+\boldsymbol{q'}=0\end{gathered}$$

For products of momenta we get (notice $|\boldsymbol{p}|=\sqrt{\omega^2-m^2}\overset{m\approx 0}{\simeq}\omega$)

$$\begin{aligned}(p\cdot q)&=\omega^2-\boldsymbol{pq}=\omega^2|\boldsymbol{p}|^2=2\omega^2-m^2\overset{m\approx 0}{\simeq}2\omega^2\\ (p'\cdot q')&=\omega^2-\boldsymbol{pq}=\omega^2|\boldsymbol{p}|^2=2\omega^2-M^2\\ (p\cdot p')&=\omega^2-\boldsymbol{pp'}=\omega^2-|\boldsymbol{p}||\boldsymbol{p'}|\cos\theta=(q\cdot q')\overset{m\approx 0}{\simeq}\omega(\omega-|\boldsymbol{p'}|\cos\theta)\\ (p\cdot q')&=\omega^2+\boldsymbol{pp'}=\omega^2+|\boldsymbol{p}||\boldsymbol{p'}|\cos\theta=(q\cdot p')\overset{m\approx 0}{\simeq}\omega(\omega+|\boldsymbol{p'}|\cos\theta)\\ |\boldsymbol{p'}|^2&=\omega^2-M^2=\omega^2\left(1-\frac{M^2}{\omega^2}\right)=\frac{s}{4}\left(1-\frac{4M^2}{s}\right)\\ |\boldsymbol{p}|^2&=\omega^2-m^2\overset{m\approx 0}{\simeq}\omega^2=\frac{s}{4}\end{aligned}$$

Now we can rewrite $\left|\overline{\mathcal{M}_{fi}}\right|^2$ in terms of ω and θ. If $m\approx 0$ is valid (e.g. $m_e/m_\mu=1/200$)

$$\begin{aligned}\left|\overline{\mathcal{M}_{fi}}\right|^2&\overset{m\approx 0}{\simeq}\frac{8q^4}{s^2}\left[(\omega^4-2\omega^3|\boldsymbol{p'}|\cos\theta+\omega^2|\boldsymbol{p'}|^2\cos^2\theta)+(\omega^4+2\omega^3|\boldsymbol{p'}|\cos\theta+\omega^2|\boldsymbol{p'}|^2\cos^2\theta)+2M^2\omega^2\right]\\ &=\frac{8q^4}{s^2}\left[\frac{2s}{4}\left(\frac{s}{4}+\frac{s}{4}\left(1-\frac{4M^2}{s}\right)\cos^2\theta\right)+2M^2\frac{s}{4}\right]\\ &=q^4\left[\left(1+\frac{4M^2}{s}\right)+\left(1-\frac{4M^2}{s}\right)\cos^2\theta\right]\end{aligned}$$

Cross section formula in the CM frame reads

$$\begin{aligned}\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)_{CM}&=\frac{1}{64\pi^2}\ \frac{1}{s}\ \frac{|p'|}{|q'|}\ \left|\overline{\mathcal{M}}\right|^2_{CM}\\ &=\frac{q^4}{64\pi^2}\ \frac{1}{s}\ \left(1-\frac{4M^2}{s}\right)^{1/2}\left[\left(1+\frac{4M^2}{s}\right)+\left(1-\frac{4M^2}{s}\right)\cos^2\theta\right]\\ &=\frac{\alpha^2}{4s}\left(1-\frac{4M^2}{s}\right)^{1/2}\left[\left(1+\frac{4M^2}{s}\right)+\left(1-\frac{4M^2}{s}\right)\cos^2\theta\right]\quad\text{with }\alpha\equiv\frac{e^2}{4\pi}\approx\frac{1}{137}\end{aligned}\tag{3.2}$$

Integrating over $\mathrm{d}\Omega$ we find the total unpolarized cross section

$$(\bar{\sigma})_{TOT} = \int \mathrm{d}\Omega \left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)_{CM} = \frac{4\pi\alpha^2}{3s}\left(1-\frac{4M^2}{s}\right)^{1/2}\left(1+\frac{2M^2}{s}\right) + o(\alpha^3) \tag{3.3}$$

which is frame independent.

Again we see that we have the physical constraint $s = 4\omega^2 > 4M^2$, meaning that the initial energy must be large enough to create the masses of the final particles, $\omega > M$. Moreover we see that for $\omega \approx M$ the cross section vanish and (see eq. (3.2)) there is almost no dependence on the angle.

The term $\sigma(\alpha^3)$ contains all corrections at higher order, which at the present we are neglecting, since we are considering only tree level Feynman diagrams.

We used approximation $m \approx 0$ because usually in experiments $\omega \approx 1GeV$ and $m_\mu \simeq 105MeV \approx 200m_e$. For very high energy experiments, $\omega \approx TeV$, we can omit also f mass, $M \approx 0$. This is called **ultra relativistic regime** (only energies are taken into account)

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)^{UR}_{CM} = \frac{\alpha^2}{4s}\underbrace{(1+\cos\theta)}_{\substack{\text{scattering amplitude}\\ \text{is higher for small angles}}} \qquad (\bar{\sigma})^{UR}_{TOT} = \frac{4\pi\alpha^2}{3s} + o\left(\frac{M^2}{s}\right) \tag{3.4}$$

Let's summarize how we obtained these results. The method extends in a straightforward way to the calculation of unpolarized cross section for other QED processes at lowest order. The general procedure is the following:

(1) draw the diagram(s) for the desired process;

(2) use Feynman rules to write down the amplitude $\mathcal{M}_{fi}$;

(3) square the amplitude and average or sum over spins, using completeness relations (for processes involving photons in the final state there is an analogous completeness relation, we will derive it in Compton scattering);

(4) evaluate traces using γ-matrices proprieties, collect terms and simplify the answer as much as possible;

(5) specialize to a particular frame of reference, and draw a picture of the kinematic variable in that frame; express all 4-momentum vectors in terms of suitably chosen set of variables such as E and θ

(6) plug the resulting expression for $\left|\overline{\mathcal{M}_{fi}}\right|^2$ into the cross-section formula and integrate over phase space variables that are not measured to obtain a differential cross section in the desired form.

Exercise 1

Calculate cross sections of following precesses ($f \neq e$) (unpolarized)

- $e^- f^- \to e^- f^-$ (t-channel)
- $e^- f^+ \to e^- f^+$ (u-channel)

3.4.2 Polarized scattering relations between helicity and chiarality - The ultra relativistic limit and helicity amplitudes

Peskin sec 5.2; Schwartz sec 13.3; Mandl sec 8.4

In the previous calculation we obtained the polarized amplitude

$$\mathcal{M}_{fi} = \frac{iq^2}{s}\left[\bar{u}_{r'}(p')\gamma^\mu v_{s'}(q')\right]\left[\bar{v}_s(q)\gamma_\mu u_r(p)\right]$$

Now we want to perform the computation of the different contributions of $\left|\overline{\mathcal{M}}\right|^2$ due to the different polarizations that constitutes our physical states.

We must choose a basis of polarization states. The best choice is to quantize each spin along the direction of particle's motion, that is, to use states of definite helicity.

In general, helicity projectors are hard to be used for lower energies, so we work in the ultra relativistic limit. Recall that in the massless limit, the left- and right-handed helicity states of a Dirac particle live in different representations of the Lorentz group. We might expect them to behave independently, and in fact they do.

We would like to use the spin sum identities to write the squared amplitude in term of traces as before, ever though we now want to consider only one set of polarizations at a time. We note that in the ultra-relativistic limit, helicity is related to chirality, so we can use chirality projectors, that are much simpler. In particular, let $\Lambda_\pm$ be energy projectors, $\Pi_\pm$ be helicity projectors, $P_{L,R}$ be chirality projectors, and h be the helicity eigenvalue, then following relations holds:

$$\Pi_\pm \Lambda_+ = P_{R,L}\Lambda_+ \qquad \Pi_\pm \Lambda_- = P_{L,R}\Lambda_-$$

where

$$P_R = \frac{1+\gamma_5}{2} \qquad P_L = \frac{1-\gamma_5}{2}$$

We notice that for antiparticles the relations between chirality and helicity is inverted. This can be easily interpreted using Dirac's Holes Theory, for example an antiparticle with positive helicity is the hole left by a particle with positive helicity (since both momenta and spin change sign), and so particles and antiparticles with same helicity must have inverse chirality.

Using chirality projectors properties we can write Dirac currents as

$$\begin{aligned}\bar{v}\gamma^\mu u &= \bar{v}(P_L + P_R)\gamma^\mu\,(P_L + P_R)u \\ &= \bar{v}P_L\,\gamma^\mu\,P_L\,u + \bar{v}P_R\,\gamma^\mu\,P_R\,u + \bar{v}P_L\,\gamma^\mu\,P_R\,u + \bar{v}P_R\,\gamma^\mu\,P_L\,u\end{aligned}$$

Recall the commuting relations between γ matrices and chirality projectors

$$P_{R,L}\,\gamma^\mu = \frac{1\pm\gamma^5}{2}\gamma^\mu = \frac{\gamma^\mu \pm \gamma^5\,\gamma^\mu}{2} = \frac{\gamma^\mu \pm \overbrace{\{\gamma^5,\gamma^\mu\}}^{=0} \mp \gamma^\mu\,\gamma^5}{2} = \gamma^\mu\,\frac{1\mp\gamma^5}{2} = \gamma^\mu\,P_{L,R}$$

then we can rewrite

$$\begin{aligned}\bar{v}\gamma^\mu\,u &= \bar{v}\gamma^\mu\,P_R\,P_L\,u + \bar{v}\gamma^\mu\,P_L\,P_R\,u + \bar{v}\gamma^\mu\,P_R\,P_R\,u + \bar{v}\gamma^\mu\,P_L\,P_L\,u \\ &= \bar{v}\gamma^\mu\,P_R\,u + \bar{v}\gamma^\mu\,P_L\,u = \bar{v}\gamma^\mu_R\,u + \bar{v}\gamma^\mu_L\,u = J^\mu_R + J^\mu_L\end{aligned}$$

where we are defined $\gamma^\mu_{R,L} = \gamma^\mu\,P_{R,L}$ and the left- and right-currents:

$$J^\mu_L = \bar{v}\,\gamma^\mu_L\,u \qquad J^\mu_R = \bar{v}\,\gamma^\mu_R\,u$$

We defined right- and left-handed spinors as follows:

$$\begin{aligned}u_L &= P_L\,u \quad u_R = P_R\,u \\ v_L &= P_L\,v \quad v_R = P_R\,v\end{aligned}$$

and them conjugated as (let w be a generic spinors, either u or v)

$$\begin{aligned}w_L = P_L\,\psi \quad &\to \quad \bar{\psi}_L = \bar{\psi}\,P_R \\ w_R = P_R\,\psi \quad &\to \quad \bar{\psi}_R = \bar{\psi}\,P_L\end{aligned}$$

With this notation I can rewrite left- and right-handed currents as

$$\begin{aligned}\bar{v}\,\gamma^\mu\,u &= \bar{v}\,\gamma^\mu_R\,u + \bar{v}\,\gamma^\mu_L\,u = \bar{v}_R\gamma^\mu\,u_R + \bar{v}_L\gamma^\mu\,u_L \\ \bar{u}\,\gamma^\mu\,v &= \bar{u}\,\gamma^\mu_R\,v + \bar{u}\,\gamma^\mu\,v = \bar{u}_R\,\gamma^\mu\,v_R + \bar{u}_L\,\gamma^\mu\,v_L\end{aligned}$$

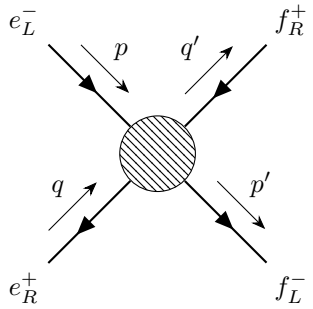
Here is evident that a Dirac current can be divided in its left handed and right handed components.

An important point to highlight is a notational problem: a particle with positive helicity is associated to the spinor with positive helicity and positive energy u_R, while an antiparticle with positive helicity is associated to the spinor with positive helicity and negative energy v_L. Nevertheless, often particles and antiparticles with positive helicity are denoted with R, such as $e^- R/e^+_R$, while particles with negative helicity are denoted with L, such as e^-_L/e^+_L. While this choice makes no problem with particles, in the case of antiparticles the situation is more trickily: an antiparticle with positive helicity is denoted with R, even though it is always left-handed and associated with v_L; and an antiparticle with negative helicity is denoted with L, even though it is always right-handed and associated to v_R.

Going back to the Feynman amplitude

$$\begin{aligned}\mathcal{M}_{fi} = \frac{iq^2}{s}\Big\{ &\big[\bar{u}(p')\,\gamma^\mu_L\,v(q')\big]\big[\bar{v}(q)\,\gamma^L_\mu\,u(p)\big] + && \to\ \mathcal{M}_{LL}\\ &+\big[\bar{u}(p')\,\gamma^\mu_L\,v(q')\big]\big[\bar{v}(q)\,\gamma^R_\mu\,u(p)\big] + && \to\ \mathcal{M}_{LR}\\ &+\big[\bar{u}(p')\,\gamma^\mu_R\,v(q')\big]\big[\bar{v}(q)\,\gamma^L_\mu\,u(p)\big] + && \to\ \mathcal{M}_{RL}\\ &+\big[\bar{u}(p')\,\gamma^\mu_R\,v(q')\big]\big[\bar{v}(q)\,\gamma^R_\mu\,u(p)\big]\Big\} && \to\ \mathcal{M}_{RR}\end{aligned}$$

If I didn't used the UR limit, I would have obtained 16 independent terms, instead of 4. Each factor in the latter, corresponds to a different Feynman diagram with different left- and right-handed, initial and final particles, for instance

$\mathcal{M}_{LL} =$ 

$= \mathrm{UR}\left[e^-\left(h=-\frac{1}{2}\right)+e^+\left(h=+\frac{1}{2}\right)\to f^-\left(h=-\frac{1}{2}\right)+f^+\left(h=+\frac{1}{2}\right)\right]$

Having inserted these projection operators, we are now free to sum over the electron and positron spins in the squared amplitude. As a consequence of projector operators, we saw that the amplitude vanishes (in UR limit) unless the electron and positron have opposite helicity, or equivalently, unless their spinors have the same chirality.

Let's calculate the squared Feynman amplitude of $\mathcal{M}_{LL}$, summing over spins

$$\big|\overline{\mathcal{M}_{LL}}\big|^2 = \sum_{\text{spins}} \underbrace{\big|\bar{u}_{r'}(p')\,\gamma^\mu_L\,v_{s'}(q')\big|^2}_{\mathbf{B}}\underbrace{\big|\bar{v}_s(q)\,\gamma^L_\mu\,u_r(p)\big|^2}_{\mathbf{A}}$$

with

$$\begin{aligned}\mathbf{A} &= \sum_{rs}\big(\bar{v}_s(q)\,\gamma^L_\mu\,u_r(p)\big)\big(\bar{u}_r(p)\,\gamma^l_\nu\,v_s(q)\big)\\ &= \mathrm{Tr}\big[\not{q}\,\gamma^L_\mu\,\not{p}\,\gamma^L_\nu\big] = \mathrm{Tr}\big[q^\rho\,\gamma_\rho\,\gamma_\mu\,P_L\,p^\sigma\,\gamma_\sigma\,\gamma_\nu\,P_L\big]\\ &= \mathrm{Tr}\big[q^\rho\,\gamma_\rho\,\gamma_\mu\,p^\sigma\,\gamma_\sigma\,P_R\gamma_\nu\,P_L\big] = \mathrm{Tr}\left[\not{q}\,\gamma_\mu\,\not{p}\,\gamma_\nu\left(\frac{1-\gamma^5}{2}\right)\right]\\ &= \frac{1}{2}\,\mathrm{Tr}\big[\not{q}\,\gamma_\mu\,\not{p}\,\gamma_\nu\big] - \frac{1}{2}\,\mathrm{Tr}\big[\not{q}\,\gamma_\mu\,\not{p}\,\gamma_\nu\,\gamma_5\big]\end{aligned}$$

and then using

$$\begin{gathered}\mathrm{Tr}\big[\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma\big] = 4\big(g^{\mu\nu}\,g^{\rho\sigma} - g^{\mu\rho}\,g^{\nu\sigma} + g^{\mu\sigma}\,g^{\nu\rho}\big)\\ \mathrm{Tr}\big[\gamma^\mu\,\gamma^\nu\,\gamma^\rho\,\gamma^\sigma\,\gamma^5\big] = -4i\varepsilon^{\mu\nu\rho\sigma}\end{gathered}$$

we can rewrite (for $\mathbf{B}$ the computation is analogous)

$$\mathbf{A} = 2\big(q_\mu p_\nu - q\cdot p g_{\mu\nu} + q_\nu p_\mu - i\varepsilon_{\alpha\mu\beta\nu}\, q^\alpha p^\beta\big)$$
$$\mathbf{B} = 2\big(p'^\mu q'^\nu - q'\cdot p' g^{\mu\nu} + p'^\nu q'^\mu - i\varepsilon^{\rho\mu\sigma\nu} p'_\rho q'_\sigma\big)$$

We finally obtain

$$\begin{aligned}\big|\overline{\mathcal{M}_{LL}}\big|^2 &= \frac{4q^2}{s^2}\big[2(p\cdot p')(q\cdot q') + 2(p\cdot q')(p'\cdot q) - \varepsilon^{\rho\mu\sigma\nu}\,\varepsilon_{\alpha\mu\beta\nu}\, q^\alpha p^\beta p'_\rho q'_\sigma\big] \\ &= \frac{8q^2}{s^2}\big[(p\cdot p')(q\cdot q') + (p\cdot q')(p'\cdot q) - (p\cdot p')(q\cdot q') + (p\cdot q')(q\cdot p')\big] \\ &= \frac{16q^2}{s^2}(p\cdot q')(p'\cdot q)\end{aligned}$$

In a generic frame

$$u = (p-q')^2 = (q-p')^2 = 2m^2 - 2(p\cdot q') \overset{m\approx 0}{\simeq} -2(p\cdot q') = -2(q\cdot p')$$

and in the center of mass frame

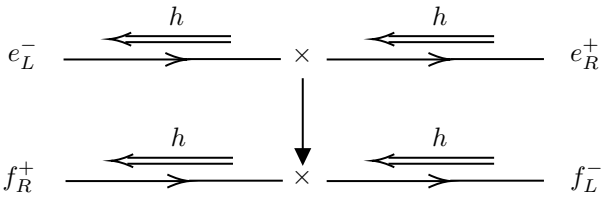
$$(p\cdot q') = (q\cdot p') = \omega^2(1+\cos\theta)$$

so the squared amplitude in the center of mass frame reads

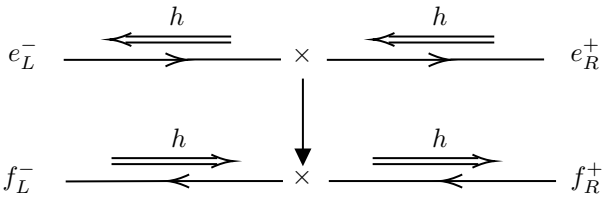
$$|\mathcal{M}_{LL}|^2 = \frac{16q^4}{s^2}\omega^4(1+\cos\theta)^2 = q^4(1+\cos\theta)^2 = q^4\frac{u^2}{s^2}$$

In particular

(i) $|\mathcal{M}_{LL}|^2$ is max when $\theta = 0$



(ii) $|\mathcal{M}_{LL}|^2$ is zero when $\theta = \pm\pi$



We see that the dependence on θ of the scattering amplitude is somehow related with the helicities of the initial and final states, for instance the amplitude vanishes when helicity states are orthogonal. This is what we would expect, since for $\theta = \pi$ the total angular momentum of the final state is opposite to that of the initial state.

Moreover as we pointed out before, the total helicity of the particles must vanish, while the chiralities sum up giving the polarization of the photon which intermediates the interaction.

Exercise 2

Compute $|\mathcal{M}_{RR}|^2, |\mathcal{M}_{LR}|^2, |\mathcal{M}_{RL}|^2$.

Solution:

$$|\mathcal{M}_{RR}|^2 = q^4(1+\cos\theta)^2 = q^4\frac{u^2}{s^2}$$
$$|\mathcal{M}_{LR}|^2 = |\mathcal{M}_{RL}|^2 = q^4(1-\cos\theta)^2 = q^4\frac{t^2}{s^2}$$

Notice that $|\mathcal{M}_{LL}|^2 = |\mathcal{M}_{RR}|^2$ and $|\mathcal{M}_{LR}|^2 = |\mathcal{M}_{RL}|^2$ are consequences of parity invariance.

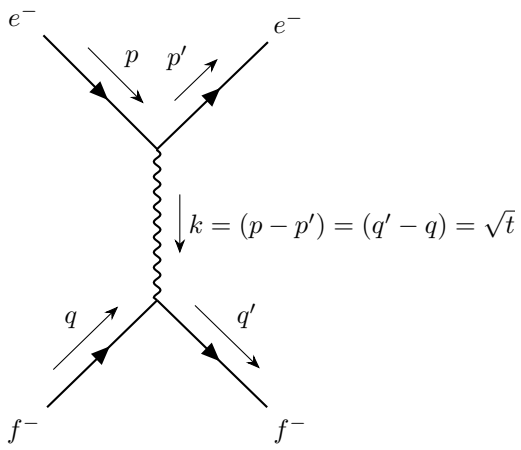
Adding up all four contributions, and dividing by 4 to average over the electron and positron states, we recover the unpolarized cross section derived previously in the massless limit

$$\begin{aligned}|\mathcal{M}|^2 &= \frac{1}{4}\left(|\mathcal{M}_{LL}|^2 + |\mathcal{M}_{RR}|^2 + |\mathcal{M}_{RL}|^2 + |\mathcal{M}_{LR}|^2\right)\\ &= q^4(1+\cos^2\theta)\end{aligned}$$

3.4.3 Mandelstam variables and crossing symmetries

Peskin, sec 5.4

Let's consider a different but closely related QED process: $e^-\, f^- \to e^-\, f^-$



The Feynman amplitude of this process (in Feynmann gauge) reads

$$\mathcal{M}_{fi} = \frac{iq^2}{t}\left(\bar{u}(p')\,\gamma^\mu\,u(p)\right)\left(\bar{u}(q')\,\gamma_\mu\,u(q)\right)$$

The relation between this process and $e^+e^- \to f^+f^-$ becomes clear when we compute the squared amplitude, average and summed over spins:

$$\left|\overline{\mathcal{M}_{fi}}\right|^2 = \frac{q^4}{4t^2}\,\mathrm{Tr}\left[(\not{p}+m)\gamma^\nu(\not{p}'+m)\gamma^\mu\right]\,\mathrm{Tr}\left[(\not{q}+m)\gamma_\nu(\not{q}'+M)\gamma_\mu\right]$$

Recall that for $e^-e^+ \to f^-f^+$

$$\left|\overline{\mathcal{M}_{fi}}\right|^2_{e^-e^+} = \frac{q^4}{4s^2}\,\mathrm{Tr}\left[(\not{p}+m)\gamma^\nu(\not{q}-m)\right]\,\mathrm{Tr}\left[(\not{q}'-m)\gamma_\nu(\not{p}'+m)\gamma_\mu\right]$$

Notice that following symmetry relations hold

$$\begin{aligned} e^+e^- &\leftrightarrow e^-f^- \\ p &\leftrightarrow p \\ q &\leftrightarrow -p' \\ p' &\leftrightarrow q' \\ q' &\leftrightarrow -q \end{aligned} \qquad \xrightarrow{\text{in Mandelstam variables}} \qquad \begin{aligned} e^+e^- &\leftrightarrow e^-f^- \\ s = (p+q)^2 &\leftrightarrow t = (p-p')^2 \\ t = (p-p')^2 &\leftrightarrow u = (p-q')^2 \\ u = (p-q')^2 &\leftrightarrow s = (p+q)^2 \end{aligned}$$

So instead of evaluating traces from scratch, we can just make the same replacement in our previous result. Setting $m = 0$ we find

$$\begin{aligned} \left|\overline{\mathcal{M}_{fi}}\right|^2 =& \frac{8q^4}{t^2}\left((p\cdot q')(p'\cdot q) + (p\cdot q)(p'\cdot q') - M^2(p\cdot p')\right) \\ \underset{\substack{\text{UR limit}\\ n=0}}{\approx}& 2q^4\left(\frac{t^2+u^2}{s^2}\right) \end{aligned}$$

Where we used following relation for the UR limit

$$\begin{aligned} s &= 2pq = 2p'q' \\ t &= -2pp' = -2qq' \\ u &= -2pq' = -2p'q \end{aligned}$$

In the center of mass frame, in the UR limit, we have

$$\begin{aligned} \left|\overline{\mathcal{M}_{fi}}\right|^2_{CM} &= 2e^4\left[\frac{4+(1+\cos\theta)^2}{(1-\cos\theta)^2}\right] \\ \left(\frac{\overline{\mathrm{d}\sigma}}{\mathrm{d}\Omega}\right)_{CM} &= \frac{\alpha^2}{2s}\left[\frac{4+(1+\cos\theta)^2}{(1-\cos\theta)^2}\right] \end{aligned}$$

Where θ is the angle between f^-_{init} and e^-_{fin}

Diagramma foglio 11

3.4.4 Crossing symmetry

The trick we made use of here, namely the relation between the two processes $e^+e^- \to f^+f^-$ and $e^-f^- \to e^-f^-$, is our first example of a type of relation known as **crossing symmetry**:

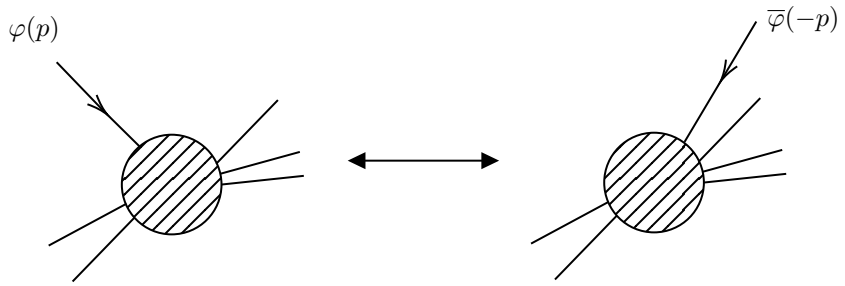
> *The S-matrix for any process involving a particle with momentum p in the initial state is equal to the S-matrix for an otherwise identical process but with an antiparticle of momentum $k = -p$ in the final state*
>
> $$\mathcal{M}\big(\phi(p) + \cdots \to \ldots\big) = \pm\mathcal{M}\big(\cdots \to \cdots + \bar{\phi}(k)\big)$$

Note that there is no value of p for which p and k are both physically allowed, since the particle must have $p^0 > 0$ and the antiparticle $k^0 > 0$. So technically, we should say that either amplitude can be obtained from the other by analytic continuation

previous relation follows directly from Feynman rules. Relative sign between amplitude is not predicted by Feynman rules, but it can be obtained from commuting/anticommuting proprieties of field. For example, for spinors fields

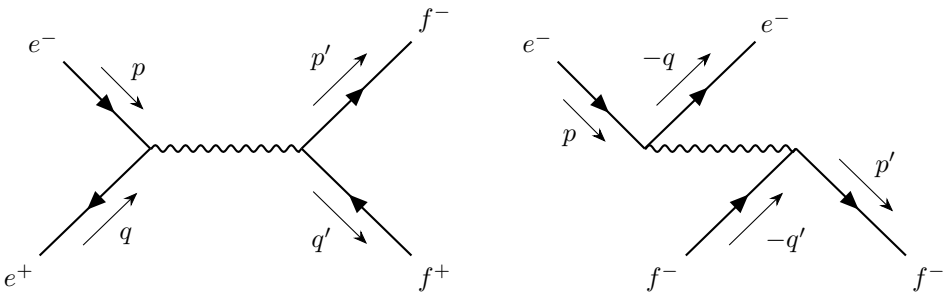
$$\begin{cases} \text{odd permutations} & \to & \text{minus sign} \\ \text{ever permutations} & \to & \text{plus sign} \end{cases}$$

For squared amplitudes the relative sign is irrelevant but it's important when we have to sum different diagrams contributions
The diagrams that contribute to the two amplitudes fall into a natural one-to-one correspondence, where corresponding diagrams differ inly by changing the incoming ϕ into the outgoing $\bar{\phi}$



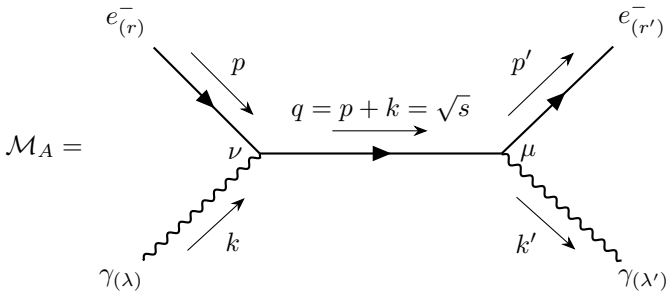
For example, in the $e^-e^+ \to f^+f^-$ and $e^-f^- \to e^-f^-$ correspondence we have

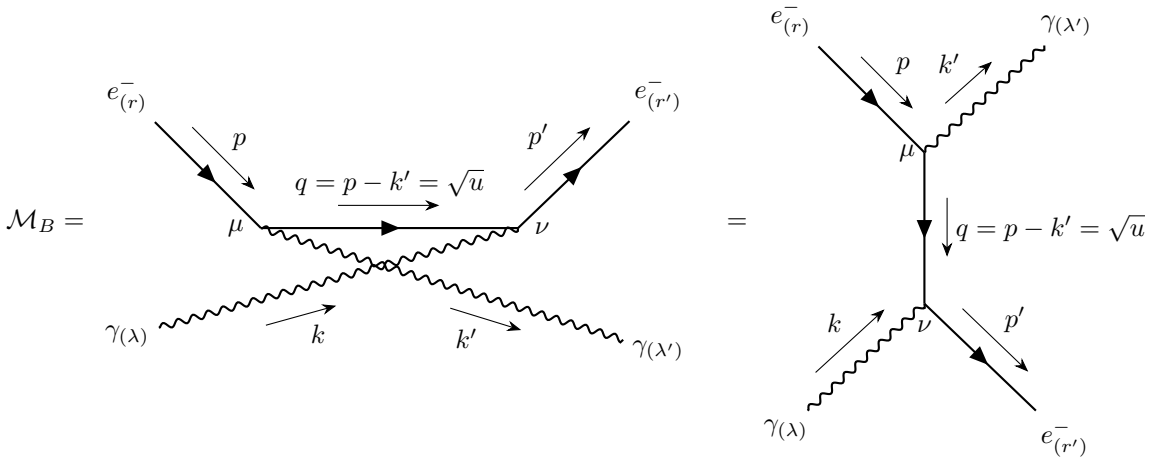
Controlla se sono giusti



3.5 $e^-\gamma \to e^-\gamma$ (Compton)

See Peskin, sec 5.5
Let's examine a process with external bosons: *Compton scattering*, or $e^-\gamma \to e^-\gamma$. This process is described by two independent diagrams, since they are topologically different:





We wrote the diagram of $\mathcal{M}_B$ in two topologically equivalent forms: in the first one is clear the topological relation with diagram of $\mathcal{M}_A$ (this is useful to find the relative sign between diagrams A and B: it's clear that diagrams differs for the permutation of two bosons), while in the second one is clear that it describes a u-channel.

Amplitudes reads, using Feynman rules

$$\begin{aligned}\mathcal{M}_A &= \bar{u}_{r'}(p')(-iq\gamma^\mu)\varepsilon^{\lambda'*}_\mu(k')\tilde{S}_F(p+k)(-iq\gamma^\nu)\varepsilon^\lambda_\nu(k)u_r(p)\\ &= -q^2\varepsilon^{\lambda'*}_\mu(k')\varepsilon^\lambda_\nu(k)\left[\bar{u}_{r'}(p')\gamma^\mu\tilde{S}_F(p+k)\gamma^\nu u_r(p)\right]\end{aligned}$$

$$\mathcal{M}_B = -q^2\varepsilon^{\lambda'*}_\mu(k')\varepsilon^\lambda_\nu(k)\left[\bar{u}_{r'}(p')\gamma^\nu\tilde{S}_F(p-k')\gamma^\mu u_r(p)\right]$$

(Recall that $\tilde{S}_F$ is a matrix, so elements in the squared bracket must be written in this order)

Because of anticommuting relations for bosons, these amplitudes must be summed up in the total amplitude. The explicit form of Feynman propagator for the Dirac field reads

$$\tilde{S}_F(p) = \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon} = \frac{i}{\not{p}-m+i\varepsilon}$$

so total amplitude is

$$\mathcal{M} = -iq^2\varepsilon^{\lambda'*}_\mu(k')\varepsilon^\lambda_\nu(k)\bar{u}_{r'}(p')\left[\frac{\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu}{(p+k)^2-m^2}+\frac{\gamma^\nu(\not{p}-\not{k}'+m)\gamma^\mu}{(p-k')^2-m^2}\right]u_r(p)$$

We make some simplifications before squaring this expression. Since $p^2 = m^2$ and $k^2 = 0$:

$$(p+k)^2-m^2 = 2p\cdot k \qquad (p-k')^2-m^2 = -2p\cdot k'$$

To simplify numerators, I can use Dirac algebra:

$$\begin{aligned}(\not{p}+m)\gamma^\nu u(p) &= (p_\mu\gamma^\mu\gamma^\nu + m\gamma^\nu)u(p) = (2g^{\mu\nu}p_\mu - p_\mu\gamma^\nu\gamma^\mu + m\gamma^\nu)u(p)\\ &= 2p^\nu u(p) - \gamma^\nu\underbrace{(\not{p}-m)}_{2m\,\Lambda_-(p)}u(p) = 2p^\nu u(p)\end{aligned}$$

Using these tricks we obtain

$$\mathcal{M} = -iq^2\varepsilon^{\lambda'*}_\mu(k')\varepsilon^\lambda_\nu(k)\bar{u}_{r'}(p')\left[\frac{\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k}+\frac{-\gamma^\nu\not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{-2p\cdot k'}\right]u_r(p)$$

3.5.1 The Ward Identities and sum over the photon polarizations

See Mandl, sec 8.3

The next step in the calculation will be to square this expression for $\mathcal{M}$ and sum or average over electron and photon polarization states. The sum over electron polarizations can be performed as before, using $\sum u(p)\bar{u}(p) = \not{p} + m$. Fortunately, there is a similar trick for summing over photons polarization vectors. Gauge invariance of the theory implies the gauge invariance of the matrix elements, i.e. of the Feynman amplitudes. It is, of course, only the matrix element itself, corresponding to the sum of all possible Feynman graphs in a given order of perturbation theory, which must be gauge invariant. For example, for the Compton scattering, the individual amplitudes $\mathcal{A}$ and $\mathcal{B}$ are not gauge invariants, but their sum $\mathcal{M}$ is.

For any process involving external photons, the Feynman amplitude $\mathcal{M}$ is of the form

$$\mathcal{M} = \varepsilon_\alpha^{\lambda_1}(k_1)\varepsilon_\beta^{\lambda_2}(k_2)\dots L^{\alpha\beta\cdots}(k_1, k_2, \dots) \tag{3.5}$$

with one polarization vector $\varepsilon^{\lambda_i}(k_i)$ for each external photon, and the tensor amplitude $L^{\alpha\beta\cdots}(k_1, k_2, \dots)$ independent of these polarization vectors.

The polarization vectors are of course gauge dependent. For example, for a free photon described in the Lorentz gauge by the plane wave

$$A^\mu(x) = \text{const} \cdot \varepsilon_\lambda^\mu(k) e^{\pm ikx}$$

the gauge transformation

$$A^\mu \to A'^\mu(x) = A^\mu(x) + \partial^\mu \alpha(x) \qquad \text{with} \quad \alpha(x) = \tilde{\alpha}(k) e^{\pm ikx}$$

implies

$$\varepsilon_\lambda^\mu(k) \to \varepsilon'^\mu_\lambda(k) = \varepsilon_\lambda^\mu(k) \pm ik^\mu \tilde{\alpha}(k)$$

Invariance of the amplitude Eq.(3.5) under this transformation requires

$$k_1^\alpha L_{\alpha,\beta,\dots}(k_1, k_2, \dots) = k_1^\beta L_{\alpha,\beta,\dots}(k_1, k_2, \dots) = \dots = 0$$

i.e. when any external photon polarization vector is replaced by the corresponding four momentum, the amplitude must vanish. This is the statement of the *Ward Identity*:

> *If $\mathcal{M}(k) = \varepsilon_\mu(k) L^\mu(k)$ is the amplitude for some QED process involving an external photon with momentum k, then this amplitude vanishes if we replace ε_μ with k_μ:*
>
> $$k_\mu L^\mu(k) = 0$$

Exercise 3

Verify explicitly the Ward Identity for the Feynman amplitude of Compton scattering

See Peskin, sec 5.5

Returning to our derivation of the polarization sum formula for squared scattering amplitude. Writing in general

$$\mathcal{M} = \varepsilon_\mu^{(\lambda)}(k) L^\mu(k)$$

then the sum over polarizations of the photon with momentum k reads

$$\sum_{\lambda=1,2} |\mathcal{M}|^2 = \sum_{\lambda=1,2} \varepsilon_\mu^{(\lambda)}(k) \varepsilon_\nu^{(\lambda)*}(k) L^\mu(k) L^{\nu\dagger}(k)$$

Because of the covariance of the theory we can do the calculation in a specific frame. In order to simplify the analysis we choose the frame where the photon moves along the $\hat{z}$ axis:

$$k^\mu = (|k|, 0, 0, |k|)$$

In this case the Ward Identity reads

$$0 = k_\mu L^\mu = |k| \left(L^0 - L^3\right) \qquad \longrightarrow \qquad L^0 = L^3$$

Recall that in this frame

$$\varepsilon_\mu^{(1)}(k) = (0,1,0,0) \qquad \varepsilon_\mu^{(2)}(k) = (0,0,1,0)$$

So we have

$$\sum_{\lambda=1,2} \varepsilon_\mu^{(\lambda)}(k)\varepsilon_\nu^{(\lambda)*}(k)L^\mu(k)L^{\nu\dagger}(k) = \left|L^1\right|^2 + \left|L^2\right|^2 = \left|L^1\right|^2 + \left|L^2\right|^2 + \left|L^3\right|^2 - \left|L^0\right|^2 = -g_{\mu\nu}L^\mu L^\nu$$

So we obtained the general rule to simplify photons polarization sum[III]

$$\boxed{\sum_{\lambda=1,2} \varepsilon_\mu^{(\lambda)}(k)\varepsilon_\nu^{(\lambda)*}(k)L^\mu(k)L^{\nu\dagger}(k) \quad \longrightarrow \quad -g_{\mu\nu}}$$

3.5.2 The Klein-Nishima formula and the Thomson scattering

See Peskin, sec. 5.5

To obtain the unpolarized cross section for Compton scattering, we use the covariant method described in the previous section. Writing

$$\mathcal{M} = \varepsilon_\mu^{\lambda'*}(k')\varepsilon_\nu^{\lambda}(k)\left(L^{\mu\nu}(k,k')\right)_{r,r'}$$

with

$$\left(L^{\mu\nu}(k,k')\right)_{r,r'} = -iq^2\bar{u}_{r'}(p')\left[\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{-\gamma^\nu \not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{-2p\cdot k'}\right]u_r(p)$$

we obtain

$$\begin{aligned}
|\bar{\mathcal{M}}|^2 &= \frac{1}{4}\left(\sum_{\lambda'} \varepsilon_\mu^{(\lambda')*}(k')\varepsilon_\rho^{(\lambda')}(k')\right)\left(\sum_{\lambda} \varepsilon_\nu^{(\lambda)*}(k)\varepsilon_\sigma^{(\lambda)}(k')\right)\sum_{r,r'}\left(L^{\mu\nu}\right)_{r,r'}\left(L^{\rho\sigma}\right)^\dagger_{r,r'} \\
&= \frac{1}{4}g_{\mu\rho}g_{\nu\sigma}\sum_{r,r'}\left(L^{\mu\nu}\right)_{r,r'}\left(L^{\rho\sigma}\right)^\dagger_{r,r'} = \frac{1}{4}\left(L^{\mu\nu}\right)_{r,r'}\left(L_{\mu\nu}\right)^\dagger_{r,r'} \\
&= \frac{q^4}{4}\,\mathrm{Tr}\left[(\not{p}' + m)\left(\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{\gamma^\nu \not{k}'\gamma^\mu - 2\gamma^\nu p^\mu}{2p\cdot k'}\right)\times\right. \\
&\qquad \left.\times(\not{p} + m)\left(\frac{\gamma_\nu \not{k}\gamma_\mu + 2\gamma_\mu p_\nu}{2p\cdot k} + \frac{\gamma_\mu \not{k}'\gamma_\nu - 2\gamma_\nu p_\mu}{2p\cdot k'}\right)\right] \\
&= \frac{q^4}{4}\left\{\frac{T_{AA}}{(2p\cdot k)^2} + \frac{T_{BB}}{(2p\cdot k')^2} + \frac{T_{AB}+T_{BA}}{(2p\cdot k)(2p\cdot k')}\right\}
\end{aligned}$$

where

$$\begin{aligned}
T_{AA} &= \mathrm{Tr}\left[(\not{p}' + m)(\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu)(\not{p} + m)(\gamma_\nu \not{k}\gamma_\mu + 2\gamma_\mu p_\nu)\right] \\
T_{BB} &= \mathrm{Tr}\left[(\not{p}' + m)(\gamma^\nu \not{k}'\gamma^\mu - 2\gamma^\nu p^\mu)(\not{p} + m)(\gamma_\mu \not{k}'\gamma_\nu - 2\gamma_\nu p_\mu)\right] \\
T_{AB} &= \mathrm{Tr}\left[(\not{p}' + m)(\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu)(\not{p} + m)(\gamma_\mu \not{k}'\gamma_\nu - 2\gamma_\nu p_\mu)\right] \\
T_{BA} &= \mathrm{Tr}\left[(\not{p}' + m)(\gamma^\nu \not{k}'\gamma^\mu - 2\gamma^\nu p^\mu)(\not{p} + m)(\gamma_\nu \not{k}\gamma_\mu + 2\gamma_\mu p_\nu)\right]
\end{aligned}$$

[III] Notice that we could prove (see Peskin) that even if we took $\lambda = 0,1,2,3$, we could have obtained that the unphysical time-like and longitudinal photons can be consistently omitted from QED calculations, since in any event the squared amplitudes for producing these states cancel to give zero total probability.

Notice that $T_{BB} = T_{AA}(k \leftrightarrow -k')$ and $T_{BA} = T_{AB}(k \leftrightarrow -k')$, we need therefore only calculate T_{AA} and T_{AB}.
Considering T_{AA}, there are 16 terms inside the trace, but half contains an odd number of γ matrices and therefore vanishes. Other terms are

$$
\begin{aligned}
(1) &= \mathrm{Tr}\left[\not p' \gamma^\mu \not k \gamma^\nu \not p \gamma_\nu \not k \gamma_\mu\right] \\
(2) &= 2\,\mathrm{Tr}\left[\not p' \gamma^\mu \not k \gamma^\nu \not p \gamma_\mu p_\nu\right] = 2\,\mathrm{Tr}\left[\not p' \gamma^\mu \not k \not p \not p \gamma_\mu\right] \\
(3) &= 2\,\mathrm{Tr}\left[\not p' \gamma^\mu p^\nu \not p \gamma_\nu \not k \gamma_\mu\right] = 2\,\mathrm{Tr}\left[\not p' \gamma^\mu \not p \not p \not k \gamma_\mu\right] \\
(4) &= 4\,\mathrm{Tr}\left[\not p' \gamma^\mu p^\nu \not p \gamma_\mu p_\nu\right] = 4p^2\,\mathrm{Tr}\left[\not p' \gamma^\mu \not p \gamma_\mu\right] \\
(5) &= m^2\,\mathrm{Tr}[\gamma^\mu \not k \gamma^\nu \gamma_\nu \not k \gamma_\mu] \\
(6) &= 2m^2\,\mathrm{Tr}[\gamma^\mu \not k \gamma^\nu \gamma_\mu p_\nu] = 2m^2\,\mathrm{Tr}\left[\gamma^\mu \not k \not p \gamma_\mu\right] \\
(7) &= 2m^2\,\mathrm{Tr}[\gamma^\mu p^\nu \gamma_\nu \not k \gamma_\mu] = 2m^2\,\mathrm{Tr}\left[\gamma^\mu \not p \not k \gamma_\mu\right] \\
(8) &= 4m^2\,\mathrm{Tr}[\gamma^\mu p^\nu \gamma_\mu p_\nu] = 4m^2 p^2\,\mathrm{Tr}[\gamma^\mu \gamma_\mu]
\end{aligned}
$$

In order to simplify above formulas we recall the proprieties of contractions of γ matrices, i.e. products in the form $\gamma^\mu A \gamma^\mu$ where A is a matrix:

(i) $\gamma^\mu \gamma_\mu = 4\mathbb{1}$

(ii) $\gamma^\mu \not p \gamma_\mu = -2\not p$

(iii) $\gamma^\mu \not p \not q \gamma_\mu = 4p \cdot q$

(iv) $\gamma^\mu \not p \not q \not k \gamma_\mu = -2\not k \not q \not p$

Using these proprieties, cyclicity of the trace and anticommuting proprieties of gamma matrices[IV], we obtain (remember that $p^2 = m^2$ and $k^2 = 0$):

$$
\begin{aligned}
(1) &= \mathrm{Tr}\left[\not p' \gamma^\mu \not k \gamma^\nu \not p \gamma_\nu \not k \gamma_\mu\right] = -2\,\mathrm{Tr}\left[\not p' \gamma^\mu \not k \not p \not k \gamma_\mu\right] = 4\,\mathrm{Tr}\left[\not p' \not k \not p \not k\right] = -4\,\mathrm{Tr}\left[\not p' \not k^2 \not p\right] + 8(p \cdot k)\,\mathrm{Tr}\left[\not p' \not k\right] = 32(p \cdot k)(p' \cdot k) \\
(2) &= 2\,\mathrm{Tr}\left[\not p' \gamma^\mu \not k \not p \not p \gamma_\mu\right] = -4\,\mathrm{Tr}\left[\not p' \not p \not p \not k\right] = -4m^2\,\mathrm{Tr}\left[\not p' \not k\right] = -16m^2(p' \cdot k) \\
(3) &= 2\,\mathrm{Tr}\left[\not p' \gamma^\mu \not p \not p \not k \gamma_\mu\right] = 2m^2\,\mathrm{Tr}\left[\not p' \gamma^\mu \not k \gamma_\mu\right] = -4m^2\,\mathrm{Tr}\left[\not p' \not k\right] = -16m^2(p' \cdot k) \\
(4) &= 4p^2\,\mathrm{Tr}\left[\not p' \gamma^\mu \not p \gamma_\mu\right] = -8m^2\,\mathrm{Tr}\left[\not p' \not p\right] = -32m^2(p' \cdot p) \\
(5) &= m^2\,\mathrm{Tr}[\gamma^\mu \not k \gamma^\nu \gamma_\nu \not k \gamma_\mu] = 4m^2\,\mathrm{Tr}[\gamma^\mu \not k \not k \gamma_\mu] = 0 \\
(6) &= 2m^2\,\mathrm{Tr}\left[\gamma^\mu \not k \not p \gamma_\mu\right] = 8m^2(k \cdot p)\,\mathrm{Tr}[\mathbb{1}] = 32m^2(k \cdot p) \\
(7) &= 2m^2\,\mathrm{Tr}\left[\gamma^\mu \not p \not k \gamma_\mu\right] = 8m^2(p \cdot k)\,\mathrm{Tr}[\mathbb{1}] = 32m^2(p \cdot k) \\
(8) &= 4m^2 p^2\,\mathrm{Tr}[\gamma^\mu \gamma_\mu] = 16m^4\,\mathrm{Tr}[\mathbb{1}] = 64m^4
\end{aligned}
$$

At the end we find

$$
\begin{aligned}
T_{AA} &= 16\left(4m^4 - 2m^2 p \cdot p' + 4m^2 p \cdot k - 2m^2 p' \cdot k + 2(p \cdot k)(p' \cdot k)\right) \\
&= 16\left(2m^4 + m^2(s - m^2) - \frac{1}{2}(s - m^2)(u - m^2)\right)
\end{aligned}
$$

[IV] $\not A \not B = A_\mu B_\nu \gamma^\mu \gamma^\nu = A_\mu B_\nu (2g^{\mu\nu}\mathbb{1} - \gamma^\nu \gamma^\mu) = 2(A \cdot B)\,\mathbb{1} - \not B \not A \qquad \rightarrow \qquad \not A \not A = A^2\,\mathbb{1}$
$\mathrm{Tr}\left[\not A \not B\right] = 2(A \cdot B)\ \mathrm{Tr}[\mathbb{1}] - \mathrm{Tr}\left[\not B \not A\right] = 8(A \cdot B) - \mathrm{Tr}\left[\not A \not B\right] \qquad \rightarrow \qquad \mathrm{Tr}\left[\not A \not B\right] = 4(A \cdot B)\mathbb{1}$

where we introduced Mandelstam variables:

$$\begin{aligned} s &= (p+k)^2 = 2p\cdot k + m^2 = 2p'\cdot k' + m^2 \\ t &= (p'-p)^2 = -2p\cdot p' + 2m^2 = -2k\cdot k' \\ u &= (k'-p)^2 = -2k'\cdot p + m^2 = -2k\cdot p' + m^2 \end{aligned}$$

Sending $k \leftrightarrow -k'$ ($s \leftrightarrow u$) we can immediately write

$$\begin{aligned} T_{BB} &= 16\left(4m^4 - 2m^2 p\cdot p' - 4m^2 p\cdot k' + 2m^2 p'\cdot k' + 2(p\cdot k')(p'\cdot k')\right) \\ &= 16\left(2m^4 + m^2(u-m^2) - \frac{1}{2}(u-m^2)(s-m^2)\right) \end{aligned}$$

Exercise 4

Compute the elements T_{AB} and T_{BA}

Evaluating the traces in T_{AB} and T_{BA} requires about the same amount of work as we have just done. The answer is

$$\begin{aligned} T_{AB} = T_{BA} &= -16\left(4m^4 + m^2(p\cdot k - p\cdot k')\right) \\ &= -16\left(2m^4 + \frac{m^2}{2}((s-m^2) - (u-m^2))\right) \end{aligned}$$

Putting together the pieces of the unpolarized Feynman amplitude for Compton scattering we obtain

$$\begin{aligned} |\bar{\mathcal{M}}|^2 &= 2q^4\left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right] \\ &= 2q^4\left[-\left(\frac{u-m^2}{s-m^2} + \frac{s-m^2}{u-m^2}\right) + 4m^2\left(\frac{1}{s-m^2} + \frac{1}{u-m^2}\right) + 4m^4\left(\frac{1}{s-m^2} + \frac{1}{u-m^2}\right)^2\right] \end{aligned} \tag{3.6}$$

To turn this expression into a cross section we must decide a frame of reference and draw a piture of the kinematics. We will analyze two different frames

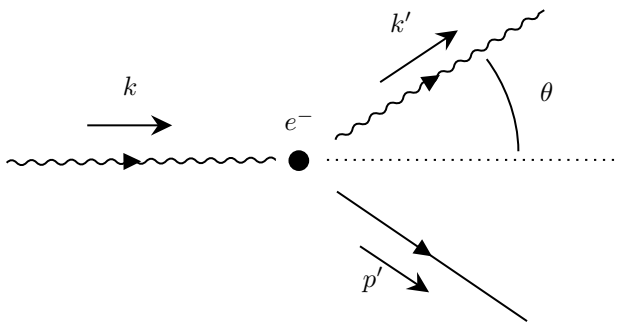
(i) *"Lab" frame*, in which the electron is initially at rest, this frame is useful for low energy incoming photons: $\omega_\gamma \ll m_e$;

(ii) *c.o.m. frame*, in which the center of mass is at rest, this frame is useful for high energy incoming photons: $\omega_\gamma \gg m_e$, where we can set $m_e = 0$

3.5.3 Lab frame - Low energy photon

See also Mandl sec. 8.6

In the low energy case, I can verify if QED prediction agrees with Thomason law for low energies scattering.



$p = (m_e, 0)$
$k = (\omega_k, \boldsymbol{k})$
$p' = (E', \boldsymbol{p}')$
$k' = (\omega_{k'}, \boldsymbol{k}')$

We will express the cross section in terms of ω and θ. We can find ω', the energy of the final photon, using the following trick:

$$\begin{aligned} m^2 = (p')^2 = (p+k-k')^2 &= p^2 + 2p\cdot(k-k') - 2k\cdot k' \\ &= m^2 + 2m(\omega_k - \omega_{k'}) - 2\omega_k\omega_{k'}(1-\cos\theta) \end{aligned}$$

hence, we obtain Compton's formula for the shift in the photon wavelength:

$$\Delta\lambda = \left(\frac{1}{\omega_{k'}} - \frac{1}{\omega_k}\right) = \frac{1-\cos\theta}{m}$$

For our purposes, however, is more useful to solve for $\omega_{k'}$:

$$\omega_{k'} = \frac{\omega_k}{1 + \frac{\omega_k}{m}(1-\cos\theta)} \tag{3.7}$$

The unpolarized amplitude in the Lab frame is

$$\begin{aligned} |\bar{\mathcal{M}}|^2_{\text{LAB}} &= 2q^4\left[\left(\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}}\right) + 2m\left(\frac{1}{\omega_k} - \frac{1}{\omega_{k'}}\right) + m^2\left(\frac{1}{\omega_k} - \frac{1}{\omega_{k'}}\right)^2\right] \\ &= 2q^4\left[\left(\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}}\right) - \sin^2\theta\right] \end{aligned}$$

The covariant flux factor reads

$$I_{\text{LAB}} = [(p\cdot k)^2 - m_e^2 m_\gamma^2]^{1/2} = |p\cdot k| = m_e\omega_k$$

The 2-body phase space

$$\begin{aligned} \int \mathrm{d}\Phi_{(2)} &= \int \frac{\mathrm{d}^3k'}{(2\pi)^3 2\omega_{k'}}\frac{\mathrm{d}^3p'}{(2\pi)^3 2E'}(2\pi)^4\delta^4(k'+p'-k-p) = \int \frac{\omega_{k'}^2 \mathrm{d}\omega_{k'}\mathrm{d}\Omega}{(2\pi)^2}\frac{1}{4\omega_{k'}E'}\delta(\omega_{k'}+E'-\omega_k-m) \\ &= \int \frac{\omega_{k'}^2 \mathrm{d}\omega_{k'}\mathrm{d}\Omega}{(2\pi)^2}\frac{1}{4\omega_{k'}E'}\frac{\delta(\omega_{k'}-|\boldsymbol{k}'|)}{\left|\frac{\partial(\omega_{k'}+E'-\omega_k-m)}{\partial|k'|}\right|_{\omega_{k'}=|\boldsymbol{k}'|}} = \int \mathrm{d}\Omega \frac{|\boldsymbol{k}'|^2}{16\pi^2\omega_{k'}E'}\left|\frac{\partial(\omega_{k'}+E')}{\partial|k'|}\right|^{-1}_{\omega_{k'}=|\boldsymbol{k}'|} \end{aligned}$$

where

$$E' = \left(m^2 + (\boldsymbol{k}-\boldsymbol{k}')^2\right)^{1/2} = \left[m^2 + \omega_k^2 + \omega_{k'}^2 - 2\omega_{k'}\omega_k\cos\theta\right]^{1/2}$$

$$\frac{\partial E'}{\partial|k'|} = \frac{\omega_{k'} - \omega_k\cos\theta}{E'}$$

and

$$\left|\frac{\partial(\omega_{k'}+E')}{\partial|k'|}\right|_{\omega_{k'}=|\boldsymbol{k}'|} = \left|1 + \frac{\omega_{k'}-\omega_k\cos\theta}{E'}\right| = \frac{m\omega_k}{E'\omega_{k'}}$$

Controlla l'ultimo passaggio

So the unpolarized cross section is

$$\begin{aligned} \left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)_{\text{LAB}} &= \frac{\overline{|\mathcal{M}|}^2_{\text{LAB}}}{4I_{\text{LAB}}}\frac{\mathrm{d}\Phi_{(2)}}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\frac{|\boldsymbol{k}'|^2}{I_{\text{LAB}}\omega_{k'}E'}\left|\frac{\partial(\omega_{k'}+E')}{\partial|k'|}\right|^{-1}\overline{|\mathcal{M}|}^2_{\text{LAB}} \\ &= \frac{q^4}{32\pi^2}\frac{1}{m^2}\left(\frac{\omega_{k'}}{\omega_k}\right)^2\left(\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}} - \sin^2\theta\right) \\ &= \frac{\alpha^2}{2}\frac{1}{m^2}\left(\frac{\omega_{k'}}{\omega_k}\right)^2\left(\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}} - \sin^2\theta\right) \end{aligned}$$

where $\omega_{k'}/\omega_k$ is given by (3.7) and in the last step we used $\alpha = e^2/(4\pi)$. Writing $\mathrm{d}\Omega = (2\pi)\mathrm{d}\cos\theta$ we obtain

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\cos\theta}\right)_{\text{LAB}} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega_{k'}}{\omega_k}\right)^2\left(\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}} - \sin^2\theta\right) \tag{3.8}$$

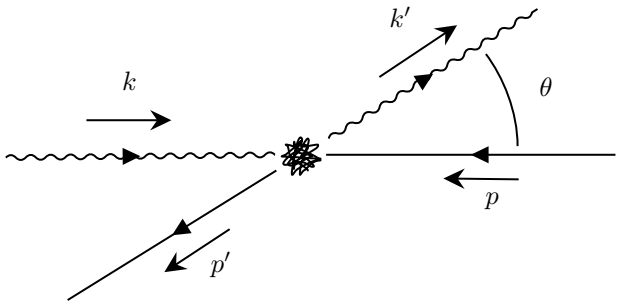
This is the (spin-averaged) *Klein-Nishina formula*. In the low energy limit $\omega_k \ll m$, from (3.7) we have $\omega_{k'} \approx \omega_k$, i.e. the kinetic energy of the recoil electron is negligible, and Eq.(3.8) reduces to the familiar Thomson cross-section for scattering of classical electromagnetic radiation by a free electron:

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\cos\theta}\right)_{\mathrm{LAB}} \overset{\omega_k \ll m}{=} \frac{\pi\alpha^2}{m^2}\left(1+\cos^2\theta\right) \quad \rightarrow \quad (\bar{\sigma})_{\mathrm{LAB}} = \frac{8\pi\alpha^2}{3m^2} \equiv \frac{8}{3}\pi r_e^2$$

We have calculated the full relativistic corrections for the Thomson formula.

3.5.4 C.o.M. frame - High energy photon

See Peskin sec. 5.5, and Schwartz sec. 13.5.4 To analyze the high-energy behaviour of the Compton scattering cross section, it is easiest to work in the center-of-mass frame.



$p = (E, -\boldsymbol{k})$
$k = (\omega_k, \boldsymbol{k})$
$p' = (E', -\boldsymbol{k}')$
$k' = (\omega'_k, \boldsymbol{k}')$

The kinematics of the reaction in the high energy limit ($m \approx 0$) looks like

$$E = \sqrt{\boldsymbol{k}^2 + m^2} \overset{m=0}{\approx} |\boldsymbol{k}| = \omega_k$$
$$E' = \sqrt{\boldsymbol{k}'^2 + m^2} \overset{m=0}{\approx} |\boldsymbol{k}'| = \omega_{k'}$$

$$p \cdot k = E\omega_k + |\boldsymbol{k}|^2 = \omega_k(E+\omega_k) \overset{m=0}{\approx} 2\omega_k^2$$
$$p \cdot p' = E\,E' - \boldsymbol{k}\boldsymbol{k}' = E\,E' - |\boldsymbol{k}||\boldsymbol{k}'|\cos\theta = E\,E' - \omega_k\omega_{k'}\cos\theta \overset{m=0}{\approx} \omega_k\omega_{k'}(1-\cos\theta)$$
$$p \cdot k' = E\,\omega_{k'} + \boldsymbol{k}\boldsymbol{k}' = E\,\omega_{k'} + |\boldsymbol{k}||\boldsymbol{k}'|\cos\theta = \omega_{k'}(E+\omega_k\cos\theta) \overset{m=0}{\approx} \omega_k\omega_{k'}(1+\cos\theta)$$

We also have

$$s = (p+k)^2 = m^2 + 2p\cdot k = m^2 + 2\omega_k(E+\omega_k) \overset{m=0}{\approx} 4\omega_k^2 \quad \rightarrow \quad \omega_k \overset{m=0}{\approx} \frac{\sqrt{s}}{2}$$
$$s = (p'+k')^2 = m^2 + 2p'\cdot k' = m^2 + 2\omega_{k'}(E'+\omega'_k) \overset{m=0}{\approx} 4\omega_{k'}^2 \quad \rightarrow \quad \omega_{k'} \overset{m=0}{\approx} \frac{\sqrt{s}}{2}$$

Plugging these values into Eq.(3.6)

$$\overline{|\mathcal{M}|}^2 = 2q^4\left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right]$$

for c.o.m. frame with $E \gg m$ we have

$$\overline{|\mathcal{M}|}^2_{\mathrm{CM}} \approx 2q^4\left(\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'}\right) \approx 2q^4\left(\frac{1+\cos\theta}{2} + \frac{2}{1+\cos\theta}\right)$$

we notice that the term $p\cdot k/p\cdot k'$ becomes divergent when the electron is emitted in the backward direction ($\theta \approx \pi$), while other terms are all of $\mathcal{O}(1)$ or smaller.

Notice that two initial diagrams $\mathcal{M}_A$, s-channel, and $\mathcal{M}_B$, u-channel, give contributions to the total amplitude proportional to[V]

$$\mathcal{M}_A \quad \to \quad \frac{1}{2p\cdot k} = \frac{1}{s-m^2} \qquad \mathcal{M}_B \quad \to \quad \frac{1}{2p\cdot k'} = \frac{1}{u-m^2}$$

Here is clear the relation between the momentum of the channel and the contribution to the total Feynman amplitude. The divergent contribution is due to the square of the u-channel diagram, we can see that for $\theta = \pi$ we have $u = (p-k')^2 = m^2 - 2p\cdot k' \approx m^2 - 2\omega_k^2(1+\cos\theta) \approx m^2$ i.e. the divergent contribution is related to the situation where the initial electron emits a photon with all its kinetic energy and then absorbs all the energy of the initial photon. The amplitude is large at $\theta \approx \pi$ because the denominator of the propagator is then small ($\sim m^2$) compared to s. This kind of divergence is called *Infra-Red divergence*. We can correct the divergent term (unphysical) considering higher terms in the Taylor expantion of E in m:

This is a Infra-Red divergence, related to Sudakov logs. Add some comment. See Mandl 8.9

$$E = \sqrt{\boldsymbol{k}^2 + m^2} = |\boldsymbol{k}| + \frac{m^2}{2|\boldsymbol{k}|} + o(m^3) \overset{m\approx 0}{\approx} \omega_k + \frac{m^2}{2\omega_k}$$

$$p\cdot k' = \omega_{k'}(E + \omega_k\cos\theta) \overset{m=0}{\approx} \omega_{k'}\left(\omega_k + \frac{m^2}{2\omega_k} + \omega_k\cos\theta\right) = \omega_k^2\left(1+\cos\theta+\frac{m^2}{2\omega_k^2}\right)$$

$$\overline{|\mathcal{M}|}^2_{\mathrm{CM}} \approx 2q^4\left(\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'}\right) \approx 2q^4\left(\frac{1+\cos\theta}{2} + \frac{2}{1+\cos\theta+\frac{m^2}{2\omega_k^2}}\right)$$

Now we can compute the cross section in the CM frame (we can use the formula for elastic scattering):

$$\left(\frac{\mathrm{d}\bar\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} = \frac{1}{64\pi^2}\frac{\overline{|\mathcal{M}|}^2_{\mathrm{CM}}}{s} \approx \frac{q^4}{32\pi^2 s}\left(\frac{1+\cos\theta}{2} + \frac{2}{1+\cos\theta+\frac{m^2}{2\omega_k^2}}\right)$$
$$\approx \frac{\alpha^2}{2s}\left(\frac{1+\cos\theta}{2} + \frac{2}{1+\cos\theta+\frac{m^2}{2\omega_k^2}}\right)$$

or

$$\left(\frac{\mathrm{d}\bar\sigma}{\mathrm{d}(\cos\theta)}\right)_{\mathrm{CM}} \approx \frac{\pi\alpha^2}{s}\left(\frac{1+\cos\theta}{2} + \frac{2}{1+\cos\theta+\frac{m^2}{2\omega_k^2}}\right)$$

Recall that the electron mass m can be neglected completely in this formula if it were not necessary to cutoff the singularity for $\theta = 0$.
The total Compton scattering cross section reads:

$$\bar\sigma_{\mathrm{total}} = \int_{-1}^{1}\mathrm{d}(\cos\theta)\left(\frac{\mathrm{d}\bar\sigma}{\mathrm{d}(\cos\theta)}\right)_{\mathrm{CM}} \approx \frac{\pi\alpha^2}{s}\int_{-1}^{1}\mathrm{d}(\cos\theta)\left(\frac{1+\cos\theta}{2} + \frac{2}{1+\cos\theta+\frac{m^2}{2\omega_k^2}}\right)$$
$$= \frac{\pi\alpha^2}{s} + \frac{2\pi\alpha^2}{s}\log\left(\frac{s}{m^2}\right)$$

The main dependence α^2/s follows from dimensional analysis. But the singularity associated to backward scattering of photons leads to an enhancement by an extra logarithm of the energy, called *Sudakov logarithm*.

Inserire diagrammi di Feynman pag 168 Peskin (ruotati appropriatamente)

Exercise 5: Pair Annihilation into Photons

Find the total cross section of the annihilation process $e^+e^- \to 2\gamma$

[V] You can easily verify this statement looking at the calculation on the beginning of this section.

3.6 Scattering by an external E.M. field and the Rutherford formula

3.6.1 $e^- p \to e^- p$ Rutherford scattering

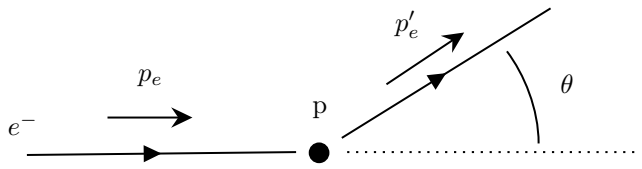
See Schwartz sec. 13.4

Now let us go back to the problem of scattering of an electron by a Coulomb potential. Recall the classical Rutherford scattering formula,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m_e^2 e^4}{4|\boldsymbol{p}_i|^4 \sin^4 \frac{\theta}{2}}$$

where $|\boldsymbol{p}_i| = |\boldsymbol{p}_f|$ is the magnitude of the incoming electron momentum, which. is the same as the magnitude of the outgoing electron momentum for elastic scattering. Rutherford calculated this using classical mechanics to describe how an electron would get deflected in a central potential, as from atomic nucleus.

We study the process in the CM frame for the proton.



$$\begin{aligned} p_e &= (\omega_e, \boldsymbol{p_e}) \\ p_p &= (M_p, 0) \\ p'_e &= (\omega'_e, \boldsymbol{p_e}') \\ p'_p &= (\omega'_p, \boldsymbol{p_p}') \end{aligned}$$

We neglect the recoil of the proton because of its huge mass ($\boldsymbol{p_p}' \approx 0$) (we are not considering high energy case).

$$(\omega'_p, \boldsymbol{p_p}') = \left(M + \frac{|\boldsymbol{p_p}'|^2}{2M} + o\left(|\boldsymbol{p_p}'|^3\right), \boldsymbol{p_p}' \right) \approx (M, 0)$$

If we can neglect the electron mass ($m_e \approx 0$) we also have

$$\omega_e = \sqrt{m^2 + \boldsymbol{p}_e^2} = |\boldsymbol{p}_e| + \frac{m^2}{2|\boldsymbol{p}_e|} + o(m^3) \approx |\boldsymbol{p}_e|$$

$$\omega'_e = \sqrt{m^2 + \boldsymbol{p}_e'^2} = |\boldsymbol{p}'_e| + \frac{m^2}{2|\boldsymbol{p}'_e|} + o(m^3) \approx |\boldsymbol{p}'_e|$$

Then energy conservation reads

$$\omega_e + M_p = \omega'_e + \omega'_p \quad \to \quad \omega_e = \omega'_e + \frac{|\boldsymbol{p_p}'|^2}{2M_p} + o\left(|\boldsymbol{p_p}'|^3\right) \quad \to \quad \omega_e \approx \omega'_e$$

The only quantity that shows a remarkable variation is the angle θ (variation can be $\mathcal{O}(1)$):

$$\begin{aligned} (p_e - p'_e)^2 &= 2m^2 - 2p_e \cdot p'_e = 2m^2 - 2\omega_e\omega'_e + 2|\boldsymbol{p}_e||\boldsymbol{p}'_e|\cos\theta \approx -2|\boldsymbol{p}_e|^2(1-\cos\theta) \\ (p_p - p'_p)^2 &= 2M^2 - 2p_p \cdot p'_p = 2M^2 - 2M\omega'_p \approx -|\boldsymbol{p}'_p|^2 \end{aligned}$$

and because of momentum conservation $p_e + p_p = p'_e + p'_p$ we have

$$-2|\boldsymbol{p}_e|^2(1-\cos\theta) \approx -|\boldsymbol{p}'_p|^2 \quad \to \quad \cos\theta \approx 1 - \frac{1}{2}\frac{|\boldsymbol{p}'_p|^2}{|\boldsymbol{p}_e|^2}$$

In order to give a description of this process using QED, we modify the QED Lagrangian, so that we introduce also protons in our theory. We consider a low energy process, where the proton can be consider as a fundamental particle, described as a spin 1/2 fermion. Then we can do the same trick we used for QED flavours. The modified Lagrangian reads:[VI]

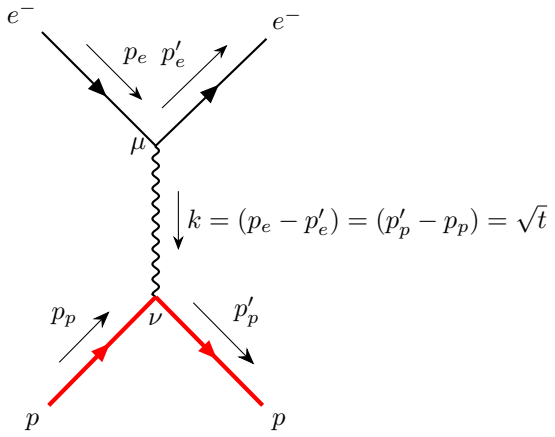
$$\mathcal{L} = \bar{\psi}_e(i\partial\!\!\!/ - m_e)\psi_e + \bar{\psi}_p(i\partial\!\!\!/ - M_p)\psi_p + \underbrace{q_e\bar{\psi}_e A\!\!\!/\psi_e + q_p\bar{\psi}_p A\!\!\!/\psi_p}_{\mathcal{L}_{\text{int}}}$$

Notice that $\mathcal{L}_{\text{int}}$ in order to obtain a description of Rutherford scattering we need to consider at least 2-nd order processes, i.e. 2 vertex. The two-vertex S matrix element becomes

$$\begin{aligned} S_{(2)} &= \frac{(-i)^2}{2!}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{T}\left\{\mathrm{N}\left[q_e\bar{\psi}_e A\!\!\!/\psi_e + q_p\bar{\psi}_p A\!\!\!/\psi_p\right]_x \mathrm{N}\left[q_e\bar{\psi}_e A\!\!\!/\psi_e + q_p\bar{\psi}_p A\!\!\!/\psi_p\right]_y\right\} \\ &= \frac{1}{2}(-iq_e)(-iq_p)\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{T}\left\{\mathrm{N}\left[\bar{\psi}_e A\!\!\!/\psi_e\right]_x \mathrm{N}\left[\bar{\psi}_p A\!\!\!/\psi_p\right]_y + \mathrm{N}\left[\bar{\psi}_e A\!\!\!/\psi_e\right]_y \mathrm{N}\left[\bar{\psi}_p A\!\!\!/\psi_p\right]_x\right\} \\ &= (-iq_e)(-iq_p)\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{T}\left\{\mathrm{N}\left[\bar{\psi}_e A\!\!\!/\psi_e\right]_x \mathrm{N}\left[\bar{\psi}_p A\!\!\!/\psi_p\right]_y\right\} \end{aligned}$$

where in the first step we omitted products in the integral that does not describe interaction between electron and proton, and in the second step we change integral variables in the second term of the time product.

The $e^-p \to e^-p$ process is similar to the process $e^-f^- \to e^-f^-$. Both processes are described by a t-channel



Feynman amplitude for this process is (we consider the gauge fixing term for EM field with $\zeta = 1$)

$$\mathcal{M} = i\frac{q_e q_p}{t}\left(\bar{u}_{r'}(p'_e)\gamma^\mu u_r(p_e)\right)_e\left(\bar{u}_{s'}(p'_p)\gamma_\mu u_s(p_p)\right)_p$$

where lower indices means that spinors are related respectively to electron and proton fields. Since proton is at rest its spinors takes a simple form

$$u_s(p_e = (M,0)) = \sqrt{2M}\begin{pmatrix}\xi_s\\0\end{pmatrix} \qquad \bar{u}_s(p'_e = (M,0)) = \sqrt{2M}\begin{pmatrix}\xi_s^\dagger & 0\end{pmatrix}$$

and the proton current reads[VII]

$$\left(\bar{u}_{s'}(p'_p)\gamma_\mu u_s(p_p)\right)_p = 2M\begin{pmatrix}\xi_s'^\dagger & 0\end{pmatrix}\gamma_\mu\begin{pmatrix}\xi_s\\0\end{pmatrix} = 2Mg_{\mu 0}\delta_{ss'}$$

so we obtain

[VI] See Schwartz sec. 5.2

[VII] The last identity can be easly proved using an explicit representation of gamma matrices.

$$\mathcal{M}_{\text{LAB}} = i\frac{q_e q_p}{t}\left(\bar{u}_{r'}(p'_e)g_{\mu 0}\gamma^\mu u_r(p_e)\right)_e 2M\delta_{ss'} = i\frac{q_e q_p}{t}\left(\bar{u}_{r'}(p'_e)\gamma^0 u_r(p_e)\right)_e 2M\delta_{ss'}$$

The unpolarized squared amplitude in the lab frame reads

$$\begin{aligned}\overline{|\mathcal{M}|}^2_{\text{LAB}} &= \left(\frac{q_e q_p}{t}\right)^2\left(\sum_{s,s'} 4M^2\delta_{ss'}\right)\frac{1}{2}\sum_{rr'}\left(\bar{u}_{r'}(p'_e)\gamma^0 u_r(p_e)\right)_e\left(\bar{u}_r(p_e)\gamma_0 u_{r'}(p'_e)\right)_e \\ &= \left(\frac{q_e q_p}{t}\right)^2 2M^2\,\mathrm{Tr}\left[(\not{p}_e + m)\gamma^0(\not{p}'_e + m)\gamma_0\right] = \cdots = 16\left(\frac{q_e q_p}{t}\right)^2 M^2\omega_e^2\left(1 - v^2\sin^2\left(\frac{\theta}{2}\right)\right)\end{aligned}$$

where we define the magnitude of the speed of the electron

Verify last step, probably is wrong See both Schwartz and Mandl. Also next calculation can be wrong

$$v = \frac{|\boldsymbol{p}_e|}{m}$$

We also have

$$t = -4\omega_e^2 v^2\sin^2\left(\frac{\theta}{2}\right)$$

Putting pieces together we obtain

$$\overline{|\mathcal{M}|}^2_{\text{LAB}} = (q_e q_p)^2\frac{\omega_e^2 M^2}{|\boldsymbol{p}_e|^4}\left(\frac{1 - v^2\sin^2\left(\frac{\theta}{2}\right)}{\sin^4\left(\frac{\theta}{2}\right)}\right)$$

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\omega}\right)_{\text{LAB}} = \frac{\alpha^2\left(1 - v^2\sin^2\left(\frac{\theta}{2}\right)\right)}{4\omega_e^2 v^4\sin^4\left(\frac{\theta}{2}\right)} \qquad (q_p = -q_e = e)$$

The latter is known as *Mott formula*. In the non-relativistic limit we can use $v \ll 1$ and $p_e \ll \omega_e \sim m_e$, thus

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\omega}\right)^{\text{non-rel.}}_{\text{LAB}} = \frac{\alpha^2}{4\omega_e^2 v^4\sin^2\left(\frac{\theta}{2}\right)}$$

which is the Rutherford formula. We can consider a generic nucleous with atomic number Z, then

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\omega}\right)^{\text{non-rel.}}_{\text{LAB}} = \frac{\alpha^2 Z^2}{4\omega_e^2 v^4\sin^2\left(\frac{\theta}{2}\right)}$$

3.6.2 Generic external E.M. field

See Mandl sec. 8.7

We can use a more general approach, where instead of considering the proton as a fundamental particle, we consider it as a source of E.M. field, considered as an external field in our theory.

So far, the electromagnetic field has been described by a quantized field, involving photon creation and annihilation operators. In some problem, where the quantum fluctuations are unimportant, it may be adequate to describe the field as a purely classical function of the space-time coordinates. In cases such as Rutherford scattering, we consider an applied external electromagnetic field $A^{\text{ext}}(x)$, such as the Coulomb field of a heavy nucleus. More generally, one may have to consider both types of field, replacing A by the sum of the quantized and the classical fields, $A(x) + A^{\text{ext}}(x)$.

In the last section, we have seen that matrix element of the scattering amplitude reads

$$\begin{aligned}S_{fi} &= \langle f|\,S_{(2)}\,|i\rangle = \left\langle e^-_{r'}(p'_e)p_{(s')}(M)\right|S_{(2)}\left|e^-_r(p_e)p_{(s)}(M)\right\rangle \\ &= \int \mathrm{d}^4x\left\langle e^-_{r'}(p'_e)\right|(-iq_e)\left[\bar{\psi}_e\gamma^\mu\psi_e\right]_x\left|e^-_r(p_e)\right\rangle\int \mathrm{d}^4y\left\langle p_{(s')}(M)\right|(-iq_p)\left[\bar{\psi}_p\gamma^\nu\psi_p\right]_y\left|p_{(s)}(M)\right\rangle iD^F_{\mu\nu}(x-y) \\ &= -iq_e\int \mathrm{d}^4x\left\langle e^-_{r'}(p'_e)\right|\left[\bar{\psi}_e\gamma^\mu\psi_e\right]_x\left|e^-_r(p_e)\right\rangle\int \mathrm{d}^4y D^F_{\mu\nu}(x-y)\underbrace{q_p\left\langle p_{(s')}(M)\right|\left[\bar{\psi}_p\gamma^\nu\psi_p\right]_y\left|p_{(s)}(M)\right\rangle}_{J^\nu_p(y)}\end{aligned}$$

where $J_p^\nu(y)$ is a 4-current density, associated to the proton field. We can therefore define the classic field (i.e. a function)

$$A_\mu^{\text{ext}}(x) = \int \mathrm{d}^4 y D_{\mu\nu}^F(x-y) J_p^\nu(y)$$

which Maxwell equation reads[VIII]

$$\Box A_{\text{ext}}^\mu(x) = \int \mathrm{d}^4 y \left(\Box D_{\mu\nu}^F(x-y)\right) J_p^\nu(y) = J_p^\nu(x)$$

i.e. $A_{\text{ext}}^\mu(x)$ can be interpreted as a classic Maxwell field with external source $J_p^\nu(x)$. With this notation we obtain

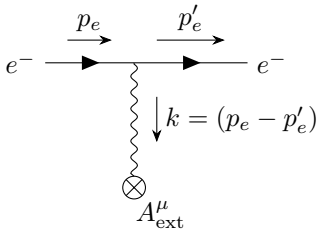
$$S_{fi} = -iq_e \int \mathrm{d}^4 x \left\langle e_{r'}^-(p_e') \right| \left[\bar\psi_e \not{A}_{\text{ext}} \psi_e\right]_x \left| e_r^-(p_e) \right\rangle$$

We obtained the same result if we would have defined the QED interaction Lagrangian as

$$\mathcal{L}_{\text{int}} = -q_e \bar\psi \gamma^\mu \psi (A_\mu + A_\mu^{\text{ext}})$$

and we considered its expantion at the first order (the field A_μ vanishes in the matrix element because of the sandwich with initial and final states). Notice that A_μ is a quantum field (i.e. an operator) while A_μ^{ext} is a classic field (i.e. a function).

From a "graphical" point of view we introduce a new symbol, marking external source by a crossed circle



As a simple example, let's assume that A_{ext}^μ is a static potential.[IX] In this case we obtain same result as classic Rutherford scattering. First we express the external field in momentum space[X]

$$A_{\text{ext}}^\mu(x) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3 k \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \varepsilon_{\text{ext}}^\mu(\boldsymbol{k})$$

Then the S matrix element at first order reads

$$\begin{aligned}
S_{fi} &= -iq_e \int \mathrm{d}^4 x \left\langle e_{r'}^-(p_e') \right| \bar\psi_e \not{A}_{\text{ext}} \psi_e \left| e_r^-(p_e) \right\rangle \\
&= -iq_e \int \mathrm{d}^4 x \left\langle e_{r'}^-(p_e') \right| \bar\psi_e \gamma^\mu \psi_e \left| e_r^-(p_e) \right\rangle \frac{1}{(2\pi)^3} \int \mathrm{d}^3 k \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \varepsilon_\mu^{\text{ext}}(\boldsymbol{k}) \\
&= -iq_e \frac{1}{(2\pi)^3} \int \mathrm{d}^4 x \, \mathrm{d}^3 k \, \bar u(p_e') \gamma^\mu u(p_e) \, \varepsilon_\mu^{\text{ext}}(\boldsymbol{k}) \, e^{-ip\cdot x + ip'\cdot x + i\boldsymbol{k}\cdot\boldsymbol{x}} \\
&= -iq_e \int \mathrm{d}x \, \mathrm{d}^3 k \, \delta^3(\boldsymbol{k} + \boldsymbol{p} - \boldsymbol{p}') \bar u(p_e') \gamma^\mu u(p_e) \, \varepsilon_\mu^{\text{ext}}(\boldsymbol{k}) \, e^{-i(\omega_e - \omega_e')x} \\
&= -iq_e (2\pi) \delta(\omega_e - \omega_e') \bar u(p_e') \gamma^\mu u(p_e) \, \varepsilon_\mu^{\text{ext}}(\boldsymbol{p}' - \boldsymbol{p}) \\
&= (2\pi) \delta(\omega_e - \omega_e') \mathcal{M}_{fi}^{\text{ext}}
\end{aligned}$$

where

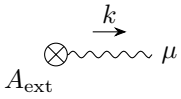
$$\mathcal{M}_{fi}^{\text{ext}} = -iq_e \bar u(p_e') \gamma^\mu u(p_e) \, \varepsilon_\mu^{\text{ext}}(k)$$

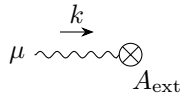
[VIII] Here we used proprieties of Green functions

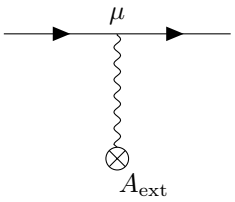
[IX] This is not a restrictive assumption.

[X] Notice that $\varepsilon_{\text{ext}}^\mu(\boldsymbol{k}) = \tilde A_{\text{ext}}^\mu(\boldsymbol{k})$, but we use this notiation in analogy with the quantum Maxwell field expantion.

Analysing $\mathcal{M}_{fi}^{\text{ext}}$, we notice that the Feynman rules for QED with external field are the usual ones, using factors $\tilde{A}^{\mu}_{\text{ext}}(k)$ instead of polarization vector for the Maxwell field. Set again $\varepsilon_{\mu}^{\text{ext}}(k) = \tilde{A}_{\mu}^{\text{ext}}(k)$ we have the following rules

$\varepsilon_{\mu}^{\text{ext}}(k) =$ 

$\varepsilon_{\mu}^{\text{ext}\,*}(k) =$ 

$-iq_e\gamma^{\mu} =$ 

Other rules are unchanged.

Let's consider the cross section for the incoming electron[XI]

$$\begin{aligned}
\mathrm{d}\omega_{fi} &= |S_{fi}^{CN}|^2 \frac{V\mathrm{d}^3p_e'}{(2\pi)^3} = (2\pi)^2\delta(\omega_e-\omega_e')\frac{T}{2\pi}\left(\frac{V\mathrm{d}^3p_e'}{(2\pi)^3}\right)|M_{fi}^{\text{ext}}|^2 \\
&= (2\pi)\delta(\omega_e-\omega_e')T\frac{1}{2\omega_e V}\frac{1}{2\omega_e' V}\left(\frac{V\mathrm{d}^3p_e'}{(2\pi)^3}\right)|\mathcal{M}_{fi}^{\text{ext}}|^2 \\
&= (2\pi)\delta(\omega_e-\omega_e')\frac{T}{2\omega_e V}\left(\frac{\mathrm{d}^3p_e'}{(2\pi)^3\,2\omega_e'}\right)|\mathcal{M}_{fi}^{\text{ext}}|^2
\end{aligned}$$

and (in lab frame, with number densities $n_i = 1/V$)

$$\mathrm{d}\sigma = \frac{\mathrm{d}\omega_{fi}}{n_e n_p v_e VT} = \frac{V}{Tv_e}\mathrm{d}\omega_{fi} = \frac{1}{16\pi^2}\delta(\omega_e-\omega_e')\frac{|\mathcal{M}_{fi}^{\text{ext}}|^2}{\omega_e\omega_e' v_e}\mathrm{d}^3p_e'$$

Writing $\mathrm{d}^3p_e' = |p_e'|^2\,\mathrm{d}|p_e'|\,\mathrm{d}\Omega \approx \omega_e'^2\,\mathrm{d}\omega_e'\,\mathrm{d}\Omega$, integrating over the magnitude of final momentum $|p_e'|$, and averaging over polarizations, we obtain ($|p_e'| \approx \omega_e'$)

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)_{\text{LAB}} = \frac{1}{16\pi^2}\frac{|p_e'|^2}{\omega_e v_e}\overline{|\mathcal{M}_{fi}^{\text{ext}}|}^2_{\text{LAB}}$$

The squared unpolarized amplitude reads

$$\begin{aligned}
\overline{|\mathcal{M}_{fi}^{\text{ext}}|}^2_{\text{LAB}} &= \frac{1}{2}\sum_{rr'} q_e^2\varepsilon_{\mu}^{\text{ext}}(k)\varepsilon_{\nu}^{\text{ext}\,*}(k)\bar{u}_{r'}(p_e')\gamma^{\mu}u_r(p_e)\bar{u}_r(p_e)\gamma^{\nu}u_{r'}(p_e') \\
&= \frac{1}{2}q_e^2\varepsilon_{\mu}^{\text{ext}}(k)\varepsilon_{\nu}^{\text{ext}\,*}(k)\,\mathrm{Tr}\Big[(\not{p}_e'+m)\gamma^{\mu}(\not{p}_e+m)\gamma^{\nu}\Big]
\end{aligned}$$

For example, in the Rutherford scattering, the external field is a Coulomb field, i.e.

$$A_{\text{ext}}^{\mu}(x) = \left(\frac{Zq_p}{4\pi|\boldsymbol{x}|}, 0, 0, 0\right)$$

Taking its Fourier transform

$$\varepsilon_{\text{ext}}^{\mu}(x) = \left(\frac{Zq_p}{|\boldsymbol{k}|^2}, 0, 0, 0\right)$$

we obtain

Complete calculations, see also previous section. Notice that mandl use wrong normalization

[XI] Respect to the calculations done in chapter 2, we must make the replace $(2\pi)^4\delta^4(P_f - P_i) \to (2\pi)\delta(\omega_e' - \omega_e)$. Notice that in the first line the amplitude is canonically normalized.

$$\overline{|\mathcal{M}_{fi}^{\text{ext}}|^2}_{\text{LAB}} = \frac{q_e^2 q_p^2 Z^2}{2|\boldsymbol{k}|^4} \operatorname{Tr}\Big[(\not{p}'_e + m)\gamma^0(\not{p}_e + m)\gamma^0\Big] = \dots$$

Introducing scattering angle we obtain again the Mott formula

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)_{\text{LAB}} = \dots = \frac{(\alpha Z)^2}{4\omega_e^2 v_e^4}\left(\frac{1 - v_e^2 \sin^2(\theta/2)}{\sin^4(\theta/2)}\right)$$

for scattering of relativistic electrons by a Coulomb field. In the non-relativistic limit ($v_e \ll 1$) this reduces to the Rutherford scattering formula

$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)_{\text{LAB}} = \frac{(\alpha Z)^2}{4m^2 v_e^4 \sin^4(\theta/2)}$$

We have here only considered the nucleus as a point charge. We only mention that the treatment is easily modified to the realistic case of a nucleus whose charge is distributed over a finite volume. For high energy electrons, this leads to an important method of investigating nuclear charge distribution.

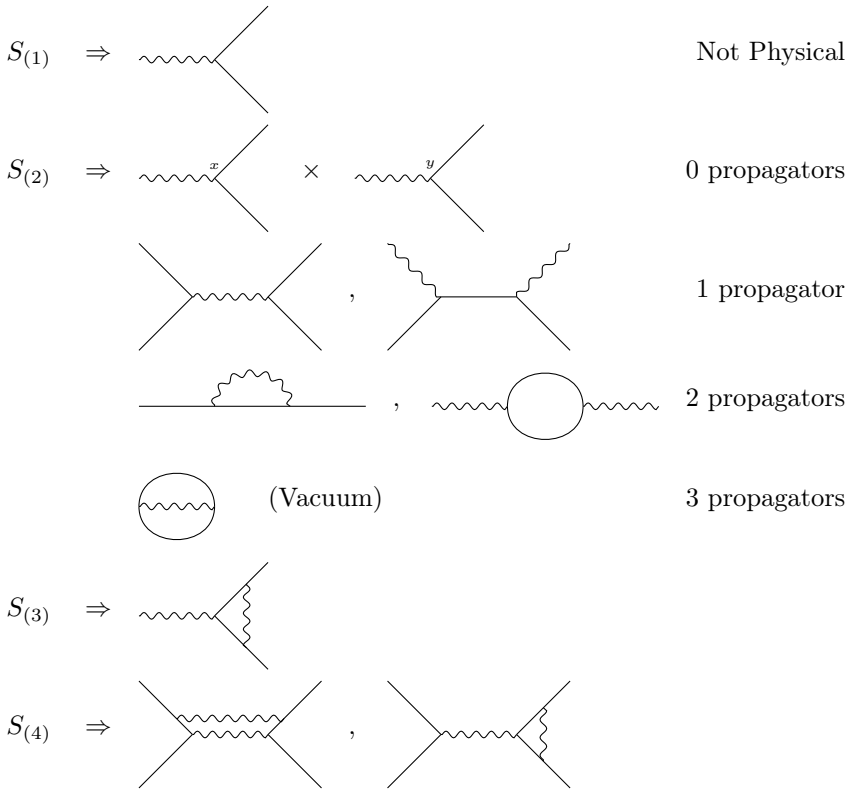
Chapter 4

QED processes at higher order

4.1 Beyond the tree-level

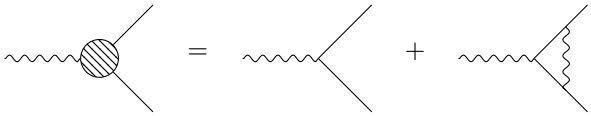
Up to now we have calculated QED processes in the lowset order of perturbation theory. On taking higher order into account we expect correction contributions from both real and virtual radiation. However, these corrections drag with them divergences in the corresponding integrals which make the results unphysical. In order to remove these inherent effects in the higher order perturbative approach, we can consider adopting the tools provided by regularization and renormalization theory based on the concept of modifying quantities to keep the finite and well-defined feature of the physical theory.

Feynman diagrams representing higher order corrections contain additional vertices, compared with those describing the process in the lowest order of perturbation theory. In the following we write down some of these diagrams:

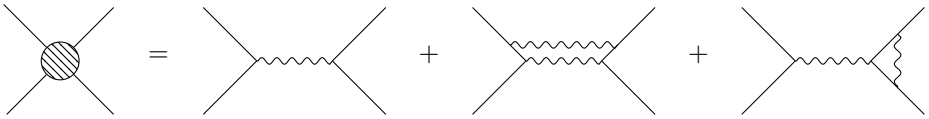


The 1 loop diagrams with 2 external particles are higher order corrections of the free propagator:

Diagrams at third order loops modifies tree level interactions



And at fourth order loops affects the scattering amplitude



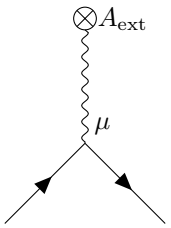
I call processes without loops (*tree*) α_{EM}**-order processes**, then processes with n loops are $\alpha_{EM} \circ (\alpha_{EM}^n)$ **order processes**.

We have so far shown how to calculate processes in the order $\alpha_{EM}(1+o(\alpha_{EM})+\dots)$. Since QED is a perturbative theory, theoretical result we obtained so far (considering only tree-level diagrams) coincides with experimental results only up to the first order in α_{EM}. In order to obtain better results we must consider higher order contibutions. See next example:

Example 4: Magnetic Dipole Momentum of e^-

Mandl, sec 9.6.1

The magnetic moment of a particle shows up through the scattering of the particle by a magnetic field. For this reason we shall one more study the elastic scattering of an electron by a static potential. We considered this process in lowest order in sec.3.6.2, whose diagram at lowest order is:



We can rewrite the lowest order scattering amplitude as follows (using *Gordon Identity*)[a]

$$iq_e\bar{u}(p')\gamma^\mu u(p) = i\left(\frac{q_e}{2m_e}\right)\bar{u}(p')\left\{(p'+p)^\mu + 2i\Sigma^{\mu\nu}(p'-p)_\nu\right\}u(p) \tag{4.1}$$

Physically this means that γ^μ (related to QED interaction) using Dirac e.o.m. can be rewritten in terms of

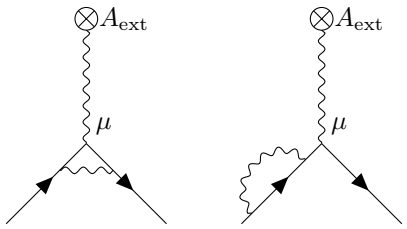
(i) $(p'+p)^\mu$ (which is related to e^- coupling)

(ii) $g_e\Sigma^{\mu\nu}$ (where $\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ is the Lorentz generator and $g_e = 2$ is the gyromagnetic ratio)

(iii) $(p'-p)_\nu$ (which is related to spin coupling)

The the non-relativistic limit of slowing moving particles and static magnetic field, the second term in (4.1) is just the amplitude for the scattering of a spin 1/2 particle with magnetic moment $(-e/2m)$, i.e. with gyromagnetic ratio $g_e = 2$. In QM the factor $g_e = 2$ is given by experimental results, while QFT predict this value at lowest order.

At higher orders, QED predicts $o(\alpha_{EM})$ corrections to g_e. Let's consider higher order diagrams, i.e. diagrams with one loop, for example[b]



All possible diagrams with one loop gives a $o(\alpha_{EM})$ correction. In 1948 Schwinger did the calculation and obtained another term in the amplitude of the process, namely

$$i\left(\frac{q_e}{2m}\right)\bar{u}(p')\left\{\left(\frac{\alpha}{2\pi}\right)2i\Sigma^{\mu\nu}(p'-p)_\nu\right\}u(p)$$

Then the QED prediction at $o(\alpha_{EM})$ is

$$g_e = 2\left(1+\frac{\alpha}{2\pi}\right)+o(\alpha^2) \quad \rightarrow \quad \left(\frac{g_e-2}{2}\right)_{TH} = \frac{\alpha}{2\pi} = 0.00116$$

Kusch and Foley (respectively in 1947 and 1948) done experiments and found

$$\left(\frac{g_e-2}{2}\right)_{EXP} = \frac{\alpha}{2\pi} = 0.00119 \pm 0.00005$$

This was the first prove that we need QED to describe EM interactions with high precision.

Subsequently, both theory and experiment have been greatly refined. Theoretically, the high order corrections of order α^2, α^3 and α^4 have been calculated. The result of these heavy calculations is

$$10^{12}\left(\frac{g_e-2}{2}\right)_{TH} = 1159652183 \pm 8$$

while the experimental value is

$$10^{12}\left(\frac{g_e-2}{2}\right)_{EXP} = 1159652182 \pm 7$$

The agreement can only be described as remarkable.

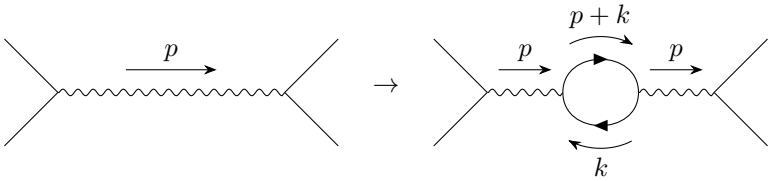
[a] See Mandl for explicit derivation.
[b] All possible diagrams with one loop are shown in Mandl.

When calculating diagrams with loops often one finds divergences. Consider for example the photon self-energy

Example 5: Photon Self-Energy

Mandl sec. 9.2

Consider the effect of the photon self-energy insertion in the photon propagator



In the Feynman amplitudes, the insertion of the photon propagator corresponds to the replace-

ment

$$D_{F\alpha\beta}(p) \quad \to \quad D_{F\alpha\mu}(p)\left(-iq_e\right)^2\Pi^{\mu\nu}(p,m)\right)D_{F\nu\beta}(p)$$

where

$$(-iq_e)^2\Pi^{\mu\nu}(p,m) = (-1)(-iq_e)^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[((\not{p}+\not{k})+m)\gamma^\mu(\not{k}+m)\gamma^\nu\right]}{[(p+k)^2-m^2][k^2-m^2]} \tag{4.2}$$

This formula express the possibility of creating virtual electrons with any momentum k. This leads to some problem since the integral is divergent. Take $\Lambda > 0$. Then:

$$\Pi^{\mu\nu}(p,m) = \int_0^\Lambda(\dots) + \lim_{\Lambda_0\to\infty}\int_\Lambda^{\Lambda_0}(\dots) \equiv \Pi^{\mu\nu}_{Fin}(p,m) + \Pi^{\mu\nu}_{Div}(p,m)$$

Suppose $\Lambda \gg p, m$, then

$$\begin{aligned}\Pi^{\mu\nu}_{Div}(p,m) &= \lim_{\Lambda_0\to\infty}\int_\Lambda^{\Lambda_0}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}[\not{k}\gamma^\mu\not{k}\gamma^\nu]}{k^4}\\ &\propto \lim_{\Lambda_0\to\infty}\int_\Lambda^{\Lambda_0}\mathrm{d}^4k\frac{k^\mu k^\nu}{k^4}\\ &\simeq \lim_{\Lambda_0\to\infty}\Lambda_0^2\end{aligned}$$

i.e. the integral is quadratic divergent.

4.2 Superficial degree of divergence and renormalizability condition on the coupling constant

Peskin sec. 10.1

The divergence shown in the previous example (for $k\to\infty$) is called **ultraviolet divergence**. We can also say that the **superficial degree of divergence** D of this diagram is $D = 2$.

Exercise 6

Verify that following diagrams are superficially divergent ($D \geq 0$):

Verify that following diagrams are not superficially divergent ($D < 0$):

There is a simple formula to determinate the superficial degree of divergence of a diagram. First we introduce some notation for diagrams:

(i) D is the superficial degree of divergence

(ii) L is the number of loops (i.e. the number of independent momentum in Feynman graph)

(iii) $E_{B/F}$ is the number of external bosons/fermions

(iv) $n_{B/F}$ is the number of bosons/fermions attached to each vertex

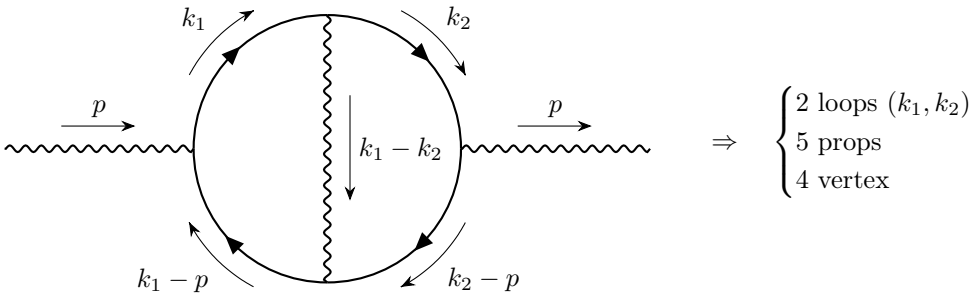
(v) $P_{B/F}$ is the number of bosonic/fermionic propagators

(vi) V is the number of vertex in the diagram

First, notice that the following topological propriety holds:

$$L = P_B + P_F - V + 1$$

Example 6



Since the number of bosons and the number of fermions attached to each vertex is constant,[I] then the number of vertices can also be express in terms of external particles:

$$V = \frac{2P_B + E_B}{n_B} = \frac{2P_F + E_F}{n_f}$$

Notice that each boson propagator decrease the degree of divergence by 2 since it's related to a factor $\frac{1}{k^2}$, while for fermionic propagators I have a factor $\frac{1}{\not{k}} = \frac{\not{k}}{k}$ so it decrease the degree of divergence by 1. Then I obtain

$$D = 4L - 2P_B - P_F$$

Putting all these relations together I obtain:

$$D = 4 - \left(4 - n_B - \frac{3}{2}n_F\right)V - E_B - \frac{3}{2}E_F$$

Notice that this formula holds for any theory involving bosons and fermions, since we didn't made any restrictive assumption on the theory. The dimension of the lagrangian is 4, while dimensions of bosonic and fermionic fields are respectively 1 and 3/2, therefore let $[g]$ be the dimension of the coupling constant g of my theory, I have

$$[g] = 4 - n_B - \frac{3}{2}n_F$$

and then

$$\boxed{D = 4 - [g]V - E_B - \frac{3}{2}E_F} \tag{4.3}$$

Example 7

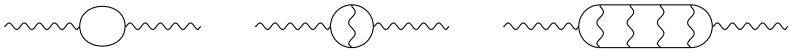
In QED $n_B = 1$ and $n_F = 2$, so $[g] = 4 - n_B - \frac{3}{2}n_F = 0$ as we expected since interaction lagrangian is $\mathcal{L}_{int} = q\bar{\psi}\not{A}\psi$ and $[q] = 0$.

Notice that from (4.3) follows that in theories with $[g] = 0$ (therefore QED is included) the degree divergence is independent by the number of vertices in diagrams.

Example 8

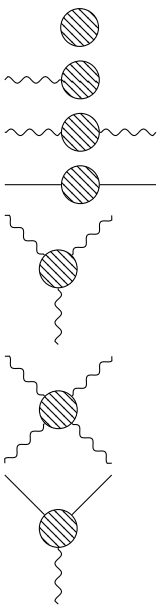
[I] For any theory, n_B and n_F are fixed by the interaction lagrangian. For example, in QED $n_B = 1$ and $n_F = 2$

Following diagrams have all $D = 2$:



A necessary condition for renormalizability is that $[g] \leq 0$, otherwise if $[g] < 0$ then D increases with the number of vertex and, therefore, with the order of the perturbative expansion, making perturbative expansion useless.

In spite of the superficial degree of divergente, doing explicit calculations we find that actually the degree of divergence of diagrams may be smaller that the one we obtained through the topological analysis, for example in QED we have these seven amplitudes whose superficial degree of divergence is ≥ 0:



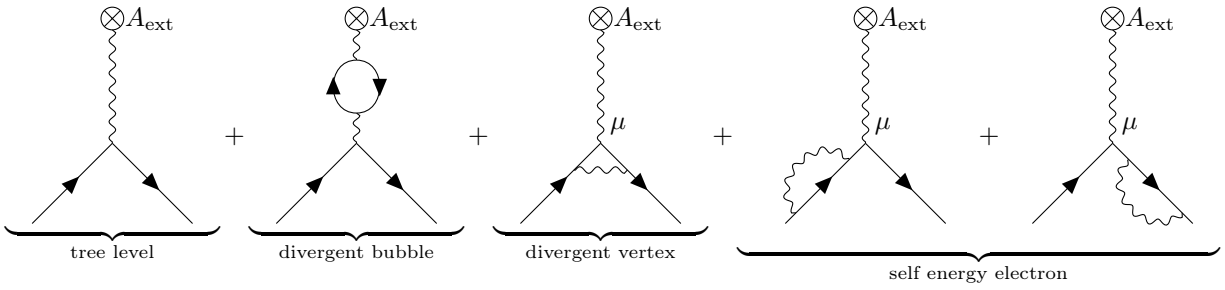
= Vacuum	$D = 4$	Irrelevant
= Tadpole	$D = 3$	Vanishes
= Photon Self-Energy	$D = 2$	Log divergence
= Electron Self-Energy	$D = 1$	Log divergence
$= 3\gamma$	$D = 1$	Vanishes
$= 4\gamma$	$D = 0$	Finite
= Vertex	$D = 0$	Log Divergence

This is due to the additional degrees of freedom of QED. When a theory has additional symmetries (as gauge symmetry for QED $\rightarrow$ Ward Identity) the divergence can be smaller than superficial divergence D.

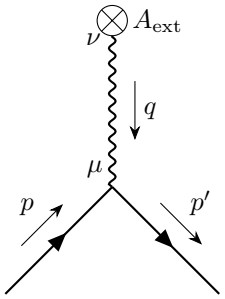
4.3 Basic idea behind the renormalization procedure

Halzen sec. 7.2; Mandl sec. 9.1, 9.2

Let's consider again the scattering by an external potential problem. Now we consider higher corrections in the total amplitude (in the previous analysis we considered only the first diagram):

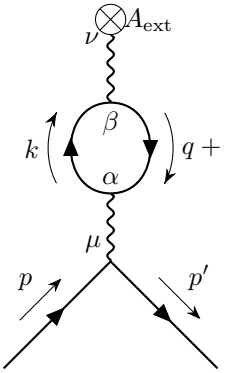


We already know Feynman amplitude for the first diagram

$\mathcal{M}_0 =$ 

$= (-ie)\bar{u}(p')\gamma^{\mu}u(p)g_{\mu\nu}\varepsilon^{\nu}_{\text{ext}}(q)$

For the diagram with the photon self-energy we can use previous computation obtaining

$\mathcal{M}_1 =$ 

$= (-ie)\bar{u}(p')\gamma^{\mu}u(p)\left(-i\frac{g_{\mu\alpha}}{q^2}\right)(-ie)^2\Pi^{\alpha\beta}(q,m)\left(-i\frac{g_{\beta\nu}}{q^2}\right)\varepsilon^{\nu}_{\text{ext}}(q)$

where $\Pi^{\alpha\beta}(q,m)$ was defined in (4.2):

$$\Pi^{\alpha\beta}(q,m) = (-1)\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[(\not{q}+\not{k}+m)\gamma^{\alpha}(\not{k}+m)\gamma^{\beta}\right]}{[(q+k)^2-m^2][k^2-m^2]}$$

This integral is quadratically divergent for large k. In order to handle it, we must regularize it, that is, we most modify it so that it becomes a well-defined finite integral, that is, we must modify it so that it becomes a well-defined finite integral. For example, this could be archived by multiplying int integrand in the previous equation by the convergence factor

$$\left(\frac{-\Lambda_{\infty}^2}{k^2-\Lambda_{\infty}^2}\right)^2$$

Here, Λ_{∞} is a **cut-off** parameter. For large, but finite, values of Λ_{∞}, the integral now behaves like $\int \mathrm{d}^4k/k^6$ for large k, and is well- defined and convergent. For $\Lambda_{\infty}\to\infty$, the factor tends to unity, and the original theory is restored. One can think of such convergence factor either as a mathematical device, introduced to overcome a very unsatisfactory feature of QED, or as a genuine modification of QED at very high energies, i.e. at very small distances, which should show up in experiments at sufficently high energies.

Let assume that the theory has been already regularized in this way, so that all expression are well-defined, finite and gauge invariant.

After regularization $\Pi^{\alpha\beta}(q^2)$ expression can be simplified. It follows from Lorentz invariance that $\Pi^{\alpha\beta}(q^2)$ must be of the form

$$e^2\Pi^{\alpha\beta}(q^2) = -ig^{\alpha\beta}I(q^2) + iq^{\alpha}q^{\beta}K(q^2)$$

since this is the most general second-rank tensor which can be formed using only the four-vector q^{μ}. From the Ward Identity (i.e. gauge invariance) follows that the second term proportional to the photon momentum q give vanishing contributions, hence, it can be omitted. $I(q^2)$ takes the form

$$I(q^2) = \frac{\alpha}{3\pi}\int_{m^2}^{\Lambda_{\infty}}\frac{\mathrm{d}k^2}{|k|^2} - \frac{2\alpha}{\pi}\int_0^1 \mathrm{d}z(1-z)\log\left(1-\frac{q^2z(1-z)}{m^2}\right)$$

where m is the mass of the electron and Λ_∞ is the cutoff parameter that we introduced in order to regularize the integral. Here we notice that the divergence is due to the first integral, and has logarithmic behaviour. After we will have solved the integral we have to take the limit $\Lambda \to \infty$. With this procedure we avoid the calculation of indeterminated divergent integrals.

In the approximation $q^2 \ll m^2$ we obtain following formula for $I(q^2)$:

$$I(q^2) \simeq \frac{\alpha}{3\pi} \log\left(\frac{\Lambda_\infty^2}{m^2}\right) + \frac{\alpha}{15\pi}\frac{q^2}{m^2} + o\left(\frac{q^2}{m^2}\right)$$

Unless we can dispose of the infinite part of $I(q^2)$ the result will not be physically meaningful.

The way to proceed is best explained by returning to Rutherford scattering. Recall $\varepsilon^\nu_{\text{ext}}(q) = (Ze/q^2, 0, 0, 0)$, including the loop contribution to the tree-order amplitude, we obtain for the small q^2 limit the amplitude

$$\mathcal{M} = -i\frac{Ze^2}{q^2}\left(1 - \frac{e^2}{12\pi^2}\log\left(\frac{\Lambda_\infty^2}{m^2}\right) - \frac{e^2}{60\pi^2}\frac{q^2}{m^2} + o\left(\alpha^2, \frac{q^4}{m^4}\right)\right)$$

Now, notice that the parameter e is just a theoretical parameter, called **bare parameter** $e_B = e$, that cannot be measured. Then I can introduce a new parameter called **renormalized parameter**

$$e_R \equiv e_B\left(1 - \frac{e^2}{12\pi^2}\log\frac{\Lambda_\infty^2}{m^2}\right)^{1/2} \tag{4.4}$$

With this new variable Feynman amplitude is

$$\mathcal{M} = -i\frac{Ze_R^2}{q^2}\left(1 - \frac{e_R^2}{60\pi^2}\frac{q^2}{m^2} + o\left(\alpha^2, \frac{q^4}{m^4}\right)\right)$$

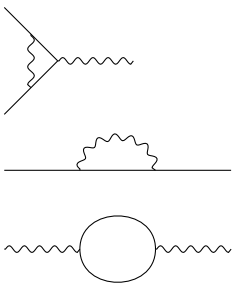
In this way we obtained a copmletely physical amplitude. Since the measurable quantities are scattering amplitudes, I solved the problem of divergent contributions in my theory due to photons self-energy (in the first order).

4.3.1 Renormalizable theories

Up to now we saw that a sufficent condition for a theory to be renormalizable is that all the divergences can be absorbed into redefinitions of physical parameters.

In QED I have only two kind of particles: photons and electrons (and positrons). Therefore only parameters are the charge and the mass of the electron, which are the only parameter I can use to renormalize the theory.

I have three kind of divergences:



$$= \quad -ieZ_1\gamma^\mu$$

$$= \quad \frac{+iZ_2}{p^2 - m^2}$$

$$= \quad -i\frac{g^{\mu\nu}}{p^2}Z_3$$

i.e. I have 3 divergent terms to be removed: Z_1, Z_2, Z_3. Since I have only 2 parameters I can use to remove these divergent terms, it seems that QED is not renormalizable. Luckily, doing explicit calculations, I obtain that thanks to gauge invariance (Ward Identities) I obtain $Z_1 = Z_2$ and therefore QED is renormalizable using opportune definitions of free parameters:

$$e_R = e_B\left(1 - \frac{\delta e}{e}\right) \qquad\qquad m_R = m_B\left(1 - \frac{\delta m}{m}\right)$$

where $\delta e/e$ and $\delta m/m$ are divergent terms. We obtained that QED is renormalizable at any order.

Chapter 5

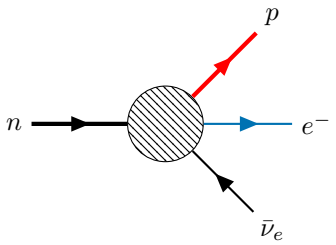
Weak Interactions

Mandl sec. 16.1; Halzen sec. 12.Introduction; [I]

The Weak interactions theory was born after, in the study of β-deday[II], scientist discovered a process that couldn't be described by QED, namely the neutron decay:

$$n \to p + e^- + \bar{\nu}_e$$

where n is the neutron, p is the proton, e^- is the electron and $\bar{\nu}_e$ is the antineutrino. From the energy spectrum of out-coming e^- particles scientist understood that this is a 3 body decay. Notice also that netrunos (ν_e and $\bar{\nu}_e$) are fermions that couldn't be discovered by detectors since they have no charge. This is the sketch of this interaction:[III]



This new interaction can't be described using QED and two photons that mediate the interactions because of the charge and the spin conservation.

Before proceeding, we note that it is conventional to divide weak interaction processes into three categories, depending on whether leptons and/or hadrons are involved. Accordingly, we distinguish:

(i) *purely leptonic* processes like

$$\mu^- \to e^- \bar{\nu}_e \nu_\mu$$

which involve only the charged leptons $e^p m$, $\mu^\pm$, $\tau^\pm$ and their associated neutrinos ν_e, ν_μ, ν_τ and antineutrinos $\bar{\nu}_e$, $\bar{\nu}_\mu$, $\bar{\nu}_\tau$

(ii) *semileptonic* processes involving both hadrons and lepton, like neutron β-decay

$$n \to p e^- \bar{\nu}_e$$

and

$$\pi^\mp \to \mu^\mp \overset{(-)}{\nu}_\mu$$

[I] Beside bibliography, for this chapter some good references can be found in the following sites:
https://www.imsc.res.in/~graj/genGRaja.pdf
http://polywww.in2p3.fr/~paganini/ch6.pdf
These reference are used in the following sections even if not cited.

[II] The theory of β-decay was developed by W. Pauli in 1934. To solve the puzzle of the continuous energy spectrum of the electrons emitted in the beta decay of nuclei, Pauli had suggested that along with the electron, an almost massless neutral particle also was emitted. This particle was named by Fermi as *neutrino*.

[III] We used different colors for different charge currents and different line thickness to distinguish nucleons by smaller fermions.

(iii) *purely hadronic* processes like the Λ decay

$$\Lambda \to p\pi^-$$

and

$$K^+ \to \pi^+\pi^0$$

Unfortunately we have only a limited understanding of hadrons structure, since the strong forces which bind the wearks act over distances of order 1 fm, where they cannot be treated in perturbation theory. In contrast, we belive that perturbation theory is valid for weak and electromagnetic interactions, and that we understand the latter. Consequently purely leptonic processes afford an unambiguous and far simpler field for studying weak interactions, and we shall restric ourselves to purely leptonic processes. This is analogous to our treatment of QED, where we also did not consider hadrons.

5.1 Four Fermions Fermi Theory and $V - A$ Theory

Halzen sec. 12.1; Maggiore sec. 8.1

In 1934 Fermi proposed his theory to describe β-decay process $n \to p + e^- + \bar{\nu}_e$. In QED the electromagnetic current $q(\bar{e}\gamma^\mu e)$ of the charged particle like the electron interacts with the electromagnetic vector potential A_μ, which becomes the field operator for the photon:

$$\mathcal{L} = q(\bar{e}\gamma^\mu e)A_\mu$$

The electric current and the vector potential are vectors and so Fermi adopted the vector form for the weak currrents also. In Fermi's theory of weak interactions, the weak current of the proton-neutron pair interacts with the weak current of the electron-neutrino pair:

$$\mathcal{L}_{\rm F} = \frac{G_F}{\sqrt{2}} (\bar{p}\gamma^\mu n) (\bar{e}\gamma_\mu \nu_e)$$

where the particle symbols represent the corresponding field operators. The strength of the weak interaction is characterized by the constant G_F called **Fermi constant**. The value of G_F is very small, and therefore this interaction is called "weak". This lagrangian describes 4 fermions interaction, and therefore is called **Four Fermions Fermi Lagrangian**. A new coupling constant meant a new force was born. Moreover, the currents in the latter Lagrangian are charged while in QED they are always neutral. Perturbation theory based on this Lagrangian described pretty well the β-decay.

Since lagrangian has to be 4-dimensional and all these fields have dimension 3/2 the Fermi constant must have dimension

$$[G_F] = 4 - 6 = -2$$

i.e. this theory is not renormalizable since $D = 2 + 2V - E_B - 3E_F/2 > 0$. We will use the SM to obtain a renormalizable theory of weak interactions.

5.1.1 Modification of Fermi's theory: the V - A current

Mandl sec. 16.2, Maggiore sec. 8.2

This theory of weak interactions proposed by Fermi almost 80 years ago purely on an intuitive basis stood the test of time inspite of many amendments that were incorporated into Fermi's theory successfully. One important amendment came after in 1957 Wu found a Parity violation ($\not{P}$) in Weak interaction, since the e^- produced in the neutron decay is mostly left handed. Fermi's theory survived even this fundamental revolution and the only modification was to replace the vector current of Fermi by an equal mixture of vector currents (V)

$$j_V^\alpha = \bar{\psi}\gamma^\alpha\phi$$

and axial currents (A)

$$j_A^\alpha = \bar{\psi}_L\gamma^\alpha\phi = \bar{\psi}\gamma_L^\alpha\phi = \bar{\psi}\gamma^\alpha\phi_L$$

Vectors and axial currents behave differently when we go from left to right-handed coordinate systems and hence the parity violation. The new current takes the form:

$$j^{\alpha}_{\text{V-A}} = c_V j^{\alpha}_V - c_A j^{\alpha}_A = \bar{\psi}\gamma^{\alpha}(c_V - c_A\gamma_5)\phi$$

where constant c_V and c_A must be fixed in such a way our Lagrangian is consistent with parity proprieties of our theory. Since e^- has both chiralities[IV] then I obtain that neutrino has to be only left-handed. Usually I chose $c_V = c_A = 1/2$ in order to obtain $(c_V - c_A\gamma_5) = P_L$. Introducing this current in the Fermi Lagrangian I obtain the Four Fermions V-A[V] interaction for the neutron decay:

$$\mathcal{L}^{n}_{\text{V-A}} = -\frac{G_F}{\sqrt{2}}\left(\bar{p}\gamma^{\alpha}\left(1 - c_A\gamma_5\right)n\right)\left(\bar{e}\gamma_{\alpha}\left(1-\gamma_5\right)\nu_e\right)$$

where we modified both hadronic and leptonic currents. Parameter c_A correspond to a free parameter in the lagrangian that can be fixed in further experiments. For example, if $c_A = 0$ we obtain a non-chiral current for hadrons.

For leptonic processes involving electrons, muons, and them corresponding neutrinos (such as muon decay $\mu^- \to e^-\bar{\nu}_e\nu_\mu$) we can obtain a similar solution:

$$\mathcal{L}^{\mu^-}_{\text{V-A}} = -\frac{G_F}{\sqrt{2}}\left(\bar{\nu}_\mu\gamma^{\alpha}(1-\gamma_5)\mu\right)\left(\bar{e}\gamma_{\alpha}(1-\gamma_5)\nu_e\right)$$

The general form for V-A Lagrangian responsible for leptonic processes is constructed from bilinear forms of the lepton field operators, i.e. from charged[VI] currents:

$$\mathcal{L}_{CC} = -\frac{4G_F}{\sqrt{2}}\left(\sum_{l=e,\mu,\tau} J^{\alpha}_{CC}(l)\right)^{\dagger}\left(\sum_{l=e,\mu,\tau} J_{CC,\alpha}(l)\right) \tag{5.1}$$

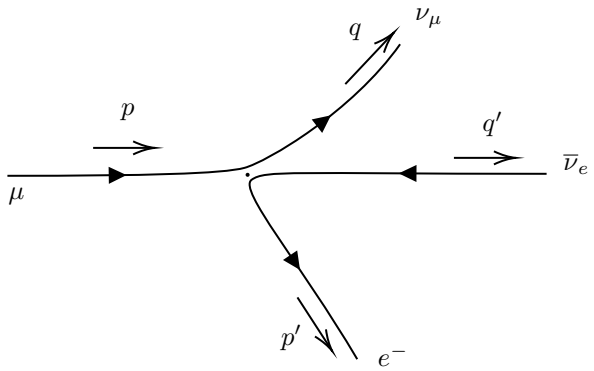
with

$$J^{\alpha}_{CC}(l) = \bar{l}\gamma^{\alpha}P_L\nu_l \tag{5.2}$$

5.1.2 μ-decay and 3-body final state

Maggiore problem 8.1; Mandl sec. 16.6.1; Halzen sec. 12.5

Let's consider muon decay rate using V-A lagrangian, $\mu^- \to e^-\bar{\nu}_e\nu_\mu$. Here there is a sketch of the process, where lines are related to conserved currents in the lagrangian:



[IV] In QED $\bar{e}\gamma^{\alpha}e = \bar{e}_R\gamma^{\alpha}e_R + \bar{e} + L\gamma^{\alpha}e_L$

[V] It is said "V minus A".

[VI] Since l describes a charged particle while ν_l an uncharged one. This means that (in the future development of the weak theory) this interaction must be mediated by a charged particle.

Instead of using Feynman diagrams, we prefer to use direct computation from the interaction lagrangian, since Feynman diagrams involve more work due to the product of two different diagrams. Therefore:

$$\begin{aligned} S_{fi}^{(\mu)} &= \langle e(p')\bar{\nu}_e(q')\nu_\mu(q)| \left(-i\int \mathrm{d}^4x \mathcal{L}_{\text{V-A}}(x)\right)|\mu(p)\rangle \\ &= -i\frac{G_F}{\sqrt{2}}\int \mathrm{d}^4x \,\langle e(p')\bar{\nu}_e(q')\nu_\mu(q)|\left(\bar{\nu}_\mu\gamma_L^\alpha\mu\right)_x\left(\bar{e}\gamma_{\alpha,L}\nu_e\right)_x|\mu(p)\rangle \\ &= (2\pi)^4\delta^4(p-p'-q'-q)\mathcal{M}_{fi} \end{aligned}$$

with

$$\mathcal{M}_{fi} = -i\frac{4G_F}{\sqrt{2}}\left(\bar{u}_{\nu_\mu}(q)\gamma_L^\alpha u_\mu(p)\right)\left(\bar{u}_e(p')\gamma_{\alpha,L}v_{\nu_e}(q')\right)$$

where we introduced Dirac spinors u_f and $\bar{u}_f$ for each fermion field f. The unpolarized amplitude reads[VII]

$$\overline{|\mathcal{M}|}^2 = \frac{1}{2}\left(\frac{4G_F}{\sqrt{2}}\right)^2 \mathrm{Tr}\Big[(\not{p}+m_\mu)\gamma_L^\beta \not{q}\gamma_L^\alpha\Big]\,\mathrm{Tr}\big[(\not{p}'+m_e)\gamma_{\alpha,L}\not{q}'\gamma_{\beta,L}\big]$$

We can assume vanishing neutrinos masses since $m_{\nu_e} \simeq m_{\nu_\mu} \approx 10^{-6}m_e$:

$$\overline{|\mathcal{M}|}^2 = \frac{1}{8}\left(\frac{4G_F}{\sqrt{2}}\right)^2 \underbrace{\mathrm{Tr}\big[\not{p}\gamma^\beta\not{q}\gamma^\alpha(1-\gamma_5)\big]}_{\text{(A)}}\underbrace{\mathrm{Tr}\big[\not{p}'\gamma_\alpha\not{q}'\gamma_\beta(1-\gamma_5)\big]}_{\text{(B)}}$$

Using traces proprieties the result can be simplified using a simmetric tensor S and a completely anti-simmetric (Levi-Civita) tensor:

$$\begin{aligned} \text{(A)} &= 4\left(S^{\rho\beta\sigma\alpha} - i\varepsilon^{\rho\beta\sigma\alpha}\right)p_\rho q_\sigma \\ \text{(B)} &= 4\left(S_{\rho'\alpha\sigma'\beta} - i\varepsilon_{\rho'\alpha\sigma'\beta}\right)(p')^{\rho'}(q')^{\sigma'} \end{aligned}$$

Performing the contraction we obtain

$$|\mathcal{M}|^2 = 64G_F^2(p\cdot q')(q\cdot p')$$

We can now compute the differential decay rate in the lab frame:

$$\left(\mathrm{d}\overline{\Gamma}\right)_{\text{LAB}} = \frac{\overline{|\mathcal{M}|}^2_{\text{LAB}}}{2m_\mu}\mathrm{d}\Phi_{(3)}$$

where $\mathrm{d}\phi_{(3)}$ is the three body phase space:

$$\mathrm{d}\Phi_{(3)} = (2\pi)^4\delta^4(p-p'-q'-q)\left(\frac{\mathrm{d}^3p'}{(2\pi)^3 2\omega_e'}\right)\left(\frac{\mathrm{d}^3q'}{(2\pi)^3 2\omega_{\nu_e}'}\right)\left(\frac{\mathrm{d}^3q}{(2\pi)^3 2\omega_{\nu_\mu}}\right)$$

Notice that since I cannot see neutrino experimentally, I can't measure the energies of neutrinos separately. Therefore neutrino energies must be eliminated in the phase space computation. If we define

$$k = (p-p') = (q+q')$$

then

$$\left(\mathrm{d}\overline{\Gamma}\right)_{\text{LAB}} = \frac{4G_F^2}{(2\pi)^5}\frac{p_\alpha p'_\beta}{m_\mu}\frac{\mathrm{d}^3p'}{\omega_e'}I^{\alpha\beta}(k)$$

[VII] For this process $\overline{|\mathcal{M}|}^2 = \frac{1}{2}|\mathcal{M}|^2$.

where we introduced the following tensor

$$I_{\alpha\beta}(k) = \int \frac{\mathrm{d}^3 q'}{\omega'_{\nu_e}} \frac{\mathrm{d}^3 q}{\omega_{\nu_\mu}} \delta^4(p - p' - q' - q) q'_\alpha q_\beta$$
$$= g_{\alpha\beta} A(k^2) + \frac{k_\alpha k_\beta}{k^2} B(k^2)$$

for some functions $A(k^2)$ and $B(k^2)$. Last step is a consequence of Lorentz invariance, that implies that most general formula of $I_{\alpha\beta}$ is the one showed above. Using tensor proprieties we have

$$\begin{cases} g_{\alpha\beta} I^{\alpha\beta} &= 4A(k^2) + B(k^2) \quad \equiv I_0(k^2) \\ k_\alpha k_\beta I^{\alpha\beta} &= k^2 (A(k^2) + B(k^2)) \equiv I_1(k^2) \end{cases} \tag{5.3}$$

Let's calculate I_0 and I_1. For the former we have

$$I_0(k^2) = \int \frac{\mathrm{d}^3 q'}{\omega'_{\nu_e}} \frac{\mathrm{d}^3 q}{\omega_{\nu_\mu}} \delta^4(k - q - q')(q \cdot q')$$

Since this object is a scalar, then it's a Lorentz invariant. I can calculate in it any frame, for convenience we use the c.o.m. frame for the neutrinos. In this case following kinematic relations holds

$$q' = (\omega'_{\nu_e}, \boldsymbol{q}') \qquad q = (\omega_{\nu_\mu}, -\boldsymbol{q}') \qquad |\boldsymbol{q}| = |\boldsymbol{q}'| \qquad \omega'_{\nu_e} = \omega_{\nu_\mu} \equiv \omega'$$

and we have[VIII]

$$I_0(k^2) = \int \frac{\mathrm{d}^3 q' \mathrm{d}^3 q}{\omega'^2} \frac{k^2}{2} \delta^4(k - q - q') = \frac{k^2}{2} \int \frac{\mathrm{d}^3 q'}{\omega'^2} \delta(k_0 - 2\omega')$$
$$= \frac{k^2}{2} \int \frac{\omega'^2 \mathrm{d}\omega'}{\omega'^2} \mathrm{d}\Omega \, \frac{1}{2} \delta\left(\omega' - \frac{k_0}{2}\right) = \pi k^2$$

Now we calculate I_1

$$I_1(k^2) = \int \frac{\mathrm{d}^3 q'}{\omega'_{\nu_e}} \frac{\mathrm{d}^3 q}{\omega_{\nu_\mu}} (k \cdot q)(k \cdot q') \delta^4(k - q - q')$$

Again, this is a scalar and we calculate it in the c.o.m. frame for neutrinos, we have

$$\begin{cases} k \cdot q &= (q + q') \cdot q &= k_0 \omega_{\nu_\mu} \\ k \cdot q' &= (q + q') \cdot q' &= k_0 \omega'_{\nu_e} \end{cases}$$

where we used $\boldsymbol{q} + \boldsymbol{q}' = 0$ in c.o.m. frame. Therefore $k = (k_0, 0, 0, 0)$ and $k^2 = k_0^2$. We obtain

$$I_1(k^2) = k^2 \int \mathrm{d}^3 q' \mathrm{d}^3 q \delta^4(k - q - q') = \frac{\pi}{2} k^4$$

Using (5.3) we obtain

$$\begin{cases} 4A(k^2) + B(k^2) &= \pi k^2 \\ k^2 (A(k^2) + B(k^2)) &= \frac{\pi}{2} k^4 \end{cases} \qquad \rightarrow \qquad \begin{cases} A(k^2) = \frac{\pi}{6} k^2 \\ B(k^2) = \frac{\pi}{3} k^2 \end{cases}$$

and finally

$$I_{\alpha\beta} = g_{\alpha\beta} \frac{\pi}{6} k^2 + k_\alpha k_\beta \frac{\pi}{3}$$

The differential cross section is

$$\left(\mathrm{d}\overline{\Gamma}\right)_{\mathrm{LAB}} = \frac{G_F^2}{48\pi^4 m_\mu} [k^2 (p \cdot p') + 2(p \cdot k)(p' \cdot k')] \underbrace{\frac{\omega_e'^2 \mathrm{d}\omega'_e}{\omega'_e} \mathrm{d}\Omega_e}_{\mathrm{d}^3 p' / \omega'_e}$$

[VIII] In the first step we used $k^2 = (p \cdot p')^2 = (q + q')^2 = 2qq'$ and the fact that k^2 value is fixed by the δ^4 function and therefore is independent from the integrating variable q'.

We notice that differential term above (and below) the brace depends only on electron energy and angular distribution. In the μ rest frame

$$p = (m_\mu, 0) \quad \rightarrow \quad \begin{cases} p \cdot p' &= m_\mu \omega'_e \\ p \cdot k &= m_\mu (m_\mu - \omega'_e) \\ p' \cdot k &= m_\mu \omega'_e \\ k^2 &= m_\mu (m_\mu - 2\omega'_e) \end{cases}$$

so

$$\left(\frac{\mathrm{d}\overline{\Gamma}}{\mathrm{d}\omega'_e} \right)_{\mathrm{LAB}} = \frac{G_F^2}{12\pi^3} \omega'^2_e m_\mu (3m_\mu - 4\omega'_e)$$

Now we consider the total decay rate. First notice that minimum value for ω'_e is $(\omega'_e)_{\mathrm{MIN}} = 0$ and in this case all energies are taken by two neutrinos, on the other side its maximum value is $(\omega'_e)_{\mathrm{MAX}} = m_\mu/2$ and this happen when direction of e is opposite to the direction of neutrinos. Therefore the total decay rate is

$$\left(\overline{\Gamma}_{\mathrm{TOT}}\right)_{\mathrm{LAB}} = \int_{\omega_{min}}^{\omega_{max}} \mathrm{d}\omega'_e \left(\frac{\mathrm{d}\overline{\Gamma}}{\mathrm{d}\omega'_e} \right)_{\mathrm{LAB}} = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

From the latter we see that only relevant parameter in Lab frame are G_F and m_μ. The lifetime in the Lab frame is[IX]

$$\tau_\mu = \frac{1}{\overline{\Gamma}_{\mathrm{TOT}}} = 2.2 \cdot 10^{-6}\mathrm{s} = 3.467 \cdot 10^{18}\ \mathrm{GeV}^-1$$

while muon mass is

$$m_\mu = 0.105\ \mathrm{GeV}$$

In this way we obtain the value for G_F:

$$G_F = \left(\frac{192\pi^3}{\tau_\mu m_\mu^5} \right)^{1/2} = 1.16 \cdot 10^{-5}\ \mathrm{GeV}$$

The muon lifetime only determines the coupling constant G_F. This coupling constant is much smaller that interaction coupling in QED ($\alpha = 0.007$). This is the reason why this kind of interaction are called *Weak*.

Exercise 7

Find other physical processes that can be described by the V-A lagrangian

Hint: For example the process $\nu_\mu e^- \rightarrow \nu_e \mu^-$ can be described by the lagrangian

$$\mathcal{L}_{\text{V-A}} = -\frac{4G_F}{\sqrt{2}} \left(\bar{\nu}_\mu \gamma_L^\alpha \mu\right) \left(e^- \gamma_{\alpha,L} \nu_e\right) - \frac{4G_F}{\sqrt{2}} \left(\bar{\mu} \gamma_L^\alpha \nu_\mu\right) \left(\bar{\nu}_e \gamma_{\alpha,L} e\right)$$

Exercise 8

Compute the cross section for the process $\nu_\mu e^- \rightarrow \nu_e \mu^-$.

Solution:

$$\overline{|\mathcal{M}|}^2 = 16 G_F^2 s^2$$
$$(\overline{\sigma})_{\mathrm{CM}} = \frac{G_F}{\pi} s$$

[IX] This is an experimental result

Exercise 9

Compute the cross section for the process $\nu_e e^- \to \nu_\mu \mu^-$.
Solution:

$$\overline{|\mathcal{M}|}^2 = 16 G_F^2 u^2$$
$$(\overline{\sigma})_{\text{CM}} = \frac{G_F}{3\pi} s$$

5.1.3 Problems of Fermi Lagrangian

The four-fermion contact interaction with V-A currents described well the weak interaction processes known in the 1950s-1960s, namely at low energies. Rapidly, it was realized that this theory was only an approximation, i.e. an effective theory in the low energy range. The theory we presented have some problems, some of them related to high energy behaviour. This leads to the interpretation of $V - A$ lagrangian as the low-energy limit of a wider theory.

Processes that cannot be described

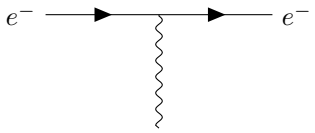
With the Fermi Lagrangian we are not able to describe following processes:

$$\begin{aligned} \nu_\mu e^- &\to \nu_\mu e^- \\ \bar{\nu}_\mu e^- &\to \bar{\nu}_\mu e^- \end{aligned}$$

Currents in the lagrangian

We have seen that $V - A$ lagrangian for leptonic interactions contains charged currents eq.(5.2), for example

In order to obtain a charge conservation in these processes, and exchange of charge must be somehow mediated during the interactions between currents. Recall that in QED interactions are mediated through photon, i.e. an uncharged spin-1 boson



In order to mediate interaction between charged currents we cannot used an uncharged particle, therefore it is clear that we can't use the photon as mediator for these interactions. We will see in the next section that this mediator is the $W^\pm$ charged boson.

Neutral Currents

Halzen sec. 12.9, 12.10; Maggiore sec. 8.2

In 1973 at CERN, neutral currents in weak interactions were discovered. This discovery has its own intrinsic importance because it opened up a whole new class of weak interactions which had remained undetected in all the 70 years'history of weak interactions. From the point of view of electroweak theory it has an added significance since the neutral-current (NC) interaction acts as a bridge between electrodynamics and the old charged-current (CC) weak interaction, but involves a massive boson Z like the W involved in the CC interaction.

Anyhow, the neutral current can be described by the $V-A$ Lagrangian too. In this case the lagrangian takes the form:

$$\mathcal{L}_{NC} = -\frac{4G'_F}{\sqrt{2}} \left(\sum_{l=e,\mu,\tau,\nu_e,\nu_\mu,\nu_\tau} J^\alpha_{NC}(l) \right)^\dagger \left(\sum_{l=e,\mu,\tau,\nu_e,\nu_\mu,\nu_\tau} J_{NC,\alpha}(l) \right) \tag{5.4}$$

with

$$J^\alpha_{NC}(l) = \bar{l}\gamma^\alpha(c_L P_L + c_R P_R)l \tag{5.5}$$

We will see that the coupling G'_F is actually different from the coupling for charged currents G_F.

Very general arguments show that the range of any force is given by the Compton wavelength of the particle transmitting it. This implies that neutral currents cannot be mediated through the photon either, since photon is massless and therefore describes infinite-range interaction. But experiments shows that weak interactions are low range interactions, and therefore must be mediated through massive particles. So we have introduce a mediator for interactions between neutral currents as well, in this case we have to introduce a massive neutral boson Z.

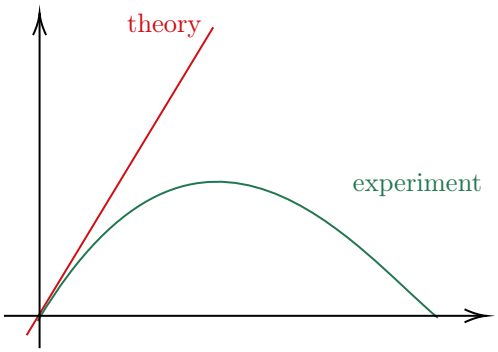
Unitarity

Halzen sec. 12.7

If we compute the cross section for the process $\nu_\mu e^- \to \nu_e \mu^-$ with $\mathcal{L}_{\text{V-A}}$ we obtain

$$(\bar{\sigma})_{CM} = \frac{G_F^2 s}{\pi}$$

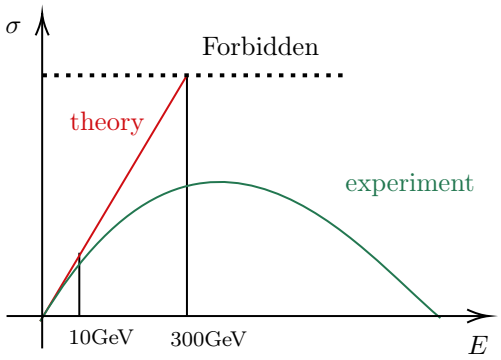
If initial energy s approaches infinity then cross section goes to ∞. Obviously this is wrong, since leads to an unitarity problem. Experimentally we have



This implies that the theory does not work for high energies. In order to have an unitary theory a requirement is

$$\bar{\sigma} \leq \frac{4\pi}{s}$$

i.e. V-A theory preserves unitariety only if $\sqrt{s} \leq \sqrt{2\pi/G_F} \simeq 700\text{GeV}$. This implies that initial energies of particles in c.o.m. have to be smaller than $\approx 300\text{GeV}$.



Non renormalizabililty

We saw that Fermi theory is not renormalizable, since $[G_F] = -2$ implies ultaviolet divergence at higher orders.

5.2 Interacting Vector Bosons

Mandl chap. 16.1, 16.2; Maggiore sec. 8.2

We have already drawn attention to the analogy between weak interactions and QED which Fermi exploited in constructing his theory. The analogy with QED that he banked upon not only made him to choose the correct form of the weak interactions - the vector form in contrast to the scalar or tensor form that were introduced later but then rejected - but also served as a fruitful analogy in the searcch for a more complete theory of weak interactions. This is what we shall describe in this section.

In beta decay, Fermi had imagined the $n-p$ line and the $e-\nu$ line interacting at the same space-time point. But clearly the correspondence with QED is greatly enhanced if the two pairs of lines are separated and an exchange of a mediator W between the $n-p$ and $e-\nu$ lines is inserted. The proprieties of this new particle must be the following:

(i) W has to be charged, in contrast to the photon, in order to conserve the electric charge;

(ii) just like the photon, the W particle also has a spin angular momentum of one unit;

(iii) in contrast to the photon, the W boson has to be a very massive object. For, the weak interaction has a short range unlike the infinite-ranged electromagnetic interaction

This considerations (with also later correction) leads to the theory of Weak Interactions that involves interacting bosons. Like the strong and electromagnetic interactions, the weak interactions is also associated with the exchange of elementary spin-1 bosons that act as 'force carriers' between quarks and/or leptons. However, in constrast to photons and gluons, there are very massive particles. There are three such 'intermediate vector boson': the charged bosons W^+ and W^- and the neutral Z boson, with masses

$$M_W = 80.40\text{GeV} \qquad M_Z = 91.19\text{GeV}$$

These larges masses have two obvious consequences.

Firstly, we already said that the range of any force is given by the Compton wavelength of the particle transmitting it. Consequently, the range of the weak interactions between quarks and leptons is of order 10^{-3}fm and at low energies the weak interaction can be treated as a zero range interactions. This is the reason why Fermi theory works very well to describe weak interactions in the low energy limit.

Secondly, very large energies are required to produce $W^\pm$ and Z bosons in the laboratory, so that they were not discovered until 1983, long after their existence and masses had been theoretically predicted.

The idea that weak interactions were due to the exchange of massive charged bosons seems to have been first by Klein in 1938, and until 1973 all observed weak interactions were consistent with the hypotesis that they were mediated by the exchange of heavy charged bosons $W^\pm$ only. However, in the 1960s,

Glashow, Salam and Weinberg developed a theory that unified electromagnetic and weak interactions in a way that is often compared to the unification of electric and magnetic interactions by Faraday and Maxwell a century earlier. This new theory made several remarkable predictions, including the existence of a neutral vector boson Z^0, and of the weak reactions arising from its exchange. In particular, neutral reactions of the type

$$\nu_\mu + N \to \nu_\mu + X$$

were predicted, where N is a nucleon and X is any set of hadrons allowed by the conservation laws. Such reactions were finally observed at CERN in 1973, as we stated in the previous section.

The prediction of the existence and proprieties of neutral currents, prior to their discovery, is only one of the many spectacular successes of the unified theory of electromagnetic and weak interactions. Others include the prediction of the existence of the charmed quark, prior to its discovery in 1974; and the prediction of the masses of the $W^\pm$ and Z^0 bosons prior to the long-awaited detection of these particles in 1983. In general, the theory is in complete areement with all the data on both weak and electromagnetic interactions, which are now usually referred to collectively as the *electroweak interaction*. However the new unification only becomes manifest at very high energies, and at lower energies, weak and electromagnetic interactions can still be clearly separated, as we shall see.

In the previous section we constructed $V-A$ lagrangian using vector currents, but the forms eq. (5.1) and eq. (5.4) are not the unique ways of costructing interactions from the currents eq. (5.2) and eq. (5.5). The most general renormalizable coupling with 2 fermions and an interacting particle is

$$\bar{\psi}\gamma^\mu(c_L P_L + c_R P_R)\psi V_\mu$$

where $c_{L,R}$ are adimensional factors $[c_L] = [c_R] = 0$. Notice that when $c_L = c_R$ we obtain a vector-like coupling (like in QED) while if $c_L \neq c_R$ we obtain an axial coupling (this is the case of Weak interactions as we know after Wu experiment: even if ν_R exists, it doesn't couple in Weak Interactions).

Let's introduce two charged vector bosons $W^\pm$ described by a complex Proca field[X] W_α and neutral vector boson Z with its neutral Proca field Z_α. Their corresponding field strengths are

$$W_{\alpha\beta} = \partial_\alpha W_\beta + \partial_\beta W_\alpha$$
$$Z_{\alpha\beta} = \partial_\alpha Z_\beta + \partial_\beta Z_\alpha$$

Then we can define the **Interacting Vector Boson (IVB) Lagrangian**:

$$\mathcal{L}_{IVB} = \mathcal{L}_{KB} + \mathcal{L}_{KF} + \mathcal{L}_{int} \tag{5.6}$$

where

$$\mathcal{L}_{KB} = -\frac{1}{2}W^\dagger_{\alpha\beta}W^{\alpha\beta} + M_W^2 W^\dagger_\alpha W^\alpha - \frac{1}{4}Z_{\alpha\beta}Z^{\alpha\beta} + \frac{1}{2}M_Z^2 Z_\alpha Z^\alpha \tag{5.7}$$

$$\mathcal{L}_{KF} = \bar{e}(i\partial\!\!\!/ - m_e)e + \bar{\mu}(i\partial\!\!\!/ - m_\mu)\mu + \bar{\nu}_e(i\partial\!\!\!/)\nu_e + \bar{\nu}_\mu(i\partial\!\!\!/)\nu_\mu \tag{5.8}$$

$$\begin{aligned}\mathcal{L}_{int} = {} & \frac{g}{\sqrt{2}}W^\alpha\left[\bar{e}\gamma_\alpha P_L\nu_e + \bar{\mu}\gamma_\alpha P_L\nu_\mu + \bar{\nu}_e\gamma_\alpha P_L e + \bar{\nu}_\mu\gamma_\alpha P_L\mu\right] + \\ & + \frac{g}{c_W}Z^\alpha[\bar{e}\gamma_\alpha(c_L^e P_L + c_R^e P_R)e + \bar{\mu}\gamma_\alpha(c_L^\mu P_L + c_R^\mu P_R)\mu + \\ & + \bar{\nu}_e\gamma_\alpha c_L^{\nu_e}P_L\nu_e + \bar{\nu}_\mu\gamma_\alpha c_L^{\nu_\mu}P_L\nu_\mu]\end{aligned} \tag{5.9}$$

where g and c_W are dimensionless coupling constant. In eq.(5.8) and eq.(5.9) we omitted fields corresponding to τ and ν_τ, understanding how to introduce them in the lagrangian is trivial. Notice that in (5.9) I introduced the interaction only for right-handed spinors, since left-handed spinors does not interact. We assumed (it's not the general case) that all fermions have same coupling (g for the coupling with W_α and g/c_W for the coupling with Z_α).

[X] These fields are coupled with leptonic vector currents, hence they must be vector fields. These vector fields satisfies the relation $\partial^\alpha W_\alpha = 0 = \partial^\alpha Z_\alpha$ and therefore are vector bosons with spin 1. See Mandl sec. 16.3 the proof of this statement.

Feynman Rules for the IVB Lagrangian

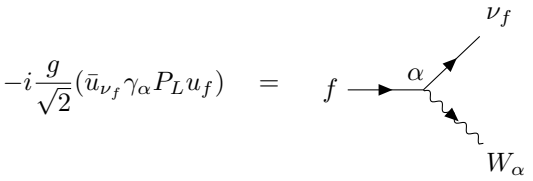
Mandl sec. 16.4

We can easily write down the Feynman rules for treating leptonic processes between charged currents (i.e. mediated through $W^{\pm}$ bosons). Recall that these interactions are described by first line of eq.(5.9), we have that for each vertex we write a factor

$$-i\frac{g}{\sqrt{2}}\gamma_\alpha P_L$$

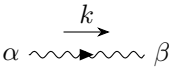
Feynman rules for external lines are obtained from QED ones only requiring a trivial relabeling for all possible fermions, for example

$-i\frac{g}{\sqrt{2}}(\bar{u}_{\nu_f}\gamma_\alpha P_L u_f)$ =



For each internal W boson line, labelled by momentum k, write a factor

$$D_W^{\alpha\beta}(k) = \frac{-i}{k^2 - M_W^2}\left(g^{\alpha\beta} - \frac{k^\alpha k^\beta}{M_W^2}\right) =$$

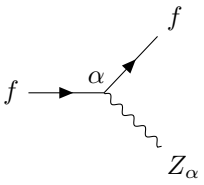


For processes involving neutral currents (i.e. mediated through Z boson) each vertex gives a factor

$$-i\frac{g}{c_W}\gamma_\alpha(c_L P_L + c_R P_R)$$

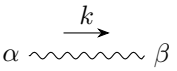
Analogously to the rules for charged bosons we have (recall that for neutrinos we have $c_R = 0$)

$-i\frac{g}{c_W}(\bar{u}_f\gamma_\alpha(c_L P_L + c_R P_R)u_f)$ =



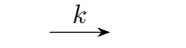
For each internal Z boson line, labelled by momentum k, write a factor

$$D_Z^{\alpha\beta}(k) = \frac{-i}{k^2 - M_W^2}\left(g^{\alpha\beta} - \frac{k^\alpha k^\beta}{M_Z^2}\right) =$$

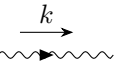


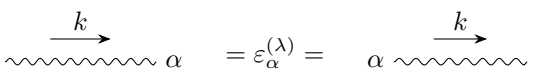
For each external initial or final boson line, write a factor $\varepsilon_\alpha^{(\lambda)}$ for each initial boson and a factor $\varepsilon_\alpha^{(\lambda)*}$ for each final boson. For the field Z_μ, which is real, $\varepsilon_\alpha^{(\lambda)} = \varepsilon_\alpha^{(\lambda)*}$:

$\varepsilon_\alpha^{(\lambda)}$ =



$\varepsilon_\alpha^{(\lambda)*}$ =





Notice that λ describes one of the possible polarization of the boson. For massive boson all possible polarization are 3.

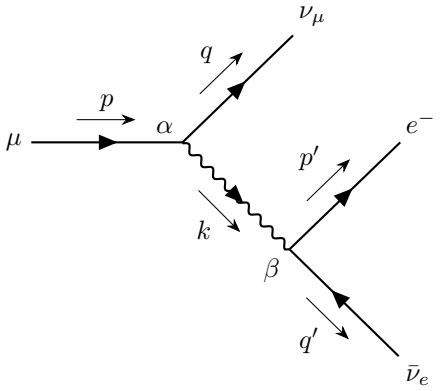
5.2.1 Fermi Lagrangian as low energy limit of a massive vector boson Lagrangian

Mandl sec. 16.6.1; Halzen sec. 12.2

In the section 5.1 we said that $V-A$ lagrangian can describe Weak interactions in the low energy limit. Indeed, the full structure of the electroweak theory (described by the Standard Model) is only revealed at energies comparable to the masses of the bosons $W^{\pm}$ and Z^0 that, together with the photon, mediate the electroweak interactions. Since $m_W = 80.425(38)\text{GeV}$ and $m_Z = 91.1876(21)\text{GeV}$, the weak decays of particles with masses between a few hundred MeV and a few GeV, as for instance the muons, the pions, the kaons, the neutron, charmed mesons like the D^0, etc., can be studied in a low-energy approximation to the SM using the $V-A$ Lagrangian. For instance in the β-decay of the free neutron, $n \to pe^-\delta\nu_e$, we have a mass difference $m_n - m_p \simeq 1.29$ MeV. Therefore, even if at the fundamental level the decay is mediated by the W-boson, the fact that the maximum momentum transfer is much smaller than m_W allows un to use a low-energy effective theory. The same approximation holds for nuclear β-decays.

This is the reason why we used the low-energy theory in order to study a scattering process mediated by weak interactions, as for instance the muon decay, at center-of-mass energies well below m_W.

Now we have to prove that IVB Lagrangian in the low energy limit leads to the results we obtained using Four Fermions model. Let's consider again the muon decay width, this time using IVB lagrangian in the limit of $\omega_e, \omega_\mu \ll M_W$. There is only 1 Feynman diagram in the lower order of S matrix expantion (i.e. second order):



Obviously kinematics constraints impose $k = p - q = p' + q'$. The Feynman amplitude is

$$\mathcal{M} = \left(-i\frac{g}{\sqrt{2}}\right)^2 \left(\bar{u}_{\nu_\mu}\gamma^\alpha P_L u_\mu\right)\left(\bar{u}_e\gamma^\beta P_L v_{\bar{\nu}_e}\right) D^W_{\alpha\beta}(p-q)$$

Using muon rest frame we have $k^2 = (m_\mu^2 - 2m_\mu\omega_{\nu_\mu})$ and in the limit $m_\mu \ll M_W$ we have $k^2 \ll M_W^2$, therefore we can expand the propagator $D^W_{\alpha\beta}(k)$ in terms of k^2:

$$D^W_{\alpha\beta}(k) = \frac{i}{M_W^2}g_{\alpha\beta} + \frac{i}{M_W^2}g_{\alpha\beta}\frac{k^2}{M_W^2} + o\left(\frac{k^4}{M_W^4}\right)$$

In this limit our Feynman amplitude reads

$$\mathcal{M} = -\frac{g^2}{2M_W^2}\left(\bar{u}_{\nu_\mu}\gamma^\alpha P_L u_\mu\right)\left(\bar{u}_e\gamma^\beta P_L v_{\bar{\nu}_e}\right)$$

Then, If I compare with the amplitude I obtained using $V-A$ amplitude, I can identify this relation between coefficents G_F and g:

$$\frac{4G_F}{\sqrt{2}} = \frac{g^2}{2M_W^2} \qquad \to \qquad G_F = \frac{\sqrt{2}g^2}{8M_W^2} \tag{5.10}$$

For the moment we do not know exactly the value of g, but we know that $g/\sqrt{2}$ must be lower than 1 in order to obtain a well-defined perturbative expantion. This means that in the low energy limit I have an information about the mass of the $W^{\pm}$ boson:

$$M_W \lesssim 120\text{GeV}$$

This calculation finally prove that $V-A$ Lagrangian is the low energy limit of the IVB lagrangian.

Exercise 10

Calculate the μ decay rate using the full W propagator
Hint: We expect a correcting factor term in the order $o(m_\mu^2/M_W^2)$.

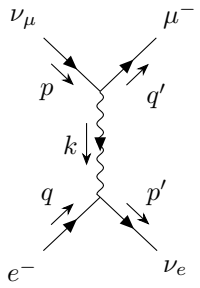
5.2.2 Charge Current (CC) and Neutral Current (NC) processes in IVB

One can try to compute the cross section for the process $\nu_\mu e^- \to \nu_e \mu^-$ in order to verify if IVB Lagrangian has same unitarity problem shown for the $V-A$ Lagrangian. In order to fix unitarity problem, we expect neutrino scattering process at sufficiently high energies to exhibit the effects of the intermediate vector boson. We leave the full calculation as an exercize, and we just write down the solutions in order to discuss them

Exercise 11: $\nu_\mu e^- \to \nu_e \mu^-$ cross section

Mandl sec. 16.6.2

Calculate $\nu_\mu e^- \to \nu_e \mu^-$ cross section using the full propagator in the unitary limit (i.e. at high energies $s \gg m_e^2, M_W^2$)



Solution:
The solution is easily obtained from the results we derived using $V-I$ lagrangian, just using the following substitution

$$\frac{G_F}{\sqrt{2}} \quad \to \quad \frac{g^2}{8}\frac{1}{u-M_W^2}$$

where $u = (p-q')^2 = k^2$. Notice that factor $(k_\alpha k_\beta)/M_W^2$ does not contribute when I neglect fermions mass. Therefore I obtain

$$|\mathcal{M}|^2_{V-A} = 16 G_F^2 s^2 \quad \to \qquad |\mathcal{M}|^2_{IVB} = \frac{g^4}{2}\frac{s^2}{(u-M_W^2)^2}$$

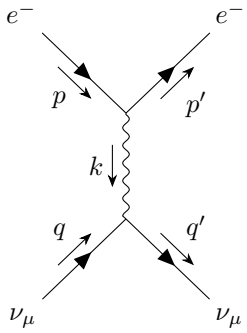
$$\left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)^{V-A}_{CM} = \frac{G_F^2 s}{4\pi^2} \quad \to \quad \left(\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\Omega}\right)^{IVB}_{CM} = \frac{g^4}{128\pi^2}\frac{s}{(u-M_W^2)^2} \simeq \frac{g^4}{128\pi^2}\frac{1}{s}$$

Now the cross section is finite at high energies, and then is clear that the introduction of the massive boson W cure the unitarity problem.

Another problem of the Four Fermions model were related to the impossibility of the description of processes $\nu_\mu e^- \to \nu_\mu e^-$. Using the IVB Lagrangian is now possible to describe these processes. Let's consider its description in low energy limit ($s \ll M_Z^2$).

Example 9: Neutral current contributions to $\nu_\mu e^- \to \nu_\mu e^-$ cross section

At the lowest order this process not include the charged part of the Lagrangian



Obviously kinematics impose $k = (p - p') = (q' - q)$. The Feynman amplitude reads

$$\mathcal{M} = \left(-i\frac{g}{c_W}\right)^2 \left(\bar{u}_{\nu_\mu}\gamma^\alpha c_L^\nu P_L u_{\nu_\mu}\right)\left(\bar{u}_e\gamma^\beta(c_L^e P_L + c_R^e P_R)u_e\right) D^Z_{\alpha\beta}(p-p')$$

In the low energy limit we have

$$\mathcal{M} = -i\left(\frac{g}{c_W M_Z}\right)^2 \left(\bar{u}_{\nu_\mu}\gamma^\alpha c_L^\nu P_L u_{\nu_\mu}\right)\left(\bar{u}_e\gamma^\beta(c_L^e P_L + c_R^e P_R)u_e\right)$$

Comparing this relation with eq.(5.4) we notice that following relation between g and G'_F:

$$\frac{4G'_F}{\sqrt{2}} = \frac{g^2}{c_W^2 M_Z^2} \qquad \to \qquad G'_F = \frac{\sqrt{2}g^2}{4c_W^2 M_Z^2}$$

Therefore the coupling G_F of the charged current in the Fermi theory is actually different than the coupling G'_F of the neutral current, since eq.(5.10) is different than the latter relation. The unpolarized amplitude reads

$$\overline{|\mathcal{M}|}^2 = \frac{1}{4}\left(\frac{g}{c_W M_Z}\right)^4 \mathrm{Tr}\left[\not{q}'(c_L\gamma_\alpha P_L)\not{q}(c_L\gamma_\beta P_L)\right]\mathrm{Tr}\left[\not{p}'(c_L\gamma^\alpha P_L + c_R\gamma^\alpha P_R)\not{p}(c_L\gamma^\beta P_L + c_R\gamma^\beta P_R)\right]$$

If we consider the high energies approximation $m_e = m_{\nu_\mu} = 0$ (notice that features related to the presence of Z boson are related to the high energy behaviour of processes) then

$$\overline{|\mathcal{M}|}^2 = \frac{4g^4}{c_W^4 M_Z^4}(c_L^\nu)^2\left[c_L^2(p\cdot q)(p'\cdot q') + c_R^2(p\cdot q')(p'\cdot q)\right]$$

Now I can take the rest frame and go on with the computation of the unpolarized scattering amplitude.

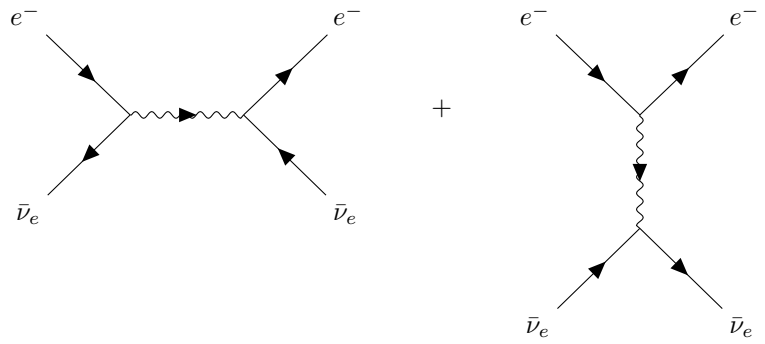
Exercise 12

Complete computation of the cross section in the previous example.

Exercise 13

Calculate the $\bar{\nu}_e e^- \to \bar{\nu}_e e^-$ amplitude.

Hint / Solution: The process is described by following diagrams:

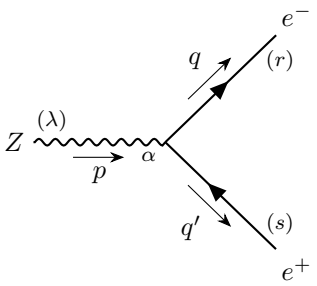


5.2.3 Decay rates of charged and neutral massive vector bosons

In this section it is shown how to compute the decay rate of the Z boson. Same procedure can be used for the W boson.

Example 10: Z boson decay

We consider the leptonic decay process of the Z boson $Z \to f\bar{f}$ (for example the process $Z \to e^- e^+$ can be described as follows)



Using Feynman rules we obtain following amplitude

$$\mathcal{M} = -i\frac{g}{c_W}\bar{u}_r(q)(c_L\gamma^\alpha P_L + c_R\gamma^\mu P_R)v_s(q')\varepsilon_\alpha^{(\lambda)}(p)$$

When we compute the unpolarized amplitude we have to introduce a factor $1/3$ in order to average over all possible initial polarizations. Therefore:

$$\begin{aligned}
\overline{|\mathcal{M}|}^2 &= \left(\frac{g}{c_W}\right)^2 \left(\frac{1}{3}\sum_\lambda \varepsilon_\alpha^{(\lambda)}\varepsilon_\beta^{(\lambda)*}\right) \sum_{r,s} \bar{u}_r(c_L\gamma^\alpha P_L + c_R\gamma^\alpha P_R)v_s\bar{v}_r(c_L\gamma^\beta P_L + c_R\gamma^\beta P_R)u_s \\
&= \frac{1}{3}\left(\frac{g}{c_W}\right)^2 \left(-g_{\alpha\beta} + \frac{p_\alpha p_\beta}{M_Z^2}\right) \mathrm{Tr}\Big[(\not{q}' - M_Z)(c_L\gamma_L^\beta + c_R\gamma_R^\beta)(\not{k} + m_f)(c_L\gamma_L^\alpha + c_R\gamma_R^\alpha)\Big] \\
&= \frac{1}{3}\left(\frac{g}{c_W}\right)^2 \left(2(c_L^2 + c_R^2)\left(q\cdot q' + \frac{2(p\cdot q)(p\cdot q')}{M_Z^2}\right) + 12m_f^2 c_L c_R\right)
\end{aligned}$$

If we consider the lab frame, i.e. the rest frame for the initial boson, we have following kinematic

relations

$$\begin{cases} p^2 = M_Z^2 = (q+q')^2 = 2m_f^2 + 2q\cdot q' & \to \quad q\cdot q' = \dfrac{M_Z^2}{2} - m_f \\ (p-q)^2 = M_Z^2 + m_f^2 - 2p\cdot q = q'^2 = m_f^2 & \to \quad p\cdot q = \dfrac{M_Z^2}{2} \\ (p-q')^2 = q^2 & \to \quad p\cdot q' = \dfrac{M_Z^2}{2} \end{cases}$$

So we obtain

$$\overline{|\mathcal{M}|}^2_{\text{LAB}} = \left(\frac{g}{c_W}\right)^2 \frac{M_Z}{3}\left(2(c_L^2+c_R^2)\left(1-\frac{m_f^2}{M_Z^2}\right) + 12c_Lc_R\frac{m_f^2}{M_Z^2}\right)$$

We can see that there is no dependence on momentum and angles for this process. In order to obtain final formula for the decay rate, let's consider the phase space $\mathrm{d}\Phi_{(2)}$:

$$\begin{aligned} \int \mathrm{d}\Phi_{(2)} &= \int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\mathrm{d}^3q'}{(2\pi)^3}\frac{1}{4E_qE_{q'}}(2\pi)^4\delta^4(p-q-q') \\ &= \int \frac{\mathrm{d}^3q}{(2\pi)^2}\frac{1}{4E_q^2}\delta(M_Z-2E_q) \\ &= \int \frac{\mathrm{d}\Omega}{4\pi}\frac{|\boldsymbol{q}|^2\,\mathrm{d}|\boldsymbol{q}|}{4\pi E_q^2}\delta(M_Z-2E_q) \\ &= \frac{1}{4\pi}\int \mathrm{d}|\boldsymbol{q}|\,\frac{|\boldsymbol{q}|^2}{E_q^2}\frac{1}{2}\delta\left(E_q-\frac{M_Z}{2}\right) \\ &= \frac{1}{8\pi}\left(1-\frac{4m_f^2}{M_Z^2}\right) \end{aligned}$$

Where in the fourth line we performed the integration over the solid angle since there is no dependence over the angle in the unpolarized amplitude. In the last step we used

$$\left(\frac{M_Z}{2}\right)^2 = E_q^2 = m_f^2 + |\boldsymbol{q}|^2 \qquad \to \quad |\boldsymbol{q}|^2 = \frac{M_Z^2}{4} - m_f^2$$

Finally we can obtain the decay rate

$$\begin{aligned} \left(\mathrm{d}\overline{\Gamma}_Z\right)_{\text{LAB}} &= \frac{\overline{|\mathcal{M}|}^2_{\text{LAB}}}{2M_Z}\mathrm{d}\Phi_{(2)} \\ \left(\overline{\Gamma}_Z\right)_{\text{LAB}} &= \left(\frac{g}{c_W}\right)^2\frac{M_Z}{48\pi}\left(1-\frac{4m_f^2}{M_Z^2}\right)^{1/2}\left(2(c_L^2+c_R^2)\left(1-\frac{m_f^2}{M_Z^2}\right)+12c_Lc_R\frac{m_f^2}{M_Z^2}\right) \end{aligned}$$

In the special case where f is a neutrino, we have $c_R = 0$, $c_L = 1/2$, $m_\nu = 0$ and therefore the decay rate reads

$$\left(\overline{\Gamma}_Z^{\nu}\right)_{\text{LAB}} = \left(\frac{g}{c_W}\right)^2\frac{M_Z}{96\pi}$$

We stress the fact that this decay rate correspond to the decay process into a specific neutrino and its antineutrino. Recall that neutrinos cannot be detached, therefore we cannot "see" experimentally the process $Z \to \nu\,\bar{\nu}$. Let's define

$$\left(\overline{\Gamma}_Z^{inv}\right)_{\text{LAB}} = N_\nu\left(\frac{g}{c_W}\right)^2\frac{M_Z}{96\pi}$$

the decay rate of Z into *inv*isible particles. Experimentally $N_\mu \simeq 3.01$, therefore we can interpret this result as the fact that the only invisible products of Z decay are neutrinos, i.e. there is no other invisible particle coupled with the field Z_α into $\mathcal{L}_{\text{int}}$ (eq.(5.9))

Exercise 14

Mandl sec. 16.6.3

Calculate the decay rate of the charged boson W
Solution:

$$\left(\overline{\Gamma}_Z\right)_{\text{LAB}} = \frac{g^2}{48\pi^2} M_W \left(1 - \frac{m_e^2}{M_W^2}\right)^2 \left(1 + \frac{m_e^2}{2M_W^2}\right)$$

5.2.4 Problems with IVB theory

This model describes very well Weak Interaction at lowest orders, and solve most of the problems of the Fermi theory, but still it cannot be renormalizable. Indeed, in IVB theory we have $[g] = 0$, but in the naive dimensional counting we assumed that the propagator dacayes as $1/k^2 \sim D_A(k)$. But for massive vector bosons this is no more true since

$$D_{W,Z} = \frac{1}{k^2 - M^2}\left(g_{\alpha\beta} - \frac{k_\alpha k_\beta}{M^2}\right) \sim \frac{1}{k^2} + c$$

The additional term c has as consequence that the naive dimensional analysis of the superficial degree of divergence D in this case reads

(i) for $M = 0$ boson

$$D = 4 - [g]V - E_B - \frac{3}{2}E_F = 4 - E_B - \frac{3}{2}E_F$$

(ii) for $M \neq 0$ boson

$$D = 4 - [g]V + n_B V - 2E_B - \frac{3}{2}E_F = 4 + n_B V - 2E_B - \frac{3}{2}E_F$$

We can see that $M \neq 0$ case there is an additional term $n_B V$ that counts the number of vertices, and therefore the theory cannot be renormalizable. In order to obtain a renormalizable theory with vector mediators required conditions are $[g] \geq 0$ and $M_B = 0$ (these two conditions guaranteers $D \sim 1/k^2$).

Therefore in order to obtain a renormalizable gauge theory[XI] we need to find a trick to define a theory with massless boson that also in some level produces massive bosons without breaking the renormalizability. This is what we will do in the next chapter.

We can also see this requirement as follows: in order to obtain a Lagrangian that exhibits gauge symmetry we cannot have massive terms for the interacting bosons, otherwise any term proportional to $M^2 A^\mu A_\mu$ will break the symmetry.

[XI] All interacting theories are defined in terms of gauge theories.

Chapter 6

Non abelian Gauge Theories

Maggiore sec. 10.1; Peskin sec. 15.1; Schwartz sec. 25.Intro; Mandl sec. 11.Intro, 11.1

In this chapter we introduce non-abelian gauge theories, or Yang-Mills theories. Their importance stems from the fact that strong interactions are described by a non-abelian gauge theory with gauge group $SU(3)$, known as quantum chromodynamics or QCD, while the electromagnetic and weak interactions are unified in a gauge theory with gauge group $SU(2) \times U(1)$, the electroweak theory. Together, QCD and the electroweak theory form the Standard Model, which to date reproduces all known experimental results of particle physics, up to energies of the order of a few hundred GeV.

A full presentation of the Standard Model is beyond the scope of this course. In this and the next chapter we will however introduce two of its main ingredients, namely Yang-Mills theories and the Higgs mechanism.

Non-abelian gauge theories[I], beside having an extraordinary experimental success, have also a very rich theoretical structure, at the classical and especially at the quantum level. Within the scope of this course, we can only limit ourself to just a few elementary aspects, in particular, we will discuss how to generalize gauge transformations to non-abelian groups and how to write the corresponding invariant Lagrangians.

As a first step, it is useful to rewrite the abelian gauge transformation of electrodynamics in a form more suitable for generalization. We already know that the Lagrangian of QED is invariant under a very large group of transformations, allowing an independent symmetry transformation at every point in spacetime. This invariance is the famous *gauge symmetry* of QED. From the modern viewpoint, however, gauge symmetry is not an incidental curiosity, but rather the fundamental principle that determines the form of the Lagrangian. Let us now review the elements of the theory, taking a modern viewpoint.

We begin with the complex-valued Dirac field $\psi(x)$, and stipulate that our theory should be invariant under the transformation

$$\psi(x) \quad \to \quad e^{iq\alpha(x)}\psi(x) = \Omega(x)\psi(x)$$

where $\Omega(x) = e^{iq\alpha(x)}$ is the representation of the $U(1)$ transformation. The transformation of the gauge field instead is

$$A_\mu(x) \quad \to \quad A_\mu(x) - \partial_\mu\alpha(x) = \Omega(x)A_\mu(x)\Omega^\dagger(x) - \frac{i}{q}(\partial_\mu\Omega)\Omega^\dagger$$

where in the last step we used the propriety $\Omega^{-1}(x) = \Omega^\dagger(x)$ that holds since we arae considering the unitary group $U(1)$. The coupling between A_μ and ψ is obtained using the *covariant derivative*, which in the representation to which ψ belongs takes the form

$$D_\mu = (\partial_\mu + iqA_\mu)$$

The important propriety of the covariant derivative is that, even under x-dependent transformations, it transforms in the same way as ψ:

$$D_\mu\psi \quad \to \quad \Omega(D_\mu\psi)$$

Using covariant derivatives there is a very simple way to construct a theory with local $U(1)$ invariance: we start from a theory with global $U(1)$ invariance and we just replace all the ordinary derivatives with

[I] For a description of Gauge groups, Lie Algebras and their representation, see Peskin sec. 15.4; Schwartz sec. 25.1

covariant derivatives. This method of coupling matter to the electromagnetic field is know as *minimal coupling*. Non-minimal coupling are also possible, but they are characterized by coupling constant with dimensions of the inverse powers of mass. If we consider the example of the Fermi theory, we understand that couplings with inverse mass dimensions are less fundamental than dimensionless couplings, and emerge as the low-energy limit of some more fundamental dimensionless coupling. Therefore, it is the minimal coupling that we want to generalize.
Notice that, since covariant derivative $D_\mu\psi$ has same proprieties of the field ψ, we have

$$D_\mu D_\nu\psi \quad \to \quad \Omega(D_\mu D_\nu\psi)$$

However, the commutator between covariant derivatives is not a derivative at all, since in general (for any gauge theory) it acts multiplicatively on the field ψ:

$$-\frac{i}{q}[D_\mu, D_\nu]\psi = F_{\mu\nu}\psi \tag{6.1}$$

where we introduced the structure $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Let's consider the transformation propriety of this structure:

$$F_{\mu\nu}\psi \quad \to \quad -\frac{i}{q}\Omega[D_\mu, D_\nu]\psi = \Omega F_{\mu\nu}\psi$$

this means that

$$F_{\mu\nu} \quad \to \quad \Omega F_{\mu\nu}\Omega^\dagger \tag{6.2}$$

and therefore this structure is a tensor.

If the simple geometrical construction we have just presented yield Maxwell's theory of electrodynamics, then surely it must be possible to construct other interesting theories by starting from more general geometrical principles. We want to generalize the above transformations to the case where Ω belong to a non-abelian group G, rather then just to $U(1)$, and we want to construct a Lagrangian invariant under such local transformations. We will limit ourselves to the case $G = SU(N)$, although the construction is very general; G is called the **gauge group**.

We start considering a fermion field[II] in N-dimensional fundamental representation[III]

$$\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}$$

The fact that Ψ transform in the N-dimensional fundamental representation means that, under a gauge transformation

$$\Psi(x) \quad \to \quad \Psi'(x) = \Omega(x)\Psi(x)$$

The transformation matrix $\Omega \in SU(N)$ can be written as

$$\Omega(x) = \exp\{ig\hat{\alpha}(x)\}$$

where $\hat{\alpha}$ is a function that can be written in terms of the $N^2 - 1$ generators T^a of the group $SU(N)$

$$\hat{\alpha}(x) = \alpha^a(x)T^a \tag{6.3}$$

To construct an invariant Lagrangian, we introduce a set of gauge fields A^a_μ labeled by an index a, with one gauge field for each generator of the gauge group; the A^a_μ are called **non-abelian gauge fields**. In particular, $SU(N)$ has $N^2 - 1$ generators, so we have three gauge fields for $SU(2)$ and eight gauge fields for $SU(3)$. We introduce the matrix field

$$\hat{A}_\mu(x) = \sum_{a=1}^{N^2-1} A^a_\mu(x)T^a \tag{6.4}$$

[II] For definiteness we take ψ_i to be Dirac fermions, but all the subsequent considerations are very general and apply to any matter fields, e.g. to bosonic fields or to Weyl fermions.

[III] Notice that this construction is similar to the one we used for flavours. Nevertheless, this time we require a local symmetry, not only global.

The summation over a will be omitted from now on, it should be introduced each time there are two indeces a in the same term.

We now introduce the covariant derivative on the field ψ. The most general form is

$$\hat{D}_\mu \Psi = (\mathbb{1}_N \partial_\mu + igA^a_\mu(x)T^a)\Psi = (\hat{\partial}_\mu + ig\hat{A}_\mu(x))\Psi \tag{6.5}$$

Fields A^a_μ do not depend on the specific representation we choose[IV] for the field Ψ and neither do the covariant derivative $\hat{D}_\mu$. Even if we use non-fundamental representation Ψ_R of the field (and therefore different generators T^a_R) equation eq. (6.5) does not change.

6.1 The Yang-Mills Lagrangian

Maggiore sec. 10.2; (Peskin sec. 15.2)

The free Dirac Lagrangian

$$\mathcal{L}_{\text{free}} = \bar{\Psi}(i\hat{\not{\partial}} - m)\Psi \equiv \sum_{\alpha=1}^{N} \bar{\psi}^\alpha (i(\hat{\not{\partial}}\psi)^\alpha - m\psi^\alpha)$$

is invariant under global $SU(N)$ transformations, since if $\Psi \to \Omega\Psi$ then $\bar{\Psi} \to \Psi\Omega^\dagger$ and, if Ω is independent of x, it goes through ∂_μ and cancels against $\Omega^\dagger$. However, if Ω depends on x, performing the transformation we also get a term proportional to $\partial_\mu\Omega$ and this Lagrangian is no longer invariant.

To construct an invariant Lagrangian, we have to use the covariant derivative, just replacing $\partial_\mu \to D_\mu$ in the free theory, that is

$$\boxed{\mathcal{L}_D = \bar{\Psi}(i\hat{\not{D}} - m)\Psi \equiv \sum_{\alpha=1}^{N} \bar{\psi}^\alpha (i(\hat{\not{D}}\psi)^\alpha - m\psi^\alpha)} \tag{6.6}$$

We can obtain the gauge transformation of A_μ just requiring the invariance of the Dirac Lagrangian. We impose that

$$\hat{D}_\mu \Psi \quad \to \quad \Omega(\hat{D}_\mu \Psi)$$

We have

$$\begin{aligned}(\hat{D}_\mu \Psi)' &= (\hat{\partial}_\mu + ig\hat{A}'_\mu(x))\Omega\Psi = ((\hat{\partial}_\mu\Omega) + \Omega\hat{\partial}_\mu + ig\hat{A}'_\mu(x))\Psi \\ &\overset{!}{=} \Omega(\hat{D}_\mu\Psi) = \Omega(\hat{\partial}_\mu + ig\hat{A}_\mu(x))\Psi\end{aligned}$$

and therefore the requirement implies the following gauge transformation for the field $\hat{A}_\mu$:

$$\boxed{\hat{A}_\mu \quad \to \quad \Omega\hat{A}_\mu\Omega^\dagger + \frac{1}{g}(\hat{\partial}_\mu\Omega)\Omega^\dagger} \tag{6.7}$$

The Lagrangian eq. (6.6) contains the fermionic field and its interaction with the gauge fields. The interaction term, which is hidden in the covariant derivative, is

$$\mathcal{L}_{\text{int}} = g\bar{\psi}^\alpha \not{A}^a_\mu (T^a)_{\alpha\beta}\psi^\beta$$

and we see that g is a couling constant. We also need a kinetic term for the gauge fields. One might try to define the field strength tensor of each of the gauge fields A^a_μ as $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$, but it is immediate to verify that this quantity does not satisfy any simple transformation property under gauge transformation under eq. (6.7). Instead, a straightforward computation shows that the quantity

$$\boxed{\hat{F}_{\mu\nu} = \hat{\partial}_\mu\hat{A}_\nu - \hat{\partial}_\nu\hat{A}_\mu - ig[\hat{A}_\mu, \hat{A}_\nu]} \tag{6.8}$$

[IV] Just as in electromagnetism the gauge field, and therefore its transformation proprieties, does not know anything about about the representation of the matter field we consider.

transforms as

$$F_{\mu\nu} \quad \to \quad \Omega(x) F_{\mu\nu} \Omega^\dagger(x) \tag{6.9}$$

and satisfy the relation

$$-\frac{i}{g}[\hat{D}_\mu, \hat{D}_\nu]\Psi = F_{\mu\nu}\Psi$$

which are the trivial generalizations of eq. (6.2) and eq. (6.1). Therefore, using the definition eq. (6.8) the tensor $\hat{F}_{\mu\nu}$ is the generalization of the field strength tensor $F_{\mu\nu}$ we defined for the Maxwell's field. The object $\hat{F}_{\mu\nu}$ is called **non-abelian field strength tensor**. From eq. (6.8) ad eq. (6.4) we see that we can rewrite $\hat{F}_{\mu\nu}$ as

$$\hat{F}_{\mu\nu} = F^a_{\mu\nu} T^a$$

with[V]

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{6.10}$$

Now it is easy to construct a gauge-invariant kinetic term for the gauge field; it is given by

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2}\,\text{Tr}[F_{\mu\nu}F^{\mu\nu}] = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a$$

where $F_{\mu\nu}$ has be taken in the fundamental representation, and we used the fact that $\text{Tr}\big(T^a T^b\big) = \delta^{ab}/2$. Under gauge transformations $\text{Tr}\, F_{\mu\nu}F^{\mu\nu} \to \text{Tr}\big(\Omega F_{\mu\nu} F^{\mu\nu} \Omega^\dagger\big) = \text{Tr}\, F_{\mu\nu} F^{\mu\nu}$ due to the cyclic property of the trace.

The complete Lagrangian of the $SU(N)$ Yang-Mills theory with Dirac fermions is therefore

$$\boxed{\mathcal{L}_{YM} = \bar{\Psi}(i\not{D} - m)\Psi - \frac{1}{2}\,\text{Tr}\, F_{\mu\nu}F^{\mu\nu}} \tag{6.11}$$

This Lagrangian is called **Yang-Mills Lagrangian**. The kinetic term for the gauge field can be rewritten using the vector potential $\hat{A}_\mu$ as follows

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2}\,\text{Tr}\Big[(\partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu)^2\Big] \tag{6.12a}$$

$$- ig\,\text{Tr}\Big[(\partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu)[A^\mu, A^\nu]\Big] \tag{6.12b}$$

$$+ \frac{g^2}{2}\,\text{Tr}\Big[[\hat{A}_\mu, \hat{A}_\nu][\hat{A}^\mu, \hat{A}^\nu]\Big] \tag{6.12c}$$

Observe, from eq. (6.12), that the term F^2 contains not only the standard kinetic term of the gauge fields, but also an interaction vertex with three gauge bosons, proportional to g, and a vertex with four gauge bosons, proportional to g^2. This interactions represented in following figures:

Observe also that gauge invariance has fixed the three-boson, four-boson, and boson-fermion-fermion vertices in terms of a single parameter, the gauge coupling g.

Adjoint representation of the field strength tensor

Maggiore sec. 10.4

Finally, notice that eq. (6.9) means that the tensor $\hat{F}_{\mu\nu}$ transforms with the adjoint representation of the group. Let's take an infinitesimal gauge transformation, i.e. $\Omega(x) \simeq \mathbb{1} + ig\hat{\alpha}(x) + o(\hat{\alpha}^2(x))$, then

$$\Psi \quad \to \quad \Omega\Psi = \left(e^{ig\alpha^a T^a}\right)\Psi \simeq \Psi + ig\alpha^a T^a \Psi \tag{6.13}$$

[V] Recall that *structure constant* are defined as $[T^a, T^b] = i f^{abc} T^c$.

Under the infinitesimal transformation eq. (6.9) reads[VI]

$$\begin{aligned}\hat{F}_{\mu\nu} \quad \xrightarrow{\sim} \quad & \hat{F}_{\mu\nu} + ig[\hat{\alpha}, \hat{F}_{\mu\nu}] = \hat{F}_{\mu\nu} + ig\alpha^a F^b_{\mu\nu}[T^a, T^b] \\ & = \hat{F}_{\mu\nu} - g\alpha^a F^b_{\mu\nu} f^{abc} T^c = \hat{F}_{\mu\nu} + g\alpha^a F^b_{\mu\nu} f^{acb} T^c \\ & = \hat{F}_{\mu\nu} + g\alpha^b F^c_{\mu\nu} f^{bac} T^a = \hat{F}_{\mu\nu} + ig\alpha^b F^c_{\mu\nu} (T^b_{\text{adj}})^{ac} T^a\end{aligned}$$

where in the fourth step we used the total asymmetry of structure constant f^{abc}, in the fifth we just renamed indexes, and in the last the relation $(T^a_{\text{adj}})^{bc} = -if^{abc}$. Using the decomposition $\hat{F}_{\mu\nu} = F^a_{\mu\nu} T^a$:

$$F^a_{\mu\nu} \quad \xrightarrow{\sim} \quad F^a_{\mu\nu} + ig\alpha^b (T^b_{\text{adj}})^{ac} F^c_{\mu\nu} \tag{6.14}$$

If we introduce the following representation

$$\boldsymbol{F}_{\mu\nu} = \begin{pmatrix} F^1_{\mu\nu} \\ \vdots \\ F^{N^2-1}_{\mu\nu} \end{pmatrix}$$

the eq. (6.14) can be rewritten (neglecting $o(\hat{\alpha}^2)$ terms):

$$\boldsymbol{F}_{\mu\nu} \quad \xrightarrow{\sim} \quad \boldsymbol{F}_{\mu\nu} + ig\alpha^a T^a_{\text{adj}} \boldsymbol{F}_{\mu\nu} \simeq \left(e^{ig\alpha^a T^a_{\text{adj}}} \right) \boldsymbol{F}_{\mu\nu} \tag{6.15}$$

We notice that compared with the transformation in the fundamental representation eq. (6.13), the transformation for the field strength eq. (6.15) is obtained just substituting the generators T^a with them adjoints T^a_{adj}. This means that the field $F_{\mu\nu}$ transforms in the adjoint representation of $SU(N)$. We could expected this result since the representation of $\hat{F}_{\mu\nu}$ is (N^2-1)-dim., and the dimension of the adjoint representation of a fundamental N-dim. representation has N^2-1 dimensions.

6.2 The Strong sector of the Standard Model: the $SU(3)$ example

Schwartz 25.1.1, 26.Intro, 26.1

Let's consider the specific case of the $SU(3)$ group. The $SU(3)$ has $N^2-1 = 8$ hermitian[VII] generators that must satisfy the algebra

$$[T^a, T^b] = if^{abc} T^c \qquad \text{with} \quad a, b, c = 1, \dots, 8$$

where the *structure constant* f_{abc} must be a completely antisymmetric tensor. The values of f_{abc} are conveniently chosen to be

$$\begin{aligned} f^{123} &= 1 \\ f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} &= \frac{1}{2} \\ f^{458} = f^{678} &= \frac{\sqrt{3}}{2} \end{aligned}$$

and all other f^{abc} not related to these by permuting indices are zero. If we define **Gell-Mann matrices** as follows:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

[VI] The symbol $\xrightarrow{\sim}$ indicate an approximated transformation, since we neglect $o(\hat{\alpha}^2)$ terms.
[VII] The Lie algebra of $SU(N)$ is made of hermitian matrices.

then the generators T^a in the fundamental representation are given by

$$T^a = \frac{1}{2}\lambda^a$$

They satisfy this normalization propriety

$$\mathrm{Tr}\big[T^a T^b\big] = \frac{1}{2}\delta_{ab} \tag{6.16}$$

Let's introduce eight non-abelian gauge fields G^a_μ. Adopting the previous fundamental representation of $SU(3)$ the gauge matrix field become

$$\hat{G}_\mu(x) \equiv \sum_{a=1}^{8} G^a_\mu(x) T^a = \frac{1}{2}\begin{pmatrix} G^3_\mu + \frac{1}{\sqrt{2}}G^8_\mu & G^1_\mu - iG^2_\mu & G^4_\mu - iG^5_\mu \\ G^1_\mu + iG^2_\mu & G^3_\mu + \frac{1}{\sqrt{3}}G^8_\mu & G^6_\mu - iG^7_\mu \\ G^4_\mu + iG^5_\mu & G^6_\mu + iG^7_\mu & -\frac{2}{\sqrt{3}}G^8_\mu \end{pmatrix} \tag{6.17}$$

The matrix $\hat{G}_\mu$ shows 2 gauge bosons in the diagonal sector and 6 gauge bosons in the off-diagonal sector. These gauge bosons associated to each field $G^a_\mu(x)$ are called **gluons**.

6.2.1 The QCD Yang-Mills Lagrangian

Schwartz sec. 26.1; Mandl sec. 11.2, 11.3.1

Matter fields in QCD describe particles called **quarks**. Quark's fields are 3-dim complex Dirac fields, that can be represented as

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \psi_R \\ \psi_G \\ \psi_B \end{pmatrix}$$

where components of this fields are the three possible "color states" called respectively **red** (R), **green** (G), **blue** (B) quarks. The Dirac Lagrangian takes the form

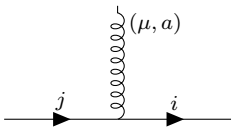
$$\mathcal{L}_D = \bar{\Psi}(i\not{D} - m)\Psi = \bar{\Psi}(i\not{\partial} - m)\Psi - g_s\bar{\Psi}\gamma^\mu\hat{G}_\mu\Psi \tag{6.18}$$

Gluons couplings with quarks

Let's consider the interacting part of eq. (6.18):

$$\mathcal{L}_{int} = -g_S\bar{\Psi}\gamma^\mu\hat{G}_\mu\Psi = -g_s\bar{\psi}_i(\hat{G}_\mu)_{ij}\gamma^\mu\psi_j \tag{6.19}$$

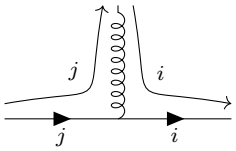
where in the second step we wrote explicitly the components. We can immediately write down Feynman's rule for vertices in the form $f\bar{f}g$ (where f indicate fermions and g the gluon):



$$= -ig_s\gamma^\mu(T^a)_{ij}$$

Indexes i, j describes the color of external quarks, while indexes μ and a are referred to the gluon field. In particular, μ indicate the component of the gluon field and a indicate which gluon is involved in the process.

When we consider eq. (6.19) with eq. (6.17) we notice that colors of quarks attached to each vertex define which gluons can be involved in the process. For example, if incoming and outgoing quarks has same color $i = j$, then only gluons in the diagonal sector of eq. (6.17) are involved, namely G^3_μ and G^8_μ. Differently, if I take $i = 1$ and $j = 2$ only gluons G^1_μ and G^2_μ are involved. We can interpret this feature of the theory as the fact gluons carries colors. They have to carry out the color of the incoming quark and provide the color of the outgoing gluon. This (heuristic) description of the interaction between gluons and quarks is here represented pictorially:



Gluon propagators and gluon self-interactions

The Feynman propagator for the gluon field takes the following form

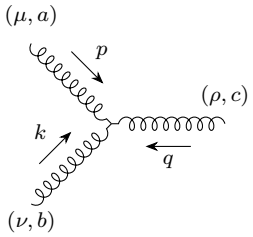
$$(\mu,a) \quad\quad (\nu,b) \qquad = D^{ab}_{\mu\nu}(k) = -\frac{i}{k^2}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\delta_{ab} \tag{6.20}$$

where the term $(1-\xi)$ is the gauge fixing term needed for the quantization of the field. When we consider amplitudes we can chose $\xi = 1$ in order to simplify computation. Again, indexes μ, ν indicate components of the gluon field, while indexes a, b indicate color carried by the gluon. We notice the absence of mass terms, since gauge symmetry requires massless gauge fields.

Let's consider 3 gluons interaction given by the gauge field kinetic term proportional to g_s (eq. (6.12b))

$$-ig_s \,\mathrm{Tr}\Big[(\partial_\mu \hat{G}_\nu - \partial_\nu \hat{G}_\mu)[\hat{G}^\mu, \hat{G}^\nu]\Big] = g_s f^{abc}(\partial_\mu G^a_\nu)G^{\mu,b}G^{\nu,c}$$

where we used eq. (6.16) to obtain the right side term. We shall prove the following Feynman rule for 3 gluons interactions



$$= g_s f^{abc}(g_{\mu\nu}(p-k)_\rho - g_{\nu\rho}(k-q)_\mu + g_{\rho\mu}(q-p)_\nu) \tag{6.21}$$

Exercise 15: Proof of eq. (6.21)

Let's show how to derive the Feynman rule eq. (6.21). The interaction is given by the interaction Lagrangian

$$g_s f^{ijk}(\partial_\sigma G^i_\tau)G^{\sigma,j}G^{\tau,k}$$

Then we have

$$\begin{aligned}&\langle 0| - i\int \mathrm{d}^4x \mathcal{H}^{\mathrm{int}}(x)\,|(\mu,a,p),(\nu,b,k),(\rho,c,q)\rangle = \\ &= \langle 0|\, ig_s f^{ijk}\int \mathrm{d}^4x \left[(\partial_\sigma G^i_\tau)G^{\sigma,j}G^{\tau,k}\right]_x a^\dagger(\mu,a,p)a^\dagger(\nu,b,k)a^\dagger(\rho,c,q)\,|0\rangle\end{aligned} \tag{6.22}$$

Notice that I have $3! = 6$ ways to associate these fields with outgoing gluon states. First consider the case where

(i) the term $(\partial_\sigma G^i_\tau)$ is applied to gluon (μ, a, p);

(ii) the term $G^{\sigma,j}$ is applied to gluon (ν, b, k);

(iii) the term $G^{\tau,k}$ is applied to gluon (ρ, c, q).

In this case we can make following substitution into eq. (6.22)

$$(\partial_\sigma G^i_\tau)G^{\sigma,j}G^{\tau,k} \quad\to\quad \left(g_{\tau\mu'}\partial_\sigma G^{\mu',i}\right)\left(g^{\sigma\nu'}G^j_{\nu'}\right)\left(g^{\tau\rho'}G^k_{\rho'}\right)$$

The contribution to the amplitude is

$$ig_s f^{abc}\left(g_{\tau\mu}p_\sigma \varepsilon^{\mu,a}\right)\left(-ig^{\sigma\nu}\varepsilon^b_\nu\right)\left(g^{\tau\rho}\varepsilon^c_\rho\right)$$

Terms in the form $\varepsilon^{\mu,a}$ are given by Feynman rules of incoming gluons, therefore must be omitted. We now have

$$g_s f^{abc} g_{\tau\mu} p_\sigma g^\sigma_\nu g^\tau_\rho = g_s f^{abc} g_{\mu\rho} p_\nu$$

This is the contribution given by only one of the six possible configurations. When we swap (ν, b, k) and (ρ, c, q) the contribution is given by a simple exchange of indexes

$$g_s f^{acb} g_{\mu\nu} p_\rho = -ig_s f^{abc} g_{\mu\nu} p_\rho$$

Therefore the choice "the term $(\partial_\sigma G^i_\tau)$ is applied to gluon (μ, a, p)" give a contribution

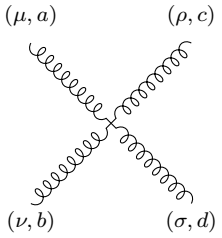
$$g_s f^{abc}(g_{\mu\rho}p_\nu - g_{\mu\nu}p_\rho)$$

Using cyclic permutations $(\mu, a, p) \to (\nu, b, k) \to (\rho, c, q) \to (\mu, a, p)$ we can sum over all possible choices for $(\partial_\sigma G^i_\tau)$ and finally obtain eq. (6.21).

Let's consider 4 gluons interaction given by the gauge field kinetic term proportional to g_s^2 (eq. (6.12c))

$$\frac{g^2}{2}\operatorname{Tr}\left[[\hat{A}_\mu, \hat{A}_\nu][\hat{A}^\mu, \hat{A}^\nu]\right] = \frac{g^2}{4} f^{abc} f^{cde} G^a_\mu G^b_\nu G^{\mu,c} G^{\nu,d}$$

where we used eq. (6.16) to obtain the right side term. We shall prove the following Feynman rule for 4 gluons interactions



$$\begin{aligned} &= g_s^2\{f^{abc}f^{cde}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) + \\ &\quad + f^{abc}f^{cde}(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\sigma}g_{\nu\rho}) + \\ &\quad + f^{abc}f^{cde}(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma})\} \end{aligned} \tag{6.23}$$

Notice that for 4 gluons interactions there is no dependence on the momenta od gauge particles.

6.3 Topics in QCD

Peskin sec. 17.1

Quantum chromodynamics is a non-Abelian gauge theory with gauge group $SU(3)$, and describes strong interactions between quarks and gluons.

Our current theoretical picture of the strong interactions began with the identification of the elementary fermions that make up the proton and other hadrons. As the properties of these fermions became better understood, the nature of their interactions became tightly constrained, in a way that led eventually to a unique candidate theory.

In 1963 Gell-Mann and Zweig proposed a model that explained the spectrum of strongly interacting particles in terms of elementary constituents called **quarks**. Meson were expected to be quark-antiquark bound states. Indeed the lightest mesons have just the correct quantum numbers to justify the interpretation; they are spin-0 and spin-1 states of odd parity. Baryons were interpreted as bound states of three quarks. To explain the electric charges and other quantum numbers of hadrons, Gell-Mann and Zweig needed to assume three species of quarks, **up** (u), **down** (d), and **strange** (s). Additional hadrons discovered since that time require the existence of three more species: **charm** (c), **bottom** (b), and **top** (t). To make baryons with integer charges, the quark needed to be assigned fractional electric charge: $+2/3$ for u, c, t, and $-1/3$ for d, s, b. Then, for example, the proton would be a bound state of uud, while the neutron would be a bound state of udd. The six types of quarks are conventionally referred to as **flavors**.

Quark	up	down	charm	strange	top	bottom
Mass (GeV)	2.15	4.70	1270	93.5	163000	4180
E-M Charge	+2/3	-1/3	+2/3	-1/3	+2/3	-1/3

Despite the phenomenological success of the original quark model, it had two serious problems. First, despite considerable effort, free particles with fractional charge could not be found. Second, the spectrum of baryons required the assumption that the wave functions of the three quarks be totally symmetric under the interchange of the quark spin and flavour quantum numbers, contradicting the expectation that quarks, which must have spin 1/2, should obey Fermi statistics.

To reconcile the baryon spectrum with the spin-statistics theorem, was proposed that quarks carry an additional, unobserved quantum number, called **color**, and that baryon wave functions must be totally antisymmetric in color quantum numbers. Then, if the quark wave functions are totally symmetric in spin and flavour, they are totally antisymmetric overall, in agreement with Fermi statistics. The simplest model of color would be to assign quarks to the fundamental representation of a new, internal $SU(3)$ global symmetry.

Suppressing for a moment the spin and flavor quantum numbers, we can represent quarks by q_i, where $i = 1, 2, 3$ is the color index. Thus quarks transform under the fundamental, or "3", representation of the color $SU(3)$ symmetry. Antiquarks, $\bar{q}^i$, transforms in the $\bar{3}$ representation. In the inner product of a 3 and a $\bar{3}$ is an invariant of $SU(3)$. One can also form a invariant by using the totally antisymmetric combination of three 3's, ε_{ijk}: this object is invariant under $SU(3)$ transformations. Under the postulate that all hadrons wave functions must be invariant under $SU(3)$ symmetry transformations, these two types of combinations are the only simple ones allowed:

$$\bar{q}^i q_i \qquad \varepsilon^{ijk} q_i q_j q_k \qquad \varepsilon_{ijk} \bar{q}^i \bar{q}^j \bar{q}^k$$

That is, the assumption that physical hadrons are singlets under color implies that the only possible light hadrons are the **mesons** $\bar{q}^i q_i$, **baryons** $\varepsilon^{ijk} q_i q_j q_k$, **antibaryons** $\varepsilon_{ijk} \bar{q}^i \bar{q}^j \bar{q}^k$. All these color-neutral bound states are called **hadrons**.

Like the original quark model, the color hypothesis was phenomenologically successful but raised additional questions: Why should quarks have this seemingly superfluous property, and what mechanism insures that all hadron wave functions are color singlets? The answer to these questions came from deep-inelastic scattering experiments and the ensuing search for a theory of parton binding with the property of asymptotic freedom. When it was discovered that non-Abelian gauge theories have this property, all that remained was to identify the correct gauge group, with the colors being the gauge quantum numbers of the quarks. This reasoning resulted in a model of the strong interaction as a system of quarks, of the various flavours, each assigned to the fundamental representation of the local gauge group $SU(3)$. The quanta of the $SU(3)$ gauge field are called **gluons**, and th e theory is known as Quantum Chromodynamics of QCD.

Wilson showed that for sufficiently strong coupling, QCD exhibits *confinement of color*: the only finite-energy asymptotic states of the theory are those that are singlets of color $SU(3)$. Thus the assumption that explains the spectrum of hadrons turns out to be a consequence of the non-Abelian gauge theory coupling to color.

The short-distance limit of QCD can be readily studied using Feynman diagrams. Here asymptotic freedom makes the coupling weak, and there is a sensible diagrammatic perturbation theory that begins from the model of free quarks and gluons.

6.3.1 QCD vs QED running constant: confinement and asymptotic freedom

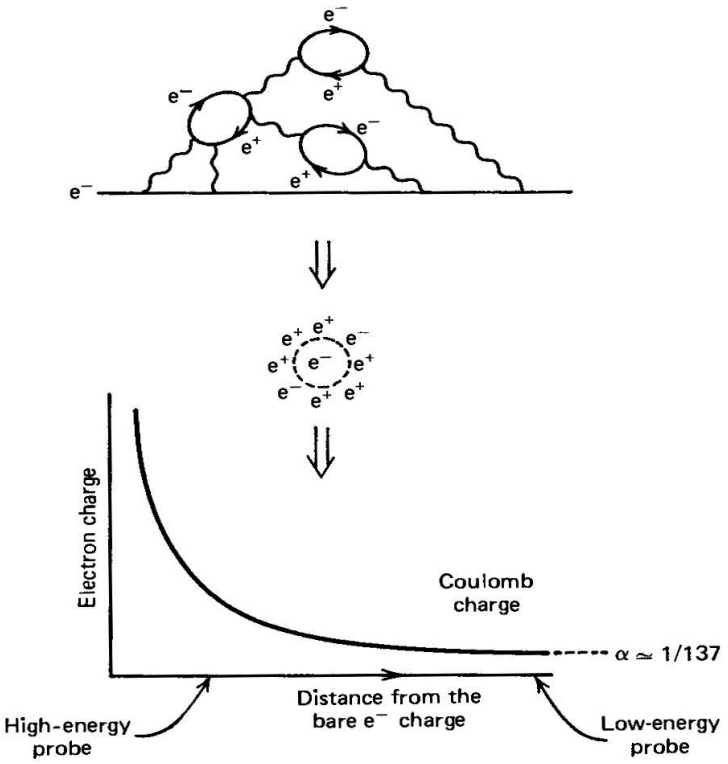
Halzen sec. 1.3, 7.8, 7.9

The existence of direct coupling between gluons in QCD has dramatic implications that become evident if oue contrasts the effects of *charge screening* in both QED and QCD.

Charge screening in QED

In QED, the electron is not just an electron, but it can suddenly emit a photon, or it can emit a photon that subsequently annihilates into an electron-posititron pair, and so on. Hence, any electron is surrounded

by $e^- e^+$ pairs and, because opposite charges attract, the positrons will be preferentially closer to the electron. Therefore, the electron is surrounded by a cloud of charges which polarized in such a way that the positive charges are closer to the electron; the negative charge of the electron is thus screened, as shown in the next figure:



Suppose that we want to determine the charge of the electron by measuring the Coulomb force experienced by a test charge. The result will depend on where we place the test charge: when moving the test charge closer to the electron, we penetrate the cloud of positrons that screens the electron's charge. Therefore, the closer one approaches the electron, the larger is the charge one measures. In QFT the vacuum surrounding the electron has become a polarizable medium. The situation is analogous to that of a negative charge in a dielectric medium: the electron-positron pairs respond to the presence of the electron like the polarized molecules do in the dielectric. This effect is known as charge screening; as a result the "measured charge" depends on the distance one is probing the electron.

In the QED we discussed electron-proton scattering and we derived that this can be described using a Coulomb potential in the form

$$V(r) = \frac{\alpha}{r}$$

where α_0 is the usual E-M coupling constant. We seen that when we renormalize QED (compare for instance with eq. (4.4)) the charge is modified by vacuum polarization loop in the photon propagator. Indeed, more in general, for higher order we can describe the interaction in the renormalized theory through a modified coulomb potential, where α is substituted by

$$\alpha_{EM}(s) = \frac{\alpha}{1 - \frac{\alpha}{3\pi}\log(s/\mu_0^2)}$$

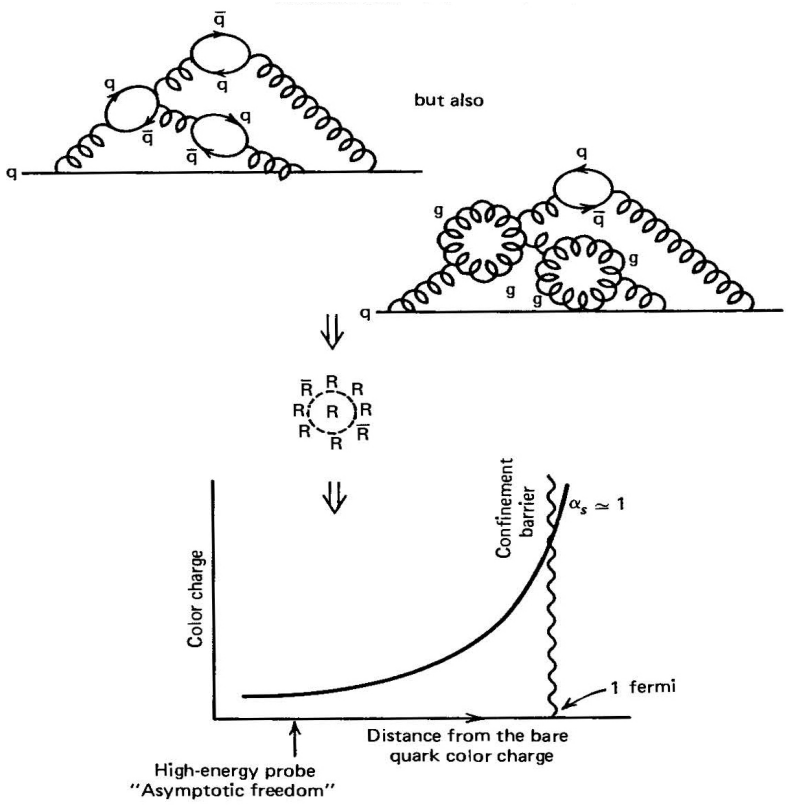
where μ_0 is some term given by renormalization called renormalization or reference momentum, and s is the Mandelstam variable associated to the momentum of the propagator.

The term $\alpha_{EM}(s)$ is called **running coupling constant** since it depends on the energy of the photon, and describes how the effective charge depends on the separation of the two charged. Then, by summing part of all orders of perturbation theory, we have obtained the charge screening of electrodynamics. As s increases, the photon sees more and more charge until, at some astronomically large but finite s, the

coupling $\alpha_{EM}(s)$ is infinite. such point is called **Landau pole** and corresponds usually to $\sqrt{s} = 10^{286}$eV. However, by inserting numerical values, we find that for all practically attaindable s, the variation of α_{EM} with s is extremely small; α_{EM} increases from 1/137 very slowly as s increases. Of course, as s increases, other loops (formed, for example, by a $\mu^+\mu^-$-pair or a $\bar{u}u$-quark pair) will also contribute to the variation. The energy

Confinement and asymptotic freedom in QCD

One can carry through the same calculation for the color charge of a quark. Color screening would be a carbon copy of charge screening if it were not for the new configurations involving gluons turning into pairs of gluons, as shown in next figure:



The gluons, themselves carriers of color, also spread out the effective color charge of the quark. It turns out that the additional diagrams reverse the familiar result of quantum electrodynamics: a red charge is preferentially surrounded by other red charges. We now repeat the measure made with the probe for color changes. By moving out test probe closer to the original red quark, the probe penetrates a sphere of predominantly red charge and the amount of red charge measured decreases. The resulting "antiscreening" effect is referred to as **asymptotic freedom**. Asymptotically, two red quarks interact (i.e. for very small separations) through color fields of reduced strength and approach a state where they behave as essentially free, noninteracting particles.

Doing explicit calculation in QCD, after we renormalized the theory, the running coupling term for strong interactions shows following dependence on s (the label s in α_s indicate that this is the coupling for the $s$trong interaction):

$$\alpha_s(s) = \frac{\alpha_s}{1 + \frac{\alpha_s}{12\pi}(33 - 2n_f)\log(s/\mu_0^2)}$$

where n_f is the number of quark flavours at energy s (in our world $n_f \leq 6$). Only in a world with more that 16 quarks flavours is the sign of the coefficient the same as in QED. Then α_s decreases with increasing s and therefore becomes small for short-distance interactions. We say that the theory is "**asymptotically free**". One parameter, μ_0, with the dimension of mass, remains as a relic of the renormalization. We see that at sufficiently low s, the effective coupling will become large. This implies that in such case

the perturbative approach doesn't work. It is costumary to denote the s scale at which this happens by Λ^2_{QCD}. Typical scale of this value is $\Lambda_{QCD} \simeq 200MeV$ and corresponds to a distance of $\lambda \simeq 1$fm.

For s values much larger than Λ^2_{QCD}, the effective coupling is small and perturbative description in terms of quarks and gluons interacting weakly makes sense. For s of order Λ^2_{QCD}, we cannot make such a picture, since quarks and gluons will arrange themselves into strongly bound cluster, namely, hadrons. Thus, we can think of Λ_{QCD} as marking the boundary between a world of quasi-free quarks and gluons, and the world of pions, protons, and so on.

6.3.2 $e^+e^- \to$ hadrons production

Peskin 5.1; Halzen, sec. 11.1

The simples reaction involving quarks is the production of quarks pairs in e^+e^- annihilation, which at most elementary level is a straightforward generalization of the process $e^+e^- \to f^+f^-$. Indeed, according to QCD, the simplest e^+e^- process that ends in hadrons is

$$e^+e^- \to q\bar{q}$$

the annihilation of an electron and a positron, through a virtual photon, into a quark-antiquark pair. After they are created, the quarks interact with one another through their strong forces, producing more quark pairs. Eventually the quark and antiquarks combine to form some number of mesons and baryons.

To adapt the result we obtained for $e^+e^- \to f^+f^-$ scattering in sec. 3.4 to handle the case of quarks, we must make three modifications:

(i) replace the f charge with the quark charge;

(ii) count each quark three times, one for each color;

(iii) include the effects of the strong interactions of the produced quark and antiquark

Counting colors is necessary because experiments measure only the total cross section for production of all three colors (since hadrons are detected colorless). Surely the third modification is extremely difficult to make, but in high-energy limit, the effect of the strong interaction on the quark production process can be completely neglected, thanks to asymptotic freedom of the theory. The only effect of the strong interaction in this limit is to dress up the final-state quarks into bunches of hadrons.

Starting from the total scattering amplitude in the UR regime for the $e^+e^- \to f^+f^-$ given by eq. (3.4) (we consider only first order terms):

$$\sigma_{e^+e^- \to f^+f^-} = \frac{4\pi\alpha^2_{EM}}{3s}$$

it is conventional to define, in the asymptotic limit

$$R = \frac{\sigma_{e^+e^- \to \text{hadrons}}}{\sigma_{e^+e^- \to f^+f^-}} = N_c \sum_q (q_q^2)$$

where N_c is the number of colors for quarks (in our theory $N_c = 3$) and the sum runs over all quarks whose masses are smaller that $E_{cm}/2$ and q_q is the fractional charge of these quarks (respect to the electron one). For instance

$$\begin{aligned} R &= 3\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right) = 2 && \text{for } u,\, d,\, s \\ &= 2 + 3\left(\frac{2}{3}\right)^2 = \frac{10}{3} && \text{for } u,\, d,\, s,\, c \\ &= \frac{10}{3} + 3\left(\frac{1}{3}\right)^2 = \frac{11}{3} && \text{for } u,\, d,\, s,\, c,\, b \end{aligned}$$

In fig. 6.1 these prediction are compared to the measurements of R. The value $R \simeq 2$ is apparent below the threshold for producing charmed particles at $\sqrt{s} = 2(m_c + m_u) \simeq 3.7$GeV. Above the threshold for

all five quark flavors ($\sqrt{s} > m_b \simeq 10\text{GeV}$),$R \simeq \frac{11}{3}$ as predicted. These measurements confirm that there are three color of quarks, since $R = \frac{11}{3}$ would be reduced by a factor 3 if there was only one color.

When $E_{cm}/2$ is in the vicinity of one of quark masses, the strong interaction cause large deviation from this formula. The most dramatic such effect is the appearance of bound states just below $E_{cm} = 2m_q$, manifested as very sharp spikes in the cross section.

These results for R will be modified when we take into account correction due to the running coupling. In particular, the result we obtained holds only at leading order.

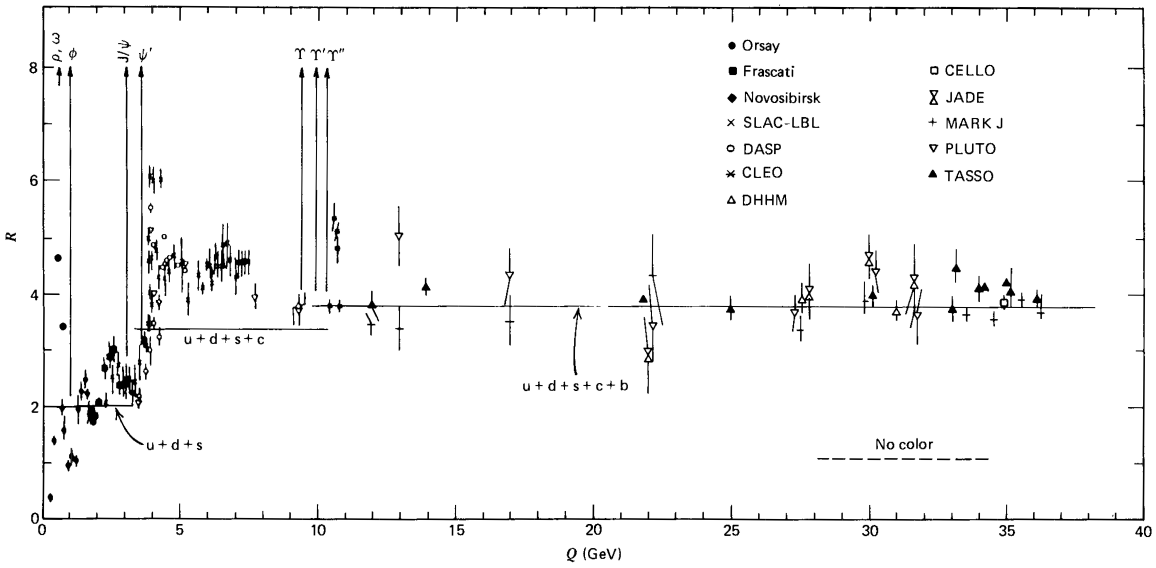


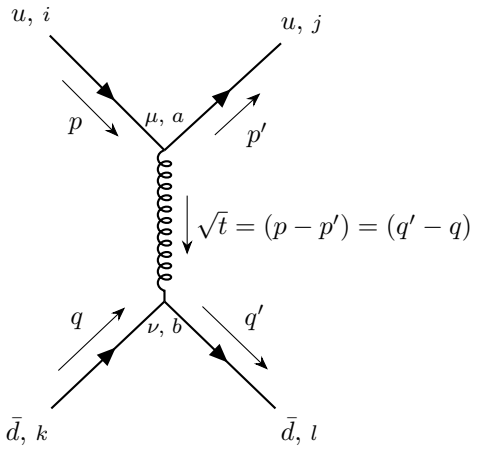
Figure 6.1: Ratio R as a function of the total e^-e^+ center-of-mass energy. (The sharp peaks correspond to the production of narrow resonance just below or near the flavor thresholds)

6.3.3 Quark-quark scattering hard process and QCD potential

Schwartz, sec. 26.2; Peskin, sec. 17.4

To get a feeling for how QCD differs from QED, we examine the tree-level potential between quarks. In QED, we saw in sec.3.6 that the potential could be extracted from Coulomb scattering $e^-p^+ \to e^-p^+$, which has a contributions from t-channel photons. Let us now consider the process $u\bar{d} \to u\bar{d}$ where u and d are up and down quarks. These are Dirac spinors transforming in the fundamental representation of QCD. The sign and the strength of the potential extracted from the t channel exchange will tell us whether the transition is attractive or repulsive, and thus whether bound states can exist.

The tree-level diagram for elastic $u\bar{d} \to u\bar{d}$ scattering in QCD is



where labels a, b and i, j, k, l indicate colors respectively of quarks and gluons

Notice that a similar diagram, with a photon instead of the gluon, is given by QED, but comparing coupling constant for typical values of energies we see that QED contributions can be neglected

$$\alpha_s(s) \simeq 0.12 \quad \gg \quad \alpha_{EM}(s) \simeq 0.007$$

The Ferynman amplitude for this QCD process is given by

$$\mathcal{M} = (ig_s)^2 \left(\bar{u}_j(p')\gamma^\mu u_i(p)\right)\left(v_k(q)\gamma^\nu v_l(q')\right) \sum_{ab} \left(T^a\right)_{ji} \left(T^b\right)_{kl} D^{ab}_{\mu\nu}(t)$$

And using eq. (6.20) with $\xi = 1$ we obtain

$$\mathcal{M} = \frac{ig_s^2}{t} \sum_a \left(T^a\right)_{ji} \left(T^a\right)_{kl} \left(\bar{u}_j(p')\gamma^\mu u_i(p)\right)\left(v_k(q)\gamma^\nu v_l(q')\right)$$

Notice that $\frac{1}{t}$ behaviour of the amplitude is the same as for process $e^+e^- \to e^+e^-$ in QED, therefore also the interaction given by gluon is Coulomb like.

To understand the $\left(T^a\right)_{ji}\left(T^b\right)_{kl}$ coefficient, let us consider different choices for the color of the incoming u and $\bar{d}$ quarks. For example, suppose the incoming u is red and the incoming $\bar{d}$ is anti-green ($i = 1$, $k = 2$). Then, by explicit computation using Gell-Mann matrices we obtain

$$\sum_a \left(T^a\right)_{j1} \left(T^a\right)_{2l} = \begin{pmatrix} 0 & 1/6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{jl} = -\frac{1}{6}\delta_{j1}\delta_{2l}$$

so that $j = 1$ and $l = 2$. That is, the final state must also have a red quark and a green anti-quark. Of course, just because color is conserved. Since $\sum_a \left(T^a\right)_{11} \left(T^a\right)_{22} = -\frac{1}{6}$, the t-channel diagram has the opposite sign from the e^-p^+ case and the potential is therefore repulsive (as it would be for say e^+p^+ scattering). Indeed it is impossible to have stable colored states since in this case potential is repulsive.

On the other hand, suppose both u is red and $\bar{d}$ is anti-red. Then the initial state is a color singlet (color and anti-color cancel each other). In this case $i = 1$ and $k = 1$ and

$$\sum_a \left(T^a\right)_{j1} \left(T^a\right)_{2l} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}_{jl} \tag{6.24}$$

so the final state can be red/anti-red, blue/anti-blue or green/anti-green. In any of these cases, the color factor will have a positive coefficient and the potential will be attractive.

To get the overall strength of the potential, let's consider the most general color singlet state:

$$|1\rangle = \frac{1}{\sqrt{3}}\left(\,|r\bar{r}\rangle + |b\bar{b}\rangle + |g\bar{g}\rangle\,\right)$$

Summing over all the possibilities for $|r\bar{r}\rangle \to$ anything gives a factor $4/3$ from the trace of eq. (6.24). We then get three times this (for the three collors) multiplied by the normalization $\left(\frac{1}{\sqrt{3}}\right)^2$. Therefore the potentia is

$$V_s(r) = -\frac{N_c}{(\sqrt{3})^2} \underbrace{\mathrm{Tr}\left[\sum_a T^a T^a\right]}_{4/3} \frac{\alpha_s}{r} = -\frac{4}{3}\frac{\alpha_s}{r}$$

Then only the color singlet channel is attractive is consistent with the observational fact we do not find colored mesons in nature.

Let's compute the total scattering amplitude in the high energy limit. The unpolarized Feyman amplitude must average also over all possible color of initial states:

$$\begin{aligned} |\overline{\mathcal{M}}|^2 &= \left(\frac{1}{3}\right)^2 \sum_{\text{color}} \left(\frac{1}{2}\right)^2 \sum_{\text{spin}} \left(\frac{g_s^2}{t}\right)^2 |(u\text{-current})(\bar{d}\text{-current})|^2 |\sum_a \left(T^a\right)_{ji} \left(T^a\right)_{kl}|^2 \\ &= \frac{1}{4}\left(\frac{g_s^2}{t}\right)^2 \mathrm{Tr}[u\text{-current}]\,\mathrm{Tr}\left[\bar{d}\text{-current}\right] \frac{1}{9} \sum_{ab}\sum_{ij} \left(T^a\right)_{ji} \left(T^b\right)_{ij} \sum_{kl} \left(T^a\right)_{kl} \left(T^b\right)_{lk} \end{aligned}$$

where we used $(T^a)^*_{ij} = (T^a)_{ji}$. We can simplify calculation using the result we already know from QED process $e^-\mu^+ \to e^-\mu^+$

$$\begin{aligned}
|\overline{\mathcal{M}}|^2 &= \frac{1}{9}\sum_{ab} \mathrm{Tr}\left[T^a T^b\right] \mathrm{Tr}\left[T^a T^b\right] \left(\frac{\alpha_s}{\alpha_{EM}}\right)^2 |\overline{\mathcal{M}}|^2_{e^-\mu^+} \\
&= \frac{1}{9}\sum_{ab} \left(\frac{1}{2}\delta_{ab}\right)\left(\frac{1}{2}\delta_{ab}\right) \left(\frac{\alpha_s}{\alpha_{EM}}\right)^2 |\overline{\mathcal{M}}|^2_{e^-\mu^+} \\
&= \frac{1}{9}\frac{8}{4}\left(\frac{\alpha_s}{\alpha_{EM}}\right)^2 |\overline{\mathcal{M}}|^2_{e^-\mu^+} = \frac{2}{9}\left(\frac{\alpha_s}{\alpha_{EM}}\right)^2 |\overline{\mathcal{M}}|^2_{e^-\mu^+}
\end{aligned}$$

Finally (recall that we are working in the high energy limit, i.e. $m_f = 0$):

$$|\overline{\mathcal{M}}|^2_{u\bar{d}} = \frac{64\pi^2}{9}\alpha_s^2\left(\frac{s^2+u^2}{t^2}\right)$$

and

$$\left(\frac{\mathrm{d}\overline{\sigma}}{\mathrm{d}\Omega}\right)_{u\bar{d}} = \frac{\alpha_s^2}{9t}\left(\frac{s^2+u^2}{t^2}\right)$$

Chapter 7

Spontaneous Symmetry Breaking (SSB)

Spontaneous symmetry breaking is one of the most important concepts in quantum field theory. The distinction between spontaneous and explicit symmetry breaking is that with spontaneous symmetry breaking the Lagrangian is invariant under the symmetry, but the ground state of the theory is not. With explicit symmetry breaking, there was never an exact symmetry to begin with. One usually associates spontaneous symmetry breaking with phase transitions. The amazing thing about spontaneous symmetry breaking is that one can say a tremendous amount about the broken phase with an effective theory whose input is the symmetry that was broken - no detailed microscopic description is needed.

A simple example is given by a ferromagnet. The action governing its microscopic dynamic is invarian under spatial rotations. Above a critical temperature a ferromagnet has a unique ground state, with zero magnetization. Of course this state respects the rotational invariance, since on it the expectation value of the magentization $\boldsymbol{M}$ vanishes, and therefore no preferred direction is selected. Below a critical temperature instead it becomes thermodynamically favourable to develop a non-zero magnetization, and in this new vacuum $\boldsymbol{M} \neq 0$ and the full $SO(3)$ rotational symmetry is broken to the subgroup $SO(2)$ of rotations around the magnetization axis.

The original invariance of the Lagrangian is now reflected in the fact that, instead of a single vacuum state, there is a whole family of vacua related to each other by rotations, since the magnetization can in principle develop in any direction. However, the system will choose one of these states as its vacuum state. The symmetry is then said to be *spontaneously broken* by the choice of a vacuum.

For $T < T_C$ it is helpful to write $\boldsymbol{M}(x) = \boldsymbol{\mu} + \boldsymbol{\sigma}(s)$, where $\boldsymbol{\mu}$ is the expectation value of $\boldsymbol{M}$ in the vacuum ($T = 0$) and $\boldsymbol{\sigma}$ are the excitations around this minimum. The field $\boldsymbol{\sigma}(x)$ encodes spin waves whose quanta are called Goldstone bosons.

For continuous global symmetry, such as $\phi(x) \to e^{i\alpha}\phi(x)$, the breaking of the symmetry automatically implies the existence of long-range correlations and associated massless particles. This is Goldstone's theorem, and the massless particles are called Goldstone bosons. If the symmetry is gauged, as for $\phi(x) \to \phi^{i\alpha(x)}\phi(x)$ with an associated massless gauge field $A_\mu(x)$, then in the broken phase the gauge boson will acquire a mass. This is known as the Higgs mechanism, and allows to describes Weak interactions (which are mediated throw massive bosons $W^\pm$ and Z) as a spontaneously broken (renormalizable) gauge theory (an explicit presence of a mass term in the Lagrangian broke the gauge symmetry, since any term in the form $M^2 A_\mu A^\mu$ is not gauge invariant because of the transformation rule for gauge fields).

7.1 Spontaneous Symmetry Breaking of global symmetries

Maggiore sec. 11.1-11.2; Peskin sec. 11.1; Schwartz sec. 28.1-28.2; Mandl sec. 18.1; Halzen sec. 14.6-14.7

7.1.1 The renormalizable complex scalar potential and the sign of the quadratic term μ^2

The simplest example of a field theory exhibiting spontaneous symmetry breaking is the *Goldstone model*, i.e. a complex scalar theory with $U(1)$ symmetry. The Lagrangian of this model is

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - V(\phi^\dagger \phi) \qquad \text{with} \qquad V(\phi^\dagger \phi) = \lambda \left(\phi^\dagger \phi + \frac{\mu^2}{2} \right)^2$$

This theory has a global $U(1)$ symmetry $\phi(x) \to e^{i\alpha}\phi(x)$ for constant $\alpha \in \mathbb{R}$. The Hamiltonian for this theory is

$$H = \int \mathrm{d}^3x \left\{ |\pi|^2 + |\boldsymbol{\nabla}\phi|^2 + V(\phi^\dagger \phi) \right\}$$

hence we require $\lambda > 0$ in such a way that the energy is bounded from below. We can see that no conditions is required for the parameter μ, since in any case the potential is well defined. The Lagrangian can also be written in terms of two real fields ϕ_1 and ϕ_2 such that $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - V(\phi_1^2 + \phi_2^2) \qquad \text{with} \qquad V(\phi_1^2 + \phi_2^2) = \frac{\lambda}{4}\left(\phi_1^2 + \phi_2^2 + \mu^2 \right)^2$$

In this representation the symmetry of the Lagrangian is $SO(2) \simeq U(1)$.

Let us discuss the proprieties of the ground state (vacuum) of the system. We define the vacuum expectation values as

$$\langle \phi \rangle_0 = \langle 0 | \, \phi \, | 0 \rangle = \phi_0$$

where ϕ is a quantum field (i.e. an operator) considered in the vacuum configuration while ϕ_0 is a classical field. The ground state must satisfy following requirements:

(1) ϕ_0 is a constant field configuration: $\partial_\mu \phi_0(x,t) = 0$ (we are asking that the kinetic term in the Lagrangian is zero for ϕ_0). Hence the vacuum has no energy and no 3-momentum: $p^\mu = 0$.

(2) ϕ_0 must be a minimum for the potential V. Then ϕ_0 is a stationary point for $V(\phi^\dagger \phi)$:

$$\begin{cases} \left. \frac{\partial V}{\partial \phi} \right|_0 = 2\phi_0 \left(\phi_0^\dagger \phi_0 + \frac{\mu^2}{2} \right) = 0 \\ \left. \frac{\partial V}{\partial \phi^*} \right|_0 = 2\phi_0 \left(\phi_0^\dagger \phi_0 + \frac{\mu^2}{2} \right) = 0 \end{cases}$$

Moreover we want the Hessian of V (respect to ϕ and $\phi^\dagger$) to be positive definite in ϕ_0. (Notice that for non-interacting theories (2) is already satisfied when (1) is satisfied).

Two different situations occur, depending on the sign of μ^2:

(a) $\mu^2 > 0$. In this case two terms in V are positive definite. In this case the only minima of the potential is given by $\phi_0 = \phi_0^\dagger = 0$ and the vacuum state is non-degenerate. The term $\lambda|\phi|^4$ will be treat by perturbation theory, and in the quantized theory represents a self-interaction of the particles. The field ϕ represent a charged field for a particle with mass $m_\phi^2 = \lambda\mu^2$.

(b) $\mu^2 < 0$. In this case for $\phi(x) = 0$ the Hessian is negative definite, and the minima is given by a whole circle

$$\phi(x) = \phi_0 = \left(\frac{-\mu^2}{2\lambda} \right)^{1/2} e^{i\theta} \qquad 0 \le \theta < 2\pi$$

where the phase angle θ defines a direction in the complex ϕ-plane. We see that the state of lowest energy, the vacuum state, is not unique in this case. This arbitrariness in the direction θ is analogous to that in the direction of the magnetization $\boldsymbol{M}$ of a ferromagnet. Spontaneous symmetry breaking will occur if we choose one particular direction θ to represent the vacuum ground state. Because of the invariance of the Lagrangian density under the global phase transformations the value of θ chosen is not significant and we shall take $\theta = 0$, so that

$$\phi_0 = \left(\frac{-\mu^2}{2\lambda} \right)^{1/2} = \frac{v}{\sqrt{2}} \quad (> 0)$$

is purely real.

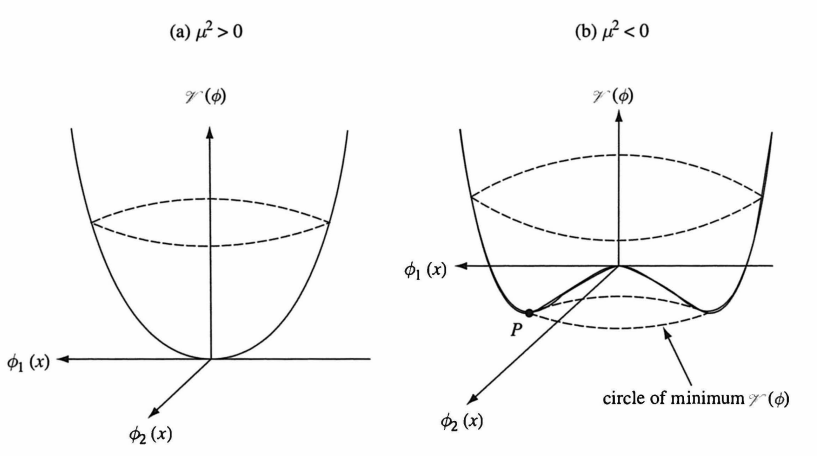


Figure 7.1: Representation of $V(\phi_1^2 + \phi_2^2)$ for different signs of μ^2

7.1.2 The minimum of the theory and the choice of the vacuum configuration: the $U(1)$ example

Suppose

$$\phi_0 = \frac{v}{\sqrt{2}} \qquad v > 0$$

Let's redefine our complex field introducing two real fields $\sigma(x)$ and $\pi(x)$:

$$\phi(x) = \frac{\sigma(x)}{\sqrt{2}} e^{\frac{i\pi(x)}{v}}$$

then the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{\sigma^2}{2v^2}(\partial_\mu \pi)^2 - V(\sigma, \pi)$$

where

$$V(\sigma, \pi) = \frac{\lambda}{4}(\sigma^2 - v^2)^2$$

The first order derivatives of this Lagrangian is given by

$$\begin{cases} \left.\frac{\partial V}{\partial \sigma}\right|_0 = \frac{\lambda}{2}\sigma_0(\sigma_0^2 - v^2) \\ \left.\frac{\partial V}{\partial \pi}\right|_0 = 0 \end{cases}$$

hence the vacuum state is given by $\sigma_0 = v$ ($\sigma_0 = 0$ corresponds to negative definite Hessian), no condition over $\pi(x)$ is required. In order to write explicitly fluctuations about the vacuum state, let's define the field $\tilde{\sigma}(x)$ as follows

$$\sigma(x) = \tilde{\sigma}(x) + v$$

in such a way the expected value of $\tilde{\sigma}(x)$ (i.e. the fluctuation) in the vacuum is $\langle \tilde{\sigma} \rangle_0 = 0$ The Lagrangian can be written in terms of fields $\tilde{\sigma}$ and π:

$$\mathcal{L}_{\text{SSB}} = \frac{1}{2}(\partial_\mu \tilde{\sigma})^2 - \frac{(2\lambda v^2)}{2}\tilde{\sigma}^2 - \lambda v \tilde{\sigma}^3 - \frac{\lambda}{4}\tilde{\sigma}^4 + \frac{1}{2}(\partial_\mu \pi)^2 + \left(\frac{2v\tilde{\sigma} + \tilde{\sigma}^2}{2}\right)(\partial_\mu \pi)^2 \tag{7.1}$$

This Lagrangian shows kinetic terms for both $\tilde{\sigma}$ and π, a massive term only for $\tilde{\sigma}$:

$$\begin{cases} m_{\tilde{\sigma}}^2 = 2\lambda v^2 \\ m_\pi^2 = 0 \end{cases}$$

Lot of new interactions appears, depending on $\tilde{\sigma}$ and $\partial_\mu \pi$. Notice that the the fact that π has only derivative interactions is a typical consequence of SSB. Anyhow, this is not the most general Lagrangian depending on $\tilde{\sigma}$ and π, indeed it depends only on 2 parameters λ and v^2, and self-interacting couplings

$$d_3 := \lambda v \qquad \text{and} \qquad d_4 := \frac{\lambda}{4}$$

have a relation with the mass of the real scalar field $\tilde{\sigma}$.

Notice that eq. (7.1) shows a hidden/residual symmetry called "shift symmetry" given by

$$\pi(x) \quad \longrightarrow \quad \pi(x) + \alpha \qquad \text{for} \quad \alpha \in \mathbb{R}$$

We can heuristically interpret this symmetry as a redefinition of the field ϕ by a phase $e^{i\alpha}$. This is a $U(1)$ symmetry hidden in the SSB, inherited by π from the original $U(1)$ symmetry of ϕ.

7.1.3 The Nambu-Goldstone bosons and the Goldstone theorem

The example in the previous section is a particular case of a general theorem, the Goldstone theorem, which states that, given a field theory which is Lorentz invariant, local, and has a Hilbert space witha. a positive definite scalar product, if a continuous global simmetry is spontaneously broken, then in the expansion around the symmetry-breaking vacuum there appears a massless particle for each generator that breaks the symmetry. This particle is called a Goldstone (or Nambu-Goldstone) particle. In the previous example the particle was given by the field $\pi(x)$.

The emergence of massless particles corresponds to the possibility of moving, in field space, in the direction of the manifold of vacua. The dimensionality of the manifold of vacua is equal to the number of generators which break the symmetry. In fact, setting the vacuum energy to zero, by definition we have $H|0\rangle = 0$. Since T^a is the generator of a symmetry transformation, it satisfies $[T^a, H] = 0$ and therefore

$$H(T^a|0\rangle) = T^a H|0\rangle = 0$$

So, if $T^a|0\rangle \neq 0$ (and if it is not proportional to $|0\rangle$ itself) we have found a new state with the minimum energy, i.e. another vacuum state. This is the origin of the fact that we have a Goldstone particle for each generator which breaks the symmetry.

Goldstone theorem

Let's consider a global group of symmetries (compact) G, with dimension d_G. Let's define a field ϕ in the (real) n-th dimensional representation of G: $\phi = (\phi_1, \dots, \phi_n)^T$. Consider a real Lagrangian for ϕ:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^T)(\partial^\mu \phi) - V(\phi^T \phi)$$

with a non-trivial ground state $\langle \phi \rangle_0 = \boldsymbol{v} \neq 0$, in particular

$$\left.\frac{\partial V}{\partial \phi_i}\right|_{\langle \phi_i \rangle = v_i} = 0$$

The choice of a vacuum state spontaneously broke down the symmetry group G to a subgroup H, that describes the *residual global symmetry*, with $\dim H = d_H$. The vacuum is then invariant under H, but not under the remaining elements of G, which are denoted as a *coset* and written as G/H. The coset is not a subgroup of G (for example, it does not contain the identity element, moreover in general do not have closure property).

In particular we can consider N generators of G, $\{T^a\} = \{X^{\hat{a}}, Y^{\bar{a}}\}$, where $a = 1, \dots, d_G$, $\bar{a} = 1, \dots, d_H$ and $\hat{a} = 1, \dots, d_G - d_H$. The set $\{Y^{\bar{a}}\}$ is given by the *unbroken generators*

$$Y^{\bar{a}} \boldsymbol{v} = 0$$

and generates the group H. Meanwhile, $\{X^{\hat{a}}\}$ is given by *broken generators*

$$X^{\hat{a}} \boldsymbol{v} \neq 0$$

and generates the coset G/H.

For any broken generator corresponds a massless particle in the lagrangian

$$\pi_{\hat{a}}(x) \qquad \hat{a} = 1, \dots, d_G - d_H$$

and all these massless particles are called *Goldstone bosons.*

Proof:

Consider an infinitesimal transformation of the original symmetry of the Lagrangian

$$\phi' = \phi + i\hat{\alpha}\phi \qquad \longleftrightarrow \qquad \phi'_i = \phi_i + i\alpha_a T^a_{ij}\phi_j$$

Under such a transformation the variation of the potential is zero because of the symmetry of the Lagrangian:

$$\delta V(\phi) = \frac{\partial V}{\partial \phi_i}\delta\phi_i = i\alpha_a \frac{\partial V}{\partial \phi_i} T^a_{ik}\phi_k = 0$$

Let's derive with respect to ϕ_j:

$$i\alpha_a \left\{ \frac{\partial^2 V}{\partial\phi_i \partial\phi_j} T^a_{ik}\phi_k + \frac{\partial V}{\partial\phi_j} T^a_{ij} \right\} = 0$$

And for $\phi = \phi_0$, so that $\langle\phi\rangle_0 = \boldsymbol{v}$, the second term in the l.h.s. of the previous equation vanishes since ϕ_0 is a minimum of V and since the equality must holds for any symmetry transformation (i.e. for each $\hat{a}$) we obtain

$$\left(\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right)_{\langle\phi_i\rangle_0 = v_i} T^a_{ik} v_k = 0 \qquad \forall\, a = 1, \dots, d_G$$

I define the mass term M^2_{ij} as the coefficient of the quadratic term in the expansion of $V(\phi)$ about the vacuum state:

$$M^2_{ij} := \left(\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right)_{\langle\phi_i\rangle_0}$$

and

$$w^a_i := T^a_{ij} v_j$$

Then I obtain the following condition:

$$M^2 w^a = 0$$

hence the matrix M has some zero eigenvalues for some eigenvectors w^a. If we distinguish between eigenvectors of the zero eigenvalue and other eigenvectors we obtain the distinction between broken and unbroken generators described in the statement of the theorem:

$$W^{\bar{a}} = Y^{\bar{a}} v = 0 \qquad \text{and} \qquad W^{\hat{a}} = X^{\hat{a}} v \neq 0$$

with $\bar{a} = 1, \dots, d_H$ and $\hat{a} = 1, \dots, d_G - d_H$. Hence M^2 has $d_G - d_H$ zero eigenvectors.

The Goldstone bosons are given by

$$\pi^{\hat{a}} = i(X^{\hat{a}}\boldsymbol{v})\phi = i(X^{\hat{a}}\boldsymbol{v})_i\phi_i$$

Example 11: $SO(3) \to SO(2)$ **SSB**

Take three scalar fields $\phi = (\phi_1, \phi_2, \phi_3)$ in the fundamental representation of $SO(3)$, with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu)^2 - V(\phi^\dagger\phi)$$

where

$$V(\phi^\dagger\phi) = \frac{\lambda}{4}(\phi^\dagger\phi - v^2)^2 \qquad v^2 > 0$$

The minima condition is given by $\phi^\dagger\phi = v^2$. One possible choice of the vacuum is $\boldsymbol{v} = (0, 0, v)$, i.e.

$$\langle\phi_1\rangle_0 = \langle\phi_2\rangle_0 = 0 \qquad \langle\phi_3\rangle_v$$

The generators of $SO(3)$ are

$$T_1 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad T_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

in particular T_1 is the unbroken generator, while T_2 and T_3 are the broken generators:

$$T_1\boldsymbol{v} = 0 \qquad T_2\boldsymbol{v} \neq 0 \qquad T_3\boldsymbol{v} \neq 0$$

Physical fields are the ones that in the vacuum state has zero expectation value, and by defining

$$\phi_3 = \tilde{\phi}_3 + v$$

we obtain that physical fields are ϕ_1, ϕ_2 and $\tilde{\phi}_3$.

The Goldstone bosons are

$$\pi_1 = \frac{i}{v}(T_2\boldsymbol{v})\phi = \phi_1 \qquad \pi_2 = \frac{i}{v}(T_3\boldsymbol{v})\phi = \phi_2$$

which are massless fields, while expanding the Lagrangian in terms of physical fields we obtain that the mass of the particle described by $\tilde{\phi}_3$ is $m_3^2 = 2\lambda v$.

Another example of SSB is given by the spontaneous symmetry breaking of $SU(2) \times SU(2)$, whose Goldstone bosons are the pions.

7.2 Spontaneous Symmetry Breaking of a local symmetry

Schwartz sec. 28.3-28.4; Peskin sec. 20.1, 21.1; Mandl sec. 18.2; Halzen sec. 14.8-14.9

What happens if we include both local gauge invariance and spontaneous symmetry breaking in the same theory? In this section we will find that this combination of ingredients leads to new possibilities for the construction of quantum field theory models. We will see that spontaneous symmetry breaking requires gauge vector bosons to acquire mass. However, the interactions of these massive bosons are still constrained by the underlying gauge symmetry, and these constraints can have observable consequences.

In elementary particle physics, the principal application of spontaneously broken local symmetry is in the currently accepted model of weak interactions. We will see that it makes a number of precise and successful predictions for weak interactions phenomena. Remarkably, this model also unifies the weak interactions with electromagnetism in a single larger gauge theory.

7.2.1 $U(1)$ gauge theory minimally coupled to a complex scalar field with a non-trivial scalar potential

As our first example, consider a complex scalar field couple both to itself and to an electromagnetic field:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi^\dagger\phi)$$

with $D_\mu = \partial_\mu + igA_\mu$. This Lagrangian is invariant under the local $U(1)$ transformation

$$\phi(x) \rightarrow e^{ig\alpha(x)}\phi(x) \qquad\qquad A_\mu(x) \rightarrow A_\mu - \partial_\mu\alpha(x)$$

which implies

$$(D_\mu\phi) \rightarrow e^{ig\alpha(x)}(D_\mu\phi)$$

If we choose the potential in $\mathcal{L}$ to be of the form

$$V(\phi^\dagger\phi) = \lambda\left(\phi^\dagger\phi - \frac{v^2}{2}\right)^2$$

with $v^2 > 0$, the field ϕ will acquire a vacuum expectation value and the $U(1)$ global symmetry will be spontaneously broken.

7.2.2 The minimum of the theory and the choice of the vacuum configuration

Let us consider the requirements for the vacuum configurations. The Hamiltonian for this theory is

$$\mathcal{H} = \int \mathrm{d}^3x \left\{ \frac{1}{2}(F_{0i})^2 + \frac{1}{4}(F_{ij})^2 + (D_0\phi)^\dagger(D_0\phi) + (D_i\phi)^\dagger(D_i\phi) + V(\phi^\dagger\phi) \right\} > E_{\min}$$

hence the vacuum configuration must satisfy the followings:

(1) A^μ is a pure gauge configuration $A^0_\mu = \partial_\mu\alpha(x)$, this implies $F_{0i} = F_{ij} = 0$

(2) ϕ_0 is a constant field configuration: $(D_\mu\phi_0) = 0$, this is satisfied by taking $\phi_0 = \frac{c}{\sqrt{2}}e^{ig\alpha}$ for some constant c and $0 \leq \alpha < 2\pi$

(3) $V(\phi_0^\dagger\phi_0)$ is a minimum for V, hence $\phi_0^*\phi_0 = \frac{v^2}{2}$

If we choose $\alpha(x) = 0$ we obtain the vacuum state

$$\langle A_\mu\rangle_0 = A^0_\mu = 0 \qquad\qquad \langle\phi\rangle_0 = \phi_0 = \frac{v}{\sqrt{2}}$$

Let's define fluctuations about the vacuum configuration:

$$\phi(x) = \frac{v+\tilde{\sigma}}{\sqrt{2}}e^{i\pi(x)/v} \qquad \rightarrow \qquad \langle\sigma\rangle_0 = v \qquad \langle\tilde{\sigma}\rangle_0 = 0$$

In terms of $\tilde{\sigma}$ and π the Lagrangian density becomes

$$\begin{aligned}\mathcal{L}_{\mathrm{SSB}} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g^2}{2}(v+\tilde{\sigma})^2\left(A_\mu + \frac{1}{gv}\partial_\mu\pi\right)^2 \\ &+ \frac{1}{2}(\partial_\mu\tilde{\sigma})(\partial^\mu\tilde{\sigma}) - \frac{1}{2}m_{\tilde{\sigma}}^2\tilde{\sigma}^2 - \frac{\lambda}{4}(4v\tilde{\sigma}^3 + \tilde{\sigma}^4)\end{aligned} \tag{7.2}$$

where $m_{\tilde{\sigma}}^2 = 2\lambda v^2$ is the mass term for $\tilde{\sigma}$ field.

7.2.3 The Would Be Goldstone Boson (WBGB) and the longitudinal vector boson d.o.f.

The direct interpretation of eq. (7.2) leads to difficulties. It contains the equation for a real Klein-Gordon field with mass $\sqrt{2\lambda v^2}$. However, the mixed term $A^\mu\partial_\mu\pi$ shows that A^μ and π are not independent normal coordinates, and one cannot interpret A_μ and π as massive vector bosons and a massless scalar boson respectively. This difficulty also shoes up if we count degrees of freedom: two from the complex scalar field ϕ and two from the real massless vector field A^μ (i.e. for massless photons there are only two independent polarization states, the third is eliminated by gauge invariance). In $\mathcal{L}_{\mathrm{SSB}}$ the real scalar fields $\tilde{\sigma}$ and π each represent one degree and the real massive vector field A^μ contributes three degrees (corresponding to three independent polarization states), i.e. the SSB Lagrangian appears to have five degrees of freedom. Of course, a change of variables cannot alter the number of degrees of freedom of a system. We most conclude that the Lagrangian density contains an unphysical field which does not represent real particles and which can be eliminated.

The field π can be eliminated through an appropriate choice of a gauge, in particular we can define the following boson vector field:

$$B_\mu := A_\mu + \frac{1}{gv}(\partial_\mu\pi) \qquad\qquad B_{\mu\nu} := \partial_\mu B_\nu - \partial_\nu B_\mu$$

and introducing this field in the Lagrangian we obtain

$$\begin{aligned}\mathcal{L}_{\mathrm{SSB}} = &-\frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{1}{2}m_B^2B^\mu B_\mu + \frac{g^2}{2}(2v\tilde{\sigma} + \tilde{\sigma}^2)B^\mu B_\mu \\ &+ \frac{1}{2}(\partial_\mu\tilde{\sigma})(\partial^\mu\tilde{\sigma}) - \frac{1}{2}m_{\tilde{\sigma}}^2\tilde{\sigma}^2 - \frac{\lambda}{4}(4v\tilde{\sigma}^3 + \tilde{\sigma}^4)\end{aligned} \tag{7.3}$$

where we introduce the mass term for the B^μ field $m_B^2 := g^2v^2$. Hence the SSB of a gauge theory leads to the introduction to a massive vector boson, which "eats" the Goldstone boson π and the initial massless gauge field A^μ. The the theory of SSB for gauge theories the field π is called *Would Be Goldstone Boson (WBGB)*.

By giving a mass to A_μ introducing B_μ, we have clearly introduced a longitudinal polarization for the gauge field.

7.2.4 The Higgs mechanism

What we obtained is a remarkable result. Having started from a complex scalar field and a massless real vector field, we have ended up with the Lagrangian density for a real scalar field and a massive real vector field. The number of degrees of freedom is four in both cases. Of the two degrees of freedom of the comple field ϕ, one has be taken up by the vector field A_μ, which as become massive (B_μ) in the process; the other shows up as the real field $\tilde{\sigma}$.

This phenomenon by which a vector boson acquires mass without destroying the gauge invariance of the Lagrangian density is known as the *Higgs mechanism*, and the massive spin-0 boson associated with the field $\tilde{\sigma}$ is called a Higgs boson or a Higgs scalar. The Higgs mechanism does not generate Goldstone bosons, in contrast to the spontaneous symmetry breaking of the global phase invariance of the Goldstone model. In essence, the field π which in the Goldstone model was associated with the massless Goldstone boson, has been eliminated by gauge invariance, and the degree of freedom of π has been transferred to the vector field B_μ.

7.2.5 Feynman rules in the Unitary gauge and in the R_ξ gauge

We notice that eq. (7.2) still exhibits a local $U(1)$ symmetry:

$$\tilde{\sigma}(x) \to \tilde{\sigma}(x) \qquad \pi(x) \to \pi(x) + gv\alpha(x) \qquad A_\mu(x) \to A_\mu(x) - \partial_\mu\alpha(x)$$

even though in this case I have a mass term in the boson part of the Lagrangian. This is a consequence of the original gauge symmetry of the Lagrangian. Moreover, notice that $\tilde{\sigma}$ is invariant under this transformation, so it is a singlet under $U(1)$.

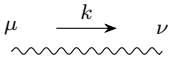
The field B_μ is a specific gauge choice for such a symmetry, called *unitary gauge*. In the unitary gauge the Lagrangian of a SSB theory shows explicitly the physical fields.

In some cases, especially if we want to quantize such a theory, we may use different gauge choices, introducing (non-unitary) gauge fixing terms in the Lagrangian, which corresponds to non-physical d.o.f. of the theory (e.g. the longitudinal polarization for the Maxwell field). In particular, the so called R_ξ *gauge choice* corresponds to the introduction of the following gauge fixing term in eq. (7.2):

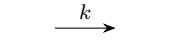
$$\mathcal{L}_{\text{GF}} := -\frac{1}{2\xi}(\partial_\mu A^\mu - \xi(gv)\pi)^2 = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \frac{1}{2}(\xi g^2 v^2)\pi^2 + gv(\partial_\mu A^\mu)\pi \tag{7.4}$$

which corresponds to the introduction of a mass term $m_\pi^2 = \xi^2 g^2 v^2 = \xi^2 m_B^2$ for the field π and a cross mixing term $(\partial_\mu A^\mu)\pi = -A^\mu(\partial_\mu\pi) + \partial_\mu(A^\mu\pi)$ which cancels the mixing term in eq. (7.2).

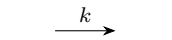
If we consider the full Lagrangian with R_ξ gauge fixing term $\mathcal{L} = (7.2) + (7.4)$ we can quantize the SSB theory and obtain following Feynman rules ($M_A^2 = g^2v^2$):

A_μ : 

$$D^A_{\mu\nu}(k) = -\frac{i}{k^2 - M_A^2}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi M_A^2}\right) \tag{7.5a}$$

$\tilde{\sigma}$: 

$$D^{\tilde{\sigma}}(k) = -\frac{i}{k^2 - m_{\tilde{\sigma}}^2} \tag{7.5b}$$

π : 

$$D^{\pi}(k) = -\frac{i}{k^2 - \xi M_A^2} \tag{7.5c}$$

Notice that for $\xi \to \infty$ I obtain the unitary gauge eq. (7.3), in this case the propagator for the gauge field becomes ($A_\mu = B_\mu$)

$$D^B_{\mu\nu}(k) = -\frac{i}{k^2 - M_B^2}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{M_B^2}\right)$$

so we obtain (as we expected) the propagator of a Proca field, i.e. a massive vector propagator. Moreover, for $\xi \to \infty$ we have $D^\pi = 0$, and the non-physical Goldstone boson disappear from Feynman diagrams.

As we had disclosed, the gauge R_ξ it is useful to describe the renormalization of SSB gauge theories. Indeed, for $k \to \infty$ the degree of divergence of $D^A_{\mu\nu}(k)$ is the same as the one of the photon propagator in QED:

$$D^A_{\mu\nu}(k) = -\frac{i}{k^2 - M_A^2}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi M_A^2}\right) \approx -\frac{i}{k^2}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$$

Since the renormalizability does not depend on the gauge choice, SSB gauge theories are renormalizable.

7.2.6 Generalization of the Higgs mechanism to generic (compact) group

In this section we discuss the SSB of a generic gauge theory. For definiteness we consider the case $G = SU(N)$, but the following derivation holds in general. Consider a complex scalar field ϕ in the N-dimensional fundamental representation of G. Suppose that the Lagrangian shows a potential with a non-trivial minimum condition in the form $V(\phi^\dagger\phi) = \lambda(\phi^\dagger\phi - \frac{\boldsymbol{v}^2}{2})^2$ and take as vacuum state $\langle\phi\rangle_0 = \phi_0 = (0, \dots, v_i, \dots, 0)$ for some $v_i \neq 0$.

The choice of a vacuum state will broke doen the symmetry group G to a subgroup H. As we seen in general, d_H generators $\{Y^{\bar{a}}\}$ of G are unbroken, while $d_G - d_H$ generators $\{X^{\hat{a}}\}$ will be broken by the choice of a vacuum state.

Consider the minimal coupling of the Lagrangian:

$$(D_\mu\phi)^\dagger(D^\mu\phi) = [(\partial_\mu + ig\hat{A}_\mu)\phi]^\dagger[(\partial^\mu + ig\hat{A}^\mu)\phi]$$

If we define the fluctuation $\tilde{\phi}$ as $\phi = \tilde{\phi} - \boldsymbol{v}$ we can express the previous component of the Lagrangian in terms of this field, obtaining

$$(D_\mu\tilde{\phi})^\dagger(D^\mu\tilde{\phi}) - ig\left\{(\partial^\mu\tilde{\phi})^\dagger\hat{A}_\mu(\boldsymbol{v}+\tilde{\phi}) - (\boldsymbol{v}+\tilde{\phi})^\dagger A^\mu(\partial_\mu\tilde{\phi})\right\} + g^2(\boldsymbol{v}+\tilde{\phi})^\dagger\hat{A}^\mu\hat{A}_\mu(\boldsymbol{v}+\tilde{\phi})$$

so I obtained a mass term for gauge bosons:

$$g^2\boldsymbol{v}^\dagger\hat{A}^\mu\hat{A}_\mu\boldsymbol{v} = g^2 A^\mu_a A^b_\mu \underbrace{(T^a\boldsymbol{v})^\dagger(T^b\boldsymbol{v})}_{(M_B^2)_{ab}}$$

The matrix M_B^2 is positive definite and has exactly $d_G - d_H$ non-zero eigenvalues due to $(X^{\hat{a}}\boldsymbol{v}) \neq 0$ and $(Y^{\bar{a}}\boldsymbol{v}) = 0$.

Chapter 8

The EW sector of the Standard Model

Schwartz sec. 29.1, 29.3, 29.5; Mandl sec. 18.3, chap. 19; Peskin chap. 20

Electro-weak interactions can be (phenomenologically) be explained by introducing appropriate (renormalizable) couplings $\bar{\psi}\gamma^{\nu}\psi\nu_{\mu}$. Indeed

- the photon is massless, then QED can be described by an unbroken $U(1)$ gauge theory
- the weak bosons are massive ($\sim$ 100GeV), and can be described by a SSB gauge theory, through the Higgs mechanism
- phenomenologically we know that $\alpha_{EM} \neq \alpha_W$ (since $e \neq g$) so we cannot embed the 4 gauge bosons in a single group (as for example $U(2)$)

8.1 The $SU(2)_L \times U(1)_Y$ gauge sector

Let's assume that the gauge group of the Standard Model is $\boxed{G_{SM} \equiv SU(2)_L \times U(1)_Y}$. The Yang-Mills Lagrangian for G_{SM} reads

$$\boxed{\mathcal{L}_{YM} = -\frac{1}{2}\,\mathrm{Tr}\Big[\hat{W}^{\mu\nu}\hat{W}_{\mu\nu}\Big] - \frac{1}{4}B^{\mu\nu}B_{\mu\nu}}$$

with following field-strength tensors

$$\begin{cases} \hat{W}_{\mu\nu} = \partial_\mu \hat{W}_\nu - \partial_\mu + ig\left[\hat{W}_\mu, \hat{W}_\nu\right] & \to \quad SU(2) \\ B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu & \to \quad U(1) \end{cases}$$

Notice that elements of the $W_{\mu\nu}$ gauge vector boson are

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g\varepsilon^{abc}W^b_\mu W^c_\nu$$

For phenomenological reasons we have to set $U(1)_Y \neq U(1)_{EM}$,[I] which implies

$$\begin{cases} B_\mu \neq A_\mu \\ W^3_\mu \neq Z_\mu \end{cases}$$

This is required by the fact that W^3_μ has only left-handed interactions, while Z_μ (which is related to the weak neutral current) has both left and right-handed couplings.

The YM Lagrangian is invariant under the local transformations given by G_{SM}, which are determinated by $\{T_i\}$ and Y, generators respectively of $SU(2)_L$ and $U(1)_Y$:

$$\begin{cases} \Omega(x) = e^{ig\hat{\alpha}(x)} & \to \qquad \hat{\alpha}(x) = \alpha_i(x)T^i \quad \in \quad SU(2)_L \\ \omega(x) = e^{ig'\hat{\beta}(x)} & \to \qquad \hat{\beta}(x) = \beta(x)Y \quad \in \quad U(1)_Y \end{cases}$$

[I] The Noether charged associated to $U(1)_{EM}$ is the *electric charge*, on the other side $U(1)_Y$ correspond to the *hypercharge* Y we will define in the following.

with the gauge fields transforming as:

$$\Omega(x) \quad \rightarrow \quad \begin{cases} \hat{W}'_\mu = \Omega \hat{W}_\mu \Omega^\dagger + \frac{i}{g}(\partial_\mu \Omega)\Omega^\dagger \\ B'_\mu = B_\mu \qquad (B_\mu \text{ invariant under } SU(2)_L) \end{cases}$$

$$\omega(x) \quad \rightarrow \quad \begin{cases} \hat{W}'_\mu = \hat{W}_\mu \qquad (\hat{W}_\mu \text{ invariant under } U(1)_y) \\ B'_\mu = \omega B_\mu \omega^\dagger + \frac{i}{g'}(\partial_\mu W)\omega^\dagger = B_\mu - \partial_\mu \beta(x) \end{cases}$$

A generic G_{SM} transformation then reads

$$g_{SM} = \omega \cdot \Omega$$

8.2 The Gauge-Higgs Lagrangian and SSB of the EW gauge group

To induce the SSB mechanism we need to introduce a scalar sector with a non-trivial potential. The Lagrangian of the minimally coupled complex scalar field reads

$$\boxed{\mathcal{L}_H = (D_\mu \phi)^\dagger (D^\mu \phi) - \lambda \left(\phi^\dagger \phi - \frac{v^2}{2} \right)^2}$$

with a covariant derivative defined in such a way that gives a coupling with G_{SM}:

$$D_\mu \phi = (\partial_\mu + ig\hat{W}_\mu + ig'\hat{B}_\mu)\phi \quad \rightarrow \quad \begin{cases} \hat{W}_\mu = W^i_\mu T_i \\ \hat{B}_\mu = B_\mu Y \end{cases}$$

with T_i and Y the generators of $SU(2)_L \times U(1)_Y$. Let's choose ϕ to be in the fundamental representation fo $SU(2)_L$ and with eigenvalue for the hypercharge $Y = \frac{1}{2}$ (i.e. it is a doublet). Then

$$T_i = \frac{1}{2}\sigma_i \qquad \text{and} q \qquad Y = \frac{1}{2}\mathbb{1}_2$$

There are several way we can parametrize the complex doublet ϕ. Having a non trivial potential let's choose the *exponential notation*:

$$\phi = \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} \overset{\text{(SSB)}}{\equiv} e^{i\hat{\pi}/v} \begin{pmatrix} 0 \\ \frac{h+v}{\sqrt{2}} \end{pmatrix} \qquad \rightarrow \qquad \begin{cases} \hat{\pi}(x) = \pi_i(x) T^i \\ h(x) \end{cases}$$

where φ_+, φ_- are complex fields and we chosen the following *vacuum configuration*

$$\langle \phi \rangle_0 = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.1}$$

hence $\hat{\pi}(x)$ and $h(x)$ describes the fluctuations around $\langle \phi \rangle_0$, in particular $\pi_i(x)$ are Goldstone bosons and h is the Higgs field..

All the $SU(2)_L \times U(1)_Y$ generators of the basis are broken by this choice of the vacuum

$$T_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq 0 \qquad \text{and} \qquad Y \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq 0$$

but one particular combination of generators is unbroken, indeed

$$Q = T_3 + Y = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \rightarrow \quad Q \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

The spontaneous symmetry breaking induced by the scalar field with non trivial vacuum expectation valus is then

$$SU(2)_L \times U(1)_Y \quad \rightarrow \quad U(1)_{EM}$$

and we identify Q as the generator of the $U(1)_{EM}$ as we want the photon to remain massless. Indeed only unbroken symmetries corresponds to preserved symmetries after SSB, and they are related to massless bosons.

From the representation of Q we obtain that φ_+ is a charged field: $Q\begin{pmatrix}\varphi_+\\0\end{pmatrix} = +\begin{pmatrix}\varphi_+\\0\end{pmatrix}$, while ϕ_0 is the neutral field ($Q\begin{pmatrix}0\\\varphi_-\end{pmatrix} = 0$). This also implies that h is a real neutral scalar field.

The final theory will possess an explicit gauge invariance

$$U(1)_{EM} \quad \to \quad \omega_{EM} = e^{ieQ\alpha(x)} \qquad \text{with} \quad \hat{\alpha} = Q\alpha \quad \text{and} \quad Q = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

The physical component h corresponds to the *EM neutral scalar*, indeed corresponds to the d.o.f. in the choice of the vacuum expectation value in eq. (8.1).

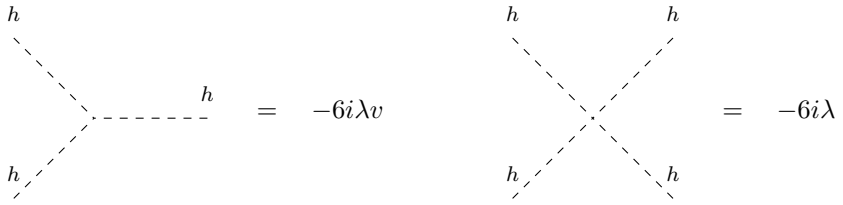
Let's study the *Gauge-Higgs Lagrangian* in the *unitary gauge* by setting $\pi_i = 0$, i.e. all Goldstone bosons are "eaten" by the gauge boson.

i) Higgs Potential

The *Higgs potential* contains the $h(x)$ mass term and self couplings:

$$\boxed{V(\phi^\dagger\phi) = \frac{\lambda}{4}\left((v+h)^2 - v^2\right)^2 = \frac{1}{2}(2\lambda v^2)h^2 + \lambda v h^3 + \frac{\lambda}{4}h^4}$$

i.e. the physical scalar has mass $m_H^2 = 2\lambda v^2 = (125\text{GeV})^2$ and we have self-interactions for the field h described by the following Feynman rules:



where the first symmetry factors are 3! for the triple and 4! for the quartic Higgs self-interactions.

ii) Gauge-Higgs interaction

The *gauge-Higgs interactions* are in the kinetic part of the Lagrangian, given by the covariant derivative

$$(D_\mu\phi) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\\partial_\mu h\end{pmatrix} + \frac{v+h}{\sqrt{2}}\begin{pmatrix}\frac{1}{2}g(W_\mu^1 - iW_\mu^2)\\\frac{1}{2}(g'B_\mu - gW_\mu^3)\end{pmatrix}$$

To have diagonal mass terms let's do the following field redefinitions (rotations of fields in the neutral sector):

$$W_\mu^\pm = \frac{W_\mu^1 \mp iW_\mu^2}{\sqrt{2}} \qquad A_\mu = \frac{gB_\mu + g'W_\mu^3}{\sqrt{g^2+g'^2}} \qquad Z_\mu = \frac{gW_\mu^3 - g'B_\mu}{\sqrt{g^2+g'^2}}$$

where a_μ and Z_μ are physical gauge bosons. Then the Goldstone boson has benn eaten by the gauge boson and the $(D_\mu\phi)$ term reads (unitary gauge + physical bosons)

$$(D_\mu\phi) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\\partial_\mu h\end{pmatrix} + \frac{v+h}{\sqrt{2}}\begin{pmatrix}\frac{1}{\sqrt{2}}gW_\mu^+\\-\frac{1}{2}\sqrt{g^2+g'^2}Z_\mu\end{pmatrix}$$

Notice again that the upper part is charged while the lower part is neutral under $U(1)_{EM}$. We obtained the following form for the kinetic part of the Lagrangian in the unitary gauge

$$\boxed{(D_\mu\phi)^\dagger(D^\mu\phi) = \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + \left(1+\frac{h}{v}\right)^2\left\{\left(\frac{g^2v^2}{4}\right)^2 W_\mu^+W_-^\mu + \frac{1}{2}\left(\frac{(g^2+g'^2)v^2}{4}\right)Z^\mu Z_\mu\right\}}$$

and contains the W, Z interactions with h and h^2 and mass terms for both $W_\mu^\pm$ and Z_μ.

$$M_W^2 = \frac{g^2 v^2}{4} \qquad M_Z^2 = \frac{(g^2+g'^2)v^2}{4}$$

We can express the neutral boson rotation in term of the **Weinberg angle** θ_w:

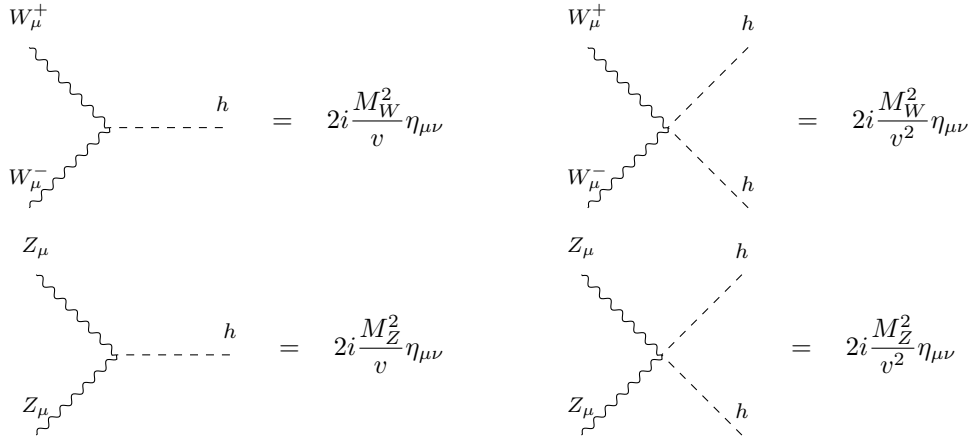
$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} c_w & s_w \\ -s_w & c_w \end{pmatrix} \begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix} \quad \rightarrow \quad \begin{cases} c_w = \frac{g}{\sqrt{g^2+g'^2}} \\ s_w = \frac{g'}{\sqrt{g^2+g'^2}} \end{cases}$$

where $c_w \equiv \cos(\theta_w)$ and $s_w \equiv \sin(\theta_w)$. Then the mass term can be written as

$$M_W^2 = \frac{g^2 v^2}{4} \qquad M_Z^2 = \frac{g^2 v^2}{4c_\omega^2} = \frac{M_W^2}{c_w^2}$$

This implies that if $A_\mu \neq B_\mu$ then $c_w \neq 1$ and $M_W < M_Z$ (this is a prediction of the model). Notice that the transformation of the field we used must be a rotation, otherwise Z_μ and A_μ kinetic terms will not be canonically normalized.

Finally we can write the Feynman rules for hVV and h^2VV interactions:



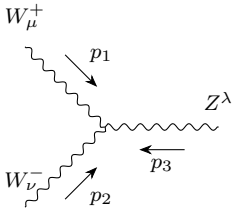
8.2.1 Diagonalization of the gauge bosons mass matrix and the physical bosons

Now that we have discussed the symmetry breaking and identified the physical gauge boson one can derive the gauge self-interactions Feynman rules. The *gauge sector* in the *physical basis* can be written in the following way (we set $e \equiv g s_w$, we will see that this is actually the electric charge in EM):

$$\begin{aligned}
\mathcal{L}_{YM} &= -\frac{1}{4} W^a_{\mu\nu} W_a^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \\
&= -\frac{1}{2}(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+)(\partial^\mu W_-^\nu - \partial^\nu W_-^\mu) - \frac{1}{4} Z^{\mu\nu} Z_{\mu\nu} - \\
&\quad - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \left[M_W^2 W_\mu^+ W_-^\mu + \frac{1}{2} M_Z^2 Z^\mu Z_\mu \right] + \\
&\quad + igc_w \left(Z^{\mu\nu} W_\mu^+ W_\nu^- - W_{\mu\nu}^+ Z^\mu W_-^\nu + W_{\mu\nu}^- Z^\mu W_+^\nu \right) + \\
&\quad + ie \left(F^{\mu\nu} W_\mu^+ W_\nu^- - W_{\mu\nu}^+ A^\mu W_-^\nu + W_{\mu\nu}^- A^\mu W_+^\nu \right) + \\
&\quad + \frac{g^2}{2} \left(W_\mu^+ W_+^\mu W_\nu^- W_-^\nu - W_\mu^+ W_-^\mu W_\nu^+ W_-^\nu \right) + \\
&\quad + e^2 \left(A^\mu W_\mu^+ A^\nu W_\nu^- - A^\mu A_\mu W_+^\nu W_\nu^- \right) + \\
&\quad + g^2 c_w^2 \left(Z^\mu W_\mu^+ Z^\nu W_\nu^- - Z^\mu Z_\mu W_+^\nu W_\nu^- \right) + \\
&\quad + egc_w \left(W_+^\mu W_-^\nu A_\mu Z_\nu + W_-^\mu W_+^\nu A_\mu Z_\nu - 2 W_+^\mu W_\mu^- Z^\nu A_\nu \right)
\end{aligned}$$

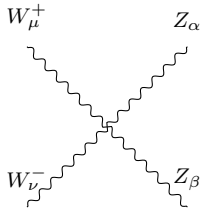
where the term in squared brackets is the mass term given by the gauge-Higgs interaction.

One can derive the corresponding Feynman rules for this Lagrangian. For example the trilinear gauge vertex WWZ reads



$$= \quad -igc_w\left[\eta_{\mu\nu}(p_1-p_2)_\lambda + \eta_{\nu\lambda}(p_2-p_3)_\mu + \eta_{\lambda\mu}(p_3-p_1)_\nu\right]$$

while the quartic gauge vertex reads



$$= \quad ig^2c_w^2\left(\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu} - 2\eta_{\alpha\beta}\eta_{\mu\nu}\right)$$

All the other rules can be obtained by accordingly change the coupling constant (e, gc_w,etc.).

Note that we can show the unbroken $U(1)_{EM}$ symmetry of the SSB Lagrangian. We can see it in two ways

(i) Assigning a charge ±1 to $W_\mu^\pm$ and charge 0 to A_μ and Z_μ. Then all terms in the Lagrangian have total charge 0, i.e. we have a global $U(1)$ symmetry.

(ii) Formally one can assign the following $U(1)_{EM}$ gauge transformation properties to the gauge boson fields

$$\begin{cases} A_\mu & \to & A_\mu - \partial_\mu\alpha \\ Z_\mu & \to & Z_\mu \\ W_\mu^\pm & \to & e^{\pm iq\alpha}W_\mu^\pm \\ h & \to & h \end{cases}$$

We can notice that the Z_μ and h fields are invariant under this symmetry since are neutral fields, moreover the transformation for the fields $W_\mu^\pm$ is actually a phase transformation for the complex fields.

From the expanded Lagrangian the $U(1)_{EM}$ symmetry looks cumbersome due to the presence of couplings with derivative terms. One should instead collect terms together defining a covariant derivative for the $W_\mu^\pm$ fields

$$\begin{cases} \mathcal{D}_\mu W_\nu^\pm \equiv (\partial_\mu \pm iqA_\mu)W_\nu^\pm \\ W_{\mu\nu}^\pm = \mathcal{D}_\mu W_\nu^\pm - \mathcal{D}_\nu W_\mu^\pm \mp igc_w(Z_\mu W_\nu^\pm - Z_\nu W_\mu^\pm) \end{cases}$$

We define $\mathcal{D}_\mu$ to be the *covariant derivative for $W_\mu^\pm$ fields.*

Note that in the previous Lagrangian we set $e \equiv gs_w$ to assign charge ±1 to the $W_\mu^\pm$ fields. From this definition one can rewrite the s_w and c_w of the neutral sector rotation as

$$\begin{aligned} s_w &\equiv \frac{g'}{\sqrt{g^2+g'^2}} = \frac{e}{g} \\ c_w &\equiv \frac{g}{\sqrt{g^2+g'^2}} = \frac{e}{g'} \end{aligned} \quad \to \quad e \equiv \frac{gg'}{\sqrt{g^2+g'^2}}$$

so finally one has

$$e = gs_w = g'c_w$$

Moreover we found an explicit relation between the couplings e, g and g' (one can use only two of them to describe the theory). We will verify in the following that this definition is consistent with the fermionic charge (EM) assignment.

8.3 The gauge-fermion sector

We have defined QED and QCD as vector like interactions, i.e. $\bar{\psi}\gamma^\mu\psi V_\mu$. When discussing weak interactions we saw that left and right chiralities have different couplings with weak bosons, i.e. are described by chiral currents in the form $\bar{\psi}(c_L\gamma^\mu_L + c_R\gamma^\mu_R)\psi V_\mu$ with $c_L \neq c_R$. In other words we saw that weak interactions are a chiral theory. Then in building the SM interactions between fermions and gauge boson we have to assume

(i) $SU(2)_L$ acts only on the left chiral fermionic component (i.e. charged currents are only left handed)

(ii) $U(1)_Y$ acts on both left and right chiral components (but in the general the action on the left component is different from the action on the right component)

As left fermions transform under $SU(2)$ while right fermions do not, this gives a requirement on the representation of these fields:

(i) Left fermions are doublets of $SU(2)$ (i.e. are in the fundamental representation)

(ii) Right fermions are singlets of $SU(2)$ (i.e. are in the trivial representation)

Let's consider the first family of fermions containing the lightest fermion of each type

$$\text{leptons } = (\nu_e, e) \qquad\qquad \text{quarks } = (u, d)$$

Recall that families differs only because of masses.

Let's assign the following quantum numbers (i.e. charges) to the first fermionic family:

	$SU(2)$	T_3	$U(1)_Y$	$Q = T_3 + Y$
$l_L \equiv \begin{pmatrix}\nu_e\\ e\end{pmatrix}_L$	2	$+1/2$ $-1/2$	-1/2	0 -1
ν^e_R	1	0	0	0
e_R	1	0	-1	-1
$q_L \equiv \begin{pmatrix}u\\ d\end{pmatrix}_L$	2	$+1/2$ $-1/2$	$+1/6$	$+2/3$ $-1/3$
u_R	1	0	$+2/3$	$+2/3$
d_R	1	0	-1/3	-1/3

where T_3 is the diagonal generator of $SU(2)_L$. Hence all right-handed fermions have $T_3 = 0$, in this way they are left invariant under $SU(2)_L$. In general quantum numbers are fixed watching how these particles interacts with gauge bosons, moreover hypercharge and isospin eigenvalues (i.e. T_3 and Y) are fixed in order to satisfy $Q = T_3 + Y$. In this way $Q = T_3 + Y$ has the same definition of the generator of $U(1)_{EM}$ we gave before. For this reason we gave zero Q eigenvalue for the neutrino. Notice that both ν_e and e must have same isospin eigenvalue in order to preserve gauge invariance, same statement holds also for u and d. For the adjoint field $\bar{\psi}$ the quantum numbers are the same as for ψ but multiplied by -1.

Knowing all the fermionic charges we can couple fermions to gauge bosons via minimal coupling. Each fermion component has its own D_μ dependency on its charges:

$$\begin{aligned}
D_\mu l_L &= \left(\partial_\mu + igW^a_\mu T_a + ig'B_\mu\left(-\frac{1}{2}\right)\mathbb{1}\right)\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L\\
D_\mu q_L &= \left(\partial_\mu + igW^a_\mu T_a + ig'B_\mu\left(+\frac{1}{6}\right)\mathbb{1}\right)\begin{pmatrix}u\\ d\end{pmatrix}_L\\
D_\mu \nu_R &= \partial_\mu \nu_R\\
D_\mu e_R &= \left(\partial_\mu + ig'B_\mu\,(-1)\,\mathbb{1}\right)e_R\\
D_\mu u_R &= \left(\partial_\mu + ig'B_\mu\left(\frac{2}{3}\right)\mathbb{1}\right)u_R\\
D_\mu d_R &= \left(\partial_\mu + ig'B_\mu\left(-\frac{1}{3}\right)\mathbb{1}\right)d_R
\end{aligned}$$

We can see that neutrinos does not interact with weak gauge bosons.

The *fermionic-gauge Lagrangian* then can be rewritten using covariant derivatives:

$$\boxed{\mathcal{L}_F = \bar{\psi}_i(i\not{D})\psi_i = \bar{l}_L i\not{D} l_L + \bar{q}_L i\not{D} q_L + \bar{\nu}_R i\not{\partial}\nu_R + \bar{e}_R i\not{D} e_R + \bar{u}_R i\not{D} u_R + \bar{d}_R i\not{D} d_R}$$

and contains both kinetic part and gauge interactions. It is invariant under a $SU(2)_L \times U(1)_Y$ gauge transformations $\Omega \cdot \omega$:

$$SU(2)_L \quad \to \quad \begin{cases} L' = \Omega L \\ e'_R = e_R \end{cases} \qquad\qquad U(1)_Y \quad \to \quad \begin{cases} L' = \omega_L L \\ e'_R = \omega_R e_R \end{cases}$$

In terms of the physical gauge bosons (i.e. using mass eigenvectors) we have

$$W^{\pm}_{\mu} = \frac{W^1_{\mu} \mp i W^2_{\mu}}{\sqrt{2}} \qquad \text{with} \qquad T_{\pm} = T_1 \pm i T_2$$

and

$$\begin{pmatrix} A_{\mu} \\ Z_{\mu} \end{pmatrix} = \begin{pmatrix} c_w & s_w \\ -s_w & c_w \end{pmatrix} \begin{pmatrix} B_{\mu} \\ W^3_{\mu} \end{pmatrix}$$

so the Lagrangian reads

$$\boxed{\begin{aligned} \mathcal{L}_F = \mathcal{L}_{\text{KIN}} &+ \frac{g}{\sqrt{2}} \left\{ (\bar{l}_L \gamma^{\mu} T_+ l_L + \bar{q}_L \gamma^{\mu} T_+ q_L) W^+_{\mu} + \text{h.c.} \right\} + \\ &+ \Big\{ \bar{l}_L \gamma^{\mu} (g s_w T_3 + g' c_w Y_{l_L}) l_L + \bar{q}_L \gamma^{\mu} (g s_w T_3 + g' c_w Y_{q_L}) q_L + \\ &\quad + \bar{e}_R \gamma^{\mu} (g' c_w Y_{e_R}) e_R + \bar{u}_R \gamma^{\mu} (g' c_w Y_{u_R}) u_R + \bar{d}_R \gamma^{\mu} (g' c_w Y_{d_R}) d_R \Big\} A_{\mu} \\ &+ \Big\{ \bar{l}_L \gamma^{\mu} (g c_w T_3 + g' s_w Y_{l_L}) l_L + \bar{q}_L \gamma^{\mu} (g c_w T_3 + g' s_w Y_{q_L}) q_L + \\ &\quad + \bar{e}_R \gamma^{\mu} (-g' s_w Y_{e_R}) e_R + \bar{u}_R \gamma^{\mu} (-g' s_w Y_{u_R}) u_R + \bar{d}_R \gamma^{\mu} (-g' s_w Y_{d_R}) d_R \Big\} Z_{\mu} \end{aligned}}$$

where now the couplings with the photon has to reconstruct the EM charge

$$\begin{cases} g s_w T_3 + g' c_w Y = eQ & \text{for the left component} \\ g' c_w Y = eQ & \text{for the right component} \end{cases}$$

For example for the leptons one has (use quantum numbers shown in the previous table)

$$\begin{cases} g s_w \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} + g' c_w Y_L \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \\ g' c_w Y_{e_R} = -e \\ g' c_w Y_{\nu_R} = 0 \end{cases}$$

that with the following assignments for Y (notice $Y_{\nu_L} + Y_{e_L} = -1/2$ as shown in the table)

$$Y_{\nu_L} = -\frac{1}{2} \qquad Y_{e_L} = -1/2 \qquad Y_{\nu_R} = 0 \qquad Y_{e_R} = -1$$

gives the following relations

$$\begin{cases} g s_w = g' c_w \\ g' c_w = e \\ \frac{1}{2}(g s_w + g' c_w) = e \end{cases}$$

Then in order to recover the EM charge one has

$$g s_w = g' c_w = e$$

hence previous assignments are consistent.

Note that assuming left fermions in the fundamental representation and right fermions in the trivial representation, from the previous equations one recover the *hypercharge* assignment (compatible with EM)

$$Q = T_3 + Y$$

8.3.1 Couplings with fermions: EW, Charged and Neutral currents

By using previous definitions the fermionic Lagrangian can be written

$$\boxed{\mathcal{L}_F = \mathcal{L}_{\rm KIN} + \frac{g}{\sqrt{2}}(Y^\mu_{\rm CC} W^+_\mu + \text{h.c.}) + eJ^\mu_{\rm EM} A_\mu + \frac{g}{c_w} J^\mu_Z Z_\mu}$$

with the following currents

$$J^\mu_{\rm CC} = \bar\nu_L \gamma^\mu e_L + \bar u_L \gamma^\mu d_L$$
$$J^\mu_{\rm EM} = \sum_i q_i \bar\psi_i \gamma^\mu \psi_i \qquad \begin{pmatrix}\psi_i \\ q_i\end{pmatrix} = \begin{pmatrix} e & u & d \\ -1 & 2/3 & -1/3\end{pmatrix}$$
$$J^\mu_Z = \sum_i \bar\psi_i (c^i_L \gamma^\mu_L + c^i_R \gamma^\mu_R)\psi_i \qquad \psi_i = (\nu, e, u, d)$$

where we defined the coupling constants

$$c^i_L = t^i_3 - s^2_w q_i \qquad c^i_R = -s^2_w q_i$$

with q_i charges of ψ_i and t^i_3 eigenvalues of T_3. As we required we have $c_L \neq c_R$ for chiral couplings and $c_L = c_R$ for vector couplings.

This Lagrangian is exactly the same as IVB. Now we can obtain the boson-fermion couplings in terms of the $SU(2) \times U(1)$ quantum numbers.

Notice that fermionic mass terms are missing, indeed fermion mass terms are forbidden in a chiral theory by gauge invariance:

(i) In vector-like gauge theories like QED ($U(1)_{\rm EM}$) or QCD ($SU(3)_C$) mass terms are allowed: if we start from a Lagrangian ($\bar\psi \not{A} \psi = \bar\psi_R \not{A} \psi_R + \bar\psi_L \not{A} \psi_L$)

$$\mathcal{L}_{\rm vector} = \bar\psi(i\not{D} - M)\psi$$

the theory is invariant under gauge transformations ($\Omega \in$ gauge group)

$$\psi' = \Omega\psi \qquad (D_\mu\psi)' = \Omega(D_\mu\psi)$$

indeed

$$\mathcal{L}'_{\rm vector} = \bar\psi' i(\not{D}\psi)' - M\bar\psi' \mathbb{1}\psi' = \bar\psi(i\not{D} - M\mathbb{1})\psi = \mathcal{L}_{\rm vector}$$

(ii) In chiral gauge theories like $SU(2)_L \times U(1)_Y$ mass terms are forbidden: if we take a generic chiral Lagrangian with a mass term for $\psi = (\nu, e)$ and covariant derivatives $D^L_\mu = \partial_\mu + ig\hat W_\mu$ and $D^R_\mu = \partial_\mu$:

$$\mathcal{L}_{\rm chiral} = \bar\psi_L i\not{D}_L \psi_L + \bar\psi_R i\not{D}_R \psi_R - M(\bar\psi_L \psi_R + \bar\psi_R \psi_L)$$

and for simplicity we consider only a $SU(2)_L$ transformation ($\Omega \in SU(2)_L$)

$$\psi_L = \begin{pmatrix}\nu_L \\ e_L\end{pmatrix} \quad \to \quad \psi'_L = \Omega\psi_L \qquad \psi_R = \begin{pmatrix}\nu_R \\ e_R\end{pmatrix} \quad \to \quad \psi'_R = \psi_R$$

then the Lagrangian becomes

$$\begin{aligned}\mathcal{L}'_{\rm chiral} &= \bar\psi'_L i(\not{D}_L \psi_L)' + \bar\psi_R i\not{\partial}\psi_R - M(\bar\psi'_L \psi_R + \bar\psi_R \psi'_L) \\ &= \bar\psi_L i\not{D}_L \psi_L + \bar\psi_R i\not{\partial}\psi_R - M(\bar\psi_L \Omega^\dagger \psi_R + \bar\psi_R \Omega \psi_L)\end{aligned}$$

and the mass term is clearly not invariant.

8.4 The Higgs-fermion sectors and mass terms for chiral fermions (1 family)

We can give a mass term to fermions by using the Higgs SSB mechanism, coupling fermions to ϕ. To make an $SU(2)_L \times U(1)_Y$ coupling between SM fermion and Higgs we have to match the $SU(2)_L$ and $U(1)_Y$ charges:

$$\phi = (2, 1/2) \quad \begin{matrix} l_L = (2, -1/2) & e_R = (1, -1) & \nu_R = (1, 0) \\ q_L = (2, 1/6) & u_R = (1, 2/3) & d_R = (1, -1/3) \end{matrix}$$

In particular we use a Lagrangian containing **Yukawa interactions** i.e. with interacting terms which involves two fermions and one scalar. In this way we obtain the following gauge invariant Lagrangian

$$\boxed{\mathcal{L}_{\text{YUK}} = -y_u \bar{q}_L \tilde{\phi} u_R - y_d \bar{q}_L \phi d_R - y_e \bar{l}_L \phi e_R - y_\nu \bar{l}_L \tilde{\phi} \nu_R}$$

where we have defined

$$\tilde{\phi} = i\sigma_2 \phi^* = \frac{v+h}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \rightarrow \quad \tilde{\phi} = (2, -1/2)$$

To check the $SU(2)_L \times U(1)_Y$ invariance one has to check the invariance of $\mathcal{L}_{\text{YUK}}$ under both these sets of transformation for fields (verify the invariance of the Lagrangian as exercize):

$$SU(2)_L \quad \rightarrow \quad \begin{cases} q'_L = \Omega q_L & l'_L = \Omega l_L \\ \phi' = \Omega \phi & \tilde{\phi}' = \Omega \tilde{\phi} \end{cases}$$

$$U(1)_Y \quad \rightarrow \quad \begin{cases} \begin{cases} q'_L = \omega_{1/6} q_L & l'_L = \omega_{-1/2} l_L \\ \phi' = \omega_{1/2} \phi & \tilde{\phi}' = \omega_{-1/2} \tilde{\phi} \end{cases} \\ \begin{cases} \nu'_R = \nu_R & e'_R = \omega_{-1} e_R \\ u'_R = \omega_{2/3} u_R & d'_R = \omega_{-1/3} d_R \end{cases} \end{cases}$$

Notice that the sum of both $SU(2)_L$ and $U(1)_Y$ eigenvalues in each term of the previous Lagrangian gives a zero sum, for instance the sum of $U(1)_Y$ eigenvalues in $\bar{q}_L \tilde{\phi} u_R$ gives $-1/6 - 1/2 + 2/3 = 0$. This ensures the invariance under $U(1)_Y$ transformations we just described.

Note that in the original definition of the SM ν_R was not introduced, since it has no effect in gauge interactions ($\nu_R = (1, 0)$). Actually according to the Yukawa Lagrangian in principle it can have Yukawa interactions.

When the Higgs field acquires a vacuum expectation value, in the unitary gauge we have

$$\phi = \frac{v+h}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \tilde{\phi} = \frac{v+h}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and the Yukawa Lagrangian reads

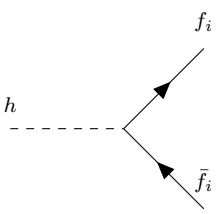
$$\mathcal{L}_{\text{YUK}} = -\sum m_i \left(1 + \frac{h}{v}\right) \left(\bar{\psi}^i_L \psi^i_R + \bar{\psi}^i_R \psi^i_L\right)$$

for $\psi_i = (\nu_e, e, u, d)$. In this way we obtained a mass term for each fermion field:

$$m_i \equiv \frac{y_i v}{\sqrt{2}}$$

and is called **Yukawa mass term**.

The Feynman rules associated to this interaction are very symple:



$$= \quad i\frac{m_i}{v} = i\frac{y_i}{\sqrt{2}}$$

8.5 Summary of the $SU(2)_L \times U(1)_Y$ Lagrangian for 1 family

(1) The (EW) SM Lagrangian describes the interactions between *gauge bosons*, *fermions* and *Higgs scalars*:

(i) $SU(2)_L \times U(1)_Y$ gauge bosons: $W^{\pm}_{\mu}$, Z_μ, A_μ

(ii) 1 family of fermions: $\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L$, ν_R, e_R, $\begin{pmatrix}u\\ d\end{pmatrix}_L$, u_R, d_R

(iii) Higgs complex scalar doublet $\phi = \begin{pmatrix}\varphi_+\\ \varphi_-\end{pmatrix} = e^{i\hat{\pi}/v}\begin{pmatrix}0\\ \frac{v+h}{\sqrt{2}}\end{pmatrix}$

$$\boxed{\mathcal{L}_{\rm SM} = \mathcal{L}_{\rm YM} + \mathcal{L}_{\rm H} + \mathcal{L}_{\rm F} + \mathcal{L}_{\rm YUK}}$$

(2) The Lagrangian is $SU(2)_L \times U(1)_Y$ gauge invariant

(3) The non-trivial potential force the Higgs to acquire a non-vanishing vacuum expectation value $\langle\phi\rangle_0 = \frac{v}{\sqrt{2}}\begin{pmatrix}0\\ 1\end{pmatrix}$ that breaks spontateously the gauge symmetry

$$SU(2)_L \times U(1)_Y \quad \xrightarrow{\rm SSB} \quad U(1)_{\rm EM}$$

This gives the EM charge described by the generator $Q = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$.

(4) The non-vanishing vacuum expectation value is the origin of all masses of the SM fields

(i) Weak bosons from $(D_\mu\phi)^\dagger(D^\mu\phi)$: $M_W^2 = \frac{g^2v^2}{4}$ and $M_Z^2 = \frac{(g^2+g'^2)v^2}{4}$

(ii) The Higgs mass from the Higgs potential $V(\phi^\dagger\phi)$: $m_H^2 = 2\lambda v^2$

(iii) The fermion masses from $\mathcal{L}_{\rm YUK}$: $m_i = \frac{y_i v}{\sqrt{2}}$

8.6 The 3-families case and general 3×3 Yukawa sector and flavor violation in the CC sector: the $V_{\rm CKM}$ and the $V_{\rm PMNS}$ matrices

In the previous sections we have discussed all the SM construction for 1 family (the lightest one):

$$q_L^1 = \begin{pmatrix}u\\ d\end{pmatrix}_L \quad \begin{matrix}u_R\\ d_R\end{matrix} \qquad l_L^1 = \begin{pmatrix}\nu_e\\ e\end{pmatrix}_L \quad \begin{matrix}\nu_{e,R}\\ e_R\end{matrix}$$

In nature exist other 2 replicas of this family structure, which differs only by the masses of particles

$$q_L^2 = \begin{pmatrix}c\\ s\end{pmatrix}_L \quad \begin{matrix}c_R\\ s_R\end{matrix} \qquad l_L^2 = \begin{pmatrix}\nu_\mu\\ \mu\end{pmatrix}_L \quad \begin{matrix}\nu_{\mu,R}\\ \mu_R\end{matrix}$$

$$q_L^3 = \begin{pmatrix}t\\ b\end{pmatrix}_L \quad \begin{matrix}t_R\\ b_R\end{matrix} \qquad l_L^3 = \begin{pmatrix}\nu_\tau\\ \tau\end{pmatrix}_L \quad \begin{matrix}\nu_{\tau,R}\\ \tau_R\end{matrix}$$

There are some theoretical hints about the need of a complete family (*anomaly*) but why we have exactly three families is an open question.

To deal with 3 families we have to add a 3 dimensional flavor structure:

$$u_{L,R} \to U_{L,R} = \begin{pmatrix}u\\ c\\ t\end{pmatrix}_{L,R} \qquad d_{L,R} \to D_{L,R} = \begin{pmatrix}d\\ s\\ b\end{pmatrix}_{L,R}$$

$$e_{L,R} \to E_{L,R} = \begin{pmatrix}e\\ \mu\\ \tau\end{pmatrix}_{L,R} \qquad \nu_{L,R} \to N_{L,R} = \begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix}_{L,R}$$

$$q_L \to Q_L = \begin{pmatrix}U\\ D\end{pmatrix}_L \qquad l_L \to L_L = \begin{pmatrix}N\\ E\end{pmatrix}_L$$

The gauge-fermion Lagrangian

The gauge-fermions Lagrangian with the flavor structure now reads

$$\boxed{\mathcal{L}_F = \mathcal{L}_{\text{KIN}} + \frac{g}{\sqrt{2}}(J^\mu_{CC} W^+_\mu + \text{h.c.}) + \frac{g}{c_w} J^\mu_{NC} Z_\mu + e J^\mu_{\text{EM}} A_\mu}$$

where we define following currents

$$\begin{aligned} J^\mu_{\text{CC}} &= \bar{U}_L \gamma^\mu \mathbb{1} D_L + \bar{N}_L \gamma^\mu \mathbb{1} E_L \\ J^\mu_{\text{EM}} &= q_U \bar{U} \gamma^\mu \mathbb{1} U + q_D \bar{D} \gamma^\mu D + q_E \bar{E} \gamma^\mu \mathbb{1} E \\ J^\mu_Z &= \bar{U}(c^u_L \gamma^\mu_L + c^u_R \gamma^\mu_R)\mathbb{1} U + \bar{D}(c^d_L \gamma^\mu_L + c^d_R \gamma^\mu_R)\mathbb{1} D + \bar{N}(c^\nu_L \gamma^\mu_L + c^\nu_R \gamma^\mu_R)\mathbb{1} N + \bar{E}(c^e_L \gamma^\mu_L + c^e_R \gamma^\mu_R)\mathbb{1} E \end{aligned}$$

We define the **flavor (interaction) basis** the basis where the interactions are diagonal. We assumed *universal couplings*, i.e. are family independent. With these definitions the interaction gauge-fermion Lagranian has an $SU(3)$ global symmetry.

The Higgs-fermion Lagrangian

The Higgs-fermion Lagrangian with the flavor structure now reads

$$-\mathcal{L}_{\text{YUK}} = +\bar{Q}_L \tilde{\phi} Y_u U_R + \bar{Q}_L \phi Y_d D_R + \bar{L}_L \phi Y_e E_R + \bar{L}_L \tilde{\phi} Y_\nu N_R + \text{h.c.}$$

In the interaction (flavor) basis the **Yukawa matrices** Y_i are arbitrary 3×3 complex matrices.

Remember that fermion masses are proportional to the Yukawa couplings Y, indeed if we set $\phi = \frac{v+h}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then:

$$M_U = \frac{Y_u v}{\sqrt{2}} \qquad M_D = \frac{Y_d v}{\sqrt{2}} \qquad M_N = \frac{Y_\nu v}{\sqrt{2}} \qquad M_E = \frac{Y_e v}{\sqrt{2}}$$

In general the mass matrices are not to be assumed diagonal in the flavor basis. According to **CKM ansatz**, *all sources of $SU(3)$ flavor violation are encoded in the Yukawa matrices.* With an appropriate rotation of the fields we can go to the mass basis.

Diagonalization of the mass matrices

Even if sometimes is useful to work in the flavor basis, the physical basis is where the physical observables (propagators) has to be calculated. This means that physical states are given by mass eigenvalues.

A general 3×3 matrix can be always diagonalized by a biunitary transformation. Given M we can always build two different Hermitian matrices $MM^\dagger$, $M^\dagger M$ and diagonalize them with unitary transformations

$$\begin{cases} MM^\dagger & \to \quad \mathcal{U}^\dagger (MM^\dagger)\mathcal{U} = M^2_{\text{diag}} \\ M^\dagger M & \to \quad \mathcal{V}^\dagger (M^\dagger M)\mathcal{V} = M^2_{\text{diag}} \end{cases}$$

As obviously the eigenvalues are the same (real) and we can write ($\mathcal{U}\mathcal{U}^\dagger = \mathcal{V}\mathcal{V}^\dagger = \mathbb{1}$)

$$M^2_{\text{diag}} = (\mathcal{U}^\dagger M \mathcal{V})(\mathcal{V}^\dagger M^\dagger \mathcal{U}) = (\mathcal{V}^\dagger M^\dagger \mathcal{U})(\mathcal{U}^\dagger M \mathcal{V}) = M^\dagger_{\text{diag}} M_{\text{diag}} = M_{\text{diag}} M^\dagger_{\text{diag}}$$

hence we can set (the other choice is exactly equivalent)

$$M_{\text{diag}} \equiv \mathcal{U}^\dagger M \mathcal{V} \qquad\qquad M^\dagger_{\text{diag}} \equiv \mathcal{V}^\dagger M^\dagger \mathcal{U}$$

Now let's rotate all fermion triplets in flavor space by appropriate $SU(3)$ elements $R_{L,R}, S_{L,R}, T_{L,R}, X_{L,R} \in SU(3)$ (in general we have to treat differently left and right components since we are considering a chiral theory)

$$\begin{cases} U'_L = R_L U_L & D'_L = S_L D_L \\ U'_R = R_R U_R & D'_R = S_R D_R \end{cases} \qquad \begin{cases} N'_L = T_L N_L & E'_L = X_L E_L \\ N'_R = T_R N_R & E'_R = X_R E_R \end{cases} \tag{8.2}$$

By these rotations the Yukawa Lagrangian becomes

$$\boxed{\begin{aligned}-\mathcal{L}_{\rm YUK} &= \left\{\bar{D}'_L S_L^\dagger M_D S_R D'_R + \bar{U}'_R R_L^\dagger M_U R_R U'_R + \bar{E}'_L X_L^\dagger M_E X_R E'_R + \bar{N}'_L T_L^\dagger M_N T_R N'_R + \text{h.c.}\right\}\left(1+\frac{h}{v}\right)\\ &= \left\{\bar{D}'_L M_D^{\rm diag} D'_R + \bar{U}'_R M_U^{\rm diag} U'_R + \bar{E}'_L M_E^{\rm diag} E'_R + \bar{N}'_L M_N^{\rm diag} N'_R + \text{h.c.}\right\}\left(1+\frac{h}{v}\right)\end{aligned}}$$

where we introduced the diagonal mass matrices

$$M_U^{\rm diag} \equiv R_L^\dagger M_U R_R = \begin{pmatrix} m_u & 0 & 0\\ 0 & m_c & 0\\ 0 & 0 & m_t\end{pmatrix} \qquad M_D^{\rm diag} \equiv S_L^\dagger M_D S_R = \begin{pmatrix} m_d & 0 & 0\\ 0 & m_s & 0\\ 0 & 0 & m_b\end{pmatrix}$$

$$M_N^{\rm diag} \equiv T_L^\dagger M_N T_R = \begin{pmatrix} m_{\nu_e} & 0 & 0\\ 0 & m_{\nu_\mu} & 0\\ 0 & 0 & m_{\nu_\tau}\end{pmatrix} \qquad M_E^{\rm diag} \equiv X_L^\dagger M_E X_R = \begin{pmatrix} m_e & 0 & 0\\ 0 & m_\mu & 0\\ 0 & 0 & m_\tau\end{pmatrix}$$

The gauge-fermion interaction Lagrangian

The gauge-fermion interaction Lagrangian has also to be rotated and written in the mass basis.

Notice that the neutral current sector is not affected by the change of basis eq. (8.2). Indeed all the pieces are in the form

$$c_L \bar{U}_L \gamma^\mu U_L + c_R \bar{U}_R \gamma^\mu U_R + \dots$$

hence are transformed into

$$c_L \bar{U}'_L \gamma^\mu U'_L + c_R \bar{U}'_R \gamma^\mu U'_R + \dots$$

Hence neutral currents are diagonal and universal in the mass basis. The neutral sector has an $SU(3)_F$ symmetry, i.e. flavor is not broken by neutral currents.

The charged current sector however is not left diagonal by the change of basis as it involves different fields:

$$\mathcal{L}_{CC} = \frac{g}{\sqrt{2}}(\bar{D}_L \gamma^\mu D_L + \bar{E}_L \gamma^\mu E_L) W_\mu^+ + \text{h.c.}$$

and in the mass basis reads

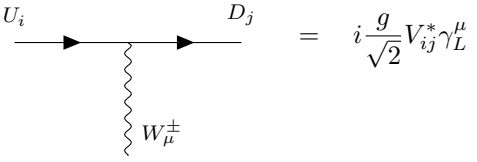
$$\boxed{\begin{aligned}\mathcal{L}_{CC} &= \frac{g}{\sqrt{2}}(\bar{D}'_L \gamma^\mu (S_L R_L^\dagger) U'_L + \bar{E}'_L \gamma^\mu (X_L T_L^\dagger) N_L) W_\mu^+ + \text{h.c.}\\ &= \frac{g}{\sqrt{2}}(\bar{D}'_L \gamma^\mu V_{\rm CKM}^\dagger U'_L + \bar{E}'_L \gamma^\mu V_{\rm PMNS}^\dagger N_L) W_\mu^+ + \text{h.c.}\end{aligned}}$$

where we defined the following 3×3 unitary matrices[II]

$$V_{\rm CKM} \equiv R_L S_L^\dagger \qquad\qquad V_{\rm PMNS} \equiv T_L X_L^\dagger$$

which in general are not diagonal in the mass basis. These matrices contain all the flavor (and CP) violation of the SM. In the neutral sector there are no flavor changing couplings, only flavor violation appears in the charged sector.

The associated Feynman rules read, for instance



$$= \quad i\frac{g}{\sqrt{2}} V_{ij}^* \gamma_L^\mu$$

where u_i and d_j are elements of D and U respectively, with indices i and j. Reverting the role of U and D in the diagram the matrix element V_{ij}^* must be substituted with V_{ij}.

[II] CKM=Cabibbo-Kobayashi-Maskawa
PMNS=Pontecorvo-Maki-Nakagawa-Sakata

Notice that the CC sector brokes both $SU(3)$ and $U(1)^3$ flavor symmetries. There is only a residual global $U(1)$ symmetry, which corresponds to the conservation of the **Baryon number**, which is unbroken by $V_{\rm CKM}$ and $V_{\rm PMNS}$, as we will see later. Flavor changing couplings of the CC can mix quarks/leptons of different families, for instance the probability amplitude for $W^+_\mu \to u\bar{s}$ is proportional to V_{us}, which in general is non-zero.

Matrices $V_{\rm CKM}$ and $V_{\rm PMNS}$ are general unitary matrices, so they can be complex and we must be careful when we write down Feynman rules, since in general $V_{ij} \neq V^*_{ij}$.

8.7 Counting of physical parameters in the Yukawa sector

Take for instance the quark sector, an analogous discussion holds also for the leptonic sector. In the Yukawa Lagrangian in the flavor basis we introduced complex matrices Y_U and Y_D. The number of parameters of 2 generic complex 3×3 matrices is $2 \times 2N^2 = 36$. However not all of these degrees of freedom are physical. To count the number of physical parameters one has to go to the mass basis.

The number of physical parameters in the Yukawa Lagrangian is given by $M_U^{\rm diag} = {\rm diag}(m_u, m_c, m_t)$ and $M_D^{\rm diag} = {\rm diag}(m_d, m_s, m_b)$, i.e. we have 3+3=6 degrees of freedom. Moreover we have d.o.f. related to $V_{\rm CKM}$ in the charged sector, in particular a generic 3×3 unitary matrix contains $\frac{1}{2}N(N-1) = 3$ free moduli and $\frac{1}{2}N(N+1) = 6$ free phases.

Not all the phases in the $V_{\rm CKM}$ are physical, in the sense that one has always the freedom to redefine (complex) spinor by an arbitrary phase (indeed $U\bar{U}$ is invariant under such redefinition). Let's redefine the U and D triplets as

$$\hat{U} = \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & e^{i\alpha_3} \end{pmatrix} U' = \phi_U U' \qquad\qquad \hat{D} = \begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & e^{i\beta_3} \end{pmatrix} D' = \phi_D D'$$

Notice that for each spinor we have 3 d.o.f. in the choice of the phases. The neutral sector is invariant under these redefinition, hence for the neutral sector this redefinition is actually a $U(1)^6$ symmetry transformation. On the other side, the charged sector is not invariant under this $U(1)^6$ transformation (e.g. it mixes up and down quarks)

$$\bar{U}'_L \gamma^\mu V_{\rm CKM} D'_L \quad \to \quad \hat{\bar{U}}'_L \gamma^\mu (\phi_U V_{\rm CKM} \phi_D^\dagger) \hat{D}'_L$$

This phases redefinition modify the $V_{\rm CKM}$ matrix

$$V_{\rm CKM} \quad \to \quad \hat{V}_{\rm CKM} \equiv \phi_U V_{\rm CKM} \phi_D^\dagger$$

and one can choose the ϕ_U and ϕ_D phases in such a way to cancel some of the $V_{\rm CKM}$ phases. Such cancellation reduces the number of degrees of freedom in $V_{\rm CKM}$.

Note that the charged sector (and all the EW Lagrangian) is invariant under the following global $U(1)_B$ transformation:

$$U' \quad \to \quad \hat{U}' = e^{i\alpha} U' \qquad\qquad D' \quad \to \quad \hat{D}' = e^{i\alpha} D'$$

This $U(1)_B$ symmetry[III] is the one associated to the baryon number conservation. In particular all quarks and antiquarks have $1/3$ and $-1/3$ $U(1)_B$ charge respectively.

Due to the $U(1)_B$ global symmetry 1 phase is arbitrary, i.e. only $2N - 1 = 5$ phases can be removed by spinorial phase redefinition.

Finally, the CKM matrix has $\frac{1}{2}N(N-1) = 3$ d.o.f. in moduli and $\frac{1}{2}N(N+1) - (2N-1) = 1$ d.o.f. in the phase. Then number of independent parameters in the Yukawa (quark) sector is then

$$\begin{cases} 2N & \text{masses} \to 6 \\ \frac{1}{2}N(N-1) & \text{moduli} \to 3 \\ \frac{1}{2}(N-1)(N-2) & \text{phases} \to 1 \end{cases} \qquad \Longrightarrow \qquad 10 \text{ independent parameters}$$

Hence there are 10 independent parameters in the quark sector, instead of the 36 ones we discussed before. With the presence of ν_R there are 10 independent parameters also in the leptonic sector.

The fact that $V_{\rm CKM}$ and $V_{\rm PMNS}$ are complex implies that SM violates CP symmetry. The only discrete symmetry of SM is CPT.

[III] B = Baryon

Chapter 9

Discrete Symmetries

9.1 The C, P and T discrete symetries for scalar, vector and Dirac fermion fields

Schwartz sec. 11.3-11.6; Maggiore sec. 4.2.3; Peskin sec. 3.6

Up to now when discussing properties related to the Lorentz group, we have considered only local transformation near the identity, i.e. transformations in the proper orthochronus subgroup

$$\mathcal{L}_+^\uparrow \equiv \{\Lambda^0{}_0 > 0\,,\, \det\Lambda = 1\}$$

We wanto to discuss now the role of discrete symmetries, and in particular C, P, T transformations:

- P and T are transformations of the Lorentz group not belonging to the Λ_{++} subgruoup
- C is a symmetry transformation that has no analogue in the non-relativistic theories, and is called *charge conjugation*

We will see that C, P, T even if are symmetries of QED and QCD they are not symmetries of the SM. On the other side their combination CPT is instead a symmetry of nature, as it is described by the CPT theorem.

9.1.1 Discrete symmetries and scalar fields

For simplicity we define C, P and T for a scalar fields and then we extend them for spinors and vector fields.

Parity

Parity is a transformation that belongs to the *improper orthochronus set* : $\Lambda_0)^0 > 0$, $\det\Lambda = -1$. Parity transformation is given by the inversion of space-like coordinates:

$$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad \text{such that} \qquad x^\mu \overset{P}{\to} x'^\mu = x_\mu = \left\{ \begin{array}{c} x'^0 = x_0 \\ x'^i = x_i \end{array} \right\} = \tilde{x} = (x_0, -\boldsymbol{x})$$

In classical mechanics parity acts as

$$\boldsymbol{x} \overset{P}{\to} -\boldsymbol{x} \qquad \boldsymbol{p} \overset{P}{\to} -\boldsymbol{p} \qquad \boldsymbol{J} \overset{P}{\to} \boldsymbol{J}$$

where in the last relation we can see that axial vectors such as the angular momentum $\boldsymbol{J}$ are invariant under parity. We see that parity invert vectors but leaves invariant pseudovectors.

A scalar field ϕ under P transforms as

$$\phi(x,t) \xrightarrow{P} \eta_P \phi(-x,t) = \eta_P \phi(\tilde{x})$$

where $\eta_P^2 = 1$ and if the field ϕ is scalar then $\eta_P = 1$, whereas if ϕ is a pseudoscalar then $\eta_P = -1$. The quantity η_P is called *intrinsic parity* of the scalar / pseudoscalar field.

Example 12

The $\lambda(\phi^\dagger\phi)^2$ scalar field theory is P-invariant, indeed starting from its Lagrangian

$$\mathcal{L} = \mathcal{L}(x) = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 \phi^\dagger \phi - \frac{\lambda}{4!}(\phi^\dagger \phi)^2$$

(in the first equality we showed explicitly the coordinate dependence of the Lagrangian) we can see that the Lagrangian is not invariant under parity:

$$\mathcal{L}(x) \xrightarrow{P} \mathcal{L}(\tilde{x}) = \mathcal{L}(-x,t) \neq \mathcal{L}(x)$$

but the corresponding action is invariant as the measure compensates the transformation of the Lagrangian:

$$S = \int_{M_4} \mathrm{d}^4x\, \mathcal{L}(x,t) = \int_{M_4} \mathrm{d}^4x\, \mathcal{L}(-x,t) = S_P$$

where in the second step we performed the coordinates transformation $x \to -x$.

Time Reversal

Time reversal is a transformation that belong to the improper non-orthochronus set: $\Lambda_0)^0 < 0$, $\det \Lambda = -1$. Parity transformation is given by the inversion of time-like coordinate:

$$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad \text{such that} \qquad x^\mu \xrightarrow{T} x'^\mu = -x_\mu = \left\{ \begin{array}{c} x'^0 = -x^0 \\ x'^i = x^i \end{array} \right\} = -\tilde{x}$$

In classical mechanics time reversal acts as

$$\boldsymbol{x} \xrightarrow{T} \boldsymbol{x} \qquad \boldsymbol{p} \xrightarrow{T} -\boldsymbol{p} \qquad \boldsymbol{J} \xrightarrow{T} -\boldsymbol{J}$$

so that the helicity is left invariant, as $\boldsymbol{J} \cdot \boldsymbol{p} \xrightarrow{T} \boldsymbol{J} \cdot \boldsymbol{p}$.

A scalar field ϕ under T transforms as

$$\phi(x,t) \xrightarrow{T} \eta_T \phi(x,-t) = \eta_T \phi(-\tilde{x})$$

where similarly to η_T we have $\eta_T^2 = 1$.

Exercise 16

Show that the $\lambda(\phi^\dagger\phi)^2$ theory is T-invariant. As in the previous example, you will see that the Lagrangian is not invariant, but the action is.

Charge conjugation

Charge conjugation is a new symmetry that cannot be described by a Lorentz transformation and has no equivalent in non-relativistic theories.

A scalar field ϕ under C transforms as

$$\phi(x,t) \xrightarrow{C} \eta_C \phi^*(x,t) \equiv \phi^C$$

with $\eta_C^2 = 1$. This means that the charge conjugations transforms a (complex) field of charge $+q$ into a field of charge $-q$.

Exercise 17

Show that the $\lambda(\phi^\dagger\phi)^2$ theory is C-invariant.

9.1.2 Discrete symmetries and vector bosons

C, P, T transformations can be easily generalized to the vector boson case.

A vector field V_μ under parity transform as:

$$V^\mu(x,t) \xrightarrow{T} \eta_P V_\mu(-x,t) = \eta_P V_\mu(\tilde{x})$$

with $\eta_P^2 = 1$. If $\eta_P = +1$ then V_μ is a *vector* while if $\eta_P = -1$ then V_μ is said a *pseudovector* or *axial vector*. Notice that lowering the μ index we changed the sign of the space-like components of the vector field.

A vector field V_μ under time reversal trasnforms as

$$V^\mu(x,t) \xrightarrow{T} -\eta_T V_\mu(x,-t) = -\eta_T V_\mu(-\tilde{x})$$

with $\eta_T^2 = 1$. Combining the minus sign with the lowered index we actually changed the sign of the time-like component of the vector field.

Notice that previous transformations are more or less obvious from the Lorentz transformations of a vector:

$$x^\mu \xrightarrow{P} x_\mu \qquad \text{and} \qquad x^\mu \xrightarrow{T} -x_\mu$$

The sign difference in V_μ is due to the fact that T acts as as *antiunitary operator*[I]

A vector field V_μ under C transforms as

$$V^\mu(x,t) \xrightarrow{C} \eta_C (V^\mu(x,t))^\dagger$$

with $\eta_C^2 = 1$. Again we define C as the transformation that sends a field in its conjugated.

Exercise 18

Show that YM theories (for example $SU(N)$) are invariant under C, P, and T.

9.1.3 Discrete symmetries and Dirac fields

The action of discrete symmetries on spinorial fields involves also transformations of them spinorial indices, as each components of spinors have an intrinsic meaning that may be related to these transformations.

A Dirac field under parity transforms as

$$\psi(x,t) \xrightarrow{P} \eta_P \gamma^0 \psi(-x,t) = \eta_P \gamma^0 \psi(\tilde{x}) = \mathcal{P}\psi(\tilde{x})$$

with $\eta_P^4 = 1$ since ψ appear in terms of bilinears and $\mathcal{P} = \eta_P \gamma^0$. For the adjoint field we have then

$$\bar{\psi}(x,t) \xrightarrow{P} \eta_P^* \bar{\psi}(-x,t)\gamma^0 = \eta_P^* \bar{\psi}(\tilde{x})\gamma^0 = \bar{\psi}(\tilde{x})\mathcal{P}^\dagger$$

[I] See Schwartz discussion in sec. 11.6.

A Dirac field under time reversal transforms as

$$\psi(x,t) \xrightarrow{T} \eta_T \gamma_1\gamma_3 \psi(x,-t) = \mathcal{T}\psi(-\tilde{x})$$

with $\eta_T^4 = 1$ and $\mathcal{T} = \eta_T\gamma_1\gamma_3$ in the standard parametrization of the gamma-matrices. For the adjoint field:

$$\bar{\psi}(x,t) \xrightarrow{T} \eta_T^* \bar{\psi}(x,-t)\gamma_3\gamma_1 = \bar{\psi}(-\tilde{x})\mathcal{T}^\dagger$$

where $\mathcal{T}^\dagger = \mathcal{T}^{-1}$ as $\gamma_i^\dagger = -\gamma_i$ for $i = 1,2,3$.

A Dirac field under charge conjugation transform as

$$\psi(x,t) \xrightarrow{C} \eta_C i\gamma_2\gamma_0\bar{\psi}^T = \mathcal{C}\bar{\psi}^T \equiv \psi^C$$

with $\eta_C^4 = 1$ and $\mathcal{C} = i\eta_C\gamma_2\gamma_0$. The conjugated field transforms as

$$\bar{\psi}(x,t) \xrightarrow{C} -\psi^T\mathcal{C}^\dagger = \bar{\psi}^C$$

Moreover one can verify the following properties:

$$\mathcal{C}^\dagger = \mathcal{C}^{-1} = \mathcal{C}^T = -\mathcal{C} \qquad \text{and} \qquad \mathcal{C}^{-1}\gamma^\mu\mathcal{C} = -\gamma_\mu^T$$

Notice that parity and time reversal transformations can be proven using the generators of the Lorentz group, while this is not possible for the charge conjugation. Indeed there is not an unique parametrization for the charge conjugation, and in literature different parametrization of this transformations can be found.

In the following exercize one can check that the transformations defined for ψ and $\bar{\psi}$ under charge conjugation are consistent:

Exercise 19

Derive the E.M. coupled Dirac equation for ψ^C.

Solution: Let's set $m = 0$ (not need to discuss $\pm m$), then

$$i\not{D}\psi = (i\not{\partial} - q\not{A})\psi = 0 \quad \longleftrightarrow \quad \bar{\psi}i\overleftarrow{\not{D}} = \bar{\psi}(i\overleftarrow{\not{\partial}} + q\not{A}) = 0$$

By transposing the last equation and multiplying by $-\mathcal{C}$ we obtain

$$-\mathcal{C}(i\not{\partial}^T + q\not{A}^T)\bar{\psi}^T = (i\not{\partial} + q\not{A})\psi^C = 0$$

Hence we see that ψ^C has opposite charge of ψ, as ψ^C obeys the Dirac equation with the sign of the charge inverted respect to the Dirac equation of ψ.

9.2 CPT transformations (and bilinears)

As we always have to deal with bilinears of Dirac fields, let's study their properties under C,P, T transformations. We define the following bilinears:

Scalar	$S(x) = \bar{\psi}(x)\psi(x)$
Pseudoscalar	$P(x) = \bar{\psi}(x)\gamma_5\psi(x)$
Vector	$V^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$
Axial vector (or Pseudovector)	$A^\mu(x) = \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$
Rank 2 antisymmetric tensor	$T^{\mu\nu}(x) = \bar{\psi}(x)\sigma^{\mu\nu}\psi(x)$

Notice that the set of 16 matrices ($\gamma^\mu = 4$ matrices, $\sigma^{\mu\nu} = 6$ matrices)

$$\Gamma = \{\mathbb{1}, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu}\}$$

gives a basis for all possibile 4×4 matrices which corresponds to all possibile Dirac bilinears.

One can verify the following transformations rules

	$S(x)$	$P(x)$	$V^\mu(x)$	$A^\mu(x)$	$T^{\mu\nu}(x)$
P	$S(\tilde{x})$	$-P(\tilde{x})$	$V_\mu(\tilde{x})$	$-A_\mu(\tilde{x})$	$T_{\mu\nu}(\tilde{x})$
T	$S(-\tilde{x})$	$-P(-\tilde{x})$	$V_\mu(-\tilde{x})$	$A_\mu(-\tilde{x})$	$-T^{\mu\nu}(-\tilde{x})$
C	$S(x)$	$P(x)$	$-V^\mu(x)$	$A^\mu(x)$	$-T^{\mu\nu}(x)$
CPT	$S(-x)$	$P(-x)$	$-V^\mu(-x)$	$-A^\mu(-x)$	$T^{\mu\nu}(-x)$

where again $\tilde{x} = (x_0, -\boldsymbol{x})$ and CPT is the operator that combines the 3 symmetries. In the case of complex bilinears one have to add the conjugation in all the terms in the third and in the fourth rows.

Exercise 20

Show that bilinears transform exactly as the corresponding fields.

Solution:

$$\bar{\psi}(x)\psi(x) \xrightarrow{P} \bar{\psi}(\tilde{x})\mathcal{P}^\dagger\mathcal{P}\psi(\tilde{x}) = \bar{\psi}(\tilde{x})\psi(\tilde{x}) \qquad \text{(recall that } |\eta_P|^2 = 1 \text{ and } \mathcal{P}^\dagger\mathcal{P} = \mathbb{1})$$

$$\bar{\psi}(x)\gamma^\mu\psi(x) \xrightarrow{P} \bar{\psi}(\tilde{x})\mathcal{P}^\dagger\gamma^\mu\mathcal{P}\psi(\tilde{x}) = \bar{\psi}(\tilde{x})\gamma_\mu\psi(\tilde{x}) \qquad \text{(recall that } \gamma_0\gamma^\mu\gamma_0 = \gamma_\mu)$$

$$\bar{\psi}(x)\psi(x) \xrightarrow{C} \bar{\psi}^C(x)\psi^C(x) = -\psi^T\mathcal{C}^\dagger\mathcal{C}\bar{\psi}^T = -\psi^T\gamma_0\psi^* = \bar{\psi}(x)\psi(x) \qquad \text{(recall that } \{\psi, \psi^\dagger\} = 0)$$

$$\bar{\psi}(x)\gamma^\mu\psi(x) \xrightarrow{C} \bar{\psi}^C(x)\gamma^\mu\psi^C(x) = -\psi^T\mathcal{C}^\dagger\gamma^\mu\mathcal{C}\bar{\psi}^T = -\psi^T(-\gamma_\mu^\dagger)\bar{\psi}^T = -(\bar{\psi}\gamma_\mu\psi)^T = -\bar{\psi}(x)\gamma^\mu\psi(x)$$

One can similarly prove all the other bilinears.

9.3 The CPT theorem

The *CPT theorem* tells us that any relativistic QFT described by Hermitian and Lorentz invariant (under $\Lambda_+^\uparrow$) Lagrangian preserves the CPT symmetry.

We don't give a formal proof. But notice that the Lagrangian is a real scalar quantity, consequently

$$\mathcal{L}(x) \overset{CPT}{\to} \mathcal{L}(-x)$$

hence the Lagrangian is in general not invariant under CPT, whereas the CPT theorem tells us that the CPT is a symmetry of the action

$$S(x) \overset{CPT}{\to} S(x)$$

Moreover notice that CPT theorem does not hold for C, P, T independently (even though our theory may be invariant under some of these symmetries.

9.4 QED, QCD and EW properties under C, P, T transformations

Peskin sec. 20.3

Exercise 21

Show that QED and QCD are invariant under C,P,T transformations.
Solution:

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi - q\bar{\psi}A\!\!\!/\psi$$

Under a parity transformation one has:

$$\begin{aligned} F^{\mu\nu}(x) &\xrightarrow{P} F_{\mu\nu}(\tilde{x}) & \qquad A^{\mu}(x) &\xrightarrow{P} A_{\mu}(\tilde{x}) \\ \bar{\psi}\gamma^{\mu}\psi &\xrightarrow{P} \bar{\psi}(\tilde{x})\gamma_{\mu}\psi(\tilde{x}) & \qquad \partial_{\mu} &\xrightarrow{P} \partial^{\mu} \end{aligned}$$

$$\mathcal{L}^{P}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}(\tilde{x})F^{\mu\nu}(\tilde{x}) + \bar{\psi}(\tilde{x})(i\partial^{\mu}\gamma_{\mu})\psi(\tilde{x}) - q\bar{\psi}(\tilde{x})\gamma_{\mu}\psi(\tilde{x})A^{\mu}(\tilde{x}) = \mathcal{L}_{\text{QED}}(\tilde{x})$$

$$S^{P}_{\text{QED}} = \int \mathrm{d}^4x\, \mathcal{L}_{\text{QED}}(\tilde{x}) = \int \mathrm{d}^4x\, \mathcal{L}_{\text{QED}}(x) = S_{\text{QED}}$$

For the invariance under time reversal the proof is analogous. Under a charge conjugation transformation one has:

$$\begin{aligned} F^{\mu\nu}(x) &\xrightarrow{C} -F^{\mu\nu}(x) & \qquad A^{\mu}(x) &\xrightarrow{C} -A^{\mu}(x) \\ \bar{\psi}\gamma^{\mu}\psi &\xrightarrow{C} -\bar{\psi}(x)\gamma^{\mu}\psi(x) & \qquad \partial_{\mu} &\xrightarrow{C} \partial_{\mu} \end{aligned}$$

$$\mathcal{L}^{C}_{\text{QED}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi - q\bar{\psi}A\!\!\!/\psi$$

so $\mathcal{L}_{\text{QED}}$ is invariant.

9.4.1 Violation of CP of the EW sector of the Standard Model Lagrangian

Exercise 22

Let's study the $\mathcal{P}$ and $\mathcal{C}$ transformations for chiral currents:

$$\begin{aligned} \bar{\psi}_1\gamma^{\mu}P_L\psi_2 &\xrightarrow{P} \bar{\psi}_1(\tilde{x})\mathcal{P}^{\dagger}\gamma^{\mu}P_L\mathcal{P}\psi_2(\tilde{x}) = \bar{\psi}_1(\tilde{x})\gamma_{\mu}P_R\psi_2(\tilde{x}) \\ \bar{\psi}_1\gamma^{\mu}P_L\psi_2 &\xrightarrow{C} -\psi_1^T\mathcal{C}^{\dagger}\gamma^{\mu}P_L\mathcal{C}\bar{\psi}_2^T = -\psi_1^T\mathcal{C}^{\dagger}\gamma^{\mu}\mathcal{C}P_L\bar{\psi}_2^T = \psi_1^T\gamma^{\mu,T}P_L\bar{\psi}_2^T = \\ &\quad = -(\bar{\psi}_2P_L\gamma^{\mu}\psi_1) = -\bar{\psi}_2\gamma^{\mu}P_R\psi_1 \end{aligned}$$

Hence we have seen that under $\mathcal{C}$ and $\mathcal{P}$ transformations L, R currents are exchanged:

$$\begin{cases} L \xrightarrow{P} R \\ L \xrightarrow{C} -R \end{cases}$$

This implies that weak interactions violates both P (since left-handed particles are swept with right handed ones) and C (since in general right handed and left handed particles have different couplings), as we'll see better in this section.

From the first measure of Wu is known that weak interactions are C and P violating. Let's consider for example the charge current interactions in the mass basis

$$\mathcal{L}_{\text{CC}} = \frac{g}{\sqrt{2}}\left(\bar{D}'_L\gamma^{\mu}V^{\dagger}_{\text{CKM}}U'_L + \bar{E}'_L\gamma^{\mu}V^{\dagger}_{\text{PMNS}}N'_L\right)W^-_{\mu} + h.c.$$

By a P transformation one has

$$\frac{g}{\sqrt{2}}\left(\bar{D}'\gamma^{\mu}V^{\dagger}_{\text{CKM}}U'\right)_x \xrightarrow{P} \frac{g}{\sqrt{2}}\left(\bar{D}'\gamma^{\mu}_R V^{\dagger}_{\text{CKM}}U'\right)_{\tilde{x}}$$

and similarly by C transformation one has

$$\frac{g}{\sqrt{2}}\left(\bar{D}'\gamma^{\mu}_L V^{\dagger}_{\text{CKM}}U'_L\right)_x \xrightarrow{C} -\frac{g}{\sqrt{2}}\left(\bar{U}'\gamma^{\mu}_R V^{\dagger}_{\text{CKM}}D'\right)_x$$

hence we see that CC Lagrangian in not invariant under C, P, i.e. is $\not{C}, \not{P}$. For the NC sector as $c_L \neq c_R$ one obtains the same conclusion. Indeed, only the QED and QCD sectors are C and P since they are vector like. Conversely, chiral interactions violate both C and P.

If $V_{\rm CKM} \neq V_{\rm CKM}^\dagger$ then also CP is violated in the charged current sector. Let's single out the quark current (CC)

$$\mathcal{L}_{\rm CC} = \frac{g}{\sqrt{2}} \left(\bar{D}' \gamma_L^\mu V_{\rm CKM}^\dagger U' W_\mu^- + \bar{U}' \gamma_L^\mu V_{\rm CKM} D' W_\mu^+ \right)_x$$

where the h.c. piece has been exploited. Under a P transformation the CC Lagrangian becomes

$$\mathcal{L}_{\rm CC}^P = \frac{g}{\sqrt{2}} \left(\bar{D}' \gamma_R^\mu V_{\rm CKM}^\dagger U' W_\mu^- + \bar{U}' \gamma_R^\mu V_{\rm CKM} D' W_\mu^+ \right)_{\tilde{x}}$$

Applying now a C transformation one obtains

$$\mathcal{L}_{\rm CC}^{CP} = \frac{g}{\sqrt{2}} \left(\bar{U}' \gamma_L^\mu V_{\rm CKM}^\dagger D' W_\mu^+ + \bar{D}' \gamma_L^\mu V_{\rm CKM} U' W_\mu^- \right)_{\tilde{x}}$$

Hence $V_{\rm CKM} \neq V_{\rm CKM}^\dagger$ implies $\mathcal{L}_{\rm CC}^{CP} \neq \mathcal{L}_{\rm CC}$, i.e. the $V_{\rm CKM}$ and $V_{\rm PMNS}$ phases (if non-zero) are the sources of CP violation, $\not{C}\not{P}$. Moreover notice that all flavor and CP violation of the Standard model originates from the Yukawa sector, and this is called *CKM Ansatz*.

Exercise 23

Verify that the EW neutral sector beside being flavor conserving is also CP conserving.

Notice that for $N = 2$ families the Standard Model could not have any $\not{C}\not{P}$, since both $V_{\rm CKM}$ and $V_{\rm PMNS}$ are real matrices (the number of phases is $(N-1)(N-2)/2$).

The $\not{C}\not{P}$ is a fundamental ingredient for understanding the *baryon asymmetry* of the universe, i.e. why the universe is mainly made of matter rather than antimatter (CP transforms matter particles in anti-matter particles and vice versa). However the amount of $\not{C}\not{P}$ of the SM is not enough to explain all the universe baryon asymmetry. This problem might have two possible solutions: either the existence of new sources of CP violations in theories beyond the Standard Model or new paradigm of universe, where the big bang didn't provided only energy.

Chapter 10

The $V_{\rm CKM}$ and $V_{\rm PMNS}$ matrices

Schwartz sec. 29.3.2 - 29.3.4

The $V_{\rm CKM}$ and $V_{\rm PMNS}$ matrices are 3×3 unitary matrices that after the spinorial phases redefinitions depend on

$$\begin{cases} \dfrac{N(N-1)}{2} \overset{N=3}{=} & 3\,\text{moduli / angles} \\ \dfrac{(N-1)(N-2)}{2} \overset{N=3}{=} & 1\,\text{phase} \end{cases}$$

The standard parametrization of $V_{\rm CKM}$ and $V_{\rm PMNS}$

Let's define the following 3×3 rotations:

$$R_{12}(\theta_{12}) = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_{13}(\theta_{13},\delta) = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \qquad R_{23}(\theta_{23}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}$$

where $\theta_{ij}\in[0,\pi/2)$ and $\delta\in[0,2\pi)$.Then, conventionally one defines

$$V_{\rm CKM} = R_{23}(\theta_{23})\cdot R_{13}(\theta_{13},\delta)\cdot R_{12}(\theta_{12})$$

Notice that the order of rotations and the position of the $C\!\!\!/P$ phase is conventional.

For the leptonic sector there are some complications. If νs are Dirac spinors then leptonic and hadronic sectors are replicas. However, neutral νs can also be Majorana spinors and in such case two extra phases can appear in the mixing matrix:

$$U_{\rm PMNS} = \begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{i\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix} V_{\rm PMNS} = \phi_M(\alpha,\beta)\cdot R_{23}(\theta_{23})\cdot R_{13}(\theta_{13},\delta)\cdot R_{12}(\theta_{12})$$

where ϕ_M is a matrix which describes Majorana phases, and becomes the identity if we have Dirac particles.

The 2×4 parameters in $V_{\rm CKM}$ and $V_{\rm PMNS}$ are arbitrary parameters that have to be determined experimentally (at the same level of masses and gauge couplings).

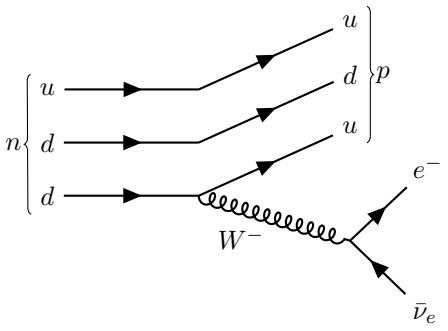
In the following we'll discuss the main phenomenological aspects of $V_{\rm CKM}$ and $V_{\rm PMNS}$.

10.1 The quark mixing matrix $V_{\rm CKM}$

The $V_{\rm CKM}$ entries are usually dubbed with the nouns of the up-down quarks it connects:

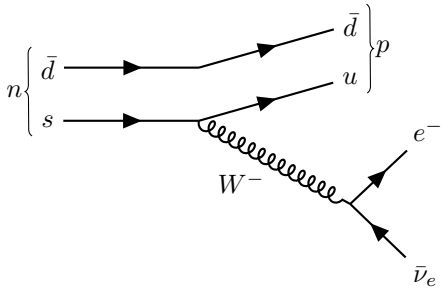
$$V_{\rm CKM} \equiv R_L S_L^\dagger = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = R_{23}\cdot R_{13}\cdot R_{12}$$

To determine the $V_{\rm CKM}$ entries one has to measure the corresponding meson/baryon decays. For instance, the measure of V_{ud} can be done through the measurement of the β-decay amplitude:



$$\propto |V_{ud}|^2 g^4$$

while the measurement of V_{us} can be done observing the K^0 decay:



$$\propto |V_{us}|^2 g^4$$

Notice that inside mesons / baryons the quarks are in the mass eigenstates, so we have to use the mass basis Feynman rules. From all available experimental determinations we obtain the following matrix, where phases are omitted:

$$|V_{\rm CKM}| = \begin{pmatrix} 0.97446 \pm 0.00010 & 0.22452 \pm 0.00044 & 0.00365 \pm 0.00012 \\ 0.22438 \pm 0.00044 & 0.97359 \pm 0.00010 & 0.04214 \pm 0.00076 \\ 0.00856 \pm 0.00024 & 0.04133 \pm 0.00074 & 0.999105 \pm 0.000032 \end{pmatrix}$$

Notice that the matrix is quasi diagonal (hence the angles of rotation are small) and is quasi-symmetric. In particular it gives the following determination of the standard parameters

$$\begin{aligned} \theta_{12} &= 13.02^\circ \pm 0.04^\circ \qquad & \theta_{23} &= 2.36^\circ \pm 0.08^\circ \\ \theta_{13} &= 0.20^\circ \pm 0.02^\circ \qquad & \delta_{(13)} &= 69^\circ \pm 5^\circ \end{aligned}$$

Since $\delta \neq 0, \pi$ we see that $V_{\rm CKM}$ violates the CP symmetry.

10.1.1 The Wolfestein parametrization

From the experimental results one can notice a well defined pattern/hierarchy between the coefficients V_{ij} in $V_{\rm CKM}$:

(i) the diagonal elements are $O(1)$

(ii) elements V_{12} and V_{21} are $O(10^{-1})$,

(iii) elements V_{23} and V_{32} are $O(10^{-2})$,

(iv) elements V_{13} and V_{31} are $O(10^{-3})$,

and, since

$$R_{23}(\theta_{23}) \cdot R_{13}(\theta_{13}, \delta) \cdot R_{12}(\theta_{12}) = \begin{pmatrix} c_{12}c_{13} & s_{12}s_{13} & s_{13}e^{-i\delta} \\ \bullet & \bullet & s_{23}c_{13} \\ \bullet & \bullet & c_{23}c_{13} \end{pmatrix}$$

(where $\bullet$ denots more complex terms, omitted), consequently

$$s_{13} \ll s_{23} \ll s_{12} < 1$$

To made explicit this hierarchical structure on V_{CKM} it is often convenient to use the **Wolfenstein parametrization**:

$$s_{12} = \frac{|V_{us}|}{(|V_{ud}|^2 + |V_{us}|^2)^{1/2}} \equiv \lambda \qquad s_{23} = \lambda \frac{|V_{cb}|}{|V_{us}|} \equiv A\lambda^2 \qquad s_{13} e^{i\delta} = V_{ub}^* \equiv A\lambda^3(\rho + i\eta)$$

Then V_{CKM} can be written in terms of the 4 independent parameters $(A, \lambda, \rho\eta)$ as

$$V_{\text{CKM}} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4)$$

Exercise 24

Prove that up to $O(\lambda^4)$ the V_{CKM} matrix is approximated by the Wolfenstein parametrization.

If we write $\lambda = s_{12} \equiv \sin(\theta_C)$ then the angle θ_C is said **Cabibbo angle**. Usually we call λ itself the Cabibbo angle.

Experimental fits give the following determination of the Wolfenstein parameters

$$\begin{aligned} \lambda &= 0.22453 \pm 0.00044 & A &= 0.836 \pm 0.015 \\ \rho &= 0.122 \pm 0.018 & \eta &= 0.355 \pm 0.012 \end{aligned}$$

The hierarchy of V_{CKM} is then explicitated in terms of the small Cabibbo angle:

(i) the diagonal elements are $O(1)$

(ii) elements V_{12} and V_{21} are $O(\lambda)$,

(iii) elements V_{23} and V_{32} are $O(\lambda^2)$,

(iv) elements V_{13} and V_{31} are $O(\lambda^3)$.

10.1.2 V_{CKM} and unitary triangles

One of the best constraints of V_{CKM} comes from imposing unitary conditions. Notice that *CKM ansatz* says that all flavor and CP violation in the SM comes from V_{CKM} (and V_{PMNS}). By construction V_{CKM} (and V_{PMNS}) is a unitary 3×3 matrix:

$$V_{\text{CKM}}^\dagger V_{\text{CKM}} = \mathbb{1}_{3\times 3} = V_{\text{CKM}} V_{\text{CKM}}^\dagger$$

Multiplying rows and columns of a 3×3 unitary matrix one obtains 12 relations:

(i) 6 normalization conditions:

$$\sum_{i=1}^{3} |V_{ij}|^2 = 1 = \sum_{j=1}^{3} |V_{ij}|^2$$

(ii) 6 orthogonality conditions

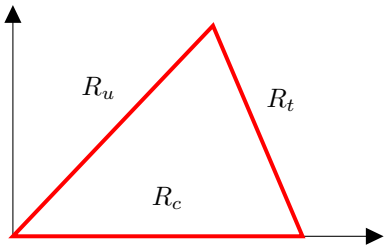
$$\sum_{i=1}^{3} V_{ij} V_{ik}^* = 0 = \sum_{j=1}^{3} V_{ij} V_{kj}^*$$

Orthogonality conditions are particularly useful as one can precisely check the Standard Model Yukawa sector and eventually have hints of physics beyond the SM.

Orthogonality relations have a useful graphical representation. Lets consider $\sum_{i=1}^{3} V_{ij}V_{ik}^* = 0$ with $j = 1, k = 3$. This relation corresponds to multiply the 3^{rd} row of $V_{\text{CKM}}^\dagger$ with the 1^{st} columns of V_{CKM}:

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0 = R_u + R_c + R_t$$

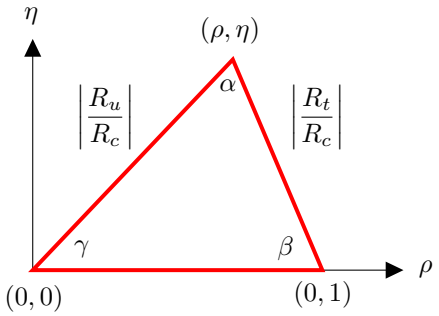
In the complex space the values R_u, R_c, R_t can be regarded as vectors whose sum vanishes, so pictorially the unitary conditions can be represented by a triangle whose edges are these three vectors:



In the Wolfenstein parametrization one has

$$\begin{cases} R_u = V_{ud}V_{ub}^* = \left(1 - \frac{\lambda^2}{2}\right)(A\lambda^3)(\rho + i\eta) \simeq A\lambda^3(\rho + i\eta) \\ R_c = V_{cd}V_{cb}^* = -A\lambda^3 \\ R_t = V_{td}V_{tb} = A\lambda^3(1 - \rho - i\eta) \end{cases}$$

hence by renormalizing with respect to $|R_c|$ the previous triangle reads



The measure of the 2 sides (the third is chosen to be 1 by normalization) and the 3 angles provides a stringent check of the unitary of V_{CKM}

$$\left|\frac{R_u}{R_c}\right| = \left|\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right| \simeq \frac{1}{\lambda}\left|\frac{V_{ub}}{V_{cb}}\right| = [\rho^2 + \eta^2]^{1/2}$$

$$\left|\frac{R_t}{R_c}\right| = \left|\frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*}\right| \simeq \frac{1}{\lambda}\left|\frac{V_{td}}{V_{cb}}\right| = [(1-\rho)^2 + \eta^2]^{1/2}$$

$$\alpha = \text{Arg}\left[-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right]$$

$$\beta = \text{Arg}\left[-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right]$$

$$\gamma = \text{Arg}\left[-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right]$$

together with the constraints $\alpha + \beta + \gamma = \pi$. In terms of ρ and η one can also write

$$\sin 2\beta = \frac{2\eta(1-\rho)}{(1-\rho)^2 + \eta^2} \qquad \sin 2\gamma = \frac{2\eta\rho}{\rho^2 + \eta^2}$$

which shows that if $\eta = 0$ then $\beta = \gamma = 0$. If either $\sin 2\beta \neq 0$ or $\sin 2\gamma \neq 0$ then $\eta \neq 0$ and we have a CP violation.

By measuring all the $V_{\rm CKM}$ elements (and all their combinations) we over-constrained the unitary triangle and one have a consistency check. Any deviation from the unitary constraints is would be a signal of new physics coming out.

10.2 The lepton mixing matrix $V_{\rm PMNS}$

If neutrinos are Dirac fermions, all the discussion done for the quarks on the mixing matrix can be extended to the $V_{\rm PMNS}$:

$$V_{\rm PMNS} \equiv R_{23}(\theta_{23})R_{13}(\theta_{13}, \delta)R_{12}(\theta_{12})$$

However the fact that neutrinos are practically massless makes very complicated the measure of the $V_{\rm PMNS}$ entries. Let's recall the $\mathcal{L}_{\rm CC}$ interaction term in the mass basis:

$$\bar{N}'_L \gamma^\mu V_{\rm PMNS} E'_L W^+_\mu + h.c. = \sum_{i,j} \bar{\nu}'_i V_{ij} e'_j W^+_\mu + h.c.$$

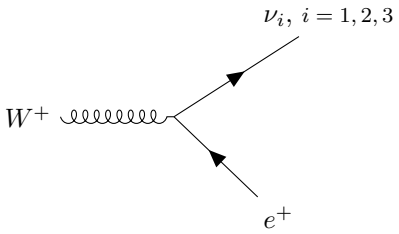
and let's make the following definitions:

(i) $\nu'_i \equiv (\nu_1, \nu_2, \nu_3)$ are **neutrino mass eigenstates**;

(ii) $\nu_\alpha \equiv (\nu_e, \nu_\mu, \nu_\tau)$ are **neutrino flavor eigenstates**;

(iii) $\ell'_i \equiv (e, \mu, \tau)$ are the physical leptons we defined in QED (mass eigenstates)

Let's consider the $W^+_\mu \longrightarrow e^+ \nu_i$, the W^+ decay into physical (mass) states. If we are not able to identify νs mass eigenstates the process we measure is



$$= \left(\sum_{i=1}^{3} \bar{\nu}_i V_{ie} \right) \gamma^\mu_L \, e^+ W^+_\mu$$

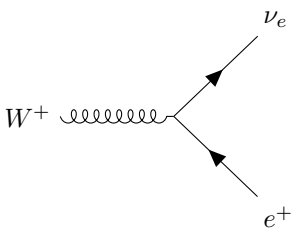
since we are not able to identify which neutrino is produced in the final state. In other words we measure only the interaction between the electron and the ν_e flavor eigenstate, which means:

$$\nu_e \equiv \sum_{i=1}^{3} V^*_{ei} \nu_i$$

while more in general

$$\nu_\alpha \equiv \sum_{i=1}^{3} V^*_{\alpha i} \nu_i \qquad \text{where } \alpha = (e, \mu, \tau)$$

This means that neutrinos are produced and detected only as flavor eigenstates..



$$= \bar{\nu}_e \gamma^\mu_L e^+ W^+_\mu$$

Remember that if $m_{\nu_i} = 0$ then $V_{\rm PMNS} = \mathbb{1}$. We just seen that $V_{\rm PMNS}$ parameters cannot be measured in weak interactions (such as $W^\pm$ decays etc.).

10.2.1 Neutrino oscillations (in vacuum)

The only way we can at present have access to neutrinos (differences of masses and) mixing is thanks to **neutrino oscillations**. To describe this phenomena, one can use pure quatum mechanics description of waves propagations. Let's suppose that flavors are different than mass eigenstates:

(i) $|\nu_\alpha\rangle \equiv (\nu_e, \nu_\mu, \dots)$ is the **flavor basis**,

(ii) $|\nu_i\rangle \equiv (\nu_1, \nu_2, \dots)$ is the **mass basis**.

The flavor state can be described as a linear combination of mass eigenstates

$$|\nu_\alpha\rangle = \sum U_{\alpha i} |\nu_i\rangle$$

Notice that if neutrinos are Majorana particles then $U = \phi_M^\dagger \cdot V_{\text{PMNS}}^\dagger$. However as we will see neutrino oscillations do not depend on the Majorana phases ϕ_M.

Suppose that a ν_α is produced at $t = 0$. We want to measure how many ν_β are detected at a $t = t_0 \neq 0$. The evolution of ν_α can be obtained by solving the ν_i Schrödinger equation

$$\begin{cases} i\frac{\mathrm{d}}{\mathrm{d}t} |\nu_i\rangle = \mathcal{H} |\nu_i\rangle = E_i |\nu_i\rangle \\ |\nu_i(t)\rangle = \exp\{-iE_i t\} |\nu_i(0)\rangle \end{cases}$$

together with

$$|\nu_\alpha(t)\rangle = \sum U_{\alpha i} \exp\{-iE_i t\} |\nu_i(0)\rangle$$

The probability to detect a flavor β (by weak interactions) at a certain $t \neq 0$ is then

$$\begin{aligned} P_{\alpha\beta}(t) \equiv |\langle \nu_\beta | \nu_\alpha(t)\rangle|^2 &= \Big| \sum_i U_{\alpha i} \exp\{-iE_i t\} \langle \nu_\beta | \nu_i(0)\rangle \Big|^2 \\ &= \Big| \sum_{ij} U^*_{\beta j} \exp\{-iE_i t\} U_{\alpha i} \langle \nu_j(0) | \nu_i(0)\rangle \Big|^2 = \Big| \sum_i U^*_{\beta i} \exp\{-iE_i t\} U_{\alpha i} \Big|^2 \end{aligned}$$

Exercise 25

Show that $P_{\alpha\beta}$ does not depend on the diagonal Majorana phases matrices ϕ_M.

The general probability formula can be easily obtained as follows

$$\begin{aligned} P_{\alpha\beta} &= \Big| \sum_i U^*_{\beta i} \exp\{-iE_i t\} U_{\alpha i} \Big|^2 = \sum_{ij} \exp\{-i(E_i - E_j)t\} U^*_{\beta i} U_{\alpha_i} U_{\beta j} U^*_{\alpha j} \\ &= \sum_{ij} W^{ij}_{\alpha\beta} \exp\{-i(E_i - E_j)t\} = \sum_{i=j} W^{ij}_{\alpha\beta} + \sum_{i<j} \left(\mathrm{Re}\left(W^{ij}_{\alpha\beta}\right) \cos(\Delta E_{ij} t) + \mathrm{Im}\left(W^{ij}_{\alpha\beta}\right) \sin(\Delta E_{ij}) \right) \\ &= \sum_{i=j} W^{ij}_{\alpha\beta} + \sum_{i\neq j} \mathrm{Re}\left(W^{ij}_{\alpha\beta}\right) - 4 \sum_{i<j} \mathrm{Re}\left(W^{ij}_{\alpha\beta}\right) \sin^2\left(\frac{\Delta E_{ij} t}{2}\right) + 2 \sum_{i<j} \mathrm{Im}\left(W^{ij}_{\alpha\beta}\right) \sin(\Delta E_{ij} t) \end{aligned}$$

where $W^{ij}_{\alpha\beta} = U^*_{\beta i} U_{\alpha_i} U_{\beta j} U^*_{\alpha j}$. The fist two terms in the last line give $\delta_{\alpha\beta}$ due to unitarity of $U_{\alpha i}$. The other pieces can be rewritten using the following relation:

$$\Delta E_{ij} = E_i - E_j \simeq \left(p + \frac{1}{2}\frac{m_1^2}{p}\right) - \left(p + \frac{1}{2}\frac{m_2^2}{p}\right) \simeq \frac{\Delta m^2_{ij}}{2E}$$

where $p \simeq E$ as $m_i \ll p$ (notice that the 3 mass eigenstates have same value fo p, since we are propagating just one wave). As $c = 1$ we can also replace $t \to L$. The probability of oscillations then reads

$$P_{\alpha\beta} = \delta_{\alpha\beta} - 4 \sum_{i<j} \mathrm{Re}\left(W^{ij}_{\alpha\beta}\right) \sin^2\left(\frac{\Delta m^2_{ij} L}{4E}\right) \pm 2 \sum_{i<j} \mathrm{Im}\left(W^{ij}_{\alpha\beta}\right) \sin\left(\frac{\Delta m^2_{ij} L}{2E}\right)$$

The $\pm$ sign refers either to neutrino $(+)$ or antineutrino $(-)$. In fact under CP one has $U \to U^\dagger$ and so $W^{ij}_{\alpha\beta} \to W^{ij\,*}_{\alpha\beta}$, then the minus sign comes out due to $\mathrm{Im}(\bullet)$ after the conjugation.

Measuring the oscillation probability one has access to

(i) V_{PMNS} parameters through $W^{ij}_{\alpha\beta}$,

(ii) Δm^2_{ij} mass difference.

The explicit 3 neutrinos oscillation probability without any approximation is quite large. We can also show this result for 2 families.

Exercise 26

Derive $P_{\alpha\beta}$ for two neutrinos.
Solution:

The V_{PMNS} matrix for $N = 2$ depends only on 1 angle and 0 phases:

$$\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} = \begin{pmatrix}U_{e1} = c_\theta & U_{e2} = s_\theta\\ U_{\mu 1} = -s_\theta & U_{\mu 2} = c_\theta\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} \quad \to \quad \begin{cases}\nu_e(+) = c_\theta e^{-iE_1 t}\nu_1 + s_\theta e^{-iE_2 t}\nu_2\\ \nu_\mu(+) = -s_\theta e^{-iE_1 t}\nu_1 + c_\theta e^{-iE_2 t}\nu_2\end{cases}$$

The $P(\nu_\mu \to \nu_e)$ appearance probability is

$$\begin{aligned}P_{21} &= |\,\langle\nu_e|\nu_\mu(+)\rangle\,|^2 = \left|-s_\theta c_\theta e^{-iE_1 t} + s_\theta c_\theta e^{-iE_2 t}\right|^2 = 2s^2_\theta c^2_\theta - 2s^2_\theta c^2_\theta \cos(\Delta E_{12} t)\\ &= 2s^2_\theta c^2_\theta - 2s^2_\theta c^2_\theta + 4s^2_\theta c^2_\theta \sin^2\left(\frac{\Delta E_{12} t}{2}\right) = \sin^2 2\theta \sin^2\left(\frac{\Delta m^2_{12} L}{4E}\right)\end{aligned}$$

The $P(\nu_\mu \to \nu_e)$ disappearance probability is, due to unitarity (conservation of probability),

$$P_{11} = |\,\langle\nu_e|\nu_e(+)\rangle\,|^2 = 1 - P_{21} = 1 - \sin^2 2\theta \sin^2\left(\frac{\Delta m^2_{12} L}{4E}\right)$$

As for $N = 2$ the matrix V_{PMNS} is real, it is immediate to recover the antineutrino probability

$$\begin{cases}P(\bar\nu_\mu \to \bar\nu_e) = P(\nu_\mu \to \nu_e)\\ P(\bar\nu_e \to \bar\nu_e) = P(\nu_e \to \nu_e)\end{cases}$$

10.2.2 Experimental determination of V_{PMNS}

One can use many sources of neutrino fluxes:

(i) solar neutrinos (from the Sun are mainly ν_e),

(ii) atmospherical neutrinos (from cosmic rays are mainly ν_μ),

(iii) accelerators and reactors (all the three types ν_e, ν_μ, ν_τ).

By measuring all appearance and disappearance probabilities of neutrino oscillations one can obtain the following results:

(i) $\Delta m^2_{21} = \Delta m^2_{\mathrm{Sun}} = (7.53 \pm 0.18) \times 10^{-5}\mathrm{eV}$

(ii) $\Delta m^2_{31} = \Delta m^2_{\mathrm{Atm}} = (2.44 \pm 0.06) \times 10^{-3}\mathrm{eV}$

(iii) $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$

(iv) $\sin^2(2\theta_{23}) > 0.92$

(v) $\sin^2(2\theta_{13}) = 0.093 \pm 0.008$

(vi) δ is not determined yet

Notice that angles θ_{12} and θ_{23} are big, while θ_{13} is small. Since the values for V_{PMNS} are different from values of V_{CKM}, in this case we cannot use Wolfenstein parametrization.

If $m_{\nu_i} = 0$ then one could not have observed neutrino oscillations (neither if $m_{\nu_i} = m_{\nu_j}$). If the mass basis is the same as the flavor basis we should have $\theta_{ij} = 0$ as $V_{\text{PMNS}} = \mathbb{1}$ and all $P(\nu_\alpha \to \nu_\beta) = 0$ for $\alpha \neq \beta$. This prove that neutrinos are massive, and their masses are different.

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