

Notes on Electromagnetism

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Lesson 1: Coulomb's Law

Coulomb's Law (Legge di Coulomb) describes the electrostatic force (forza elettrostatica) between two static charged particles (particelle cariche). The force \vec{F}_{12} that a charge (carica) q_1 exerts on a second charge q_2 is directly proportional to the product of the charges and inversely proportional to the square of the distance between them. The force is directed along the line connecting the two charges.

The vector form of the equation is:

$$\vec{F}_{12} = k \frac{q_1 q_2}{R^2} \hat{R}_{12} \quad (1)$$

Here, k is Coulomb's constant (costante di Coulomb), R is the distance between the charges, and \hat{R}_{12} is the unit vector (versore) pointing from q_1 to q_2 . The law is reciprocal, meaning the force exerted by q_2 on q_1 is equal in magnitude and opposite in direction, so $\vec{F}_{12} = -\vec{F}_{21}$.

Coulomb's constant is expressed in terms of the vacuum permittivity (permittività del vuoto), ϵ_0 , as:

$$k = \frac{1}{4\pi\epsilon_0} \quad (2)$$

Following the 2019 redefinition of the SI base units (Unità di base del SI), the elementary charge (carica elementare) e is a defining constant. As a result, ϵ_0 is now an experimentally determined value with an associated uncertainty, not a defined quantity.

The electric force is significantly stronger than the gravitational force. For instance, the electric force (forza elettrica) F_E between a proton and an electron is many orders of magnitude greater than the gravitational force (forza gravitazionale) F_G between them:

$$F_G = G \frac{m_p m_e}{R^2} \approx 10^{-47} N \quad F_E = \frac{1}{4\pi\epsilon_0} \frac{e^2}{R^2} \approx 10^{-7} N \quad (3)$$

Lesson 2: Electric Flux and Gauss's Law

The concept of electric flux (Flusso Elettrico) quantifies the flow of an electric field (campo elettrico) through a surface. For a uniform electric field \vec{E} and a flat surface area \vec{A} , the flux is the scalar product $\Phi_E = \vec{E} \cdot \vec{A}$. For non-uniform fields and curved surfaces, the flux is found by integrating the field over the surface:

$$\Phi_E = \int_S \vec{E} \cdot d\vec{A} \quad (4)$$

This integral effectively sums the component of the electric field perpendicular to the surface at every point.

Gauss's Law (Legge di Gauss), a cornerstone of electrostatics (elettrostatica) and one of the four fundamental Maxwell's Equations (Equazioni di Maxwell), relates the electric flux through a closed surface (superficie chiusa) to the net electric charge (carica elettrica) enclosed within it. The law states that the net electric flux is directly proportional to the enclosed charge.

In its integral form, Gauss's Law is expressed as:

$$\Phi_E = \oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} \quad (5)$$

The integral is performed over a conceptual closed surface known as a Gaussian surface (superficie Gaussiana), and Q_{enc} is the total charge it encloses. By applying the divergence theorem (teorema della divergenza), which connects the surface integral of a vector field to the volume integral of its divergence, we can derive the differential form of Gauss's Law:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad (6)$$

Here, ρ represents the volume charge density (densità di carica volumica). This form locally relates the divergence of the electric field to the charge density at that point.

A key application of Gauss's Law is to calculate electric fields in situations with high symmetry. By choosing a Gaussian surface that mirrors the symmetry of the charge distribution (distribuzione di carica), the flux integral becomes simple to solve. For example, to find the field from an infinite line of charge with a uniform linear charge density (densità di carica lineare) λ , we use a cylindrical Gaussian

surface. The symmetry dictates that the electric field must point radially outward. The flux through the top and bottom caps of the cylinder is zero, and the flux through the side wall is $E \cdot (2\pi rL)$. The enclosed charge is λL . Applying Gauss's Law, $E(2\pi rL) = \lambda L/\epsilon_0$, yields the field:

$$E = \frac{\lambda}{2\pi\epsilon_0 r} \quad (7)$$

Lesson 3: Applications of Gauss's Law

Gauss's Law is a powerful tool for calculating the electric field for charge distributions with a high degree of symmetry, such as spherical, cylindrical, or planar symmetry. The symmetry of the charge density ρ is inherited by the electric field \vec{E} , which simplifies the calculation.

A classic example is the field of a uniformly charged sphere (Sfera Carica Uniformemente) of radius R and total charge Q . Due to spherical symmetry (simmetria sferica), the electric field must be radial, $\vec{E} = E(r)\hat{r}$. By choosing a spherical Gaussian surface of radius r , the flux integral simplifies to $\oint_S \vec{E} \cdot d\vec{S} = E(r) \cdot (4\pi r^2)$. Outside the sphere ($r > R$), the enclosed charge is the total charge Q . Applying Gauss's law gives $E(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}$, which is identical to the field of a point charge (carica puntiforme) Q at the origin. Inside the sphere ($r \leq R$), the enclosed charge is only the fraction of charge within the radius r , $Q_{\text{int}} = Q \frac{r^3}{R^3}$. Gauss's law then yields $E(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^3} r$, showing that the field increases linearly from the center.

This method can be extended to other symmetric configurations. For a charged spherical shell (Guscio Sferico Carico), the electric field inside the inner cavity is zero. For a sphere with an off-center cavity, the superposition principle (principio di sovrapposizione) can be used by treating the cavity as a superposition of a sphere with negative charge density over a larger sphere with positive charge density. This reveals a uniform electric field inside the cavity. Gauss's law also efficiently re-derives the fields for an infinite line of charge (Linea Infinita di Carica) and an infinite plane of charge (Piano Infinito di Carica).

The concept of divergence (divergenza) of a vector field (campo vettoriale) measures the field's tendency to originate from or converge to a point. For an electric field, divergence signifies the presence of a charge source or sink. The Divergence Theorem (Teorema della Divergenza) relates the flux of a field through a closed surface to the integral of its divergence over the enclosed volume. Combining this theorem with Gauss's Law leads to its differential form:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad (8)$$

This is the first of Maxwell's Equations, stating that the divergence of the electric field at any point is proportional to the charge density at that same point.

Lesson 4: Superposition, Conductors, and Plasma Oscillations

The superposition principle (principio di sovrapposizione) is a fundamental concept for calculating electric fields from complex charge distributions. A powerful illustration is the case of two overlapping spheres of the same radius R , one with uniform positive charge density $+\rho_0$ and the other with uniform negative charge density $-\rho_0$. The total electric field is the vector sum of the fields from each sphere. Using the known result for the field inside a uniformly charged sphere, $\vec{E} = \frac{\rho_0}{3\epsilon_0} \vec{r}$, we can calculate the field in the overlapping region. If the centers of the spheres are separated by a vector $\vec{\delta}$, the field in the overlapping region is found to be uniform and constant:

$$\vec{E}_{\text{overlap}} = -\frac{\rho_0}{3\epsilon_0} \vec{\delta} \quad (9)$$

This model provides an excellent analogy for the behavior of conductors (conduttori) in an external electric field. In a conductor, conduction electrons are free to move. When an external field \vec{E}_{ext} is applied, the "sea" of electrons displaces slightly, creating a charge separation, or polarization (polarizzazione). This separation induces an internal electric field (campo elettrico interno) \vec{E}_{int} that opposes the external field. The displacement continues until the net electric field inside the conductor is zero, a condition known as electrostatic equilibrium (equilibrio elettrostatico). At this point, $\vec{E}_{\text{ext}} + \vec{E}_{\text{int}} = 0$. The conductor effectively shields its interior from the external static electric field.

If the external field is suddenly removed, the displaced electron sea is no longer in equilibrium and experiences a restoring force from the fixed positive ions. This force acts like a spring, leading to oscillations. The equation of motion for the electron sea is that of a simple harmonic oscillator (oscillatore armonico semplice). This collective, high-frequency oscillation of the electrons is a quantum mechanical effect known as a plasma oscillation (oscillazione di plasma). The characteristic angular frequency of this oscillation is the plasma frequency (ω_p), a property of the material that depends on the density of conduction electrons.

$$\omega_p^2 = \frac{n_e e^2}{m_e \epsilon_0} \quad (10)$$

This phenomenon is crucial in understanding the optical properties of metals.

Lesson 5: Maxwell's First Equation and Electrostatic Potential

This lesson begins by deriving the first of Maxwell's Equations, which concerns the divergence of the electric field. Starting from the divergence theorem (teorema della divergenza), which relates the flux of a vector field through a closed surface to the integral of its divergence over the enclosed volume:

$$\oint_S \vec{E} \cdot d\vec{S} = \int_\tau (\nabla \cdot \vec{E}) d\tau \quad (11)$$

We can combine this with Gauss's Law (Teorema di Gauss), which states that the flux of the electric field through a closed surface is proportional to the enclosed charge Q_{int} :

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{\text{int}}}{\epsilon_0} \quad (12)$$

By equating these two expressions for the flux, and expressing the enclosed charge as the integral of the volume charge density (densità volumetrica di carica) ρ , we get:

$$\int_\tau (\nabla \cdot \vec{E}) d\tau = \int_\tau \frac{\rho}{\epsilon_0} d\tau \quad (13)$$

Since this equality must hold for any arbitrary volume τ , the integrands themselves must be equal. This gives us the differential form of Gauss's Law, also known as the first of Maxwell's Equations for the electric field in a vacuum:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad (14)$$

This is a local equation (equazione locale) because it relates the divergence of the electric field at a specific point in space to the charge density at that very same point. This is in contrast to the integral form of Gauss's Law, which is non-local as it relates the field on a surface to the total charge contained within it.

To handle point charges (cariche puntiformi) within the framework of continuous charge densities, we introduce the Dirac delta function (delta di Dirac), which is technically a distribution (distribuzione). It is defined as being zero everywhere except at the origin, where it is infinite, yet its integral over all space is one. This allows us to define the volume charge density for a point charge q_0 located at position \vec{r}_0 as:

$$\rho(\vec{r}) = q_0 \delta(\vec{r} - \vec{r}_0) \quad (15)$$

Next, we explore the line integral (integrale di linea) of the electrostatic field between two points, A and B. For the field generated by a single point charge, the result of this integral depends only on the start and end points, not on the path taken. A field with this property is called a conservative field (campo conservativo). A direct consequence is that the line integral of the electrostatic field around any closed loop is zero:

$$\oint \vec{E} \cdot d\vec{l} = 0 \quad (16)$$

This conservative nature allows us to define a scalar field called the electrostatic potential (potenziale elettrostatico), V . The potential difference (differenza di potenziale) between two points is related to the line integral of the electric field:

$$V(A) - V(B) = \int_A^B \vec{E} \cdot d\vec{l} \quad (17)$$

From this relationship, we can derive the differential form, which connects the electric field to the potential via the gradient (gradiente):

$$\vec{E} = -\nabla V \quad (18)$$

The electric field points from regions of higher potential to regions of lower potential.

Surfaces on which the potential V is constant are known as equipotential surfaces (superfici equipotenziali). The electric field lines are always perpendicular to these surfaces. No work (lavoro) is done when moving a charge along an equipotential surface, as the displacement is always orthogonal to the electric force.

Finally, we define the electrostatic potential energy (energia potenziale elettrostatica) U of a point charge q placed in a potential V as $U = qV$. This energy can be interpreted as the work done by an external force to bring the charge from a reference point (where the potential is defined as zero) to its current position. The unit of potential is the Volt (Volt), defined as a Joule per Coulomb. In atomic and nuclear physics, a convenient unit of energy is the electron-volt (elettronvolt), which is the energy gained by an electron when it moves through a potential difference of one volt.

Lesson 6: Electrostatic Potential Energy and The Electric Dipole

Electrostatic Potential Energy

We continue our discussion on electrostatic potential energy. For a single point charge Q , the electrostatic energy is given by the product of the charge and the potential V at its location: $U = Q \cdot V$. This corresponds to the negative of the work done by the field's forces to bring the charge from infinity to that point, denoted as L_∞ .

Let's extend this to a system of two point charges, Q_1 and Q_2 , subject to the Coulomb force. To assemble this configuration, we can imagine bringing the charges from infinity. Placing the first charge, Q_1 , requires no work as there is no pre-existing electric field. However, positioning the second charge, Q_2 , involves work done by the electric field generated by Q_1 .

The work L_{field} done by the field to bring Q_2 from infinity to a distance R_{12} from Q_1 is calculated as:

$$L_{\text{field}} = \int_{\infty}^{R_{12}} \vec{F} \cdot d\vec{l} = \int_{\infty}^{R_{12}} \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} \hat{R} \cdot d\vec{l}$$

Since the electrostatic field is conservative, the integral is path-independent. We can choose a radial path, so $d\vec{l} = dR \hat{R}$.

$$L_{\text{field}} = \int_{\infty}^{R_{12}} \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} dR = \left[-\frac{Q_1 Q_2}{4\pi\epsilon_0 R} \right]_{\infty}^{R_{12}} = -\frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}}$$

The external work L_{ext} required to assemble the configuration at a constant, near-zero velocity (implying zero change in kinetic energy, $\Delta K = 0$) is the negative of the work done by the field: $L_{\text{ext}} = -L_{\text{field}}$.

$$L_{\text{ext}} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}}$$

The potential energy U_2 of the two-charge system is defined as this external work, which is equivalent to the negative of the work done by the field:

$$U_2 = -L_{\text{field}} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}}$$

For a system of three point charges (Q_1, Q_2, Q_3), the total potential energy is the sum of the work required to assemble them. We first bring in Q_1 (zero work), then Q_2 (work U_2), and finally Q_3 in the presence of the fields from Q_1 and Q_2 . The external work to bring in Q_3 is:

$$L_3 = \frac{Q_1 Q_3}{4\pi\epsilon_0 R_{13}} + \frac{Q_2 Q_3}{4\pi\epsilon_0 R_{23}}$$

The total potential energy U_3 of the system is the sum of the energies of all pairs:

$$U_3 = U_2 + L_3 = \frac{1}{4\pi\epsilon_0} \left(\frac{Q_1 Q_2}{R_{12}} + \frac{Q_1 Q_3}{R_{13}} + \frac{Q_2 Q_3}{R_{23}} \right)$$

This can be generalized for a system of N point charges. The total electrostatic potential energy is the sum over all unique pairs of charges:

$$U_N = \frac{1}{2} \sum_{i=1}^N \sum_{j=1, j \neq i}^N \frac{Q_i Q_j}{4\pi\epsilon_0 R_{ij}}$$

The factor of $\frac{1}{2}$ is included to correct for double-counting each pair in the summation (e.g., the pair Q_1, Q_2 is counted as both $i=1, j=2$ and $i=2, j=1$). This expression can be rewritten as:

$$U_N = \frac{1}{2} \sum_{i=1}^N Q_i \left(\sum_{j=1, j \neq i}^N \frac{Q_j}{4\pi\epsilon_0 R_{ij}} \right)$$

The term in the parenthesis is the electric potential V_i at the position of charge Q_i due to all other charges. Therefore:

$$U_N = \frac{1}{2} \sum_{i=1}^N Q_i V_i$$

This represents the interaction energy (energia di interazione) of the N point charges. It is the energy required to assemble the configuration of charges, assuming they are pre-existing.

For continuous charge distributions, the sum becomes an integral. The potential energy U for a volume charge distribution with density ρ is:

$$U = \frac{1}{2} \int_V \rho(\vec{r}) V(\vec{r}) d\tau$$

where the integral is over the volume V where the charge density ρ is non-zero. For surface and linear distributions, the expressions are analogous:

$$U = \frac{1}{2} \int_S \sigma V dS \quad \text{and} \quad U = \frac{1}{2} \int_L \lambda V dl$$

This formulation for continuous distributions represents the total electrostatic potential energy, which includes both the interaction energy between different infinitesimal charge elements and the self-energy (autoenergia) required to assemble each infinitesimal charge element itself.

As an example, the energy of a uniformly charged sphere of radius R and total charge Q can be calculated. The result is:

$$U = \frac{1}{4\pi\epsilon_0} \frac{3}{5} \frac{Q^2}{R}$$

This result can be used to define a conventional radius for the electron, known as the classical electron radius (raggio classico dell'elettrone). By equating the electrostatic energy of a sphere with charge e to the electron's rest energy $m_e c^2$, we can solve for the radius. Omitting the $\frac{3}{5}$ factor for an order-of-magnitude estimate:

$$m_e c^2 \approx \frac{1}{4\pi\epsilon_0} \frac{e^2}{r_e} \implies r_e = \frac{1}{4\pi\epsilon_0} \frac{e^2}{m_e c^2} \approx 2.8 \times 10^{-15} \text{ m}$$

We can express the electrostatic energy in terms of the electric field. Starting from $U = \frac{1}{2} \int \rho V d\tau$ and using Maxwell's first equation, $\nabla \cdot \vec{E} = \rho/\epsilon_0$, we substitute $\rho = \epsilon_0 \nabla \cdot \vec{E}$:

$$U = \frac{\epsilon_0}{2} \int_V (\nabla \cdot \vec{E}) V d\tau$$

Using the vector identity $\nabla \cdot (V \vec{E}) = (\nabla V) \cdot \vec{E} + V(\nabla \cdot \vec{E})$ and the relation $\vec{E} = -\nabla V$, we get $V(\nabla \cdot \vec{E}) = \nabla \cdot (V \vec{E}) + E^2$. Substituting this into the energy integral:

$$U = \frac{\epsilon_0}{2} \int_V (\nabla \cdot (V \vec{E}) + E^2) d\tau = \frac{\epsilon_0}{2} \int_V \nabla \cdot (V \vec{E}) d\tau + \frac{\epsilon_0}{2} \int_V E^2 d\tau$$

Applying the divergence theorem to the first term converts the volume integral into a surface integral over the boundary S of the volume V :

$$\int_V \nabla \cdot (V \vec{E}) d\tau = \oint_S (V \vec{E}) \cdot d\vec{S}$$

If we extend the integration volume to all of space, the surface S goes to infinity. At large distances r from a finite charge distribution, V falls off as $1/r$ and E as $1/r^2$, while the surface area S grows as r^2 . The product $V\vec{E}$ thus decreases as $1/r^3$, and the surface integral $\oint (V\vec{E}) \cdot d\vec{S} \propto (1/r^3) \cdot r^2 = 1/r$, which tends to zero as $r \rightarrow \infty$. Therefore, the first term vanishes, and the total energy is:

$$U = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 d\tau$$

This allows us to define the electrostatic energy density (densità di energia elettrostatica) as:

$$u_e = \frac{1}{2} \epsilon_0 E^2$$

This interpretation is conceptually powerful, as it associates energy with the electric field itself, distributed throughout space. The total energy is the integral of this density over all space where the field exists.

The Electric Dipole

An electric dipole (dipolo elettrico) is a system of two equal and opposite point charges, $+Q$ and $-Q$, separated by a small distance vector $\vec{\delta}$, which points from the negative to the positive charge by convention. The electric dipole moment (momento di dipolo elettrico) is defined as:

$$\vec{p} = Q\vec{\delta}$$

The unit is Coulomb-meter (C·m). Dipoles are fundamental in physics and chemistry. Many molecules, like water (H₂O), are polar molecules, meaning they have a permanent electric dipole moment. This explains phenomena like water's high boiling point and its effectiveness as a solvent. The forces between dipoles, such as hydrogen bonds (legami idrogeno), are crucial for the structure of DNA and proteins.

The electric potential V at a point P far from a dipole ($r \gg \delta$) can be approximated. The exact potential is the sum of the potentials from the two charges:

$$V(\vec{r}) = \frac{Q}{4\pi\epsilon_0 r_+} - \frac{Q}{4\pi\epsilon_0 r_-} = \frac{Q}{4\pi\epsilon_0} \frac{r_- - r_+}{r_- r_+}$$

For large distances r , we can approximate $r_- r_+ \approx r^2$ and $r_- - r_+ \approx \delta \cos \alpha$, where α is the angle between the dipole moment \vec{p} and the position vector \vec{r} .

$$V(\vec{r}) \approx \frac{Q\delta \cos \alpha}{4\pi\epsilon_0 r^2} = \frac{p \cos \alpha}{4\pi\epsilon_0 r^2}$$

This can be written compactly using the dot product:

$$V(\vec{r}) = \frac{\vec{p} \cdot \vec{r}}{4\pi\epsilon_0 r^3}$$

The potential of a dipole falls off as $1/r^2$. The electric field \vec{E} is found by taking the negative gradient of the potential, $\vec{E} = -\nabla V$. In spherical coordinates, with \vec{p} along the z-axis, the potential is $V(r, \theta) = \frac{p \cos \theta}{4\pi\epsilon_0 r^2}$. The field components are:

$$\begin{aligned} E_r &= -\frac{\partial V}{\partial r} = \frac{2p \cos \theta}{4\pi\epsilon_0 r^3} \\ E_\theta &= -\frac{1}{r} \frac{\partial V}{\partial \theta} = \frac{p \sin \theta}{4\pi\epsilon_0 r^3} \\ E_\phi &= 0 \end{aligned}$$

The electric field of a dipole falls off as $1/r^3$, which is faster than the $1/r^2$ decay of a single point charge. This is because, from far away, the two opposite charges appear to cancel each other out.

A dipole placed in a uniform external electric field \vec{E}_{ext} experiences no net force, as the forces on the positive and negative charges are equal and opposite ($\vec{F}_{\text{net}} = Q\vec{E}_{\text{ext}} - Q\vec{E}_{\text{ext}} = 0$). However, it does experience a torque (momento delle forze):

$$\vec{\tau} = \vec{p} \times \vec{E}_{\text{ext}}$$

This torque tends to align the dipole moment with the external field.

In a non-uniform electric field, the forces on the two charges are no longer equal, resulting in a net force on the dipole. This force can be expressed as:

$$\vec{F} = (\vec{p} \cdot \nabla) \vec{E}_{\text{ext}}$$

This is also equal to the gradient of the scalar product of \vec{p} and \vec{E} :

$$\vec{F} = \nabla(\vec{p} \cdot \vec{E}_{\text{ext}})$$

The potential energy U of a dipole in an external electric field is the work required to orient it in the field. It is given by:

$$U = -\vec{p} \cdot \vec{E}_{\text{ext}}$$

The energy is minimized ($U = -pE$) when the dipole is aligned with the field ($\theta = 0$), which is a stable equilibrium. The energy is maximized ($U = +pE$) when it is anti-aligned ($\theta = \pi$), which is an unstable equilibrium. The force on the dipole can also be derived from the potential energy: $\vec{F} = -\nabla U = \nabla(\vec{p} \cdot \vec{E}_{\text{ext}})$.

Lesson 8: Curl, Stokes' Theorem, and Conductors

The Curl of a Vector Field

Today we introduce the curl (rotore), a vector differential operator that, when applied to a vector field, yields another vector field. To build intuition, consider the velocity field of a fluid. The curl of the velocity field describes the fluid's circulation density (densità di circolazione). If we place a small, rough ball in the fluid, fixed at its center but free to rotate, it will start spinning if the curl at that point is non-zero. The axis of rotation gives the direction of the curl vector, determined by the right-hand rule. A non-zero curl indicates the presence of vortices in the fluid; ideal, non-viscous fluids are irrotational (their curl is zero everywhere).

In Cartesian coordinates, the curl of a vector field $\vec{F} = (F_x, F_y, F_z)$ is written as $\nabla \times \vec{F}$ and can be calculated as the formal determinant of a matrix:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{k}$$

An important identity in vector calculus is that the curl of the gradient of any scalar field V is always zero, provided the second partial derivatives are continuous:

$$\nabla \times (\nabla V) = 0$$

This is because the mixed partial derivatives cancel out (e.g., $\frac{\partial^2 V}{\partial y \partial z} = \frac{\partial^2 V}{\partial z \partial y}$).

Stokes' Theorem

Stokes' theorem (teorema di Stokes), also known as the curl theorem, relates the circulation of a vector field along a closed path to the flux of its curl. Consider an oriented closed curve γ and any open surface S that has γ as its boundary. Stokes' theorem states that for a vector field \vec{V} :

$$\oint_{\gamma} \vec{V} \cdot d\vec{l} = \int_S (\nabla \times \vec{V}) \cdot d\vec{S}$$

The left side is the circulation of \vec{V} along the closed path γ . The right side is the flux of the curl of \vec{V} through the surface S . The orientation of the surface normal $d\vec{S}$ is related to the orientation of the path γ by the right-hand rule: if your fingers curl in the direction of the path, your thumb points in the direction of the normal.

This theorem provides a coordinate-independent definition of the curl. The component of the curl along a normal vector \hat{n} can be defined as the limit of the circulation per unit area as the area shrinks to a point:

$$(\nabla \times \vec{V}) \cdot \hat{n} = \lim_{S \rightarrow 0} \frac{1}{S} \oint_{\gamma} \vec{V} \cdot d\vec{l}$$

The Third Maxwell's Equation for Electrostatics

We know that the electrostatic field \vec{E} is conservative. This means its circulation around any closed path γ is zero:

$$\oint_{\gamma} \vec{E} \cdot d\vec{l} = 0$$

By applying Stokes' theorem, we can relate this to the curl of \vec{E} :

$$\oint_{\gamma} \vec{E} \cdot d\vec{l} = \int_S (\nabla \times \vec{E}) \cdot d\vec{S} = 0$$

Since this must hold for any arbitrary closed path γ and any surface S bounded by it, the integrand itself must be zero. This gives us the third of Maxwell's equations for static fields:

$$\nabla \times \vec{E} = 0$$

This is a local equation, stating that the curl of the electrostatic field is zero at every point in space. A field with zero curl is called an irrotational field (campo irrotazionale). This result is consistent with the fact that $\vec{E} = -\nabla V$, because the curl of a gradient is always zero.

Conductors in Electrostatic Equilibrium

Conductors (conduttori) are materials containing mobile charges (e.g., electrons) that are free to move under the influence of an electric field. Electrostatic equilibrium is the state where there is no net motion of charge within the conductor.

For this to be true, the electric field inside a conductor must be zero. If it were not, the free charges would experience a force and move, contradicting the definition of equilibrium.

$$\vec{E}_{\text{inside}} = 0$$

A direct consequence, via Gauss's Law, is that the net charge density ρ inside the conductor must also be zero. If we take any Gaussian surface entirely within the conductor, the flux through it is zero because $\vec{E} = 0$. Therefore, the enclosed charge must be zero. Any net charge placed on a conductor must reside entirely on its outer surface in a very thin layer. We describe this with a surface charge density σ .

At the interface between two media, the tangential component of the electrostatic field is continuous. Consider a small rectangular loop straddling the interface. The circulation $\oint \vec{E} \cdot d\vec{l} = 0$. By making the sides perpendicular to the interface infinitesimally short, their contribution vanishes. The circulation is then due to the two long sides parallel to the interface:

$$E_{1,t}\Delta l - E_{2,t}\Delta l = 0 \implies E_{1,t} = E_{2,t}$$

For the interface between a conductor (medium 2) and a vacuum (medium 1), we have $\vec{E}_2 = 0$. Therefore, the tangential component of the electric field just outside the conductor must be zero: $E_{1,t} = 0$. This means the electric field at the surface of a conductor is always perpendicular to the surface. Since the electric field lines are always orthogonal to equipotential surfaces, it follows that the surface of a conductor in electrostatic equilibrium is an equipotential surface.

Coulomb's Theorem and Conductor Properties

Coulomb's Theorem (Teorema di Coulomb) relates the electric field just outside a conductor to the local surface charge density σ . By applying Gauss's law to a small cylindrical "pillbox" that straddles the surface, we find:

$$\oint \vec{E} \cdot d\vec{S} = EA = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0} \implies E = \frac{\sigma}{\epsilon_0}$$

The field is directed along the outward normal \hat{n} , so $\vec{E} = \frac{\sigma}{\epsilon_0} \hat{n}$.

The potential is constant throughout the entire conductor (both surface and interior). There is, however, a potential difference between the interior (V_i) and the exterior surface (V_e), related to the work function (funzione lavoro), $W = e(V_i - V_e)$, which is the energy required to extract an electron from the conductor. In our problems, when we speak of the "potential of a conductor," we refer to the potential of its surface, V_e .

A cavity inside a conductor is shielded from external static electric fields. This is the principle of the Faraday cage (gabbia di Faraday). If a conductor with an empty cavity is placed in an external field, charges will rearrange on the outer surface to make $\vec{E} = 0$ inside the conductor material. Applying Gauss's law to a surface within the conductor material that encloses the cavity shows that the net charge on the inner surface of the cavity must be zero. Further analysis shows the field inside the empty cavity is also zero.

The power of points (effetto delle punte) describes the tendency of charge to accumulate on parts of a conductor with a small radius of curvature (i.e., sharp points). This leads to a much stronger electric field at these points. For two conducting spheres of radii R_1 and R_2 connected by a wire, they will be at the same potential. This implies $V = \frac{Q'_1}{4\pi\epsilon_0 R_1} = \frac{Q'_2}{4\pi\epsilon_0 R_2}$. The surface charge densities are related by $\sigma'_1/\sigma'_2 = R_2/R_1$. The sphere with the smaller radius has the higher charge density and thus a stronger electric field at its surface ($E \propto \sigma$). This is why electrical discharges, like sparks, tend to occur from sharp points.