Classical Mechanics Notes

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Class Information

First Semester

Legranges Mechanics, Tensors, Small Variations, Small Osilations and Hamiltonian Mechanics

Second Semester

Restricted relativity and Statistical Mechanics

Office Hours

Monday at 16:30 at 161 first floor, building C

1 Recap

1.1 Kinematics

The position of a particle is described by a vector $\mathbf{x}(t)$, which is a function of time. The velocity of the particle is the time derivative of the position:

$$\mathbf{v}(t) = \frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}}(t) \tag{1}$$

The acceleration is the time derivative of the velocity:

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \ddot{\mathbf{x}}(t) \tag{2}$$

1.2 Angular Momentum

2 Dynamical Systems of N Particles

For a system of N particles, the position of the a-th particle is given by $\mathbf{x}_a(t)$, where $a = 1, \ldots, N$. The equation of motion for the a-th particle is given by Newton's second law:

$$m_a \ddot{\mathbf{x}}_a = \mathbf{F}_a \tag{3}$$

where m_a is the mass of the a-th particle and $\mathbf{F}_a$ is the total force acting on it.

The total force $\mathbf{F}_a$ can be split into external forces, $\mathbf{F}_a^e$, and internal forces, $\mathbf{F}_{ab}$, which are the forces exerted by particle b on particle a:

$$\mathbf{F}_a = \mathbf{F}_a^e + \sum_{b \neq a} \mathbf{F}_{ab} \tag{4}$$

The total momentum of the system is the sum of the individual momenta:

$$\mathbf{P} = \sum_{a=1}^{N} \mathbf{p}_a = \sum_{a=1}^{N} m_a \dot{\mathbf{x}}_a \tag{5}$$

The time derivative of the total momentum is equal to the sum of all external forces acting on the system. Assuming that the internal forces cancel out (due to Newton's third law, $\mathbf{F}_{ab} = -\mathbf{F}_{ba}$):

$$\frac{d\mathbf{P}}{dt} = \sum_{a=1}^{N} \mathbf{F}_a^e \tag{6}$$

Lesson 2: Simple Pendulum with a Moving Pivot

We consider a simple pendulum of length l and mass m. Its pivot point is not fixed but moves along the horizontal axis according to a given function of time, $f(t)$. This is an example of a system with a rheonomic constraint (vincolo reonomo), as the constraint depends explicitly on time.

Let θ be the angle the pendulum makes with the vertical. The position of the mass m can be described by the coordinates (x, y):

$$\begin{aligned} x(t) &= f(t) + l\sin\theta(t) \\ y(t) &= -l\cos\theta(t) \end{aligned}$$

This system has one degree of freedom, which can be described by the generalized coordinate (coordinata generalizzata) θ.

The constraint equation is $(x - f(t))^2 + y^2 = l^2$. Since this is an algebraic equation relating the coordinates, the constraint is holonomic (olonomo). Because it contains an explicit time dependence through $f(t)$, it is also rheonomic. A constraint independent of time is called scleronomic (scleronomo). Constraints that cannot be written as an algebraic equation are called non-holonomic (anolonomi).

To find the equation of motion, we can use the Lagrangian approach. First, we find the kinetic energy (energia cinetica), T. The velocity components are:

$$\begin{aligned} \dot{x} &= \dot{f}(t) + l\dot{\theta}\cos\theta \\ \dot{y} &= l\dot{\theta}\sin\theta \end{aligned}$$

The kinetic energy is $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$:

$$\begin{aligned} T &= \frac{1}{2}m\left((\dot{f} + l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2\right) \\ &= \frac{1}{2}m\left(\dot{f}^2 + 2l\dot{f}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta\right) \\ &= \frac{1}{2}m\left(\dot{f}^2 + 2l\dot{f}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right) \end{aligned}$$

The potential energy (energia potenziale), U, due to gravity is $U = mgy$:

$$U = -mgl\cos\theta$$

The Lagrangian (Lagrangiana) is $L = T - U$:

$$L = \frac{1}{2}m(\dot{f}^2 + 2l\dot{f}\dot{\theta}\cos\theta + l^2\dot{\theta}^2) + mgl\cos\theta$$

The equation of motion is given by the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

We calculate the partial derivatives:

$$\begin{aligned} \frac{\partial L}{\partial\dot{\theta}} &= ml\dot{f}\cos\theta + ml^2\dot{\theta} \\ \frac{\partial L}{\partial\theta} &= -ml\dot{f}\dot{\theta}\sin\theta - mgl\sin\theta \end{aligned}$$

Taking the total time derivative of the first expression:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = m(l\ddot{f}\cos\theta - l\dot{f}\dot{\theta}\sin\theta + l^2\ddot{\theta})$$

Substituting into the Lagrange equation:

$$m(l\ddot{f}\cos\theta - l\dot{f}\dot{\theta}\sin\theta + l^2\ddot{\theta}) - (-ml\dot{f}\dot{\theta}\sin\theta - mgl\sin\theta) = 0$$

$$ml^2\ddot{\theta} + ml\ddot{f}\cos\theta + mgl\sin\theta = 0$$

Dividing by ml^2, we get the final equation of motion:

$$\ddot{\theta} + \frac{g}{l}\sin\theta + \frac{\ddot{f}(t)}{l}\cos\theta = 0$$

This equation describes the oscillation of the pendulum, driven by the acceleration of its pivot point.

Lesson 3: The Double Pendulum and Generalized Coordinates

We now consider a more complex system: the double pendulum. It consists of a mass m_1 suspended from a fixed pivot by a massless rod of length l_1, and a second mass m_2 suspended from m_1 by a massless rod of length l_2. The system moves in a vertical plane.

This system has two degrees of freedom. We can choose the angles θ_1 and θ_2 as our generalized coordinates (coordinate generalizzate). θ_1 is the angle the first rod makes with the vertical, and θ_2 is the angle the second rod makes with the vertical.

The Cartesian positions of the two masses can be expressed in terms of these generalized coordinates. Let the pivot be at the origin $(0,0)$. The position of the first mass, m_1, is:

$$x_1 = l_1 \sin\theta_1$$
$$y_1 = -l_1 \cos\theta_1$$

The position of the second mass, m_2, is found by adding the displacement from m_1:

$$x_2 = x_1 + l_2 \sin\theta_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$
$$y_2 = y_1 - l_2 \cos\theta_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$$

The entire state of the system at any time is completely determined if we know the values of $\theta_1(t)$ and $\theta_2(t)$.

The goal of Lagrangian mechanics is to describe the dynamics of the system using these generalized coordinates. The method involves two key scalar functions: the kinetic energy (energia cinetica), T, and the potential energy (energia potenziale), U.

The kinetic energy is the sum of the kinetic energies of the two masses, $T = T_1 + T_2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$. To calculate this, we need the velocities, which are the time derivatives of the positions. The velocities will depend on $\theta_1, \theta_2, \dot\theta_1, \dot\theta_2$. Therefore, the total kinetic energy is a function of the generalized coordinates and their time derivatives:

$$T = T(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2)$$

The potential energy depends on the vertical positions of the masses, $U = U_1 + U_2 = m_1 g y_1 + m_2 g y_2$. It is a function of the generalized coordinates only:

$$U = U(\theta_1, \theta_2)$$

In classical mechanics, the dynamics of many systems can be elegantly described by a single function called the Lagrangian (Lagrangiana), L, which is defined as the difference between the kinetic and potential energy:

$$L = T - U$$

For the double pendulum, the Lagrangian is $L(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2)$.

The equations of motion for the system are obtained from the Lagrangian using the principle of stationary action, which leads to the Euler-Lagrange equations. There will be one equation for each generalized coordinate. The derivation and application of these equations will be the subject of the next lesson.

Lesson 4: The Lagrangian Equations of Motion

In this lesson, we derive the Euler-Lagrange equations, which form the foundation of Lagrangian mechanics. The derivation starts from D'Alembert's principle (Principio di d'Alembert). For a system of particles, this principle states that the total virtual work done by the forces of inertia and the applied forces is zero:

$$\sum_i (m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i) \cdot \delta\mathbf{r}_i = 0$$

where $\delta\mathbf{r}_i$ is a virtual displacement (spostamento virtuale) of the i-th particle, consistent with the constraints of the system. For ideal constraints (vincoli ideali), the constraint forces do no work, so $\mathbf{F}_i$ can be replaced by the applied (or active) forces, $\mathbf{F}_i^a$.

We express the positions $\mathbf{r}_i$ and displacements $\delta\mathbf{r}_i$ in terms of n generalized coordinates q_α:

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \ldots, q_n, t) \quad \Longrightarrow \quad \delta\mathbf{r}_i = \sum_{\alpha=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \delta q_\alpha$$

D'Alembert's principle becomes:

$$\sum_{\alpha=1}^{n}\left(\sum_i m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_\alpha}-\sum_i\mathbf{F}_i^a\cdot\frac{\partial\mathbf{r}_i}{\partial q_\alpha}\right)\delta q_\alpha=0$$

Since the δq_α are independent, each term in the parenthesis must be zero. The second part defines the generalized force (forza generalizzata), Q_α:

$$Q_\alpha=\sum_i\mathbf{F}_i^a\cdot\frac{\partial\mathbf{r}_i}{\partial q_\alpha}$$

The first part involving acceleration can be rewritten in terms of the kinetic energy, $T=\sum_i\frac{1}{2}m_i\dot{\mathbf{r}}_i^2$. Through a standard but lengthy derivation, one can show that:

$$\sum_i m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_\alpha}=\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_\alpha}\right)-\frac{\partial T}{\partial q_\alpha}$$

Combining these results gives the Lagrange equations of motion:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_\alpha}\right)-\frac{\partial T}{\partial q_\alpha}=Q_\alpha$$

If the applied forces are conservative, they can be derived from a potential energy function $U(q_1,\ldots,q_n)$, such that $Q_\alpha=-\frac{\partial U}{\partial q_\alpha}$. The equations become:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_\alpha}\right)-\frac{\partial T}{\partial q_\alpha}=-\frac{\partial U}{\partial q_\alpha}$$

By defining the Lagrangian $L=T-U$, and noting that U does not depend on generalized velocities $\dot{q}_\alpha$, we have $\frac{\partial L}{\partial\dot{q}_\alpha}=\frac{\partial T}{\partial\dot{q}_\alpha}$ and $\frac{\partial L}{\partial q_\alpha}=\frac{\partial T}{\partial q_\alpha}-\frac{\partial U}{\partial q_\alpha}$. Substituting these into the equation yields the celebrated Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_\alpha}\right)-\frac{\partial L}{\partial q_\alpha}=0$$

As an example, let's reconsider the simple pendulum. The generalized coordinate is $q=\theta$. The kinetic and potential energies are:

$$T=\frac{1}{2}ml^2\dot{\theta}^2,\quad U=-mgl\cos\theta$$

The Lagrangian is $L=T-U=\frac{1}{2}ml^2\dot{\theta}^2+mgl\cos\theta$. Applying the Lagrange equation:

$$\frac{\partial L}{\partial\dot{\theta}}=ml^2\dot{\theta}$$
$$\frac{\partial L}{\partial\theta}=-mgl\sin\theta$$

$$\frac{d}{dt}(ml^2\dot{\theta})-(-mgl\sin\theta)=0\quad\Longrightarrow\quad ml^2\ddot{\theta}+mgl\sin\theta=0$$

This simplifies to the familiar equation $\ddot{\theta}+\frac{g}{l}\sin\theta=0$.

For the double pendulum, the Lagrangian $L=T-U$ can be constructed with:

$$T=\frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2+\frac{1}{2}m_2l_2^2\dot{\theta}_2^2+m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)$$
$$U=-(m_1+m_2)gl_1\cos\theta_1-m_2gl_2\cos\theta_2$$

Applying the Lagrange equation for each coordinate, θ_1 and θ_2, yields a system of two coupled, non-linear differential equations that describe the complex motion of the double pendulum.

Lesson 5: Symmetries and Conservation Laws

The Lagrangian formulation is not only elegant but also provides deep insights into the connection between symmetries and conservation laws.

First, we note a certain freedom in the definition of the Lagrangian. The equations of motion are not changed if we add a total time derivative of a function of coordinates and time, $F(q,t)$, to the Lagrangian. Let $L' = L + \frac{dF}{dt}$. The action integral changes by a constant:

$$S' = \int_{t_1}^{t_2} L' dt = \int_{t_1}^{t_2} L dt + \int_{t_1}^{t_2} \frac{dF}{dt} dt = S + F(q(t_2), t_2) - F(q(t_1), t_1)$$

Since the variation δS depends on fixed endpoints, the additional terms vanish, and the principle of stationary action $\delta S' = 0$ yields the same equations of motion. This is a gauge symmetry (simmetria di gauge) of the Lagrangian. Furthermore, multiplying the Lagrangian by a non-zero constant also leaves the equations of motion unchanged.

This connection between symmetries and physical laws is made precise by Noether's Theorem (Teorema di Noether). The theorem states that for every continuous symmetry of the action, there is a corresponding conserved quantity. A symmetry is a transformation of the coordinates that leaves the equations of motion invariant.

Let's consider a continuous transformation of the generalized coordinates parameterized by a small parameter ϵ:

$$q_\alpha(t) \to q'_\alpha(t) = q_\alpha(t) + \epsilon \psi_\alpha(q)$$

If the Lagrangian is invariant under this transformation (i.e., $\delta L = 0$), Noether's theorem guarantees that the quantity

$$I = \sum_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} \psi_\alpha$$

is conserved, meaning $\frac{dI}{dt} = 0$.

Let's examine some fundamental symmetries and their corresponding conservation laws:

Homogeneity of Time: If the Lagrangian does not explicitly depend on time ($\frac{\partial L}{\partial t} = 0$), the system is symmetric under time translation ($t \to t + \epsilon$). The corresponding conserved quantity is the energy of the system, which is defined by the Hamiltonian (Hamiltoniana):

$$H = \sum_\alpha \dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} - L$$

The conservation of energy, $\frac{dH}{dt} = 0$, is a direct consequence of the laws of physics being the same today as they were yesterday.

Homogeneity of Space: If the Lagrangian is invariant under a translation in space (e.g., $\mathbf{r}_i \to \mathbf{r}_i + \boldsymbol{\epsilon}$ for all particles), the system has translational symmetry. This occurs when there is no external potential, or the potential depends only on relative positions. The conserved quantity is the total linear momentum (quantità di moto). In the Lagrangian formalism, if a coordinate q_k does not appear in the Lagrangian (it is a cyclic or ignorable coordinate), then $\frac{\partial L}{\partial q_k} = 0$. The Lagrange equation for q_k becomes:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

This implies that the generalized momentum conjugate (momento coniugato) to q_k, defined as $p_k = \frac{\partial L}{\partial \dot{q}_k}$, is a constant of motion.

Isotropy of Space: If the Lagrangian is invariant under rotations, the system has rotational symmetry. This happens for central potentials, for example. The corresponding conserved quantity is the total angular momentum (momento angolare).

Noether's theorem is a cornerstone of modern physics, linking fundamental principles of symmetry to the conservation laws that govern the physical world.