Linear Algebra

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Pre-Det

Spaces (Spazi Vettoriali)

"I have not failed. I've just found 10,000 ways that won't work."

Basic Definitions

Subsets (Sottoinsiemi): This is a set where all the elements are contained within another set. For linear algebra it could be a collection of vectors within a vector space e.g $\mathbb{R}^3$

$$S = \{(x, y, 0) \,|x, y \in \mathbb{R}\}$$

Proper Subsets (Sottoinsiemi Propri): This is a subset that is always smaller than the original set. The exact definition is:

$$A \subsetneq B \tag{1}$$

if:

1. Containment: "Every element of A is also a element of B"

$$\forall x (x \in A \Longmapsto x \in B) \tag{2}$$

2. Not equal: "There exists at least one element in B that is not A"

$$\exists y (y \in B \wedge y \notin A) \tag{3}$$

Supersets (Soprainsiemi): This is the opposite of a subset. If $A \subset B$, then B is a superset of A

Linear Independence

Let V be a vector space over a field F. $S \subseteq V$ is linearly independent if the following conditions are true:

For a collection of vectors $\{v_1, v_2, ..., v_n\} \subseteq S$ and scalars $\{a_1, a_2, ..., a_n\} \in F$

$$a_1v_1 + a_2v_2 + ... + a_nv_n = 0 \tag{4}$$

i.e every single variant of a up to n has to equal zero. If it does not, it is linearly dependent

Dimensions of a space

The dimension (dimensione) of a vector space (spazio vettoriale) is the number of vectors in any basis (base) for that space. A basis (base) is a set of vectors that are linearly independent (linearmente indipendenti) and span (generano) the entire space.

Sum of subspace (Somma di sottospazi)

Let U and V be subspaces (sottospazi) of a vector space (spazio vettoriale) V over a field $\mathbb{F}$. The sum (somma) can be defined as:

$$U + V = \{u + v | u \in U, v \in V\} \tag{5}$$

Direct Sum (Somma diretta)

U+V is called a direct sum, if the intersection of U and V is trivial ???- (Further desc required) then:

$$U \cap V = \{0\}$$

The direct sum can be denoted as $U \oplus V$:–to be expanded upon

$$z = u + v$$

For subspaces $U_1, U_2, \ldots, U_n \subseteq V$, the sum $U_1 + U_2 + \cdots + U_n$ is a *direct sum* (somma diretta) if for each i,

$$U_i \cap \left(\sum_{j=1,\ j\neq i}^{n} U_j \right) = \{0\}.$$

The direct sum is denoted $U_1 \oplus U_2 \oplus \cdots \oplus U_n$, and every $z \in U_1 \oplus U_2 \oplus \cdots \oplus U_n$ has a unique representation:

$$z = u_1 + u_2 + \cdots + u_n \quad with u_i \in U_i \ \forall i.$$

Formal Uniqueness Condition: For $U \oplus V$,

$$\forall z \in U \oplus V,\ \exists!\,(u, v) \in U \times V such that z = u + v.$$

Here, $\exists!$ denotes "there exists exactly one."

Post-Det