Analisi 1

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Pre-Derivatives

Theorems, functions and axioms

“Calvin: You know, I don’t think math is a science, I think it’s a religion.

Hobbes: A religion?

Calvin: Yeah. All these equations are like miracles. You take two numbers and when you add them, they magically become one NEW number! No one can say how it happens. You either believe it or you don’t. [Pointing at his math book] This whole book is full of things that have to be accepted on faith! It’s a religion!”

Exponent and Logarithms

I dont really know what to do with this section. I might add actual explanations, but for not Ill just keep properties

Exponential Properties

For $a > 0$, $b > 0$, and $x, y \in \mathbb{R}$:

$$a^x a^y = a^{x+y}$$
$$\frac{a^x}{a^y} = a^{x-y}$$
$$(a^x)^y = a^{xy}$$
$$a^0 = 1$$
$$a^{-x} = \frac{1}{a^x}$$
$$(ab)^x = a^x b^x$$
$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

Def of Logarithms

For $a > 0$ $(a \neq 1)$, $b > 0$:

$$\log_a b = x \iff a^x = b$$

$$\log_a 1 = 0$$

$\log_a a = 1$

$a^{\log_a b} = b$

Logarithm Properties

For $a > 0$ $(a \neq 1)$, $x, y > 0$, $k \in \mathbb{R}$:

$\log_a(xy) = \log_a x + \log_a y$

$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

$\log_a(x^k) = k \log_a x$

$\log_a a^x = x$

Change of Base

For $a, b > 0$ $(a, b \neq 1)$, $c > 0$:

$$\log_a c = \frac{\log_b c}{\log_b a}$$

Special case: $\log_a b = \frac{1}{\log_b a}$

Natural Exponential and Logarithm

$e^x = \exp(x)$

$\ln x = \log_e x$

$e^{\ln x} = x$ for $x > 0$

$\ln(e^x) = x$ for $x \in \mathbb{R}$

Exponential-Logarithmic Equations

Key solving techniques:

If $a^x = a^y$ then $x = y$

If $\log_a x = \log_a y$ then $x = y$

To solve $a^{f(x)} = b$: take logarithms of both sides

To solve $\log_a f(x) = b$: rewrite as $f(x) = a^b$

Complex numbers

A complex number is defined, by $x, y \in \mathbb{R}$ and i as the imaginary unit.

$$z = x + iy \tag{1}$$

Imagine $\mathbb{R}$ covering the whole x axis, and $\mathbb{C}$ covering the whole y axis, that's the complex plane.

Operations

Addition

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \tag{2}$$

Subtraction

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2) \tag{3}$$

Multiplication

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \tag{4}$$

Complex Conjugate

$$\bar{z} = x - iy \tag{5}$$

Modulus

$$|z| = \sqrt{x^2 + y^2} \tag{6}$$

Inverse

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2} \tag{7}$$

Sums and Sequences

Series

Post-Derivatives

Derivatives

"The Difference Between the Almost Right Word and the Right Word Is Really a Large Matter—'Tis the Difference Between the Lightning Bug and the Lightning" - Mark Twain

$$f'(x) = lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{8}$$

Methods for Differentiation

Product rule

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x) \tag{9}$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \tag{10}$$

Chain Rule

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \tag{11}$$

Integrals

"If you're gonna shoot an elephant Mr. Schneider, you better be prepared to finish the job."

— Gary Larson, The Far Side

Riemann Integral

A Riemann integral is the the limit $f : [a, b] \to \mathbb{R}$ of all the Riemann sums between the points a and b. Defined as:

$$\int_a^b f(x)dx = lim_{||P|| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

Where:

n is the sub-intervals

$\Delta x_i$is the width of the i sub-interval

$c_i$is a sample point

*this exact definition will likely never be used in

Properties,

$$\text{Linearity:} \quad \int_a^b (\alpha f(x) + \beta g(x))\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx$$

$$\text{Additivity:} \quad \int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx \quad \text{for } c \in (a,b)$$

$$\text{Monotonicity:} \quad f(x) \le g(x)\ \forall x \in [a,b] \implies \int_a^b f(x)\, dx \le \int_a^b g(x)\, dx$$

$$\text{Bounds:} \quad m(b-a) \le \int_a^b f(x)\, dx \le M(b-a) \quad \text{where } m = \inf_{[a,b]} f,\ M = \sup_{[a,b]} f$$

$$\text{Integrability:} \quad f \text{ is Riemann integrable } \iff f \text{ bounded } \wedge\ f \text{ continuous a.e. on } [a,b]$$

$$\text{Fundamental Theorem:} \quad F' = f \implies \int_a^b f(x)\, dx = F(b) - F(a)$$

Fundamental theorem of calculus

Methods for integration

Substitution, if $u = f(x)$, then $du = g'(x)dx$

$$\int f(g(x))g'(x)dx = \int f(u)du \tag{12}$$

Integration by parts

$$\int udv = uv - \int vdu \tag{13}$$

Product rule

$$\tag{14}$$

Chain Rule

Lebiz rule for differentiation. If f(t) is continuous and g(x) is differentiable

$$f(x) = \int_a^{g(x)} f(t)dt$$

If this is true, then:

$$F(x)' = f\left(g(x)\right) \cdot g'(x) \tag{15}$$

If both the limits are dependent on x:

$$I(x) = \int_{g_1(x)}^{g_2(x)} f(t)dt$$

then:

$$I'(x) = f\left(g_2(x)\right) g_2'(x) - f\left(g_1(x)\right) g_1'(x) \tag{16}$$

Limit under a derviative

Differential Equations

"Would you tell me, please, which way I ought to go from here?"
"That depends a good deal on where you want to get to," said the Cat.

ODEs and PDEs

Ordinary Differential Equations (ODEs) are a differential equation which has a single variable. ODEs have a general form:

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^ny}{dx^n}\right) = 0 \tag{17}$$

where
- x independent
- y dependent

Partial Differential Equations (PDEs) are a Differential equation which has multiple independent variables. Instead of using the standard d, they use partial derivatives (∂) to show the change with respect for multiple variables. The general form is:

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, ...\right) = 0 \tag{18}$$

where
- x,y independent
- u(x,y) dependent

Fundamentally image a ODE as a means to track a single car, while PDE track all the traffic in the city.

Types of ODEs

ODEs are usually classified by 2 primary things. their order, aka the degree of their derivative, and by whether they are linear or non-linear.

First order ODEs are pretty self explanatory, they involve only the 1st derivative. Here is a basic first order ODE:

$$\frac{dy}{dx} + y = x \tag{19}$$

Second order ODEs involve UP to the 2nd derivative. Here is a example:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \tag{20}$$

Higher order ODEs involve everything 3rd derivative or higher. It is unlikely to ever appear in a 1st year analisis exam, but you never know.

Liniar and Non-liniear ODEs. An ODE is linear if the dependent variable and the derivatives are in a linear form. Basically: they are not multiplied together. Anything else is considered non-liniear. A linear ODE can be written in the form:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \tag{21}$$

- a(x) is a function of x

Here are some basic examples. We will go in much more detail when solving ODEs.

$\frac{dy}{dx} + 3y = x$, and $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = sinx$

$$Ineedtolearnbetterformating :)$$

A non-linear ODE is any ordinary differential equation that cannot be written in the linear form shown earlier. This is because the dependent variable or its derivatives are not linear. I will cover this more later on in the chapter.

The weird classifications of ODEs

A Homogeneous ODE, is a differential equation where L is a linear differential operator.

$$L[y] = 0$$

A Non-Homogeneous ODE, is a differential equation where f(x) $\neq 0$

$$L[y] = f(x)$$

A Autonomous ODE, is a differential equation where the independent variable (usually x or t) does not appear in the equation.

$$\frac{d^ny}{dx^n} = F\left(y, y', ..., y^{(n-1)}\right)$$

A Non-Autonomous ODE, is a differential equation if the independent variable appears eq (1).

*You can use multiple types at once, just use common sense to make sure its right

Basic existence

Dealing with ODEs

Separable ODE, can be written as

$$\frac{dy}{dx} = f(x)g(y)$$

Divide both sides by g(y), and multiply both sides by dx

$$\frac{dy}{g(y)} = f(x)dx$$

Integrate both sides, make sure to keep the constant on the RHS

$$\int \frac{dy}{g(y)} = \int f(x)dx$$

Integrating Method. Format the equation to fit the following before using the method

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Find the function $\mu(x)$ that will help simplify the problem. (Symplify as much as possible here it will help a lot later on)

$$\mu(x) = e^{\int P(x)dx}$$

Multiply every term by the function $\mu(x)$ and using the product rule calculate the derivative of the LHS

$$\frac{d}{dx}(\mu(x)y) = \mu(x)Q(x)$$

Integrate both sides, remember the constant!

$$y = \frac{1}{\mu(x)} \int \mu(x)Q(x)dx + C$$

Quick note on integrating factor. A integrating factor is the method used to solve derivative equation above. The µ(x) also called the integrating factor, works for any, first-order linear differential equations. A function is derived by multiplying the equation with µ(x), which makes the left-hand side a derivative of µ(x)y.

Exact Method If a DE is exact, which can be found if it is in this form

$$M(x, y)dx + N(x, y)dy = 0$$

After this, calculate the partial derivative of M in respect to y and the partial derivative of N with respect to x. If these are equivalent the DE is exact.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Lets make a function that we will call Ψ such that, $\mathit{\Psi}_x = M(x,y)$ and $\Psi_y = N(x,y)$

Therefore we can write this now as

$$\Psi_x + \Psi_y \frac{dy}{dx} = 0$$

we can start to find this function Ψ. So we will start to integrate M with respect to x. (h(y) is a function of y)

$$\Psi(x,y) = \int M dx + h(y)$$

We can now differentiate Ψ with respect to y

$$\frac{\partial \Psi}{\partial y} = \frac{\partial}{\partial y}\left(\int M dx\right) + h'(y) = N$$

Solve the integral for h(y)

Types of Problems

Initial Value Problem

Generally, IVPs are a DE and a initial condition or condition's which when used in unison they can be used to solve a function, that will also fit the DE. The steps are pretty straight forward.

1. Solve the DE

$$y(x) = \int f(x)dx + C$$

2. Use the initial condition, lets say that $y(x_0) = y_0$

$$y_0 = \int f(x_0)dx + C$$

Where C is:

$$C = y_0 - V$$

*V is the value of the integral at x_0, therefore if we replace C, the final answer is

$$y(x) = \int f(x)dx + (y_0 - V)$$

Proving existence and uniqueness

Theorem Definition: If f(x,y) and the partial derivative $\frac{\partial f}{\partial y}$ are continuous in:

$$D = \{(x, y)||x - x_0| \leq a, |y - y_0| \leq b\}$$

around the point (x_0, y_0) therefore, there exits a interval between x and $x_0$where there is at least one solution of y(x). And it proves that this solution is unique on the interval.

Steps: Write the ODE in standard form, aka:

$$\frac{dy}{dx} = f(x, y)$$

use intuition to check that the function f is continuous between the points you want. Then if there are any points where the function is non continuous, then be sure to mark it such that it is clear, using $\leq \geq > <$. This is the first rule.

Now, do a partial derivative of y such that:

$$\frac{\partial f}{\partial y} = f(x, y)$$

Remember to treat x as a constant in this case!!!

If the result is continuous in D between x and x_0 then this proves that there is at least a solution.

Interpreting answers

Interval of validity for Linear DE. The interval of validity for a Linear DE is largest around $x_0$where $p(x)$ and $q(x)$ are continuous. Make sure to exclude discontinuities.

Interval of validity for Non-Linear DE, Solutions may be exponential to infinity or become undefined despite $f(x, y)$ being smooth. Therefore check all the points where the solution becomes undefined. Extend the interval on both sides of x_0 till it is no longer possible.

Interpreting answers

(this whole section is a bit useless, I may remove it if I don't find any use for it son)

Explicit Solution, is when the dependent variable is isolated and in terms of the independent variable. e.g

$$y = x^2 + C$$

(soluzione esplicita)

Implicit Solution, is when the dependent variable is not explicitly isolated from the independent. e.g

$$x^2 + y^2 = C$$

(soluzione implicita)

General Solution, is the solution containing all the possible solutions for the differential equation, ie it keeps the constants, its the trivial form. (soluzione generale)

Particular Solution, is the solution which is a specific solution by locking the constants by using the initial conditions, ie the C has a fixed value. (soluzione particolare)

Equilibrium Solution, is a solution which is constant because the dependent variable does not change and therefore the derivative is zero. (soluzione di equilibrio)

Parametric Solution, is a solution represented using a parameter like (t,u,z..) instead of using x and y. eg.

$$y(t) = \sqrt{t^2 + C}$$

(soluzione parametrica)

Second Order Equations– temporary latex code which isnt my own for the exam

Solving Second-Order Differential Equations

A second-order differential equation has the general form:

$$F(y'', y', y, x) = 0$$

Below are methods for solving linear and nonlinear cases.

1. Linear Homogeneous Equations with Constant Coefficients General form:

$$ay'' + by' + cy = 0 \quad (a \neq 0)$$

Solution procedure:

1. Solve the **characteristic equation**:

$$ar^2 + br + c = 0$$

2. **Case analysis for roots r_1, r_2:**

- **Distinct real roots:** If $r_1 \neq r_2$,
$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$
- **Repeated real root:** If $r_1 = r_2 = r$,
$$y(x) = (C_1 + C_2 x)e^{rx}$$
- **Complex conjugate roots:** If $r = \alpha \pm i\beta$,
$$y(x) = e^{\alpha x}\left[C_1 \cos(\beta x) + C_2 \sin(\beta x)\right]$$

2. Linear Nonhomogeneous Equations **General form:**

$$y'' + p(x)y' + q(x)y = g(x)$$

Method 1: Undetermined Coefficients

1. Find the complementary solution y_c (solve the homogeneous equation).
2. Assume a particular solution y_p based on $g(x)$ (e.g., $g(x) = e^{kx} \Rightarrow y_p = Ae^{kx}$).
3. If $g(x)$ matches part of y_c, multiply y_p by x (or x^n for repeated roots).
4. Substitute y_p into the DE and solve for coefficients.
5. General solution: $y = y_c + y_p$.

Method 2: Variation of Parameters

1. Find $y_c = C_1 y_1 + C_2 y_2$.
2. Compute the Wronskian:
$$W = y_1 y_2' - y_1' y_2$$
3. Particular solution:
$$y_p = -y_1 \int \frac{y_2 g(x)}{W}\,dx + y_2 \int \frac{y_1 g(x)}{W}\,dx$$
4. General solution: $y = y_c + y_p$.

3. Cauchy-Euler Equations **General form:**

$$x^2 y'' + bxy' + cy = 0$$

Solution steps:

1. Assume $y = x^r$. Substitute to get:
$$r^2 + (b-1)r + c = 0$$
2. Solve for r. The solution mirrors constant-coefficient cases:
   - Real distinct roots: $y = C_1 x^{r_1} + C_2 x^{r_2}$
   - Repeated root: $y = x^r(C_1 + C_2 \ln x)$
   - Complex roots $r = \alpha \pm i\beta$: $y = x^\alpha\left[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)\right]$

4. Reduction of Order **When one solution y_1 is known:**

1. Let $y_2 = v(x)y_1$.
2. Substitute y_2 into the DE and solve for $v(x)$.
3. The second solution is $y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$.

Special cases:

- **Equation missing y:** Let $p = y'$, reducing to $p' = f(x, p)$.
- **Equation missing x:** Let $p = y'$, then $y'' = p\frac{dp}{dy}$, reducing to $p\frac{dp}{dy} = f(y, p)$.

5. Series Solutions Near Ordinary Points For $P(x)y''+Q(x)y'+R(x)y = 0$ with ordinary point at x_0:

1. Assume $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$.
2. Substitute into DE and equate coefficients of like powers.
3. Derive a recurrence relation for a_n.

6. Laplace Transform for Initial Value Problems **Procedure:**

1. Take Laplace transform of the DE:

$$\mathcal{L}\{y''\} + a\mathcal{L}\{y'\} + b\mathcal{L}\{y\} = \mathcal{L}\{g(x)\}$$

2. Use:

$$\mathcal{L}\{y'\} = sY(s) - y(0), \quad \mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$$

3. Solve for $Y(s)$, then compute $y(x) = \mathcal{L}^{-1}\{Y(s)\}$.

Nonlinear Second-Order DEs No universal method exists. Common approaches:

- Substitutions to reduce order (e.g., $p = y'$)
- Exact equations (identify integrable combinations)
- Numerical methods (e.g., Runge-Kutta)

Homogeneous Equations with constant coefficients have the general form:

$$y'' + ay' + by = 0$$

Non-homogeneous Equations: Method of Undetermined Coefficients

Finished

Nth order Differential Equation

0.0.1 Solving n-th Order Differential Equations

Linear Homogeneous Equations with Constant Coefficients Consider the equation:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Solution Steps:

1. **Form characteristic equation**:

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

2. **Find roots** $r_1, r_2, \ldots, r_n$

3. **Construct general solution**:

   - *Distinct real roots*:

$$y_h = \sum_{i=1}^{n} C_i e^{r_i x}$$

   - *Repeated real root r with multiplicity k*:

$$e^{rx} \left(C_1 + C_2 x + \cdots + C_k x^{k-1} \right)$$

   - *Complex conjugate pairs $\alpha \pm \beta i$*:

$$e^{\alpha x} \left[C_1 \cos(\beta x) + C_2 \sin(\beta x) \right]$$

   For repeated pairs (multiplicity m):

$$x^{m-1} e^{\alpha x} \left[C_1 \cos(\beta x) + C_2 \sin(\beta x) \right]$$

Linear Nonhomogeneous Equations For equations:

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = g(x)$$

General Solution:

$$y = y_h + y_p$$

where y_h = homogeneous solution, y_p = particular solution.

Method of Undetermined Coefficients Use when $g(x)$ is polynomial, exponential, sine, cosine, or combinations:

1. Assume y_p with same form as $g(x)$
2. If any term matches y_h, multiply by x^s (s = smallest integer eliminating duplication)
3. Substitute y_p into DE and solve for coefficients

Variation of Parameters General method for arbitrary $g(x)$:

1. Find fundamental set $\{y_1, \ldots, y_n\}$ from y_h
2. Compute Wronskian:

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

3. Find $u_i' = \frac{W_i}{W}$ where $W_i =$ Wronskian with i-th column replaced by $\begin{bmatrix} 0 \\ \vdots \\ g(x) \end{bmatrix}$
4. Integrate to get u_i, then:

$$y_p = \sum_{i=1}^{n} u_i y_i$$

Variable Coefficient Equations For $y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_0(x)y = Q(x)$:

Reduction of Order If solution y_1 is known, let:

$$y = y_1 \int v(x)dx$$

Substitute to reduce equation order by 1.

Cauchy-Euler Equations Form: $x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + a_0 y = 0$

1. Assume solution $y = x^m$
2. Substitute to get characteristic equation:

$$m(m-1)\cdots(m-n+1) + \sum_{k=0}^{n-1} a_k m(m-1)\cdots(m-k+1) = 0$$

3. Handle roots as with constant coefficient equations

Nonlinear Equations

Order Reduction Techniques

- *Missing* y: Let $v = y'$, reduces order by 1
- *Missing* x: Let $v = y'$, then $y'' = v\frac{dv}{dy}$, reduces to 1st order in v

Exact Equations If equation can be written as:

$$\frac{d}{dx}\,[\text{Lower order expression}] = 0$$

Integrate successively to solve.

Integrating Factors Find $\mu(x)$ or $\mu(y)$ such that:

$$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0$$

becomes exact.

Special Forms

- *Bernoulli*: $y' + P(x)y = Q(x)y^n$, use $z = y^{1-n}$
- *Riccati*: $y' = P(x)y^2 + Q(x)y + R(x)$, use $y = y_1 + \frac{1}{v}$ if particular solution y_1 known

Systems of Differential Equations

Solving Systems of Differential Equations

Linear Systems with Constant Coefficients For the system $\mathbf{x}' = A\mathbf{x}$ where A is an $n \times n$ constant matrix:

Solution Method:

1. **Find eigenvalues** λ by solving:
$$\det(A - \lambda I) = 0$$
2. **Find eigenvectors** ξ for each eigenvalue by solving:
$$(A - \lambda I)\xi = 0$$
3. **Construct general solution**:
   - *Real distinct eigenvalues*:
   $$\mathbf{x}(t) = \sum_{i=1}^{n} C_i e^{\lambda_i t}\xi_i$$
   - *Complex eigenvalues* $\alpha \pm \beta i$:
   $$\mathbf{x}(t) = C_1 e^{\alpha t}[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)] + C_2 e^{\alpha t}[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)]$$
   where $\mathbf{a} + i\mathbf{b}$ is the complex eigenvector
   - *Repeated eigenvalues*:
     - If geometric multiplicity = algebraic multiplicity: proceed as distinct eigenvalues
     - If deficient eigenvectors: use generalized eigenvectors

Nonhomogeneous Systems For $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$:

General Solution:

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)$$

Variation of Parameters

1. Find fundamental matrix $\Phi(t)$ from homogeneous solutions
2. Compute particular solution:

$$\mathbf{x}_p(t) = \Phi(t) \int \Phi^{-1}(t)\mathbf{g}(t)dt$$

Method of Undetermined Coefficients Use when $\mathbf{g}(t)$ contains polynomials, exponentials, or trigonometric functions:

1. Assume $\mathbf{x}_p$ with same form as $\mathbf{g}(t)$
2. Adjust for resonance if any term matches homogeneous solution
3. Substitute and solve for coefficients

Matrix Exponential Method For $\mathbf{x}' = A\mathbf{x}$:

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

where e^{At} can be computed via:

- Diagonalization: $A = PDP^{-1} \Rightarrow e^{At} = Pe^{Dt}P^{-1}$
- Jordan form for defective matrices
- Taylor series expansion for simple cases

Nonlinear Systems

Linearization Near Critical Points

1. Find equilibrium points $\mathbf{x}_0$ where $\mathbf{f}(\mathbf{x}_0) = 0$
2. Compute Jacobian matrix:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\mathbf{x}_0}$$

3. Analyze eigenvalues of J to determine stability

Phase Plane Analysis (2D Systems)

- Classify critical points: node, spiral, saddle, center
- Use nullclines and direction fields
- Lyapunov functions for stability (when applicable)

Conversion to First-Order Systems Any n-th order DE can be converted to a system:

1. Let $x_1 = y$, $x_2 = y'$, ..., $x_n = y^{(n-1)}$
2. Create system:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_n' = F(t, x_1, \ldots, x_n) \end{cases}$$

Important Special Cases

Coupled Oscillators

$$\begin{cases} m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 x_2'' = -k_2(x_2 - x_1) \end{cases}$$

Solve by diagonalizing the coefficient matrix

Competing Species Model

$$\begin{cases} x' = x(a - by) \\ y' = y(c - dx) \end{cases}$$

Analyze using linearization and phase plane methods

Special theorems and Problems

Picard–Lindelöf theorem

Picard–Lindelöf Theorem (Existence & Uniqueness)

Theorem Statement Consider the initial value problem (IVP):

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

where $f : D \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. If:

- f is **continuous** in t on rectangle $R = [t_0 - a, t_0 + a] \times \overline{B}(y_0, b)$
- f is **Lipschitz continuous** in y:

$$\exists L > 0 \text{ s.t. } \|f(t, y_1) - f(t, y_2)\| \leq L\|y_1 - y_2\|, \ \forall (t, y_1), (t, y_2) \in R$$

Then $\exists \tau > 0$ such that the IVP has a **unique solution** $y(t)$ on $[t_0 - \tau, t_0 + \tau]$.

Proof Outline (Method of Successive Approximations)

1. Reformulate IVP as integral equation:

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds$$

2. Define Picard iterations:

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))ds$$

   Starting with $y_0(t) \equiv y_0$

3. Show $\{y_n\}$ converges uniformly to solution y:
   - Use Lipschitz condition to prove $\|y_{n+1} - y_n\| \leq \frac{M}{L} \frac{(L|t-t_0|)^{n+1}}{(n+1)!}$
   - Apply Banach fixed-point theorem in complete metric space

Implementation Steps To apply the theorem:

1. Verify continuity of $f(t, y)$ in t
2. Check Lipschitz condition in y:
   - If $\frac{\partial f}{\partial y}$ exists and bounded $\Rightarrow$ Lipschitz
   - For scalar case: $|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$
3. Determine existence interval $\tau = \min\left(a, \frac{b}{M}\right)$ where:

$$M = \max_{(t,y) \in R} \|f(t, y)\|$$

Example Application For IVP $y' = y, \ y(0) = 1$:

- Picard iterations:

$$
\begin{aligned}
y_0(t) &= 1 \\
y_1(t) &= 1 + \int_0^t y_0(s)ds = 1 + t \\
y_2(t) &= 1 + \int_0^t (1+s)ds = 1 + t + \frac{t^2}{2} \\
&\vdots \\
y_n(t) &= \sum_{k=0}^{n} \frac{t^k}{k!} \to e^t
\end{aligned}
$$

Important Notes

- **Local vs Global**: Theorem guarantees local solution - need additional conditions for global existence
- **Sharpness**: τ estimate often conservative
- **Failure Cases**:
  - f not Lipschitz $\Rightarrow$ possible non-uniqueness (e.g., $y' = \sqrt{|y|}$)
  - Discontinuous f $\Rightarrow$ solutions may not exist

Cauchy's Problem

Here we have a nth order ODE

$$y^{(n)}(t) = f\left(t, y, y', ..., y^{(n-1)}\right)$$

The Cauchy problem also known as the