Mathematical Analysis 1

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Pre-Derivatives

Theorems, functions and axioms

"Calvin: You know, I don't think math is a science, I think it's a religion.
Hobbes: A religion?
Calvin: Yeah. All these equations are like miracles. You take two numbers and when you add them, they magically become one NEW number! No one can say how it happens. You either believe it or you don't. [Pointing at his math book] This whole book is full of things that have to be accepted on faith! It's a religion!"

Logic

Propositional Logic

A proposition is a statement. Statements in math are either true or false, if you combined multiple propositions there are multiple outcomes depending on the validity of each proposition in the statement. I call them compound statements (but I dont think its the actual name).

Negation (Negazione): Represented by a $\neg$. Inverts the value of a statement, e.g (Bob went to the store), the opposite can be represented as $\neg$(Bob went to the store).

Disjunction (Disgiunzione): Represented by a $\vee$. Statement is true if at least one of the propositions is also true.

Conjunction (Congiunzione): Represented by a $\wedge$. Statement is true only if both propositions are also true.

Implication (Implicazione): Represented by a $\rightarrow$. Its formally defined as : $\neg P \vee Q$.

Biconditional (Bicondizionale): Represented by a $\leftrightarrow$. Statement is true if both of the propositions share the same value.

Predicate Logic

Universal: It is represented by $\forall$. "For every ..."

Existence: It is represented by $\exists$. "There exists ..."

Axioms

Axioms (Assiomi): An axiom is a postulate, more commonly known as assumption. It is a statement that is held as always true in regards to the problem or proof needed to solve. There are different types of axioms which are briefly stated below:

Logical Axioms: A universal truth in all of Mathematics applicable in both the Physics Notes and the Linear Algebra Notes.

Non Logical Axioms: Domain specific assumptions, such as Axioms only applicable in $\mathbb{R}$(Gross oversimplification)

Group Theory (Theoria degli insiemi)

Definition of a Group

A group is a set, that has the following requirements:

Closure: For all $a, b \in G, a * b \in G$

Associativity: $a, b, c \in G, (a * b) * c = a * (c * b)$

Identity: Lets say e exists $e \in G$ such that $e * a = a * e = a$ for all $a \in G$

Inverse: If there exists a a, such that $a \in G$ therefore there exists a inverses

Proofs (Dimostrazione)

In Math, every theorem and formula needs to be able to be proven in a proof. There are multiple types of proofs which are used to show that a theorem and or formula is valid. In these notes, very few proofs will be done, however in the written blue book. Each proof I have written in there has a classification. If a proof is referenced it will have a corresponding code written next to it referencing the written proof in the blue notebook.

Direct Proof (Dimostrazione Diretta). By using known definitions, axioms and theorems, a sequence of logical steps can be used to directly demonstrate whether or not the statement is correct.

Proof by Contradiction (Dimostrazione per Assurdo). Assume that a statement is false and connect it to a logical contradiction.

Induction (Induzione). Used for statements involving natural numbers

Base Case: Verify the statement holds for the initial value

Inductive Step: Assume it holds for a n=k, then prove it holds for n=k+1.

Constructive Proof (Dimostrazione Costruttiva). Make a identity with the exact desired property. More formally ''Demonstrates the existence of an object by explicitly constructing it''

Functions

A **function** (*funzione*) is a very common conecept in math that formalizes the relationship between two sets by assigning each element of the first set to exactly one element of the second set. Formally, a function $f : A \to B$ consists of:

- A **domain** (*dominio*) A, the set of all possible inputs.
- A **codomain** (*codominio*) B, the set into which all outputs are mapped.
- A rule or correspondence that links each element $x \in A$ to a unique element $f(x) \in B$.

Formal Definition A function f is a subset of the Cartesian product $A \times B$ such that for every $x \in A$, there exists exactly one $y \in B$ where $(x, y) \in f$. This is denoted as $y = f(x)$.

Key Properties

1. **Injectivity** (*iniettiva*): A function is injective if distinct inputs map to distinct outputs:
$$\forall x_1, x_2 \in A, \quad f(x_1) = f(x_2) \implies x_1 = x_2.$$
2. **Surjectivity** (*suriettiva*): A function is surjective if every element in B is an output for some input:
$$\forall y \in B, \quad \exists x \in A \text{ such that } y = f(x).$$
3. **Bijectivity** (*biiettiva*): A function is bijective if it is both injective and surjective, establishing a one-to-one correspondence between A and B.

Natural Numbers

The **natural numbers** (*numeri naturali*) are the standard version of numbers in maths, used for counting and ordering. Formally, the set of natural numbers $\mathbb{N}$ is defined as:

$$\mathbb{N} = \{1, 2, 3, \ldots\} \quad \text{(sometimes including 0 depending on context).}$$

They are characterized by their discrete, non-negative integer values and form the foundation for number theory and arithmetic.

Formal Definition (Peano Axioms) The properties of natural numbers are axiomatically defined by the **Peano axioms** (*assiomi di Peano*):

1. 1 (or 0) is a natural number.
2. Every natural number n has a unique successor $S(n)$, which is also a natural number.
3. 1 (or 0) is not the successor of any natural number.
4. Distinct natural numbers have distinct successors: $S(m) = S(n) \implies m = n$.
5. **Induction**: If a property holds for 1 (or 0) and holds for $S(n)$ whenever it holds for n, then it holds for all natural numbers.

Key Properties

- **Closure under addition and multiplication**: For all $a, b \in \mathbb{N}$, $a+b \in \mathbb{N}$ and $a \cdot b \in \mathbb{N}$.
- **Non-closure under subtraction and division**: Subtraction $a - b$ or division a/b may not result in a natural number.
- **Well-ordering principle** (*principio del buon ordinamento*): Every non-empty subset of $\mathbb{N}$ has a least element.
- **Infinite cardinality**: $\mathbb{N}$ is countably infinite.

Number Theory Natural numbers are central to number theory, which studies:

1. **Prime numbers** (*numeri primi*): Natural numbers > 1 with no divisors other than 1 and themselves:
$$\mathbb{P} = \{2, 3, 5, 7, 11, \ldots\}.$$
2. **Divisibility**: A number a divides b ($a \mid b$) if $\exists k \in \mathbb{N}$ such that $b = a \cdot k$.
3. **Mathematical induction** (*induzione matematica*): A proof technique leveraging the Peano axioms.
4. **Modular arithmetic** (*aritmetica modulare*): Operations on residues modulo n, e.g., $7 \equiv 2 \mod 5$.

Whole Numbers

Whole Numbers (*numeri interi non negativi*) are an extension of the natural numbers that include zero, forming the set $\mathbb{W} = \{0, 1, 2, 3, \ldots\}$. They are used for counting discrete objects and represent non-negative integers without fractions or decimals.

Formal Definition The set $\mathbb{W}$ satisfies the **Peano axioms** (*assiomi di Peano*) with zero as the base element:

- 0 is a whole number.
- Every whole number n has a unique successor $S(n) \in \mathbb{W}$.
- 0 is not the successor of any whole number.
- Distinct numbers have distinct successors: $S(a) = S(b) \implies a = b$.
- **Induction**: If a property holds for 0 and for $S(n)$ whenever it holds for n, it holds for all $\mathbb{W}$.

Key Properties

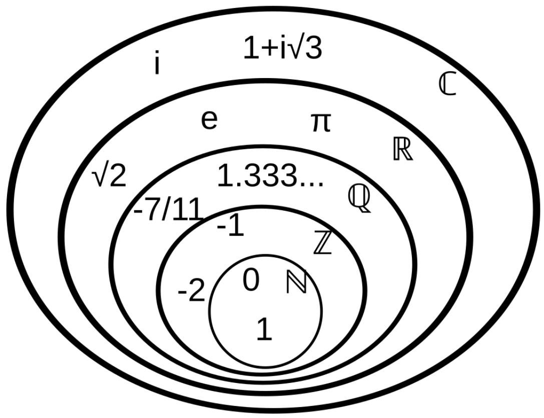
- **Closure under addition and multiplication**: For $a, b \in \mathbb{W}$, $a + b \in \mathbb{W}$ and $a \cdot b \in \mathbb{W}$.
- **Non-closure under subtraction**: $a - b \in \mathbb{W}$ only if $a \geq b$.
- **Additive identity**: $0 + a = a$ for all $a \in \mathbb{W}$.
- **Well-ordering principle** (*principio del buon ordinamento*): Every non-empty subset of $\mathbb{W}$ has a least element.

Representation Whole numbers are represented in numeral systems such as:

- **Decimal**: $0, 1, 2, \ldots$
- **Binary**: $0_2 = 0_{10}, 1_2 = 1_{10}, 10_2 = 2_{10}$
- **Unary**: 0 (often represented as an absence of marks), $| = 1, || = 2$.

Differences from Natural Numbers Unlike natural numbers (*numeri naturali*), which sometimes exclude zero, whole numbers explicitly include 0. This makes $\mathbb{W}$ the set $\mathbb{N} \cup \{0\}$ in contexts where natural numbers start at 1.

Below, is a image of all the relevant groups of numbers and how they are related. Not all of them have been stated in this point in the notes, but they are all relevant for analysis 1



Exponential Properties

Exponential Functions

For $a > 0$ and $x \in \mathbb{R}$, the *exponential function* with base a is defined as the function $f(x) = a^x$. This represents continuous growth (when $a > 1$) or decay (when $0 < a < 1$) processes. The fundamental identity $a^x = e^{x \ln a}$ relates all exponentials to the natural base $e \approx 2.71828$, where $\exp(x) = e^x$ is the unique solution to $y' = y$ with $y(0) = 1$. Exponentials map additive changes to multiplicative scaling: $a^{x+y} = a^x \cdot a^y$.

Exponential Properties

For $a > 0$, $b > 0$, and $x, y \in \mathbb{R}$:

$$a^x a^y = a^{x+y}$$
$$\frac{a^x}{a^y} = a^{x-y}$$
$$(a^x)^y = a^{xy}$$
$$a^0 = 1$$
$$a^{-x} = \frac{1}{a^x}$$
$$(ab)^x = a^x b^x$$
$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

Def of Logarithms

For $a > 0$ $(a \neq 1)$, $b > 0$:

$$\log_a b = x \iff a^x = b$$

$$\log_a 1 = 0$$

$$\log_a a = 1$$

$$a^{\log_a b} = b$$

Logarithm Properties

For $a > 0$ ($a \neq 1$), $x, y > 0$, $k \in \mathbb{R}$:

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a(x^k) = k \log_a x$$

$$\log_a a^x = x$$

Change of Base

For $a, b > 0$ ($a, b \neq 1$), $c > 0$:

$$\log_a c = \frac{\log_b c}{\log_b a}$$

Special case: $\log_a b = \frac{1}{\log_b a}$

Natural Exponential and Logarithm

$$e^x = \exp(x)$$

$$\ln x = \log_e x$$

$e^{\ln x} = x$ for $x > 0$

$\ln(e^x) = x$ for $x \in \mathbb{R}$

Exponential-Logarithmic Equations

Key solving techniques:

If $a^x = a^y$ then $x = y$

If $\log_a x = \log_a y$ then $x = y$

To solve $a^{f(x)} = b$: take logarithms of both sides

To solve $\log_a f(x) = b$: rewrite as $f(x) = a^b$

Trigonometric Functions

Trigonometric Functions (*funzioni trigonometriche*) are periodic functions that relate angles in a right triangle or on the unit circle to ratios of side lengths. The primary trigonometric functions are sine ($\sin$), cosine ($\cos$), tangent ($\tan$), and their reciprocals: cosecant ($\csc$), secant ($\sec$), and cotangent ($\cot$).

Formal Definition (Unit Circle) For an angle θ measured counterclockwise from the positive x-axis on the unit circle ($x^2 + y^2 = 1$):

$$\sin\theta = y, \quad \cos\theta = x, \quad \tan\theta = \frac{y}{x} \quad (x \neq 0).$$

The reciprocals are defined as:

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{x}{y} \quad (y \neq 0).$$

Key Properties

- **Periodicity** (*periodicità*): $\sin\theta$ and $\cos\theta$ have period 2π; $\tan\theta$ and $\cot\theta$ have period π.
- **Range**:

  $$\sin\theta, \cos\theta \in [-1, 1]; \quad \tan\theta \in \mathbb{R} \text{ (excluding asymptotes).}$$

- **Parity**: $\sin\theta$ and $\tan\theta$ are odd functions; $\cos\theta$ is even:

  $$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta.$$

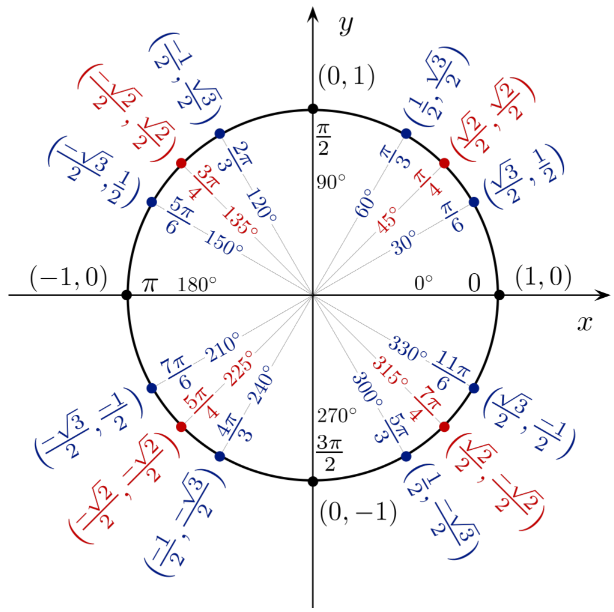
- **Pythagorean Identity** (*identità pitagorica*):

  $$\sin^2\theta + \cos^2\theta = 1.$$

Differences from Other Functions Unlike polynomial or exponential functions, trigonometric functions:

- Are periodic and bounded (except $\tan\theta$ and $\cot\theta$).
- Model oscillatory behavior (e.g., waves, circular motion).
- Require angular input (radians or degrees) rather than purely scalar quantities.

Unit Circle For Reference:



Fundamental Identities (*identità fondamentali*)

- **Reciprocal Relations**:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}.$$

- **Extended Pythagorean Identities**:

$$1 + \tan^2\theta = \sec^2\theta, \quad 1 + \cot^2\theta = \csc^2\theta.$$

- **Co-Function Identities** (*identità complementari*):

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \quad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \quad \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta.$$

Angle Addition & Subtraction For any angles α and β:

- $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
- $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
- $\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

Multiple-Angle & Half-Angle Identities

- **Double-Angle**:

$$\sin(2\theta) = 2\sin\theta\cos\theta, \quad \cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta.$$

- **Triple-Angle**:

$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta, \quad \cos(3\theta) = 4\cos^3\theta - 3\cos\theta.$$

- **Half-Angle** (sign depends on quadrant):

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}, \quad \cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}.$$

Product-to-Sum & Sum-to-Product

- **Product-to-Sum**:

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)], \quad \cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)].$$

- **Sum-to-Product**:

$$\sin\alpha \pm \sin\beta = 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right), \quad \cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right).$$

Triangle Relations (Laws) For any triangle with sides a, b, c opposite angles A, B, C:

- **Law of Sines**:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \quad (R = \text{circumradius}).$$

- **Law of Cosines**:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

- **Law of Tangents**:

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}.$$

Complex Numbers introduction

The largest domain covered in these notes: $\mathbb{C}$. Complex numbers are an extension of real numbers, defined by the imaginary number i With this strange property where $i^2 = -1$. They are used to resolve polynomial equations unsolvable in real numbers, as shown in the fundamental theorem of algebra (Might need a section on this). Below is a complex number with iy as the imaginary component and x as the real component.

$$z = x + iy \tag{1}$$

Imagine $\mathbb{R}$ covering the whole x axis, and $\mathbb{C}$ covering the whole y axis, that's the complex plane.

Operations

Addition

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \tag{2}$$

Subtraction

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2) \tag{3}$$

Multiplication

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \tag{4}$$

Complex Conjugate

$$\bar{z} = x - iy \tag{5}$$

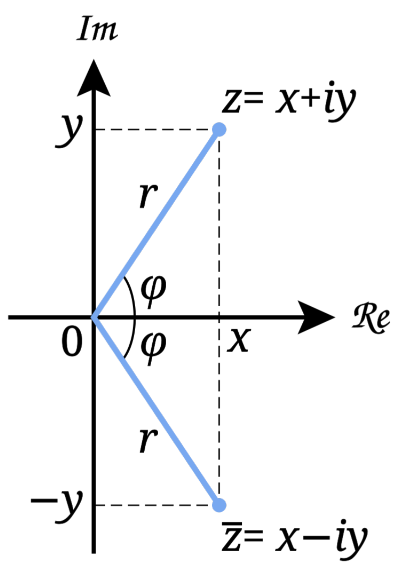
Modulus

$$|z| = \sqrt{x^2 + y^2} \tag{6}$$

Inverse

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2} \tag{7}$$

A image of the $\mathbb{C}$ plane for reference, showing the inverse modulus as well:



Polar Coordinates

Polar coordinates are a alternative to the commonly used Cartesian coordinate system (x,y). It is mesured using two metrics:

Radial Distance (r): Which is the distance from the origin which is also the hypotenuse of the triangle.

Angular Coordinate (θ): Which is the angle between the radial distance and the +x axis. Which increases counterclockwise.

Basic Conversion between Cartesian and Polar For Cartesian to Polar it is:

$$r = \sqrt{x^2 + y^2}, \theta = arctan\left(\frac{y}{x}\right)$$

And for Polar to Cartesian it is:

$$x = rcos\theta, y = rsin\theta$$

Basis vectors Basis vectors represented in polar coordinates:

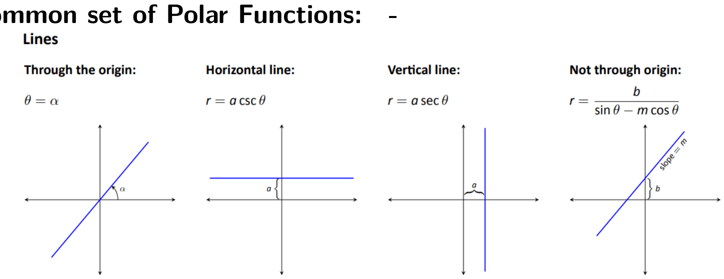
$$\hat{\mathbf{r}} = cos\theta\hat{\mathbf{i}} + sin\theta\hat{\mathbf{j}}, \hat{\theta} = -sin\theta\hat{\mathbf{i}} + cos\theta\hat{\mathbf{j}}$$

Polar Functions A polar function is represented as:

$$r = f(\theta)$$

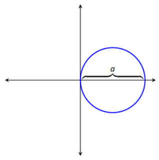
Much like a regular function it has a domain and a range, expressed below:
Domain: $\theta \in [0, 2\pi)$
Range: $r \in \mathbb{R}$

Common set of Polar Functions: -

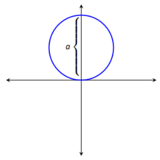


Circles

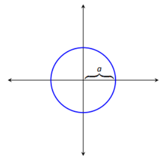
Centered on x-axis:
$r = a\cos\theta$



Centered on y-axis:
$r = a\sin\theta$

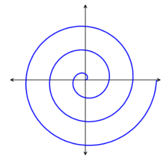


Centered on origin:
$r = a$



Spiral

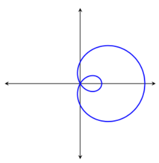
Archimedean spiral
$r = \theta$



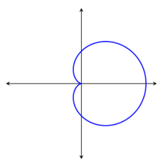
Limaçons

Symmetric about x-axis: $r = a \pm b\cos\theta$;   Symmetric about y-axis: $r = a \pm b\sin\theta$; $a, b > 0$

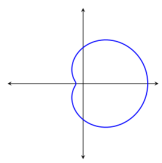
With inner loop:
$\frac{a}{b} < 1$



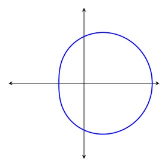
Cardiod:
$\frac{a}{b} = 1$



Dimpled:
$1 < \frac{a}{b} < 2$



Convex:
$\frac{a}{b} > 2$



Sums and Sequences

"La situazione è grave ma non è seria."

Limits

How its represented. Any sequence that converges to a limit is represented as:

$$lim_{n \to \infty} a_n = L$$

Where $\{a_n\}$is the sequence.

Formal Definition.

$$\lim_{n \to \infty} a_n = L \quad \iff \quad \forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \geq N,\ |a_n - L| < \varepsilon$$

"For every positive number, there exists a natural number such that, for all integers, the distance between and L is less than ε."

Sequence

Definition: A sequence/sucession is simply a list of $\mathbb{N}$ while following a set of rules defined by a function. The function then maps each $\mathbb{N}$ to a corresponding $\mathbb{R}$ following the rules defined by the function.

Representation: It is often shown as a random letter (this case a) a_n with n representing the number of the term.

The first term, a_1 is called the initial term (termine iniziale)

The terms after the first, a_{1+n} is called the recursive formula (formula ricorsiva)

Types: There are 2 specific categories which will be covered in more detail. 1. Whether its bounded (limitata) or unbounded (illimitata). 2. Whether its convergent (convergente), divergent (divergente) or oscillatory (oscillante).

Bounded Successions (Successioni Limitate)

Definition: A bounded sequence (successione limitata), is a sequence a_n that exists within a range such that if $b \in \mathbb{R}$, b is greater than a_n, and there exists $c \in \mathbb{R}$ that is less than a_n, it is a bounded sequence. A bounded sequence is may suggest convergences and can be proven by using the Bolzano-Weierstrass Theorem.

Intervals:

Open Interval

$$(a, b) := \{x \in \mathbb{R} | a < x < b\}$$

A open interval includes all $\mathbb{R}$ numbers between a and b, excluding the endpoints. Represented by a () and $<$

Closed Interval

$$[a, b] := \{x \in \mathbb{R} | a \leq x \leq b\}$$

A closed interval includes all $\mathbb{R}$ numbers between a and b, including the endpoints. Represented by a [] and $\leq$.

Empty Interval Denoted as $\emptyset$, it contains no numbers.

Degenerate Interval A single point, $[a, a] = \{a\}$. By technicality it is always closed.

Half Intervals It is possible to have intervals which are open at one end and closed at the other, and vice versa. e.g

$$(a, b] := \{x \in \mathbb{R} | a < x \leq b\}$$

Infinite Intervals There are also intervals which are infinite on one side and open or closed on the other. e.g

$$(a, +\infty) := \{x \in \mathbb{R} | a < x\}$$

The infinite can be negative as well on the other side. (Not sure if the infinite has to be in a open interval, because I have not seen any which are not in a open interval)

Types of bounds:

Upper bound: If a upper bound exists we call it bounded from above.

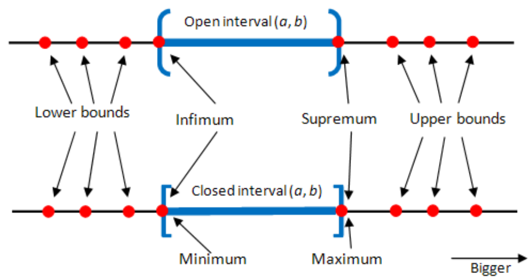
Lower bound: If a lower bound exists we call it bounded from below.
If both upper and lower bound exist the set is bounded.

Bound properties: There can be multiple upper and lower bound (As clearly shown in the diagram below).

Supremum and Infimum The supremum and infimum can only exist for a interval with at least one open point. Its the smallest possible upper bound (If its a supremum) or the largest possible lower bound (If its a infimum). It can NEVER reach the interval. The Supermum can be written as $supM$ and the Infium can be written as $infM$

Minimum and Maximum For a Minimum or a Maximum you must have at least one closed interval point. (as shown in the diagram below). The minimum or maximum is the point a or b that hold the interval.

Diagram of open and closed sequences:



Monotone Sequences

Definition: A sequence that is monotone is either non-decreasing or non-increasing. Therefore, by extention a constant sequence is simultaneously non-decreasing and non-increasing. Therefore it is monotone. The strictly increasing/decreasing are simply subsets.

Growth of Sequences:

Non-decreasing sequence (successioni crescenti): Is a sequence that increases if each term is greater than or equal to the previous term:

$$\forall n \in \mathbb{N}, a_{n+1} \geq a_n$$

Non-increasing sequences (successioni decrescenti): Is a sequence that it decreases if each term is less than or equal to the previous term.

$$\forall n \in \mathbb{N}, a_{n+1} \leq a_n$$

Strictly increasing sequences (successioni strettamente crescenti): Is a sequence that strictly increasing if each term is strictly greater than the previous term.

$$\forall n \in \mathbb{N}, a_{n+1} > a_n$$

Strictly decreasing sequences (successioni strettamente decrescenti): Is a sequence that strictly decreasing if each term is strictly less than the previous term.

$$\forall n \in \mathbb{N}, a_{n+1} < a_n$$

Convergent and Divergent Sequences

Convergent

A sequence is convergent if it approaches a finite limit (L) as the n approaches infinity. Formally this can be defined:

A sequence $\{a_n\}$ converges to L if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} | \forall n \geq N, |a_n - L| < \varepsilon$$

Accumulation Points Link A convergent sequence has exactly one accumulation point which is its limit. This is different from the divergent sequence that will have multiple accumulation values or many improper ones.

Limit sup and inf For convergent sequences:

$$lim_{n\to\infty} sup a_n = lim_{n\to\infty} inf a_n = L$$

Note on Bounded Sequences: While boundedness is a requirement for a sequence to be convergent it is not the only requirement.

Monotone Convergence Theorem: Every bounded monotonic sequence converges

Limit properties for convergence: If b is a sequence that converges to M and a is a sequence that converges to L then

$$lim_{n\to\infty}(a_n \pm b_n) = L \pm M$$

$$lim_{n\to\infty}(a_n \times b_n) = L \times M$$

$$lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M} (M \neq 0)$$

Divergent

A sequence is divergent if it does not converge to any limit L. There are two primary times of divergence, divergence to infinity(may also be refered to improper divergence) or oscillatory divergence. The formal definitions for divergence to infinity is listed below:

$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N}$$

Such that $\forall n \geq N, a_n > M$, and if it diverging to negative infinity it is $\forall n \geq N, a_n < M$

And for Oscillations:

Bounded Oscillations: Terms change between bounds with the lim sum not equalling the lim inf

Unbounded oscillation: Terms grow wile changing and slowly diverging in absolute value but don't have a directional convergence.

Unbounded While it is not guaranteed that every bounded sequence is convergent, it is guaranteed that every unbounded sequence is divergent by definition. However not all divergent sequences are unbounded.

Sub sequences If a sub sequence of a sequence diverges to infinity then the also sequence will diverge.

If two sub sequences converge to different limits, the original sequence diverges.

Limit properties If $lim_{n\to\infty}a_n = +\infty$ and the sequence b is bounded below then

$$lim_{n\to\infty}(a_n + b_n) = +\infty$$

If $lim_{n\to\infty}a_n = +\infty$ and $lim_{n\to\infty}b_n = c > 0$ then

$$lim_{n\to\infty}(a_n \times b_n) = +\infty$$

Cauchy Sequence

A Cauchy sequence is a sequence whose terms become arbitrarily close to one another as the sequence progresses, regardless of whether the sequence converges to a specific limit.

Formal Definition Every sequence with this definition is a Cauchy Sequence

$$\forall\epsilon > 0, \exists N \in \mathbb{N} \Rightarrow \forall m, n \geq N, |a_m - a_n| < \epsilon$$

Properties

Convergence In R every Cauchy sequence by definition must be convergent, due to the completeness property.

Boundedness Every Cauchy Sequence is bounded.

Subsequence If a Cauchy sequence has a convergent subsequence, the entire sequence converges to the same limit.

Notes on convergence All convergent sequences are Cauchy, not all Cauchy sequences are convergent IN INCOMPLETE SPACES

Napier's Constant (Costante di Nepero)

This may also be known as e or exponential or exp. It has 3 separate definitions: Limit, Series, and integral. However I will only do the explanation for limit definition here. The integral explanation will be done in the integral chapter.

Limit Definition:

$$e = lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \tag{8}$$

Base of the Natural Logarithm e is the base of the natural logarithm, denoted as $\ln(x)$.

Approximate Value $e \approx 2.71828$.

Irrational Number e cannot be expressed as a ratio of two integers.

Transcendental Number e is not a root of any non-zero polynomial with rational coefficients.

Limit Definition

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Euler's Identity

$$e^{i\pi} + 1 = 0$$

Connecting e, imaginary numbers, and π

Complex Exponential (l'esponenziale Complesso)

The complex exponential function denoted as e^z or exp(z). The vast majority of definitions of complex exponential are the exact same as the regular exponential function. Including the proof of their equivalence. There are several ways to define a complex exponential function, and I will list them below.

Power series expansion

$$e^z = \sum_{k=0}^{\infty}\frac{z^k}{k!}$$

Limit Definition

$$e^z = lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n$$

Additive Property (Kinda definition)

$$e^{w+z} = e^w e^z$$

Complex log

$$log(e^z) = \{z + 2\pi ik | k \in \mathbb{Z}\}$$

Key Properties

Non zero $\frac{1}{e^z} = e^{-z}$ and e^z does not equal zero for all C

Periodic

$$e^{z+2\pi} = e^z$$

Identita di Euler For $e^{i\theta} = cos\theta + i * sin\theta$

$$e^{i\pi} = -1$$

Conjugate $\overline{e^z} = e^{\overline{z}}$

Modulus $|e^z| = e^{\mathbb{R}(z)}$

Handling Infinity

Limit Superior

For a sequence (a_n), the limit superior which is written as

$$limsup_{n\to\infty} a_n$$

is the supremum of all the accumulation points (cluster points) of the sequence. For example if a sequence diverges to infinity its only accumulation point is infinity therefore

$$limsup_{n\to\infty} a_n = \infty$$

The main diffrence between the supremum is that the supremum

Lets say we have a sequence (a_n) which is given by $a_n = n$, the limit superior of (a_n) is the largest accumulation value of (a_n). Whether it is improper or proper accumulation value. It is represented as:

$$a = limsup_{n\to\infty} a_n$$

The difference between the limit superior and superior is that the limit superior is ALWAYS smaller than the

Limit Inferior The limit inferior of (a_n) is the smallest improper accumulation value of (a_n), and its represented by the notation

$$a = liminf_{n \to \infty} a_n$$

Accumulation Values and Divergence A value $a \in \mathbb{R} \cup \{+\infty, -\infty\}$ is called a accumulation value of a_n, if there exists a sub sequence a_{n_k} of a_n such that the limit $k \to \infty | a_{n_k} = a$

For improper cases:

$+\infty$ is a accumilation value if the sequences is unbounded above, meaning for every M∈R infinit

Improper accumulation values Any sequence that has no accumulation values has at least one improper accumulation value

Bolzano-Weierstrass Theorem (Teorema di Bolzano-Weierstrass)

- **Statement (Teorema di Bolzano-Weierstrass):** Every bounded sequence has a convergent subsequence. Formally:

$$(\exists K > 0 \text{ such that } \forall n \in \mathbb{N}, |a_n| \leq K) \implies \exists \{a_{n_k}\} \subseteq \{a_n\} \text{ and } \exists L \in \mathbb{R} \text{ such that } \lim_{k \to \infty} a_{n_k} = L$$

- **Key requirements:**
  - Bounded sequence $(\exists K > 0 : \forall n \in \mathbb{N}, |a_n| \leq K)$
- **Consequence:** Applies in $\mathbb{R}^n$ (via metodo degli intervalli incapsulati or Heine-Borel)

Theorem of Zero

Trigonometric Functions and π

Complex Polynomials

Fundamental Theorem of Algebra

Series

''I'm not lost, I am just taking a scenic route to understand it''

Series (Serie)

A series is the sum of the all the terms in a sequence. Formally, if a is a sequence and we want the infinite sum of the series:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...$$

Partial Sum (Somma Parziale)

A series does not have to necessarily have to be the sum of infinite terms. A partial sum, allows just the sum of the k terms:

$$S_k = \sum_{n=1}^{k} a_n$$

If $\sum a_n$ converges, then the lim

This series converges to S if $\lim_{k\to\infty} S_k = S$ otherwise, as mentioned in previous sections it diverges.

Necessary Condition for Convergence (Divergence Test)

If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
Equivalently, if $\lim_{n\to\infty} a_n \neq 0$ (or the limit does not exist), the series diverges.

Types of Series

Geometric (Serie Geometrica)

It is a series with the standard form $\sum_{n=0}^{\infty} ar^n$ where r is the common ratio. There are 3 primary types of geometric series:

Partial sum (Finite geometric series) For a finite number k, the partial sum

$$S_k = \sum_{n=0}^{n-1} ar^n = a + ar + ar^2 + ... + ar^{n-1}$$

Formula

$$S_k = a \times \frac{1 - r^n}{1 - r}$$

valid only if $r \neq 1$. If r=1 then

$$S_n = a \times n$$

There is also a alternate notation for the partial sum formula that is pretty commonly used.

$$S_{n+1} = \sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}$$

Infinite geometric series Theorem:
if $\sum_{n=0}^{\infty} ar^n$ converges then $|r| < 1$ and

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

Divergence The series is divergent if $|r| \geq 1$, and the terms do not approach zero

Telescoping (Serie Telescopica)

A telescopic series is a series where a lot of the terms cancel out when written in partial sums

$$\sum_{n=1}^{\infty}(a_n - a_{n+1})$$

whose partial sum will telescope to

$$S_k = \sum_{n=1}^{k}(a_n - a_{n+1}) = a_1 - a_{k+1}$$

When handling a limit to L

$$\sum_{n=1}^{\infty}(a_n - a_{n+1}) = a_1 - L$$

Note: This requires $\lim_{n\to\infty} a_n = L$.
Example: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ converges to 1.

Harmonic Series (Serie armonica)

A harmonic series is a series that diverges despite its terms tending to zero, it is represented as:

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

A small proof is listed below:

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+...+\frac{1}{8}\right)+... \geq 1+\frac{1}{2}+2\left(\frac{1}{4}\right)+4\left(\frac{1}{8}\right)+... = \sum_{k=0}^{\infty}\frac{1}{2} = \infty$$

This shows that $a_n \to 0$ is necessary for convergence, but not sufficient to prove it.

P series Harmonic series are a type of p series. P series is often used in comparison to check if series converge or not. It is represented as

$$\sum_{n=1}^{\infty}\frac{1}{n^p}$$

where it is said that if $p \leq 1$ the series diverges, and if $p > 1$ the series **converges**.

Associativity of series (Associatività della somma di serie)

The associativity of series, or the property that grouping terms in different ways does not affect the sum, holds for convergent series but not necessarily for divergent ones.

Series with varying signs (Serie a segno variabile)

Power Series (Serie di potenze)

- **Definition:** A series of the form:

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

  where c_n are coefficients, a is the center, and x is the variable.

- **Radius of Convergence (R):** Determined by:

$$R = \frac{1}{L} \quad \text{where} \quad L = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$$

  (Cauchy-Hadamard formula) or when it exists:

$$R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$$

- **Interval of Convergence:** The set $x \in (a-R, a+R)$ where:
  - Converges absolutely for $|x-a| < R$
  - May converge or diverge at endpoints $x = a \pm R$
  - Diverges for $|x-a| > R$

- **Properties:**
  - **Continuity:** Continuous on $(a - R, a + R)$
  - **Differentiation:** Can differentiate term-by-term:

    $$f'(x) = \sum_{n=1}^{\infty} nc_n(x - a)^{n-1}$$

    with same radius R
  - **Integration:** Can integrate term-by-term:

    $$\int f(x)dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x - a)^{n+1}$$

    with same radius R
- **Examples:**
  - Exponential: $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (R = \infty)$
  - Geometric: $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (R = 1)$
  - Arctangent: $\arctan x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad (R = 1)$
- **Important Theorem:** If $\sum c_n(x - a)^n = 0$ for all x in some interval, then $c_n = 0$ for all n.

Linearity of Series Sum (Teorema linearità somma)

- **Statement:** Given two convergent series $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, and constants $\alpha, \beta \in \mathbb{R}$, then:

  $$\sum_{n=1}^{\infty} (\alpha a_n + \beta b_n) = \alpha A + \beta B$$

- **Proof Outline:**
  - Let $s_N = \sum_{n=1}^{N} a_n \to A$ and $t_N = \sum_{n=1}^{N} b_n \to B$
  - The partial sum of the combined series:

    $$\sigma_N = \sum_{n=1}^{N} (\alpha a_n + \beta b_n) = \alpha s_N + \beta t_N$$

– By limit laws:

$$\lim_{N\to\infty} \sigma_N = \alpha \lim_{N\to\infty} s_N + \beta \lim_{N\to\infty} t_N = \alpha A + \beta B$$

- **Key Requirements:**
  - Both series must converge individually
  - Linearity fails for divergent series (counterexample: $a_n = (-1)^n$, $b_n = (-1)^{n+1}$)
- **Extension:** For finitely many convergent series $\sum a_n^{(k)} = S_k$ and constants c_k:

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{m} c_k a_n^{(k)} \right) = \sum_{k=1}^{m} c_k S_k$$

- **Warning:** Does *not* apply to conditionally convergent series rearrangements (Riemann series theorem).

Cesàro Summability (Convergenza alla Cesàro)

- **Definition:** A sequence (a_n) is Cesàro summable to limit L if the arithmetic mean of its first n partial sums converges to L:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} s_k = L$$

  where $s_k = a_1 + a_2 + \cdots + a_k$ are the partial sums.
- **Equivalent Form:** For the sequence (a_n) itself:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k = L$$

- **Key Properties:**
  - **Regularity:** If $\lim_{n\to\infty} a_n = L$ exists, then (a_n) is Cesàro summable to L
  - **Strictly Weaker:** Cesàro summability doesn't imply convergence (e.g., $a_n = (-1)^n$)
  - **Linearity:** If $a_n \to_{\mathsf{C}} A$ and $b_n \to_{\mathsf{C}} B$, then:

$$\alpha a_n + \beta b_n \to_{\mathsf{C}} \alpha A + \beta B$$

- **Example (Divergent Sequence):** Consider $a_n = (-1)^{n+1}$:

$$\text{Sequence: } 1, -1, 1, -1, 1, -1, \ldots$$

$$\text{Partial sums: } s_n = \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$\text{Cesàro mean: } \sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k \to \frac{1}{2}$$

Thus (a_n) is Cesàro summable to $1/2$ despite divergence.

Weierstrass M-Test (Criterio M di Weierstrass)

- **Statement:** Let $\{f_n\}$ be a sequence of functions on $E \subseteq \mathbb{R}$. If $\exists \{M_n\} \subset \mathbb{R}$ such that:

$$\text{(i) } |f_n(x)| \le M_n \quad \forall x \in E, \quad \forall n \in \mathbb{N}$$

$$\text{(ii) } \sum_{n=1}^{\infty} M_n < \infty$$

Then $\sum_{n=1}^{\infty} f_n$ converges uniformly on E.

- **Key requirements:**
  - Dominating series $\sum M_n$ must converge (absolutely)
  - M_n independent of $x \in E$
  - $M_n \ge 0$ (non-negative majorants)
- **Consequences:**
  - Preserves continuity: If $f_n \in \mathcal{C}(E)$, then $\sum f_n \in \mathcal{C}(E)$
  - Term-by-term integration:

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)dx$$

  - Term-by-term differentiation (requires additional convergence of $\sum f_n'$)

Special Case: Weierstrass Factorization

- **Statement:** Every entire function f can be represented as:

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right)$$

where a_n are non-zero zeros, m is zero multiplicity at origin, g entire, and E_p are elementary factors.

Linearity of Series Sum (Teorema linearità somma)

- **Statement:** Given two convergent series $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, and constants $\alpha, \beta \in \mathbb{R}$, then:

$$\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n) = \alpha A + \beta B$$

- **Proof Outline:**
  - Let $s_N = \sum_{n=1}^{N} a_n \to A$ and $t_N = \sum_{n=1}^{N} b_n \to B$
  - The partial sum of the combined series:

$$\sigma_N = \sum_{n=1}^{N}(\alpha a_n + \beta b_n) = \alpha s_N + \beta t_N$$

  - By limit laws:

$$\lim_{N\to\infty} \sigma_N = \alpha \lim_{N\to\infty} s_N + \beta \lim_{N\to\infty} t_N = \alpha A + \beta B$$

- **Key Requirements:**
  - Both series must converge individually
  - Linearity fails for divergent series (counterexample: $a_n = (-1)^n$, $b_n = (-1)^{n+1}$)
- **Extension:** For finitely many convergent series $\sum a_n^{(k)} = S_k$ and constants c_k:

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{m} c_k a_n^{(k)}\right) = \sum_{k=1}^{m} c_k S_k$$

- **Warning:** Does *not* apply to conditionally convergent series rearrangements (Riemann series theorem).

~~Cauchy Criterion~~

~~A series a converges if and and only if for every epsilon that is grater than zero there exists a N such that for all $m > n \geq N$~~

$$\cancel{\left|\sum_{k=n+1}^{m} a_k\right| < \epsilon}$$

~~**Absolute and Conditional Convergence**~~

~~**Using Cauchy criterion in problems**~~

~~**For a Sequence**~~ ~~There are 4 main steps:~~

~~1. Assign a epsilon that is grater than zero~~
~~2. Find a integer N such that for all m, n > N, the inequality~~ $\cancel{|a_m - a_n| < \epsilon}$ ~~holds.~~

~~**For a Series** The steps are very similar to a sequence~~
~~1. Assign a epsilon that is grater than zero~~
~~2. Find a integer N such that for all m>n>N, the inequality~~ $\cancel{|\sum_{k=n+1}^{m} a_k| < \epsilon}$

Handling Convergence in Problems

Comparison (Criterio del confronto)

To find convergence using comparison, there are two primary methods used. Direct comparison test and the Limit comparison test.

Direct Comparison Test For two series $\sum a_n$ and $\sum b_n$ with $0 \leq a_n \leq b_n$ for all $n \geq N$
If $\sum b_n$ converges then $\sum a_n$ will converge
If $\sum a_n$ diverges then $\sum b_n$ will diverge

Limit Comparison Test For two series $\sum a_n$ and $\sum b_n$ with $a_n > 0, b_n > 0$

$$L = lim_{n \to \infty} \frac{a_n}{b_n}$$

If L>0 $\sum a_n$ and $\sum b_n$ share the same convergence type
If L=0 and $\sum b_n$ converges then $\sum a_n$ also converges
If L=∞ and $\sum b_n$ diverges, then $\sum a_n$ also diverges

Ratio (Criterio del rapporto)

A series $\sum a_n$, calculate the limit

$$L = lim_{n \to \infty} |\frac{a_{n+1}}{a_n}|$$

If L $<$ 1 the series converges
If L $>$ 1 the series diverges
if L $=$ 1 this test doesn't give enough info do determine a result

Root (Criterio della radice)

A series $\sum a_n$, calculate the limit

$$L = lim_{n \to \infty} \sqrt[n]{|a_n|}$$

If L $<$ 1 the series converges
If L $>$ 1 the series diverges
if L $=$ 1 this test doesn't give enough info do determine a result

Integral (Criterio del integrale)

A series $\sum_{n=N}^{\infty} a_n$ with continuous, positive and decreasing terms, $a_n = f(n)$ for $n \geq N$

If the integral $\int_N^{\infty} f(x)dx$ converges, then the series $\sum a_n$ converges

If the integral diverges then $\sum a_n$ diverge

Cauchy Condensation Test (it is another way to say the same thing as above, I may remove this)

If $a_n \geq 0$ and a_n is decreasing, then $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.

Alternating Series Test (Leibniz)

An alternating series $\sum(-1)^n b_n$ or $\sum(-1)^{n+1} b_n$ converges if:

1. **$b_n \geq 0$ and monotonically decreasing: $b_{n+1} \leq b_n$**
2. **$\lim_{n\to\infty} b_n = 0$**

This implies conditional convergence (converges but not absolutely).

Post-Derivatives

Differential Quotient/Derivative

“The Difference Between the Almost Right Word and the Right Word Is Really a Large Matter—’Tis the Difference Between the Lightning Bug and the Lightning” - Mark Twain

Fundamental Formula for derivatives

The fundamental goal of a derivative is to measure a functions behaviour at a single point. For a function to be differentiable, it must be continuous & ....

Secant (Retta Secante) Take a function $f : \mathbb{R} \to \mathbb{R}$, and pick two points on its graph, called x and x1. The direct line that connects x and x1 is called the secant. It is calculated like so:

$$\frac{f(x) - f(x_1)}{x - x_1}$$

Consider that the denominator is the horizontal difference, and the nominator is the vertical difference. The secant is useful for describing how a function changes over two set distances, but it cannot be used to show more precise info regarding the graph.

Tangent (Retta Tangente) The tangent is almost identical in concept to the secant, with one key difference. The tangent applies a limit from $x \to x_1$. So as x moves closer to x1, eventually using a limit they show the functions behaviour at a single point. More formally its calculated like so:

$$lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1}$$

If this limit does not exist then the function is ‘’non differentiable’’, if instead the limit does exists it defines the slope of the tangent at the point x.

Formal Definition The differential quotient is more formally represented as:

$$f'(x) = lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Where h:

$$h = x - x_1$$

which means:

$$f'(x) = lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1}$$

Differentiablility implies continuity (Property)

$$f(x) = f(x_1) + f'(x_1)(x - x_1) + (x - x_1)r(x)$$

where r(x) -> 0 since x -> x1, taking the limits:

$$lim_{x \to x_1} f(x) = f(x_1) + 0 + 0 = f(x_1)$$

∴f is continuous at x1

WARNING; THE REVERSE IS NOT TRUE, not all functions that are continuous are differentiable.

Methods for Differentiation

Sum Rule

$$\frac{d}{dx}\left(f + g\right) = f'\left(x\right) + g'\left(x\right) \tag{9}$$

Product rule

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x) \tag{10}$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \tag{11}$$

Chain Rule

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) \tag{12}$$

Inverse derivatives

Local Extreme and Rolle's Theorem

If a function $f : [a, b] \in \mathbb{R}$ is defined on a closed interval [a,b], we can identify four distinct features: local maximum, local minimum, global maximum, and global minimum. The local minimum and local maximum are points within [a,b] where the function reaches its lowest or highest value in a small neighborhood around them. They can be found at tangent = 0, the derivative is undefined, at endpoints like a,b. The global maximum and global minimum are the absolute highest and lowest points on the entire interval [a,b]. These always exist for continuous functions on closed intervals.

Rolle's Theorem

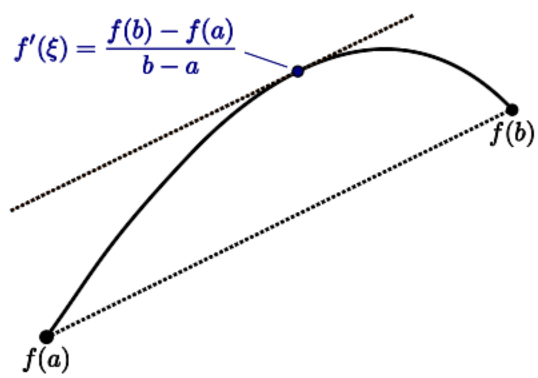
Rolle's theorem states that, ''If any differentiable function crosses the same point on the y axis more than once, there must always be at least point where the tangent is zero, this point is often known as a stationary point or as $\hat{x}$. More formally it is defined as:

$f : [a, b] \to \mathbb{R}$is continuous and differentiable on (a,b) and f(a)=f(b), then there exists $\hat{x} \in (a, b)$ with $f'(x) = 0$

Rolle's Theorem fails for non-differentiable functions

Mean Value Theorem

The mean value theorem involves two points on the same function f. The mean value theorem states, that by taking the secant of those two points (a,b) you can calculate what the tangent of a point on the same function and in between a and b. See image below as reference.



[1]

Uses: Assume that $f : [a, b] \to \mathbb{R}$ be differentiable

Strictly Monotonically Increasing If $f'(x) > 0$ for all $x \in [a, b]$: then $\hat{x} \in (x_1, x_2)$ with $f'(\hat{x}) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$

Rewritten then

$$f(x_2) - f(x_1) = \underbrace{f'(\hat{x})}_{>0} \cdot \underbrace{(x_2 - x_1)}_{>0} > 0$$

This shows it is strictly monotonically increasing

Strictly Monotonically Decreasing If $f'(x) < 0$ for all $x \in [a, b]$

This shows it is strictly monotonically decreasing.

Follow similar steps as above to show <0

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Monotonically Increasing If $f'(x) \geq 0$ for all $x \in [a, b]$
This shows monotonically increasing

Follow similar steps as above to show $>= 0$

Monotonically Decreasing If $f'(x) \leq 0$ for all $x \in [a, b]$
This shows monotonically decreasing

Follow similar steps as above to show $=< 0$

Auxiliary Function for proof

$$h(x) = f(x) - \left[f(a) + \underbrace{\frac{f(b) - f(a)}{b - a}}_{slope} (x - a) \right]$$

f(a): anchors the secant line to a
(x-a): represents horizontal displacement

Extended Mean Value Theorem

There are two functions $f, g : [a, b] \to \mathbb{R}$ which need to be differentiable and $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists $\hat{x} \in (a, b)$

Giving the following formula:

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\hat{x})}{g'(\hat{x})}$$

EMVT auxiliary function

$$h(x) = f(x) - \left[f(a) + \underbrace{\frac{f(b) - f(a)}{g(b) - g(a)}}_{Slope(k)} (g(x) - g(a)) \right]$$

f(a): anchors the secant line
(g(x)-g(a)): replaces (x-a) with displacement measured by g

L'Hôpital's rule

Is a theorem, which is primarily used to check limits by comparing the derivatives of the limits.

Definition f and g are functions which are differentiable on a open interval. Assuming that,

$lim_{x \to a} f(x) = lim_{x \to a} g(x) = 0$, $g'(x) \neq 0$ and $lim_{x \to a} \frac{f'(x)}{g'(x)}$

Then:

$$lim_{x \to a} \frac{f(x)}{g(x)} = lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Higher-Order Derivatives

The n-th derivative is defined recursively:

$$f^{(n)}(x) = \frac{d}{dx} \left[f^{(n-1)}(x) \right]$$

Other notations include

$$f^n = \frac{d^n f}{dx^n} = \frac{d^n}{dx^n} f$$

Min max test Assuming that $f'(x_0) = 0$ and that $f''(x_0)$ exists then

Local Minimum $f''(x_0) > 0$ Local min at x0

Local Maximum $f''(x_0) < 0$ Local max at x0

If $f''(x_0) = 0$ the test fails. Note: Any derivative higher than 2 is valid as well.

Taylor's Theorem (Teorema di Taylor)

Taylor's theorem provides a way to approximate a function f with a polynomial near a point x_0, called the expansion point (punto di espansione). For a function that is differentiable $(n+1)$ times on an interval I, its value at $x = x_0 + h$ can be written as:

$$f(x_0 + h) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} h^k}_{T_n(h)} + \underbrace{R_n(h)}_{\text{remainder (resto)}}$$

$T_n(h)$ is the n-th order Taylor polynomial (polinomio di Taylor), our approximation of the function. The term $R_n(h)$ is the remainder, which is the error of the approximation, given by:

$$R_n(h) = \frac{f^{(n+1)}(c)}{(n+1)!} h^{n+1}$$

where c is a point between x_0 and $x_0 + h$. The theorem can also be expressed in terms of x:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + R_n(x), \quad \text{with } R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$$

Key Properties

The Taylor polynomial $T_n(h)$ is the unique polynomial of its degree that matches the function's value and its first n derivatives at x_0. This is expressed as $T_n^{(k)}(0) = f^{(k)}(x_0)$ for $k = 0, 1, \ldots, n$.

The remainder $R_n(h)$ becomes very small as h approaches 0, ensuring the approximation grows more accurate the closer we are to x_0. Specifically, the error decreases faster than h^n.

To estimate the error, we can set a bound. If the absolute value of the $(n+1)$-th derivative, $\left|f^{(n+1)}(z)\right|$, is less than or equal to a constant M for all z between x_0 and x, then the remainder is bounded by:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}.$$

Application Procedure

To approximate $f(x)$ near x_0, the process is straightforward. First, compute derivatives (calcolare derivate) of f at x_0 up to the n-th order. Second, construct T_n by assembling the polynomial:

$$T_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Third, bound the remainder to check the approximation's accuracy. Find a value M that is greater than or equal to $\left|f^{(n+1)}(z)\right|$ for all z between x_0 and x. The error is then given by:

$$|f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}.$$

Exponential/Logarithmic Derivatives

- $\frac{d}{dx}[e^x] = e^x$
- $\frac{d}{dx}[a^x] = a^x \ln a$
- $\frac{d}{dx}[\ln x] = \frac{1}{x}$
- $\frac{d}{dx}[\log_a x] = \frac{1}{x \ln a}$

Trigonometric Derivatives

- $\frac{d}{dx}[\sin x] = \cos x$
- $\frac{d}{dx}[\cos x] = -\sin x$
- $\frac{d}{dx}[\tan x] = \sec^2 x$
- $\frac{d}{dx}[\cot x] = -\csc^2 x$

- $\frac{d}{dx}[\sec x] = \sec x \tan x$
- $\frac{d}{dx}[\csc x] = -\csc x \cot x$

Inverse Trig Derivatives

- $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}[\cos^{-1} x] = -\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$
- $\frac{d}{dx}[\cot^{-1} x] = -\frac{1}{1+x^2}$
- $\frac{d}{dx}[\sec^{-1} x] = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}[\csc^{-1} x] = -\frac{1}{|x|\sqrt{x^2-1}}$

Implicit Differentiation

Example for $x^2 + y^2 = 1$:

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Parametric Derivatives

For $x = x(t)$, $y = y(t)$:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{d^2y}{dx^2} = \frac{d/dt(dy/dx)}{dx/dt}$$

Hyperbolic Derivatives

- $\frac{d}{dx}[\sinh x] = \cosh x$
- $\frac{d}{dx}[\cosh x] = \sinh x$
- $\frac{d}{dx}[\tanh x] = \text{sech}^2 x$

Inverse Function Derivative

If f and g are inverses:

$$g'(x) = \frac{1}{f'(g(x))}$$

Integrals

"If you're gonna shoot an elephant Mr. Schneider, you better be prepared to finish the job."
— Gary Larson, The Far Side

The Riemann Integral (Integrale di Riemann)

The Riemann integral represents the signed area between a function's graph and the x-axis on an interval $[a, b]$. It is calculated by partitioning the interval and summing the areas of rectangles, known as a step function (funzione a gradino).

$$\int_a^b \phi(x)\, dx := \sum_{j=1}^{n} c_j \cdot (x_j - x_{j-1})$$

General Riemann Integral

For a general function f, the integral is found by approximating it with step functions. A function is Riemann integrable if the approximations from above and below meet at the same value. Continuous and monotonic functions are always Riemann integrable.

Properties of the Riemann Integral

The integral is a linear and monotonic map.

For a point c between a and b, an integral can be split:

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

By convention, reversing the integration bounds negates the value:

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

Mean Value Theorem for Integration (Teorema della media integrale)

This theorem states that for a continuous function f on $[a, b]$, there exists a point $x_{hat} \in [a, b]$ where the function's value is the average value over the interval. This means the area under the curve is equal to the area of a rectangle with height $f(x_{hat})$. This concept is a key step in proving the Fundamental Theorem.

$$\int_a^b f(x)\, dx = f(x_{hat}) \cdot (b - a)$$

The Fundamental Theorem of Calculus (Teorema fondamentale del calcolo integrale)

This theorem connects differentiation and integration, showing they are inverse operations. First, an antiderivative (primitiva) is a function F whose derivative is f, so $F' = f$.

First Fundamental Theorem of Calculus: This part shows that the derivative of an integral function is the original function.

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Second Fundamental Theorem of Calculus: This part provides a method to calculate definite integrals using an antiderivative F.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Integration Rules

Integration by Substitution (Integrazione per sostituzione)

This rule, also known as change of variables (cambiamento di variabili), simplifies integrals of composite functions and is the integral counterpart to the chain rule. For a continuous function f and a continuously differentiable function ϕ, the substitution $x = \phi(t)$ transforms the integral:

$$\int_a^b f(\phi(t))\phi'(t)\,dt = \int_{\phi(a)}^{\phi(b)} f(x)\,dx$$

The relation $dx = \phi'(t)\,dt$ is used to change the differential, and the limits of integration are updated from $[a, b]$ to $[\phi(a), \phi(b)]$.

When applying the rule in reverse, the function ϕ must be invertible (bijective). The substitution $x = \phi(t)$ transforms the integral as follows:

$$\int_a^b f(x)\,dx = \int_{\phi^{-1}(a)}^{\phi^{-1}(b)} f(\phi(t))\phi'(t)\,dt$$

Here, the new integration bounds are found using the inverse function, ϕ^{-1}.

Partial Fraction Decomposition (Decomposizione di fratti semplici)

Given a rational function $f(x) = \frac{P(x)}{Q(x)}$ where the degree of $P(x)$ is strictly less than the degree of $Q(x)$. The decomposition of $f(x)$ depends on the nature of the roots of the denominator $Q(x)$.

Distinct Real Roots

This case applies when the denominator $Q(x)$ of degree n has n distinct real roots, denoted $x_1, x_2, \ldots, x_n$. The denominator can be factored as:

$$Q(x) = (x - x_1)(x - x_2)\ldots(x - x_n)$$

The partial fraction decomposition is a sum of n fractions:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - x_1} + \frac{A_2}{x - x_2} + \cdots + \frac{A_n}{x - x_n}$$

The coefficients $A_1, A_2, \ldots, A_n$ are real constants that must be determined.

Repeated Real Roots

This case applies when $Q(x)$ has fewer than n distinct real roots, meaning at least one root has a multiplicity greater than 1. Let the distinct roots be $x_1, \ldots, x_k$ with corresponding multiplicities $\alpha_1, \ldots, \alpha_k$, such that $\sum_{i=1}^{k} \alpha_i = n$. A root x_i with multiplicity α_i contributes α_i terms to the decomposition:

$$\frac{A_{i,1}}{x - x_i} + \frac{A_{i,2}}{(x - x_i)^2} + \cdots + \frac{A_{i,\alpha_i}}{(x - x_i)^{\alpha_i}}$$

The full decomposition is the sum of these groups of terms for all distinct roots.

Complex Roots

This case applies when $Q(x)$ has at least one pair of complex conjugate roots. The procedure is analogous to the real root cases, but the roots x_i and coefficients A_i can be complex numbers. The calculations follow the same algebraic rules. For real-valued rational functions, the final expression can be simplified to avoid imaginary units by combining conjugate terms.

Note on Finding Coefficients To find the unknown coefficients (the A_i values), one typically uses the method of equating coefficients. After setting up the decomposition and multiplying both sides by the original denominator $Q(x)$, an identity between two polynomials is formed.

$$P(x) = \text{Polynomial in } x \text{ with coefficients as functions of } A_i$$

For this identity to hold for all x, the coefficients of corresponding powers of x on both sides of the equation must be equal. This creates a system of linear equations which can be solved for the unknown coefficients.

Improper Integrals (Integrali Impropri)

Improper integrals extend the concept of the Riemann integral to cases where the domain of integration is unbounded or the function itself is unbounded within the domain. These are evaluated by taking a limit of a definite integral as an endpoint of the integration interval approaches a specific real number or infinity.

Type 1: Unbounded Domains (Domini Illimitati)

This type of improper integral applies when the interval of integration extends to infinity. For a function f that is Riemann integrable on every interval $[a, b]$ for $b > a$, the integral over $[a, \infty)$ is defined as the limit:

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty} \int_a^b f(x)\,dx$$

The improper integral is said to converge if this limit exists and is finite. A similar definition applies for intervals of the form $(-\infty, b]$ or $(-\infty, \infty)$.

Type 2: Unbounded Functions (Funzioni Illimitate)

This type addresses functions that have a vertical asymptote, or "pole," at some point within the integration interval. If a function f is unbounded at the endpoint a of the interval $(a, b]$ but is Riemann integrable on every subinterval $[a + \epsilon, b]$ for $\epsilon > 0$, the integral is defined as:

$$\int_a^b f(x)\,dx := \lim_{\epsilon\to 0^+} \int_{a+\epsilon}^b f(x)\,dx$$

If the limit exists and is finite, the integral converges. A corresponding definition holds if the function is unbounded at the upper endpoint b.

Comparison Test (Criterio del Confronto)

The convergence of an improper integral can often be determined by comparing it to an integral whose convergence properties are known. Let f and g be two functions integrable on any closed subinterval of $[a, \infty)$.

If $0 \le f(x) \le g(x)$ for all $x \ge a$, then the convergence of the integral of the majorant function g implies the convergence of the integral of f.

$$\int_a^\infty g(x)\,dx < \infty \implies \int_a^\infty f(x)\,dx < \infty$$

Conversely, the divergence of the integral of the minorant function f implies the divergence of the integral of g.

Integral Test for Series (Criterio dell'Integrale per le Serie)

This test provides a direct link between the convergence of an infinite series and an improper integral. For a function f that is continuous, positive, and monotonically decreasing on the interval $[N, \infty)$, the series $\sum_{n=N}^\infty f(n)$ converges if and only if the improper integral $\int_N^\infty f(x)\,dx$ converges.

$$\sum_{n=N}^\infty f(n) < \infty \iff \int_N^\infty f(x)\,dx < \infty$$

This relationship allows the use of integration techniques to analyze the convergence of series.

Cauchy Principal Value (Valore Principale di Cauchy)

When a function f has a discontinuity at a point p inside the interval $[a, b]$, the integral is typically split into two parts. The integral converges only if both limits exist independently:

$$\int_a^b f(x)\,dx = \lim_{\epsilon_1 \to 0^+} \int_a^{p-\epsilon_1} f(x)\,dx + \lim_{\epsilon_2 \to 0^+} \int_{p+\epsilon_2}^b f(x)\,dx$$

However, a weaker form of convergence can be defined using a symmetric limit. The Cauchy Principal Value is defined as:

$$\text{P.V.} \int_a^b f(x)\,dx := \lim_{\epsilon \to 0^+} \left(\int_a^{p-\epsilon} f(x)\,dx + \int_{p+\epsilon}^b f(x)\,dx \right)$$

This value may exist even when the individual limits do not, providing a method for assigning a value to certain divergent integrals.

BUFFER ZONE FROM OLD NOTES

Improper Integral Convergence Criteria

For $\int_a^\infty f(x)dx$ or $\int_a^b f(x)dx$ with singularity at b:

Type	Condition
$\int_1^\infty x^{-p}dx$	Converges iff $p > 1$
$\int_0^1 x^{-p}dx$	Converges iff $p < 1$
$\int_a^b \lvert x-c\rvert^{-p}dx$	Converges iff $p < 1$
Oscillatory $\int_1^\infty g(x)dx$	Converges if $\int_1^\infty \lvert g(x)\rvert dx < \infty$ (absolute conv.) or by Dirichlet test (bounded antiderivative, monotone $\to 0$)

Limit Comparison Test Framework

Given $f(x) \sim g(x)$ as $x \to c$ (singularity or ∞):

1. Compute $L = \lim_{x\to c} \frac{f(x)}{g(x)}$
2. If $0 < L < \infty$, then $\int f$ and $\int g$ converge/diverge together
3. For $L = 0$: $\int g < \infty \Rightarrow \int f < \infty$
4. For $L = \infty$: $\int g = \infty \Rightarrow \int f = \infty$

Singularity Analysis Toolkit

Near singular point c:

$$\begin{aligned}
|\sin x| &\sim |x - k\pi| \quad \text{near } x = k\pi \\
|\ln x| &\sim |x - 1| \quad \text{near } x = 1 \\
\sqrt{x^2 + a} - x &\sim \frac{a}{2x} \quad \text{as } x \to \infty \\
e^x - 1 - x &\sim \frac{x^2}{2} \quad \text{near } x = 0
\end{aligned}$$

Parameter-Dependent Integrals Strategy

For $\int_a^b h(x, \alpha, \beta)dx$ with $\alpha, \beta > 0$:

Problem Type	Solution Approach
$\int_0^1 x^{-\alpha}\lvert\ln x\rvert^\beta dx$	1. Near 0: compare to $x^{-\alpha+\epsilon}$ 2. Near 1: compare to $\lvert x-1\rvert^\beta$ $\boxed{\text{Converges iff } \alpha < 1}$
$\int_0^\infty \frac{dx}{x^\alpha \lvert\sin x\rvert^\beta}$	1. Near $k\pi$: $\lvert\sin x\rvert^{-\beta} \sim \lvert x-k\pi\rvert^{-\beta}$ 2. At ∞: periodic singularities $\Rightarrow$ sum test $\boxed{\text{Finite interval: } \beta < 1}$ $\boxed{\text{Infinite interval: } \beta < 1 \text{ and } \alpha > 1}$

Advanced Convergence Techniques

1. **Integration by Parts:** For $\int f'g$, requires $\lim_{x\to c} f(x)g(x)$ exists and $\int fg' < \infty$
2. **Series Comparison:** For $\int_1^\infty$, decompose into $\sum \int_n^{n+1}$ and compare to series
3. **Exponential Domination:** $x^p e^{-ax} \to 0$ for all p when $a > 0$

Parameter Limit Computations

For $F(a) = \int_0^\infty g(x, a)dx$:

1. Justify limit-interchange via Dominated Convergence or uniform bounds
2. For $F_a(x) = \int_0^{x^a} h(t)dt$:

$$\lim_{x\to 0^+} F_a(x) = \begin{cases} 0 & a > 0 \\ \int_0^\infty h(t)dt & a < 0 \end{cases}$$

$$F_a'(0) = \begin{cases} 0 & a > 1 \\ \text{DNE} & a \le 1 \end{cases}$$

Theoretical Result Template

Given $f \in C^1((0, 1])$ with $\int_0^1 \sqrt{x}|f'(x)|dx < \infty$:

1. $\sqrt{\epsilon}f(\epsilon) \to 0$ as $\epsilon \to 0^+$ by:

$$\left|\sqrt{\epsilon}f(\epsilon)\right| \le \sqrt{\epsilon}|f(1)| + \int_\epsilon^1 \sqrt{t}|f'(t)|\frac{\sqrt{\epsilon}}{\sqrt{t}}dt$$

2. $\int_0^1 \frac{f(x)}{\sqrt{x}}dx < \infty$ via integration by parts:

$$\int_\epsilon^1 \frac{f(x)}{\sqrt{x}}dx = \left[2\sqrt{x}f(x)\right]_\epsilon^1 - 2\int_\epsilon^1 \sqrt{x}f'(x)dx$$

Differential Equations

"Would you tell me, please, which way I ought to go from here?"
"That depends a good deal on where you want to get to," said the Cat.

ODEs and PDEs

Ordinary Differential Equations (ODEs) are a differential equation which has a single variable. ODEs have a general form:

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^ny}{dx^n}\right) = 0 \tag{13}$$

where
- x independent
- y dependent

Partial Differential Equations (PDEs) are a Differential equation which has multiple independent variables. Instead of using the standard d, they use partial derivatives (∂) to show the change with respect for multiple variables. The general form is:

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, ...\right) = 0 \tag{14}$$

where
- x,y independent
- u(x,y) dependent

Fundamentally image a ODE as a means to track a single car, while PDE track all the traffic in the city.

Types of ODEs

ODEs are usually classified by 2 primary things. their order, aka the degree of their derivative, and by whether they are linear or non-linear.

First order ODEs are pretty self explanatory, they involve only the 1st derivative. Here is a basic first order ODE:

$$\frac{dy}{dx} + y = x \tag{15}$$

Second order ODEs involve UP to the 2nd derivative. Here is a example:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \tag{16}$$

Higher order ODEs involve everything 3rd derivative or higher. It is unlikely to ever appear in a 1st year analysis exam, but you never know.

Liniar and Non-liniear ODEs. An ODE is linear if the dependent variable and the derivatives are in a linear form. Basically: they are not multiplied together. Anything else is considered non-liniear. A linear ODE can be written in the form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \tag{17}$$

- a(x) is a function of x

Here are some basic examples. We will go in much more detail when solving ODEs.

$\frac{dy}{dx} + 3y = x$, and $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = sinx$

$$Ineedtolearnbetterformating :)$$

A non-linear ODE is any ordinary differential equation that cannot be written in the linear form shown earlier. This is because the dependent variable or its derivatives are not linear. I will cover this more later on in the chapter.

The weird classifications of ODEs

A Homogeneous ODE, is a differential equation where L is a linear differential operator.

$$L[y] = 0$$

A Non-Homogeneous ODE, is a differential equation where f(x) $\neq$ 0

$$L[y] = f(x)$$

A Autonomous ODE, is a differential equation where the independent variable (usually x or t) does not appear in the equation.

$$\frac{d^n y}{dx^n} = F\left(y, y', ..., y^{(n-1)}\right)$$

A Non-Autonomous ODE, is a differential equation if the independent variable appears eq (1).

*You can use multiple types at once, just use common sense to make sure its right

Basic existence

Dealing with ODEs

Separable ODE, can be written as

$$\frac{dy}{dx} = f(x)g(y)$$

Divide both sides by g(y), and multiply both sides by dx

$$\frac{dy}{g(y)} = f(x)dx$$

Integrate both sides, make sure to keep the constant on the RHS

$$\int \frac{dy}{g(y)} = \int f(x)dx$$

Integrating Method. Format the equation to fit the following before using the method

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Find the function $\mu(x)$ that will help simplify the problem. (Symplify as much as possible here it will help a lot later on)

$$\mu(x) = e^{\int P(x)dx}$$

Multiply every term by the function $\mu(x)$ and using the product rule calculate the derivative of the LHS

$$\frac{d}{dx}(\mu(x)y) = \mu(x)Q(x)$$

Integrate both sides, remember the constant!

$$y = \frac{1}{\mu(x)} \int \mu(x)Q(x)dx + C$$

Quick note on integrating factor. A integrating factor is the method used to solve derivative equation above. The μ(x) also called the integrating factor, works for any, first-order linear differential equations. A function is derived by multiplying the equation with μ(x), which makes the left-hand side a derivative of μ(x)y.

Exact Method If a DE is exact, which can be found if it is in this form

$$M(x, y)dx + N(x, y)dy = 0$$

After this, calculate the partial derivative of M in respect to y and the partial derivative of N with respect to x. If these are equivalent the DE is exact.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Lets make a function that we will call Ψ such that, $\mathit{\Psi}_x = M(x, y)$ and $\Psi_y = N(x, y)$

Therefore we can write this now as

$$\Psi_x + \Psi_y \frac{dy}{dx} = 0$$

we can start to find this function Ψ. So we will start to integrate M with respect to x. (h(y) is a function of y)

$$\Psi(x, y) = \int M dx + h(y)$$

We can now differentiate Ψ with respect to y

$$\frac{\partial \Psi}{\partial y} = \frac{\partial}{\partial y}\left(\int M dx\right) + h'(y) = N$$

Solve the integral for h(y)

Types of Problems

Initial Value Problem

Generally, IVPs are a DE and a initial condition or condition's which when used in unison they can be used to solve a function, that will also fit the DE. The steps are pretty straight forward.

1. Solve the DE

$$y(x) = \int f(x)dx + C$$

2. Use the initial condition, lets say that $y(x_0) = y_0$

$$y_0 = \int f(x_0)dx + C$$

Where C is:

$$C = y_0 - V$$

*V is the value of the integral at x_0, therefore if we replace C, the final answer is

$$y(x) = \int f(x)dx + (y_0 - V)$$

Proving existence and uniqueness

Theorem Definition: If f(x,y) and the partial derivative $\frac{\partial f}{\partial y}$ are continuous in:

$$D = \{(x,y)||x-x_0| \leq a, |y-y_0| \leq b\}$$

around the point (x_0, y_0) therefore, there exits a interval between x and $x_0$where there is at least one solution of y(x). And it proves that this solution is unique on the interval.

Steps: Write the ODE in standard form, aka:

$$\frac{dy}{dx} = f(x,y)$$

use intuition to check that the function f is continuous between the points you want. Then if there are any points where the function is non continuous, then be sure to mark it such that it is clear, using $\leq \geq > <$. This is the first rule.

Now, do a partial derivative of y such that:

$$\frac{\partial f}{\partial y} = f(x,y)$$

Remember to treat x as a constant in this case!!!

If the result is continuous in D between x and x_0 then this proves that there is at least a solution.

Interpreting answers

Interval of validity for Linear DE. The interval of validity for a Linear DE is largest around $x_0$where $p(x)$ and $q(x)$ are continuous. Make sure to exclude discontinuities.

Interval of validity for Non-Linear DE, Solutions may be exponential to infinity or become undefined despite $f(x,y)$ being smooth. Therefore check all the points where the solution becomes undefined. Extend the interval on both sides of x_0 till it is no longer possible.

Interpreting answers

(this whole section is a bit useless, I may remove it if I don't find any use for it son)

Explicit Solution, is when the dependent variable is isolated and in terms of the independent variable. e.g

$$y = x^2 + C$$

(soluzione esplicita)

Implicit Solution, is when the dependent variable is not explicitly isolated from the independent. e.g

$$x^2 + y^2 = C$$

(soluzione implicita)

General Solution, is the solution containing all the possible solutions for the differential equation, ie it keeps the constants, its the trivial form. (soluzione generale)

Particular Solution, is the solution which is a specific solution by locking the constants by using the initial conditions, ie the C has a fixed value. (soluzione particolare)

Equilibrium Solution, is a solution which is constant because the dependent variable does not change and therefore the derivative is zero. (soluzione di equilibrio)

Parametric Solution, is a solution represented using a parameter like (t,u,z..) instead of using x and y. eg.

$$y(t) = \sqrt{t^2 + C}$$

(soluzione parametrica)

Homogeneous Equations with constant coefficients have the general form:

$$y'' + ay' + by = 0$$

Non-homogeneous Equations: Method of Undetermined Coefficients

$$Finished$$

Nth order Differential Equation

Systems of Differential Equations

Special theorems and Problems

Picard–Lindelöf theorem

Function Analysis

These questions test your ability to analyze specific function properties, often without requiring a full graphical study.

Strategy for Max/Min

1. **Identify Domain:** Determine the function's domain. Pay close attention to the boundaries.
2. **First Derivative:** Calculate the first derivative, $f'(x)$.
3. **Critical Points:** Find the critical points by solving $f'(x) = 0$ and identifying where $f'(x)$ is undefined.
4. **Monotonicity:** Study the sign of $f'(x)$ to find intervals where the function is increasing or decreasing.
5. **Evaluate:** Calculate the value of $f(x)$ at the critical points and at the boundaries of the domain.
6. **Conclude:** The largest value found is the absolute maximum, and the smallest is the absolute minimum.

Common Pitfall: Forgetting to evaluate the function at the domain's boundaries.

Strategy for Inequalities (e.g., prove $f(x) > g(x)$)

1. **Define Auxiliary Function:** Create a new function $h(x) = f(x) - g(x)$. The goal is now to prove $h(x) > 0$.
2. **Find Minimum:** Analyze the sign of $h'(x)$ to find the absolute minimum of $h(x)$ on the given interval.
3. **Check Minimum's Value:** If the minimum value, h_{min}, is greater than 0, the inequality is proven.
4. **Tangent Point Case:** Often, the minimum is a point x_0 where $h(x_0) = 0$. In this case, you must show that x_0 is a minimum (e.g., by checking that $h'(x_0) = 0$ and $h''(x_0) > 0$, or by studying the sign of $h'(x)$ around x_0). This proves that $h(x) \geq 0$, and is only 0 at that single point.

Strategy for Number of Solutions to $f(x) = k$

1. **Graphical Mindset:** This is a graphical problem. You are looking for the number of intersections between the graph of $y = f(x)$ and the horizontal line $y = k$.
2. **Analyze Variation:**
   - Calculate $f'(x)$ and study its sign to determine monotonicity.
   - Find all local maximum and minimum values.
   - Calculate the limits of $f(x)$ at the boundaries of its domain (e.g., at $\pm\infty$).

3. **Conclude:** Based on the values of the local extrema and the limits, you can determine how many times the line $y = k$ intersects the graph for different ranges of k. For example, if a local max has value M and a local min has value m, for any k such that $m < k < M$, there might be multiple solutions.

1 Convergence of Series $\sum a_n$

1. **Necessary Condition:** Calculate $\lim_{n\to\infty} a_n$. If the limit is NOT 0, the series **diverges**.
2. **Check for Positive Terms ($a_n \geq 0$):**
   - **Asymptotic Comparison Test (Most Common):** Find a simpler series b_n (typically $1/n^p$) and evaluate $L = \lim_{n\to\infty} \frac{a_n}{b_n}$. If L is a finite, non-zero number, then $\sum a_n$ behaves exactly like $\sum b_n$.
   - *How to find b_n?* Use Taylor expansions for terms like $\sin(1/n), \log(1 + 1/n), e^{1/n} - 1$, etc., around 0.
   - **Ratio Test:** Best for factorials and powers. If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, it converges. If > 1, it diverges. If $= 1$, it's inconclusive.
3. **Check for Alternating Terms ($\sum(-1)^n a_n$):**
   - **Leibniz Test:** If $a_n > 0$, a_n is decreasing, and $\lim_{n\to\infty} a_n = 0$, the series converges.
   - **Absolute Convergence:** Check if $\sum |a_n|$ converges (using the tests for positive series). If it does, the original series converges absolutely.

2 Improper Integrals $\int_a^b f(x)dx$

1. **Identify Problem Points:** Find the points where the integral is improper. This occurs if a bound is $\pm\infty$ or if $f(x)$ is unbounded at some point $c \in [a, b]$.
2. **Analyze Each Point Separately:**
   - **At ∞:** Use asymptotic comparison with $g(x) = 1/x^p$. The integral $\int_a^\infty f(x)dx$ converges if the integral of $g(x)$ converges, which happens for $p > 1$.
   - **At a point c:** Use asymptotic comparison with $g(x) = 1/|x - c|^p$. The integral converges if $p < 1$.
3. **Key Tool:** To find the asymptotic behavior, use Taylor expansions. For a problem at $x = c$, expand $f(x)$ for $x \to c$.

3 Limits

1. **Indeterminate Form:** First, try direct substitution. If you get an indeterminate form $(0/0, \infty/\infty, 1^\infty$, etc.), proceed.
2. **Taylor Expansions (for $x \to 0$):** This is the most reliable method.
   - Replace functions with their expansions: $e^x \approx 1+x+\frac{x^2}{2!}$, $\sin x \approx x-\frac{x^3}{3!}$, $\cos x \approx 1-\frac{x^2}{2!}$, $\log(1+x) \approx x-\frac{x^2}{2}$.
   - Keep enough terms to ensure the lowest-order terms in the numerator and denominator do not cancel to zero.
3. **For 1^∞ forms:** Rewrite $f(x)^{g(x)}$ as $e^{g(x)\ln(f(x))}$. Calculate the limit of the exponent, $L = \lim g(x)\ln(f(x))$. The final result is e^L. This often simplifies by using $\ln(1+t) \sim t$ for $t \to 0$.

4 Sequences by Recurrence ($a_{n+1} = f(a_n)$)

1. **Find Fixed Points:** These are the candidates for the limit. Solve the equation $f(x) = x$.
2. **Study Monotonicity:**
   - Analyze the sign of $f(x) - x$.
   - If $f(x) > x$, the sequence is increasing ($a_{n+1} > a_n$).
   - If $f(x) < x$, the sequence is decreasing ($a_{n+1} < a_n$).
3. **Check for Boundedness:**
   - Find an interval I such that if $a_n \in I$, then $a_{n+1} \in I$. This is called an invariant interval.
   - If the initial term a_0 is in I, all subsequent terms will be as well, proving the sequence is bounded.
4. **Conclude:** A monotonic and bounded sequence always converges to a limit L. This limit L must be one of the fixed points found in step 1. Use the bounds and monotonicity to select the correct one.

5 Differential Equations: A Detailed Guide

This section provides a step-by-step guide for the types of differential equations that appear in the exams. The key is to first identify the type of the equation and then apply the corresponding standard method.

Type 1: First-Order Separable Equations

- **How to Identify:** The equation can be written in the form $\frac{dy}{dx} = f(x) \cdot g(y)$. You can algebraically separate all x terms from all y terms.

- **Step-by-Step Solution:**

  1. Rewrite y' as $\frac{dy}{dx}$.
  2. Separate the variables: move all y terms to one side and all x terms to the other. This yields:

  $$\frac{1}{g(y)} dy = f(x) dx$$

  3. Integrate both sides:

  $$\int \frac{1}{g(y)} dy = \int f(x) dx + C$$

     Remember to add the constant of integration, C, on the side with the independent variable (x).

  4. Solve for y explicitly, if possible.

- **Important Note on Constant Solutions:** Before separating, check if there are any values y_c for which $g(y_c) = 0$. If so, then $y(x) = y_c$ is a constant (or stationary) solution. These are important and can sometimes be the specific solution to a Cauchy problem.

Type 2: Second-Order Linear Homogeneous with Constant Coefficients

- **How to Identify:** The equation has the form $ay'' + by' + cy = 0$, where a, b, c are real numbers.

- **Step-by-Step Solution:**

  1. Write the **Characteristic Equation**: This is a quadratic equation formed by replacing y'' with λ^2, y' with λ, and y with 1:

  $$a\lambda^2 + b\lambda + c = 0$$

  2. Solve for λ. The form of the general solution depends on the roots (λ_1, λ_2):

  Case A: Two distinct real roots, $\lambda_1 \neq \lambda_2$. The general solution is:

  $$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

Case B: One repeated real root, $\lambda_1 = \lambda_2 = \lambda$. The general solution is:

$$y(x) = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$$

(Notice the extra x in the second term).

Case C: Two complex conjugate roots, $\lambda = \alpha \pm i\beta$. The general solution is:

$$y(x) = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x))$$

Type 3: Second-Order Linear Non-Homogeneous

- **How to Identify:** The equation has the form $ay'' + by' + cy = f(x)$, where $f(x)$ is not zero.
- **General Principle:** The final solution is the sum of two parts:

$$y(x) = y_h(x) + y_p(x)$$

  - $y_h(x)$ is the solution to the corresponding **homogeneous** equation ($ay''+by' + cy = 0$), found using the method from Type 2.
  - $y_p(x)$ is a **particular solution** that depends on the form of $f(x)$.
- **Finding the Particular Solution $y_p(x)$ (Method of Undetermined Coefficients):**
  1. First, find the homogeneous solution $y_h(x)$.
  2. Make a guess for $y_p(x)$ based on the form of $f(x)$:
     - If $f(x)$ is a polynomial of degree n, guess a polynomial of degree n: $y_p(x) = Ax^n + ... + D$.
     - If $f(x) = Ke^{kx}$, guess $y_p(x) = Ae^{kx}$.
     - If $f(x) = K\sin(kx)$ or $K\cos(kx)$, guess $y_p(x) = A\sin(kx) + B\cos(kx)$.
  3. **CRITICAL - The Resonance Rule:** If your guess for $y_p(x)$ (or any part of it) is already present in the homogeneous solution $y_h(x)$, you must multiply your entire guess by x. If it's still present, multiply by x again.
  4. Differentiate your guess to find y_p' and y_p''.
  5. Substitute y_p, y_p', y_p'' into the original non-homogeneous equation.
  6. Solve for the coefficients (e.g., A, B) by equating the coefficients on both sides.
  7. Your final solution is $y(x) = y_h(x) + y_p(x)$.

The Final Step for All DEs: Solving the Cauchy Problem

- **What it is:** A Cauchy Problem gives you the differential equation plus one or more initial conditions, like $y(x_0) = y_0$ or $y'(x_0) = y_1$. Your task is to find the values of the constants of integration (C, or C_1 and C_2) in your general solution.

- **The Universal Method:**

  1. **Find the General Solution First:** Follow the methods for the appropriate DE type (Separable, 2nd Order Homogeneous, etc.) to find the solution with the constants still in it. Let's call it $y_{gen}(x)$.
  2. **Apply the First Condition ($y(x_0) = y_0$):**
     - Take your general solution $y_{gen}(x)$.
     - Substitute x_0 for every x and set the entire expression equal to y_0.
     - This gives you your first equation involving the constants. For a first-order equation, you can solve for C directly.
  3. **Apply the Second Condition ($y'(x_0) = y_1$) (for 2nd order DEs):**
     - Take your general solution $y_{gen}(x)$ and calculate its derivative, $y'_{gen}(x)$. **Important:** Differentiate first, then substitute.
     - Now substitute x_0 for every x in the derivative and set the expression equal to y_1.
     - This gives you your second equation involving the constants.
  4. **Solve for the Constants:** You now have a system of one or two linear equations for your constants (C_1, C_2). Solve this system to find their specific numerical values.
  5. **Write the Final Solution:** Substitute the values you found for the constants back into your general solution. This final expression is the unique solution to the Cauchy Problem.

Fin :)