Mathematical Analysis 1

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Pre-Derivatives

Theorems, functions and axioms

"Calvin: You know, I don't think math is a science, I think it's a religion.

Hobbes: A religion?

Calvin: Yeah. All these equations are like miracles. You take two numbers and when you add them, they magically become one NEW number! No one can say how it happens. You either believe it or you don't. [Pointing at his math book] This whole book is full of things that have to be accepted on faith! It's a religion!"

Logic

Propositional Logic

A proposition is a statement. Statements in math are either true or false, if you combined multiple propositions there are multiple outcomes depending on the validity of each proposition in the statement. I call them compound statements (but I dont think its the actual name).

Negation (Negazione): Represented by a $\neg$. Inverts the value of a statement, e.g (Bob went to the store), the opposite can be represented as $\neg$(Bob went to the store).

Disjunction (Disgiunzione): Represented by a $\vee$. Statement is true if at least one of the propositions is also true.

Conjunction (Congiunzione): Represented by a $\wedge$. Statement is true only if both propositions are also true.

Implication (Implicazione): Represented by a $\rightarrow$. Its formally defined as : $\neg P \vee Q$.

Biconditional (Bicondizionale): Represented by a $\leftrightarrow$. Statement is true if both of the propositions share the same value.

Predicate Logic

Universal: It is represented by $\forall$. "For every ..."

Existence: It is represented by $\exists$. "There exists ..."

Axioms

Axioms (Assiomi): An axiom is a postulate, more commonly known as assumption. It is a statement that is held as always true in regards to the problem or proof needed to solve. There are different types of axioms which are briefly stated below:

Logical Axioms: A universal truth in all of Mathematics applicable in both the Physics Notes and the Linear Algebra Notes.

Non Logical Axioms: Domain specific assumptions, such as Axioms only applicable in $\mathbb{R}$(Gross oversimplification)

Group Theory (Theoria degli insiemi)

Definition of a Group

A group is a set, that has the following requirements:

Closure: For all $a, b \in G, a * b \in G$

Associativity: $a, b, c \in G, (a * b) * c = a * (c * b)$

Identity: Lets say e exists $e \in G$ such that $e * a = a * e = a$ for all $a \in G$

Inverse: If there exists a a, such that $a \in G$ therefore there exists a inverses

Proofs (Dimostrazione)

In Math, every theorem and formula needs to be able to be proven in a proof. There are multiple types of proofs which are used to show that a theorem and or formula is valid. In these notes, very few proofs will be done, however in the written blue book. Each proof I have written in there has a classification. If a proof is referenced it will have a corresponding code written next to it referencing the written proof in the blue notebook.

Direct Proof (Dimostrazione Diretta). By using known definitions, axioms and theorems, a sequence of logical steps can be used to directly demonstrate whether or not the statement is correct.

Proof by Contradiction (Dimostrazione per Assurdo). Assume that a statement is false and connect it to a logical contradiction.

Induction (Induzione). Used for statements involving natural numbers

Base Case: Verify the statement holds for the initial value

Inductive Step: Assume it holds for a n=k, then prove it holds for n=k+1.

Constructive Proof (Dimostrazione Costruttiva). Make a identity with the exact desired property. More formally ''Demonstrates the existence of an object by explicitly constructing it''

Functions

A **function** (*funzione*) is a very common conecept in math that formalizes the relationship between two sets by assigning each element of the first set to exactly one element of the second set. Formally, a function $f : A \to B$ consists of:

- A **domain** (*dominio*) A, the set of all possible inputs.
- A **codomain** (*codominio*) B, the set into which all outputs are mapped.
- A rule or correspondence that links each element $x \in A$ to a unique element $f(x) \in B$.

Formal Definition A function f is a subset of the Cartesian product $A \times B$ such that for every $x \in A$, there exists exactly one $y \in B$ where $(x, y) \in f$. This is denoted as $y = f(x)$.

Key Properties

1. **Injectivity** (*iniettiva*): A function is injective if distinct inputs map to distinct outputs:
$$\forall x_1, x_2 \in A, \quad f(x_1) = f(x_2) \implies x_1 = x_2.$$
2. **Surjectivity** (*suriettiva*): A function is surjective if every element in B is an output for some input:
$$\forall y \in B, \quad \exists x \in A \text{ such that } y = f(x).$$
3. **Bijectivity** (*biiettiva*): A function is bijective if it is both injective and surjective, establishing a one-to-one correspondence between A and B.

Natural Numbers

The **natural numbers** (*numeri naturali*) are the standard version of numbers in maths, used for counting and ordering. Formally, the set of natural numbers $\mathbb{N}$ is defined as:

$$\mathbb{N} = \{1, 2, 3, \ldots\} \quad \text{(sometimes including 0 depending on context).}$$

They are characterized by their discrete, non-negative integer values and form the foundation for number theory and arithmetic.

Formal Definition (Peano Axioms) The properties of natural numbers are axiomatically defined by the **Peano axioms** (*assiomi di Peano*):

1. 1 (or 0) is a natural number.
2. Every natural number n has a unique successor $S(n)$, which is also a natural number.
3. 1 (or 0) is not the successor of any natural number.
4. Distinct natural numbers have distinct successors: $S(m) = S(n) \implies m = n$.
5. **Induction**: If a property holds for 1 (or 0) and holds for $S(n)$ whenever it holds for n, then it holds for all natural numbers.

Key Properties

- **Closure under addition and multiplication**: For all $a, b \in \mathbb{N}$, $a + b \in \mathbb{N}$ and $a \cdot b \in \mathbb{N}$.
- **Non-closure under subtraction and division**: Subtraction $a - b$ or division a/b may not result in a natural number.
- **Well-ordering principle** (*principio del buon ordinamento*): Every non-empty subset of $\mathbb{N}$ has a least element.
- **Infinite cardinality**: $\mathbb{N}$ is countably infinite.

Number Theory Natural numbers are central to number theory, which studies:

1. **Prime numbers** (*numeri primi*): Natural numbers > 1 with no divisors other than 1 and themselves:
$$\mathbb{P} = \{2, 3, 5, 7, 11, \ldots\}.$$
2. **Divisibility**: A number a divides b ($a \mid b$) if $\exists k \in \mathbb{N}$ such that $b = a \cdot k$.
3. **Mathematical induction** (*induzione matematica*): A proof technique leveraging the Peano axioms.
4. **Modular arithmetic** (*aritmetica modulare*): Operations on residues modulo n, e.g., $7 \equiv 2 \mod 5$.

Whole Numbers

Whole Numbers (*numeri interi non negativi*) are an extension of the natural numbers that include zero, forming the set $\mathbb{W} = \{0, 1, 2, 3, \ldots\}$. They are used for counting discrete objects and represent non-negative integers without fractions or decimals.

Formal Definition The set $\mathbb{W}$ satisfies the **Peano axioms** (*assiomi di Peano*) with zero as the base element:

- 0 is a whole number.
- Every whole number n has a unique successor $S(n) \in \mathbb{W}$.
- 0 is not the successor of any whole number.
- Distinct numbers have distinct successors: $S(a) = S(b) \implies a = b$.
- **Induction**: If a property holds for 0 and for $S(n)$ whenever it holds for n, it holds for all $\mathbb{W}$.

Key Properties

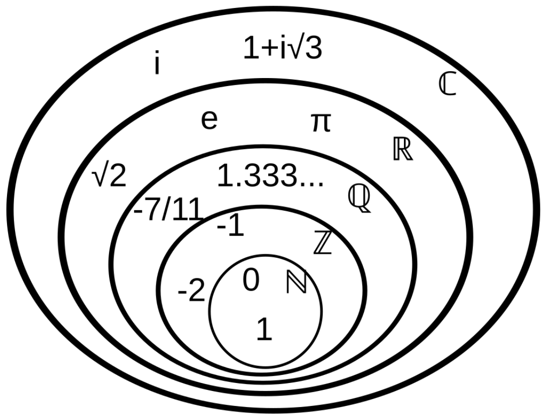
- **Closure under addition and multiplication**: For $a, b \in \mathbb{W}$, $a + b \in \mathbb{W}$ and $a \cdot b \in \mathbb{W}$.
- **Non-closure under subtraction**: $a - b \in \mathbb{W}$ only if $a \geq b$.
- **Additive identity**: $0 + a = a$ for all $a \in \mathbb{W}$.
- **Well-ordering principle** (*principio del buon ordinamento*): Every non-empty subset of $\mathbb{W}$ has a least element.

Representation Whole numbers are represented in numeral systems such as:

- **Decimal**: $0, 1, 2, \ldots$
- **Binary**: $0_2 = 0_{10}, 1_2 = 1_{10}, 10_2 = 2_{10}$
- **Unary**: 0 (often represented as an absence of marks), $| = 1, || = 2$.

Differences from Natural Numbers Unlike natural numbers (*numeri naturali*), which sometimes exclude zero, whole numbers explicitly include 0. This makes $\mathbb{W}$ the set $\mathbb{N} \cup \{0\}$ in contexts where natural numbers start at 1.

Below, is a image of all the relevant groups of numbers and how they are related. Not all of them have been stated in this point in the notes, but they are all relevant for analysis 1



Trigonometric Functions

Trigonometric Functions (*funzioni trigonometriche*) are periodic functions that relate angles in a right triangle or on the unit circle to ratios of side lengths. The primary trigonometric functions are sine ($\sin$), cosine ($\cos$), tangent ($\tan$), and their reciprocals: cosecant ($\csc$), secant ($\sec$), and cotangent ($\cot$).

Formal Definition (Unit Circle) For an angle θ measured counterclockwise from the positive x-axis on the unit circle ($x^2 + y^2 = 1$):

$$\sin\theta = y, \quad \cos\theta = x, \quad \tan\theta = \frac{y}{x} \quad (x \neq 0).$$

The reciprocals are defined as:

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{x}{y} \quad (y \neq 0).$$

Key Properties

- **Periodicity** (*periodicità*): $\sin\theta$ and $\cos\theta$ have period 2π; $\tan\theta$ and $\cot\theta$ have period π.
- **Range**:

$$\sin\theta, \cos\theta \in [-1, 1]; \quad \tan\theta \in \mathbb{R} \text{ (excluding asymptotes)}.$$

- **Parity**: $\sin\theta$ and $\tan\theta$ are odd functions; $\cos\theta$ is even:

$$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta.$$

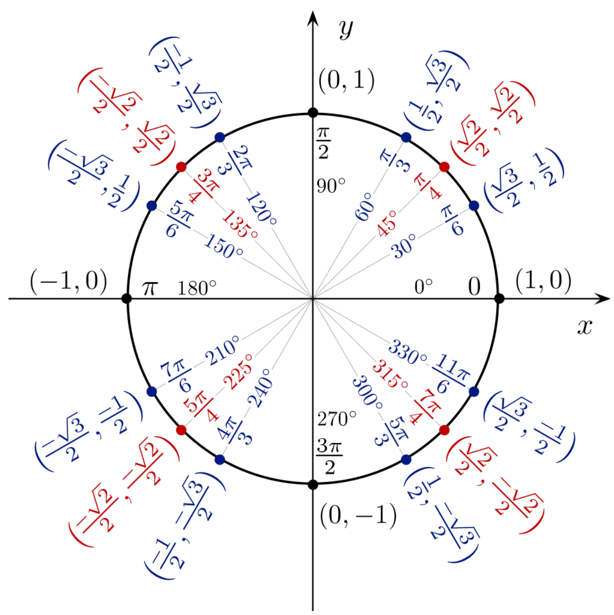
- **Pythagorean Identity** (*identità pitagorica*):

$$\sin^2\theta + \cos^2\theta = 1.$$

Differences from Other Functions Unlike polynomial or exponential functions, trigonometric functions:

- Are periodic and bounded (except $\tan\theta$ and $\cot\theta$).
- Model oscillatory behavior (e.g., waves, circular motion).
- Require angular input (radians or degrees) rather than purely scalar quantities.

Unit Circle For Reference:



Fundamental Identities (*identità fondamentali*)

- **Reciprocal Relations**:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}.$$

- **Extended Pythagorean Identities**:

$$1 + \tan^2\theta = \sec^2\theta, \quad 1 + \cot^2\theta = \csc^2\theta.$$

- **Co-Function Identities** (*identità complementari*):

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \quad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \quad \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta.$$

Angle Addition & Subtraction For any angles α and β:

- $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
- $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
- $\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

Multiple-Angle & Half-Angle Identities

- **Double-Angle**:

$$\sin(2\theta) = 2\sin\theta\cos\theta, \quad \cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta.$$

- **Triple-Angle**:

$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta, \quad \cos(3\theta) = 4\cos^3\theta - 3\cos\theta.$$

- **Half-Angle** (sign depends on quadrant):

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}, \quad \cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}.$$

Product-to-Sum & Sum-to-Product

- **Product-to-Sum**:

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)], \quad \cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)].$$

- **Sum-to-Product**:

$$\sin\alpha \pm \sin\beta = 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right), \quad \cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right).$$

Triangle Relations (Laws) For any triangle with sides a, b, c opposite angles A, B, C:

- **Law of Sines**:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \quad (R = \text{circumradius}).$$

- **Law of Cosines**:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

- **Law of Tangents**:

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}.$$

Complex Numbers introduction

The largest domain covered in these notes: $\mathbb{C}$. Complex numbers are an extension of real numbers, defined by the imaginary number i With this strange property where $i^2 = -1$. They are used to resolve polynomial equations unsolvable in real numbers, as shown in the fundamental theorem of algebra (Might need a section on this).

Below is a complex number with iy as the imaginary component and x as the real component.

$$z = x + iy \tag{1}$$

Imagine $\mathbb{R}$ covering the whole x axis, and $\mathbb{C}$ covering the whole y axis, that's the complex plane.

Operations

Addition

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \tag{2}$$

Subtraction

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2) \tag{3}$$

Multiplication

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \tag{4}$$

Complex Conjugate

$$\bar{z} = x - iy \tag{5}$$

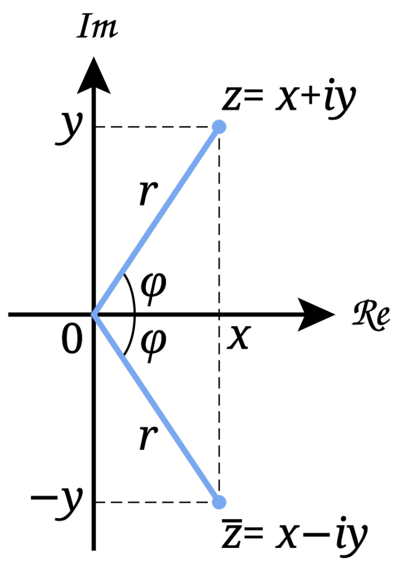
Modulus

$$|z| = \sqrt{x^2 + y^2} \tag{6}$$

Inverse

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2} \tag{7}$$

A image of the $\mathbb{C}$ plane for reference, showing the inverse modulus as well:



Polar Coordinates

Polar coordinates are a alternative to the commonly used Cartesian coordinate system (x,y). It is mesured using two metrics:

Radial Distance (r): Which is the distance from the origin which is also the hypotenuse of the triangle.

Angular Coordinate (θ): Which is the angle between the radial distance and the +x axis. Which increases counterclockwise.

Basic Conversion between Cartesian and Polar For Cartesian to Polar it is:

$$r = \sqrt{x^2 + y^2}, \theta = arctan\left(\frac{y}{x}\right)$$

And for Polar to Cartesian it is:

$$x = rcos\theta, y = rsin\theta$$

Basis vectors Basis vectors represented in polar coordinates:

$$\hat{\mathbf{r}} = cos\theta\hat{\mathbf{i}} + sin\theta\hat{\mathbf{j}}, \hat{\theta} = -sin\theta\hat{\mathbf{i}} + cos\theta\hat{\mathbf{j}}$$

Polar Functions A polar function is represented as:

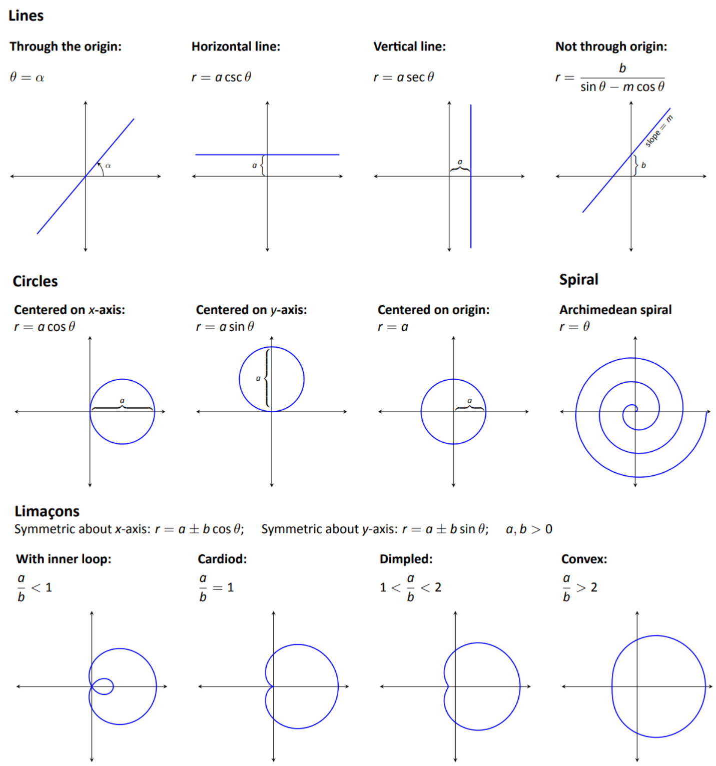
$$r = f(\theta)$$

Much like a regular function it has a domain and a range, expressed below:

Domain: $\theta \in [0, 2\pi)$

Range: $r \in \mathbb{R}$

Common set of Polar Functions: -



Sums and Sequences

"La situazione è grave ma non è seria."

Limits

How its represented. Any sequence that converges to a limit is represented as:

$$lim_{n \to \infty} a_n = L$$

Where $\{a_n\}$is the sequence.

Formal Definition.

$$\lim_{n\to\infty} a_n = L \quad \iff \quad \forall\varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \geq N,\ |a_n - L| < \varepsilon$$

"For every positive number, there exists a natural number such that, for all integers, the distance between and L is less than ε."

Sequence

Definition: A sequence/sucession is simply a list of $\mathbb{N}$ while following a set of rules defined by a function. The function then maps each $\mathbb{N}$ to a corresponding $\mathbb{R}$ following the rules defined by the function.

Representation: It is often shown as a random letter (this case a) a_n with n representing the number of the term.

The first term, a_1 is called the initial term (termine iniziale)

The terms after the first, a_{1+n} is called the recursive formula (formula ricorsiva)

Types: There are 2 specific categories which will be covered in more detail. 1. Whether its bounded (limitata) or unbounded (illimitata). 2. Whether its convergent (convergente), divergent (divergente) or oscillatory (oscillante).

Bounded Successions (Successioni Limitate)

Definition: A bounded sequence (successione limitata), is a sequence a_n that exists within a range such that if $b \in \mathbb{R}$, b is greater than a_n, and there exists $c \in \mathbb{R}$ that is less than a_n, it is a bounded sequence. A bounded sequence is may suggest convergences and can be proven by using the Bolzano-Weierstrass Theorem.

Intervals:

Open Interval

$$(a, b) \coloneqq \{x \in \mathbb{R} | a < x < b\}$$

A open interval includes all $\mathbb{R}$ numbers between a and b, excluding the endpoints. Represented by a () and $<$

Closed Interval

$$[a, b] \coloneqq \{x \in \mathbb{R} | a \leq x \leq b\}$$

A closed interval includes all $\mathbb{R}$ numbers between a and b, including the endpoints. Represented by a [] and $\leq$.

Empty Interval Denoted as $\emptyset$, it contains no numbers.

Degenerate Interval A single point, $[a, a] = \{a\}$. By technicality it is always closed.

Half Intervals It is possible to have intervals which are open at one end and closed at the other, and vice versa. e.g

$$(a, b] := \{x \in \mathbb{R} | a < x \leq b\}$$

Infinite Intervals There are also intervals which are infinite on one side and open or closed on the other. e.g

$$(a, +\infty) := \{x \in \mathbb{R} | a < x\}$$

The infinite can be negative as well on the other side. (Not sure if the infinite has to be in a open interval, because I have not seen any which are not in a open interval)

Types of bounds:

Upper bound: If a upper bound exists we call it bounded from above.

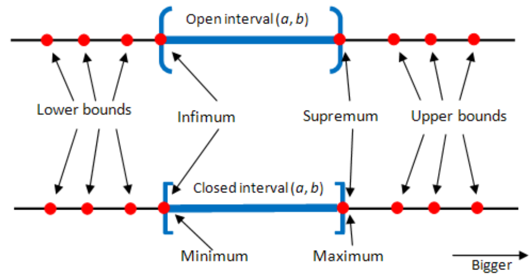
Lower bound: If a lower bound exists we call it bounded from below.
If both upper and lower bound exist the set is bounded.

Bound properties: There can be multiple upper and lower bound (As clearly shown in the diagram below).

Supremum and Infimum The supremum and infimum can only exist for a interval with at least one open point. Its the smallest possible upper bound (If its a supremum) or the largest possible lower bound (If its a infimum). It can NEVER reach the interval. The Supermum can be written as $supM$ and the Infium can be written as $infM$

Minimum and Maximum For a Minimum or a Maximum you must have at least one closed interval point. (as shown in the diagram below). The minimum or maximum is the point a or b that hold the interval.

Diagram of open and closed sequences:



Monotone Sequences

Definition: A sequence that is monotone is either non-decreasing or non-increasing. Therefore, by extention a constant sequence is simultaneously non-decreasing and non-increasing. Therefore it is monotone. The strictly increasing/decreasing are simply subsets.

Growth of Sequences:

Non-decreasing sequence (successioni crescenti): Is a sequence that increases if each term is greater than or equal to the previous term:

$$\forall n \in \mathbb{N}, a_{n+1} \geq a_n$$

Non-increasing sequences (successioni decrescenti): Is a sequence that it decreases if each term is less than or equal to the previous term.

$$\forall n \in \mathbb{N}, a_{n+1} \leq a_n$$

Strictly increasing sequences (successioni strettamente crescenti): Is a sequence that strictly increasing if each term is strictly greater than the previous term.

$$\forall n \in \mathbb{N}, a_{n+1} > a_n$$

Strictly decreasing sequences (successioni strettamente decrescenti): Is a sequence that strictly decreasing if each term is strictly less than the previous term.

$$\forall n \in \mathbb{N}, a_{n+1} < a_n$$

Convergent and Divergent Sequences

Convergent

A sequence is convergent if it approaches a finite limit (L) as the n approaches infinity. Formally this can be defined:

A sequence $\{a_n\}$ converges to L if

$$\forall\varepsilon > 0, \exists N \in \mathbb{N} | \forall n \geq N, |a_n - L| < \varepsilon$$

Accumulation Points Link A convergent sequence has exactly one accumulation point which is its limit. This is different from the divergent sequence that will have multiple accumulation values or many improper ones.

Limit sup and inf For convergent sequences:

$$lim_{n\to\infty} sup a_n = lim_{n\to\infty} inf a_n = L$$

Note on Bounded Sequences: <u>While boundedness is a requirement for a sequence to be convergent it is not the only requirement.</u>

Monotone Convergence Theorem: Every bounded monotonic sequence converges

Limit properties for convergence: If b is a sequence that converges to M and a is a sequence that converges to L then

$$lim_{n\to\infty}(a_n \pm b_n) = L \pm M$$

$$lim_{n\to\infty}(a_n \times b_n) = L \times M$$

$$lim_{n\to\infty}\frac{a_n}{b_n} = \frac{L}{M}(M \neq 0)$$

Divergent

A sequence is divergent if it does not converge to any limit L. There are two primary times of divergence, divergence to infinity(may also be refered to improper divergence) or oscillatory divergence. The formal definitions for divergence to infinity is listed below:

$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N}$$

Such that $\forall n \geq N, a_n > M$, and if it diverging to negative infinity it is $\forall n \geq N, a_n < M$

And for Oscillations:

Bounded Oscillations: Terms change between bounds with the lim sum not equalling the lim inf

Unbounded oscillation: Terms grow wile changing and slowly diverging in absolute value but don't have a directional convergence.

Unbounded While it is not guaranteed that every bounded sequence is convergent, it is guaranteed that every unbounded sequence is divergent by definition. However not all divergent sequences are unbounded.

Sub sequences If a sub sequence of a sequence diverges to infinity then the also sequence will diverge.

If two sub sequences converge to different limits, the original sequence diverges.

Limit properties If $lim_{n\to\infty}a_n = +\infty$ and the sequence b is bounded below then

$$lim_{n\to\infty}(a_n + b_n) = +\infty$$

If $lim_{n\to\infty}a_n = +\infty$ and $lim_{n\to\infty}b_n = c > 0$ then

$$lim_{n\to\infty}(a_n \times b_n) = +\infty$$

Cauchy Sequence

A Cauchy sequence is a sequence whose terms become arbitrarily close to one another as the sequence progresses, regardless of whether the sequence converges to a specific limit.

Formal Definition Every sequence with this definition is a Cauchy Sequence

$$\forall\epsilon > 0, \exists N \in \mathbb{N} \Rightarrow \forall m, n \geq N, |a_m - a_n| < \epsilon$$

Properties

Convergence In R every Cauchy sequence by definition must be convergent, due to the completeness property.

Boundedness Every Cauchy Sequence is bounded.

Subsequence If a Cauchy sequence has a convergent subsequence, the entire sequence converges to the same limit.

Notes on convergence All convergent sequences are Cauchy, not all Cauchy sequences are convergent IN INCOMPLETE SPACES

Napier's Constant (Costante di Nepero)

This may also be known as e or exponential or exp. It has 3 separate definitions: Limit, Series, and integral. However I will only do the explanation for limit definition here. The integral explanation will be done in the integral chapter.

Limit Definition:

$$e = lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \tag{8}$$

Base of the Natural Logarithm e is the base of the natural logarithm, denoted as $\ln(x)$.

Approximate Value $e \approx 2.71828$.

Irrational Number e cannot be expressed as a ratio of two integers.

Transcendental Number e is not a root of any non-zero polynomial with rational coefficients.

Limit Definition

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Euler's Identity

$$e^{i\pi} + 1 = 0$$

Connecting e, imaginary numbers, and π

Complex Exponential (l'esponenziale Complesso)

The complex exponential function denoted as e^z or exp(z). The vast majority of definitions of complex exponential are the exact same as the regular exponential function. Including the proof of their equivalence. There are several ways to define a complex exponential function, and I will list them below.

Power series expansion

$$e^z = \sum_{k=0}^{\infty}\frac{z^k}{k!}$$

Limit Definition

$$e^z = lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n$$

Additive Property (Kinda definition)

$$e^{w+z} = e^w e^z$$

Complex log

$$log(e^z) = \{z + 2\pi ik | k \in \mathbb{Z}\}$$

Key Properties

Non zero $\frac{1}{e^z} = e^{-z}$ and e^z does not equal zero for all C

Periodic

$$e^{z+2\pi} = e^z$$

Identita di Euler For $e^{i\theta} = cos\theta + i * sin\theta$

$$e^{i\pi} = -1$$

Conjugate $\overline{e^z} = e^{\overline{z}}$

Modulus $|e^z| = e^{\mathbb{R}(z)}$

Handling Infinity

Limit Superior

For a sequence (a_n), the limit superior which is written as

$$limsup_{n \to \infty} a_n$$

is the supremum of all the accumulation points (cluster points) of the sequence. For example if a sequence diverges to infinity its only accumulation point is infinity therefore

$$limsup_{n \to \infty} a_n = \infty$$

The main diffrence between the supremum is that the supremum

Lets say we have a sequence (a_n) which is given by $a_n = n$, the limit superior of (a_n) is the largest accumulation value of (a_n). Whether it is improper or proper accumulation value. It is represented as:

$$a = limsup_{n \to \infty} a_n$$

The difference between the limit superior and superior is that the limit superior is ALWAYS smaller than the

Limit Inferior The limit inferior of (a_n) is the smallest improper accumulation value of (a_n), and its represented by the notation

$$a = liminf_{n\to\infty} a_n$$

Accumulation Values and Divergence A value $a \in \mathbb{R} \cup \{+\infty, -\infty\}$ is called a accumulation value of a_n, if there exists a sub sequence a_{n_k} of a_n such that the limit $k \to \infty | a_{n_k} = a$

For improper cases:

$+\infty$ is a accumilation value if the sequences is unbounded above, meaning for every M∈R infinit

Improper accumulation values Any sequence that has no accumulation values has at least one improper accumulation value

Bolzano Vistras Theorem

Formal Definition: 1. In $\mathbb{R}$ every bounded sequence of real numbers contains a convergent subsequence

2. In $\mathbb{R}^n$ ever bounded sequence has a convergent subsequence

Proof: The proof can be found at 200.20.05.10

Theorem of Zero

Trigonometric Functions and π

Exponential Properties

Below are some old section from a scrapped chapter. Will probably be useful here

Exponential Properties

For $a > 0$, $b > 0$, and $x, y \in \mathbb{R}$:

$a^x a^y = a^{x+y}$
$\frac{a^x}{a^y} = a^{x-y}$
$(a^x)^y = a^{xy}$
$a^0 = 1$
$a^{-x} = \frac{1}{a^x}$
$(ab)^x = a^x b^x$
$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

Def of Logarithms

For $a > 0$ ($a \neq 1$), $b > 0$:

$$\log_a b = x \iff a^x = b$$

$$\log_a 1 = 0$$

$$\log_a a = 1$$

$$a^{\log_a b} = b$$

Logarithm Properties

For $a > 0$ ($a \neq 1$), $x, y > 0$, $k \in \mathbb{R}$:

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a(x^k) = k \log_a x$$

$$\log_a a^x = x$$

Change of Base

For $a, b > 0$ ($a, b \neq 1$), $c > 0$:

$$\log_a c = \frac{\log_b c}{\log_b a}$$

Special case: $\log_a b = \frac{1}{\log_b a}$

Natural Exponential and Logarithm

$$e^x = \exp(x)$$

$$\ln x = \log_e x$$

$e^{\ln x} = x$ for $x > 0$

$\ln(e^x) = x$ for $x \in \mathbb{R}$

Exponential-Logarithmic Equations

Key solving techniques:

If $a^x = a^y$ then $x = y$

If $\log_a x = \log_a y$ then $x = y$

To solve $a^{f(x)} = b$: take logarithms of both sides

To solve $\log_a f(x) = b$: rewrite as $f(x) = a^b$

Complex Polynomials

Weierstrass Factorization Theorem

Fundamental Theorem of Algebra

Series

Series (Serie)

A series is the sum of the all the terms in a sequence. Formally, if a is a sequence and we want the infinite sum of the series:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...$$

Partial Sum (Somma Parziale)

A series does not have to necessarily have to be the sum of infinite terms. A partial sum, allows just the sum of the k terms:

$$S_k = \sum_{n=1}^{k} a_n$$

If $\sum a_n$ converges, then the lim

This series converges to S if $\lim_{k\to\infty} S_k = S$ otherwise, as mentioned in previous sections it diverges.

Necessary Condition for Convergence (Divergence Test)

If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
Equivalently, if $\lim_{n\to\infty} a_n \neq 0$ (or the limit does not exist), the series diverges.

Types of Series

Geometric (Serie Geometrica)

It is a series with the standard form $\sum_{n=0}^{\infty} ar^n$ where r is the common ratio. There are 3 primary types of geometric series:

Partial sum (Finite geometric series) For a finite number k, the partial sum

$$S_k = \sum_{n=0}^{n-1} ar^n = a + ar + ar^2 + ... + ar^{n-1}$$

Formula

$$S_k = a \times \frac{1 - r^n}{1 - r}$$

valid only if $r \neq 1$. If r=1 then

$$S_n = a \times n$$

There is also a alternate notation for the partial sum formula that is pretty commonly used.

$$S_{n+1} = \sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}$$

Infinite geometric series Theorem:
if $\sum_{n=0}^{\infty} ar^n$ converges then $|r| < 1$ and

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

Divergence The series is divergent if $|r| \geq 1$, and the terms do not approach zero

Telescoping (Serie Telescopica)

A telescopic series is a series where a lot of the terms cancel out when written in partial sums

$$\sum_{n=1}^{\infty}(a_n - a_{n+1})$$

whose partial sum will telescope to

$$S_k = \sum_{n=1}^{k}(a_n - a_{n+1}) = a_1 - a_{k+1}$$

When handling a limit to L

$$\sum_{n=1}^{\infty}(a_n - a_{n+1}) = a_1 - L$$

Note: This requires $\lim_{n\to\infty} a_n = L$.
Example: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ converges to 1.

Harmonic Series (Serie armonica)

A harmonic series is a series that diverges despite its terms tending to zero, it is represented as:

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

A small proof is listed below:

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+...+\frac{1}{8}\right)+... \geq 1+\frac{1}{2}+2\left(\frac{1}{4}\right)+4\left(\frac{1}{8}\right)+... = \sum_{k=0}^{\infty}\frac{1}{2} = \infty$$

This shows that $a_n \to 0$ is necessary for convergence, but not sufficient to prove it.

P series Harmonic series are a type of p series. P series is often used in comparison to check if series converge or not. It is represented as

$$\sum_{n=1}^{\infty}\frac{1}{n^p}$$

where it is said that if $p \leq 1$ the series diverges, and if $p > 1$ the series **converges**.

Associativity of series (Associatività della somma di serie)

The associativity of series, or the property that grouping terms in different ways does not affect the sum, holds for convergent series but not necessarily for divergent ones.

Series with varying signs (Serie a segno variabile)

Summation by parts (Somma per parti)

~~Cauchy Criterion~~

~~A series a converges if and and only if for every epsilon that is grater than zero there exists a N such that for all~~ $\cancel{m > n \geq N}$

$$\cancel{|\sum_{k=n+1}^{m} a_k| < \epsilon}$$

~~Absolute and Conditional Convergence~~

~~Using Cauchy criterion in problems~~

~~**For a Sequence**~~ ~~There are 4 main steps:~~

~~1. Assign a epsilon that is grater than zero~~

~~2. Find a integer N such that for all m, n > N, the inequality~~ $\cancel{|a_m - a_n| < \epsilon}$ ~~holds.~~

~~**For a Series**~~ ~~The steps are very similar to a sequence~~

~~1. Assign a epsilon that is grater than zero~~

~~2. Find a integer N such that for all m>n>N, the inequality~~ $\cancel{|\sum_{k=n+1}^{m} a_k| < \epsilon}$

Handling Convergence in Problems

Comparison (Criterio del confronto)

To find convergence using comparison, there are two primary methods used. Direct comparison test and the Limit comparison test.

Direct Comparison Test For two series $\sum a_n$ and $\sum b_n$ with $0 \leq a_n \leq b_n$ for all $n \geq N$
If $\sum b_n$ converges then $\sum a_n$ will converge
If $\sum a_n$ diverges then $\sum b_n$ will diverge

Limit Comparison Test For two series $\sum a_n$ and $\sum b_n$ with $a_n > 0, b_n > 0$

$$L = lim_{n\to\infty}\frac{a_n}{b_n}$$

If L>0 $\sum a_n$ and $\sum b_n$ share the same convergence type
If L=0 and $\sum b_n$ converges then $\sum a_n$ also converges
If L=∞ and $\sum b_n$ diverges, then $\sum a_n$ also diverges

Ratio (Criterio del rapporto)

A series $\sum a_n$, calculate the limit

$$L = lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|$$

If L $<$ 1 the series converges
If L $>$ 1 the series diverges
if L = 1 this test doesn't give enough info do determine a result

Root (Criterio della radice)

A series $\sum a_n$, calculate the limit

$$L = lim_{n\to\infty}\sqrt[n]{|a_n|}$$

If L $<$ 1 the series converges
If L $>$ 1 the series diverges
if L = 1 this test doesn't give enough info do determine a result

Integral (Criterio del integrale)

A series $\sum_{n=N}^{\infty} a_n$ with continuous, positive and decreasing terms, $a_n = f(n)$ for $n \geq N$
If the integral $\int_N^\infty f(x)dx$ converges, then the series $\sum a_n$ converges
If the integral diverges then $\sum a_n$ diverge

Cauchy Condensation Test (it is another way to say the same thing as above, I may remove this)

If $a_n \geq 0$ and a_n is decreasing, then $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.

Alternating Series Test (Leibniz)

An alternating series $\sum(-1)^n b_n$ or $\sum(-1)^{n+1} b_n$ converges if:

1. $b_n \geq 0$ **and monotonically decreasing:** $b_{n+1} \leq b_n$
2. $\lim_{n\to\infty} b_n = 0$

This implies conditional convergence (converges but not absolutely).

L'Hôpital's rule

Is a theorem, which is primarily used to check limits by comparing the derivatives of the limits.

Definition f and g are functions which are differentiable on a open interval. Assuming that,

$lim_{x\to a} f(x) = lim_{x\to a} g(x) = 0$, $g'(x) \neq 0$ and $lim_{x\to a} \frac{f'(x)}{g'(x)}$

Then:

$$lim_{x\to a} \frac{f(x)}{g(x)} = lim_{x\to a} \frac{f'(x)}{g'(x)}$$

THIS WILL GO HERE BECAUSE I DONT KNOW WHERE ELSE TO PUT IT

Domain Analysis (Analisi del Dominio)

- Determine where $f(x)$ is defined
- Identify interval boundaries and discontinuities

Closed Interval Check (Intervalli Chiusi)

- For $[a, b]$, apply Extreme Value Theorem (Teorema dei Valori Estremi):

$$\exists\ x_{\mathsf{max}}, x_{\mathsf{min}} \in [a, b] \quad \text{where} \quad f(x_{\mathsf{max}}) \geq f(x) \geq f(x_{\mathsf{min}})$$

First Derivative (Derivata Prima)

- Compute $f'(x)$
- Find critical points (punti critici):

$$\begin{cases} f'(x) = 0 \quad \text{(stationary points)} \\ f'(x) \text{ undefined} \quad \text{(cusps/corners)} \end{cases}$$

Classification (Classificazione)

Second Derivative Test (Test della Derivata Seconda)

- Compute $f''(x)$

$$\begin{cases} f''(x_0) > 0 \Rightarrow \text{local min (minimo locale)} \\ f''(x_0) < 0 \Rightarrow \text{local max (massimo locale)} \\ f''(x_0) = 0 \Rightarrow \text{inconclusive} \end{cases}$$

First Derivative Test (Test della Derivata Prima)

- Analyze sign changes:

$$\begin{array}{ll} + \to - & \Rightarrow \text{local max} \\ - \to + & \Rightarrow \text{local min} \\ \text{No change} & \Rightarrow \text{not extremum} \end{array}$$

Endpoint Evaluation (Valutazione agli Estremi)

- For closed intervals:

$$\text{Compare} \quad f(a),\ f(b),\ f(x_{\text{crit}})$$

Post-Derivatives

Derivatives

"The Difference Between the Almost Right Word and the Right Word Is Really a Large Matter—'Tis the Difference Between the Lightning Bug and the Lightning" - Mark Twain

Derivatives (Derivate)

Fundamental Concept

The derivative of a function f at a point a is the instantaneous rate of change of the function with respect to its input. Geometrically, it represents the slope of the tangent line to the graph of f at the point $(a, f(a))$, providing the best linear approximation of the function near a.

Notation (Notazione)

Two primary notations exist for derivatives:

- Leibniz notation: $\frac{df}{dx}$ or $\frac{dy}{dx}$
- Prime notation: $f'(x)$

Higher-order derivatives (derivate di ordine superiore) extend this notation with additional differentials or prime marks, representing repeated differentiation.

Limit Definition (Definizione di Limite)

A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable (derivabile) at $a \in \mathbb{R}$ if either of the following equivalent limits exists:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

This definition requires f to be defined on an open interval containing a. In real analysis, differentiability signifies that f admits a linear approximation at a:

$$f(x) = f(a) + f'(a)(x - a) + r(x)$$

where the remainder term $r(x)$ satisfies $\lim_{x \to a} \frac{r(x)}{|x-a|} = 0$. The ε-δ formalization states:

$$\forall \varepsilon > 0,\ \exists \delta > 0\ :\ 0 < |x - a| < \delta \implies \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \varepsilon$$

Geometric Interpretation (Interpretazione Geometrica)

The derivative $f'(a)$ is the limit of slopes of secant lines through points $(a, f(a))$ and $(x, f(x))$ as $x \to a$. When this limit exists, it defines the slope of the unique tangent line (retta tangente) to f at a.

Continuity and Differentiability

Differentiability implies continuity (continuità): f differentiable at $a \implies f$ continuous at a. The converse does not hold, as demonstrated by functions with discontinuities in their rate of change, such as those containing corners (punti angolosi) or vertical tangents (tangenti verticali). In real analysis, this is formalized through the linear approximation condition: A function fails to be differentiable at points where the remainder $r(x)$ does not vanish faster than $|x - a|$.

Methods for Differentiation

Product rule

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x) \tag{9}$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \tag{10}$$

Chain Rule

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) \tag{11}$$

Higher-Order Derivatives

The n-th derivative is defined recursively:

$$f^{(n)}(x) = \frac{d}{dx}\left[f^{(n-1)}(x)\right]$$

Additional Differentiation Rules

Basic Rules

- Constant Rule: $\frac{d}{dx}[c] = 0$
- Power Rule: $\frac{d}{dx}[x^n] = nx^{n-1}$
- Constant Multiple: $\frac{d}{dx}[cf(x)] = cf'(x)$
- Sum/Difference: $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Higher-Order Derivatives

Second derivative: $f''(x)$ or $\frac{d^2y}{dx^2}$. Recursive formula:

$$f^{(n)}(x) = \frac{d}{dx}\left[f^{(n-1)}(x)\right]$$

Exponential/Logarithmic Derivatives

- $\frac{d}{dx}[e^x] = e^x$
- $\frac{d}{dx}[a^x] = a^x \ln a$
- $\frac{d}{dx}[\ln x] = \frac{1}{x}$
- $\frac{d}{dx}[\log_a x] = \frac{1}{x \ln a}$

Trigonometric Derivatives

- $\frac{d}{dx}[\sin x] = \cos x$
- $\frac{d}{dx}[\cos x] = -\sin x$
- $\frac{d}{dx}[\tan x] = \sec^2 x$
- $\frac{d}{dx}[\cot x] = -\csc^2 x$
- $\frac{d}{dx}[\sec x] = \sec x \tan x$
- $\frac{d}{dx}[\csc x] = -\csc x \cot x$

Inverse Trig Derivatives

- $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}[\cos^{-1} x] = -\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$
- $\frac{d}{dx}[\cot^{-1} x] = -\frac{1}{1+x^2}$
- $\frac{d}{dx}[\sec^{-1} x] = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}[\csc^{-1} x] = -\frac{1}{|x|\sqrt{x^2-1}}$

Implicit Differentiation

Example for $x^2 + y^2 = 1$:

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Parametric Derivatives

For $x = x(t)$, $y = y(t)$:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{d^2y}{dx^2} = \frac{d/dt(dy/dx)}{dx/dt}$$

Hyperbolic Derivatives

- $\frac{d}{dx}[\sinh x] = \cosh x$
- $\frac{d}{dx}[\cosh x] = \sinh x$
- $\frac{d}{dx}[\tanh x] = \text{sech}^2 x$

Inverse Function Derivative

If f and g are inverses:

$$g'(x) = \frac{1}{f'(g(x))}$$

Taylor Series (Serie di Taylor) The Taylor series of a smooth function $f : \mathbb{R} \to \mathbb{R}$ expanded about $a \in \mathbb{R}$ is:

$$T_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

where $f^{(n)}(a)$ is the n-th derivative of f at a.

Taylor's Theorem (Teorema di Taylor) For $N \geq 0$, the function decomposes as:

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_N(x),$$

with the Lagrange remainder:

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1} \quad \text{for some } c \in (a, x).$$

Analyticity (Analiticità)

- The series converges to $f(x)$ near a if $\lim_{N\to\infty} R_N(x) = 0$.
- The **radius of convergence** R satisfies:

$$\frac{1}{R} = \limsup_{n\to\infty} \left| \frac{f^{(n)}(a)}{n!} \right|^{1/n}.$$

- Smoothness$\not\Longrightarrow$ analyticity: The series may diverge or converge to a different function.

Multi-Variable Case (Caso Multivariato) For $f : \mathbb{R}^k \to \mathbb{R}$, the series generalizes using multi-indices $\alpha = (\alpha_1, \dots, \alpha_k)$:

$$T_f(\mathbf{x}) = \sum_{|\alpha| \geq 0} \frac{\partial^{|\alpha|} f(a)}{\alpha!} (\mathbf{x} - \mathbf{a})^\alpha,$$

with $\alpha! = \alpha_1! \cdots \alpha_k!$ and $\partial^{|\alpha|} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_k^{\alpha_k}}$.

Key Distinctions (Differenze Fondamentali)

- **Analytic** $\iff$ Taylor series converges to f locally.
- **Uniqueness:** Analytic functions have unique power series representations.
- **Global vs. Local:** Analyticity extends local derivative data to a neighborhood.

General Taylor Series Application (Applicazione Generale della Serie di Taylor) Consider a smooth function $f : \mathbb{R} \to \mathbb{R}$ and a point $a \in \mathbb{R}$.

Expansion Process (Processo di Sviluppo)

- Compute derivatives $f^{(n)}(a)$ for $n \geq 0$.
- Construct the Taylor series:

$$T_f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

Approximation and Error (Approssimazione ed Errore) For a fixed N, the N-th Taylor polynomial approximates f:

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x - a)^n,$$

with error bounded by the Lagrange remainder:

$$|R_N(x)| \leq \frac{\max_{c \in [a,x]} |f^{(N+1)}(c)|}{(N+1)!} |x - a|^{N+1}.$$

Convergence Analysis (Analisi della Convergenza)

- Calculate the radius of convergence R using:

$$\frac{1}{R} = \limsup_{n \to \infty} \left| \frac{f^{(n)}(a)}{n!} \right|^{1/n}.$$

- If $\lim_{N \to \infty} R_N(x) = 0$ for $|x - a| < R$, then $T_f(x) = f(x)$ in this interval.

Interpretation (Interpretazione) This constructs a polynomial sequence $\{P_N(x)\}$ that:

- Matches f's derivatives at a up to order N.
- Converges to $f(x)$ locally if f is analytic at a.

Partial Derivatives (Derivate Parziali)

Fundamental Concept

The partial derivative extends the concept of differentiation to functions of several variables. It measures the instantaneous rate of change of the function with respect to one variable while holding all other variables constant. For a function $f : \mathbb{R}^n \to \mathbb{R}$, the partial derivative at a point $\mathbf{a} = (a_1, \ldots, a_n)$ with respect to the i-th variable provides the slope of the tangent line to the graph in the direction of the i-th coordinate axis.

Notation (Notazione)

Common notations for partial derivatives include:

- Leibniz notation: $\frac{\partial f}{\partial x_i}$
- Subscript notation: f_{x_i} or $\partial_{x_i} f$
- Operator notation: $D_i f$

For functions $z = f(x, y)$, we write $\frac{\partial z}{\partial x}$ or f_x for the partial with respect to x.

Limit Definition (Definizione di Limite)

The partial derivative of f with respect to x_i at $\mathbf{a}$ is defined as:

$$\frac{\partial f}{\partial x_i}(\mathbf{a}) = \lim_{h \to 0} \frac{f(a_1, \ldots, a_i + h, \ldots, a_n) - f(a_1, \ldots, a_n)}{h}$$

provided this limit exists. This definition requires f to be defined in an open neighborhood of $\mathbf{a}$. In terms of real analysis, this represents the ordinary derivative of the single-variable function:

$$g(x_i) = f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n)$$

evaluated at $x_i = a_i$.

Geometric Interpretation (Interpretazione Geometrica)

Geometrically, $\frac{\partial f}{\partial x_i}(\mathbf{a})$ represents the slope of the tangent line to the curve formed by intersecting the graph of f with the plane where all variables except x_i are held constant at their values in $\mathbf{a}$. For functions of two variables, $\frac{\partial f}{\partial x}(a, b)$ and $\frac{\partial f}{\partial y}(a, b)$ give the slopes of the tangent lines in the x- and y-directions, respectively, at the point $(a, b, f(a, b))$.

Higher-Order Partial Derivatives (Derivate Parziali di Ordine Superiore)

Partial derivatives can be iterated to obtain higher-order derivatives. The second-order partial derivative with respect to x_i and x_j is denoted:

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$

Under continuity conditions (Clairaut's theorem), the order of differentiation is interchangeable:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

for all i, j when both derivatives exist and are continuous in an open neighborhood.

Differentiability for Multivariable Functions (Derivabilità per Funzioni a Più Variabili)

The existence of partial derivatives does not guarantee differentiability. A function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a}$ if there exists a linear transformation $L : \mathbb{R}^n \to \mathbb{R}$ such that:

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - L(\mathbf{h})}{\|\mathbf{h}\|} = 0$$

where L is the total derivative (differenziale totale). The linear map L is represented by the gradient $\nabla f(\mathbf{a}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{a}), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{a})\right)$ when f is differentiable. Sufficient conditions for differentiability include the continuity of all partial derivatives in a neighborhood of $\mathbf{a}$.

Integrals

"If you're gonna shoot an elephant Mr. Schneider, you better be prepared to finish the job."

— Gary Larson, The Far Side

Riemann Integral

A Riemann integral is the the limit $f : [a, b] \to \mathbb{R}$ of all the Riemann sums between the points a and b. Defined as:

$$\int_a^b f(x)dx = lim_{||P||\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

Where:

n is the sub-intervals

$\Delta x_i$is the width of the i sub-interval

$c_i$is a sample point

*this exact definition will likely never be used in

Properties,

$$\text{Linearity:} \quad \int_a^b (\alpha f(x) + \beta g(x))\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx$$

$$\text{Additivity:} \quad \int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx \quad \text{for } c \in (a,b)$$

$$\text{Monotonicity:} \quad f(x) \leq g(x)\ \forall x \in [a,b] \implies \int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$$

$$\text{Bounds:} \quad m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a) \quad \text{where } m = \inf_{[a,b]} f,\ M = \sup_{[a,b]} f$$

$$\text{Integrability:} \quad f \text{ is Riemann integrable} \iff f \text{ bounded} \wedge f \text{ continuous a.e. on } [a,b]$$

$$\text{Fundamental Theorem:} \quad F' = f \implies \int_a^b f(x)\, dx = F(b) - F(a)$$

Fundamental theorem of calculus

Methods for integration

Substitution, if $u = f(x)$, then $du = g'(x)dx$

$$\int f(g(x))g'(x)dx = \int f(u)du \tag{12}$$

Integration by parts

$$\int udv = uv - \int vdu \tag{13}$$

Product rule

(14)

Chain Rule

Leibniz rule for differentiation. If f(t) is continuous and g(x) is differentiable

$$f(x) = \int_a^{g(x)} f(t)dt$$

If this is true, then:

$$F(x)' = f\left(g(x)\right) \cdot g'(x) \tag{15}$$

If both the limits are dependent on x:

$$I(x) = \int_{g_1(x)}^{g_2(x)} f(t)dt$$

then:

$$I'(x) = f\left(g_2(x)\right) g_2'(x) - f\left(g_1(x)\right) g_1'(x) \tag{16}$$

Differential Equations

"Would you tell me, please, which way I ought to go from here?"
"That depends a good deal on where you want to get to," said the Cat.

ODEs and PDEs

Ordinary Differential Equations (ODEs) are a differential equation which has a single variable. ODEs have a general form:

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^ny}{dx^n}\right) = 0 \tag{17}$$

where
- x independent
- y dependent

Partial Differential Equations (PDEs) are a Differential equation which has multiple independent variables. Instead of using the standard d, they use partial derivatives (∂) to show the change with respect for multiple variables. The general form is:

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, ...\right) = 0 \tag{18}$$

where
- x,y independent
- u(x,y) dependent

Fundamentally image a ODE as a means to track a single car, while PDE track all the traffic in the city.

Types of ODEs

ODEs are usually classified by 2 primary things. their order, aka the degree of their derivative, and by whether they are linear or non-linear.

First order ODEs are pretty self explanatory, they involve only the 1st derivative. Here is a basic first order ODE:

$$\frac{dy}{dx} + y = x \tag{19}$$

Second order ODEs involve UP to the 2nd derivative. Here is a example:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \tag{20}$$

Higher order ODEs involve everything 3rd derivative or higher. It is unlikely to ever appear in a 1st year analysis exam, but you never know.

Liniar and Non-liniear ODEs. An ODE is linear if the dependent variable and the derivatives are in a linear form. Basically: they are not multiplied together. Anything else is considered non-liniear. A linear ODE can be written in the form:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \tag{21}$$

- a(x) is a function of x

Here are some basic examples. We will go in much more detail when solving ODEs.

$\frac{dy}{dx} + 3y = x$, and $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = sinx$

$$Ineedtolearnbetterformating :)$$

A non-linear ODE is any ordinary differential equation that cannot be written in the linear form shown earlier. This is because the dependent variable or its derivatives are not linear. I will cover this more later on in the chapter.

The weird classifications of ODEs

A Homogeneous ODE, is a differential equation where L is a linear differential operator.

$$L[y] = 0$$

A Non-Homogeneous ODE, is a differential equation where f(x) $\neq$ 0

$$L[y] = f(x)$$

A Autonomous ODE, is a differential equation where the independent variable (usually x or t) does not appear in the equation.

$$\frac{d^ny}{dx^n} = F\left(y, y', ..., y^{(n-1)}\right)$$

A Non-Autonomous ODE, is a differential equation if the independent variable appears eq (1).

*You can use multiple types at once, just use common sense to make sure its right

Basic existence

Dealing with ODEs

Separable ODE, can be written as

$$\frac{dy}{dx} = f(x)g(y)$$

Divide both sides by g(y), and multiply both sides by dx

$$\frac{dy}{g(y)} = f(x)dx$$

Integrate both sides, make sure to keep the constant on the RHS

$$\int \frac{dy}{g(y)} = \int f(x)dx$$

Integrating Method. Format the equation to fit the following before using the method

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Find the function $\mu(x)$ that will help simplify the problem. (Symplify as much as possible here it will help a lot later on)

$$\mu(x) = e^{\int P(x)dx}$$

Multiply every term by the function $\mu(x)$ and using the product rule calculate the derivative of the LHS

$$\frac{d}{dx}(\mu(x)y) = \mu(x)Q(x)$$

Integrate both sides, remember the constant!

$$y = \frac{1}{\mu(x)} \int \mu(x)Q(x)dx + C$$

Quick note on integrating factor. A integrating factor is the method used to solve derivative equation above. The μ(x) also called the integrating factor, works for any, first-order linear differential equations. A function is derived by multiplying the equation with μ(x), which makes the left-hand side a derivative of μ(x)y.

Exact Method If a DE is exact, which can be found if it is in this form

$$M(x, y)dx + N(x, y)dy = 0$$

After this, calculate the partial derivative of M in respect to y and the partial derivative of N with respect to x. If these are equivalent the DE is exact.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Lets make a function that we will call Ψ such that, $\Psi_x = M(x, y)$ and $\Psi_y = N(x, y)$

Therefore we can write this now as

$$\Psi_x + \Psi_y \frac{dy}{dx} = 0$$

we can start to find this function Ψ. So we will start to integrate M with respect to x. (h(y) is a function of y)

$$\Psi(x, y) = \int M dx + h(y)$$

We can now differentiate Ψ with respect to y

$$\frac{\partial \Psi}{\partial y} = \frac{\partial}{\partial y}\left(\int M dx\right) + h'(y) = N$$

Solve the integral for h(y)

Types of Problems

Initial Value Problem

Generally, IVPs are a DE and a initial condition or condition's which when used in unison they can be used to solve a function, that will also fit the DE. The steps are pretty straight forward.

1. Solve the DE

$$y(x) = \int f(x)dx + C$$

2. Use the initial condition, lets say that $y(x_0) = y_0$

$$y_0 = \int f(x_0)dx + C$$

Where C is:

$$C = y_0 - V$$

*V is the value of the integral at x_0, therefore if we replace C, the final answer is

$$y(x) = \int f(x)dx + (y_0 - V)$$

Proving existence and uniqueness

Theorem Definition: If f(x,y) and the partial derivative $\frac{\partial f}{\partial y}$ are continuous in:

$$D = \{(x, y)||x - x_0| \leq a, |y - y_0| \leq b\}$$

around the point (x_0, y_0) therefore, there exits a interval between x and $x_0$where there is at least one solution of y(x). And it proves that this solution is unique on the interval.

Steps: Write the ODE in standard form, aka:

$$\frac{dy}{dx} = f(x, y)$$

use intuition to check that the function f is continuous between the points you want. Then if there are any points where the function is non continuous, then be sure to mark it such that it is clear, using $\leq\geq><$. This is the first rule.

Now, do a partial derivative of y such that:

$$\frac{\partial f}{\partial y} = f(x, y)$$

Remember to treat x as a constant in this case!!!

If the result is continuous in D between x and x_0 then this proves that there is at least a solution.

Interpreting answers

Interval of validity for Linear DE. The interval of validity for a Linear DE is largest around $x_0$where $p(x)$ and $q(x)$ are continuous. Make sure to exclude discontinuities.

Interval of validity for Non-Linear DE, Solutions may be exponential to infinity or become undefined despite $f(x, y)$ being smooth. Therefore check all the points where the solution becomes undefined. Extend the interval on both sides of x_0 till it is no longer possible.

Interpreting answers

(this whole section is a bit useless, I may remove it if I don't find any use for it son)

Explicit Solution, is when the dependent variable is isolated and in terms of the independent variable. e.g

$$y = x^2 + C$$

(soluzione esplicita)

Implicit Solution, is when the dependent variable is not explicitly isolated from the independent. e.g

$$x^2 + y^2 = C$$

(soluzione implicita)

General Solution, is the solution containing all the possible solutions for the differential equation, ie it keeps the constants, its the trivial form. (soluzione generale)

Particular Solution, is the solution which is a specific solution by locking the constants by using the initial conditions, ie the C has a fixed value. (soluzione particolare)

Equilibrium Solution, is a solution which is constant because the dependent variable does not change and therefore the derivative is zero. (soluzione di equilibrio)

Parametric Solution, is a solution represented using a parameter like (t,u,z..) instead of using x and y. eg.

$$y(t) = \sqrt{t^2 + C}$$

(soluzione parametrica)

Second Order Equations– temporary latex code which isnt my own for the exam

Solving Second-Order Differential Equations

A second-order differential equation has the general form:

$$F(y'', y', y, x) = 0$$

Below are methods for solving linear and nonlinear cases.

1. Linear Homogeneous Equations with Constant Coefficients General form:

$$ay'' + by' + cy = 0 \quad (a \neq 0)$$

Solution procedure:

1. Solve the **characteristic equation**:

$$ar^2 + br + c = 0$$

2. **Case analysis for roots r_1, r_2:**

- **Distinct real roots:** If $r_1 \neq r_2$,

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

- **Repeated real root:** If $r_1 = r_2 = r$,

$$y(x) = (C_1 + C_2 x) e^{rx}$$

- **Complex conjugate roots:** If $r = \alpha \pm i\beta$,

$$y(x) = e^{\alpha x} \left[C_1 \cos(\beta x) + C_2 \sin(\beta x) \right]$$

2. Linear Nonhomogeneous Equations **General form:**

$$y'' + p(x)y' + q(x)y = g(x)$$

Method 1: Undetermined Coefficients

1. Find the complementary solution y_c (solve the homogeneous equation).
2. Assume a particular solution y_p based on $g(x)$ (e.g., $g(x) = e^{kx} \Rightarrow y_p = Ae^{kx}$).
3. If $g(x)$ matches part of y_c, multiply y_p by x (or x^n for repeated roots).
4. Substitute y_p into the DE and solve for coefficients.
5. General solution: $y = y_c + y_p$.

Method 2: Variation of Parameters

1. Find $y_c = C_1 y_1 + C_2 y_2$.
2. Compute the Wronskian:

$$W = y_1 y_2' - y_1' y_2$$

3. Particular solution:

$$y_p = -y_1 \int \frac{y_2 g(x)}{W} \, dx + y_2 \int \frac{y_1 g(x)}{W} \, dx$$

4. General solution: $y = y_c + y_p$.

3. Cauchy-Euler Equations **General form:**

$$x^2 y'' + bxy' + cy = 0$$

Solution steps:

1. Assume $y = x^r$. Substitute to get:

$$r^2 + (b-1)r + c = 0$$

2. Solve for r. The solution mirrors constant-coefficient cases:
   - Real distinct roots: $y = C_1 x^{r_1} + C_2 x^{r_2}$
   - Repeated root: $y = x^r (C_1 + C_2 \ln x)$
   - Complex roots $r = \alpha \pm i\beta$: $y = x^\alpha \left[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x) \right]$

4. Reduction of Order **When one solution y_1 is known:**

1. Let $y_2 = v(x)y_1$.
2. Substitute y_2 into the DE and solve for $v(x)$.
3. The second solution is $y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$.

Special cases:

- **Equation missing y:** Let $p = y'$, reducing to $p' = f(x,p)$.
- **Equation missing x:** Let $p = y'$, then $y'' = p\frac{dp}{dy}$, reducing to $p\frac{dp}{dy} = f(y,p)$.

5. Series Solutions Near Ordinary Points For $P(x)y'' + Q(x)y' + R(x)y = 0$ with ordinary point at x_0:

1. Assume $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$.
2. Substitute into DE and equate coefficients of like powers.
3. Derive a recurrence relation for a_n.

6. Laplace Transform for Initial Value Problems **Procedure:**

1. Take Laplace transform of the DE:

$$\mathcal{L}\{y''\} + a\mathcal{L}\{y'\} + b\mathcal{L}\{y\} = \mathcal{L}\{g(x)\}$$

2. Use:

$$\mathcal{L}\{y'\} = sY(s) - y(0), \quad \mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$$

3. Solve for $Y(s)$, then compute $y(x) = \mathcal{L}^{-1}\{Y(s)\}$.

Nonlinear Second-Order DEs No universal method exists. Common approaches:

- Substitutions to reduce order (e.g., $p = y'$)
- Exact equations (identify integrable combinations)
- Numerical methods (e.g., Runge-Kutta)

Homogeneous Equations with constant coefficients have the general form:

$$y'' + ay' + by = 0$$

Non-homogeneous Equations: Method of Undetermined Coefficients

$Finished$

Nth order Differential Equation

0.0.1 Solving n-th Order Differential Equations

Linear Homogeneous Equations with Constant Coefficients Consider the equation:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Solution Steps:

1. **Form characteristic equation**:

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

2. **Find roots** $r_1, r_2, \ldots, r_n$
3. **Construct general solution**:
   - *Distinct real roots*:

$$y_h = \sum_{i=1}^{n} C_i e^{r_i x}$$

   - *Repeated real root r with multiplicity k*:

$$e^{rx} \left(C_1 + C_2 x + \cdots + C_k x^{k-1} \right)$$

   - *Complex conjugate pairs $\alpha \pm \beta i$*:

$$e^{\alpha x} \left[C_1 \cos(\beta x) + C_2 \sin(\beta x) \right]$$

     For repeated pairs (multiplicity m):

$$x^{m-1} e^{\alpha x} \left[C_1 \cos(\beta x) + C_2 \sin(\beta x) \right]$$

Linear Nonhomogeneous Equations For equations:

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = g(x)$$

General Solution:

$$y = y_h + y_p$$

where y_h = homogeneous solution, y_p = particular solution.

Method of Undetermined Coefficients Use when $g(x)$ is polynomial, exponential, sine, cosine, or combinations:

1. Assume y_p with same form as $g(x)$
2. If any term matches y_h, multiply by x^s (s = smallest integer eliminating duplication)
3. Substitute y_p into DE and solve for coefficients

Variation of Parameters General method for arbitrary $g(x)$:

1. Find fundamental set $\{y_1, \ldots, y_n\}$ from y_h
2. Compute Wronskian:

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

3. Find $u_i' = \frac{W_i}{W}$ where W_i = Wronskian with i-th column replaced by $\begin{bmatrix} 0 \\ \vdots \\ g(x) \end{bmatrix}$
4. Integrate to get u_i, then:

$$y_p = \sum_{i=1}^{n} u_i y_i$$

Variable Coefficient Equations For $y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_0(x)y = Q(x)$:

Reduction of Order If solution y_1 is known, let:

$$y = y_1 \int v(x)dx$$

Substitute to reduce equation order by 1.

Cauchy-Euler Equations Form: $x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + a_0 y = 0$

1. Assume solution $y = x^m$
2. Substitute to get characteristic equation:

$$m(m-1)\cdots(m-n+1) + \sum_{k=0}^{n-1} a_k m(m-1)\cdots(m-k+1) = 0$$

3. Handle roots as with constant coefficient equations

Nonlinear Equations

Order Reduction Techniques

- *Missing* y: Let $v = y'$, reduces order by 1
- *Missing* x: Let $v = y'$, then $y'' = v\frac{dv}{dy}$, reduces to 1st order in v

Exact Equations If equation can be written as:

$$\frac{d}{dx}\,[\text{Lower order expression}] = 0$$

Integrate successively to solve.

Integrating Factors Find $\mu(x)$ or $\mu(y)$ such that:

$$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0$$

becomes exact.

Special Forms

- *Bernoulli*: $y' + P(x)y = Q(x)y^n$, use $z = y^{1-n}$
- *Riccati*: $y' = P(x)y^2 + Q(x)y + R(x)$, use $y = y_1 + \frac{1}{v}$ if particular solution y_1 known

Systems of Differential Equations

Solving Systems of Differential Equations

Linear Systems with Constant Coefficients For the system $\mathbf{x}' = A\mathbf{x}$ where A is an $n \times n$ constant matrix:

Solution Method:

1. **Find eigenvalues** λ by solving:

$$\det(A - \lambda I) = 0$$

2. **Find eigenvectors** ξ for each eigenvalue by solving:

$$(A - \lambda I)\xi = 0$$

3. **Construct general solution**:
   - *Real distinct eigenvalues*:

$$\mathbf{x}(t) = \sum_{i=1}^{n} C_i e^{\lambda_i t}\xi_i$$

   - *Complex eigenvalues* $\alpha \pm \beta i$:

$$\mathbf{x}(t) = C_1 e^{\alpha t}[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)] + C_2 e^{\alpha t}[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)]$$

     where $\mathbf{a} + i\mathbf{b}$ is the complex eigenvector
   - *Repeated eigenvalues*:
     - If geometric multiplicity = algebraic multiplicity: proceed as distinct eigenvalues
     - If deficient eigenvectors: use generalized eigenvectors

Nonhomogeneous Systems For $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$:

General Solution:

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)$$

Variation of Parameters

1. Find fundamental matrix $\Phi(t)$ from homogeneous solutions
2. Compute particular solution:

$$\mathbf{x}_p(t) = \Phi(t) \int \Phi^{-1}(t)\mathbf{g}(t)dt$$

Method of Undetermined Coefficients Use when $\mathbf{g}(t)$ contains polynomials, exponentials, or trigonometric functions:

1. Assume $\mathbf{x}_p$ with same form as $\mathbf{g}(t)$
2. Adjust for resonance if any term matches homogeneous solution
3. Substitute and solve for coefficients

Matrix Exponential Method For $\mathbf{x}' = A\mathbf{x}$:

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

where e^{At} can be computed via:

- Diagonalization: $A = PDP^{-1} \Rightarrow e^{At} = Pe^{Dt}P^{-1}$
- Jordan form for defective matrices
- Taylor series expansion for simple cases

Nonlinear Systems

Linearization Near Critical Points

1. Find equilibrium points $\mathbf{x}_0$ where $\mathbf{f}(\mathbf{x}_0) = 0$
2. Compute Jacobian matrix:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\mathbf{x}_0}$$

3. Analyze eigenvalues of J to determine stability

Phase Plane Analysis (2D Systems)

- Classify critical points: node, spiral, saddle, center
- Use nullclines and direction fields
- Lyapunov functions for stability (when applicable)

Conversion to First-Order Systems Any n-th order DE can be converted to a system:

1. Let $x_1 = y$, $x_2 = y'$, ..., $x_n = y^{(n-1)}$
2. Create system:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_n' = F(t, x_1, \ldots, x_n) \end{cases}$$

Important Special Cases

Coupled Oscillators

$$\begin{cases} m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 x_2'' = -k_2(x_2 - x_1) \end{cases}$$

Solve by diagonalizing the coefficient matrix

Competing Species Model

$$\begin{cases} x' = x(a - by) \\ y' = y(c - dx) \end{cases}$$

Analyze using linearization and phase plane methods

Special theorems and Problems

Picard–Lindelöf theorem

Picard–Lindelöf Theorem (Existence & Uniqueness)

Theorem Statement Consider the initial value problem (IVP):

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

where $f : D \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. If:

- f is **continuous** in t on rectangle $R = [t_0 - a, t_0 + a] \times \overline{B}(y_0, b)$
- f is **Lipschitz continuous** in y:

  $$\exists L > 0 \text{ s.t. } \|f(t, y_1) - f(t, y_2)\| \leq L\|y_1 - y_2\|, \ \forall (t, y_1), (t, y_2) \in R$$

Then $\exists \tau > 0$ such that the IVP has a **unique solution** $y(t)$ on $[t_0 - \tau, t_0 + \tau]$.

Proof Outline (Method of Successive Approximations)

1. Reformulate IVP as integral equation:

   $$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds$$

2. Define Picard iterations:

   $$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))ds$$

   Starting with $y_0(t) \equiv y_0$

3. Show $\{y_n\}$ converges uniformly to solution y:
   - Use Lipschitz condition to prove $\|y_{n+1} - y_n\| \leq \frac{M}{L} \frac{(L|t-t_0|)^{n+1}}{(n+1)!}$
   - Apply Banach fixed-point theorem in complete metric space

Implementation Steps To apply the theorem:

1. Verify continuity of $f(t, y)$ in t
2. Check Lipschitz condition in y:
   - If $\frac{\partial f}{\partial y}$ exists and bounded $\Rightarrow$ Lipschitz
   - For scalar case: $|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$
3. Determine existence interval $\tau = \min\left(a, \frac{b}{M}\right)$ where:

   $$M = \max_{(t,y) \in R} \|f(t, y)\|$$

Example Application For IVP $y' = y, \ y(0) = 1$:

- Picard iterations:

$$
\begin{aligned}
y_0(t) &= 1 \\
y_1(t) &= 1 + \int_0^t y_0(s)ds = 1 + t \\
y_2(t) &= 1 + \int_0^t (1+s)ds = 1 + t + \frac{t^2}{2} \\
&\vdots \\
y_n(t) &= \sum_{k=0}^{n} \frac{t^k}{k!} \to e^t
\end{aligned}
$$

Important Notes

- **Local vs Global**: Theorem guarantees local solution - need additional conditions for global existence
- **Sharpness**: τ estimate often conservative
- **Failure Cases**:
  - f not Lipschitz $\Rightarrow$ possible non-uniqueness (e.g., $y' = \sqrt{|y|}$)
  - Discontinuous f $\Rightarrow$ solutions may not exist

Cauchy's Problem

Here we have a nth order ODE

$$y^{(n)}(t) = f\left(t, y, y', ..., y^{(n-1)}\right)$$

The Cauchy problem also known as the