Notes on Electromagnetism

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Class Information

Recommended Textbooks

- The Feynman Lectures on Physics
- Picasso - Lezioni di fisica

1 Coulomb's Law

Coulomb's law gives the electrostatic force between two charged particles. The force $\vec{F}_{12}$ exerted by a charge q_1 on a charge q_2 is:

$$\vec{F}_{12} = k\frac{q_1 q_2}{R^2}\hat{\vec{R}}_{12} \tag{1}$$

where k is Coulomb's constant, R is the distance between the charges, and $\hat{\vec{R}}_{12}$ is the unit vector pointing from q_1 to q_2. The force is reciprocal:

$$\vec{F}_{12} = -\vec{F}_{21} \tag{2}$$

Coulomb's constant k is:

$$k = \frac{1}{4\pi\epsilon_0} \tag{3}$$

where ϵ_0 is the permittivity of free space.

2 2019 Redefinition of SI Base Units

Since 2019, the SI base units are defined by setting the numerical values of seven defining constants, including the elementary charge, e. Consequently, the vacuum permittivity ϵ_0 is now a measured value with an associated uncertainty, rather than a defined constant.

For example, a comparison of the gravitational force F_G and the electric force F_E:

$$F_G = G\frac{m_p m_e}{R^2} \approx 10^{-47}N \tag{4}$$

$$F_E = \frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} \approx 10^{-7}N \tag{5}$$

3 The Electric Field

The electric field (*campo elettrico*) is a vector field representing the force per unit charge. It extends the concept of Coulomb's force (*forza di Coulomb*) to describe the influence of charges in space.

The electric field $\vec{E}$ at a point is defined as the force $\vec{F}$ on a test charge (*carica di prova*) q_0 at that point, divided by the charge. The limit as $q_0 \to 0$ is taken to not disturb the source charge:

$$\vec{E} = \lim_{q_0 \to 0}\frac{\vec{F}}{q_0} \tag{6}$$

The unit is Newtons per Coulomb (N/C), or Volts per meter (V/m).

For a single point charge (*carica puntiforme*) Q at the origin, the electric field is:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r} \tag{7}$$

where $\hat{r}$ is the radial unit vector (*versore*). For a charge Q_1 at $\vec{r}_1$, the field at $\vec{r}$ is:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{|\vec{r}-\vec{r}_1|^3}(\vec{r}-\vec{r}_1) \tag{8}$$

The superposition principle (*principio di sovrapposizione*) states that the total electric field from multiple charges is their vector sum.

Field lines (*linee di campo*) are used to visualize the electric field. They are tangent to the field vector at every point, originate from positive charges, and terminate on negative charges or at infinity. Their density indicates the field's strength.

For continuous charge distributions (*distribuzioni continue di carica*) (linear λ, surface σ, or volume ρ), the field is the integral of the contributions from each infinitesimal charge element. For a volume distribution:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')dV' \tag{9}$$

For an infinite straight wire (*filo rettilineo infinito*) with uniform charge density λ, the field is $E = \frac{\lambda}{2\pi\epsilon_0 z}$ at distance z. For a charged ring of radius R and charge Q, the axial field is $E_z = \frac{1}{4\pi\epsilon_0}\frac{Qz}{(z^2+R^2)^{3/2}}$. For a uniformly charged disk (*disco carico*) of radius R and density σ, the axial field is $E_z = \frac{\sigma}{2\epsilon_0}\left(1-\frac{z}{\sqrt{z^2+R^2}}\right)$. For an infinite plane, the field is uniform (*campo uniforme*): $E = \frac{\sigma}{2\epsilon_0}$.

Gauss's Law (*teorema di Gaus*) relates the electric flux (*flusso*), Φ_E, through a closed surface (*superficie chiusa*) to the enclosed charge Q_{int}:

$$\Phi_E = \oint_S \vec{E}\cdot d\vec{S} = \frac{Q_{int}}{\epsilon_0} \tag{10}$$

where Q_{int} is the total internal charge (*cariche interne*).

4 Applications of Gauss's Law

Gauss's Law states that the electric flux through a closed surface S is proportional to the enclosed charge Q_{int}:

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{Q_{\text{int}}}{\epsilon_0} \tag{11}$$

This law is most useful for calculating the electric field in problems with a high degree of symmetry (spherical, cylindrical, or planar), because the symmetries of the charge distribution ρ are inherited by the electric field $\vec{E}$.

Uniformly Charged Sphere

Consider a sphere of radius R with a total charge Q distributed uniformly throughout its volume, giving a constant charge density $\rho = Q/(\frac{4}{3}\pi R^3)$. Due to spherical symmetry, the electric field $\vec{E}$ must be purely radial and its magnitude $E(r)$ can only depend on the distance r from the center. We choose a spherical Gaussian surface of radius r.

For this surface, the flux integral simplifies:

$$\oint_S \vec{E}\cdot d\vec{S} = E(r)\cdot(4\pi r^2) \tag{12}$$

Case 1: Outside the sphere ($r > R$). The enclosed charge is the total charge Q. Applying Gauss's law: $E(r)\cdot 4\pi r^2 = Q/\epsilon_0$, which gives:

$$E(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2} \tag{13}$$

Outside the sphere, the field is identical to that of a point charge Q at the origin.

Case 2: Inside the sphere ($r \leq R$). The enclosed charge Q_{int} is only the charge within the radius r: $Q_{\text{int}} = \rho \cdot (\frac{4}{3}\pi r^3) = Q\frac{r^3}{R^3}$. Applying Gauss's law: $E(r) \cdot 4\pi r^2 = (Qr^3/R^3)/\epsilon_0$, which gives:

$$E(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}r = \frac{\rho}{3\epsilon_0}r \tag{14}$$

The field grows linearly from zero at the center to a maximum at the surface.

Charged Spherical Shell

Consider a shell with inner radius R_1 and outer radius R_2, with a uniform charge density ρ in the region between them.

- For $r < R_1$ (inside the cavity), $Q_{\text{int}} = 0$, so $E = 0$.
- For $r > R_2$ (outside the shell), the field is that of a point charge with the total charge Q of the shell.
- For $R_1 < r < R_2$ (within the shell), the enclosed charge is $Q_{\text{int}} = \rho \cdot (\frac{4}{3}\pi r^3 - \frac{4}{3}\pi R_1^3)$. The electric field can be found by applying Gauss's law with this enclosed charge.

Sphere with Off-Center Cavity

A sphere of radius R_1 and density ρ has an off-center spherical cavity of radius R_2. This problem lacks direct spherical symmetry but can be solved using the superposition principle. The system is equivalent to a full sphere of density ρ plus a smaller sphere, at the position of the cavity, with density $-\rho$. The total field is the vector sum of the fields from these two spheres, which can be calculated individually. The field inside the cavity is found to be uniform.

Infinite Line and Plane of Charge

Gauss's law can also be used to re-derive the fields for an infinite line and an infinite plane of charge, using cylindrical and pillbox-shaped Gaussian surfaces, respectively. The results are consistent with those from direct integration:

- **Infinite Line:** $E = \frac{\lambda}{2\pi\epsilon_0 r}$
- **Infinite Plane:** $E = \frac{\sigma}{2\epsilon_0}$ (uniform field)

For two parallel planes with opposite charge densities ($+\sigma$ and $-\sigma$), the superposition principle shows the field is zero outside the planes and $E = \sigma/\epsilon_0$ between them.

5 Divergence

The divergence of a vector field is a scalar field that measures the tendency of the field to 'diverge' from or 'converge' to a point. For an electric field, it indicates the presence of a source (positive charge) or a sink (negative charge). It is written as $\nabla \cdot \vec{E}$.

In Cartesian coordinates, the divergence is:

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \tag{15}$$

The Divergence Theorem

The Divergence Theorem (or Gauss's-Ostrogradsky's theorem) relates the flux of a field through a closed surface to the integral of its divergence over the enclosed volume V:

$$\oint_S \vec{E} \cdot d\vec{S} = \int_V (\nabla \cdot \vec{E})\, dV \tag{16}$$

By combining this with Gauss's law for electricity (where $Q_{\text{int}} = \int_V \rho \, dV$), we can equate the volume integrals: $\int_V (\nabla \cdot \vec{E}) \, dV = \int_V \frac{\rho}{\epsilon_0} \, dV$ Since this must hold for any volume, the integrands must be equal. This gives the differential form of Gauss's Law, which is the first of Maxwell's Equations:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{17}$$

This equation states that the divergence of the electric field at any point is directly proportional to the charge density at that same point.