Analisi 1

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Pre-Derivatives

Theorems, functions and axioms

"Calvin: You know, I don't think math is a science, I think it's a religion.

Hobbes: A religion?

Calvin: Yeah. All these equations are like miracles. You take two numbers and when you add them, they magically become one NEW number! No one can say how it happens. You either believe it or you don't. [Pointing at his math book] This whole book is full of things that have to be accepted on faith! It's a religion!"

Complex numbers

A complex number is defined, by $x, y \in \mathbb{R}$ and i as the imaginary unit.

$$z = x + iy \tag{1}$$

Imagine $\mathbb{R}$ covering the whole x axis, and $\mathbb{C}$ covering the whole y axis, that's the complex plane.

Operations

Addition

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \tag{2}$$

Subtraction

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2) \tag{3}$$

Multiplication

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \tag{4}$$

Complex Conjugate

$$\bar{z} = x - iy \tag{5}$$

Modulus

$$|z| = \sqrt{x^2 + y^2} \tag{6}$$

Inverse

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2} \tag{7}$$

Sums and Sequences

Series

Post-Derivatives

Derivatives

Integrals

"If you're gonna shoot an elephant Mr. Schneider, you better be prepared to finish the job."
— Gary Larson, The Far Side

Riemann Integral

A Riemann integral is the the limit $f : [a, b] \to \mathbb{R}$ of all the Riemann sums between the points a and b. Defined as:

$$\int_a^b f(x)dx = lim_{||P|| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

Where:
n is the sub-intervals
$\Delta x_i$is the width of the i sub-interval
$c_i$is a sample point
*this exact definition will likely never be used in

Properties,

$$\text{Linearity:} \quad \int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$
$$\text{Additivity:} \quad \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad \text{for } c \in (a,b)$$
$$\text{Monotonicity:} \quad f(x) \leq g(x)\ \forall x \in [a,b] \implies \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$
$$\text{Bounds:} \quad m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a) \quad \text{where } m = \inf_{[a,b]} f,\ M = \sup_{[a,b]} f$$
$$\text{Integrability:} \quad f \text{ is Riemann integrable} \iff f \text{ bounded} \wedge f \text{ continuous a.e. on } [a,b]$$
$$\text{Fundamental Theorem:} \quad F' = f \implies \int_a^b f(x)\,dx = F(b) - F(a)$$

Fundamental theorem of calculus

Methods for integration

Substitution, if $u = f(x)$, then $du = g'(x)dx$

$$\int f(g(x))g'(x)dx = \int f(u)du \tag{8}$$

Integration by parts

$$\int udv = uv - \int vdu \tag{9}$$

Lebiz rule for differentiation. If f(t) is continuous and g(x) is differentiable

$$f(x) = \int_a^{g(x)} f(t)dt$$

If this is true, then:

$$F(x)' = f\left(g(x)\right) \cdot g'(x) \tag{10}$$

If both the limits are dependent on x:

$$I(x) = \int_{g_1(x)}^{g_2(x)} f(t)dt$$

then:

$$I'(x) = f\left(g_2(x)\right) g_2'(x) - f\left(g_1(x)\right) g_1'(x) \tag{11}$$

Limit under a derviative

Differential Equations

"Would you tell me, please, which way I ought to go from here?"
"That depends a good deal on where you want to get to," said the Cat.

ODEs and PDEs

Ordinary Differential Equations (ODEs) are a differential equation which has a single variable. ODEs have a general form:

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^ny}{dx^n}\right) = 0 \tag{12}$$

where
- x independent
- y dependent

Partial Differential Equations (PDEs) are a Differential equation which has multiple independent variables. Instead of using the standard d, they use partial derivatives (∂) to show the change with respect for multiple variables. The general form is:

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, ...\right) = 0 \tag{13}$$

where
- x,y independent
- u(x,y) dependent

Fundamentally image a ODE as a means to track a single car, while PDE track all the traffic in the city.

Types of ODEs

ODEs are usually classified by 2 primary things. their order, aka the degree of their derivative, and by whether they are linear or non-linear.

First order ODEs are pretty self explanatory, they involve only the 1st derivative. Here is a basic first order ODE:

$$\frac{dy}{dx} + y = x \tag{14}$$

Second order ODEs involve UP to the 2nd derivative. Here is a example:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \tag{15}$$

Higher order ODEs involve everything 3rd derivative or higher. It is unlikely to ever appear in a 1st year analisis exam, but you never know.

Liniar and Non-liniear ODEs. An ODE is linear if the dependent variable and the derivatives are in a linear form. Basically: they are not multiplied together. Anything else is considered non-liniear. A linear ODE can be written in the form:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \tag{16}$$

- a(x) is a function of x

Here are some basic examples. We will go in much more detail when solving ODEs.

$\frac{dy}{dx} + 3y = x$, and $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = sinx$

$$Ineedtolearnbetterformating :)$$

A non-linear ODE is any ordinary differential equation that cannot be written in the linear form shown earlier. This is because the dependent variable or its derivatives are not linear. I will cover this more later on in the chapter.

The weird classifications of ODEs

A Homogeneous ODE, is a differential equation where L is a linear differential operator.

$$L[y] = 0$$

A Non-Homogeneous ODE, is a differential equation where f(x) $\neq 0$

$$L[y] = f(x)$$

A Autonomous ODE, is a differential equation where the independent variable (usually x or t) does not appear in the equation.

$$\frac{d^ny}{dx^n} = F\left(y, y', ..., y^{(n-1)}\right)$$

A Non-Autonomous ODE, is a differential equation if the independent variable appears eq (1).

*You can use multiple types at once, just use common sense to make sure its right

Dealing with ODEs

Separable ODE, can be written as

$$\frac{dy}{dx} = f(x)g(y)$$

Divide both sides by g(y), and multiply both sides by dx

$$\frac{dy}{g(y)} = f(x)dx$$

Integrate both sides, make sure to keep the constant on the RHS

$$\int \frac{dy}{g(y)} = \int f(x)dx$$

Integrating Method. Format the equation to fit the following before using the method

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Find the function $\mu(x)$ that will help simplify the problem. (Symplify as much as possible here it will help a lot later on)

$$\mu(x) = e^{\int P(x)dx}$$

Multiply every term by the function $\mu(x)$ and using the product rule calculate the derivative of the LHS

$$\frac{d}{dx}(\mu(x)y) = \mu(x)Q(x)$$

Integrate both sides, remember the constant!

$$y = \frac{1}{\mu(x)} \int \mu(x)Q(x)dx + C$$

Quick note on integrating factor. A integrating factor is the method used to solve derivative equation above. The μ(x) also called the integrating factor, works for any, first-order linear differential equations. A function is derived by multiplying the equation with μ(x), which makes the left-hand side a derivative of μ(x)y.

Exact Method If a DE has a exact form (aka that there is a =0 and both terms are split by a +) the form:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

OR it satisfies Clairaut's Theorem, below: (both technically mean the same thing) -btw Clairaut's Theorem is most definitely not in the syllabus

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Lets make a function that we will call Ψ such that, $\Psi_x = M(x, y)$ and $\Psi_y = N(x, y)$

Therefore we can write this now as

$$\Psi_x + \Psi_y \frac{dy}{dx} = 0$$

we can start to find this function Ψ. So we will start to integrate M with respect to x. (h(y) is a function of y)

$$\Psi(x, y) = \int M dx + h(y)$$

We can now differentiate Ψ with respect to y

$$\frac{\partial \Psi}{\partial y} = \frac{\partial}{\partial y}\left(\int M dx\right) + h'(y) = N$$

Solve the integral for h(y)

Initial Value Problem

Generally, IVPs are a DE and a initial condition or condition's which when used in unison they can be used to solve a function, that will also fit the DE. The steps are pretty straight forward.

1. Solve the DE

$$y(x) = \int f(x)dx + C$$

2. Use the initial condition, lets say that $y(x_0) = y_0$

$$y_0 = \int f(x_0)dx + C$$

Where C is:

$$C = y_0 - V$$

*V is the value of the integral at x_0, therefore if we replace C, the final answer is

$$y(x) = \int f(x)dx + (y_0 - V)$$

Interval of validity for Linear DE. The interval of validity for a Linear DE is largest around $x_0$where $p(x)$ and $q(x)$ are continuous. Make sure to exclude discontinuities.

Interval of validity for Non-Linear DE, Solutions may be exponential to infinity or become undefined despite $f(x, y)$ being smooth. Therefore check all the points where the solution becomes undefined. Extend the interval on both sides of x_0 till it is no longer possible.

Interpreting answers

(this whole section is a bit useless, I may remove it if I don't find any use for it son)

Explicit Solution, is when the dependent variable is isolated and in terms of the independent variable. e.g

$$y = x^2 + C$$

(soluzione esplicita)

Implicit Solution, is when the dependent variable is not explicitly isolated from the independent. e.g

$$x^2 + y^2 = C$$

(soluzione implicita)

General Solution, is the solution containing all the possible solutions for the differential equation, ie it keeps the constants, its the trivial form. (soluzione generale)

Particular Solution, is the solution which is a specific solution by locking the constants by using the initial conditions, ie the C has a fixed value. (soluzione particolare)

Equilibrium Solution, is a solution which is constant because the dependent variable does not change and therefore the derivative is zero. (soluzione di equilibrio)

Parametric Solution, is a solution represented using a parameter like (t,u,z..) instead of using x and y. eg.

$$y(t) = \sqrt{t^2 + C}$$

(soluzione parametrica)

Second Order Equations– needs a lot more work

Homogeneous Equations with constant coefficients have the general form:

$$y'' + ay' + by = 0$$

Non-homogeneous Equations: Method of Undetermined Coefficients

Finished

Special theorems and Problems

Cauchy's Problem