Notes on Electromagnetism

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Lesson 1: Coulomb's Law

Coulomb's Law (Legge di Coulomb) describes the electrostatic force (forza elettrostatica) between two static charged particles (particelle cariche). The force $\vec{F}_{12}$ that a charge (carica) q_1 exerts on a second charge q_2 is directly proportional to the product of the charges and inversely proportional to the square of the distance between them. The force is directed along the line connecting the two charges.

The vector form of the equation is:

$$\vec{F}_{12} = k\frac{q_1 q_2}{R^2}\hat{\vec{R}}_{12} \tag{1}$$

Here, k is Coulomb's constant (costante di Coulomb), R is the distance between the charges, and $\hat{\vec{R}}_{12}$ is the unit vector (versore) pointing from q_1 to q_2. The law is reciprocal, meaning the force exerted by q_2 on q_1 is equal in magnitude and opposite in direction, so $\vec{F}_{12} = -\vec{F}_{21}$.

Coulomb's constant is expressed in terms of the vacuum permittivity (permittività del vuoto), ϵ_0, as:

$$k = \frac{1}{4\pi\epsilon_0} \tag{2}$$

Following the 2019 redefinition of the SI base units (Unità di base del SI), the elementary charge (carica elementare) e is a defining constant. As a result, ϵ_0 is now an experimentally determined value with an associated uncertainty, not a defined quantity.

The electric force is significantly stronger than the gravitational force. For instance, the electric force (forza elettrica) F_E between a proton and an electron is many orders of magnitude greater than the gravitational force (forza gravitazionale) F_G between them:

$$F_G = G\frac{m_p m_e}{R^2} \approx 10^{-47}N \quad F_E = \frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} \approx 10^{-7}N \tag{3}$$

Lesson 2: Electric Flux and Gauss's Law

The concept of electric flux (Flusso Elettrico) quantifies the flow of an electric field (campo elettrico) through a surface. For a uniform electric field $\vec{E}$ and a flat surface area $\vec{A}$, the flux is the scalar product $\Phi_E = \vec{E} \cdot \vec{A}$. For non-uniform fields and curved surfaces, the flux is found by integrating the field over the surface:

$$\Phi_E = \int_S \vec{E} \cdot d\vec{A} \tag{4}$$

This integral effectively sums the component of the electric field perpendicular to the surface at every point.

Gauss's Law (Legge di Gauss), a cornerstone of electrostatics (elettrostatica) and one of the four fundamental Maxwell's Equations (Equazioni di Maxwell), relates the electric flux through a closed surface (superficie chiusa) to the net electric charge (carica elettrica) enclosed within it. The law states that the net electric flux is directly proportional to the enclosed charge.

In its integral form, Gauss's Law is expressed as:

$$\Phi_E = \oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} \tag{5}$$

The integral is performed over a conceptual closed surface known as a Gaussian surface (superficie Gaussiana), and Q_{enc} is the total charge it encloses. By applying the divergence theorem (teorema della divergenza), which connects the surface integral of a vector field to the volume integral of its divergence, we can derive the differential form of Gauss's Law:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{6}$$

Here, ρ represents the volume charge density (densità di carica volumica). This form locally relates the divergence of the electric field to the charge density at that point.

A key application of Gauss's Law is to calculate electric fields in situations with high symmetry. By choosing a Gaussian surface that mirrors the symmetry of the charge distribution (distribuzione di carica), the flux integral becomes simple to solve. For example, to find the field from an infinite line of charge with a uniform linear charge density (densità di carica lineare) λ, we use a cylindrical Gaussian

surface. The symmetry dictates that the electric field must point radially outward. The flux through the top and bottom caps of the cylinder is zero, and the flux through the side wall is $E \cdot (2\pi rL)$. The enclosed charge is λL. Applying Gauss's Law, $E(2\pi rL) = \lambda L/\epsilon_0$, yields the field:

$$E = \frac{\lambda}{2\pi\epsilon_0 r} \tag{7}$$

Lesson 3: Applications of Gauss's Law

Gauss's Law is a powerful tool for calculating the electric field for charge distributions with a high degree of symmetry, such as spherical, cylindrical, or planar symmetry. The symmetry of the charge density ρ is inherited by the electric field $\vec{E}$, which simplifies the calculation.

A classic example is the field of a uniformly charged sphere (Sfera Carica Uniformemente) of radius R and total charge Q. Due to spherical symmetry (simmetria sferica), the electric field must be radial, $\vec{E} = E(r)\hat{r}$. By choosing a spherical Gaussian surface of radius r, the flux integral simplifies to $\oint_S \vec{E} \cdot d\vec{S} = E(r) \cdot (4\pi r^2)$. Outside the sphere ($r > R$), the enclosed charge is the total charge Q. Applying Gauss's law gives $E(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$, which is identical to the field of a point charge (carica puntiforme) Q at the origin. Inside the sphere ($r \leq R$), the enclosed charge is only the fraction of charge within the radius r, $Q_{\text{int}} = Q\frac{r^3}{R^3}$. Gauss's law then yields $E(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}r$, showing that the field increases linearly from the center.

This method can be extended to other symmetric configurations. For a charged spherical shell (Guscio Sferico Carico), the electric field inside the inner cavity is zero. For a sphere with an off-center cavity, the superposition principle (principio di sovrapposizione) can be used by treating the cavity as a superposition of a sphere with negative charge density over a larger sphere with positive charge density. This reveals a uniform electric field inside the cavity. Gauss's law also efficiently re-derives the fields for an infinite line of charge (Linea Infinita di Carica) and an infinite plane of charge (Piano Infinito di Carica).

The concept of divergence (divergenza) of a vector field (campo vettoriale) measures the field's tendency to originate from or converge to a point. For an electric field, divergence signifies the presence of a charge source or sink. The Divergence Theorem (Teorema della Divergenza) relates the flux of a field through a closed surface to the integral of its divergence over the enclosed volume. Combining this theorem with Gauss's Law leads to its differential form:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{8}$$

This is the first of Maxwell's Equations, stating that the divergence of the electric field at any point is proportional to the charge density at that same point.

Lesson 4: Superposition, Conductors, and Plasma Oscillations

The superposition principle (principio di sovrapposizione) is a fundamental concept for calculating electric fields from complex charge distributions. A powerful illustration is the case of two overlapping spheres of the same radius R, one with uniform positive charge density $+\rho_0$ and the other with uniform negative charge density $-\rho_0$. The total electric field is the vector sum of the fields from each sphere. Using the known result for the field inside a uniformly charged sphere, $\vec{E} = \frac{\rho_0}{3\epsilon_0}\vec{r}$, we can calculate the field in the overlapping region. If the centers of the spheres are separated by a vector $\vec{\delta}$, the field in the overlapping region is found to be uniform and constant:

$$\vec{E}_{overlap} = -\frac{\rho_0}{3\epsilon_0}\vec{\delta} \tag{9}$$

This model provides an excellent analogy for the behavior of conductors (conduttori) in an external electric field. In a conductor, conduction electrons are free to move. When an external field $\vec{E}_{ext}$ is applied, the "sea" of electrons displaces slightly, creating a charge separation, or polarization (polarizzazione). This separation induces an internal electric field (campo elettrico interno) $\vec{E}_{int}$ that opposes the external field. The displacement continues until the net electric field inside the conductor is zero, a condition known as electrostatic equilibrium (equilibrio elettrostatico). At this point, $\vec{E}_{ext} + \vec{E}_{int} = 0$. The conductor effectively shields its interior from the external static electric field.

If the external field is suddenly removed, the displaced electron sea is no longer in equilibrium and experiences a restoring force from the fixed positive ions. This force acts like a spring, leading to oscillations. The equation of motion for the electron sea is that of a simple harmonic oscillator (oscillatore armonico semplice). This collective, high-frequency oscillation of the electrons is a quantum mechanical effect known as a plasma oscillation (oscillazione di plasma). The characteristic angular frequency of this oscillation is the plasma frequency (ω_p), a property of the material that depends on the density of conduction electrons.

$$\omega_p^2 = \frac{n_e e^2}{m_e \epsilon_0} \tag{10}$$

This phenomenon is crucial in understanding the optical properties of metals.

Lesson 5: Maxwell's First Equation and Electrostatic Potential

This lesson begins by deriving the first of Maxwell's Equations, which concerns the divergence of the electric field. Starting from the divergence theorem (teorema della divergenza), which relates the flux of a vector field through a closed surface to the integral of its divergence over the enclosed volume:

$$\oint_S \vec{E} \cdot d\vec{S} = \int_\tau (\nabla \cdot \vec{E})\, d\tau \tag{11}$$

We can combine this with Gauss's Law (Teorema di Gauss), which states that the flux of the electric field through a closed surface is proportional to the enclosed charge Q_{int}:

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{\text{int}}}{\epsilon_0} \tag{12}$$

By equating these two expressions for the flux, and expressing the enclosed charge as the integral of the volume charge density (densità volumetrica di carica) ρ, we get:

$$\int_\tau (\nabla \cdot \vec{E})\, d\tau = \int_\tau \frac{\rho}{\epsilon_0}\, d\tau \tag{13}$$

Since this equality must hold for any arbitrary volume τ, the integrands themselves must be equal. This gives us the differential form of Gauss's Law, also known as the first of Maxwell's Equations for the electric field in a vacuum:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{14}$$

This is a local equation (equazione locale) because it relates the divergence of the electric field at a specific point in space to the charge density at that very same point. This is in contrast to the integral form of Gauss's Law, which is non-local as it relates the field on a surface to the total charge contained within it.

To handle point charges (cariche puntiformi) within the framework of continuous charge densities, we introduce the Dirac delta function (delta di Dirac), which is technically a distribution (distribuzione). It is defined as being zero everywhere except at the origin, where it is infinite, yet its integral over all space is one. This allows us to define the volume charge density for a point charge q_0 located at position $\vec{r}_0$ as:

$$\rho(\vec{r}) = q_0 \delta(\vec{r} - \vec{r}_0) \tag{15}$$

Next, we explore the line integral (integrale di linea) of the electrostatic field between two points, A and B. For the field generated by a single point charge, the result of this integral depends only on the start and end points, not on the path taken. A field with this property is called a conservative field (campo conservativo). A direct consequence is that the line integral of the electrostatic field around any closed loop is zero:

$$\oint \vec{E} \cdot d\vec{l} = 0 \tag{16}$$

This conservative nature allows us to define a scalar field called the electrostatic potential (potenziale elettrostatico), V. The potential difference (differenza di potenziale) between two points is related to the line integral of the electric field:

$$V(A) - V(B) = \int_A^B \vec{E} \cdot d\vec{l} \tag{17}$$

From this relationship, we can derive the differential form, which connects the electric field to the potential via the gradient (gradiente):

$$\vec{E} = -\nabla V \tag{18}$$

The electric field points from regions of higher potential to regions of lower potential.

Surfaces on which the potential V is constant are known as equipotential surfaces (superfici equipotenziali). The electric field lines are always perpendicular to these surfaces. No work (lavoro) is done when moving a charge along an equipotential surface, as the displacement is always orthogonal to the electric force.

Finally, we define the electrostatic potential energy (energia potenziale elettrostatica) U of a point charge q placed in a potential V as $U = qV$. This energy can be interpreted as the work done by an external force to bring the charge from a reference point (where the potential is defined as zero) to its current position. The unit of potential is the Volt (Volt), defined as a Joule per Coulomb. In atomic and nuclear physics, a convenient unit of energy is the electron-volt (elettronvolt), which is the energy gained by an electron when it moves through a potential difference of one volt.

Lesson 6: Electrostatic Potential Energy and The Electric Dipole

Electrostatic Potential Energy

We continue our discussion on electrostatic potential energy. For a single point charge Q, the electrostatic energy is given by the product of the charge and the potential V at its location: $U = Q \cdot V$. This corresponds to the negative of the work done by the field's forces to bring the charge from infinity to that point, denoted as L_∞.

Let's extend this to a system of two point charges, Q_1 and Q_2, subject to the Coulomb force. To assemble this configuration, we can imagine bringing the charges from infinity. Placing the first charge, Q_1, requires no work as there is no pre-existing electric field. However, positioning the second charge, Q_2, involves work done by the electric field generated by Q_1.

The work L_{field} done by the field to bring Q_2 from infinity to a distance R_{12} from Q_1 is calculated as:

$$L_{\text{field}} = \int_\infty^{R_{12}} \vec{F} \cdot d\vec{l} = \int_\infty^{R_{12}} \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} \hat{R} \cdot d\vec{l}$$

Since the electrostatic field is conservative, the integral is path-independent. We can choose a radial path, so $d\vec{l} = dR\hat{R}$.

$$L_{\text{field}} = \int_\infty^{R_{12}} \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} dR = \left[-\frac{Q_1 Q_2}{4\pi\epsilon_0 R} \right]_\infty^{R_{12}} = -\frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}}$$

The external work L_{ext} required to assemble the configuration at a constant, near-zero velocity (implying zero change in kinetic energy, $\Delta K = 0$) is the negative of the work done by the field: $L_{\text{ext}} = -L_{\text{field}}$.

$$L_{\text{ext}} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}}$$

The potential energy U_2 of the two-charge system is defined as this external work, which is equivalent to the negative of the work done by the field:

$$U_2 = -L_{\text{field}} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}}$$

For a system of three point charges (Q_1, Q_2, Q_3), the total potential energy is the sum of the work required to assemble them. We first bring in Q_1 (zero work), then Q_2 (work U_2), and finally Q_3 in the presence of the fields from Q_1 and Q_2. The external work to bring in Q_3 is:

$$L_3 = \frac{Q_1 Q_3}{4\pi\epsilon_0 R_{13}} + \frac{Q_2 Q_3}{4\pi\epsilon_0 R_{23}}$$

The total potential energy U_3 of the system is the sum of the energies of all pairs:

$$U_3 = U_2 + L_3 = \frac{1}{4\pi\epsilon_0} \left(\frac{Q_1 Q_2}{R_{12}} + \frac{Q_1 Q_3}{R_{13}} + \frac{Q_2 Q_3}{R_{23}} \right)$$

This can be generalized for a system of N point charges. The total electrostatic potential energy is the sum over all unique pairs of charges:

$$U_N = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\frac{Q_iQ_j}{4\pi\epsilon_0 R_{ij}}$$

The factor of $\frac{1}{2}$ is included to correct for double-counting each pair in the summation (e.g., the pair Q_1, Q_2 is counted as both $i=1, j=2$ and $i=2, j=1$). This expression can be rewritten as:

$$U_N = \frac{1}{2}\sum_{i=1}^{N} Q_i \left(\sum_{j=1,j\neq i}^{N}\frac{Q_j}{4\pi\epsilon_0 R_{ij}}\right)$$

The term in the parenthesis is the electric potential V_i at the position of charge Q_i due to all other charges. Therefore:

$$U_N = \frac{1}{2}\sum_{i=1}^{N} Q_iV_i$$

This represents the interaction energy (energia di interazione) of the N point charges. It is the energy required to assemble the configuration of charges, assuming they are pre-existing.

For continuous charge distributions, the sum becomes an integral. The potential energy U for a volume charge distribution with density ρ is:

$$U = \frac{1}{2}\int_V \rho(\vec{r})V(\vec{r})d\tau$$

where the integral is over the volume V where the charge density ρ is non-zero. For surface and linear distributions, the expressions are analogous:

$$U = \frac{1}{2}\int_S \sigma V dS \quad \text{and} \quad U = \frac{1}{2}\int_L \lambda V dl$$

This formulation for continuous distributions represents the total electrostatic potential energy, which includes both the interaction energy between different infinitesimal charge elements and the self-energy (autoenergia) required to assemble each infinitesimal charge element itself.

As an example, the energy of a uniformly charged sphere of radius R and total charge Q can be calculated. The result is:

$$U = \frac{1}{4\pi\epsilon_0}\frac{3}{5}\frac{Q^2}{R}$$

This result can be used to define a conventional radius for the electron, known as the classical electron radius (raggio classico dell'elettrone). By equating the electrostatic energy of a sphere with charge e to the electron's rest energy m_ec^2, we can solve for the radius. Omitting the $\frac{3}{5}$ factor for an order-of-magnitude estimate:

$$m_ec^2 \approx \frac{1}{4\pi\epsilon_0}\frac{e^2}{r_e} \implies r_e = \frac{1}{4\pi\epsilon_0}\frac{e^2}{m_ec^2} \approx 2.8\times10^{-15}\,\text{m}$$

We can express the electrostatic energy in terms of the electric field. Starting from $U = \frac{1}{2}\int \rho V d\tau$ and using Maxwell's first equation, $\nabla\cdot\vec{E} = \rho/\epsilon_0$, we substitute $\rho = \epsilon_0\nabla\cdot\vec{E}$:

$$U = \frac{\epsilon_0}{2}\int_V (\nabla\cdot\vec{E})V d\tau$$

Using the vector identity $\nabla\cdot(V\vec{E}) = (\nabla V)\cdot\vec{E} + V(\nabla\cdot\vec{E})$ and the relation $\vec{E} = -\nabla V$, we get $V(\nabla\cdot\vec{E}) = \nabla\cdot(V\vec{E}) + E^2$. Substituting this into the energy integral:

$$U = \frac{\epsilon_0}{2}\int_V (\nabla\cdot(V\vec{E}) + E^2)d\tau = \frac{\epsilon_0}{2}\int_V \nabla\cdot(V\vec{E})d\tau + \frac{\epsilon_0}{2}\int_V E^2 d\tau$$

Applying the divergence theorem to the first term converts the volume integral into a surface integral over the boundary S of the volume V:

$$\int_V \nabla\cdot(V\vec{E})d\tau = \oint_S (V\vec{E})\cdot d\vec{S}$$

If we extend the integration volume to all of space, the surface S goes to infinity. At large distances r from a finite charge distribution, V falls off as $1/r$ and E as $1/r^2$, while the surface area S grows as r^2. The product $V\vec{E}$ thus decreases as $1/r^3$, and the surface integral $\oint(V\vec{E})\cdot d\vec{S} \propto (1/r^3)\cdot r^2 = 1/r$, which tends to zero as $r \to \infty$. Therefore, the first term vanishes, and the total energy is:

$$U = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 d\tau$$

This allows us to define the electrostatic energy density (densità di energia elettrostatica) as:

$$u_e = \frac{1}{2}\epsilon_0 E^2$$

This interpretation is conceptually powerful, as it associates energy with the electric field itself, distributed throughout space. The total energy is the integral of this density over all space where the field exists.

The Electric Dipole

An electric dipole (dipolo elettrico) is a system of two equal and opposite point charges, $+Q$ and $-Q$, separated by a small distance vector $\vec{\delta}$, which points from the negative to the positive charge by convention. The electric dipole moment (momento di dipolo elettrico) is defined as:

$$\vec{p} = Q\vec{\delta}$$

The unit is Coulomb-meter (C·m). Dipoles are fundamental in physics and chemistry. Many molecules, like water (H_2O), are polar molecules, meaning they have a permanent electric dipole moment. This explains phenomena like water's high boiling point and its effectiveness as a solvent. The forces between dipoles, such as hydrogen bonds (legami idrogeno), are crucial for the structure of DNA and proteins.

The electric potential V at a point P far from a dipole ($r \gg \delta$) can be approximated. The exact potential is the sum of the potentials from the two charges:

$$V(\vec{r}) = \frac{Q}{4\pi\epsilon_0 r_+} - \frac{Q}{4\pi\epsilon_0 r_-} = \frac{Q}{4\pi\epsilon_0}\frac{r_- - r_+}{r_- r_+}$$

For large distances r, we can approximate $r_- r_+ \approx r^2$ and $r_- - r_+ \approx \delta\cos\alpha$, where α is the angle between the dipole moment $\vec{p}$ and the position vector $\vec{r}$.

$$V(\vec{r}) \approx \frac{Q\delta\cos\alpha}{4\pi\epsilon_0 r^2} = \frac{p\cos\alpha}{4\pi\epsilon_0 r^2}$$

This can be written compactly using the dot product:

$$V(\vec{r}) = \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3}$$

The potential of a dipole falls off as $1/r^2$. The electric field $\vec{E}$ is found by taking the negative gradient of the potential, $\vec{E} = -\nabla V$. In spherical coordinates, with $\vec{p}$ along the z-axis, the potential is $V(r,\theta) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$. The field components are:

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}$$

$$E_\phi = 0$$

The electric field of a dipole falls off as $1/r^3$, which is faster than the $1/r^2$ decay of a single point charge. This is because, from far away, the two opposite charges appear to cancel each other out.

A dipole placed in a uniform external electric field $\vec{E}_{\text{ext}}$ experiences no net force, as the forces on the positive and negative charges are equal and opposite ($\vec{F}_{\text{net}} = Q\vec{E}_{\text{ext}} - Q\vec{E}_{\text{ext}} = 0$). However, it does experience a torque (momento delle forze):

$$\vec{\tau} = \vec{p}\times\vec{E}_{\text{ext}}$$

This torque tends to align the dipole moment with the external field.

In a non-uniform electric field, the forces on the two charges are no longer equal, resulting in a net force on the dipole. This force can be expressed as:

$$\vec{F} = (\vec{p} \cdot \nabla)\vec{E}_{\text{ext}}$$

This is also equal to the gradient of the scalar product of $\vec{p}$ and $\vec{E}$:

$$\vec{F} = \nabla(\vec{p} \cdot \vec{E}_{\text{ext}})$$

The potential energy U of a dipole in an external electric field is the work required to orient it in the field. It is given by:

$$U = -\vec{p} \cdot \vec{E}_{\text{ext}}$$

The energy is minimized ($U = -pE$) when the dipole is aligned with the field ($\theta = 0$), which is a stable equilibrium. The energy is maximized ($U = +pE$) when it is anti-aligned ($\theta = \pi$), which is an unstable equilibrium. The force on the dipole can also be derived from the potential energy: $\vec{F} = -\nabla U = \nabla(\vec{p} \cdot \vec{E}_{\text{ext}})$.

Lesson 8: Curl, Stokes' Theorem, and Conductors

The Curl of a Vector Field

Today we introduce the curl (rotore), a vector differential operator that, when applied to a vector field, yields another vector field. To build intuition, consider the velocity field of a fluid. The curl of the velocity field describes the fluid's circulation density (densità di circolazione). If we place a small, rough ball in the fluid, fixed at its center but free to rotate, it will start spinning if the curl at that point is non-zero. The axis of rotation gives the direction of the curl vector, determined by the right-hand rule. A non-zero curl indicates the presence of vortices in the fluid; ideal, non-viscous fluids are irrotational (their curl is zero everywhere).

In Cartesian coordinates, the curl of a vector field $\vec{F} = (F_x, F_y, F_z)$ is written as $\nabla \times \vec{F}$ and can be calculated as the formal determinant of a matrix:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k}$$

An important identity in vector calculus is that the curl of the gradient of any scalar field V is always zero, provided the second partial derivatives are continuous:

$$\nabla \times (\nabla V) = 0$$

This is because the mixed partial derivatives cancel out (e.g., $\frac{\partial^2 V}{\partial y \partial z} = \frac{\partial^2 V}{\partial z \partial y}$).

Stokes' Theorem

Stokes' theorem (teorema di Stokes), also known as the curl theorem, relates the circulation of a vector field along a closed path to the flux of its curl. Consider an oriented closed curve γ and any open surface S that has γ as its boundary. Stokes' theorem states that for a vector field $\vec{V}$:

$$\oint_\gamma \vec{V} \cdot d\vec{l} = \int_S (\nabla \times \vec{V}) \cdot d\vec{S}$$

The left side is the circulation of $\vec{V}$ along the closed path γ. The right side is the flux of the curl of $\vec{V}$ through the surface S. The orientation of the surface normal $d\vec{S}$ is related to the orientation of the path γ by the right-hand rule: if your fingers curl in the direction of the path, your thumb points in the direction of the normal.

This theorem provides a coordinate-independent definition of the curl. The component of the curl along a normal vector $\hat{n}$ can be defined as the limit of the circulation per unit area as the area shrinks to a point:

$$(\nabla \times \vec{V}) \cdot \hat{n} = \lim_{S \to 0} \frac{1}{S} \oint_\gamma \vec{V} \cdot d\vec{l}$$

The Third Maxwell's Equation for Electrostatics

We know that the electrostatic field $\vec{E}$ is conservative. This means its circulation around any closed path γ is zero:

$$\oint_\gamma \vec{E} \cdot d\vec{l} = 0$$

By applying Stokes' theorem, we can relate this to the curl of $\vec{E}$:

$$\oint_\gamma \vec{E} \cdot d\vec{l} = \int_S (\nabla \times \vec{E}) \cdot d\vec{S} = 0$$

Since this must hold for any arbitrary closed path γ and any surface S bounded by it, the integrand itself must be zero. This gives us the third of Maxwell's equations for static fields:

$$\nabla \times \vec{E} = 0$$

This is a local equation, stating that the curl of the electrostatic field is zero at every point in space. A field with zero curl is called an irrotational field (campo irrotazionale). This result is consistent with the fact that $\vec{E} = -\nabla V$, because the curl of a gradient is always zero.

Conductors in Electrostatic Equilibrium

Conductors (conduttori) are materials containing mobile charges (e.g., electrons) that are free to move under the influence of an electric field. Electrostatic equilibrium is the state where there is no net motion of charge within the conductor.

For this to be true, the electric field inside a conductor must be zero. If it were not, the free charges would experience a force and move, contradicting the definition of equilibrium.

$$\vec{E}_{\text{inside}} = 0$$

A direct consequence, via Gauss's Law, is that the net charge density ρ inside the conductor must also be zero. If we take any Gaussian surface entirely within the conductor, the flux through it is zero because $\vec{E} = 0$. Therefore, the enclosed charge must be zero. Any net charge placed on a conductor must reside entirely on its outer surface in a very thin layer. We describe this with a surface charge density σ.

At the interface between two media, the tangential component of the electrostatic field is continuous. Consider a small rectangular loop straddling the interface. The circulation $\oint \vec{E} \cdot d\vec{l} = 0$. By making the sides perpendicular to the interface infinitesimally short, their contribution vanishes. The circulation is then due to the two long sides parallel to the interface:

$$E_{1,t}\Delta l - E_{2,t}\Delta l = 0 \implies E_{1,t} = E_{2,t}$$

For the interface between a conductor (medium 2) and a vacuum (medium 1), we have $\vec{E}_2 = 0$. Therefore, the tangential component of the electric field just outside the conductor must be zero: $E_{1,t} = 0$. This means the electric field at the surface of a conductor is always perpendicular to the surface. Since the electric field lines are always orthogonal to equipotential surfaces, it follows that the surface of a conductor in electrostatic equilibrium is an equipotential surface.

Coulomb's Theorem and Conductor Properties

Coulomb's Theorem (Teorema di Coulomb) relates the electric field just outside a conductor to the local surface charge density σ. By applying Gauss's law to a small cylindrical "pillbox" that straddles the surface, we find:

$$\oint \vec{E} \cdot d\vec{S} = EA = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0} \implies E = \frac{\sigma}{\epsilon_0}$$

The field is directed along the outward normal $\hat{n}$, so $\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}$.

The potential is constant throughout the entire conductor (both surface and interior). There is, however, a potential difference between the interior (V_i) and the exterior surface (V_e), related to the work function (funzione lavoro), $W = e(V_i - V_e)$, which is the energy required to extract an electron from the conductor. In our problems, when we speak of the "potential of a conductor," we refer to the potential of its surface, V_e.

A cavity inside a conductor is shielded from external static electric fields. This is the principle of the Faraday cage (gabbia di Faraday). If a conductor with an empty cavity is placed in an external field, charges will rearrange on the outer surface to make $\vec{E} = 0$ inside the conductor material. Applying Gauss's law to a surface within the conductor material that encloses the cavity shows that the net charge on the inner surface of the cavity must be zero. Further analysis shows the field inside the empty cavity is also zero.

The power of points (effetto delle punte) describes the tendency of charge to accumulate on parts of a conductor with a small radius of curvature (i.e., sharp points). This leads to a much stronger electric field at these points. For two conducting spheres of radii R_1 and R_2 connected by a wire, they will be at the same potential. This implies $V = \frac{Q'_1}{4\pi\epsilon_0 R_1} = \frac{Q'_2}{4\pi\epsilon_0 R_2}$. The surface charge densities are related by $\sigma'_1/\sigma'_2 = R_2/R_1$. The sphere with the smaller radius has the higher charge density and thus a stronger electric field at its surface ($E \propto \sigma$). This is why electrical discharges, like sparks, tend to occur from sharp points.

1 Lesson 10: Electrical Capacitance and Electrostatic Pressure

1.1 Electrical Capacitance

To discuss electrical capacitance (capacità elettrica), we must first consider conductors. Capacitance is a property defined for a system of conductors. Let's imagine a single conductor in a vacuum, isolated from other charges or conductors. If this conductor holds a charge Q, we know from electrostatics that its surface is an equipotential surface. We can therefore define the potential of the conductor, V.

The potential V is directly proportional to the charge Q. This is because the electric field is proportional to the charge, and the potential is obtained by integrating the electric field. We can see this explicitly for a spherical conductor of radius R, where the potential is $V = \frac{Q}{4\pi\epsilon_0 R}$.

This proportionality holds for any conductor shape. The ratio of charge to potential is a constant that depends only on the conductor's geometry. This constant is called capacitance (capacità), denoted by C.

$$C = \frac{Q}{V}$$

The unit of capacitance is the Farad (F), which is one Coulomb per Volt. The Farad is a very large unit. For instance, a sphere with a 1-meter radius has a capacitance of about 10^{-10} F. To get a capacitance of 1 Farad, a sphere would need a radius about ten times larger than the sun. In practice, we use microfarads (μF) or nanofarads (nF).

1.1.1 System of N Conductors

For a system of N conductors, each with charge Q_i and potential V_i, the potential of a conductor 'i' is a linear combination of the charges on all conductors:

$$V_i = \sum_{j=1}^{N} P_{ij} Q_j$$

The terms P_{ij} are called potential coefficients (coefficienti di potenziale) and depend on the geometry of the conductors. This matrix relationship can be inverted to express the charges as a function of the potentials:

$$Q_i = \sum_{j=1}^{N} C_{ij} V_j$$

The terms C_{ij} are called coefficients of capacitance or, more generally, coefficients of induction (coefficienti di induzione). The diagonal terms C_{ii} are the capacitances of the individual conductors, while the off-diagonal terms C_{ij} (for $i \neq j$) are related to the phenomenon of electrostatic induction.

1.2 Capacitors

A capacitor (condensatore) is a system of two conductors between which there is complete induction (induzione completa). This means all electric field lines originating from the first conductor terminate on the second. The two conductors are called armatures (armature). Common examples include the

spherical capacitor (a conducting sphere inside a concentric spherical shell), the cylindrical capacitor (a conducting cylinder inside a coaxial cylindrical shell), and the plane capacitor (two parallel conducting plates).

In a capacitor, the two armatures hold equal and opposite charges, +Q and -Q. This creates a potential difference ΔV between them. The capacitance of the capacitor is defined as:

$$C = \frac{Q}{\Delta V}$$

where ΔV is the magnitude of the potential difference. Like the capacitance of a single conductor, a capacitor's capacitance depends only on its geometry.

1.2.1 Calculating Capacitance

To calculate the capacitance of a specific capacitor configuration, one can assume charges +Q and -Q on the armatures, then calculate the electric field E between them. By integrating E, the potential difference ΔV is found. Finally, the capacitance is calculated using $C = Q/\Delta V$.

For a plane capacitor with plate area S and separation d, the electric field is $E = \sigma/\epsilon_0 = Q/(S\epsilon_0)$. The potential difference is $\Delta V = E \cdot d$. This gives the capacitance:

$$C = \frac{\epsilon_0 S}{d}$$

For a spherical capacitor with inner radius R_1 and outer radius R_2, the capacitance is:

$$C = 4\pi\epsilon_0 \frac{R_1 R_2}{R_2 - R_1}$$

For a cylindrical capacitor of length H, inner radius R_1, and outer radius R_2, the capacitance is:

$$C = \frac{2\pi\epsilon_0 H}{\ln(R_2/R_1)}$$

1.3 Capacitor Combinations

Capacitors can be connected in circuits in two basic ways: parallel and series.

When capacitors are connected in parallel, they share the same potential difference ΔV. The total charge stored is the sum of the charges on each capacitor, $Q_{tot} = Q_1 + Q_2$. The equivalent capacitance (C_{eq}) is the sum of the individual capacitances.

$$C_{eq} = C_1 + C_2 + \dots$$

When capacitors are connected in series, they each store the same amount of charge Q. The total potential difference across the combination is the sum of the potential differences across each capacitor, $\Delta V_{tot} = \Delta V_1 + \Delta V_2$. The reciprocal of the equivalent capacitance is the sum of the reciprocals of the individual capacitances.

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \dots$$

1.4 Applications of Capacitors

Capacitors are fundamental components in nearly all electronic circuits. They also serve as the basis for various sensors.

Capacitive sensors can detect changes in physical parameters like displacement or pressure by measuring a change in capacitance. For example, a sensor can be built with one fixed plate and one movable plate; a change in the distance between them alters the capacitance. This principle is used in sensitive displacement sensors and pressure sensors where a diaphragm deforms under pressure, changing the plate separation.

Capacitive touchscreens work by creating a grid of capacitors. When a user's finger (which is conductive) touches the screen, it alters the local capacitance at that point. The device's controller detects this change in capacitance to register the touch.

1.5 Electrostatic Pressure

Electrostatic pressure (pressione elettrostatica), or electrostatic tension, is the force per unit area acting on the surface of a charged conductor. This pressure is directed outwards and arises from the mutual repulsion of like charges distributed on the surface.

Consider a small patch of the conductor's surface. The electric field E just outside the surface is the sum of the field from the patch itself (E_{patch}) and the field from the rest of the conductor (E_{rest}). The field from an infinite plane of charge is $\sigma/(2\epsilon_0)$. So, the field from the small patch is $E_{patch} = \sigma/(2\epsilon_0)$ pointing outwards. Since the total field outside is $E = \sigma/\epsilon_0$ (from Coulomb's theorem), the field from the rest of the conductor must be:

$$E_{rest} = E - E_{patch} = \frac{\sigma}{\epsilon_0} - \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{2\epsilon_0}$$

The force on the patch is the charge on the patch ($dq = \sigma dS$) multiplied by the field from the rest of the conductor in which it is immersed.

$$dF = (dq)E_{rest} = (\sigma dS)\left(\frac{\sigma}{2\epsilon_0}\right) = \frac{\sigma^2}{2\epsilon_0}dS$$

The pressure P is the force per unit area, dF/dS.

$$P = \frac{\sigma^2}{2\epsilon_0}$$

Since the electric field at the surface of the conductor is $E = \sigma/\epsilon_0$, we can also write the pressure as:

$$P = \frac{1}{2}\epsilon_0 E^2$$

This expression is exactly the electrostatic energy density (densità di energia elettrostatica) at the surface. Therefore, the electrostatic pressure on a conductor's surface is equal to the electrostatic energy density just outside the surface.

2 Lesson 11: The General Problem of Electrostatics and Image Charges

2.1 The General Problem of Electrostatics

In electrostatics, if we know the distribution of charge density ρ in a localized region of space, the electric field $\vec{E}$ and the potential V are uniquely determined. The potential V at a point, assuming $V(\infty) = 0$, is the sum of contributions from all infinitesimal charge elements:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}d\tau'$$

The electric field is the negative gradient of this potential, $\vec{E} = -\nabla V$. This field satisfies the fundamental equations of electrostatics: $\nabla \cdot \vec{E} = \rho/\epsilon_0$ and $\nabla \times \vec{E} = 0$.

By substituting $\vec{E} = -\nabla V$ into the divergence equation, we obtain the **Poisson equation**:

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

where ∇^2 is the Laplacian operator. This is a local equation, relating the second derivatives of the potential at a point to the charge density at that same point.

2.1.1 Laplace's Equation and Uniqueness

In a region of space where there is no charge ($\rho = 0$), Poisson's equation simplifies to the **Laplace equation**:

$$\nabla^2 V = 0$$

A crucial property of these equations is captured by the **uniqueness theorem**. It states that for a given volume, a solution to Poisson's (or Laplace's) equation is uniquely determined if the value of the potential V is specified on the boundary surface of that volume. This makes physical sense: if you fix the potential on all boundaries of a region, the electric field and forces within that region should be uniquely defined.

2.2 Harmonic Functions and Earnshaw's Theorem

Solutions to the Laplace equation are called **harmonic functions** (funzioni armoniche). These functions have a significant property described by the **mean value theorem**: the average value of a harmonic function over any spherical surface is equal to its value at the center of the sphere.

A direct and profound consequence of this is that a harmonic function cannot have a local maximum or minimum within its domain. If a minimum existed, a small sphere centered on it would have an average value on its surface greater than the value at the center, violating the theorem.

This leads to **Earnshaw's Theorem**: a charged particle cannot be held in a stable equilibrium by electrostatic forces alone. A stable equilibrium requires the particle to be at a point of minimum potential energy. For a positive charge, this would mean a minimum of the potential V; for a negative charge, a maximum of V. Since the potential V is a harmonic function in charge-free space, it has no such minima or maxima. Therefore, no point of stable electrostatic equilibrium can exist in a vacuum. To trap ions, one must use non-electrostatic forces or time-varying electric fields, which fall outside the realm of electrostatics.

2.3 The Method of Image Charges

The **method of image charges** (metodo delle cariche immagini) is a powerful problem-solving technique for electrostatic problems involving point charges and conductors with simple, symmetric shapes (like planes or spheres). The method relies on the uniqueness theorem.

The core idea is to simplify a problem by replacing a conductor with one or more "image charges". These fictitious charges are placed in a way that, together with the original "real" charges, they reproduce the correct boundary conditions on a surface that originally corresponded to the conductor.

2.3.1 Point Charge and a Grounded Conducting Plane

Consider a point charge $+Q$ placed in front of an infinite, grounded conducting plane. The plane is at potential $V = 0$. The charge $+Q$ induces a surface charge distribution σ on the plane. The potential in the space containing $+Q$ is created by both $+Q$ and σ.

To solve this, we remove the plane and the induced charge σ. We then find a configuration of image charges that reproduces the $V = 0$ boundary condition on the location where the plane used to be. For this geometry, the solution is an image charge $-Q$ placed at a position symmetric to the original charge, on the other side of the plane.

The potential in the region where the original charge lies is then simply the superposition of the potentials from the real charge $+Q$ and the image charge $-Q$. This solution is valid *only* in the region of the real charge. In the region behind the plane (where the conductor was), the potential is zero, a fact we knew from the start, and which is not described by the image charge model. From the potential, the electric field and the induced surface charge density on the plane can be calculated.

2.3.2 Point Charge and a Grounded Conducting Sphere

The method can also be applied to a point charge $+Q$ placed near a grounded conducting sphere of radius R. An image charge Q' is placed inside the sphere at a distance B from its center. By enforcing the condition that the potential must be zero everywhere on the sphere's surface, one can solve for the image charge's magnitude and position. The results are:

$$Q' = -Q\frac{R}{a} \quad \text{and} \quad B = \frac{R^2}{a}$$

where a is the distance of the real charge $+Q$ from the center of the sphere. As with the plane, the potential outside the sphere is the sum of the potentials from the real charge and this image charge. The solution is not valid inside the sphere, where the potential is known to be zero.