Linear Algebra & Geometry

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Pre-Det

Spaces (Spazi Vettoriali)

"I have not failed. I've just found 10,000 ways that won't work."

Basic Definitions

Vectors (Vettori): Vectors are mathematical tools which can be visualized as arrows. They hold tow primary operations, scaling and addition.

Scaling: Scaling involves multiplying a scalar quantity λ which doesn't have a direction with the vectors. This either amplifies or reduces the vector without changing the "line its on"

$$\lambda \in \mathbb{R}, v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in R^2 : \lambda \cdot v = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}$$

Addition: Addition involves adding two vectors to make a new vector

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in R^2 : v + w = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$$

Subsets (Sottoinsiemi): This is a set where all the elements are contained within another set. For linear algebra it could be a collection of vectors within a vector space e.g $\mathbb{R}^3$

$$S = \{(x, y, 0) \,|x, y \in \mathbb{R}\}$$

Proper Subsets (Sottoinsiemi Propri): This is a subset that is always smaller than the original set. The exact definition is:

$$A \subsetneq B \tag{1}$$

if:

1. Containment: "Every element of A is also a element of B"

$$\forall x(x \in A \Longmapsto x \in B) \tag{2}$$

2. Not equal: "There exists at least one element in B that is not A"

$$\exists y(y \in B \wedge y \notin A) \tag{3}$$

Supersets (Soprainsiemi): This is the opposite of a subset. If $A \subset B$, then B is a superset of A

Linear Independence

Let V be a vector space over a field F. $S \subseteq V$ is linearly independent if the following conditions are true:

For a collection of vectors $\{v_1, v_2, ..., v_n\} \subseteq S$ and scalars $\{a_1, a_2, ..., a_n\} \in F$

$$a_1 v_1 + a_2 v_2 + ... + a_n v_n = 0 \tag{4}$$

i.e every single variant of a up to n has to equal zero. If it does not, it is linearly dependent

Dimensions of a space

The dimension (dimensione) of a vector space (spazio vettoriale) is the number of vectors in any basis (base) for that space. A basis (base) is a set of vectors that are linearly independent (linearmente indipendenti) and span (generano) the entire space.

Sum of subspace (Somma di sottospazi)

Let U and V be subspaces (sottospazi) of a vector space (spazio vettoriale) V over a field $\mathbb{F}$. The sum (somma) can be defined as:

$$U + V = \{u + v | u \in U, v \in V\} \tag{5}$$

Direct Sum (Somma diretta)

U+V is called a direct sum, if the intersection of U and V is trivial ???- (Further desc required) then:

$$U \cap V = \{0\}$$

The direct sum can be denoted as $U \oplus V$:–to be expanded upon

$$z = u + v$$

Inner Products

Definition:

For $\mathbb{R}$ For a vectors $v = (v_1....v_n)$ and $w = (w_1....w_n)$ it is

$$< v, w >= v \cdot w = \sum_{i=1}^{n} v_i w_i \tag{6}$$

For $\mathbb{C}$ For a complex vectors $v = (v_1....v_n)$ and $w = (w_1....w_n)$ it is

$$< v, w >= \sum_{i=1}^{n} v_i \overline{w}_i \tag{7}$$

Specifics in $\mathbb{R}$

Proof:

Lets take a vector u and a vector v and say that they are orthogonal. Because of the orthogonality there is some λ which allows the transformation.

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}$$

(Below the $\Leftrightarrow$ is being used as a substitute for and).

$$u_1 \cdot v_1 = -\cancelto{v_2}{u_1\lambda} u_2 \Leftrightarrow u_2 \cdot v_2 = \cancelto{-v_1}{u_2\lambda} u_1$$

Since v2 and -v1 equal to those terms above they cancel out the unknown λ

$$u_1 \cdot v_1 = -v_2 \cdot u_2 \Leftrightarrow u_2 \cdot v_2 = -v_1 u_1$$

Since both sides are now the same formula, we can move them from one side to the other removing the negative equaling =0

$$u_1 v_1 + u_2 v_2 = 0$$

$$u_1 v_1 \Rightarrow < u, v >$$

Length:

Angle Between Vectors:

Projection:

Orthogonality:

Matrixes

Post-Det