# Notes on Electromagnetism

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# Lesson 1: Coulomb's Law

Coulomb's Law (Legge di Coulomb) describes the electrostatic force (forza elettrostatica) between two static charged particles (particelle cariche). The force $\vec{F}_{12}$ that a charge (carica) $q_1$ exerts on a second charge $q_2$ is directly proportional to the product of the charges and inversely proportional to the square of the distance between them. The force is directed along the line connecting the two charges.

The vector form of the equation is:

$$\vec{F}_{12} = k\frac{q_1 q_2}{R^2}\hat{\vec{R}}_{12} \tag{1}$$

Here, $k$ is Coulomb's constant (costante di Coulomb), $R$ is the distance between the charges, and $\hat{\vec{R}}_{12}$ is the unit vector (versore) pointing from $q_1$ to $q_2$. The law is reciprocal, meaning the force exerted by $q_2$ on $q_1$ is equal in magnitude and opposite in direction, so $\vec{F}_{12} = -\vec{F}_{21}$.

Coulomb's constant is expressed in terms of the vacuum permittivity (permittività del vuoto), $\epsilon_0$, as:

$$k = \frac{1}{4\pi\epsilon_0} \tag{2}$$

Following the 2019 redefinition of the SI base units (Unità di base del SI), the elementary charge (carica elementare) $e$ is a defining constant. As a result, $\epsilon_0$ is now an experimentally determined value with an associated uncertainty, not a defined quantity.

The electric force is significantly stronger than the gravitational force. For instance, the electric force (forza elettrica) $F_E$ between a proton and an electron is many orders of magnitude greater than the gravitational force (forza gravitazionale) $F_G$ between them:

$$F_G = G\frac{m_p m_e}{R^2} \approx 10^{-47} N \quad F_E \qquad = \frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} \approx 10^{-7} N \tag{3}$$

# Lesson 2: Electric Flux and Gauss's Law

The concept of electric flux (Flusso Elettrico) quantifies the flow of an electric field (campo elettrico) through a surface. For a uniform electric field $\vec{E}$ and a flat surface area $\vec{A}$, the flux is the scalar product $\Phi_E = \vec{E} \cdot \vec{A}$. For non-uniform fields and curved surfaces, the flux is found by integrating the field over the surface:

$$\Phi_E = \int_S \vec{E} \cdot d\vec{A} \tag{4}$$

This integral effectively sums the component of the electric field perpendicular to the surface at every point.

Gauss's Law (Legge di Gauss), a cornerstone of electrostatics (elettrostatica) and one of the four fundamental Maxwell's Equations (Equazioni di Maxwell), relates the electric flux through a closed surface (superficie chiusa) to the net electric charge (carica elettrica) enclosed within it. The law states that the net electric flux is directly proportional to the enclosed charge.

In its integral form, Gauss's Law is expressed as:

$$\Phi_E = \oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} \tag{5}$$

The integral is performed over a conceptual closed surface known as a Gaussian surface (superficie Gaussiana), and $Q_{enc}$ is the total charge it encloses. By applying the divergence theorem (teorema della divergenza), which connects the surface integral of a vector field to the volume integral of its divergence, we can derive the differential form of Gauss's Law:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{6}$$

Here, $\rho$ represents the volume charge density (densità di carica volumica). This form locally relates the divergence of the electric field to the charge density at that point.

A key application of Gauss's Law is to calculate electric fields in situations with high symmetry. By choosing a Gaussian surface that mirrors the symmetry of the charge distribution (distribuzione di carica), the flux integral becomes simple to solve. For example, to find the field from an infinite line of charge with a uniform linear charge density (densità di carica lineare) $\lambda$, we use a cylindrical Gaussian

surface. The symmetry dictates that the electric field must point radially outward. The flux through the top and bottom caps of the cylinder is zero, and the flux through the side wall is $E \cdot (2\pi r L)$. The enclosed charge is $\lambda L$. Applying Gauss's Law, $E(2\pi r L) = \lambda L/\epsilon_0$, yields the field:

$$E = \frac{\lambda}{2\pi\epsilon_0 r} \tag{7}$$

# Lesson 3: Applications of Gauss's Law

Gauss's Law is a powerful tool for calculating the electric field for charge distributions with a high degree of symmetry, such as spherical, cylindrical, or planar symmetry. The symmetry of the charge density $\rho$ is inherited by the electric field $\vec{E}$, which simplifies the calculation.

A classic example is the field of a uniformly charged sphere (Sfera Carica Uniformemente) of radius $R$ and total charge $Q$. Due to spherical symmetry (simmetria sferica), the electric field must be radial, $\vec{E} = E(r)\hat{r}$. By choosing a spherical Gaussian surface of radius $r$, the flux integral simplifies to $\oint_S \vec{E} \cdot d\vec{S} = E(r) \cdot (4\pi r^2)$. Outside the sphere ($r > R$), the enclosed charge is the total charge $Q$. Applying Gauss's law gives $E(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$, which is identical to the field of a point charge (carica puntiforme) $Q$ at the origin. Inside the sphere ($r \leq R$), the enclosed charge is only the fraction of charge within the radius $r$, $Q_{\text{int}} = Q\frac{r^3}{R^3}$. Gauss's law then yields $E(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}r$, showing that the field increases linearly from the center.

This method can be extended to other symmetric configurations. For a charged spherical shell (Guscio Sferico Carico), the electric field inside the inner cavity is zero. For a sphere with an off-center cavity, the superposition principle (principio di sovrapposizione) can be used by treating the cavity as a superposition of a sphere with negative charge density over a larger sphere with positive charge density. This reveals a uniform electric field inside the cavity. Gauss's law also efficiently re-derives the fields for an infinite line of charge (Linea Infinita di Carica) and an infinite plane of charge (Piano Infinito di Carica).

The concept of divergence (divergenza) of a vector field (campo vettoriale) measures the field's tendency to originate from or converge to a point. For an electric field, divergence signifies the presence of a charge source or sink. The Divergence Theorem (Teorema della Divergenza) relates the flux of a field through a closed surface to the integral of its divergence over the enclosed volume. Combining this theorem with Gauss's Law leads to its differential form:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{8}$$

This is the first of Maxwell's Equations, stating that the divergence of the electric field at any point is proportional to the charge density at that same point.

# Lesson 4: Superposition, Conductors, and Plasma Oscillations

The superposition principle (principio di sovrapposizione) is a fundamental concept for calculating electric fields from complex charge distributions. A powerful illustration is the case of two overlapping spheres of the same radius $R$, one with uniform positive charge density $+\rho_0$ and the other with uniform negative charge density $-\rho_0$. The total electric field is the vector sum of the fields from each sphere. Using the known result for the field inside a uniformly charged sphere, $\vec{E} = \frac{\rho_0}{3\epsilon_0}\vec{r}$, we can calculate the field in the overlapping region. If the centers of the spheres are separated by a vector $\vec{\delta}$, the field in the overlapping region is found to be uniform and constant:

$$\vec{E}_{overlap} = -\frac{\rho_0}{3\epsilon_0}\vec{\delta} \tag{9}$$

This model provides an excellent analogy for the behavior of conductors (conduttori) in an external electric field. In a conductor, conduction electrons are free to move. When an external field $\vec{E}_{ext}$ is applied, the "sea" of electrons displaces slightly, creating a charge separation, or polarization (polarizzazione). This separation induces an internal electric field (campo elettrico interno) $\vec{E}_{int}$ that opposes the external field. The displacement continues until the net electric field inside the conductor is zero, a condition known as electrostatic equilibrium (equilibrio elettrostatico). At this point, $\vec{E}_{ext} + \vec{E}_{int} = 0$. The conductor effectively shields its interior from the external static electric field.

If the external field is suddenly removed, the displaced electron sea is no longer in equilibrium and experiences a restoring force from the fixed positive ions. This force acts like a spring, leading to oscillations. The equation of motion for the electron sea is that of a simple harmonic oscillator (oscillatore armonico semplice). This collective, high-frequency oscillation of the electrons is a quantum mechanical effect known as a plasma oscillation (oscillazione di plasma). The characteristic angular frequency of this oscillation is the plasma frequency ($\omega_p$), a property of the material that depends on the density of conduction electrons.

$$\omega_p^2 = \frac{n_e e^2}{m_e \epsilon_0} \tag{10}$$

This phenomenon is crucial in understanding the optical properties of metals.

## Lesson 5: Maxwell's First Equation and Electrostatic Potential

This lesson begins by deriving the first of Maxwell's Equations, which concerns the divergence of the electric field. Starting from the divergence theorem (teorema della divergenza), which relates the flux of a vector field through a closed surface to the integral of its divergence over the enclosed volume:

$$\oint_S \vec{E} \cdot d\vec{S} = \int_\tau (\nabla \cdot \vec{E})\, d\tau \tag{11}$$

We can combine this with Gauss's Law (Teorema di Gauss), which states that the flux of the electric field through a closed surface is proportional to the enclosed charge $Q_{\text{int}}$:

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{\text{int}}}{\epsilon_0} \tag{12}$$

By equating these two expressions for the flux, and expressing the enclosed charge as the integral of the volume charge density (densità volumetrica di carica) $\rho$, we get:

$$\int_\tau (\nabla \cdot \vec{E})\, d\tau = \int_\tau \frac{\rho}{\epsilon_0}\, d\tau \tag{13}$$

Since this equality must hold for any arbitrary volume $\tau$, the integrands themselves must be equal. This gives us the differential form of Gauss's Law, also known as the first of Maxwell's Equations for the electric field in a vacuum:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{14}$$

This is a local equation (equazione locale) because it relates the divergence of the electric field at a specific point in space to the charge density at that very same point. This is in contrast to the integral form of Gauss's Law, which is non-local as it relates the field on a surface to the total charge contained within it.

To handle point charges (cariche puntiformi) within the framework of continuous charge densities, we introduce the Dirac delta function (delta di Dirac), which is technically a distribution (distribuzione). It is defined as being zero everywhere except at the origin, where it is infinite, yet its integral over all space is one. This allows us to define the volume charge density for a point charge $q_0$ located at position $\vec{r}_0$ as:

$$\rho(\vec{r}) = q_0 \delta(\vec{r} - \vec{r}_0) \tag{15}$$

Next, we explore the line integral (integrale di linea) of the electrostatic field between two points, A and B. For the field generated by a single point charge, the result of this integral depends only on the start and end points, not on the path taken. A field with this property is called a conservative field (campo conservativo). A direct consequence is that the line integral of the electrostatic field around any closed loop is zero:

$$\oint \vec{E} \cdot d\vec{l} = 0 \tag{16}$$

This conservative nature allows us to define a scalar field called the electrostatic potential (potenziale elettrostatico), V. The potential difference (differenza di potenziale) between two points is related to the line integral of the electric field:

$$V(A) - V(B) = \int_A^B \vec{E} \cdot d\vec{l} \tag{17}$$

From this relationship, we can derive the differential form, which connects the electric field to the potential via the gradient (gradiente):

$$\vec{E} = -\nabla V \tag{18}$$

The electric field points from regions of higher potential to regions of lower potential.

Surfaces on which the potential V is constant are known as equipotential surfaces (superfici equipotenziali). The electric field lines are always perpendicular to these surfaces. No work (lavoro) is done when moving a charge along an equipotential surface, as the displacement is always orthogonal to the electric force.

Finally, we define the electrostatic potential energy (energia potenziale elettrostatica) U of a point charge q placed in a potential V as $U = qV$. This energy can be interpreted as the work done by an external force to bring the charge from a reference point (where the potential is defined as zero) to its current position. The unit of potential is the Volt (Volt), defined as a Joule per Coulomb. In atomic and nuclear physics, a convenient unit of energy is the electron-volt (elettronvolt), which is the energy gained by an electron when it moves through a potential difference of one volt.