Linear Algebra & Geometry

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Pre-Det

Spaces (Spazi Vettoriali)

"I have not failed. I've just found 10,000 ways that won't work."

Basic Definitions

Vectors (Vettori): Vectors are mathematical tools which can be visualized as arrows. They hold tow primary operations, scaling and addition.

Scaling: Scaling involves multiplying a scalar quantity λ which doesn't have a direction with the vectors. This either amplifies or reduces the vector without changing the "line its on"

$$\lambda \in \mathbb{R}, v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in R^2 : \lambda \cdot v = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}$$

Addition: Addition involves adding two vectors to make a new vector

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in R^2 : v + w = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$$

Subsets (Sottoinsiemi): This is a set where all the elements are contained within another set. For linear algebra it could be a collection of vectors within a vector space e.g $\mathbb{R}^3$

$$S = \{(x, y, 0)\, | x, y \in \mathbb{R}\}$$

Basic proof See 2.210 in proof book

Proper Subsets (Sottoinsiemi Propri): This is a subset that is always smaller than the original set. The exact definition is:

$$A \subsetneq B \tag{1}$$

if:

1. Containment: "Every element of A is also a element of B"

$$\forall x (x \in A \Longmapsto x \in B) \tag{2}$$

2. Not equal: "There exists at least one element in B that is not A"

$$\exists y (y \in B \land y \notin A) \tag{3}$$

Supersets (Soprainsiemi): This is the opposite of a subset. If $A \subset B$, then B is a superset of A

Linear Independence

Let V be a vector space over a field F. $S \subseteq V$ is linearly independent if the following conditions are true:

For a collection of vectors $\{v_1, v_2, ..., v_n\} \subseteq S$ and scalars $\{a_1, a_2, ..., a_n\} \in F$

$$a_1 v_1 + a_2 v_2 + ... + a_n v_n = 0 \tag{4}$$

i.e every single variant of a up to n has to equal zero. If it does not, it is linearly dependent

Dimensions of a space

The dimension (dimensione) of a vector space (spazio vettoriale) is the number of vectors in any basis (base) for that space. A basis (base) is a set of vectors that are linearly independent (linearmente indipendenti) and span (generano) the entire space.

Sum of subspace (Somma di sottospazi)

Let U and V be subspaces (sottospazi) of a vector space (spazio vettoriale) V over a field $\mathbb{F}$. The sum (somma) can be defined as:

$$U + V = \{u + v | u \in U, v \in V\} \tag{5}$$

Direct Sum (Somma diretta)

U+V is called a direct sum, if the intersection of U and V is trivial ???- (Further desc required) then:

$$U \cap V = \{0\}$$

The direct sum can be denoted as $U \oplus V$:–to be expanded upon

$$z = u + v$$

Inner Products

Definition:

For $\mathbb{R}$ For a vectors $v = (v_1....v_n)$ and $w = (w_1....w_n)$ it is

$$< v, w >= v \cdot w = \sum_{i=1}^{n} v_i w_i \tag{6}$$

For $\mathbb{C}$ For a complex vectors $v = (v_1....v_n)$ and $w = (w_1....w_n)$ it is

$$< v, w >= \sum_{i=1}^{n} v_i \overline{w}_i \tag{7}$$

Specifics in $\mathbb{R}$

Proof:

Lets take a vector u and a vector v and say that they are orthogonal. Because of the orthogonality there is some λ which allows the transformation.

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}$$

(Below the ⇔ is being used as a substitute for and).

$$u_1 \cdot v_1 = -\cancelto{v_2}{u_1 \lambda} u_2 \Leftrightarrow u_2 \cdot v_2 = \cancelto{-v_1}{u_2 \lambda} u_1$$

Since v2 and -v1 equal to those terms above they cancel out the unknown λ

$$u_1 \cdot v_1 = -v_2 \cdot u_2 \Leftrightarrow u_2 \cdot v_2 = -v_1 u_1$$

Since both sides are now the same formula, we can move them from one side to the other removing the negative equaling =0

$$u_1 v_1 + u_2 v_2 = 0$$

$$u_1 v_1 \Rightarrow < u, v >$$

Length: (Needs some work)

However the euclidean length of a vector v is from the inner product of v itself:

$$\|v\| = \sqrt{< v, v >} = \sqrt{v_1^2 + v_2^2 + ... + v_n^2}$$

Angle Between Vectors:

By using the inner product and the euclidean length the angle between the two vectors is able to be calculated:

$$< v, w >= \|v\| \|w\| cos\theta$$

If we rearrange for cosθ

$$cos\theta = \frac{< v, w >}{\|v\| \|w\|}$$

Then to compute the angle use the inverse of cos which is arccos

$$\theta = arccos \left(\frac{< v, w >}{\|v\| \|w\|} \right) \tag{8}$$

Small Proof Example: Lets make v=(3,0) and w= (0,3). This is because the vectors are perpendicular.

The inner product:

$$< v, w >= (3)(0) + (0)(3) = 0$$

^ Currently lines up with normal properties

The norms (Euclidean length) are both 3.

Solving for theta

$$\theta = arccos(\frac{0}{3 \cdot 3}) = 90°$$

□

Projection:

Orthogonality:

Matrix (Matrice)

A matrix is one of the most important parts of linear algebra. Generally it is used as a means to store a array of elements which can be anything from numbers to functions. These are stored in rows (righe) and columns (colonne). Its shortened version is generally represented by a capitalized letter such as A:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Common terms: (Will be covered in more detail later)

1. Element (elementi), the individual points such as a_{ij}
2. Square matrix (matrice quadrata), a symmetrical matrix where m=n
3. Rectangular matrix (matrice rettangolare), a matrix where m is not equal to n
4. Transpose (trasposta), denoted as A^T, is simply swapping the rows and collums
5. Diagonal (diagonale), the set of elemets where i=j. Self explanatory

Basic operations

Addition/Subtraction

$$(A \pm B)_{ij} = a_{ij} \pm b_{ij}$$

Scalar Multiplication

$$(cA)_{ij} = c \cdot a_{ij}$$

Matrix Multiplication

$$c_{ik} = \sum_{j=1}^{p} a_{ij} b_{jk}$$

Specific Matrixes

Identity Matrix (matrice identità)

Zero Matrix (matrice nulla)

Diagonal Matrix (matrice diagonale)

Symmetric Matrix (matrice simmetrica)

Post-Det