**Sareh Kouchaki – Section 1**

**Statistical-Modeling-II**

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Exercise 1.1:

We pick 3 balls from an urn with r red and b blue balls. We pick a random ball and note its color, return the ball, plus another ball of the same color. It may not be obvious that the sequence A=r, B=b, C=b has the same probability as the sequence A=b, B=b, C=r since the individual probabilities of picking the red ball first or last are completely different: r/(r+b) when it is the first ball versus r/(r+b+2) when it is the last ball (since two blue balls were added in the meantime). So, the observations are not iid. Writing down the equations makes it clear that the two sequences are exchangeable due to the equal probability but not iid.

p (A=b, B=b, C=r) =

p (A=r, B=b, C=b) =

Exercise 1.2: We want to show that:

There are possibilities to select s observations from the set of first N observations for the first group. For each of these possibilities, there are possibilities to select t-s observations for the first group from the remaining M-N observations.

= =

Solution on the blackboard:

N<M

Exchangeability p(1 0 1) = p(0 1 1)

Exercise 1.3:

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By, replacing t with M:

If

Exercise 1.4:

Poisson distribution:

Exponential family form:

T(x) = x

h(x) =

If we have n independent samples:

Exponential family form:

T(x) =

h(x) =

Exercise 1.5:

Gamma distribution

joint pdf for n observations:

Exponential family form:

Natural Parameter or

Sufficient statistic1 or T1(x) = =

Natural Parameter or

Sufficient statistic1 or T2(x) =

\*\* point:

The is not the function of .

Any function of sufficient statistics is sufficient statistics. So, in this case the sufficient statistics is the minimum statistics.

Exercise 1.6: Show that the moment generating function for the sufficient statistic of an arbitrary exponential family random variable with natural parameter can be written as:

Suppose X is distributed as:

Pdf: p (x|

Exercise 1.7:

Part a) the moment generating function:

E(x) =

Part b) The cumulant generating function:

Exercise 1.8:

All exponential families have exponential likelihood, so, they have exponential posterior.

Likelihood:

Posterior:

\*\*Dividing the numerator and operator by :

This shows that is normally distributed with mean of and variance of .

Exercise 1.9: Derive the posterior distribution for .

Likelihood function:

Gamma prior:

Posterior:

Exercise 1.10: Show that the marginal distribution of x is given by a Student's t distribution.

Marginal distribution is obtained through the following:

Kernel of Gamma dist.

\*\*

\*\* t distribution format

Exercise 1.11:

a)

b)

Exercise 1.12:

Moment Generation function for a univariate normal with mean m and variance

Normal (

PDF:

Exercise 1.13:

\*\*

Exercise 1.14:

Is equal to I, because it is standard.

Exercise 1.15:

\*\*,

Exercise 1.16:

Solution 1:

Integrate out over to find the marginal of

=

Solution 2 (Blackboard):

(use transformation & combination)

If we have p number of x:

PDF:

Exercise 1.17:

Exercise 1.18:

So, we have:

Exercise 1.19:

Exercise 1.20:

Maximum likelihood estimation of :

Exercise 1.21:

Loss function =

\*\*Loss function is the same as the joint and log of normal dist. That’s why the results are similar.

Exercise 1.22:

Minimize

Reformulate this constrained optimization using a Lagrange multiplier:

\*\*Benefit of Ridge: way of regularization for avoiding overfitting of the function.

Exercise 1.23:

Normal distribution with mean of and variance of .

Exercise 1.24:

Normal distribution with mean of and variance of .

Exercise 1.25:

We need to have an unbiased estimator of .

\* M is the projection Matrix.

\*\*tr: the trace of an n-by-n square matrix A is defined to be the sum of the elements on the main diagonal.

So,

Which means that results in an unbiased estimator of .

In this question p is number of predictors = 3 and n is the number of observations = 102.

For the estimator of the standard error we have:

Code is on Github.

According to the result, the estimation of standard error is almost similar to that one obtained from lm.

Exercise 1.26:

Exercise 1.27:

Taylor Expansion of :

Constant Part

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