Guided Tour of Machine Learning in Finance

Linear Regression

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Linear Regression as a ML task

Task: predict a scalar value $y \in \mathbb{R}$ from a vector of predictors ("**features**") $\mathbf{X} = \left(X_0, X_1, \ldots, X_{D-1}\right) \in \mathbb{R}^D$ ($D \geq 1$ is the dimension of the feature space, or the number of predictors).

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Architecture: Linear

$$\hat{\mathbf{y}} = \mathbf{X}\,\mathbf{W}$$

N predicted values

$\mathbf{X}$ is a N x D design matrix

$\mathbf{W} \in \mathbb{R}^D$ is a vector of parameters

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Example: $\mathbf{y}$ is a vector of N daily returns of AMZN, and $\mathbf{X}$ is a $N \times D$ design matrix made of $D-1$ daily market index returns (S&P 500, NASDAQ, VIX, etc.)

Learning in Linear Regression

Performance measure P : mean square error (**MSE**) on the **test set**:

$$\mathrm{MSE}_{test} = \frac{1}{N_{test}} \sum_{n=1}^{N_{test}} \left(\hat{y}_i^{test} - y_i^{test} \right)^2$$

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How to optimize parameters $\mathbf{W} \in \mathbb{R}^D$?
Minimize MSE on the **training set**!
Why? Because both MSE_{train} and MSE_{test} estimate the same generalization (expected) error $\mathbb{E}\left[\left(\hat{\mathbf{y}} - \mathbf{y} \right)^2 \right]$ from the empirical distribution $\sim p_{data}$.

Note: In ML, we often minimize one function, while actually caring about minimization of another function. This makes ML different from optimization that typically focuses only on one function.

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How to optimize parameters $\mathbf{W} \in \mathbb{R}^D$? Minimize MSE on the **training set**! Why? Because both MSE_{test} and MSE_{train} estimate the generalization (expected) error.
For optimal weights $\mathbf{W}$, the gradient of MSE_{train} should be 0:

$$\nabla_W \mathrm{MSE}_{train} = \nabla_W \frac{1}{N_{train}} \left\| \hat{\mathbf{Y}}^{train} - \mathbf{Y}^{train} \right\|_2^2 = \frac{1}{N_{train}} \nabla_W \left\| \mathbf{X}^{train} \mathbf{W} - \mathbf{Y}^{train} \right\|_2^2 = 0$$

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Normal equation

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Here
$\mathbf{X} = \mathbf{X}^{train},\ \mathbf{y} = \mathbf{y}^{train}$

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Visualize the result for optimal weights $\mathbf{W}$ in the vector form

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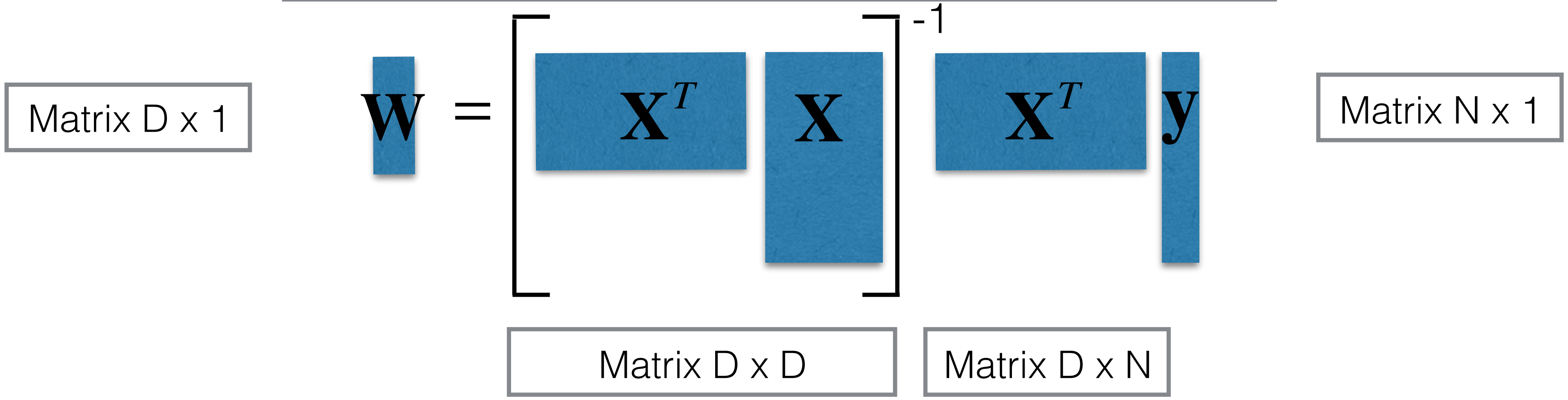

Prediction in-sample: $\hat{\mathbf{y}} = \mathbf{X}\mathbf{W} = \underbrace{\mathbf{X}\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T}_{=\mathbf{H}}\mathbf{y} \equiv \mathbf{H}\,\mathbf{y}$

The “hat” (projection) matrix

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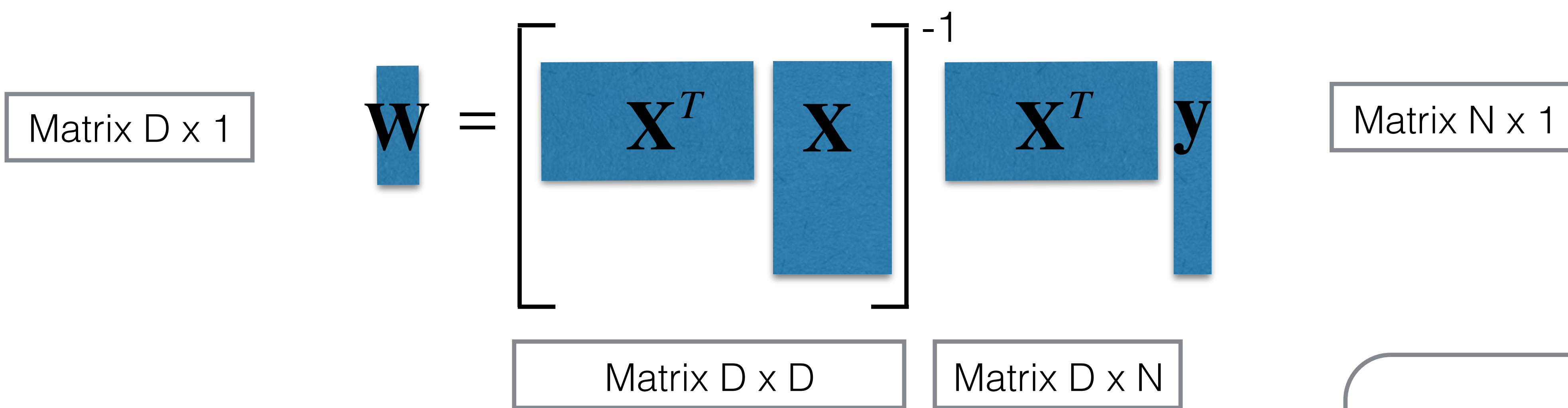

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Symmetric and idempotent: $\mathbf{H}^2 = \mathbf{H}$

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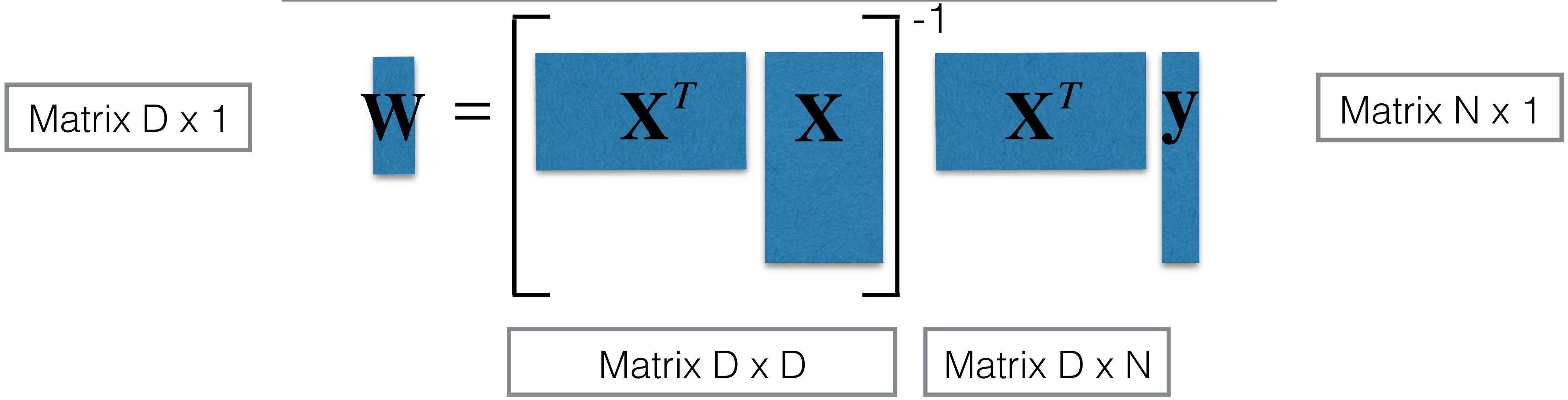

Prediction out-of-sample: $\hat{\mathbf{y}} = \mathbf{X}^{test}\mathbf{W}$

How to overfit with Linear Regression: add more features to the design matrix without a proper control!