Guided Tour of Machine Learning in Finance

The bias-variance tradeoff

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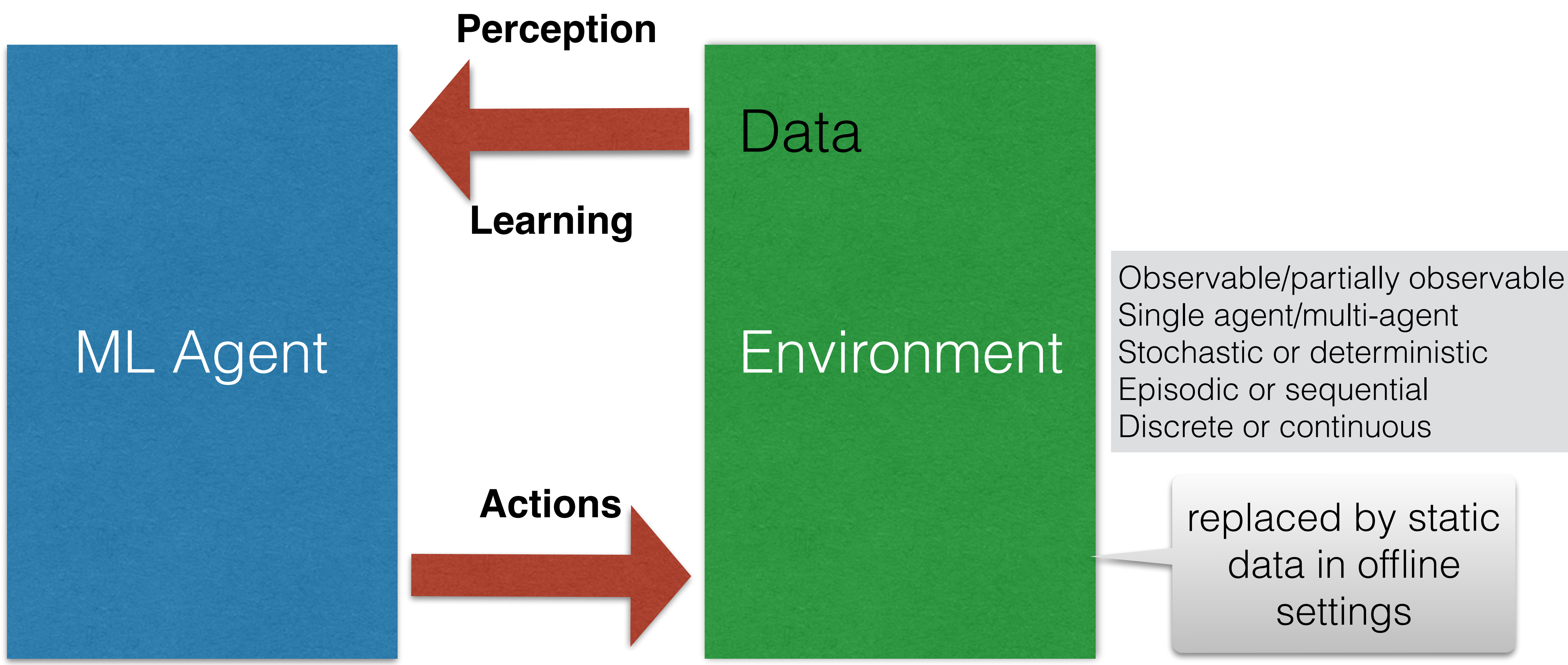
Generalization as a goal of learning



Learning = ability to generalize

= perform well on unseen data

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$$Var(y) + Var(\hat{f}) + \mathbb{E}\left[f - \hat{f}\right]^2$$

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The Bias-Variance Decomposition

This is is bias-variance decomposition:

$$\boxed{\mathbb{E}\left[\left(y - \hat{f}(\boldsymbol{x})\right)^2\right] = (bias)^2 + variance + noise}$$

Here:

$$(bias)^2 = \mathbb{E}\left[f - \hat{f}\right]^2$$

$$variance = Var(\hat{f})$$

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Noise: a property of data; beyond our control

Control question

Q: Select all correct statements

1. $variance = \mathbb{E}\left[f - \hat{f} \right]^2$ is a measure of variability of a model.

2. $variance = Var(\hat{f})$ determines sensitivity of a model $\hat{f}(\boldsymbol{x})$ with respect to the choice of a dataset.

3. $\left(bias \right)^2 = \mathbb{E}\left[f - \hat{f} \right]^2$ is a measure of flexibility of a model $\hat{f}(\boldsymbol{x})$ in matching typical data that can be observed for a given (and unknown) data-generating mechanism that corresponds to a "true" (and unknown) data model $f(\boldsymbol{x})$

Correct answers: 2, 3.

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- But it show a general decomposition of expected loss of a generic regression model.

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- Important to control **model complexity** to have an optimal tradeoff!

Control question

Q: Assume you have a complex model with a low bias and a high variance, and you want to reduce the variance. What do you think would be the right way to proceed?

A1: Use a simple model instead. Who needs complex models in Finance?

A2. Add more predictors.

A3. Try to somehow bound the values of model parameters, so that the model outputs would vary less with a variation of the input data.

Correct answer: A3.