Econometrics. Homework 1

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2023-02-20

Due date: 2 March, 2023

Problem 1 (15 Points)

Lets $n \times n$ matrix $\boldsymbol{A}$ is a function of scalar varaiable t, $\boldsymbol{A} = A(t)$, and, $\boldsymbol{x}$ - is vector of size $n \times 1$.

1. Find $\partial\left(\boldsymbol{x'Ax}\right)/\partial t$. (5 Points)
2. Find $\partial\left(\boldsymbol{x'Ax}\right)/\partial t$, if $\boldsymbol{x}$ is function of $t : \boldsymbol{x} = \boldsymbol{x}(t)$. (10 Points)

Problem 2 (20 Points)

Calculate the following expression:

$$\frac{\partial \log f(\mathbf{x} - \mathbf{y})}{\partial \mathbf{y}}$$

where f is scalar function and $\boldsymbol{x}$, $\boldsymbol{y}$ are vectors.

Sometimes the notation of gradient is used.

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \equiv \nabla f(\mathbf{x}) - \text{ the gradient of } f$$

You can use this notation in your solution. So $\nabla f(\mathbf{x} - \mathbf{y})$ means the gradient of f evaluated at $(\mathbf{x} - \mathbf{y})$.

Problem 3(20 Points)

f is a scalar function of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$f(\boldsymbol{x}, \boldsymbol{y}) = \frac{\boldsymbol{x} \cdot \boldsymbol{x}}{\boldsymbol{y} \cdot \boldsymbol{y}}$$

Calculate the derivative:

1. $\frac{\partial f}{\partial x}$(5 Points)
2. $\frac{\partial f}{\partial y}$ (15 Points)

Problem 4 (25 Points)

Suppose f is a scalar function and $\boldsymbol{x}$ is a vector. If a second-order partial derivatives of f exist, then the Hessian matrix $\mathbf{H}$ of f is a square $n \times n$ matrix

$$\mathbf{H}_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}.$$

That is, the entry of the ith row and the jth column is

$$(\mathbf{H}_f)_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

Prove that the Hessian matrix of a function f is the Jacobian matrix of the gradient of the function f; that is:

$$\mathbf{H}(f(\mathbf{x})) = \mathbf{J}(\nabla f(\mathbf{x}))$$

.

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \equiv \nabla f(\mathbf{x}) - \text{ the gradient of } f$$

The Jacobian is:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1} & \cdots & \frac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \nabla^{\mathrm{T}} f_1 \\ \vdots \\ \nabla^{\mathrm{T}} f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

where $\nabla^T f_i$ is the transpose (row vector) of the gradient of the i component.

Problem 5(20 Points)

In neural networks the Softmax function is used in back-propagation:

$$p_j = \frac{e^{x_j}}{\sum_k e^{x_k}}$$

This is used in a loss function of the form

$$L = -\sum_j y_j \log p_j,$$

where $\boldsymbol{x}$ is a some vector and $\boldsymbol{y}$ is a vector, which consist of 0 and 1 in position k:

$$y = \begin{bmatrix} 0, ..., 1, ...0 \end{bmatrix}$$

Please calculate the derivative of L with respect to $\boldsymbol{x}$.

Note: you can calculate the derivative of L by one component of $\boldsymbol{x}$:

$$\frac{\partial L}{\partial x_i}$$