Contents

Chapter 1

The real numbers

Recall the following classical sets:

- $\mathbb{N} = \{0, 1, 2, 3, \ldots, n, \ldots\}$ is the set of natural numbers.
- $\mathbb{N}^* = \mathbb{N} - \{0\} = \{1, 2, 3, \ldots, n, \ldots\}$ is the set of (strictly) positive natural numbers.
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers.
- $\mathbb{Q} = \left\{ \dfrac{a}{b}, \text{ such that } a \in \mathbb{Z}, b \in \mathbb{N}^* \right\}$ is the set of rational numbers.
  A number which is not rational is called an irrational number.
- $\mathbb{R}$ is the set of all rational and irrational numbers (called real numbers).

Exercise 1

1) **Show that every rational number can be written in the form $\frac{a}{b}$ with $a \in \mathbb{Z}$ and $b \in \mathbb{N}^*$, a and b are relatively prime.**

2) a) i) **Show that if $a \in \mathbb{Z}$ such that a^2 is even, then a is even.**
   ii) **Show that $\sqrt{2}$ is an irrational number.**
   b) i) **Show that if $a \in \mathbb{Z}$ such that a^2 is divisible by 3, then a is divisible by 3.**
   ii) **Show that $\sqrt{3}$ is an irrational number.**

3) **Prove that $\sqrt{2} + \sqrt{3}$ is an irrational number.**

4) **Prove that $\frac{\ln 2}{\ln 3}$ is an irrational number.**

Solution

1) Let $r \in \mathbb{Q}$, then there exist $u \in \mathbb{Z}$ and $v \in \mathbb{N}^*$ such that $r = \frac{u}{v}$. Let d be the positive greatest common divisor of u and v. Put $a = \frac{u}{d} \in \mathbb{Z}$ and $b = \frac{v}{d} \in \mathbb{N}^*$, then a and b are relatively prime and
$$\frac{a}{b} = \frac{\frac{u}{d}}{\frac{v}{d}} = \frac{u}{v} = r.$$

2) a) i) Let $a \in \mathbb{Z}$ such that a^2 is even. If a is odd, then there exists $k \in \mathbb{Z}$ such that $a = 2k + 1$, so

$$a^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Hence a^2 odd, which is impossible. Hence a is even.

ii) Suppose that $\sqrt{2} \in \mathbb{Q}$, then there exist $a \in \mathbb{Z}$ and $b \in \mathbb{N}^*$ such that $\sqrt{2} = \frac{a}{b}$ with a and b are relatively prime. Then $2 = \frac{a^2}{b^2}$, and therefore $a^2 = 2b^2$. So a^2 is even and therefore a is even by the part (i), so there exists $m \in \mathbb{Z}$ such that $a = 2m$, hence $a^2 = 4m^2$, so $4m^2 = 2b^2$, hence $b^2 = 2m^2$, so b^2 is even and therefore b is even by the part (i). Hence 2 is a common divisor of a and b, which is impossible since a and b are relatively prime. Thus $\sqrt{2} \notin \mathbb{Q}$, i.e., $\sqrt{2}$ is irrational.

b) i) Let $a \in \mathbb{Z}$ such that a^2 is divisible by 3. Let r be the remainder of the division of a by 3, then there exists $q \in \mathbb{Z}$ such that

$$a = 3q + r, \quad \text{with } r \in \{0, 1, 2\}.$$

Then

$$a^2 = (3q + r)^2 = 9q^2 + 6qr + r^2 = 3(3q^2 + 2qr) + r^2.$$

If $r = 1$, then $a^2 = 3(3q^2 + 2q) + 1$, so 1 is the remainder upon division of a^2 by 3, this contradicts the hypothesis.
If $r = 2$, then $a^2 = 3(3q^2 + 4q + 1) + 1$, so 1 is the remainder upon division of a^2 by 3, this contradicts the hypothesis.
Hence $r = 0$, and therefore a is divisible by 3.

ii) Suppose that $\sqrt{3} \in \mathbb{Q}$, then there exist $a \in \mathbb{Z}$ and $b \in \mathbb{N}^*$ such that $\sqrt{3} = \frac{a}{b}$ with a and b are relatively prime. Then $3 = \frac{a^2}{b^2}$, hence $a^2 = 3b^2$. So a^2 is divisible by 3 and therefore a is divisible by 3 by the part (i), so there exists $m \in \mathbb{Z}$ such that $a = 3m$, hence $a^2 = 9m^2$, so $9m^2 = 3b^2$, therefore $b^2 = 3m^2$, so b^2 is divisible by 3 and therefore b is divisible by 3 by the part (i). Hence 3 is a common divisor of a and b, which is impossible since a and b are relatively prime. Thus $\sqrt{3} \notin \mathbb{Q}$, i.e., $\sqrt{3}$ is irrational.

3) Let $\alpha = \sqrt{2} + \sqrt{3}$, then $\alpha - \sqrt{2} = \sqrt{3}$, so $(\alpha - \sqrt{2})^2 = 3$, i.e., $\alpha^2 - 2\alpha\sqrt{2} + 2 = 3$. Hence

$$\sqrt{2} = \frac{\alpha^2 - 1}{2\alpha}.$$

If $\alpha \in \mathbb{Q}$, then $\alpha^2 \in \mathbb{Q}$, and therefore $\sqrt{2} \in \mathbb{Q}$, which is impossible. Hence $\alpha \notin \mathbb{Q}$, i.e., α is an irrational number.

4) Suppose that $\frac{\ln 2}{\ln 3} \in \mathbb{Q}$, then there exist $a \in \mathbb{Z}$ and $b \in \mathbb{N}^*$ such that $\frac{\ln 2}{\ln 3} = \frac{a}{b}$ with a and b are relatively prime. So $b \ln 2 = a \ln 3$, and therefore $\ln(2^b) = \ln(3^a)$. Hence $2^b = 3^a$, which is impossible since 2^b is even and 3^a is odd. Thus $\frac{\ln 2}{\ln 3}$ is an irrational number. □

1.1 Equations and inequations in $\mathbb{R}$

The set $\mathbb{R}$, equipped with the binary relation $\leq$, satisfies the following properties:

- Reflexivity: $x \leq x$ for all $x \in \mathbb{R}$.
- Antisymmetric: for any $x, y \in \mathbb{R}$, if $x \leq y$ and $y \leq x$, then $x = y$.
- Transitivity: for any $x, y, z \in \mathbb{R}$, if $x \leq y$ and $y \leq z$, then $x \leq z$.
- The relation $\leq$ is a total order, i.e., for any $x, y \in \mathbb{R}$, $x \leq y$ or $y \geq x$.
- For any $x, y \in \mathbb{R}$, $x < y \Leftrightarrow (x \leq y$ and $x \neq y)$.
- For any $x, y \in \mathbb{R}$, $x < y \Rightarrow x \leq y$. The converse is not necessary true.
- For any $x, y, z \in \mathbb{R}$, if $x \leq y$ and $y < z$, then $x < z$. Similarly, if $x < y$ and $y \leq z$, then $x < z$.
- For any $x, y \in \mathbb{R}$, $x < y$ or $x = y$ or $y < x$.
- For any $x, y \in \mathbb{R}$, we have:
$$x \leq y \Leftrightarrow x + a \leq y + a.$$
- For any $x, y \in \mathbb{R}$, we have:
  - If $a > 0$, then
$$x \leq y \Leftrightarrow ax \leq ay.$$
  - If $a < 0$, then
$$x \leq y \Leftrightarrow ax \geq ay.$$
- For any $x, y \in \mathbb{R}$, we have:
$$xy = 0 \Leftrightarrow (x = 0 \text{ or } y = 0).$$
$$xy \neq 0 \Leftrightarrow (x \neq 0 \text{ and } y \neq 0).$$
- Let $a, b, c \in \mathbb{R}$ such that $a \neq 0$. Let's solve, in $\mathbb{R}$, the equation
$$(*) \qquad ax^2 + bx + c = 0.$$
Put $\Delta = b^2 - 4ac$, and call it the discriminant of the equation $(*)$.
  - If $\Delta = 0$, then the equation $(*)$ admits a double real root
$$x_1 = x_2 = -\frac{b}{2a}.$$
  - If $\Delta > 0$, then the equation $(*)$ admits two distinct real roots, which are:
$$x_1 = \frac{-b - \sqrt{\Delta}}{2a} \quad \text{and} \quad x_2 = \frac{-b + \sqrt{\Delta}}{2a}.$$

  - If $\Delta < 0$, then the equation $(*)$ has no real roots.

- Let $a, b, c \in \mathbb{R}$ such that $a \neq 0$.
  - If $a+b+c=0$, then the real roots of the equation $(*)$ are $x_1 = 1$ and $x_2 = \frac{c}{a}$.
  - If $a-b+c=0$, then the real roots of the equation $(*)$ are $x_1 = -1$ and $x_2 = -\frac{c}{a}$.

- Let $u, v \in \mathbb{R}$. If $S = u+v$ and $P = uv$, then u and v are the roots of the equation

$$x^2 - Sx + P = 0.$$

- Let $a, b, c \in \mathbb{R}$ such that $a \neq 0$. The sign of the trinomial ax^2+bx+c depends on the signs of a and the discriminant $\Delta = b^2-4ac$.
  - Suppose that $\Delta = 0$. Then the sign of ax^2+bx+c is the same as the sign of a except for $x = -\frac{b}{2a}$.
  - Suppose that $\Delta > 0$.
    If $a > 0$, then

$$\begin{cases} ax^2+bx+c > 0 & \text{if} \quad (x < x_1 \text{ or } x > x_2), \\ ax^2+bx+c < 0 & \text{if} \quad x_1 < x < x_2. \end{cases}$$

    If $a < 0$, then

$$\begin{cases} ax^2+bx+c > 0 & \text{if} \quad x_1 < x < x_2, \\ ax^2+bx+c < 0 & \text{if} \quad (x < x_1 \text{ or } x > x_2). \end{cases}$$

    These two cases can be written in the following way: the sign of ax^2+bx+c is the same as the sign a outside roots, and it is the opposite sign of a inside roots.

x		x_1		x_2	
sg(ax^2+bx+c)	sg(a)	0	-sg(a)	0	sg(a)

    where sg(y) is the sign of the real number y.
  - Suppose that $\Delta < 0$, then the sign of ax^2+bx+c is the same as the sign of a.

Example 1.1.1

- *Let's solve, in $\mathbb{R}$, the equation $x^2+4x-5=0$. Since $a+b+c=0$, then the solutions of this equation are $x_1 = 1$ and $x_2 = -5$.*

- *Let's solve, in $\mathbb{R}$, the equation $x^3+x^2-2x=0$. We have:*

$$\begin{aligned} x^3+x^2-2x=0 &\Leftrightarrow x(x^2+x-2)=0 \Leftrightarrow x(x-1)(x+2)=0 \\ &\Leftrightarrow (x=0 \textit{ or } x=1 \textit{ or } x=-2). \end{aligned}$$

  So the set of solutions of the equation $x^3+x^2-2x=0$ is $S = \{-2, 0, 1\}$.

- *Let's solve, in $\mathbb{R}$, the inequation $x^2 + x - 2 \leq 0$. Since $\Delta = 9 > 0$ and $a = 1 > 0$, then*
$$x^2 + x - 2 \leq 0 \Leftrightarrow -2 \leq x \leq 1.$$

Exercise 2

Show that the following proprerties are satisfied:

1) **For any $x, y \in \mathbb{R}^+$, if $x < y$, then $\sqrt{x} < \sqrt{y}$.**
2) **For any $x, y \in \mathbb{R}$, $2xy \leq x^2 + y^2$.**
3) **For any $a, b \in \mathbb{R}^+$, $\sqrt{ab} \leq \frac{a+b}{2}$.**
4) **For any $x, y, z \in \mathbb{R}$, $xy + xz + yz \leq x^2 + y^2 + z^2$.**
5) **For any $x, y, z, t \in \mathbb{R}$, we have:**
$$\Big[x \leq y \text{ and } z \leq t\Big] \Rightarrow x + z \leq y + t.$$
6) **For any $x, y, z, t \in \mathbb{R}$, we have:**
$$\Big[0 \leq x \leq y \text{ and } 0 \leq z \leq t\Big] \Rightarrow 0 \leq xz \leq yt.$$
$$\Big[0 \leq x < y \text{ and } 0 \leq z < t\Big] \Rightarrow 0 \leq xz < yt.$$
7) **For any $x, y \in \mathbb{R}$, we have:**
$$[0 \leq x \leq y] \Rightarrow x^2 \leq y^2 \quad \text{and} \quad [0 \leq x < y] \Rightarrow x^2 < y^2.$$
8) **For any $x, y \in \mathbb{R}$, we have:**
$$[0 < x < y] \Rightarrow \left(\frac{1}{y} < \frac{1}{x}\right) \quad \text{and} \quad [x < y < 0] \Rightarrow \left(\frac{1}{y} < \frac{1}{x}\right).$$
9) **For any $x, y \in \mathbb{R}$, we have:**
$$[x > 0 \text{ and } y > 0] \Rightarrow (x + y)\left(\frac{1}{x} + \frac{1}{y}\right) \geq 4.$$

Solution

1) Let $x, y \in \mathbb{R}^+$ such that $x < y$. Then, multiplying by the conjugate, we obtain:
$$\sqrt{y} - \sqrt{x} = \frac{(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x})}{\sqrt{y} + \sqrt{x}} = \frac{y - x}{\sqrt{y} + \sqrt{x}} > 0$$
since $y - x > 0$ and $\sqrt{y} + \sqrt{x} > 0$. So $\sqrt{x} < \sqrt{y}$.

2) Let $\boldsymbol{x, y \in \mathbb{R}}$. Since
$$\boldsymbol{0 \leq (x-y)^2 = x^2 + y^2 - 2xy,}$$
then $\boldsymbol{2xy \leq x^2 + y^2}$.

3) Let $\boldsymbol{a, b \in \mathbb{R}^+}$. Put $\boldsymbol{x = \sqrt{a}}$ and $\boldsymbol{y = \sqrt{b}}$, then $\boldsymbol{x^2 = a}$ and $\boldsymbol{y^2 = b}$. By the part (2), $\boldsymbol{2xy \leq x^2 + y^2}$, so $\boldsymbol{2\sqrt{a}\sqrt{b} \leq a + b}$. Hence $\boldsymbol{\sqrt{ab} \leq \frac{a+b}{2}}$.

4) Let $\boldsymbol{x, y, z \in \mathbb{R}}$. We have:
$$\begin{aligned} \boldsymbol{2(xy + xz + yz)} &= \boldsymbol{2xy + 2xz + 2yz} \\ &\leq \boldsymbol{(x^2 + y^2) + (x^2 + z^2) + (y^2 + z^2)} \quad \text{(by the part (2))} \\ &= \boldsymbol{2(x^2 + y^2 + z^2).} \end{aligned}$$
So $\boldsymbol{xy + xz + yz \leq x^2 + y^2 + z^2}$.

5) Let $\boldsymbol{x, y, z, t \in \mathbb{R}}$ such that $\boldsymbol{x \leq y}$ and $\boldsymbol{z \leq t}$, then $\boldsymbol{y - x \geq 0}$ and $\boldsymbol{t - z \geq 0}$. As
$$\boldsymbol{(y+t) - (x+z) = (y-x) + (t-z) \geq 0,}$$
then $\boldsymbol{x + z \leq y + t}$.

6) Let $\boldsymbol{x, y, z, t \in \mathbb{R}}$ such that $\boldsymbol{x \geq 0}$ and $\boldsymbol{z \geq 0}$, then $\boldsymbol{xz \geq 0}$.

   - Suppose that $\boldsymbol{0 \leq x \leq y}$ and $\boldsymbol{0 \leq z \leq t}$. To show that $\boldsymbol{xz \leq yt}$, we distinguish the following two methods:
     $\underline{\boldsymbol{1}^{\text{st}} \text{ method}}$: By multiplying the inequality $\boldsymbol{x \leq y}$ by $\boldsymbol{z \geq 0}$, we obtain $\boldsymbol{xz \leq yz}$. By multiplying the inequality $\boldsymbol{z \leq t}$ by $\boldsymbol{y \geq 0}$, we obtain $\boldsymbol{yz \leq yt}$. So $\boldsymbol{xz \leq yt}$.
     $\underline{\boldsymbol{2}^{\text{nd}} \text{ method}}$: Since $\boldsymbol{t \geq 0}$, $\boldsymbol{x \geq 0}$, $\boldsymbol{y - x \geq 0}$, $\boldsymbol{t - z \geq 0}$, then
     $$\boldsymbol{yt - xz = yt - xt + xt - xz = (y-x)t + x(t-z) \geq 0.}$$
     So $\boldsymbol{xz \leq yt}$.
   - Suppose that $\boldsymbol{0 \leq x < y}$ and $\boldsymbol{0 \leq z < t}$. By multiplying the inequality $\boldsymbol{x < y}$ by $\boldsymbol{z \geq 0}$, we obtain $\boldsymbol{xz \leq yz}$. By multiplying the inequality $\boldsymbol{z < t}$ by $\boldsymbol{y > 0}$, we obtain $\boldsymbol{yz < yt}$. So $\boldsymbol{xz < yt}$.

7) Let $\boldsymbol{x, y \in \mathbb{R}}$.

   - If $\boldsymbol{0 \leq x \leq y}$, then $\boldsymbol{0 \leq xx \leq yy}$ by the part (6), so $\boldsymbol{x^2 \leq y^2}$.
   - If $\boldsymbol{0 \leq x < y}$, then $\boldsymbol{0 \leq xx < yy}$ by the part (6), so $\boldsymbol{x^2 < y^2}$.

8) Let $\boldsymbol{x, y \in \mathbb{R}}$.

   - Suppose that $\boldsymbol{0 < x < y}$. Since $\boldsymbol{x > 0}$ and $\boldsymbol{y > 0}$, then $\boldsymbol{xy > 0}$.
     $\underline{\boldsymbol{1}^{\text{st}} \text{ method}}$: By multiplying the inequality $\boldsymbol{x < y}$ by $\boldsymbol{\frac{1}{xy} > 0}$, we obtain:
     $$\boldsymbol{\frac{x}{xy} < \frac{y}{xy}} \quad \text{i.e.,} \quad \boldsymbol{\frac{1}{y} < \frac{1}{x}.}$$
     $\underline{\boldsymbol{2}^{\text{nd}} \text{ method}}$: Since $\boldsymbol{xy > 0}$ and $\boldsymbol{y - x > 0}$, then
     $$\boldsymbol{\frac{1}{x} - \frac{1}{y} = \frac{y-x}{xy} > 0.}$$
     So $\boldsymbol{\frac{1}{y} < \frac{1}{x}}$.

- Suppose that $x < y < 0$, then $0 < -y < -x$, so $\frac{1}{-x} < \frac{1}{-y}$ by the above discussion. Hence $\frac{1}{y} < \frac{1}{x}$.

9) Let $x, y \in \mathbb{R}$ such that $x > 0$ and $y > 0$. We have:

$$(x+y)\left(\frac{1}{x}+\frac{1}{y}\right)-4=\frac{(x+y)^2}{xy}-4=\frac{x^2+y^2-2xy}{xy}=\frac{(x-y)^2}{xy}\geq 0.$$

So $(x+y)\left(\frac{1}{x}+\frac{1}{y}\right) \geq 4$. $\square$

Exercise 3

1) Show that $\frac{xy}{x+y} \leq \frac{x+y}{4}$ for any strictly positive real numbers x, y.

2) Deduce that, for any strictly positive real numbers a, b and c, we have:

$$\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ac}{a+c}\leq\frac{a+b+c}{2}.$$

Solution

1) Let x and y be two strictly positive real numbers. We have:

$$\frac{x+y}{4}-\frac{xy}{x+y}=\frac{(x+y)^2-4xy}{4(x+y)}=\frac{x^2+y^2-2xy}{4(x+y)}=\frac{(x-y)^2}{4(x+y)}\geq 0.$$

So $\frac{xy}{x+y} \leq \frac{x+y}{4}$.

2) Let a, b and c be three strictly positive real numbers. By using the part (1), we obtain:

$$\begin{aligned}\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ac}{a+c} &\leq \frac{a+b}{4}+\frac{b+c}{4}+\frac{a+c}{4}\\ &= \frac{2a+2b+2c}{4}=\frac{a+b+c}{2}. \quad \square\end{aligned}$$

Definition 1.1.1

1) A part I of $\mathbb{R}$ is said to be an interval of $\mathbb{R}$ if for all $a, b \in I$ and $x \in \mathbb{R}$, we have:

$$a \leq x \leq b \Rightarrow x \in I.$$

2) A sub-interval of an interval I of $\mathbb{R}$ is every interval of $\mathbb{R}$ contained in I.

Let $a, b \in \mathbb{R}$ such that $a \leq b$. We denote by:

- $[a, b]$ the closed interval $[a, b] = \{x \in \mathbb{R},\ a \leq x \leq b\}$.

- $]a, b[$ the open interval $]a, b[= \{x \in \mathbb{R},\ a < x < b\}$.
- $]a, b]$ the left-open interval $]a, b] = \{x \in \mathbb{R},\ a < x \leq b\}$.
- $[a, b[$ the right-open interval $[a, b[= \{x \in \mathbb{R}, a \leq x < b\}$.
- $[a, +\infty[$ the closed interval $[a, +\infty[= \{x \in \mathbb{R},\ x \geq a\}$.
- $]a, +\infty[$ the open interval $]a, +\infty[= \{x \in \mathbb{R},\ x > a\}$.
- $]-\infty, b]$ the closed interval $]-\infty, b] = \{x \in \mathbb{R},\ x \leq b\}$.
- $]-\infty, b[$ the open interval $]-\infty, b[= \{x \in \mathbb{R},\ x < b\}$.

In the exercise 16 below, we prove that they are the only intervals of $\mathbb{R}$.

Exercise 4
Let $a, b, x \in \mathbb{R}$ such that $a < b$. Show that $x \in [a, b]$ if and only if there exists $t \in [0, 1]$ such that $x = ta + (1-t)b$.

Solution

N.C. Put $t = \dfrac{x-b}{a-b}$, then we have $x = ta + (1-t)b$. As $x \in [a, b]$, then $a \leq x \leq b$, so $a - b \leq x - b \leq 0$. Dividing this double inequality by $a - b < 0$, we obtain $0 \leq t \leq 1$, i.e., $t \in [0, 1]$.
S.C. Since $1 - t \geq 0$ and $b - a \geq 0$, then $(1-t)(b-a) \geq 0$, and therefore

$$x = a + (1-t)(b-a) \geq a.$$

In the other side, since $t \geq 0$ and $b - a \geq 0$, then $t(b-a) \geq 0$, and therefore

$$x = b - t(b-a) \leq b.$$

Hence $x \in [a, b]$. □

1.2 The absolute value function

Definition 1.2.1 *For every $x \in \mathbb{R}$, we define the absolute value of x, denoted $|x|$ by:*

$$|x| = \begin{cases} +x & \text{if } x \geq 0, \\ -x & \text{if } x \leq 0. \end{cases}$$

Proposition 1.2.1 *For any $x, y \in \mathbb{R}$, the following properties are satisfied:*

1) $|x| = \max\{-x, +x\}$.
2) $x = \pm|x|$, *and therefore* $-|x| \leq x \leq +|x|$.

3) $|x| \geq 0$.

4) $|x| = 0 \Leftrightarrow x = 0$.

5) *For any* $a \in \mathbb{R}^+$, *we have:*

$$|x| = a \Leftrightarrow (x = a \text{ or } x = -a).$$

6) $|x - y| = x - y$ *if* $x \geq y$ *and* $|x - y| = y - x$ *if* $x \leq y$.

7) $|-x| = |x|$.

8) $|xy| = |x|\,|y|$.

9) $|x + y| \leq |x| + |y|$. *This inequality is called the triangular inequality. Equality holds if and only if* x *and* y *have the same sign.*

10) $|x|^2 = x^2$.

11) $|x| = |y| \Leftrightarrow x^2 = y^2$.

12) $|x| \leq |y| \Leftrightarrow x^2 \leq y^2$.

Example 1.2.1

- *Let's solve, in* $\mathbb{R}$, *the equation* $|x - 1| = 5$. *We have:*

$$|x - 1| = 5 \Leftrightarrow (x - 1 = 5 \text{ or } x - 1 = -5) \Leftrightarrow (x = 6 \text{ or } x = -4).$$

So the set of solutions of this equation is $S = \{-4, 6\}$.

- *Let's solve, in* $\mathbb{R}$, *the equation* $|x - 1| = x - 5$. *First, if* $x_0 \in \mathbb{R}$ *is a solution of this equation, then* $x_0 - 5 \geq 0$ *since* $|x_0 - 1| \geq 0$, *so* $x_0 \in [5, +\infty[$. *Hence we look for solutions of this equation in the interval* $[5, +\infty[$. *In the other side,*

$$\begin{aligned} |x - 1| = x - 5 &\Leftrightarrow \Big[x - 1 = x - 5 \text{ or } x - 1 = -(x - 5)\Big] \\ &\Leftrightarrow (-1 = -5 \text{ or } 2x = 6) \Leftrightarrow x = 3. \end{aligned}$$

As $3 \notin [5, +\infty[$, *then this equation has no solutions in* $\mathbb{R}$.

Exercise 5
Let $x \in \mathbb{R}$ and $a > 0$.

1) **Prove that $|x| \leq a \Leftrightarrow x \in [-a, a]$.**

2) **Deduce that $|x| > a \Leftrightarrow x \in]-\infty, -a[\cup]a, +\infty[$.**

Solution

1) N.C.

- If $x \geq 0$, then $|x| = x$, so $0 \leq x \leq a$, therefore $x \in [-a, a]$.
- If $x \leq 0$, then $|x| = -x$, so $-x \leq a$, therefore $x \geq -a$, therefore $-a \leq x \leq 0$, hence $x \in [-a, a]$.

S.C.

- If $x \geq 0$, then $x \in [0, a]$, so $|x| = x \leq a$.
- If $x \leq 0$, then $x \in [-a, 0]$, so $x \geq -a$, therefore $-x \leq a$, hence $|x| = -x \leq a$.

2) This equivalence can be deduced from that of the part (1) by using the fact that if P and Q are two propositions, then

$$(P \Leftrightarrow Q) \Leftrightarrow (\neg P \Leftrightarrow \neg Q). \quad \square$$

Exercise 6

1) Show that $|a-b| \leq |a-c| + |c-b|$ for all $a, b, c \in \mathbb{R}$.

2) Show that $\Big||x| - |y|\Big| \leq |x-y|$ for all $x, y \in \mathbb{R}$.

Solution

1) Let $a, b, c \in \mathbb{R}$. By using the triangular inequality, we obtain:

$$|a-b| = \Big|(a-c) + (c-b)\Big| \leq |a-c| + |c-b|\,.$$

2) Let $x, y \in \mathbb{R}$.
1^{st} method: By using the part (1) for $a = x, b = 0$ and $c = y$, we obtain:

$$|x| = |x-0| \leq |x-y| + |y-0| = |x-y| + |y|.$$

So $|x| - |y| \leq |x-y|$. Similarly, by using the part (1) for $a = y, b = 0$ and $c = x$, we obtain:

$$|y| = |y-0| \leq |y-x| + |x-0| = |x-y| + |x|.$$

So $|x| - |y| \geq -|x-y|$. Therefore

$$-|x-y| \leq |x| - |y| \leq |x-y|.$$

Hence $\Big||x| - |y|\Big| \leq |x-y|$ by the exercise 5.
2^{nd} method: Put $a = |x| - |y|$ and $b = x - y$. Since

$$\begin{aligned} a^2 - b^2 = (|x| - |y|)^2 - (x-y)^2 &= (|x|^2 + |y|^2 - 2|x||y|) - (x^2 + y^2 - 2xy) \\ &= (x^2 + y^2 - 2|xy|) - (x^2 + y^2 - 2xy) \\ &= 2(xy - |xy|) \leq 0 \quad (\text{since } xy \leq |xy|), \end{aligned}$$

then $a^2 \leq b^2$, so $|a| \leq |b|$ by the part (12) of the proposition 1.2.1. Hence $\Big||x| - |y|\Big| \leq |x-y|$. $\square$

Exercise 7
Solve, in $\mathbb{R}$, each of the following equations and inequations:

1) $|x^2+4x-5| = x^2+4x-5$.
2) $|x^2+4x-5| \leq x^2+4x-5$.
3) $|x^2+4x-5| < x^2+4x-5$.
4) $|x^2+4x-5| \geq x^2+4x-5$.
5) $|x^2+4x-5| > x^2+4x-5$.
6) $\left|\dfrac{3x-1}{2x-1}+2\right| = 1$.

Solution

1) For any $a \in \mathbb{R}$, $|a| = a$ if and only if $a \geq 0$. So

$$|x^2+4x-5| = x^2+4x-5 \Leftrightarrow x^2+4x-5 \geq 0 \Leftrightarrow x \in]-\infty,-5] \cup [1,+\infty[.$$

So the set of solutions of this equation is $S =]-\infty,-5] \cup [1,+\infty[$.

2) For any $a \in \mathbb{R}$, $|a| \leq a$ if and only if $a \geq 0$. So

$$|x^2+4x-5| \leq x^2+4x-5 \Leftrightarrow x^2+4x-5 \geq 0 \Leftrightarrow x \in]-\infty,-5] \cup [1,+\infty[.$$

So the set of solutions of this inequation is $S =]-\infty,-5] \cup [1,+\infty[$.

3) For any $a \in \mathbb{R}$, $|a| < a$ is false. Then this inequation has no solutions.

4) For any $a \in \mathbb{R}$, $|a| \geq a$. Then the set of solutions of this inequation is $\mathbb{R}$.

5) For any $a \in \mathbb{R}$, $|a| > a$ if and only if $a < 0$. Then

$$|x^2+4x-5| > x^2+4x-5 \Leftrightarrow x^2+4x-5 < 0 \Leftrightarrow x \in]-5,1[.$$

So the set of solutions of this inequation is $S =]-5,1[$.

6) We have:

$$\begin{aligned}
\left|\frac{3x-1}{2x-1}+2\right| = 1 \quad &\Leftrightarrow \quad \frac{3x-1}{2x-1}+2 = 1 \quad \text{or} \quad \frac{3x-1}{2x-1}+2 = -1 \\
&\Leftrightarrow \quad 3x-1 = -2x+1 \quad \text{or} \quad 3x-1 = -6x+3 \\
&\Leftrightarrow \quad x = \frac{2}{5} \quad \text{or} \quad x = \frac{4}{9}.
\end{aligned}$$

So the set of solutions of this equation is $\{\frac{2}{5}, \frac{4}{9}\}$. $\square$

Exercise 8
Let $x, y \in \mathbb{R}$. Prove that:

1) $|x| + |y| \leq \sqrt{2}\,\sqrt{x^2 + y^2}$**.**

2) $\sqrt{x} + \sqrt{y} \leq \sqrt{2(x+y)}$ **if** $x, y \in \mathbb{R}^+$**.**

3)

$$\max(x, y) = \frac{x + y + |x - y|}{2} \quad \text{and} \quad \min(x, y) = \frac{x + y - |x - y|}{2}.$$

4) For any $a \in \mathbb{R}^+$**, if** $|x + y| \leq a$ **and** $|x - y| \leq a$**, then** $|x| + |y| \leq a$**.**
(Hint: we may distinguish four cases according to the signs of x **and** y**).**

Solution

1) We have:

$$\begin{aligned} \left[\sqrt{2}\,\sqrt{x^2 + y^2}\right]^2 - (|x| + |y|)^2 &= 2\Big(|x|^2 + |y|^2\Big) - \Big(|x|^2 + |y|^2 + 2|xy|\Big) \\ &= |x|^2 + |y|^2 - 2|xy| \;=\; (|x| - |y|)^2 \;\geq\; 0. \end{aligned}$$

So $(|x| + |y|)^2 \leq \left[\sqrt{2}\,\sqrt{x^2 + y^2}\right]^2$. Hence $\Big||x| + |y|\Big| \leq \left|\sqrt{2}\,\sqrt{x^2 + y^2}\right|$, i.e., $|x| + |y| \leq \sqrt{2}\,\sqrt{x^2 + y^2}$.

2) Suppose that $x, y \in \mathbb{R}^+$. Put $a = \sqrt{x}$ and $b = \sqrt{y}$, then $a^2 = x$ and $b^2 = y$. By the part (1), $|a| + |b| \leq \sqrt{2}\,\sqrt{a^2 + b^2}$, so $\sqrt{x} + \sqrt{y} \leq \sqrt{2(x + y)}$.

3)
- If $x \leq y$, then

$$\max(x, y) = y, \quad \min(x, y) = x \quad \text{and} \quad |x - y| = y - x.$$

So

$$\begin{aligned} \frac{x + y + |x - y|}{2} &= \frac{x + y + y - x}{2} = y = \max(x, y) \\ \frac{x + y - |x - y|}{2} &= \frac{x + y - y + x}{2} = x = \min(x, y). \end{aligned}$$

- If $x \geq y$, then

$$\max(x, y) = x, \quad \min(x, y) = y \quad \text{and} \quad |x - y| = x - y.$$

So

$$\begin{aligned} \frac{x + y + |x - y|}{2} &= \frac{x + y + x - y}{2} = x = \max(x, y) \\ \frac{x + y - |x - y|}{2} &= \frac{x + y - x + y}{2} = y = \min(x, y). \end{aligned}$$

4) Let $a \in \mathbb{R}^+$. We distinguish the following four cases:

- Suppose that $x \geq 0$ and $y \geq 0$. Then $x + y \geq 0$, so

$$|x| + |y| = x + y = |x + y| \leq a.$$

- Suppose that $x \geq 0$ and $y < 0$, then we put $y' = -y > 0$. As $|x + y'| = |x - y| \leq a$, then, by using the first case, we obtain:
$$|x| + |y| = |x| + |y'| \leq a.$$
- Suppose that $x < 0$ and $y \geq 0$, then we put $x' = -x > 0$. As $|x' + y| = |-x + y| = |x - y| \leq a$, then, by using the first case, we obtain:
$$|x| + |y| = |x'| + |y| \leq a.$$
- Suppose that $x < 0$ and $y < 0$. Then $x + y < 0$, so
$$|x| + |y| = -x - y = -(x + y) = |x + y| \leq a. \quad \square$$

Exercise 9
Let $a, b \in \mathbb{R}$.

1) Show that if $a \leq b + \varepsilon$ for all $\varepsilon > 0$, then $a \leq b$.

2) Deduce that if $x \in \mathbb{R}$ such that $|x| \leq \varepsilon$ for all $\varepsilon > 0$, then $x = 0$.

Solution

1) Suppose that $a > b$. Put $\varepsilon = \frac{a-b}{2} > 0$, then $a \leq b + \varepsilon$ by using the hypothesis. So
$$a \leq b + \frac{a - b}{2} = \frac{a + b}{2} < \frac{a + a}{2} = a,$$
which is impossible. Hence $a \leq b$.

2) Let $x \in \mathbb{R}$ such that $|x| \leq \varepsilon$ for all $\varepsilon > 0$, Put $a = |x|$ and $b = 0$, then $a \leq b + \varepsilon$ for all $\varepsilon > 0$. So $a \leq b$, i.e., $|x| \leq 0$ by the part (1). But $|x| \geq 0$, then $|x| = 0$, and therefore $x = 0$. $\square$

1.3 Supremum and infimum

Definition 1.3.1 *Let A be a nonempty subset of $\mathbb{R}$.*

1) We say that A is bounded from above in $\mathbb{R}$ if there exists $M \in \mathbb{R}$ such that $x \leq M$ for all $x \in A$. In this case, we say that A is bounded from above by M and that M is an upper bound of A.

2) We say that A is bounded from below in $\mathbb{R}$ if there exists $m \in \mathbb{R}$ such that $m \leq x$ for all $x \in A$. In this case, we say that A is bounded from below by m and that m is a lower bound of A.

3) We say that A is bounded in $\mathbb{R}$ if A is bounded both from above and from below, i.e., if there exist $m, M \in \mathbb{R}$ such that $m \leq x \leq M$ for all $x \in A$.

4) *We say that an element* $\boldsymbol{a}$ *of* $\boldsymbol{A}$ *is a greatest element (or a maximum) of* $\boldsymbol{A}$ *if* $\boldsymbol{x} \leq \boldsymbol{a}$ *for all* $\boldsymbol{x} \in \boldsymbol{A}$.

5) *We say that an element* $\boldsymbol{a}$ *of* $\boldsymbol{A}$ *is a least element (or a minimum) of* $\boldsymbol{A}$ *if* $\boldsymbol{a} \leq \boldsymbol{x}$ *for all* $\boldsymbol{x} \in \boldsymbol{A}$.

Remark 1.3.1 *Let* $\boldsymbol{A}$ *be a nonempty subset of* $\mathbb{R}$.

- $\boldsymbol{A}$ *is bounded in* $\mathbb{R}$ *if and only if there exists a real number* $\boldsymbol{M} \geq \mathbf{0}$ *such that* $|\boldsymbol{x}| \leq \boldsymbol{M}$ *for all* $\boldsymbol{x} \in \boldsymbol{A}$.
- *A greatest element of* $\boldsymbol{A}$ *is every upper bound of* $\boldsymbol{A}$ *in* $\mathbb{R}$ *that belongs to* $\boldsymbol{A}$.
- *A least element of* $\boldsymbol{A}$ *is every lower bound of* $\boldsymbol{A}$ *in* $\mathbb{R}$ *that belongs to* $\boldsymbol{A}$.
- *If* $\boldsymbol{A}$ *has a greatest element, then it is unique, denoted by* $\mathbf{max}\, \boldsymbol{A}$.
- *If* $\boldsymbol{A}$ *has a least element, then it is unique, denoted by* $\mathbf{min}\, \boldsymbol{A}$.

Example 1.3.1

1) *The set* $\mathbb{N}$ *is bounded from below in* $\mathbb{R}$ *by* $\mathbf{0}$, *but not bounded from above in* $\mathbb{R}$. *Moreover,* $\mathbf{0}$ *is the least element of* $\mathbb{N}$. *On the other hand,* $\mathbb{N}$ *has no greatest element.*

2) *The sets* $\mathbb{Z}, \mathbb{Q}$ *and* $\mathbb{R}$ *are not bounded from above, neither bounded from below in* $\mathbb{R}$. *Moreover, they have no least and greatest elements.*

3) *Every nonempty finite subset of* $\mathbb{R}$ *is bounded and has least and greatest elements.*

Definition 1.3.2 *Let* $\boldsymbol{A}$ *be a nonempty subset of* $\mathbb{R}$.

1) *We say that a real number* $\boldsymbol{a}$ *is a supremum of* $\boldsymbol{A}$ *if it satisfies the following two conditions:*

   i) $\boldsymbol{a}$ *is an upper bound of* $\boldsymbol{A}$.

   ii) *If* $\boldsymbol{M} \in \mathbb{R}$ *is an upper bound of* $\boldsymbol{A}$, *then* $\boldsymbol{a} \leq \boldsymbol{M}$.

   In other words, a real number $\boldsymbol{a}$ *is a supremum of* $\boldsymbol{A}$ *if and only if* $\boldsymbol{a}$ *is the least upper bound of* $\boldsymbol{A}$ *in* $\mathbb{R}$.

2) *We say that a real number* $\boldsymbol{a}$ *is an infimum of* $\boldsymbol{A}$ *if it satisfies the following two conditions:*

   i) $\boldsymbol{a}$ *is a lower bound of* $\boldsymbol{A}$.

   ii) *If* $\boldsymbol{m} \in \mathbb{R}$ *is a lower bound of* $\boldsymbol{A}$, *then* $\boldsymbol{m} \leq \boldsymbol{a}$.

   In other words, a real number $\boldsymbol{a}$ *is an infimum of* $\boldsymbol{A}$ *if and only if* $\boldsymbol{a}$ *is the greatest lower bound of* $\boldsymbol{A}$ *in* $\mathbb{R}$.

Remark 1.3.2 *Let A be a nonempty subset of $\mathbb{R}$.*

- *If A has a supremum, then it is unique, denoted by $\sup_{\mathbb{R}}(A)$ or simply $\sup A$.*
- *If A has an infimum, then it is unique, denoted by $\inf_{\mathbb{R}}(A)$ or simply $\inf A$.*
- *If A has a greatest element, then A has a supremum with $\sup A = \max A$.*
- *If A has a least element, then A has an infimum with $\inf A = \min A$.*
- *If A has a supremum in A, i.e., $\sup A \in A$, then A has a greatest element with $\max A = \sup A$.*
- *If A has an infimum in A, i.e., $\inf A \in A$, then A has a least element with $\min A = \inf A$.*

Exercise 10
Let A (resp. B) be a nonempty subset having supremum and infimum in $\mathbb{R}$.

1) **Show that if $A \subseteq B$, then**

$$\sup A \leq \sup B \quad \text{and} \quad \inf A \geq \inf B.$$

2) **Prove that if $A \cap B$ has supremum and infimum in $\mathbb{R}$, then**

$$\sup(A \cap B) \leq \min\left\{\sup A, \sup B\right\} \text{ and } \inf(A \cap B) \geq \max\left\{\inf A, \inf B\right\}.$$

Verify that these two inequalities are strict for $A = [0,1] \cup \{-2,2\}$ and $B = [0,1] \cup \{-3,3\}$.

3) **Prove that $A \cup B$ has supremum and infimum in $\mathbb{R}$ and that**

$$\sup(A \cup B) = \max\left\{\sup A, \sup B\right\} \text{ and } \inf(A \cup B) = \min\left\{\inf A, \inf B\right\}.$$

Solution

1)
- Let $x \in A$. Since $A \subseteq B$, then $x \in B$, so $x \leq \sup B$. Therefore $\sup B$ is an upper bound of A in $\mathbb{R}$. Hence $\sup A \leq \sup B$ since $\sup A$ is the least upper bound of A in $\mathbb{R}$.
- Let $x \in A$. Since $A \subseteq B$, then $x \in B$, so $x \geq \inf B$. Therefore $\inf B$ is a lower bound of A in $\mathbb{R}$. Hence $\inf A \geq \inf B$ since $\inf A$ is the greatest lower bound of A in $\mathbb{R}$.

2) Suppose that $A \cap B$ has supremum and infimum in $\mathbb{R}$. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, then, by the part (1),

$$\sup(A \cap B) \leq \sup A \text{ and } \sup(A \cap B) \leq \sup B,$$

$$\inf(A \cap B) \geq \inf A \text{ and } \inf(A \cap B) \geq \inf B.$$

So

$$\sup(A \cap B) \leq \min\left\{\sup A, \sup B\right\} \text{ and } \inf(A \cap B) \geq \max\left\{\inf A, \inf B\right\}.$$

Strict inequalities: Let $A = [0,1] \cup \{-2,2\}$ and $B = [0,1] \cup \{-3,3\}$. Since 2 (resp. -2) is an upper bound (resp. lower bound) of A belonging to A, then $\sup A = 2$ and $\inf A = -2$. Similarly, since 3 (resp. -3) is an upper bound (resp. lower bound) of B belonging to B, then $\sup B = 3$ and $\inf B = -3$. In the other side, $A \cap B = [0,1]$, since 1 (resp. 0) is an upper bound (resp. lower bound) of $A \cap B$ belonging to $A \cap B$, then $\sup(A \cap B) = 1$ and $\inf(A \cap B) = 0$. Hence

$$\sup(A \cap B) = 1 < 2 = \min\left\{\sup A, \sup B\right\}$$

$$\inf(A \cap B) = 0 > -2 = \max\left\{\inf A, \inf B\right\}.$$

3) Put $M = \max\left\{\sup A, \sup B\right\}$ and $m = \min\left\{\inf A, \inf B\right\}$.

- Let $x \in A \cup B$, then $x \in A$ or $x \in B$, so $x \leq \sup A$ or $x \leq \sup B$. Therefore $x \leq M$. Hence M is an upper bound of $A \cup B$. In the other side, let $M' \in \mathbb{R}$ be an upper bound of $A \cup B$, then $x \leq M'$ for all $x \in A \cup B$. In particular, since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, then $x \leq M'$ for all $x \in A$ and $x \leq M'$ for all $x \in B$. Therefore M' is an upper bound of A and B, so $\sup A \leq M'$ and $\sup B \leq M'$, hence $M \leq M'$. Thus M is the least upper bound of $A \cup B$, i.e., $\sup(A \cup B) = M$.
- Let $x \in A \cup B$, then $x \in A$ and $x \in B$, so $x \geq \inf A$ and $x \geq \inf B$. Therefore $x \geq m$. Hence m is a lower bound of $A \cup B$. In the other side, let $m' \in \mathbb{R}$ be a lower bound of $A \cup B$, then $x \geq m'$ for all $x \in A \cup B$. In particular, since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, then $x \geq m'$ for all $x \in A$ and $x \geq m'$ for all $x \in B$. Therefore m' is a lower bound of A and B, so $\inf A \geq m'$ and $\inf B \geq m'$, hence $m \geq m'$. Thus m is the greatest lower bound of $A \cup B$, i.e., $\inf(A \cup B) = m$. □

Exercise 11
Let

$$A = \left\{\frac{1}{m} + \frac{1}{n}, \quad \text{such that } m, n \in \mathbb{Z} \setminus \{0\}\right\}.$$

Prove that A admits supremum and infimum in $\mathbb{R}$ to be determined.

Solution Let $n \in \mathbb{Z} \setminus \{0\}$. If $n \geq 1$, then $0 \leq \frac{1}{n} \leq 1$, and if $n \leq -1$, then $-1 \leq \frac{1}{n} \leq 0$. In the two cases, we obtain:

$$-1 \leq \frac{1}{n} \leq 1.$$

So, for all $m, n \in \mathbb{Z} \setminus \{0\}$, we have:

$$-2 \leq \frac{1}{m} + \frac{1}{n} \leq 2.$$

So A is bounded from above by 2 and from below by -2. In the other side, as

$$2 = \frac{1}{1} + \frac{1}{1} \in A \quad \text{and} \quad -2 = \frac{1}{-1} + \frac{1}{-1} \in A,$$

then 2 (resp. -2) is the greatest (resp. least) element of A, i.e., $\max A = 2$ (resp. $\min A = -2$). Hence A admits a supremum and an infimum in $\mathbb{R}$ with

$$\sup A = \max A = 2 \quad \text{and} \quad \inf A = \min A = -2. \quad \square$$

Theorem 1.3.1 *(Supremum criterion)*
Let $M \in \mathbb{R}$ and A be a nonempty subset of $\mathbb{R}$. Then $M = \sup A$ if and only if the following two properties are satisfied:

i) M is an upper bound of A.

ii) For any $\varepsilon > 0$, there exists $a \in A$ such that

$$M - \varepsilon < a \leq M.$$

Proof
N.C. Since $M = \sup A$, then M is an upper bound of A. In the other side, suppose, by contradiction, that the property (ii) is not satisfied, then there exists $\varepsilon > 0$ such that $x \leq M - \varepsilon$ for all $x \in A$ (we have not $x > M$ since M is an upper bound of A). So $M - \varepsilon$ is an upper bound of A. As $M = \sup A$ is the least upper bound of A, then $M \leq M - \varepsilon$, therefore $\varepsilon \leq 0$, which is impossible. Hence the property (ii) is satisfied.
S.C. Let's show that M is the least upper bound of A. Indeed, let M' be another upper bound of A in $\mathbb{R}$. Suppose, by contradiction, that $M > M'$, then put $\varepsilon = M - M' > 0$. By the property (ii), there exists $a \in A$ such that $M - \varepsilon < a \leq M$, and therefore $M' < a$, which is impossible since M' is an upper bound of A, therefore $M \leq M'$. Hence M is the least upper bound of A, i.e., $M = \sup A$. $\square$

Exercise 12
Let

$$A = \left\{ \frac{n}{n+1},\ n \in \mathbb{N} \right\}.$$

1) **Show that A is bounded in $\mathbb{R}$.**

2) **Prove that $\sup A = 1$ and $\inf A = 0$.**

Solution For every $n \in \mathbb{N}$, put $u_n = \frac{n}{n+1}$. Then $A = \{u_n,\ n \in \mathbb{N}\}$.

1) For all $n \in \mathbb{N}$, $n \leq n + 1$, so $u_n = \frac{n}{n+1} \leq 1$. Hence A is bounded from above in $\mathbb{R}$ and 1 is an upper bound of A in $\mathbb{R}$. In the other side, for all $n \in \mathbb{N}$, $u_n \geq 0$. So A is bounded from below in $\mathbb{R}$ and 0 is a lower bound of A in $\mathbb{R}$. Thus A is bounded in $\mathbb{R}$.

2)
- For $\sup A$, we will use the supremum criterion. First, 1 is an upper bound of A in $\mathbb{R}$. Let $\varepsilon > 0$. Let's find $n_0 \in \mathbb{N}$ such that
$$1 - \varepsilon < u_{n_0} \leq 1. \qquad (*)$$
Since $u_{n_0} = \frac{n_0}{n_0+1} = 1 - \frac{1}{n_0+1}$, then the double inequality $(*)$ is equivalent to the following inequality:
$$\frac{1}{n_0+1} < \varepsilon.$$
This is equivalent to
$$n_0 > \frac{1}{\varepsilon} - 1.$$
Such n_0 exists since the set of natural numbers $\mathbb{N}$ is infinite. Hence $\sup A = 1$.
- Since 0 is a lower bound of A in $\mathbb{R}$ and $0 \in A$ (since $u_0 = 0$), then 0 is the least element of A, i.e., $\min A = 0$. Hence $\inf A = \min A = 0$. □

Theorem 1.3.2 *(Infimum criterion)*
Let $m \in \mathbb{R}$ and A be a nonempty subset of $\mathbb{R}$. Then $m = \inf A$ if and only if the following two properties are satisfied:

i) m is a lower bound of A.

ii) For any $\varepsilon > 0$, there exists $a \in A$ such that
$$m \leq a < m + \varepsilon.$$

Proof

N.C. Since $m = \inf A$, then m is a lower bound of A. In the other side, suppose, by contradiction, that the property (ii) is not satisfied, then there exists $\varepsilon > 0$ such that $x \geq m + \varepsilon$ for all $x \in A$ (we have not $x < m$ since m is a lower bound of A). So $m + \varepsilon$ is a lower bound of A. As $m = \inf A$ is the greatest lower bound of A, then $m \geq m + \varepsilon$, therefore $\varepsilon \leq 0$, which is impossible. Hence the property (ii) is satisfied.
S.C. Let's show that m is the greatest lower bound of A. Indeed, let m' be another lower bound of A in $\mathbb{R}$. Suppose, by contradiction, that $m < m'$, then put $\varepsilon = m' - m > 0$. By the property (ii), there exists $a \in A$ such that $m \leq a < m + \varepsilon$, and therefore $a < m'$, which is impossible since m' is a lower bound of A, therefore $m \geq m'$. Hence m is the greatest lower bound of A, i.e., $m = \inf A$. □

Exercise 13
Let $A =]0, 1]$. Prove that $\sup A = 1$ and $\inf A = 0$.

Solution

- For all $x \in A$, $x \leq 1$. So 1 is an upper bound of A in $\mathbb{R}$. In the other side, since $1 \in A$, then 1 is the greatest element of A i.e., $\max A = 1$. Hence $\sup A = \max A = 1$.

- For $\inf A$, we will use the infimum criterion. First, for all $x \in A$, $x \geq 0$ (since $x > 0$). So 0 is a lower bound of A in $\mathbb{R}$. Let $\varepsilon > 0$. Let's find $x_0 \in A$ such that
$$0 \leq x_0 < 0 + \varepsilon.$$
If $\varepsilon \leq 1$, then we take $x_0 = \frac{\varepsilon}{2} \leq \frac{1}{2} < 1$, so $x_0 \in A$.
If $\varepsilon > 1$, then we take $x_0 = \frac{1}{2} < 1$, so $x_0 \in A$.
Hence $\inf A = 0$. □

Remark 1.3.3 *We can generalize the exercise 13 in the following way:*

- *If A is one of the intervals*
$$[a, b], \quad [a, b[, \quad]a, b], \quad]a, b[\qquad (\textit{where } a < b),$$
then $\sup A = b$ and $\inf A = a$.

- *If A is one of the intervals $[a, +\infty[$ or $]a, +\infty[$, then A is not bounded from above in $\mathbb{R}$ and $\inf A = a$.*

- *If A is one of the intervals $]-\infty, b]$ or $]-\infty, b[$, then A is not bounded from below in $\mathbb{R}$ and $\sup A = b$.*

Theorem 1.3.3 *(Supremum criterion axiom)*

- *Every nonempty bounded from above part of $\mathbb{R}$ has a supremum.*

- *Every nonempty bounded from below part of $\mathbb{R}$ has an infimum.*

Exercise 14
Let $a \in \mathbb{Q}$ and $A = \{r \in \mathbb{Q},\ r < a\}$. Prove that A admits a supremum, but A has no greatest element.

Solution Since $a - 1 \in \mathbb{Q}$ and $a - 1 < a$, then $a - 1 \in A$, so $A \neq \varnothing$. Moreover, as $r \leq a$ for all $r \in A$, then A is bounded from above in $\mathbb{R}$ by a. Hence A is a nonempty bounded from above part of $\mathbb{R}$, so A admits a supremum. In the other side, if A has a greatest element b, then $b \in A$, so $b \in \mathbb{Q}$ and $b < a$. Put $r = \frac{a+b}{2} \in \mathbb{Q}$, then $r < a$, so $r \in A$. Hence $r \leq b$ since $b = \max A$, which is impossible since $b < r$. Thus A has no greatest element. □

Exercise 15

1) **Let A and B be two nonempty and bounded from above parts of $\mathbb{R}$. Let**
$$AB = \{ab, \text{ such that } a \in A, b \in B\}.$$
Show that if $A, B \subseteq\ ]0, +\infty[$, then AB is nonempty bounded from above and
$$\sup(AB) = (\sup A)(\sup B).$$

2) Give an example of two nonempty bounded from above parts A and B of $\mathbb{R}$ such that AB is not bounded from above.

Solution

1) Since A and B are two nonempty and bounded from above parts of $\mathbb{R}$, then $\sup A$ and $\sup B$ exist. As $A \neq \varnothing$ and $B \neq \varnothing$, then there exist $a \in A$ and $b \in B$, so $ab \in AB$. Hence $AB \neq \varnothing$. Let $x \in AB$, then there exist $a \in A$ and $b \in B$ such that $x = ab$. As $0 < a \leq \sup A$ and $0 < b \leq \sup B$, then $x = ab \leq (\sup A)(\sup B)$. So AB is bounded from above by $(\sup A)(\sup B)$, and then $\sup(AB)$ exists and
$$\sup(AB) \leq (\sup A)(\sup B).$$
In the other side, if $a \in A$ and $b \in B$, then $ab \in AB$, so $ab \leq \sup(AB)$. But $b > 0$, then
$$a \leq \frac{\sup(AB)}{b}, \quad \forall\, a \in A.$$
So $\frac{\sup(AB)}{b}$ is an upper bound of A. Therefore
$$\sup A \leq \frac{\sup(AB)}{b}.$$
But $b > 0$ and $\sup A > 0$, then
$$b \leq \frac{\sup(AB)}{\sup A}, \quad \forall\, b \in B.$$
So $\frac{\sup(AB)}{\sup A}$ is an upper bound of B. Therefore
$$\sup B \leq \frac{\sup(AB)}{\sup A}.$$
Hence $(\sup A)(\sup B) \leq \sup(AB)$. Thus $\sup(AB) = (\sup A)(\sup B)$.

2) Take, for example, $A = B =]-\infty, 1] \subseteq \mathbb{R}$, they are nonempty and bounded from above by 1. Let's show that $AB = \mathbb{R}$. Indeed, $AB \subseteq \mathbb{R}$. If $x \in \mathbb{R}$, then we consider the two cases:

- if $x \leq 1$, then $x = x \times 1 \in AB$ since $x \in A$ and $1 \in B$.
- if $x > 1$, then $\sqrt{x} > 1$, so $-\sqrt{x} < -1$. Therefore $-\sqrt{x} \in A = B$ and therefore
$$x = (-\sqrt{x})(-\sqrt{x}) \in AB.$$

Hence $\mathbb{R} \subseteq AB$. Thus $AB = \mathbb{R}$ is not bounded from above. □

Exercise 16
The goal of this exercise is to prove that the only intervals of $\mathbb{R}$ are the intervals of the form:
$$\varnothing, \quad \mathbb{R}, \quad [a, +\infty[, \quad]a, +\infty[, \quad]-\infty, b], \quad]-\infty, b[,$$
$$[a, b], \quad]a, b], \quad [a, b[\quad \text{and} \quad]a, b[.$$

1) **Show that if I is an interval of $\mathbb{R}$ such that I is not bounded from below, neither bounded from above in $\mathbb{R}$, then $I = \mathbb{R}$.**

2) **Show that if I is a nonempty interval of $\mathbb{R}$ such that I is bounded from below, but not bounded from above in $\mathbb{R}$, then there exists $a \in \mathbb{R}$ such that $I = [a, +\infty[$ or $I =]a, +\infty[$.**

3) **Show that if I is a nonempty interval of $\mathbb{R}$ such that I is bounded from above, but not bounded from below in $\mathbb{R}$, then there exists $b \in \mathbb{R}$ such that $I =]-\infty, b]$ or $I =]-\infty, b[$.**

4) **Show that if I is an interval of $\mathbb{R}$ such that I is bounded from above and bounded from below in $\mathbb{R}$, then there exist $a, b \in \mathbb{R}$ such that $I = [a, b]$ or $I =]a, b]$ or $I = [a, b[$ or $I =]a, b[$.**

Solution

1) Let I be an interval of $\mathbb{R}$ such that I is not bounded from below, neither bounded from above in $\mathbb{R}$. Let $x \in \mathbb{R}$. As x is not a lower bound (resp. upper bound) of I, then there exists $a \in I$ (resp. $b \in I$) such that $a \leq x$ (resp. $x \leq b$). As I is an interval of $\mathbb{R}$ and $a \leq x \leq b$ with $a, b \in I$, then $x \in I$. So $\mathbb{R} \subseteq I$, but $I \subseteq \mathbb{R}$, then $I = \mathbb{R}$.

2) Let I be a nonempty interval of $\mathbb{R}$ such that I is bounded from below, but not bounded from above in $\mathbb{R}$. Then I admits an infimum $a = \inf I$. As a is a lower bound of I, then $a \leq x$ for all $x \in I$, so $I \subseteq [a, +\infty[$.

   - Suppose that $a \in I$. Let $x \in [a, +\infty[$, then $x \geq a$. As x is not an upper bound of I (since I is not bounded from above), then there exists $b \in I$ such that $x \leq b$. As I is an interval of $\mathbb{R}$ and $a \leq x \leq b$ with $a, b \in I$, then $x \in I$. So $[a, +\infty[\subseteq I$. Hence $I = [a, +\infty[$.
   - Suppose that $a \notin I$, then $I \subseteq]a, +\infty[$. Conversely, let $x \in]a, +\infty[$, then $x > a$. As x is not a lower bound of I (since a is the greatest lower bound of I), then there exists $a' \in I$ such that $a' \leq x$. Moreover, as x is not an upper bound of I (since I is not bounded from above), then there exists $b \in I$ such that $x \leq b$. As I is an interval of $\mathbb{R}$ and $a' \leq x \leq b$ with $a', b \in I$, then $x \in I$. So $]a, +\infty[\subseteq I$. Hence $I =]a, +\infty[$.

3) Let I be a nonempty interval of $\mathbb{R}$ such that I is bounded from above, but not bounded from below in $\mathbb{R}$. Then I admits a supremum $b = \sup I$. As b is an upper bound of I, then $x \leq b$ for all $x \in I$, so $I \subseteq]-\infty, b]$.

   - Suppose that $b \in I$. Let $x \in]-\infty, b]$, then $x \leq b$. As x is not a lower bound of I (since I is not bounded from below), then there exists $a \in I$ such that $a \leq x$. As I is an interval of $\mathbb{R}$ and $a \leq x \leq b$ with $a, b \in I$, then $x \in I$. So $]-\infty, b] \subseteq I$. Hence $I =]-\infty, b]$.
   - Suppose that $b \notin I$, then $I \subseteq]-\infty, b[$. Conversely, let $x \in]-\infty, b[$, then $x < b$. As x is not an upper bound of I (since b is the least upper bound of I), then there exists $b' \in I$ such that $x \leq b'$. Moreover, as x is not a lower bound of I (since I is not bounded from below), then il existe $a \in I$ such that $a \leq x$. As I

is an interval of $\mathbb{R}$ and $a \leq x \leq b'$ with $a, b' \in I$, then $x \in I$. So $]-\infty, b[\subseteq I$. Hence $I =]-\infty, b[$.

4) Let I be an interval of $\mathbb{R}$ such that I is bounded from above and bounded from below in $\mathbb{R}$. Then I admits an infimum $a = \inf I$ and a supremum $b = \sup I$. As a is a lower bound and b is an upper bound of I, then $a \leq x \leq b$ for all $x \in I$, so $I \subseteq [a, b]$. In each of the following four cases, we apply the same technique of the parts (2) and (3):

- If $a, b \in I$, then we get $I = [a, b]$.
- If $a \notin I$ and $b \in I$, then we get $I =]a, b]$.
- If $a \in I$ and $b \notin I$, then we get $I = [a, b[$.
- If $a \notin I$ and $b \notin I$, then we get $I =]a, b[$. □

1.4 Archimedean principle

Proposition 1.4.1 *(Archimedean principle: $\mathbb{R}$ is Archimedean)*
For any $a > 0$ and $b \in \mathbb{R}$, there exists $n \in \mathbb{N}^*$ such that $na > b$.

Proof

- If $a \geq b$, then it is sufficient to take $n = 2$ since $2a > a \geq b$.
- Suppose that $a < b$. Suppose, by contradiction, that $na \leq b$ for all $n \in \mathbb{N}^*$. Then the set
$$A = \{na, \ n \in \mathbb{N}^*\}$$
is a nonempty bounded from above (by b) part of $\mathbb{R}$. So A admits a supremum. Put $M = \sup A$. By the supremum criterion, for $\varepsilon = a > 0$, there exists $n_0 \in \mathbb{N}^*$ such that
$$M - a < n_0 a \leq M.$$
So $M < (n_0 + 1)a$. Put $m_0 = n_0 + 1 \in \mathbb{N}^*$, then $M < m_0 a$ with $m_0 a \in A$, this is in contraddiction with the fact that M is an upper bound of A. Hence there exists $n \in \mathbb{N}^*$ such that $na > b$. □

Corollary 1.4.1 *For any $\varepsilon > 0$, there exists $n \in \mathbb{N}^*$ such that $\frac{1}{n} < \varepsilon$.*

Proof Let $\varepsilon > 0$. Put $a = \varepsilon > 0$ and $b = 1 \in \mathbb{R}$. By the Archimedean principle, there exists $n \in \mathbb{N}^*$ such that $na > b$, so $n\varepsilon > 1$. Hence $\frac{1}{n} < \varepsilon$. □

Exercise 17
Let a and b be two real numbers such that $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}^*$. Show that $a \leq b$.

Solution Let $\varepsilon > 0$. By the corollary 1.4.1, there exists $n \in \mathbb{N}^*$ such that $\frac{1}{n} < \varepsilon$. So

$$a \le b + \frac{1}{n} \le b + \varepsilon.$$

Hence $a \le b$ by the exercise 9. □

Exercise 18
Show that the set $\mathbb{Z}$ is not a bounded from above subset of $\mathbb{R}$.

Solution Suppose that $\mathbb{Z}$ is a bounded from above subset of $\mathbb{R}$. As $\mathbb{R}$ is complete, then $\mathbb{Z}$ has a supremum. Let $M = \sup \mathbb{Z}$. Put $\varepsilon = 1 > 0$, then, by the supremum criterion, there exists $n \in \mathbb{Z}$ such that

$$M - 1 < n \le M.$$

So $M < n + 1$. But $n + 1 \in \mathbb{Z}$, then $n + 1 \le M$ (since M is an upper bound of $\mathbb{Z}$). Therefore $M < M$, which is impossible. Thus $\mathbb{Z}$ is not bounded from above in $\mathbb{R}$. □

Exercise 19

1) **Consider the set $A = \left\{ \frac{2}{n},\ n \in \mathbb{N}^* \right\}$.**
Prove that $\sup A = 2$ and $\inf A = 0$.

2) **Consider the set $B = \left\{ \frac{1}{n} + (-1)^n,\ n \in \mathbb{N}^* \right\}$.**
Prove that $\sup B = \frac{3}{2}$ and $\inf B = -1$.

Solution

1) For every $n \in \mathbb{N}^*$, put $u_n = \frac{2}{n}$, then $A = \{u_n,\ n \in \mathbb{N}^*\}$

- Let $n \in \mathbb{N}^*$. If $n = 1$, then $u_1 = 2 \le 2$. If $n \ge 2$, then $u_n = \frac{2}{n} \le 1 \le 2$. So 2 is an upper bound of A. Moreover, as $u_1 = 2$, then $2 \in A$. Hence 2 is the greatest element of A i.e., $\max A = 2$. Hence $\sup A = \max A = 2$.
- For all $n \in \mathbb{N}^*$, $u_n = \frac{2}{n} \ge 0$. So 0 is a lower bound of A. In the other side, let $\varepsilon > 0$, let's find $n_0 \in \mathbb{N}^*$ such that

$$0 \le u_{n_0} < 0 + \varepsilon.$$

  Since $\frac{\varepsilon}{2} > 0$, then, by the Archimedean principle, there exists $n_0 \in \mathbb{N}^*$ such that $\frac{1}{n_0} < \frac{\varepsilon}{2}$, so $\frac{2}{n_0} < \varepsilon$, i.e., $u_{n_0} < \varepsilon$. Hence $\inf A = 0$ by the infimum criterion.

2) For every $n \in \mathbb{N}^*$, put $v_n = \frac{1}{n} + (-1)^n$, then $B = \{v_n,\ n \in \mathbb{N}^*\}$.

- Let $n \in \mathbb{N}^*$. If n is even, then $n \ge 2$ and $(-1)^n = 1$, so

$$v_n = \frac{1}{n} + 1 \le \frac{1}{2} + 1 = \frac{3}{2}.$$

If n is odd, $(-1)^n = -1$, so

$$v_n = \frac{1}{n} - 1 \leq 0 \leq \frac{3}{2}.$$

Hence $\frac{3}{2}$ is an upper bound of B. Moreover, as $v_2 = \frac{1}{2} + 1 = \frac{3}{2}$, then $\frac{3}{2} \in B$. Hence $\frac{3}{2}$ is the greatest element of B i.e., $\max B = \frac{3}{2}$. Hence $\sup B = \max B = \frac{3}{2}$.

- For any $n \in \mathbb{N}^*$, as $(-1)^n \geq -1$, then

$$v_n = \frac{1}{n} + (-1)^n \geq \frac{1}{n} - 1 \geq -1.$$

So -1 is a lower bound of B. In the other side, let $\varepsilon > 0$, let's find $n_0 \in \mathbb{N}^*$ such that

$$-1 \leq v_{n_0} < -1 + \varepsilon.$$

By the Archimedean principle, there exists $n_1 \in \mathbb{N}^*$ such that $\frac{1}{n_1} < \varepsilon$.
If n_1 is odd, then we take $n_0 = n_1$, so

$$v_{n_0} = v_{n_1} = \frac{1}{n_1} + (-1)^{n_1} = \frac{1}{n_1} - 1 < -1 + \varepsilon.$$

If n_1 is even, then we take $n_0 = n_1 + 1 \in \mathbb{N}^*$ is odd, so $\frac{1}{n_0} = \frac{1}{n_1+1} < \frac{1}{n_1}$ and

$$v_{n_0} = \frac{1}{n_0} + (-1)^{n_0} = \frac{1}{n_0} - 1 < \frac{1}{n_1} - 1 < -1 + \varepsilon.$$

Hence $\inf B = -1$ by the infimum criterion. □

Exercise 20
Consider the set

$$A = \left\{ u_n, \text{ such that } u_n = \frac{3n^2+4}{n^2+1} \text{ and } n \in \mathbb{N} \right\}.$$

1) Show that $3 \leq u_n \leq 4$ for all $n \in \mathbb{N}$.

2) Determine the supremum and the infimum of A.

Solution

1) Let $n \in \mathbb{N}$. We have:

$$u_n = \frac{3n^2+4}{n^2+1} = \frac{3(n^2+1)+1}{n^2+1} = 3 + \frac{1}{n^2+1}.$$

Since $\frac{1}{n^2+1} \geq 0$, then $u_n \geq 3$.
Since $\frac{1}{n^2+1} \leq 1$, then $u_n \leq 4$.

2) By the part (1), the nonempty subset A of $\mathbb{R}$ is bounded from above by 4 and bounded from below by 3, so $\sup A$ and $\inf A$ exist in $\mathbb{R}$.

- Since $u_0 = 4$, then $4 \in A$, so 4 is the greatest element of A, i.e., $\max A = 4$. Hence $\sup A = \max A = 4$.
- Let $\varepsilon > 0$. Let's find $n_0 \in \mathbb{N}$ such that
$$3 \leq u_{n_0} < 3 + \varepsilon.$$
Indeed, by the Archimedean principle, there exists $n_0 \in \mathbb{N}^*$ such that $\frac{1}{n_0} < \varepsilon$. Since $n_0^2 + 1 > n_0^2 \geq n_0$, then $\frac{1}{n_0^2+1} < \frac{1}{n_0}$, and therefore
$$u_{n_0} = 3 + \frac{1}{n_0^2 + 1} < 3 + \frac{1}{n_0} < 3 + \varepsilon.$$
Hence $\inf A = 3$ by the infimum criterion. □

Exercise 21
Consider the set
$$A = \left\{ u_n, \text{ such that } u_n = \frac{2 + (-1)^n}{n} \text{ and } n \in \mathbb{N}^* \right\}.$$

1) **Show that A is bounded in $\mathbb{R}$.**

2) **Determine $\inf A$ and $\sup A$.**

Solution

1) Let $n \in \mathbb{N}^*$. Since $-1 \leq (-1)^n \leq 1$, then $1 \leq 2 + (-1)^n \leq 3$, so
$$0 \leq \frac{1}{n} \leq \frac{2 + (-1)^n}{n} \leq \frac{3}{n}. \qquad (*)$$
If $n \geq 2$, then $\frac{3}{n} \leq \frac{3}{2}$, so $0 \leq u_n \leq \frac{3}{2}$.
If $n = 1$, then $u_1 = 1$, so $0 \leq u_1 \leq \frac{3}{2}$.
Hence A is bounded from above by $\frac{3}{2}$ and bounded from below by 0. Thus A is bounded in $\mathbb{R}$.

2) Since A is a nonempty bounded subset of $\mathbb{R}$, then $\sup A$ and $\inf A$ exist in $\mathbb{R}$.

- Since $u_2 = \frac{3}{2}$, then $\frac{3}{2} \in A$. So $\frac{3}{2}$ is the greatest element of A, i.e., $\max A = \frac{3}{2}$. Hence $\sup A = \max A = \frac{3}{2}$.
- Since 0 is a lower bound of A, then $\inf A \geq 0$. Suppose that $\inf A \neq 0$. By the Archimedean principle, for $\varepsilon = \frac{\inf A}{3} > 0$, there exists $n_0 \in \mathbb{N}^*$ such that $\frac{1}{n_0} < \frac{\inf A}{3}$. By using $(*)$, we obtain:
$$u_{n_0} \leq \frac{3}{n_0} < \inf A,$$
which is impossible. Hence $\inf A = 0$. □

Exercise 22
Let A and B be two nonempty bounded subsets of $\mathbb{R}$. Set:

$$A + B = \Big\{a + b, \text{ such that } a \in A \text{ and } b \in B\Big\}.$$

1) **Show that $A + B$ is a nonempty bounded subset of $\mathbb{R}$, and that**

$$\sup(A + B) = \sup A + \sup B \quad \text{and} \quad \inf(A + B) = \inf A + \inf B$$

2) **Let**

$$E = \left\{\frac{2}{m} + \frac{2}{n}, \text{ such that } m, n \in \mathbb{N}^*\right\}.$$

Prove that E is a nonempty bounded subset of $\mathbb{R}$, and determine $\sup E$ and $\inf E$.

Solution

1) Since A and B are two nonempty bounded subsets of $\mathbb{R}$, then $\sup A, \sup B, \inf A$ and $\inf B$ exist. Since $A \neq \varnothing$ and $B \neq \varnothing$, then there exist $a_0 \in A$ et $b_0 \in B$, so $x_0 = a_0 + b_0 \in A + B$. Hence $A + B \neq \varnothing$. In the other side, let $x \in A + B$, then there exist $a \in A$ and $b \in B$ such that $x = a + b$. Since $\inf A \leq a \leq \sup A$ and $\inf B \leq b \leq \sup B$, then

$$\inf A + \inf B \leq x \leq \sup A + \sup B.$$

So $A + B$ is bounded from above by $\sup A + \sup B$ and bounded from below by $\inf A + \inf B$. Hence $A + B$ is a nonempty bounded subset of $\mathbb{R}$. Then $\sup(A + B)$ and $\inf(A + B)$ exist and

$$\sup(A + B) \leq \sup A + \sup B \quad \text{and} \quad \inf A + \inf B \leq \inf(A + B).$$

In order to prove the equalities, we distinguish the following two methods:
1^{st} method:

- Let $b \in B$. For any $a \in A$, $a + b \in A + B$, so $a + b \leq \sup(A + B)$, and then

$$a \leq \sup(A + B) - b.$$

So $\sup(A + B) - b$ is an upper bound of A, and then

$$\sup A \leq \sup(A + B) - b$$

since $\sup A$ is the least upper bound of A. So, for all $b \in B$,

$$b \leq \sup(A + B) - \sup A.$$

Then $\sup(A + B) - \sup A$ is an upper bound of B. Hence

$$\sup B \leq \sup(A + B) - \sup A$$

since $\sup B$ is the least upper bound of B. Hence

$$\sup A + \sup B \leq \sup(A + B).$$

Thus $\sup(A + B) = \sup A + \sup B$.

- Let $b \in B$. For any $a \in A$, $a + b \in A + B$, so $a + b \geq \inf(A + B)$, and then
$$a \geq \inf(A + B) - b.$$
So $\inf(A + B) - b$ is a lower bound of A, and then
$$\inf A \geq \inf(A + B) - b$$
since $\inf A$ is the greatest lower bound of A. So, for all $b \in B$,
$$b \geq \inf(A + B) - \inf A.$$
Hence $\inf(A + B) - \inf A$ is a lower bound of B, and then
$$\inf B \geq \inf(A + B) - \inf A$$
since $\inf B$ is the greatest lower bound of B. Hence
$$\inf A + \inf B \geq \inf(A + B).$$
Thus $\inf(A + B) = \inf A + \inf B$.

2^{nd} method: We will use the supremum and infimum criteria. Indeed,

- Let $\varepsilon > 0$. Since $\frac{\varepsilon}{2} > 0$, then, by the supremum criterion, there exist $x \in A$ and $y \in B$ such that
$$\sup A - \frac{\varepsilon}{2} < x \leq \sup A \quad \text{and} \quad \sup B - \frac{\varepsilon}{2} < y \leq \sup B.$$
Taking the sum of these two double inequalities, we obtain $x + y \in A + B$ with
$$(\sup A + \sup B) - \varepsilon < x + y \leq (\sup A + \sup B).$$
Hence $\sup(A + B) = \sup A + \sup B$ by the supremum criterion.
- Let $\varepsilon > 0$. Since $\frac{\varepsilon}{2} > 0$, then, by the infimum criterion, there exist $x \in A$ and $y \in B$ such that
$$\inf A \leq x < \inf A + \frac{\varepsilon}{2} \quad \text{and} \quad \inf B \leq y < \inf B + \frac{\varepsilon}{2}.$$
Taking the sum of these two double inequalities, we obtain $x + y \in A + B$ with
$$(\inf A + \inf B) \leq x + y < (\inf A + \inf B) + \varepsilon.$$
Hence $\inf(A + B) = \inf A + \inf B$ by the infimum criterion.

2) $E = A + A$ where
$$A = \left\{ \frac{2}{n}, \text{ such that } n \in \mathbb{N}^* \right\}.$$
By the part (1) of the exercise 19, the set A is nonempty bounded in $\mathbb{R}$ with $\sup A = 2$ and $\inf A = 0$. By the part (1) above, E is nonempty bounded in $\mathbb{R}$ with
$$\sup E = \sup A + \sup A = 4 \quad \text{and} \quad \inf E = \inf A + \inf A = 0. \quad \square$$

1.5 The integer part function

Definition 1.5.1 *For every $x \in \mathbb{R}$, denote $E(x)$ (or $\lfloor x \rfloor$), the unique integer $m \in \mathbb{Z}$ such that $m \leq x \leq m+1$, and call it the integer part of x. The function E is called the integer part function.*

For example, $E(1,3) = 1$, $E(3) = 3$, $E(\pi) = 3$, $E(-2,6) = -3$.

Proposition 1.5.1

1) *For any $x \in \mathbb{R}$, $E(x) \leq x < E(x) + 1$.*

2) *For any $x \in \mathbb{R}$,*
$$E(x) = \max\Big\{m \in \mathbb{Z},\ m \leq x\Big\}.$$

3) *$E(x) = x$ if and only if $x \in \mathbb{Z}$.*

4) *$E(E(x)) = E(x)$.*

5) *The function E is increasing on $\mathbb{R}$, i.e. if $x, y \in \mathbb{R}$ such that $x \leq y$, then $E(x) \leq E(y)$.*

6) *If $k \in \mathbb{Z}$ and $x \in \mathbb{R}$ such that $k \leq x$, then $k \leq E(x)$.*

7) *$E(x) \geq 0$ if and only if $x \geq 0$.*

8) *For any $k \in \mathbb{Z}$ and $x \in \mathbb{R}$, $E(x+k) = E(x) + k$.*

Exercise 23
By using the integer part function, give another proof of the Archimedean principle (Proposition 1.4.1).

Solution Let $a > 0$ and $b \in \mathbb{R}$.

- If $b < 0$, then $a > b$, so it is sufficient to take $n = 1 \in \mathbb{N}^*$.
- If $b \geq 0$, then put $n = E\left(\frac{b}{a}\right) + 1$, so $n \in \mathbb{N}^*$ since $E\left(\frac{b}{a}\right) \geq 0$. Since $\frac{b}{a} < E\left(\frac{b}{a}\right) + 1$, then $\frac{b}{a} < n$, so $na > b$ (since $a > 0$). □

Exercise 24
For every $x \in \mathbb{R}$, put:

$$f(x) = E\left(\frac{1}{x^2+1}\right), \quad g(x) = E\left(\frac{x}{x^2+1}\right) \quad \text{and} \quad s(x) = g(x) - g(-x).$$

1) **Calculate $f(x)$ for all $x \in \mathbb{R}$.**

2) **Calculate $g(x)$ for all $x \in \mathbb{R}$.**
(Hint: we may use the signs of the trinomials $x^2 - x + 1$ and $x^2 + x + 1$).

3) Deduce $s(x)$ for all $x \in \mathbb{R}$.

4) Deduce that, for all $x \in \mathbb{R}$,

$$|x| = x\left[E\left(\frac{x}{x^2+1}\right) - E\left(\frac{-x}{x^2+1}\right)\right].$$

Solution

1) Let $x \in \mathbb{R}$. If $x = 0$, then $f(0) = E(1) = 1$. If $x \neq 0$, then $0 < \frac{1}{x^2+1} < 1$, so $f(x) = E\left(\frac{1}{x^2+1}\right) = 0$.

2) Since the common discriminant of the trinomials $x^2 - x + 1$ and $x^2 + x + 1$ is $\Delta = -3 < 0$ and the common coefficient of x^2 is $1 > 0$, then $x^2 - x + 1 > 0$ and $x^2 + x + 1 > 0$ for all $x \in \mathbb{R}$. So $x < x^2 + 1$ and $-x < x^2 + 1$ for all $x \in \mathbb{R}$. Hence

$$-1 < \frac{x}{x^2+1} < 1, \quad \forall x \in \mathbb{R}.$$

- If $x \geq 0$, then $0 \leq \dfrac{x}{x^2+1} < 1$, so $g(x) = E\left(\frac{x}{x^2+1}\right) = 0$.
- If $x < 0$, then $-1 < \dfrac{x}{x^2+1} < 0$, so $g(x) = E\left(\frac{x}{x^2+1}\right) = -1$.

3) Let $x \in \mathbb{R}$.

- If $x = 0$, then $s(x) = g(0) - g(0) = 0$.
- If $x > 0$, then $s(x) = g(x) - g(-x) = 0 - (-1) = 1$.
- If $x < 0$, then $s(x) = g(x) - g(-x) = -1 - 0 = -1$.

4) Let $x \in \mathbb{R}$. By the part (3), $s(x)$ is the sign of x. So

$$|x| = xS(x) = x\left[E\left(\frac{x}{x^2+1}\right) - E\left(\frac{-x}{x^2+1}\right)\right]. \quad \square$$

Exercise 25
Let $n \in \mathbb{N}^*$ and $x \in \mathbb{R}$.

1) Show that $0 \leq E(nx) - nE(x) \leq n - 1$.

2) Deduce that

$$E\left(\frac{E(nx)}{n}\right) = E(x).$$

Solution

1) Since $E(x) \leq x < E(x) + 1$, then, multiplying by n, we obtain:

$$nE(x) \leq nx < nE(x) + n. \quad (*)$$

Since E is an increasing function on $\mathbb{R}$, then

$$E\Big(nE(x)\Big) \leq E(nx).$$

As $nE(x) \in \mathbb{Z}$, then $E\Big(nE(x)\Big) = nE(x)$. So $nE(x) \leq E(nx)$. Hence

$$0 \leq E(nx) - nE(x).$$

In the other side, as $E(nx) \leq nx$ and $nx < nE(x) + n$ (by $(*)$), then

$$E(nx) < nE(x) + n.$$

So $E(nx) - nE(x) < n$. But $E(nx) - nE(x) \in \mathbb{Z}$, then

$$E(nx) - nE(x) \leq n - 1.$$

2) Dividing the double inequality in the part (1) by n, we obtain:

$$0 \leq \frac{E(nx)}{n} - E(x) \leq \frac{n-1}{n} < 1.$$

So

$$E\left(\frac{E(nx)}{n} - E(x)\right) = 0.$$

As $E(x) \in \mathbb{Z}$, then

$$E\left(\frac{E(nx)}{n}\right) - E(x) = 0, \quad \text{and therefore} \quad E\left(\frac{E(nx)}{n}\right) = E(x). \quad \square$$

Exercise 26

1) Let $\lambda \in \mathbb{R}$ and A be a nonempty bounded subset of $\mathbb{R}$. Consider the set:

$$\lambda A = \{\lambda x,\ x \in A\}.$$

a) Show that

$$\sup(\lambda A) = \begin{cases} \lambda \sup A & \text{if } \lambda > 0 \\ \lambda \inf A & \text{if } \lambda < 0 \end{cases}$$

b) Deduce that

$$\inf(\lambda A) = \begin{cases} \lambda \inf A & \text{if } \lambda > 0 \\ \lambda \sup A & \text{if } \lambda < 0 \end{cases}$$

2) Consider the sets

$$A = \left\{\frac{1}{2^n},\ n \in \mathbb{N}\right\} \quad \text{and} \quad B = \left\{\frac{\sin k}{2^n},\ n \in \mathbb{N}\right\},\ \text{where } k \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right].$$

Find $\sup A$, $\inf A$, $\sup B$ and $\inf B$.

Solution

1) a) Since $\boldsymbol{A}$ is a nonempty bounded subset of $\mathbb{R}$, then $\sup \boldsymbol{A}$ and $\inf \boldsymbol{A}$ exist. As $\boldsymbol{A} \neq \varnothing$, then $\exists\, \boldsymbol{x_0} \in \boldsymbol{A}$, so $\boldsymbol{\lambda x_0} \in \boldsymbol{\lambda A}$. Hence $\boldsymbol{\lambda A} \neq \varnothing$.
Case $\boldsymbol{\lambda > 0}$: Let $\boldsymbol{y} \in \boldsymbol{\lambda A}$, then $\exists\, \boldsymbol{x} \in \boldsymbol{A}$ such that $\boldsymbol{y} = \boldsymbol{\lambda x}$. As $\boldsymbol{x} \leq \sup \boldsymbol{A}$ and $\boldsymbol{\lambda > 0}$, then

$$y = \lambda x \leq \lambda \sup A.$$

So $\lambda \sup A$ is an upper bound of λA. Hence λA is a nonempty and bounded from above subset of $\mathbb{R}$, so $\sup(\lambda A)$ exists and

$$\sup(\lambda A) \leq \lambda \sup A.$$

In order to prove the equality, we distinguish the following two methods:

- 1st method: Let $x \in A$, then $\lambda x \in \lambda A$, so $\lambda x \leq \sup(\lambda A)$. But $\lambda > 0$, then
$$x \leq \frac{\sup(\lambda A)}{\lambda}.$$
So $\frac{\sup(\lambda A)}{\lambda}$ is an upper bound of A. Hence
$$\sup A \leq \frac{\sup(\lambda A)}{\lambda}, \qquad \text{and therefore} \qquad \lambda \sup A \leq \sup(\lambda A)$$
Thus $\sup(\lambda A) = \lambda \sup A$.
- 2nd method: Put $M = \lambda \sup A$. This method uses the supremum criterion. Indeed, let $\varepsilon > 0$, let's find $y \in \lambda A$ such that
$$M - \varepsilon < y \leq M.$$
Since $\frac{\varepsilon}{\lambda} > 0$, then, by the supremum criterion, there exists $x \in A$ such that
$$\sup A - \frac{\varepsilon}{\lambda} < x \leq \sup A.$$
Multiplying this double inequality by $\lambda > 0$ and by taking $y = \lambda x \in \lambda A$, we obtain $M - \varepsilon < y \leq M$. Thus $\sup(\lambda A) = M = \lambda \sup A$.

Case $\lambda < 0$: Let $y \in \lambda A$, then there exists $x \in A$ such that $y = \lambda x$. As $x \geq \inf A$ and $\lambda < 0$, then

$$y = \lambda x \leq \lambda \inf A.$$

So $\lambda \inf A$ is an upper bound of λA. Hence λA is a nonempty and bounded from above subset of $\mathbb{R}$, so $\sup(\lambda A)$ exists and

$$\sup(\lambda A) \leq \lambda \inf A.$$

In order to prove the equality, we distinguish the following two methods:

- 1st method: Let $\boldsymbol{x} \in \boldsymbol{A}$, then $\boldsymbol{\lambda x} \in \boldsymbol{\lambda A}$, so $\boldsymbol{\lambda x} \leq \mathbf{sup}(\boldsymbol{\lambda A})$. But $\boldsymbol{\lambda} < \mathbf{0}$, then
$$\boldsymbol{x} \geq \frac{\mathbf{sup}(\boldsymbol{\lambda A})}{\boldsymbol{\lambda}}.$$
So $\frac{\mathbf{sup}(\boldsymbol{\lambda A})}{\boldsymbol{\lambda}}$ is a lower bound of $\boldsymbol{A}$. Hence
$$\mathbf{inf}\,\boldsymbol{A} \geq \frac{\mathbf{sup}(\boldsymbol{\lambda A})}{\boldsymbol{\lambda}}, \qquad \text{and therefore} \qquad \boldsymbol{\lambda}\,\mathbf{inf}\,\boldsymbol{A} \leq \mathbf{sup}(\boldsymbol{\lambda A})$$
Thus $\mathbf{sup}(\boldsymbol{\lambda A}) = \boldsymbol{\lambda}\,\mathbf{inf}\,\boldsymbol{A}$.
- 2nd method: Put $\boldsymbol{M} = \boldsymbol{\lambda}\,\mathbf{inf}\,\boldsymbol{A}$. This method uses the supremum criterion. Indeed, let $\boldsymbol{\varepsilon} > \mathbf{0}$, let's find $\boldsymbol{y} \in \boldsymbol{\lambda A}$ such that
$$\boldsymbol{M} - \boldsymbol{\varepsilon} < \boldsymbol{y} \leq \boldsymbol{M}.$$
Since $-\frac{\boldsymbol{\varepsilon}}{\boldsymbol{\lambda}} > \mathbf{0}$, by the infimum criterion, there exists $\boldsymbol{x} \in \boldsymbol{A}$ such that
$$\mathbf{inf}\,\boldsymbol{A} \leq \boldsymbol{x} < \mathbf{inf}\,\boldsymbol{A} - \frac{\boldsymbol{\varepsilon}}{\boldsymbol{\lambda}}.$$
Multiplying this double inequality by $\boldsymbol{\lambda} < \mathbf{0}$ and by taking $\boldsymbol{y} = \boldsymbol{\lambda x} \in \boldsymbol{\lambda A}$, we obtain $\boldsymbol{M} - \boldsymbol{\varepsilon} < \boldsymbol{y} \leq \boldsymbol{M}$. Thus $\mathbf{sup}(\boldsymbol{\lambda A}) = \boldsymbol{M} = \boldsymbol{\lambda}\,\mathbf{inf}\,\boldsymbol{A}$.

b) Let $\boldsymbol{B} = \boldsymbol{\lambda A}$.
Case $\boldsymbol{\lambda} > \mathbf{0}$: Put $\boldsymbol{\beta} = -\frac{\mathbf{1}}{\boldsymbol{\lambda}} < \mathbf{0}$. By the part (a), $\mathbf{sup}(\boldsymbol{\beta B}) = \boldsymbol{\beta}\,\mathbf{inf}\,\boldsymbol{B}$ and $\mathbf{sup}(-\boldsymbol{A}) = -\,\mathbf{inf}\,\boldsymbol{A}$, so
$$\mathbf{inf}(\boldsymbol{\lambda A}) = \mathbf{inf}\,\boldsymbol{B} = \frac{\mathbf{1}}{\boldsymbol{\beta}}\mathbf{sup}(\boldsymbol{\beta B}) = -\boldsymbol{\lambda}\,\mathbf{sup}(-\boldsymbol{A}) = \boldsymbol{\lambda}\,\mathbf{inf}\,\boldsymbol{A}.$$
Case $\boldsymbol{\lambda} < \mathbf{0}$: Put $\boldsymbol{\beta} = \frac{\mathbf{1}}{\boldsymbol{\lambda}} < \mathbf{0}$. By the part (a), $\mathbf{sup}(\boldsymbol{\beta B}) = \boldsymbol{\beta}\,\mathbf{inf}\,\boldsymbol{B}$, so
$$\mathbf{inf}(\boldsymbol{\lambda A}) = \mathbf{inf}\,\boldsymbol{B} = \frac{\mathbf{1}}{\boldsymbol{\beta}}\mathbf{sup}(\boldsymbol{\beta B}) = \boldsymbol{\lambda}\,\mathbf{sup}\,\boldsymbol{A}.$$

2) For all $\boldsymbol{n} \in \mathbb{N}$, $\frac{\mathbf{1}}{\mathbf{2}^{\boldsymbol{n}}} \leq \mathbf{1}$. Then $\mathbf{1}$ is an upper bound de $\boldsymbol{A}$. Moreover, as $\mathbf{1} = \frac{\mathbf{1}}{\mathbf{2}^{\mathbf{0}}} \in \boldsymbol{A}$, then $\mathbf{1}$ is the greatest element of $\boldsymbol{A}$, so $\mathbf{sup}\,\boldsymbol{A} = \mathbf{max}\,\boldsymbol{A} = \mathbf{1}$. In the other side, for all $\boldsymbol{n} \in \mathbb{N}$, $\frac{\mathbf{1}}{\mathbf{2}^{\boldsymbol{n}}} \geq \mathbf{0}$, so $\mathbf{0}$ is a lower bound of $\boldsymbol{A}$. Hence $\boldsymbol{A}$ is a nonempty and bounded from below subset of $\mathbb{R}$, so $\mathbf{inf}\,\boldsymbol{A}$ exists and $\mathbf{inf}\,\boldsymbol{A} \geq \mathbf{0}$. In order to prove the equality, we distinguish the following two methods:

- 1st method: Suppose that $\mathbf{inf}\,\boldsymbol{A} \neq \mathbf{0}$, then $\mathbf{inf}\,\boldsymbol{A} > \mathbf{0}$. By the Archimedean principle, for $\boldsymbol{\varepsilon} = \mathbf{inf}\,\boldsymbol{A} > \mathbf{0}$, there exists $\boldsymbol{n_0} \in \mathbb{N}^*$ such that $\frac{\mathbf{1}}{\boldsymbol{n_0}} < \mathbf{inf}\,\boldsymbol{A}$. As $\mathbf{2}^{\boldsymbol{n_0}} > \boldsymbol{n_0}$, then
$$\frac{\mathbf{1}}{\mathbf{2}^{\boldsymbol{n_0}}} < \frac{\mathbf{1}}{\boldsymbol{n_0}} < \mathbf{inf}\,\boldsymbol{A} \quad \text{with} \quad \frac{\mathbf{1}}{\mathbf{2}^{\boldsymbol{n_0}}} \in \boldsymbol{A},$$
which is impossible. Hence $\mathbf{inf}\,\boldsymbol{A} = \mathbf{0}$.

- $\mathbf{2^{nd}}$ method: This method uses the infimum criterion. Indeed, let $\varepsilon > 0$, let's find $n_0 \in \mathbb{N}$ such that $0 \leq \frac{1}{2^{n_0}} < 0+\varepsilon$, i.e., $2^{n_0} > \frac{1}{\varepsilon}$ (or $n_0 > -\frac{\ln \varepsilon}{\ln 2}$). It is sufficient to take
$$n_0 = \max\left\{0, E\left(-\frac{\ln \varepsilon}{\ln 2}\right) + 1\right\}.$$
Hence $\inf A = 0$.

In the other side, let $\lambda = \sin k$. Then $B = \lambda A$.

- If $k = 0$, then $\lambda = 0$, so $B = \{0\}$, and therefore $\sup B = \inf B = 0$.
- If $k \in \left]0, \frac{\pi}{2}\right]$, then $\lambda > 0$, so, by the part (1),
$$\sup B = \lambda \sup A = \lambda \times 1 = \sin k \quad \text{and} \quad \inf B = \lambda \inf A = \lambda \times 0 = 0.$$
- If $k \in \left[\frac{-\pi}{2}, 0\right[$, then $\lambda < 0$, so, by the part (1),
$$\sup B = \lambda \inf A = \lambda \times 0 = 0 \quad \text{and} \quad \inf B = \lambda \sup A = \lambda \times 1 = \sin k. \quad \square$$

1.6 Density

Theorem 1.6.1 *($\mathbb{Q}$ is dense in $\mathbb{R}$)*
Between any two real numbers, there exists always a rational number. In other words, for any $x, y \in \mathbb{R}$ such that $x < y$, there exists $r \in \mathbb{Q}$ such that $x < r < y$.

Proof Let $x, y \in \mathbb{R}$ such that $x < y$, then $y - x > 0$, then there exists $n_0 \in \mathbb{N}^*$ such that $0 < \frac{1}{n_0} < y - x$ by the corollary 1.4.1.
1^{st} method: By the Archimedean principle, there exists $n_1 \in \mathbb{N}^*$ such that $n_1 \left(\frac{1}{n_0}\right) \geq y$. Let n_1 be the smallest integer such that $\frac{n_1}{n_0} \geq y$. Put $r = \frac{n_1 - 1}{n_0} \in \mathbb{Q}$, then $r < y$ (by the choice of n_1). In the other side, if $x \geq r$, then $-x \leq -r$, so
$$y - x \leq y - r = \left(y - \frac{n_1}{n_0}\right) + \frac{1}{n_0} \leq \frac{1}{n_0} < y - x,$$
by using the fact that $y - \frac{n_1}{n_0} \leq 0$, which is impossible. Hence $x < r$, and therefore $x < r < y$.
2^{nd} method: Put $n_1 = E(n_0 x)$, then
$$n_1 \leq n_0 x < n_1 + 1.$$
So
$$\frac{n_1}{n_0} \leq x < \frac{n_1 + 1}{n_0} = \frac{n_1}{n_0} + \frac{1}{n_0} < x + (y - x) = y.$$
Put $r = \frac{n_1+1}{n_0} \in \mathbb{Q}$, then $x < r < y$. $\square$

Exercise 27

1) **Let A be a subset of $\mathbb{Z}$ having a supremum. Put $M = \sup A$.**
   (a) **Show that $M \in \mathbb{Z}$.**
   (b) **Deduce that $M \in A$ and that $\max A = \sup A$.**

2) **Let $a, b \in \mathbb{R}$ such that $a - b > 1$. Show that there exists $k \in \mathbb{Z}$ such that $b < k < a$. (Hint: we may use the set $A = \{n \in \mathbb{Z},\ n < a\}$).**

3) **Deduce that between any two real numbers, there exists always a rational number (Theorem 1.6.1).**

Solution

1) (a) Suppose that $M \notin \mathbb{Z}$, then $E(M) < M$. Put $\varepsilon = M - E(M) > 0$. By the supremum criterion, there exists $a \in A \subseteq \mathbb{Z}$ such that

$$M - \varepsilon < a \leq M \quad \text{i.e.,} \quad E(M) < a \leq M,$$

which is impossible since $a \in \mathbb{Z}$. Hence $M \in \mathbb{Z}$.

   (b) Suppose that $M \notin A$, then $x < M$ for all $x \in A$. As $M \in \mathbb{Z}$ and $A \subseteq \mathbb{Z}$, then $x \leq M - 1$ for all $x \in A$, so $M - 1$ is an upper bound of A in $\mathbb{R}$. Hence $\sup A \leq M - 1$, i.e., $M \leq M - 1$, which is impossible. Hence $M \in A$. Consequently, $\sup A \in A$, and therefore A admits a greatest element with $\max A = \sup A$.

2) Let $a, b \in \mathbb{R}$ such that $a - b > 1$. Consider the set $A = \{n \in \mathbb{Z},\ n < a\}$. Since A is a nonempty and bounded from above subset (by a) of $\mathbb{R}$, then A admits a supremum. Put $k = \sup A$. By the part (1), $k \in A$, so $k \in \mathbb{Z}$ and $k < a$. In the other side, as $E(b) + 1 \in \mathbb{Z}$ and

$$E(b) + 1 \leq b + 1 < a,$$

then $E(b) + 1 \in A$, so $E(b) + 1 \leq \sup A = k$. Hence $b < E(b) + 1 \leq k$. Thus $b < k < a$.

3) Let $x, y \in \mathbb{R}$ such that $x < y$, then $y - x > 0$. By the Archimedean principle, there exists $n \in \mathbb{N}^*$ such that $n(y - x) > 1$, i.e., $ny - nx > 1$. Put $a = ny$ and $b = nx$, then $a - b > 1$. By the part (2), there exists $k \in \mathbb{Z}$ such that $b < k < a$, then $nx < k < ny$, so $x < \frac{k}{n} < y$ with $\frac{k}{n} \in \mathbb{Q}$. □

Exercise 28

Consider the set of rational numbers between 0 and $\sqrt{2}$:

$$A = \left]0, \sqrt{2}\right[\cap \mathbb{Q} = \left\{x \in \mathbb{Q},\ 0 < x < \sqrt{2}\right\}.$$

Prove that A admits a supremum and an infimum in $\mathbb{R}$ to be determined.

Solution Since $1 \in \mathbb{Q}$ and $0 < 1 < \sqrt{2}$, then $1 \in A$, so $A \neq \varnothing$. For any $x \in A$, we have $0 \leq x \leq \sqrt{2}$. So A is bounded from above by $\sqrt{2}$ and bounded from below by 0. Hence A is bounded in $\mathbb{R}$, and then A admits a supremum and an infimum in $\mathbb{R}$.

- Let $\varepsilon > 0$. Let's find $x \in A$ such that
$$\sqrt{2} - \varepsilon < x \leq \sqrt{2}.$$
If $\sqrt{2} - \varepsilon < 1$ (i.e., $\varepsilon > \sqrt{2} - 1$), then we take $x = 1 \in A$.
If $\sqrt{2} - \varepsilon \geq 1$ (i.e., $0 < \varepsilon \leq \sqrt{2} - 1$), then, since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ such that
$$\sqrt{2} - \varepsilon < r < \sqrt{2}.$$
In this case, we take $x = r \in A$ since $0 < \sqrt{2} - \varepsilon < r < \sqrt{2}$.
Hence $\sup A = \sqrt{2}$ by the supremum criterion.

- Let $\varepsilon > 0$. Let's find $x \in A$ such that
$$0 \leq x < 0 + \varepsilon.$$
If $\varepsilon > 1$, then we take $x = 1 \in A$.
If $0 < \varepsilon \leq 1$, then, since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ such that
$$0 < r < \varepsilon.$$
In this case, we take $x = r \in A$ since $0 < r < \varepsilon \leq 1 < \sqrt{2}$.
Hence $\inf A = 0$ by the infimum criterion. □

Theorem 1.6.2 *($\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$)*
Between any two real numbers, there exists always an irrational number. In other words, for any $x, y \in \mathbb{R}$ such that $x < y$, there exists $t \in \mathbb{R} \setminus \mathbb{Q}$ such that $x \leq t \leq y$.

Proof Let $x, y \in \mathbb{R}$ such that $x < y$. Put $a = \frac{x}{\sqrt{2}} \in \mathbb{R}$ and $b = \frac{y}{\sqrt{2}} \in \mathbb{R}$, then $a < b$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ (theorem 1.6.1), there exists $r \in \mathbb{Q}$ such that $a < r < b$. So
$$x < r\sqrt{2} < y.$$
Put $t = r\sqrt{2}$, then $x < t < y$. If $t \in \mathbb{Q}$, then $\sqrt{2} = \frac{t}{r} \in \mathbb{Q}$, which is impossible, so $t \in \mathbb{R} \setminus \mathbb{Q}$. □

Exercise 29

1) **Does the product of two irrational numbers is an irrational number ?**

2) **Show that if $a \in \mathbb{R}$ such that a^2 is irrational, then a is irrational. Is the converse always true ?**

3) **Deduce that $\cos(\frac{\pi}{12})$ and $\sin(\frac{\pi}{12})$ are irrational numbers.**

Solution

1) No. For example, $\sqrt{2}$ is irrational, but $\sqrt{2}\sqrt{2} = 2$ is rational.

2) Let $a \in \mathbb{R}$ such that a^2 is irrational. If $a \in \mathbb{Q}$, then there exist $p \in \mathbb{Z}, q \in \mathbb{N}^*$ such that $a = \frac{p}{q}$, so $a^2 = \frac{p^2}{q^2}$ with $p^2 \in \mathbb{Z}$ and $q^2 \in \mathbb{N}^*$. Hence $a^2 \in \mathbb{Q}$, which is impossible. Thus $a \notin \mathbb{Q}$, i.e., a is irrational. The converse is not always true. Indeed, for example, $\sqrt{2} \notin \mathbb{Q}$, but $(\sqrt{2})^2 = 2 \in \mathbb{Q}$.

3) Since
$$\cos^2\left(\frac{\pi}{12}\right) = \frac{1+\cos(\frac{\pi}{6})}{2} = \frac{1}{2} + \frac{\sqrt{3}}{4},$$
and $\sqrt{3} \notin \mathbb{Q}$, then $\cos^2\left(\frac{\pi}{12}\right) \notin \mathbb{Q}$. So $\cos\left(\frac{\pi}{12}\right) \notin \mathbb{Q}$ by the part (2). In the other side, as
$$\sin^2\left(\frac{\pi}{12}\right) = 1 - \cos^2\left(\frac{\pi}{12}\right) = \frac{1}{2} - \frac{\sqrt{3}}{4},$$
and $\sqrt{3} \notin \mathbb{Q}$, then $\sin^2\left(\frac{\pi}{12}\right) \notin \mathbb{Q}$. So $\sin\left(\frac{\pi}{12}\right) \notin \mathbb{Q}$ by the part (2). □

Exercise 30 (A little arithmetic)

1) Let $m, n \in \mathbb{N}^*$.

(a) Let $a \in \mathbb{Z}$ and $b \in \mathbb{N}^*$ such that a and b are relatively prime. Show that if $r = \frac{a}{b} \in \mathbb{Q}$ is a root of the equation $x^n = m$, then a divides m and $b = 1$. (Hint: we may use Gauss' lemma: if $a, b, c \in \mathbb{Z}$ such that a divides bc and a is coprime with b, then a divides c).

(b) Deduce that the only possible rational solutions of the equation $x^n = m$ are integers.

2) Deduce that $\sqrt{p}$ is an irrational number for any prime number p.

3) Show that if p and q are two prime numbers, then $\sqrt{p} + \sqrt{q}$ is an irrational number (Hint: we may consider the number $\sqrt{p} - \sqrt{q}$).

Solution

1) (a) Suppose that $r = \frac{a}{b}$ is a root of the equation $x^n = m$. Then $r^n = m$, so $\frac{a^n}{b^n} = m$, and then $a^n = mb^n$. As a divides a^n, then a divides mb^n. As a is coprime with b^n, then a divides m by Gauss' lemma. In the other side, as b divides mb^n, then b divides a^n, so b is the positive gcd of b and a^n, therefore $b = 1$ since b and a^n are relatively prime.

(b) By the part (a), if $r = \frac{a}{b}$ is a root of the equation $x^n = m$, then $b = 1$, so $r = a \in \mathbb{Z}$. Hence the only possible rational solutions of the equation $x^n = m$ are integers.

2) Let p be a prime number. We know that $\sqrt{p}$ is a root of the equation $x^2 = p$. If $\sqrt{p} \in \mathbb{Q}$, then $\sqrt{p} \in \mathbb{Z}$ and $\sqrt{p}$ divides p by the part (2), so $\sqrt{p} = 1$ or $\sqrt{p} = p$, which is impossible. Hence $\sqrt{p} \notin \mathbb{Q}$.

3) Let p and q be two prime numbers. Put $a = \sqrt{p} + \sqrt{q}$ and $b = \sqrt{p} - \sqrt{q}$, then $ab = p - q$. Suppose that $a \in \mathbb{Q}$, then

$$b = \frac{p-q}{a} \in \mathbb{Q}.$$

So $a + b \in \mathbb{Q}$, i.e., $2\sqrt{p} \in \mathbb{Q}$, and then $\sqrt{p} \in \mathbb{Q}$, which is impossible by the part (2). Hence a is irrational. □

1.7 The principle of mathematical induction

1.7.1 Simple mathematical induction reasoning

Let $P(n)$ be a proposition which depends on a natural number n. Suppose that:

- The proposition $P(n_0)$ is true for a certain natural number n_0,
- The implication $P(k) \Rightarrow P(k+1)$ is true for all integers $k \geq n_0$.

Then the proposition $P(n)$ is true for all $n \geq n_0$.

Exercise 31
Use a mathematical induction to prove that:

1) **For all $n \in \mathbb{N}^*$,**

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

2) **For all $n \in \mathbb{N}^*$,**

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

3) **For all $n \in \mathbb{N}$ and $q \in \mathbb{R} \setminus \{1\}$,**

$$\sum_{k=0}^{n} q^k = 1 + q + q^2 + \cdots + q^n = \frac{1 - q^{n+1}}{1 - q}.$$

Solution

1) For every $n \in \mathbb{N}^*$, put $S_n = 1 + 2 + \cdots + n$. Consider the following proposition:

$$P(n): \quad S_n = \frac{n(n+1)}{2}.$$

- For $n = 1$, $S_1 = 1 = \frac{1(1+1)}{2}$, so $P(1)$ is true.

- Suppose that $P(k)$ is true for a certain integer $k \geq 1$ and let's show that $P(k+1)$ is true. We have:

$$\begin{aligned} S_{k+1} &= (1+2+\cdots+k)+(k+1) \;=\; S_k+(k+1) \\ &= \frac{k(k+1)}{2}+(k+1) \;=\; \frac{(k+1)(k+2)}{2} \;=\; \frac{(k+1)((k+1)+1)}{2}. \end{aligned}$$

  So $P(k+1)$ is true.

Hence $P(n)$ is true for all $n \in \mathbb{N}^*$.

2) For every $n \in \mathbb{N}^*$, put $S_n = 1^2+2^2+\cdots+n^2$. Consider the following proposition:

$$P(n): \quad S_n = \frac{n(n+1)(2n+1)}{6}.$$

- For $n=1$, $S_1 = 1 = \frac{1(1+1)(2+1)}{6}$, so $P(1)$ is true.
- Suppose that $P(k)$ is true for a certain integer $k \geq 1$ and let's show that $P(k+1)$ is true. We have:

$$\begin{aligned} S_{k+1} &= (1^2+2^2+\cdots+k^2)+(k+1)^2 \;=\; S_k+(k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6}+(k+1)^2 \;=\; \frac{(k+1)(2k^2+7k+6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \;=\; \frac{(k+1)\Big((k+1)+1\Big)\Big(2(k+1)+1\Big)}{6}. \end{aligned}$$

  So $P(k+1)$ is true.

Hence $P(n)$ is true for all $n \in \mathbb{N}^*$.

3) Let $q \in \mathbb{R} \setminus \{1\}$. For every $n \in \mathbb{N}$, put $S_n = 1+q+q^2+\cdots+q^n$. Consider the following proposition:

$$P(n): \quad S_n = \frac{1-q^{n+1}}{1-q}.$$

- For $n=0$, $S_0 = 1 = \frac{1-q}{1-q}$, so $P(0)$ is true.
- Suppose that $P(k)$ is true for a certain integer $k \geq 0$ and let's show that $P(k+1)$ is true. We have:

$$\begin{aligned} S_{k+1} &= 1+q+q^2+\cdots+q^k+q^{k+1} \;=\; S_k+q^{k+1} \\ &= \frac{1-q^{k+1}}{1-q}+q^{k+1} \;=\; \frac{1-q^{k+1}+q^{k+1}-q^{k+2}}{1-q} \;=\; \frac{1-q^{(k+1)+1}}{1-q}. \end{aligned}$$

  So $P(k+1)$ is true.

Hence $P(n)$ is true for all $n \in \mathbb{N}$. □

Exercise 32
Prove that if x is a real number such that $x > -1$, then, for all $n \in \mathbb{N}$,

$$(1+x)^n \geq 1+nx \qquad \textbf{(Bernoulli's inequality).}$$

Solution Suppose that $x > -1$. For every $n \in \mathbb{N}$, consider the following proposition:

$$P(n): \quad (1+x)^n \geq 1+nx.$$

- For $n = 0$, $(1+x)^0 = 1 \geq 1+0x$, so $P(0)$ is true.
- Suppose that $P(k)$ is true for a certain integer $k \in \mathbb{N}$ and let's show that $P(k+1)$ is true. We have

$$\begin{aligned} (1+x)^{k+1} &= (1+x)(1+x)^k \geq (1+x)(1+kx) \quad (\text{since } 1+x>0) \\ &= 1+(k+1)x+kx^2 \geq 1+(k+1)x \quad (\text{since } kx^2 \geq 0). \end{aligned}$$

  So $P(k+1)$ is true.

Hence $P(n)$ is true for all integers $n \in \mathbb{N}$. □

Exercise 33
Let $a, b \in \mathbb{R}$.

1) **Prove that if $0 \leq a < b$, then $0 \leq a^n < b^n$ for all $n \in \mathbb{N}^*$.**
2) **Deduce that:**
   i) **If $a < b \leq 0$, then $0 \leq b^n < a^n$ for all even integers $n \in \mathbb{N}^*$.**
   ii) **If $a < b$, then $a^n < b^n$ for all odd integers $n \in \mathbb{N}^*$.**
3) **Show that if $0 < a \leq 1$, then $0 < a^n \leq a$ for all $n \in \mathbb{N}^*$.**
4) **Show that if $a \geq 1$, then $a^n \geq a$ for all $n \in \mathbb{N}^*$.**

Solution

1) Suppose that $0 \leq a < b$. For every $n \in \mathbb{N}^*$, consider the following proposition:

$$P(n): \quad 0 \leq a^n < b^n.$$

   - For $n = 1$, $0 \leq a^1 < b^1$, so $P(1)$ is true.
   - Suppose that $P(k)$ is true for a certain integer $k \in \mathbb{N}^*$ and let's show that the property $P(k+1)$ is true. Since $0 \leq a < b$ and $0 \leq a^k < b^k$, then, by the part (6) of the exercise 2,

$$0 \leq aa^k < bb^k.$$

     So $0 \leq a^{k+1} < b^{k+1}$. Hence $P(k+1)$ is true.

   Thus $P(n)$ is true for all integers $n \in \mathbb{N}^*$.

2) i) Let $\boldsymbol{n} \in \mathbb{N}^*$ be an even integer. Suppose that $\boldsymbol{a < b \leq 0}$, then $\boldsymbol{0 \leq -b < -a}$. So, by the part (1),

$$\boldsymbol{0 \leq (-b)^n < (-a)^n} \quad \text{i.e.,} \quad \boldsymbol{0 \leq (-1)^n b^n < (-1)^n a^n.}$$

But $\boldsymbol{n}$ is even, then $\boldsymbol{(-1)^n = 1}$, and therefore $\boldsymbol{0 \leq b^n < a^n}$.

ii) Let $\boldsymbol{n} \in \mathbb{N}^*$ be an odd integer. Suppose that $\boldsymbol{a < b}$. We have three cases:

- If $\boldsymbol{0 \leq a < b}$, then $\boldsymbol{a^n < b^n}$ by the part (1).
- If $\boldsymbol{a < b \leq 0}$, then $\boldsymbol{0 \leq -b < -a}$. So, by the part (1),

$$\boldsymbol{0 \leq (-b)^n < (-a)^n} \quad \text{i.e.,} \quad \boldsymbol{0 \leq (-1)^n b^n < (-1)^n a^n.}$$

  But $\boldsymbol{n}$ is odd, then $\boldsymbol{(-1)^n = -1}$, and therefore $\boldsymbol{0 \leq -b^n < -a^n}$. Hence $\boldsymbol{a^n < b^n}$.
- If $\boldsymbol{a < 0 < b}$, then $\boldsymbol{a^n < 0 < b^n}$ (since $\boldsymbol{n}$ is odd), so $\boldsymbol{a^n < b^n}$.

3) Suppose that $\boldsymbol{0 < a \leq 1}$. For every $\boldsymbol{n} \in \mathbb{N}^*$, consider the following proposition:

$$\boldsymbol{P(n): \quad 0 < a^n \leq a.}$$

- For $\boldsymbol{n = 1}$, $\boldsymbol{0 < a^1 \leq a}$, so $\boldsymbol{P(1)}$ is true.
- Suppose that $\boldsymbol{P(k)}$ is true for a certain integer $\boldsymbol{k} \in \mathbb{N}^*$ and let's show that the property $\boldsymbol{P(k+1)}$ is true. Since $\boldsymbol{0 < a \leq 1}$ and $\boldsymbol{0 < a^k \leq a}$, then, by the part (6) of the exercise 2.

$$\boldsymbol{0 < aa^k \leq (1)(a).}$$

  So $\boldsymbol{0 < a^{k+1} \leq a}$. Hence $\boldsymbol{P(k+1)}$ is true.

Thus $\boldsymbol{P(n)}$ is true for all integers $\boldsymbol{n} \in \mathbb{N}^*$.

4) Suppose that $\boldsymbol{a \geq 1}$. For every $\boldsymbol{n} \in \mathbb{N}^*$, consider the following proposition:

$$\boldsymbol{P(n): \quad a^n \geq a.}$$

- For $\boldsymbol{n = 1}$, $\boldsymbol{a^1 \geq a}$, so $\boldsymbol{P(1)}$ is true.
- Suppose that $\boldsymbol{P(k)}$ is true for a certain integer $\boldsymbol{k} \in \mathbb{N}^*$ and let's show that the property $\boldsymbol{P(k+1)}$ is true. Since $\boldsymbol{1 \leq a}$ and $\boldsymbol{a \leq a^k}$, then, by the part (6) of the exercise 2.

$$\boldsymbol{(1)(a) \leq aa^k.}$$

  So $\boldsymbol{a^{k+1} \geq a}$. Hence $\boldsymbol{P(k+1)}$ is true.

Thus $\boldsymbol{P(n)}$ is true for all integers $\boldsymbol{n} \in \mathbb{N}^*$. □

Exercise 34

1) Prove that $2^n < n!$ for all integers $n \geq 4$.

2) For every $n \in \mathbb{N}$, put:

$$u_n = \sum_{k=0}^{n} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \quad \text{and} \quad v_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$

(a) Prove, by two different methods, that $u_n < 2$ for all $n \in \mathbb{N}$.

(b) Deduce that $2 < v_n < 3$ for all $n \in \mathbb{N}$.

(c) Put $A = \{u_n,\, n \in \mathbb{N}\}$. Show that $\sup A = 2$.

Solution

1) For every integer $n \geq 4$, consider the following proposition:

$$P(n): \quad 2^n < n!.$$

- For $n = 4$, $2^4 = 16 < 24 = 4!$, so $P(4)$ is true.
- Suppose that $P(k)$ is true for a certain integer $k \geq 4$ and let's show that $P(k+1)$ is true. We have $2^{k+1} = 2 \times 2^k$, and as $2^k < k!$ and

$$2 < 5 = 4 + 1 \leq k + 1,$$

then

$$2^{k+1} = 2 \times 2^k < (k+1)k! = (k+1)!.$$

So $P(k+1)$ is true.

Hence $P(n)$ is true for all integers $n \geq 4$.

2) (a) 1st method: For every $n \in \mathbb{N}$, consider the following proposition:

$$P(n): \quad u_n < 2.$$

- For $n = 0$, $u_0 = 1 < 2$, so $P(0)$ is true.
- Suppose that $P(k)$ is true for a certain integer $k \in \mathbb{N}$ and let's show that $P(k+1)$ is true. We have

$$\begin{aligned} u_{k+1} &= 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{k+1}} = 1 + \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^k}\right) \\ &= 1 + \frac{1}{2}u_k < 1 + \frac{1}{2}(2) = 2 \quad (\text{since } u_k < 2). \end{aligned}$$

So $P(k+1)$ is true.

Hence $P(n)$ is true for all integers $n \in \mathbb{N}$.

2nd method: By the part (3) of the exercise 31, for all $n \in \mathbb{N}$,

$$u_n = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k = \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^n} < 2.$$

(b) For any $n \in \mathbb{N}$,

$$v_n = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} > 2.$$

In the other side, by the part (1), for any integer $n \geq 4$, $2^n < n!$, so

$$\frac{1}{n!} < \frac{1}{2^n}.$$

So, for all $n \in \mathbb{N}$,

$$\begin{aligned} v_n &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{4!} + \cdots + \frac{1}{n!} \\ &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^4} + \cdots + \frac{1}{2^n} \quad \left(\text{since } \frac{1}{6} < \frac{1}{2^2}\right) \\ &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^n} \quad \left(\text{since } \frac{1}{2^3} > 0\right) \\ &= 1 + u_n < 1 + 2 = 3 \quad (\text{by the part (a)}). \end{aligned}$$

Hence $2 < v_n < 3$ for all $n \in \mathbb{N}$.

(c) By the part (a), $u_n \leq 2$ for all $n \in \mathbb{N}$. So 2 is an upper bound of A. Let $\varepsilon > 0$, let's find $n_0 \in \mathbb{N}$ such that

$$2 - \varepsilon < u_{n_0} \leq 2.$$

We have:

$$\begin{aligned} 2 - \varepsilon < u_{n_0} \leq 2 \quad &\Leftrightarrow \quad 2 - \varepsilon < 2 - \frac{1}{2^{n_0}} \\ &\Leftrightarrow \quad \frac{1}{2^{n_0}} < \varepsilon \;\Leftrightarrow\; 2^{n_0} > \frac{1}{\varepsilon} \;\Leftrightarrow\; n_0 > -\frac{\ln \varepsilon}{\ln 2}. \end{aligned}$$

It suffices to take $n_0 = \max\left[0, E\left(-\frac{\ln \varepsilon}{\ln 2}\right) + 1\right] \in \mathbb{N}$. Hence $\sup A = 2$ by the supremum criterion. □

1.7.2 Strong mathematical induction reasoning

Let $P(n)$ be a proposition which depends on a natural number n. Suppose that:

- The proposition $P(n_0)$ is true for a certain natural number n_0,
- The implication $\Big(\forall m \in \{n_0, \ldots, k\}\Big)\Big(P(m)\Big) \Rightarrow P(k+1)$ is true for all integers $k \geq n_0$.

Then the proposition $P(n)$ is true for all $n \geq n_0$.

Exercise 35
Prove that every natural number $n \geq 2$ has a prime divisor.

Solution For every natural number $\boldsymbol{n \geq 2}$, consider the following proposition:

$$\boldsymbol{P(n):} \quad \boldsymbol{n} \text{ has a prime divisor.}$$

- For $\boldsymbol{n = 2}$, $\boldsymbol{2}$ is a prime divisor of $\boldsymbol{n}$, so $\boldsymbol{P(2)}$ is true.
- Suppose that for any integer $\boldsymbol{m} \in \boldsymbol{\{2, \ldots, k\}}$, $\boldsymbol{P(m)}$ is true for a certain integer $\boldsymbol{k \geq 2}$ and let's show that $\boldsymbol{P(k+1)}$ is true.
  If $\boldsymbol{k+1}$ is prime, then $\boldsymbol{k+1}$ is a prime divisor of itself, so $\boldsymbol{P(k+1)}$ is true.
  If $\boldsymbol{k+1}$ is not prime, then $\boldsymbol{k+1}$ has a divisor $\boldsymbol{d} \in \mathbb{N}$ such that $\boldsymbol{d \neq 1}$ and $\boldsymbol{d \neq k+1}$, so $\boldsymbol{2 \leq d \leq k}$. As $\boldsymbol{P(d)}$ is true, then $\boldsymbol{d}$ has a prime divisor $\boldsymbol{p}$. So $\boldsymbol{p}$ is a prime divisor of $\boldsymbol{k+1}$. Hence $\boldsymbol{P(k+1)}$ is true.

Thus $\boldsymbol{P(n)}$ is true for all integers $\boldsymbol{n \geq 2}$. $\square$

Chapter 2

Real sequences

2.1 Definitions and notations

Definition 2.1.1 *Let E be a nonempty set. A sequence of elements of E is a mapping from a nonempty subset I of $\mathbb{N}$ into E:*

$$\begin{aligned} u : I &\longmapsto E \\ n &\longmapsto u(n) = u_n, \end{aligned}$$

- *The set I is called the set of indexes of the sequence.*
- *The set E can be a set of real numbers (real or numerical sequence) or complexes (sequence of complex numbers), a set of points in the plane (sequence of points), a set of vectors (sequence of vectors), a set of functions (sequence of functions), etc ...*
- *The sequence will be denoted $(u_n)_{n\in I}$ or $(u_n)_n$ or (u_n). If $I = \mathbb{N}$, then it is denoted $(u_n)_{n\geq 0}$. In general, if $I = \{n \in \mathbb{N}, n \geq n_0\}$ for a certain $n_0 \in \mathbb{N}$, then the sequence is denoted by $(u_n)_{n\geq n_0}$.*
- *The element u_n is called the general term of the sequence.*

Example 2.1.1

1) *Let $a \in \mathbb{R}$. For any integer $n \geq 0$, put $u_n = a$. The sequence $(u_n)_{n\geq 0}$ is called a constant (or stationary) real sequence.*

2) *Let $a, b, c \in \mathbb{R}$. Put $u_0 = a$, $u_1 = b$ and $u_n = c$ for every integer $n \geq 2$. The real sequence $(u_n)_{n\geq 0}$ is not necessary constant. On the other hand, the sequence $(u_n)_{n\geq 2}$ is constant. If $a = b = c$, then $(u_n)_{n\geq 0}$ is a constant sequence.*

3) *For every $n \in \mathbb{N}$, put $u_n = 2n$. The real sequence $(u_n)_{n\geq 0}$ is called the sequence of even positive integers.*

4) *For every $n \in \mathbb{N}$, put $u_n = 2n + 1$. The real sequence $(u_n)_{n\geq 0}$ is called the sequence of odd positive integers.*

5) *For every $n \in \mathbb{N}^*$, put $u_n = \frac{1}{n}$. The real sequence $(u_n)_{n\geq 1}$ is called the sequence of the inverses of positive integers.*

6) *For every* $n \in \mathbb{N}$, *put* $u_n = 2^n$. *The real sequence* $(u_n)_{n\geq 0}$ *is called the sequence of powers of* 2.

Exercise 36
Consider the real sequence $(u_n)_{n\geq 1}$ defined by:

$$u_1 = 2, u_2 = 5, u_3 = 10, u_4 = 17, \ldots.$$

Determine the general term of this sequence.

Solution $u_n = n^2 + 1$ for all $n \in \mathbb{N}^*$. □

Definition 2.1.2 *A recursive sequence is a sequence defined by its first terms, and a recursive relation allowing to calculate each term in terms of preceeding terms.*

- *A recursive sequence can be a sequence* $(u_n)_{n\in\mathbb{N}}$ *defined by:*

$$\begin{cases} u_0 = a, \\ u_{n+1} = f(u_n), \quad \forall n \in \mathbb{N}, \end{cases}$$

  where $f : D \longmapsto D$ *is a given mapping,* $a \in D$ *and* D *is a subset of* $\mathbb{R}$. *The sequence* $(u_n)_{n\in\mathbb{N}}$ *is called the recursive sequence generated by* a *and* f.

- *A recursive sequence can be also defined by:*

$$\begin{cases} u_0 = a, \\ u_1 = b, \\ u_{n+2} = f(u_n, u_{n+1}), \quad \forall n \in \mathbb{N}, \end{cases}$$

  where $f : D \times D \longmapsto D$ *is a given mapping,* $a, b \in D$ *and* D *is a subset of* $\mathbb{R}$. *The sequence* $(u_n)_{n\in\mathbb{N}}$ *is called the recursive sequence generated by* a, b *and* f.

Example 2.1.2

1) *Let* $a, r \in \mathbb{R}$. *An arithmetic sequence with first term* a *and ratio* r *is a recursive sequence defined by:*

$$\begin{cases} u_0 = a, \\ u_{n+1} = u_n + r, \quad \forall n \in \mathbb{N}. \end{cases}$$

   We have following formulas:

   - *For all* $n \in \mathbb{N}$, $u_n = a + nr$.
   - *For any* $p, n \in \mathbb{N}$ *with* $p \leq n$, *we have:*

$$u_n = u_p + (n - p)\, r.$$

$$\sum_{k=p}^{n} u_k = u_p + \cdots + u_n = \left(\frac{n - p + 1}{2}\right)(u_p + u_n).$$

2) Let $\boldsymbol{a}, \boldsymbol{r} \in \mathbb{R}$. An geometric sequence with first term $\boldsymbol{a}$ and ratio $\boldsymbol{r}$ is a recursive sequence defined by:

$$\begin{cases} u_0 = a, \\ u_{n+1} = r\, u_n, \quad \forall\, n \in \mathbb{N}. \end{cases}$$

We have following formulas:

- *For all $n \in \mathbb{N}$, $u_n = a\, r^n$.*
- *For any $p, n \in \mathbb{N}$ with $p \leq n$, we have:*

$$u_n = u_p\, r^{n-p}.$$

$$\sum_{k=p}^{n} u_k = u_p + \cdots + u_n = u_p \left(\frac{1 - r^{n-p+1}}{1 - r} \right).$$

3) The sequence of Fibonacci is the recursive sequence $(f_n)_{n \in \mathbb{N}}$ defined by:

$$\begin{cases} f_0 = 1, \\ f_1 = 1, \\ f_{n+2} = f_{n+1} + f_n, \quad \forall\, n \in \mathbb{N}. \end{cases}$$

2.2 Subsequences

Lemma 2.2.1 *If $\varphi : \mathbb{N} \longmapsto \mathbb{N}$ is a strictly increasing mapping, then $\varphi(n) \geq n$ for all $n \in \mathbb{N}$.*

Proof By mathematical induction on n. For $n = 0$, as $\varphi(0) \in \mathbb{N}$, then $\varphi(0) \geq 0$. Suppose that $\varphi(k) \geq k$ for a certain $k \in \mathbb{N}$. Since $k + 1 > k$ and φ is strictly increasing, then

$$\varphi(k + 1) > \varphi(k) \geq k.$$

So $\varphi(k + 1) > k$. But $\varphi(k + 1) \in \mathbb{N}$, then $\varphi(k + 1) \geq k + 1$. Hence $\varphi(n) \geq n$ for all $n \in \mathbb{N}$. □

Definition 2.2.1 *Let $(u_n)_{n \in \mathbb{N}}$ be a real sequence and $\varphi : \mathbb{N} \longmapsto \mathbb{N}$ be a strictly increasing mapping. For every $n \in \mathbb{N}$, put $v_n = u_{\varphi(n)}$. The sequence $(v_n)_{n \in \mathbb{N}}$ is called a subsequence of the sequence $(u_n)_{n \in \mathbb{N}}$, it will be denoted $(u_{\varphi(n)})_{n \in \mathbb{N}}$. The terms of this sequence are:*

$$u_{\varphi(0)}, u_{\varphi(1)}, u_{\varphi(2)}, \ldots, u_{\varphi(n)}, \ldots$$

Example 2.2.1 *Let $(u_n)_{n \in \mathbb{N}}$ be a real sequence.*

1) If $\varphi(n) = n$ for all $n \in \mathbb{N}$, then the sequence $(u_n)_{n \in \mathbb{N}}$ is a subsequence of itself.

2) If $\varphi(n) = 2n$ for all $n \in \mathbb{N}$, then the sequence $(u_{2n})_{n \in \mathbb{N}}$ is a subsequence of $(u_n)_{n \in \mathbb{N}}$, called the subsequence of even indexes. The terms of this sequence are:

$$u_0, u_2, u_4, u_6, \ldots, u_{100}, \ldots$$

3) *If* $\varphi(n) = 2n+1$ *for all* $n \in \mathbb{N}$, *then the sequence* $(u_{2n+1})_{n\in\mathbb{N}}$ *is a subsequence of* $(u_n)_{n\in\mathbb{N}}$, *called the subsequence of odd indexes. The terms of this sequence are:*

$$u_1, u_3, u_5, u_7, \dots, u_{99}, \dots$$

4) *If* $\varphi(n) = 3n$ *for all* $n \in \mathbb{N}$, *then the sequence* $(u_{3n})_{n\in\mathbb{N}}$ *is a subsequence of* $(u_n)_{n\in\mathbb{N}}$. *The terms of this sequence are:*

$$u_0, u_3, u_6, u_9, \dots, u_{99}, \dots$$

2.3 Supremum and infimum

Definition 2.3.1 *Let* $(u_n)_{n\in\mathbb{N}}$ *be a real sequence.*

1) *We say that* $(u_n)_{n\in\mathbb{N}}$ *is bounded from above if there exists* $M \in \mathbb{R}$ *such that* $u_n \leq M$ *for all* $n \in \mathbb{N}$. *In this case, we say that* M *is an upper bound of the sequence, and that* $(u_n)_{n\in\mathbb{N}}$ *is bounded from above by* M.

2) *We say that* $(u_n)_{n\in\mathbb{N}}$ *is bounded from below if there exists* $m \in \mathbb{R}$ *such that* $u_n \geq m$ *for all* $n \in \mathbb{N}$. *In this case, we say that* m *is a lower bound of the sequence, and that* $(u_n)_{n\in\mathbb{N}}$ *is bounded from below by* m.

3) *We say that* $(u_n)_{n\in\mathbb{N}}$ *is bounded if it is bothe bounded from above and from below, i.e., if there exist* $m, M \in \mathbb{R}$ *such that* $m \leq u_n \leq M$ *for all* $n \in \mathbb{N}$.

Example 2.3.1

1) *For every* $n \in \mathbb{N}$, *put* $u_n = \dfrac{(-1)^n}{n+1}$. *For any* $n \in \mathbb{N}$, *as* $-1 \leq (-1)^n \leq 1$, *then*

$$-1 \leq \frac{-1}{n+1} \leq u_n \leq \frac{1}{n+1} \leq 1.$$

So the sequence $(u_n)_{n\in\mathbb{N}}$ *is bounded from above by* 1 *and from below by* -1. *Hence* $(u_n)_{n\in\mathbb{N}}$ *is bounded.*

2) *For every* $n \in \mathbb{N}$, *put* $u_n = n^2$. *For any* $n \in \mathbb{N}$, $u_n \geq 0$, *so the sequence* $(u_n)_{n\in\mathbb{N}}$ *is bounded from below by* 0. *Suppose that this sequence is bounded from above, then there exists* $M \in \mathbb{R}$ *such that* $u_n \leq M$ *for all* $n \in \mathbb{N}$, *so* $n^2 \leq M$ *for all* $n \in \mathbb{N}$. *Hence* $n \leq \sqrt{M}$ *for all* $n \in \mathbb{N}$ *(*M *is positive). In particular, for* $n = E(\sqrt{M}) + 1 \in \mathbb{N}$, *we obtain* $E(\sqrt{M}) + 1 \leq \sqrt{M}$, *which is impossible. Hence the sequence* $(u_n)_{n\in\mathbb{N}}$ *is not bounded from above. Consequently, this sequence is not bounded.*

Remark 2.3.1 *Let* $(u_n)_{n\in\mathbb{N}}$ *be a real sequence and* $A = \{u_n,\ n \in \mathbb{N}\} \subseteq \mathbb{R}$.

1) *The sequence* $(u_n)_{n\in\mathbb{N}}$ *is bounded from above by a real number* M *if and only if the set* A *is bounded from above by* M.

2) *The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded from below by a real number m if and only if the set A is bounded from below by m.*

3) *The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded if and only if the set A is bounded in $\mathbb{R}$.*

4) *The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded if and only if there exists a strictly positive real number M such that $|u_n| \leq M$ for all $n \in \mathbb{N}$.*

Exercise 37
Let $(u_n)_{n\in\mathbb{N}}$ be the sequence defined by $u_n = n+(-1)^n n$. Show that this sequence is not bounded.

Solution Suppose, by contradiction, that this sequence is bounded, then there exists a strictly positive real number M such that $|u_n| \leq M$ for all $n \in \mathbb{N}$. So $|u_{2n}| \leq M$ for all $n \in \mathbb{N}$. But $u_{2n} = 4n$, then $n \leq \frac{M}{4}$ for all $n \in \mathbb{N}$. If $n_0 = E\left(\frac{M}{4}\right) + 1 \in \mathbb{N}$ (since $M > 0$), then $n_0 \leq \frac{M}{4}$, i.e., $E\left(\frac{M}{4}\right) + 1 \leq \frac{M}{4}$, which is impossible. Hence the sequence $(u_n)_{n\in\mathbb{N}}$ is not bounded. □

Exercise 38
Let $a \in \mathbb{R}_+^*$, $k \in \mathbb{N}^*$ such that $|u_n| \geq an^k$ for all $n \in \mathbb{N}$. Prove that $(u_n)_{n\in\mathbb{N}}$ is not bounded.

Solution Suppose, by contradiction, that this sequence is bounded, then there exists a strictly positive real number M such that $|u_n| \leq M$ for all $n \in \mathbb{N}$. So $an^k \leq M$ for all $n \in \mathbb{N}$. But $n \leq n^k$ for all $n \in \mathbb{N}^*$, then $an \leq an^k \leq M$ for all $n \in \mathbb{N}^*$. So $n \leq \frac{M}{a}$ for all $n \in \mathbb{N}^*$. If $n_0 = E\left(\frac{M}{a}\right) + 1 \in \mathbb{N}$ (since $\frac{M}{a} > 0$), then $n_0 \leq \frac{M}{a}$, which is impossible. Hence the sequence $(u_n)_{n\in\mathbb{N}}$ is not bounded. □

Definition 2.3.2 *Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence and $A = \{u_n,\ n \in \mathbb{N}\} \subseteq \mathbb{R}$.*

1) *We say that the sequence $(u_n)_{n\in\mathbb{N}}$ has a supremum, and we denote it $\sup\limits_{n\in\mathbb{N}} u_n$ or simply $\sup u_n$, if the set A has a supremum in $\mathbb{R}$. In this case, we write $\sup u_n = \sup A$.*

2) *We say that the sequence $(u_n)_{n\in\mathbb{N}}$ has an infimum, and we denote it $\inf\limits_{n\in\mathbb{N}} u_n$ or simply $\inf u_n$, if the set A has an infimum in $\mathbb{R}$. In this case, we write $\inf u_n = \inf A$.*

Remark 2.3.2

1) *Every bounded from above real sequence has a supremum.*

2) *Every bounded from below real sequence has an infimum.*

Example 2.3.2

1) *If* $u_n = \frac{n}{n+1}$ *for all* $n \in \mathbb{N}$*, then* $\sup u_n = 1$ *and* $\inf u_n = 0$ *by the exercise 12.*

2) *If* $u_n = \frac{1}{n} + (-1)^n$ *for all* $n \in \mathbb{N}^*$*, then* $\sup u_n = \dfrac{3}{2}$ *and* $\inf u_n = -1$ *by the exercise 19.*

3) *If* $u_n = \frac{3n^2+4}{n^2+1}$ *for all* $n \in \mathbb{N}$*, then* $\sup u_n = 4$ *and* $\inf u_n = 3$ *by the exercise 20.*

3) *If* $u_n = \frac{2+(-1)^n}{n}$ *for all* $n \in \mathbb{N}^*$*, then* $\sup u_n = \dfrac{3}{2}$ *and* $\inf u_n = 0$ *by the exercise 21.*

2.4 Monotone sequences

Definition 2.4.1 *Let* $(u_n)_{n\in\mathbb{N}}$ *be a real sequence.*

1) *The sequence* $(u_n)_{n\in\mathbb{N}}$ *is said to be increasing if* $u_{n+1} \geq u_n$ *for all* $n \in \mathbb{N}$*, i.e.,*

$$u_{n+1} - u_n \geq 0, \quad \forall n \in \mathbb{N}.$$

2) *The sequence* $(u_n)_{n\in\mathbb{N}}$ *is said to be decreasing if* $u_{n+1} \leq u_n$ *for all* $n \in \mathbb{N}$*, i.e.,*

$$u_{n+1} - u_n \leq 0, \quad \forall n \in \mathbb{N}.$$

3) *The sequence* $(u_n)_{n\in\mathbb{N}}$ *is said to be monotone if it is increasing or decreasing.*

4) *The sequence* $(u_n)_{n\in\mathbb{N}}$ *is said to be strictly increasing if* $u_{n+1} > u_n$ *for all* $n \in \mathbb{N}$*, i.e.,*

$$u_{n+1} - u_n > 0, \quad \forall n \in \mathbb{N}.$$

5) *The sequence* $(u_n)_{n\in\mathbb{N}}$ *is said to be strictly decreasing if* $u_{n+1} < u_n$ *for all* $n \in \mathbb{N}$*, i.e.,*

$$u_{n+1} - u_n < 0, \quad \forall n \in \mathbb{N}.$$

6) *The sequence* $(u_n)_{n\in\mathbb{N}}$ *is said to be strictly monotone if it is strictly increasing or strictly decreasing.*

Example 2.4.1

1) *For every* $n \in \mathbb{N}$*, put* $u_n = \frac{n}{n+1}$*. For any* $n \in \mathbb{N}$*,*

$$u_{n+1} - u_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{1}{(n+1)(n+2)} > 0.$$

So the sequence $(u_n)_{n\in\mathbb{N}}$ *is strictly increasing.*

2) *A sequence can be non increasing and non decreasing (so non monotone). For example, the sequence* $(u_n)_{n\in\mathbb{N}}$ *defined by* $u_n = (-1)^n$*.*

3) *Every strictly increasing (resp. decreasing) sequence is increasing (resp. decreasing). On the other hand, the converse is not true in general.*

4) *If $(u_n)_{n\in\mathbb{N}}$ is an increasing (resp. decreasing) sequence, then it is bounded from belaow (resp. bounded from above) by u_0.*

5) *The sequence $(u_n)_{n\in\mathbb{N}}$ is increasing if and only if the sequence $(-u_n)_{n\in\mathbb{N}}$ is decreasing.*

Exercise 39
Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence such that $u_n > 0$ for all $n \in \mathbb{N}$. Prove that $(u_n)_{n\in\mathbb{N}}$ is increasing if and only if the sequence $\left(\frac{1}{u_n}\right)_{n\in\mathbb{N}}$ is decreasing.

Solution

N.C. Let $n \in \mathbb{N}$, then $u_n > 0$. As $(u_n)_{n\in\mathbb{N}}$ is increasing, then $0 < u_n \leq u_{n+1}$. So, by the exercise 2,

$$\frac{1}{u_{n+1}} \leq \frac{1}{u_n}.$$

Hence $\left(\frac{1}{u_n}\right)_{n\in\mathbb{N}}$ is decreasing.

S.C. Let $n \in \mathbb{N}$, then $\frac{1}{u_n} > 0$. As $\left(\frac{1}{u_n}\right)_{n\in\mathbb{N}}$ is decreasing, then $0 < \frac{1}{u_{n+1}} \leq \frac{1}{u_n}$. So, by the exercise 2,

$$u_n = \frac{1}{\frac{1}{u_n}} \leq \frac{1}{\frac{1}{u_{n+1}}} = u_{n+1}.$$

Hence $(u_n)_{n\in\mathbb{N}}$ is increasing. □

Proposition 2.4.1 *Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence such that $u_n > 0$ for all $n \in \mathbb{N}$.*

1) *The sequence $(u_n)_{n\in\mathbb{N}}$ is increasing if and only if $\dfrac{u_{n+1}}{u_n} \geq 1$ for all $n \in \mathbb{N}$.*

2) *The sequence $(u_n)_{n\in\mathbb{N}}$ is decreasing if and only if $\dfrac{u_{n+1}}{u_n} \leq 1$ for all $n \in \mathbb{N}$.*

3) *The sequence $(u_n)_{n\in\mathbb{N}}$ is strictly increasing if and only if $\dfrac{u_{n+1}}{u_n} > 1$ for all $n \in \mathbb{N}$.*

4) *The sequence $(u_n)_{n\in\mathbb{N}}$ is strictly decreasing if and only if $\dfrac{u_{n+1}}{u_n} < 1$ for all $n \in \mathbb{N}$.*

Example 2.4.2

1) *For every $n \in \mathbb{N}^*$, put $u_n = n^2 > 0$. For any $n \in \mathbb{N}^*$,*

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{n^2} = \frac{n^2+2n+1}{n^2} = 1 + \frac{2n+1}{n^2} \geq 1.$$

So the sequence $(u_n)_{n\geq 1}$ is increasing.

2) For every $n \in \mathbb{N}$, put $u_n = e^{n^2} > 0$. For any $n \in \mathbb{N}$,

$$\frac{u_{n+1}}{u_n} = \frac{e^{(n+1)^2}}{e^{n^2}} = e^{2n+1} \geq e > 1.$$

So the sequence $(u_n)_{n\in\mathbb{N}}$ is strictly increasing.

Exercise 40

Let $a > 0$. Consider the sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_n = \frac{a^n}{n!}$. Prove that there exists $n_0 \in \mathbb{N}$ such that the sequence $(u_n)_{n\geq n_0}$ is strictly decreasing.

$\boxed{\text{Solution}}$ For any $n \in \mathbb{N}$, $u_n > 0$ and

$$\frac{u_{n+1}}{u_n} = \frac{a^{n+1}}{(n+1)!} \times \frac{n!}{a^n} = \frac{a}{n+1}.$$

Put $n_0 = E(a) \in \mathbb{N}$. If $n \geq n_0$, then $n > a-1$, so $n+1 > a$, hence $\frac{u_{n+1}}{u_n} = \frac{a}{n+1} < 1$. Thus the sequence $(u_n)_{n\geq n_0}$ is strictly decreasing. □

2.5 Limit of a real sequence

Example 2.5.1

1) For every $n \in \mathbb{N}^*$, put $u_n = \frac{1}{n}$. The terms of this sequence are:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{100}, \dots, \frac{1}{10452}, \dots.$$

We remak that these terms approach to 0 when n goes to infinity.

2) For every $n \in \mathbb{N}$, put $u_n = (-1)^n$. The terms of this sequence are:

$$1, -1, 1, -1, 1, -1, \dots, 1, -1, \dots, 1, -1, \dots.$$

We remak that these terms do not approach any fixed real number when n goes to infinity.

3) For every $n \in \mathbb{N}$, put $u_n = n^2$. The terms of this sequence are:

$$1, 4, 9, 16, 25, \dots, 100, 121, 144, \dots, 10000, \dots.$$

We remak that these terms approach to $+\infty$ when n goes to infinity.

Definition 2.5.1 *Let $\ell \in \mathbb{R}$ and $(u_n)_{n\in\mathbb{N}}$ be a real sequence. We say that $(u_n)_{n\in\mathbb{N}}$ has limit ℓ as n tends to infinity, if its terms approach to ℓ as n goes to infinity. In other words, $(u_n)_{n\in\mathbb{N}}$ has limit ℓ as n tends to infinity if and only if*

$$\Big(\forall\, \varepsilon > 0\Big)\Big(\exists\, n_0 \in \mathbb{N}\Big)\Bigg[\Big(\forall\, n \in \mathbb{N}\Big)\Big(n \geq n_0 \Rightarrow |u_n - \ell| < \varepsilon\Big)\Bigg],$$

i.e., for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for any $n \in \mathbb{N}$, we have:

$$n \geq n_0 \Rightarrow \ell - \varepsilon < u_n < \ell + \varepsilon.$$

Example 2.5.2

1) For every $n \in \mathbb{N}$, put $u_n = a \in \mathbb{R}$. The constant sequence $(u_n)_{n\in\mathbb{N}}$ has limit a as n tends to infinity. Indeed, let $\varepsilon > 0$, put $n_0 = 0$, for any $n \geq n_0$,

$$|u_n - a| = |a - a| = 0 < \varepsilon.$$

2) For every $n \in \mathbb{N}^*$, put $u_n = \frac{1}{n}$. Let's show that the sequence $(u_n)_{n\in\mathbb{N}^*}$ has limit 0 as n tends to infinity. Indeed, let $\varepsilon > 0$, let's find $n_0 \in \mathbb{N}^*$, such that

$$\forall\, n \geq n_0,\ |u_n - 0| < \varepsilon.$$

For any $n \in \mathbb{N}^*$, $|u_n - 0| = \frac{1}{n}$. By the Archimedean principle, there exists $n_0 \in \mathbb{N}^*$ such that $\frac{1}{n_0} < \varepsilon$. For any integer $n \geq n_0$,

$$|u_n - 0| = \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon.$$

3) Let $(u_n)_{n\in\mathbb{N}}$ be the sequence defined by $u_n = \dfrac{n-1}{n+1}$. Let's show that $(u_n)_{n\in\mathbb{N}}$ has limit 1 as n tends to infinity. Indeed, let $\varepsilon > 0$, let's find $n_0 \in \mathbb{N}$, such that

$$\forall\, n \geq n_0,\ |u_n - 1| < \varepsilon.$$

For any $n \in \mathbb{N}$, we have:

$$|u_n - 1| = \left|\frac{n-1}{n+1} - 1\right| = \frac{2}{n+1}.$$

But $\frac{2}{n+1} < \varepsilon$ if and only if $n > \frac{2}{\varepsilon} - 1$. We take $n_0 = E\left(\frac{2}{\varepsilon}\right) > \frac{2}{\varepsilon} - 1$. For any integer $n \geq n_0 > \frac{2}{\varepsilon} - 1$, we have:

$$|u_n - 1| = \frac{2}{n+1} < \varepsilon.$$

4) *For every $n \in \mathbb{N}$, put $u_n = (-1)^n$. Suppose, by contradiction, that $(u_n)_{n\in\mathbb{N}}$ has a limit $\ell \in \mathbb{R}$ as n tends to infinity. For $\varepsilon = \frac{1}{3}$, there exists $n_0 \in \mathbb{N}$ such that $|u_n - \ell| < \frac{1}{3}$. In particular, $|u_{n_0} - \ell| < \frac{1}{3}$ and $|u_{n_0+1} - \ell| < \frac{1}{3}$. So*

$$|u_{n_0+1} - u_{n_0}| = |(u_{n_0+1} - \ell) + (\ell - u_{n_0})| \leq |u_{n_0+1} - \ell| + |u_{n_0} - \ell| < \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

One of the integers n_0 and $n_0 + 1$ is even, the other is odd, then $|u_{n_0+1} - u_{n_0}| = 2$, so $2 < \frac{2}{3}$, which is impossible. Thus the sequence $(u_n)_{n\in\mathbb{N}}$ has no limit in $\mathbb{R}$ as n tends to infinity.

Remark 2.5.1 *By the definition of the limit of a real sequence, we dont modify the limit of a sequence if we modify a finite number of its terms. For example, if a real sequence $(u_n)_{n\in\mathbb{N}}$ has a limit $\ell \in \mathbb{R}$ as n tends to infinity, then the sequence $(u_n)_{n\geq n_0}$ (where $n_0 \in \mathbb{N}$) has limit ℓ as n tends to infinity.*

Proposition 2.5.1 *If a real sequence $(u_n)_{n\in\mathbb{N}}$ has a limit $\ell \in \mathbb{R}$ as n tends to infinity, then ℓ is unique. This limit will be denoted by $\lim\limits_{n\to+\infty} u_n$ or $\lim u_n$.*

Proof Suppose that $(u_n)_{n\in\mathbb{N}}$ has limits $\ell \in \mathbb{R}$ and $\ell' \in \mathbb{R}$ as n tends to infinity. Let's show that $\ell = \ell'$. Indeed, let $\varepsilon > 0$, then there exist $n_1, n_2 \in \mathbb{N}$ such that:

$$\forall\, n \geq n_1,\ |u_n - \ell| < \frac{\varepsilon}{2} \qquad \text{and} \qquad \forall\, n \geq n_2,\ |u_n - \ell'| < \frac{\varepsilon}{2}.$$

Put $n_0 = \max(n_1, n_2)$. Since $n_0 \geq n_1$ and $n_0 \geq n_2$, then $|u_{n_0} - \ell| < \frac{\varepsilon}{2}$ and $|u_{n_0} - \ell'| < \frac{\varepsilon}{2}$. So

$$|\ell - \ell'| = |\ell - u_{n_0} + u_{n_0} - \ell'| \leq |\ell - u_{n_0}| + |u_{n_0} - \ell'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $|\ell - \ell'| \leq \varepsilon$ for all $\varepsilon > 0$. Therefore $\ell - \ell' = 0$ by the exercise 9. Thus $\ell = \ell'$. $\square$

Exercise 41
By using the definition of the limit of a sequence, show that

$$\lim_{n\to+\infty} \frac{5n+2}{3n-1} = \frac{5}{3}, \qquad \lim_{n\to+\infty} \frac{n^2+n+1}{(n+1)^2} = 1 \quad \text{and} \quad \lim_{n\to+\infty} \frac{\sin(n) + 3\cos(n^2)}{\sqrt{n}} = 0.$$

Solution Consider the sequences $(u_n)_{n\geq 1}$, $(v_n)_{n\geq 1}$ and $(w_n)_{n\geq 1}$ defined by:

$$u_n = \frac{5n+2}{3n-1}, \qquad v_n = \frac{n^2+n+1}{(n+1)^2} \qquad \text{and} \qquad w_n = \frac{\sin(n) + 3\cos(n^2)}{\sqrt{n}}.$$

- Let $\varepsilon > 0$, let's find $n_0 \in \mathbb{N}$, such that $\left|u_n - \frac{5}{3}\right| < \varepsilon$ for all $n \geq n_0$.
  1st method: For any $n \geq 1$, we have:

$$\left|u_n - \frac{5}{3}\right| = \left|\frac{5n+2}{3n-1} - \frac{5}{3}\right| = \frac{11}{3(3n-1)} \leq \frac{11}{3(2n)} \leq \frac{11}{6n} \leq \frac{2}{n}.$$

  But $\dfrac{2}{n} < \varepsilon$ if and only if $n > \dfrac{2}{\varepsilon}$. We take $n_0 = E\left(\frac{2}{\varepsilon}\right) + 1 \in \mathbb{N}$. Then $n_0 > \frac{2}{\varepsilon}$. If $n \geq n_0 > \frac{2}{\varepsilon}$, then $\frac{2}{n} < \varepsilon$ and $\left|u_n - \frac{5}{3}\right| \leq \frac{2}{n} < \varepsilon$.
  2nd method: Suppose that n_0 exists. Then $\forall\, n \geq n_0 \geq 1$, we have:

$$\begin{aligned} n \geq n_0 &\Rightarrow 3n - 1 \geq 3n_0 - 1 \\ &\Rightarrow 3(3n-1) \geq 3(3n_0 - 1) \\ &\Rightarrow \frac{1}{3(3n-1)} \leq \frac{1}{3(3n_0-1)} \\ &\Rightarrow \frac{11}{3(3n-1)} \leq \frac{11}{3(3n_0-1)} \end{aligned}$$

  It is sufficient to take n_0 such that $\frac{11}{3(3n_0-1)} < \varepsilon$, i.e., $n_0 > \frac{1}{3}(\frac{11}{3\varepsilon} + 1)$, for example,

$$n_0 = E\left(\frac{11}{9\varepsilon} + \frac{1}{3}\right) + 1.$$

- Let $\varepsilon > 0$, let's find $n_0 \in \mathbb{N}$ such that $|v_n - 1| < \varepsilon$ for all $n \geq n_0$. For any $n \geq 1$,

$$|v_n - 1| = \left|\frac{n^2+n+1}{(n+1)^2} - 1\right| = \frac{n}{(n+1)^2} \leq \frac{n}{n^2} = \frac{1}{n}.$$

  But $\dfrac{1}{n} < \varepsilon$ if and only if $n > \dfrac{1}{\varepsilon}$. We take $n_0 = E(\frac{1}{\varepsilon}) + 1 \in \mathbb{N}$, then $n_0 > \frac{1}{\varepsilon}$. If $n \geq n_0$, then $n > \frac{1}{\varepsilon}$, so $\frac{1}{n} < \varepsilon$ and therefore $|v_n - 1| \leq \dfrac{1}{n} < \varepsilon$. Hence $\lim\limits_{n\to+\infty} v_n = 1$.

- Let $\varepsilon > 0$, let's find $n_0 \in \mathbb{N}$ such that $|w_n - 0| < \varepsilon$ for all $n \geq n_0$. For any $n \geq 1$,

$$|w_n - 0| = \left|\frac{\sin(n) + 3\cos(n^2)}{\sqrt{n}}\right| \leq \frac{|\sin(n)| + 3|\cos(n^2)|}{\sqrt{n}} \leq \frac{4}{\sqrt{n}}$$

  since $|\sin(n)| \leq 1$ and $|\cos(n^2)| \leq 1$. But $\dfrac{4}{\sqrt{n}} < \varepsilon$ if and only if $n > \dfrac{16}{\varepsilon^2}$. We take $n_0 = E\left(\frac{16}{\varepsilon^2}\right) + 1 \in \mathbb{N}$, then $n_0 > \frac{16}{\varepsilon^2}$. If $n \geq n_0$, then $n > \frac{16}{\varepsilon^2}$, so $\frac{4}{\sqrt{n}} < \varepsilon$ and therefore

$$|w_n - 0| \leq \frac{4}{\sqrt{n}} < \varepsilon.$$

  Hence $\lim\limits_{n\to+\infty} w_n = 0$. □

Proposition 2.5.2 *Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence and $\ell \in \mathbb{R}$. If there exists a sequence of positive terms $(a_n)_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to+\infty} a_n = 0$ and if there exists $n_0 \in \mathbb{N}$ such that*

$$|u_n - \ell| \leq a_n, \quad \forall\, n \geq n_0,$$

then $\lim\limits_{n\to+\infty} u_n = \ell$.

Proof Let $\varepsilon > 0$. Since $\lim\limits_{n\to+\infty} a_n = 0$, then there exists $n_1 \in \mathbb{N}$ such that $a_n = |a_n - 0| < \varepsilon$ for all integers $n \geq n_1$. Let $n_2 = \max(n_0, n_1)$. If $n \geq n_2$, then $n \geq n_0$ and $n \geq n_1$, so

$$|u_n - \ell| \leq a_n < \varepsilon.$$

Hence $\lim\limits_{n\to+\infty} u_n = \ell$. $\square$

Example 2.5.3 *Since $\lim\limits_{n\to+\infty} \dfrac{1}{n} = 0$ and for all $n \geq 1$,*

$$\left|\frac{\cos n}{n} - 0\right| \leq \frac{1}{n} \quad \textit{and} \quad \left|\frac{(-1)^n}{n} - 0\right| \leq \frac{1}{n},$$

then $\lim\limits_{n\to+\infty} \dfrac{\cos n}{n} = 0$ and $\lim\limits_{n\to+\infty} \dfrac{(-1)^n}{n} = 0$ by the proposition 2.5.2.

Proposition 2.5.3 *(Operations on the finite limits)*
Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two real sequences such that $\lim\limits_{n\to+\infty} u_n = \ell \in \mathbb{R}$ and $\lim\limits_{n\to+\infty} v_n = \ell' \in \mathbb{R}$. Then

1) $\lim\limits_{n\to+\infty} (u_n + v_n) = \ell + \ell'$.

2) $\lim\limits_{n\to+\infty} (a u_n) = a\ell$ *for all* $a \in \mathbb{R}$.

3) $\lim\limits_{n\to+\infty} (u_n v_n) = \ell\ell'$.

4) $\lim\limits_{n\to+\infty} |u_n| = |\ell|$.

5) $\lim\limits_{n\to+\infty} |u_n| = 0$ *if and only if* $\lim\limits_{n\to+\infty} u_n = 0$.

6) *If $\ell' \neq 0$, then there exists $n_0 \in \mathbb{N}$ such that $v_n \neq 0$, for all $n \geqslant n_0$. Then, we can consider the sequence $\left(\frac{u_n}{v_n}\right)_{n\geq n_0}$ and we have:*

$$\lim_{n\to+\infty} \frac{u_n}{v_n} = \frac{\ell}{\ell'}.$$

Corollary 2.5.1 *Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two real sequences. If $(u_n)_{n\in\mathbb{N}}$ is bounded and $\lim\limits_{n\to+\infty} v_n = 0$, then $\lim\limits_{n\to+\infty} u_n v_n = 0$.*

$\boxed{\text{Proof}}$ Since $(u_n)_{n\in\mathbb{N}}$ is bounded, then there exists a strictly positive real number M such that $|u_n| \leq M$ for all $n \in \mathbb{N}$. So

$$\big|u_n v_n - 0\big| \leq M|v_n|, \quad \forall\, n \in \mathbb{N}.$$

As $\lim\limits_{n\to+\infty} v_n = 0$, then, by using the proposition 2.5.3, we obtain:

$$\lim_{n\to+\infty} M|v_n| = M \lim_{n\to+\infty} |v_n| = 0.$$

Hence $\lim\limits_{n\to+\infty} u_n v_n = 0$ by the proposition 2.5.2. □

Definition 2.5.2 *Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence.*

1) *We say that $(u_n)_{n\in\mathbb{N}}$ has limit $+\infty$ as n tends to infinity, if its terms approach to $+\infty$ as n goes to infinity, i.e., beyond any positive real number, there exists an infinitely many terms of the sequence. In other words, $(u_n)_{n\in\mathbb{N}}$ has limit $+\infty$ as n tends to infinity if and only if*

$$\Big(\forall\, A > 0\Big)\Big(\exists\, n_0 \in \mathbb{N}\Big)\Bigg[\Big(\forall\, n \in \mathbb{N}\Big)\Big(n \geq n_0 \Rightarrow u_n > A\Big)\Bigg].$$

2) *We say that $(u_n)_{n\in\mathbb{N}}$ has limit $-\infty$ as n tends to infinity, if its terms approach to $-\infty$ as n goes to infinity, i.e., below any negative real number, there exists an infinitely many terms of the sequence. In other words, $(u_n)_{n\in\mathbb{N}}$ has limit $-\infty$ as n tends to infinity if and only if*

$$\Big(\forall\, A > 0\Big)\Big(\exists\, n_0 \in \mathbb{N}\Big)\Bigg[\Big(\forall\, n \in \mathbb{N}\Big)\Big(n \geq n_0 \Rightarrow u_n < -A\Big)\Bigg].$$

Remark 2.5.2 *Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence.*

1) *If $(u_n)_{n\in\mathbb{N}}$ has limit $+\infty$ (resp. $-\infty$) as n tends to infinity, then this limit is unique (as in the finite case in the proposition 2.5.1), i.e., $(u_n)_{n\in\mathbb{N}}$ has no limit in $\mathbb{R}$, neither $-\infty$ (resp. $+\infty$) as n tends to infinity. In this case, we write:*

$$\lim_{n\to+\infty} u_n = +\infty \quad (\textit{resp.} \ \lim_{n\to+\infty} u_n = -\infty).$$

2) *If $\lim\limits_{n\to+\infty} u_n = +\infty$ (resp. $-\infty$), then there exists $n_0 \in \mathbb{N}$ such that $u_n > 0$ (resp. $u_n < 0$) for all $n \geq n_0$.*

3) *The sequence $(u_n)_{n\in\mathbb{N}}$ tend vers $+\infty$ if and only if the sequence $(-u_n)_{n\in\mathbb{N}}$ tends to $-\infty$.*

Example 2.5.4

1) *For every $n \in \mathbb{N}$, put $u_n = n^2$. Let's show that $\lim\limits_{n\to+\infty} u_n = +\infty$. Indeed, for $A > 0$, put $n_0 = E(A) + 1 \in \mathbb{N}$. If $n \geq n_0$, then*

$$u_n = n^2 \geq n \geq n_0 > A.$$

2) For every $n \in \mathbb{N}^*$, put $u_n = \ln n$. Let's show that $\lim\limits_{n\to+\infty} u_n = +\infty$. Indeed, for $A > 0$, put $n_0 = E(e^A) + 1 \in \mathbb{N}^*$. If $n \geq n_0 > e^A$, then $n > e^A$, so $u_n = \ln n > A$.

Exercise 42

Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence such that $u_n > 0$ for all $n \in \mathbb{N}$.

1) **Prove that $\lim\limits_{n\to+\infty} u_n = 0$ if and only if $\lim\limits_{n\to+\infty} \dfrac{1}{u_n} = +\infty$.**

2) **Deduce that $\lim\limits_{n\to+\infty} u_n = +\infty$ if and only if $\lim\limits_{n\to+\infty} \dfrac{1}{u_n} = 0$.**

Solution

1) N.C. Let $A > 0$. For $\varepsilon = \frac{1}{A} > 0$, as $\lim\limits_{n\to+\infty} u_n = 0$, then there exists $n_0 \in \mathbb{N}$ such that $u_n = |u_n - 0| < \varepsilon$ for all $n \geq n_0$. So $\frac{1}{u_n} > \frac{1}{\varepsilon} = A$ for all $n \geq n_0$. Hence $\lim\limits_{n\to+\infty} \dfrac{1}{u_n} = +\infty$.

   S.C. Let $\varepsilon > 0$. For $A = \frac{1}{\varepsilon} > 0$, as $\lim\limits_{n\to+\infty} \dfrac{1}{u_n} = +\infty$, then there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{u_n} > A = \frac{1}{\varepsilon}$ for all $n \geq n_0$. So $|u_n - 0| = u_n < \varepsilon$ for all $n \geq n_0$. Hence $\lim\limits_{n\to+\infty} u_n = 0$.

2) This is a direct application of the part (1) on the real sequence $(v_n)_{n\in\mathbb{N}}$ defined by $v_n = \frac{1}{u_n} > 0$. $\square$

Proposition 2.5.4 *Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence.*

1) If there exists a real sequence $(a_n)_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to+\infty} a_n = +\infty$ and if there exists $n_0 \in \mathbb{N}$ such that

$$u_n \geq a_n, \quad \forall\, n \geq n_0,$$

then $\lim\limits_{n\to+\infty} u_n = +\infty$.

2) If there exists a real sequence $(b_n)_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to+\infty} b_n = -\infty$ and if there exists $n_0 \in \mathbb{N}$ such that

$$u_n \leq b_n, \quad \forall\, n \geq n_0,$$

then $\lim\limits_{n\to+\infty} u_n = -\infty$.

Proof

1) Let $A > 0$. Since $\lim\limits_{n\to+\infty} a_n = +\infty$, then there exists $n_1 \in \mathbb{N}$ such that $a_n > A$ for all $n \geq n_1$. Put $n_2 = \max(n_0, n_1) \in \mathbb{N}$. If $n \geq n_2$, then $n \geq n_0$ and $n \geq n_1$, so $u_n \geq a_n > A$. Hence $\lim\limits_{n\to+\infty} u_n = +\infty$.

2) Put $a_n = -b_n$ and $v_n = -u_n$ for all $n \geq n_0$. Then $v_n \geq a_n$ for all $n \geq n_0$. Since $\lim\limits_{n\to+\infty} b_n = -\infty$, then $\lim\limits_{n\to+\infty} a_n = +\infty$, so $\lim\limits_{n\to+\infty} v_n = +\infty$ by the part (1). Hence $\lim\limits_{n\to+\infty} u_n = -\infty$. □

Example 2.5.5 *Since $n + \cos n \geq n - 1$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to+\infty}(n-1) = +\infty$, then $\lim\limits_{n\to+\infty}(n + \cos n) = +\infty$ by the proposition 2.5.4.*

Exercise 43
Show that the sequence $(u_n)_{n\in\mathbb{N}^*}$ defined by

$$u_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}},$$

tends to $+\infty$.

$\boxed{\text{Solution}}$ For any $1 \leq k \leq n$, we have $\sqrt{k} \leq \sqrt{n}$, and therefore $\frac{1}{\sqrt{k}} \geq \frac{1}{\sqrt{n}}$. Hence, for any $n \in \mathbb{N}^*$,

$$\begin{aligned} u_n &= \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \\ &\geq \underbrace{\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}}}_{n\text{-terms}} = \frac{n}{\sqrt{n}} = \sqrt{n}. \end{aligned}$$

Since $\lim\limits_{n\to+\infty} \sqrt{n} = +\infty$, then $\lim\limits_{n\to+\infty} u_n = +\infty$ by the proposition 2.5.4. □

Proposition 2.5.5 *(Operations on the infinite limits)*
Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two real sequences.

1) If $\lim\limits_{n\to+\infty} u_n = \pm\infty$, then $\lim\limits_{n\to+\infty} |u_n| = +\infty$.

2) If $a \in \mathbb{R}^*$ and the sequence $(u_n)_{n\in\mathbb{N}}$ tends to $+\infty$ (resp. $-\infty$), then the sequence $(au_n)_{n\in\mathbb{N}}$ tends to $+\infty$ (resp. $-\infty$) if $a > 0$, and to $-\infty$ (resp. $+\infty$) if $a < 0$.

3) <u>Sum</u>:

- *If $\lim\limits_{n\to+\infty} u_n = +\infty$ (resp. $-\infty$) and $(v_n)_{n\in\mathbb{N}}$ is bounded, then*

$$\lim_{n\to+\infty}(u_n + v_n) = +\infty \quad (\textit{resp.} - \infty).$$

- *If $\lim\limits_{n\to+\infty} u_n = +\infty$ (resp. $-\infty$) and $\lim\limits_{n\to+\infty} v_n = \ell \in \mathbb{R}$, then*

$$\lim_{n\to+\infty}(u_n + v_n) = +\infty \quad (\textit{resp.} - \infty).$$

- *If* $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} v_n = +\infty$ *(resp.* $-\infty$*), then*

$$\lim_{n\to+\infty}(u_n + v_n) = +\infty \quad (\textit{resp.} - \infty).$$

4) *Product:*

- *If* $\lim\limits_{n\to+\infty} u_n = \ell \in \mathbb{R}$ *and* $\lim\limits_{n\to+\infty} v_n = +\infty$ *(resp.* $-\infty$*), then*

$$\lim_{n\to+\infty} u_n v_n = +\infty \quad (\textit{resp.} - \infty) \quad \textit{if } \ell > 0.$$

and

$$\lim_{n\to+\infty} u_n v_n = -\infty \quad (\textit{resp.} + \infty) \quad \textit{if } \ell < 0.$$

- *If* $\lim\limits_{n\to+\infty} u_n = \pm\infty$ *and* $\lim\limits_{n\to+\infty} v_n = \pm\infty$*, then, following the usual product of signs, we obtain:*

$$\lim_{n\to+\infty} u_n v_n = \pm\infty.$$

5) *Quotient:*

- *If* $\lim\limits_{n\to+\infty} |u_n| = +\infty$ *and* $(v_n)_{n\in\mathbb{N}}$ *is bounded, then* $\lim\limits_{n\to+\infty} \left|\dfrac{u_n}{v_n}\right| = +\infty$.
- *If* $\lim\limits_{n\to+\infty} |u_n| = +\infty$ *and* $\lim\limits_{n\to+\infty} v_n = \ell \in \mathbb{R}$*, then* $\lim\limits_{n\to+\infty} \left|\dfrac{u_n}{v_n}\right| = +\infty$.
- *If* $(u_n)_{n\in\mathbb{N}}$ *is bounded and* $\lim\limits_{n\to+\infty} |v_n| = +\infty$*, then* $\lim\limits_{n\to+\infty} \dfrac{u_n}{v_n} = 0$.
- *If* $\lim\limits_{n\to+\infty} u_n = \ell \in \mathbb{R}$ *and* $\lim\limits_{n\to+\infty} |v_n| = +\infty$*, then* $\lim\limits_{n\to+\infty} \dfrac{u_n}{v_n} = 0$.

Remark 2.5.3

1) The converse of the part (1) of the proposition 2.5.5 is not necessary true. Indeed, if $u_n = (-2)^n$*, then* $|u_n| = 2^n$*, so* $\lim\limits_{n\to+\infty} |u_n| = +\infty$*. On the other hand,* $\lim\limits_{n\to+\infty} u_n \neq \pm\infty$ *(see the proposition 2.5.6).*

2) If $\lim\limits_{n\to+\infty} u_n = +\infty$ *and* $\lim\limits_{n\to+\infty} v_n = -\infty$*, then we can say nothing about* $\lim\limits_{n\to+\infty} (u_n + v_n)$*. We are in presence of an indeterminate form* $\infty - \infty$*. For example, by using the part (3) of the proposition 2.5.5, we have:*

$$\lim_{n\to+\infty} (n^2 - n) = \lim_{n\to+\infty} n^2\left(1 - \frac{1}{n}\right) = +\infty.$$

In the other side, by multiplying by the conjugate, we obtain:

$$\lim_{n\to+\infty} (\sqrt{n+1} - \sqrt{n}) = \lim_{n\to+\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0.$$

For this reason, we call it an indeterminate form (since it can take any value).

3) If $\lim_{n\to+\infty} u_n = 0$ *and* $\lim_{n\to+\infty} v_n = \pm\infty$*, then we can say nothing about* $\lim_{n\to+\infty} u_n v_n$*. We are in presence of an indeterminate form* $0 \times \infty$*.*

4) If $\lim_{n\to+\infty} u_n = \pm\infty$ *and* $\lim_{n\to+\infty} v_n = \pm\infty$*, then we can say nothing about* $\lim_{n\to+\infty} \frac{u_n}{v_n}$*. We are in presence of an indeterminate form* $\frac{\infty}{\infty}$*. For example,*

$$\lim_{n\to+\infty} \frac{n+1}{n} = \lim_{n\to+\infty} \left(1 + \frac{1}{n}\right) = 1.$$

$$\lim_{n\to+\infty} \frac{n^2+1}{n} = \lim_{n\to+\infty} \left(n + \frac{1}{n}\right) = +\infty.$$

5) If $\lim_{n\to+\infty} u_n = 0$ *and* $\lim_{n\to+\infty} v_n = 0$*, then we can say nothing about* $\lim_{n\to+\infty} \frac{u_n}{v_n}$*. We are in presence of an indeterminate form* $\frac{0}{0}$*.*

Proposition 2.5.6 *(References limits)*

1) We have:

$$\lim_{n\to+\infty} \frac{1}{n^k} = \begin{cases} 0 & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ +\infty & \text{if } k < 0 \end{cases}$$

2) For any $a_0, a_1, \dots, a_s, b_0, b_1, \dots, b_t \in \mathbb{R}$ *such that* $a_s \neq 0$ *and* $b_t \neq 0$*, we have:*

$$\lim_{n\to+\infty} \frac{a_s n^s + \cdots + a_1 n + a_0}{b_t n^t + \cdots + b_1 n + b_0} = \left(\frac{a_s}{b_t}\right) \lim_{n\to+\infty} \frac{1}{n^{t-s}}.$$

3) We have:

$$\lim_{n\to+\infty} a^n = \begin{cases} 0 & \text{if } -1 < a < 1 \\ 1 & \text{if } a = 1 \\ +\infty & \text{if } a > 1 \\ \text{does not exist} & \text{if } a \leq -1 \end{cases}$$

4) For any $a \in]0, +\infty[$ *and* $k \in \mathbb{N}$*, we have:*

$$\lim_{n\to+\infty} \frac{\ln n}{n^a} = 0, \quad \lim_{n\to+\infty} \frac{e^n}{n^k} = +\infty \quad \text{and} \quad \lim_{n\to+\infty} \frac{a^n}{n!} = 0.$$

Consequently, there exists $n_0 \in \mathbb{N}$ *such that, for all* $n \geq n_0$*,*

$$\ln n \leq n^a \leq a^n \leq n!.$$

Exercise 44
For every $n \in \mathbb{N}$, put $u_n = \frac{n + 2\cos n}{n+1}$. Show that $\lim_{n\to+\infty} u_n = 1$.

Solution For any $n \in \mathbb{N}^*$, $u_n = \dfrac{1 + \frac{2\cos n}{n}}{1 + \frac{1}{n}}$. Since

$$\lim_{n \to +\infty} \frac{2\cos n}{n} = 0 \quad \text{and} \quad \lim_{n \to +\infty} \frac{1}{n} = 0,$$

then $\lim\limits_{n \to +\infty} u_n = \frac{1+0}{1+0} = 1$. □

2.6 Convergent sequences, divergent sequences

Definition 2.6.1 *Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence.*

1) *We say that $(u_n)_{n\in\mathbb{N}}$ is convergent if there exists $\ell \in \mathbb{R}$ such that $\lim\limits_{n\to+\infty} u_n = \ell$. In this case, we say that $(u_n)_{n\in\mathbb{N}}$ is convergent to ℓ.*

2) *We say that $(u_n)_{n\in\mathbb{N}}$ is divergent if it is not convergent, i.e., if $\lim\limits_{n\to+\infty} u_n = +\infty$ or $\lim\limits_{n\to+\infty} u_n = -\infty$ or $\lim\limits_{n\to+\infty} u_n$ does not exist in $\mathbb{R}$.*

Example 2.6.1

1) *The sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}^*}$ is convergent to 0.*

2) *The sequence $((-1)^n)_{n\in\mathbb{N}}$ is divergent since $\lim\limits_{n\to+\infty} (-1)^n$ does not exist in $\mathbb{R}$ (by the example 2.5.2).*

3) *The sequence $\left(n^2\right)_{n\in\mathbb{N}}$ is divergent since $\lim\limits_{n\to+\infty} n^2 = +\infty$ (by the example 2.5.4).*

Proposition 2.6.1 *Every convergent real sequence is bounded.*

Proof Let $(u_n)_{n\in\mathbb{N}}$ be a convergent real sequence to a limit $\ell \in \mathbb{R}$. For $\varepsilon = 1 > 0$, there exists $n_0 \in \mathbb{N}$ such that $|u_n - \ell| < 1$ for all $n \geq n_0$. So

$$\ell - 1 < u_n < \ell + 1, \quad \forall n \geq n_0.$$

Put

$$M = \max(\ell + 1, u_0, \ldots, u_{n_0}) \quad \text{and} \quad N = \min(\ell - 1, u_0, \ldots, u_{n_0}).$$

Then

$$N \leq u_n \leq M, \quad \forall n \in \mathbb{N}.$$

Hence the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded from above by M and from below by N. Thus it is bounded. □

Remark 2.6.1

1) *The converse of the proposition 2.6.1 is not necessary true. Indeed, since* $-1 \leq (-1)^n \leq 1$ *for all* $n \in \mathbb{N}$*, then the sequence* $((-1)^n)_{n\in\mathbb{N}}$ *is bounded. On the other hand, this sequence is divergent.*

2) *In general, the proposition 2.6.1 is used to show the divergence of a real sequence by using its contrapositive, i.e., if the sequence* $(u_n)_{n\in\mathbb{N}}$ *is not bounded, then it is divergent. For example, the sequence* $(u_n)_{n\in\mathbb{N}}$ *defined by*

$$u_n = n + (-1)^n n,$$

is not bounded (see exercise 37). So $(u_n)_{n\in\mathbb{N}}$ *is divergent.*

3) *If* $\lim\limits_{n\to+\infty} u_n = +\infty$ *(resp.* $-\infty$*), then the sequence* $(u_n)_{n\in\mathbb{N}}$ *is not bounded from above (resp. bounded from below), so it is not bounded.*

Proposition 2.6.2 *(Relation between the convergences of a sequence and its subsequences) A real sequence* $(u_n)_{n\in\mathbb{N}}$ *is convergent to a limit* $\ell \in \mathbb{R}$ *if and only if every subsequence of* $(u_n)_{n\in\mathbb{N}}$ *is convergent to* ℓ*.*

Proof

N.C. Let $(u_n)_{n\in\mathbb{N}}$ be a convergent real sequence to a limit $\ell \in \mathbb{R}$. Let $\left(u_{\varphi(n)}\right)_{n\in\mathbb{N}}$ be a subsequence of the sequence $(u_n)_{n\in\mathbb{N}}$ where $\varphi : \mathbb{N} \longmapsto \mathbb{N}$ is a strictly increasing mapping. Let's show that $\lim\limits_{n\to+\infty} u_{\varphi(n)} = \ell$. Indeed, let $\varepsilon > 0$, as $\lim\limits_{n\to+\infty} u_n = \ell$, then there exists $n_0 \in \mathbb{N}$ such that

$$(*) \qquad |u_n - \ell| < \varepsilon, \quad \forall\, n \geq n_0.$$

If $n \geq n_0$, then $\varphi(n) > \varphi(n_0)$ (since φ is strictly increasing), and as $\varphi(n_0) \geq n_0$ (by lemma 2.2.1), then $\varphi(n) \geq n_0$, hence $|u_{\varphi(n)} - \ell| < \varepsilon$ by $(*)$.
S.C. Since $(u_n)_{n\in\mathbb{N}}$ is a subsequence of itself, then it is convergent to ℓ. □

Proposition 2.6.3 *A real sequence* $(u_n)_{n\in\mathbb{N}}$ *is convergent to a limit* $\ell \in \mathbb{R}$ *if and only if the two subsequences* $(u_{2n})_{n\in\mathbb{N}}$ *and* $(u_{2n+1})_{n\in\mathbb{N}}$ *are convergent to* ℓ*.*

Proof

N.C. This is a direct consequence of the proposition 2.6.2.
S.C. Let $\varepsilon > 0$, as $\lim\limits_{n\to+\infty} u_{2n} = \ell$ and $\lim\limits_{n\to+\infty} u_{2n+1} = \ell$, then there exist $n_1, n_2 \in \mathbb{N}$ such that

$$(*) \qquad |u_{2n} - \ell| < \varepsilon, \quad \forall\, n \geq n_1 \quad \text{and} \quad |u_{2n+1} - \ell| < \varepsilon, \quad \forall\, n \geq n_2.$$

Let $n_0 = \max(2n_1, 2n_2 + 1) \in \mathbb{N}$. Let $n \in \mathbb{N}$ such that $n \geq n_0$, then $n \geq 2n_1$ and $n \geq 2n_2 + 1$.

- If n is even, then there exists $k \in \mathbb{N}$ such that $n = 2k$. So $2k \geq 2n_1$, and then $k \geq n_1$. Hence $|u_n - \ell| = |u_{2k} - \ell| < \varepsilon$ by $(*)$.

- If n is odd, then there exists $k \in \mathbb{N}$ such that $n = 2k+1$. So $2k+1 \geq 2n_2+1$, and then $k \geq n_2$. Hence $|u_n - \ell| = |u_{2k+1} - \ell| < \varepsilon$ by $(*)$.

Thus $\lim\limits_{n \to +\infty} u_n = \ell$. □

Example 2.6.2

1) *For every $n \in \mathbb{N}$, put $u_n = (-1)^n$. Since $u_{2n} = (-1)^{2n} = 1$ and $u_{2n+1} = (-1)^{2n+1} = -1$ for all $n \in \mathbb{N}$, then*

$$\lim_{n \to +\infty} u_{2n} = 1 \neq -1 = \lim_{n \to +\infty} u_{2n+1}.$$

So the sequence $\big((-1)^n\big)_{n \in \mathbb{N}}$ has no limit in $\mathbb{R}$.

2) *For every $n \in \mathbb{N}^*$, put $u_n = \frac{(-1)^n}{n}$. Since $u_{2n} = \frac{1}{2n}$ and $u_{2n+1} = \frac{-1}{2n+1}$ for all $n \in \mathbb{N}^*$, then*

$$\lim_{n \to +\infty} u_{2n} = 0 = \lim_{n \to +\infty} u_{2n+1}.$$

So the sequence $(u_n)_{n \in \mathbb{N}^*}$ is convergent to 0 (see the example 2.5.3).

3) *For every $n \in \mathbb{N}$, put:*

$$u_n = \begin{cases} n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Since $u_{2n} = 2n$ for all $n \in \mathbb{N}$, then $\lim\limits_{n \to +\infty} u_{2n} = +\infty$. So the sequence $(u_n)_{n \in \mathbb{N}}$ has no limit in $\mathbb{R}$. Hence this sequence is divergent.

Remark 2.6.2 *By a similar proof, the propositions 2.6.2 and 2.6.3 remain true when $\ell = \pm\infty$.*

Proposition 2.6.4 *Let $(u_n)_{n \in \mathbb{N}}$ be a convergent sequence to a limit $\ell \in \mathbb{R}$. Let $a, b \in \mathbb{R}$.*

1) *If there exists $n_0 \in \mathbb{N}$ such that $u_n \leq a$ for all $n \geq n_0$, then $\ell \leq a$.*

2) *If there exists $n_0 \in \mathbb{N}$ such that $u_n \geq b$ for all $n \geq n_0$, then $\ell \geq b$.*

Proof

1) Suppose that $\ell > a$. For $\varepsilon = \ell - a > 0$, there exists $n_1 \in \mathbb{N}$ such that $|u_n - \ell| < \varepsilon$ for all $n \geq n_1$, so $a < u_n < 2\ell - a$ for all $n \geq n_1$. Put $n_2 = \max(n_0, n_1)$. As $n_2 \geq n_0$, then $u_{n_2} \leq a$ and as $n_2 \geq n_1$, then $a < u_{n_2} < 2\ell - a$, which is impossible. Hence $\ell \leq a$.

2) For every $n \in \mathbb{N}$, put $v_n = -u_n$, then

$$\lim_{n \to +\infty} v_n = - \lim_{n \to +\infty} u_n = -\ell.$$

As $u_n \geq b$ for all $n \geq n_0$, then $v_n \leq -b$ for all $n \geq n_0$. By the part (1), $-\ell \leq -b$, and therefore $\ell \geq b$. □

Corollary 2.6.1 *Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two convergent sequences to two real numbers ℓ and ℓ' respectively.*

1) If there exists $n_0 \in \mathbb{N}$ such that $u_n \geq 0$ for all $n \geq n_0$, then $\ell \geq 0$.

2) If there exists $n_0 \in \mathbb{N}$ such that $u_n \leq v_n$ for all $n \geq n_0$, then $\ell \leq \ell'$.

Proof

1) It is sufficient to apply the part (2) of the proposition 2.6.4 for $b = 0$.

2) For every $n \in \mathbb{N}$, put $w_n = v_n - u_n$, then

$$\lim_{n\to+\infty} w_n = \lim_{n\to+\infty} v_n - \lim_{n\to+\infty} u_n = \ell' - \ell.$$

Since $u_n \leq v_n$ for all $n \geq n_0$, then $w_n \geq 0$ for all $n \geq n_0$, so $\ell' - \ell \geq 0$ by the part (1). Hence $\ell \leq \ell'$. □

Theorem 2.6.1 *(Sandwich theorem on the sequences)*
Let $(u_n)_{n\in\mathbb{N}}$, $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be real sequences. Suppose that there exists $n_0 \in \mathbb{N}$ such that

$$a_n \leq u_n \leq b_n, \quad \forall\, n \geq n_0.$$

If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are convergent to the same limit $\ell \in \mathbb{R}$, then $(u_n)_{n\in\mathbb{N}}$ is convergent to ℓ.

Proof Let $\varepsilon > 0$. Since $\lim\limits_{n\to+\infty} a_n = \ell$ and $\lim\limits_{n\to+\infty} b_n = \ell$, then there exist $n_1, n_2 \in \mathbb{N}$ such that,

$$|a_n - \ell| < \varepsilon, \quad \forall\, n \geq n_1 \qquad \text{and} \qquad |b_n - \ell| < \varepsilon, \quad \forall\, n \geq n_2.$$

Let $n_3 = \max(n_0, n_1, n_2)$. If $n \geq n_3$, then $n \geq n_0$, $n \geq n_1$ and $n \geq n_2$, so

$$\ell - \varepsilon < a_n \leq u_n \leq b_n < \ell + \varepsilon,$$

i.e., $|u_n - \ell| < \varepsilon$. Hence $\lim\limits_{n\to+\infty} u_n = \ell$ □

Remark 2.6.3 *In the proposition 2.6.4, the corollary 2.6.1 and the theorem 2.6.1, if the inequalities are strict in the given condition, then taking limits does not necessary yield to strict inequalities. For example, $\frac{1}{n} > 0$ for all $n \in \mathbb{N}^*$, but $\lim\limits_{n\to+\infty} \dfrac{1}{n} = 0$.*

Example 2.6.3

1) Let $k \in \mathbb{N}^*$. Since, for all $n \in \mathbb{N}^*$,

$$0 \leq \frac{1}{n^k} \leq \frac{1}{n},$$

and $\lim\limits_{n\to+\infty} \dfrac{1}{n} = 0$, then $\lim\limits_{n\to+\infty} \dfrac{1}{n^k} = 0$ by the Sandwich theorem.

2) For any $n \in \mathbb{N}^*$, $-1 \leq \sin n \leq 1$, *so*

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}.$$

As $\lim_{n\to+\infty} \frac{1}{n} = 0$, *then* $\lim_{n\to+\infty} \frac{\sin n}{n} = 0$ *by the Sandwich theorem.*

Exercise 45
By using the Sandwich theorem, show that, for all $a \in \mathbb{R}$,

$$\lim_{n\to+\infty} \frac{a^n}{n!} = 0.$$

Solution Let $n_0 \in \mathbb{N}$ such that $n_0 > |a|$ (take for example $n_0 = E(|a|) + 1$). If $n > n_0$, then

$$0 \leq \left|\frac{a^n}{n!}\right| = \frac{|a|\cdots|a|}{1 \times 2 \times \cdots \times n_0} \times \frac{|a|\cdots|a|}{(n_0+1)(n_0+2)\cdots(n-1)} \times \frac{|a|}{n}.$$

For any $1 \leq k \leq n - n_0 - 1$, $|a| < n_0 + k$, then $\frac{|a|}{n_0+k} < 1$. So

$$0 \leq \left|\frac{a^n}{n!}\right| = \frac{|a|^{n_0}}{n_0!} \times \frac{|a|}{n}, \qquad \forall n > n_0.$$

As $\lim_{n\to+\infty} \frac{|a|}{n} = 0$, then $\lim_{n\to+\infty} \left|\frac{a^n}{n!}\right| = 0$ by the Sandwich theorem. So $\lim_{n\to+\infty} \frac{a^n}{n!} = 0$ by the proposition 2.5.3. □

Exercise 46
Let $(u_n)_{n\in\mathbb{N}}$ be a convergent sequence to a real number ℓ.

1) **Prove that if $\ell > 0$, then there exists $n_0 \in \mathbb{N}$ such that $u_n > 0$ for all $n \geq n_0$.**

2) **Deduce that if $\ell < 0$ (resp. $\ell \neq 0$), then there exists $n_0 \in \mathbb{N}$ such that $u_n < 0$ (resp. $u_n \neq 0$) for all $n \geq n_0$.**

Solution

1) Suppose that $\ell > 0$. For $\varepsilon = \ell > 0$, there exists $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$,

$$|u_n - \ell| < \varepsilon, \qquad \text{i.e.,} \qquad \ell - \ell < u_n < \ell + \ell.$$

So $u_n > 0$ for all $n \geq n_0$.

2)
- Suppose that $\ell < 0$. Put $v_n = -u_n$ for every $n \in \mathbb{N}$, then the sequence $(v_n)_{n\in\mathbb{N}}$ is convergent to $-\ell > 0$. By the part (1), there exists $n_0 \in \mathbb{N}$ such that $v_n > 0$ for all $n \geq n_0$. So $u_n < 0$ for all $n \geq n_0$.
- If $\ell \neq 0$, then $\ell > 0$ (resp. or $\ell < 0$), so there exists $n_0 \in \mathbb{N}$ such that $u_n > 0$ (resp. or $u_n < 0$) for all $n \geq n_0$. So $u_n \neq 0$ for all $n \geq n_0$. □

Exercise 47
Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two convergent sequences to two real numbers ℓ_1 and ℓ_2 respectively. Show that if $\ell_1 < \ell_2$, then there exists $n_0 \in \mathbb{N}$ such that $u_n < v_n$ for all $n \geq n_0$.

Solution

<u>1^{st} method</u>: For every $n \in \mathbb{N}$, put $w_n = v_n - u_n$. Since the sequence $(w_n)_{n\in\mathbb{N}}$ is convergent to $\ell_2 - \ell_1 > 0$, then, by the exercise 46, there exists $n_0 \in \mathbb{N}$ such that $w_n > 0$ for all $n \geq n_0$. So $u_n < v_n$ for all $n \geq n_0$.
<u>2^{nd} method</u>: Let $\varepsilon = \frac{\ell_2-\ell_1}{2} > 0$. Since $\lim\limits_{n\to+\infty} u_n = \ell_1$ and $\lim\limits_{n\to+\infty} v_n = \ell_2$, then there exist $n_1, n_2 \in \mathbb{N}$ such that

$$(\forall\, n \geq n_1)(|u_n - \ell_1| < \varepsilon) \quad \text{and} \quad (\forall\, n \geq n_2)(|v_n - \ell_2| < \varepsilon).$$

Let $n_0 = \max(n_1, n_2) \in \mathbb{N}$. If $n \geq n_0$, then $n \geq n_1$ and $n \geq n_2$, so

$$u_n < \ell_1 + \varepsilon = \frac{\ell_1 + \ell_2}{2} = \ell_2 - \varepsilon < v_n. \quad \square$$

Exercise 48
Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence.

1) **Suppose that there exists $k \in \mathbb{R}$ such that $0 < k < 1$ and $|u_{n+1}| \leq k|u_n|$ for all $n \in \mathbb{N}$.**
   i) **Show that $|u_n| \leq k^n|u_0|$ for all $n \in \mathbb{N}$.**
   ii) **Deduce that $\lim\limits_{n\to+\infty} u_n = 0$.**
2) **Suppose that, for all $n \in \mathbb{N}$, $u_n = \frac{a^n}{n!}$ where $a \in \mathbb{R}$.**
   (a) **Show that there exists $n_0 \in \mathbb{N}$ such that $|u_{n+1}| \leq \frac{1}{2}|u_n|$ for all $n \geq n_0$.**
   (b) **Deduce that $\lim\limits_{n\to+\infty} u_n = 0$ (see the exercise 45).**
   (c) **Deduce that there exists $n_1 \in \mathbb{N}$ such that $a^n < n!$ for all $n \geq n_1$.**

Solution

1) i) By mathematical induction: for $n = 0$, $|u_0| \leq k^0|u_0| = |u_0|$. Suppose that $|u_n| \leq k^n|u_0|$, then

$$|u_{n+1}| \leq k|u_n| \leq kk^n|u_0| = k^{n+1}|u_0|$$

since $k > 0$.

   ii) We have:

$$0 \leq |u_n| \leq k^n|u_0|$$

As $0 < k < 1$, then $\lim\limits_{n\to+\infty} k^n = 0$, and therefore $\lim\limits_{n\to+\infty} k^n|u_0| = 0$. So, by the Sandwich theorem, $\lim\limits_{n\to+\infty} |u_n| = 0$. Thus $\lim\limits_{n\to+\infty} u_n = 0$.

2) (a) If $a = 0$, then $u_n = 0$ for all $n \in \mathbb{N}$ and the inequality is satisfied. Suppose that $a \neq 0$. For any $n \in \mathbb{N}$,

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{|a|^{n+1}}{(n+1)!} \times \frac{n!}{|a|^n} = \frac{|a|}{n+1}.$$

Since $\lim\limits_{n\to+\infty} \frac{|a|}{n+1} = 0$, then, for $\varepsilon = \frac{1}{2}$, there exists $n_0 \in \mathbb{N}$ such that $\frac{|a|}{n+1} < \frac{1}{2}$ for all $n \geq n_0$, so $\left|\frac{u_{n+1}}{u_n}\right| \leq \frac{1}{2}$ for all $n \geq n_0$. Hence $|u_{n+1}| \leq \frac{1}{2}|u_n|$ for all $n \geq n_0$.

(b) Put $k = \frac{1}{2}$, then $0 < k < 1$ and $|u_{n+1}| \leq k|u_n|$ for all $n \geq n_0$. By the part (1), the sequence $(u_n)_{n\geq n_0}$ is convergent to 0. Hence $\lim\limits_{n\to+\infty} u_n = 0$.

(c) Since $\lim\limits_{n\to+\infty} u_n = 0$, then, for $\varepsilon = 1$, there exists $n_1 \in \mathbb{N}$ such that $u_n < 1$ for all $n \geq n_1$. So $a^n < n!$ for all $n \geq n_1$. □

Exercise 49

Calculate the limit (if it exists) of the sequence $(u_n)_n$ in each of the following cases:

1) $u_n = \dfrac{1 + 3 + 5 + \cdots + (2n-1)}{1 + 2 + 3 + \cdots + n}$.

2) $u_n = \dfrac{n \sin n}{n^2 + 1}$.

3) $u_n = \displaystyle\sum_{k=1}^{n} \frac{n}{n^2 + k} = \frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}$.

Solution

1) Let $n \in \mathbb{N}^*$. The numerator A of u_n is the sum of n first terms of an arithmetic sequence $(a_n)_{n\geq 1}$ with ratio $r = 2$ and first term $a_1 = 1$ (where $a_n = 2n - 1$). So

$$A = a_1 + \cdots + a_n = \frac{n}{2}(a_1 + a_n) = n^2.$$

In the other side, as $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$ (by the exercise 31), then $u_n = \dfrac{2n^2}{n(n+1)}$. So $\lim\limits_{n\to+\infty} u_n = 2$.

2) For any $n \in \mathbb{N}$, as $|\sin n| \leq 1$, then $0 \leq |u_n| \leq \dfrac{n}{n^2+1}$. Since $\lim\limits_{n\to+\infty} \dfrac{n}{n^2+1} = \lim\limits_{n\to+\infty} \dfrac{1}{n} = 0$, then $\lim\limits_{n\to+\infty} |u_n| = 0$ by the Sandwich theorem, and therefore $\lim\limits_{n\to+\infty} u_n = 0$.

3) Let $n \in \mathbb{N}^*$. For any $1 \leq k \leq n$, $n^2+k \geq n^2+1$, so $\dfrac{n}{n^2+k} \leq \dfrac{n}{n^2+1}$. Hence

$$u_n \leq \frac{n}{n^2+1} + \frac{n}{n^2+1} + \cdots + \frac{n}{n^2+1} = n\left(\frac{n}{n^2+1}\right) = \frac{n^2}{n^2+1}.$$

For any $1 \leq k \leq n$, $n^2+k \leq n^2+n$, so $\dfrac{n}{n^2+k} \geq \dfrac{n}{n^2+n}$. Therefore

$$u_n \geq \frac{n}{n^2+n} + \frac{n}{n^2+n} + \cdots + \frac{n}{n^2+n} = n\left(\frac{n}{n^2+n}\right) = \frac{n}{n+1}.$$

Hence

$$\frac{n}{n+1} \leq u_n \leq \frac{n^2}{n^2+1}.$$

Since $\lim\limits_{n\to+\infty} \dfrac{n}{n+1} = \lim\limits_{n\to+\infty} \dfrac{n^2}{n^2+1} = 1$, then $\lim\limits_{n\to+\infty} u_n = 1$ by the Sandwich theorem. □

Exercise 50

Let $(u_n)_{n\in\mathbb{N}}$ be a convergent sequence to a limit $\ell \in \mathbb{R}$ such that $u_n \geq 0$ for all $n \in \mathbb{N}$.

1) **Verify that $\ell \geq 0$.**

2) **Prove that $\lim\limits_{n\to+\infty} \sqrt{u_n} = \sqrt{\ell}$.**

Solution

1) $\ell \geq 0$ by the corollary 2.6.1.

2) For any $n \in \mathbb{N}$, we have:

$$|\sqrt{u_n} - \sqrt{\ell}| = \left|\frac{(\sqrt{u_n} - \sqrt{\ell})(\sqrt{u_n} + \sqrt{\ell})}{\sqrt{u_n} + \sqrt{\ell}}\right| = \frac{|u_n - \ell|}{\sqrt{u_n} + \sqrt{\ell}} \leq \frac{1}{\sqrt{\ell}}\Big|u_n - \ell\Big|,$$

by using the fact that $\sqrt{u_n} \geq 0$. Put $a_n = \dfrac{1}{\sqrt{\ell}}\Big|u_n - \ell\Big|$ for all $n \in \mathbb{N}$. Since $\lim\limits_{n\to+\infty} u_n = \ell$, then $\lim\limits_{n\to+\infty} a_n = 0$, and as

$$|\sqrt{u_n} - \sqrt{\ell}| \leq a_n, \quad \forall\, n \geq 0,$$

then $\lim\limits_{n\to+\infty} \sqrt{u_n} = \sqrt{\ell}$ by the proposition 2.5.2. □

Theorem 2.6.2

1) *Every increasing and bounded from above sequence is convergent to its supremum, i.e., if a sequence $(u_n)_{n\in\mathbb{N}}$ is increasing and bounded from above, then it is convergent and* $\lim\limits_{n\to+\infty} u_n = \sup u_n$.

2) *Every decreasing and bounded from below sequence is convergent to its infimum, i.e., if a sequence* $(u_n)_{n\in\mathbb{N}}$ *is decreasing and bounded from below, then it is convergent and* $\lim_{n\to+\infty} u_n = \inf u_n$.

Proof

1) Since $(u_n)_{n\in\mathbb{N}}$ is bounded from above, then it has a supremum, put $\ell = \sup u_n = \sup A$ where $A = \{u_n,\ n \in \mathbb{N}\}$. Let $\varepsilon > 0$. By the supremum criterion, there exists $n_0 \in \mathbb{N}$ such that
$$\ell - \varepsilon < u_{n_0} \leq \ell.$$
Let $n \geq n_0$, then $u_{n_0} \leq u_n$ (since the sequence is increasing) and $u_n \leq \ell$ (since ℓ is an upper bound of the sequence), so
$$\ell - \varepsilon < u_n \leq \ell < \ell + \varepsilon.$$
Hence $|u_n - \ell| < \varepsilon$. Thus $\lim_{n\to+\infty} u_n = \ell$.

2) For every $n \in \mathbb{N}$, put $v_n = -u_n$. The sequence $(v_n)_{n\in\mathbb{N}}$ is increasing and bounded from above, so it is convergent and $\lim_{n\to+\infty} v_n = \sup v_n$. So
$$\lim_{n\to+\infty} u_n = -\lim_{n\to+\infty} v_n = -\sup v_n = \inf(-v_n) = \inf u_n. \quad \square$$

Example 2.6.4 *For every* $n \in \mathbb{N}$, *put* $u_n = \frac{n}{n+1}$. *The sequence* $(u_n)_{n\in\mathbb{N}}$ *is increasing (by the example 2.4.1) and bounded from above by* 1, *then it is convergent and*
$$\sup u_n = \lim_{n\to+\infty} u_n = 1.$$

Exercise 51

For every $n \in \mathbb{N}$, **put**
$$v_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \quad \text{and} \quad w_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{1}{n!} = v_n + \frac{1}{n!}.$$

1) **Prove that the two sequences** $(v_n)_{n\in\mathbb{N}}$ **and** $(w_n)_{n\in\mathbb{N}}$ **are convergent to the same limit** $e \in \mathbb{R}$.

2) **Show that** $v_n < e < w_n$ **for all** $n \in \mathbb{N}^*$.

3) **Deduce that** $2,7 < e < 2,8$.

4) **Show that** e **is irrational.**

Solution

1) The sequence $(v_n)_{n\in\mathbb{N}}$ is bounded from above by 3 (by the exercise 34). For any $n \in \mathbb{N}$, $v_{n+1} - v_n = \frac{1}{(n+1)!} \geq 0$, then $(v_n)_{n\in\mathbb{N}}$ is strictly increasing. Hence $(v_n)_{n\in\mathbb{N}}$ is convergent to $\sup v_n$. In the other side, it is obvious that the sequence $(w_n)_{n\in\mathbb{N}}$ is bounded from below by 1, and for any $n \in \mathbb{N}^*$,

$$w_{n+1} - w_n = \frac{2}{(n+1)!} - \frac{1}{n!} = \frac{1-n}{(n+1)!} \leq 0,$$

so $(w_n)_{n\in\mathbb{N}^*}$ is decreasing. Hence $(w_n)_{n\in\mathbb{N}^*}$ is convergent to $\inf\limits_{n\in\mathbb{N}^*} w_n$. Moreover, as $\lim\limits_{n\to+\infty} \frac{1}{n!} = 0$, then

$$\lim_{n\to+\infty} w_n = \lim_{n\to+\infty} v_n + \lim_{n\to+\infty} \frac{1}{n!} = \lim_{n\to+\infty} v_n.$$

2) By the part (1), $e = \sup v_n = \inf\limits_{n\in\mathbb{N}^*} w_n$. So, for any $n \in \mathbb{N}^*$,

$$v_n \leq \sup v_n = e = \inf w_n \leq w_n.$$

- Suppose that there exists a certain $n_0 \in \mathbb{N}^*$ such that $e = v_{n_0}$. Since the sequence $(v_n)_{n\in\mathbb{N}}$ is strictly increasing, then $v_{n_0} < v_{n_0+1}$, so $e < v_{n_0+1}$, which is impossible since $e = \sup v_n$. Hence $v_n < e$ for all $n \in \mathbb{N}^*$.
- Suppose that there exists a certain $n_0 \in \mathbb{N}^*$ such that $e = w_{n_0}$. Since the sequence $(w_n)_{n\geq 2}$ is strictly decreasing, then $w_{n_0} > w_{n_0+2}$, so $e > w_{n_0+2}$, which is impossible since $e = \inf\limits_{n\in\mathbb{N}^*} w_n$. Hence $e < w_n$ for all $n \in \mathbb{N}^*$.

Thus $v_n < e < w_n$ for all $n \in \mathbb{N}^*$.

3) For $n = 4$, $v_4 < e$. As $v_4 \simeq 2,7083$, then $2,7 < e$.
For $n = 3$, $e < w_3$. As $w_3 \simeq 2,833$, then $e < 2,8$.
Hence $2,7 < e < 2,8$.

4) Suppose, by contradiction, that e is rational, then there exist $p \in \mathbb{N}$ and $q \in \mathbb{N}^*$ such that $e = \frac{p}{q}$. Let $n \in \mathbb{N}$ such that $n > q$, then q divides $n!$, and therefore $\frac{n!}{q} \in \mathbb{N}$. But $v_n < e < w_n$ by the part (2), then, multiplying by $n!$, we obtain:

$$n!v_n < \left(\frac{n!}{q}\right) p < n!v_n + 1,$$

which is impossible since $n!v_n \in \mathbb{N}$ and $\left(\frac{n!}{q}\right) p \in \mathbb{N}$ (there is no natural numbers between two consecutive natural numbers). Hence e is irrational. □

Exercise 52
Consider the sequence $(u_n)_{n\in\mathbb{N}^*}$ defined by:

$$u_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}.$$

1) Show that $(u_n)_{n\in\mathbb{N}^*}$ is increasing.

2) Prove that $1 \leq u_n \leq 2$ for all $n \in \mathbb{N}^*$.

3) Deduce that $(u_n)_{n\in\mathbb{N}^*}$ is convergent to a limit ℓ satisfying $1 \leq \ell \leq 2$.

Solution

1) For any $n \in \mathbb{N}^*$, we have:

$$u_{n+1} - u_n = \frac{1}{(n+1)^2} \geq 0.$$

So the sequence $(u_n)_{n\in\mathbb{N}^*}$ is increasing.

2) For any $k \in \mathbb{N}^*$, $\dfrac{1}{(k-1)k} = \dfrac{1}{k-1} - \dfrac{1}{k}$. Let $n \in \mathbb{N}^*$, we have:

$$\begin{aligned} 1 \leq u_n &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \\ &\leq 1 + \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{(n-1)n} \\ &\leq 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &\leq 1 + 1 - \frac{1}{n} \leq 2. \end{aligned}$$

Hence $1 \leq u_n \leq 2$ for all $n \in \mathbb{N}^*$.

3) Since the sequence $(u_n)_{n\in\mathbb{N}^*}$ is increasing and bounded from above by 2, then it converges to a real number ℓ. In the other side, taking limits in the double inequality of the part (2), we obtain $1 \leq \ell \leq 2$. □

Proposition 2.6.5

1) Every increasing sequence which is not bounded from above, is divergent to $+\infty$.

2) Every decreasing sequence which is not bounded from below, is divergent to $-\infty$.

Proof

1) Let $(u_n)_{n\in\mathbb{N}}$ be an increasing non bounded from above sequence. Let $A > 0$. Since this sequence is not bounded from above by A, then there exists $n_0 \in \mathbb{N}$ such that $u_{n_0} > A$. If $n \geq n_0$, then $u_n \geq u_{n_0}$ since the sequence is increasing, and therefore $u_n > A$. Hence $\lim\limits_{n\to+\infty} u_n = +\infty$.

2) Let $(u_n)_{n\in\mathbb{N}}$ a decreasing non bounded from below sequence. Then the sequence $(-u_n)_{n\in\mathbb{N}}$ is increasing non bounded from above, so $\lim\limits_{n\to+\infty}(-u_n) = +\infty$ by the part (1). Hence $\lim\limits_{n\to+\infty} u_n = -\infty$. □

2.7 Equivalent sequences and negligible sequences

2.7.1 Equivalent sequences

Definition 2.7.1 *Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two real sequences. We say that $(u_n)_{n\in\mathbb{N}}$ is equivalent to $(v_n)_{n\in\mathbb{N}}$ (in a neighborhood of infinity), and we write $u_n \underset{+\infty}{\sim} v_n$, if there exists a real sequence $(a_n)_{n\in\mathbb{N}}$ and a certain rank $n_0 \in \mathbb{N}$ such that the following two conditions are satisfied:*

i) $\lim\limits_{n\to+\infty} a_n = 1$.

ii) $u_n = a_n v_n$ *for all* $n \geq n_0$.

Proposition 2.7.1 *Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ be real sequences.*

1) $u_n \underset{+\infty}{\sim} u_n$.

2) If $u_n \underset{+\infty}{\sim} v_n$, then $v_n \underset{+\infty}{\sim} u_n$. In this case, we say that the two sequences are equivalent (in a neighborhood of infinity).

3) If $u_n \underset{+\infty}{\sim} v_n$ and $v_n \underset{+\infty}{\sim} w_n$, then $u_n \underset{+\infty}{\sim} w_n$.

4) If $(v_n)_{n\in\mathbb{N}}$ is nonzero after a certain rank, then

$$\boxed{u_n \underset{+\infty}{\sim} v_n \Leftrightarrow \lim_{n\to+\infty} \frac{u_n}{v_n} = 1}$$

5) Suppose that $u_n \underset{+\infty}{\sim} v_n$.

i) The two sequences have the same sign after a certain rank.

ii) If one of these two sequences is nonzero after a certain rank, then the other sequence is also nonzero after a certain rank.

Proof

1) It is sufficient to take $a_n = 1$ for all $n \in \mathbb{N}$.

2) If $u_n \underset{+\infty}{\sim} v_n$, then there exists a real sequence $(a_n)_{n\in\mathbb{N}}$ and a certain rank $n_1 \in \mathbb{N}$ such that $\lim\limits_{n\to+\infty} a_n = 1$ and $u_n = a_n v_n$ for all $n \geq n_1$. As $\lim\limits_{n\to+\infty} a_n \neq 0$, then there exists $n_2 \in \mathbb{N}$ such that $a_n \neq 0$ for all $n \geq n_2$ (by the exercise 46). Put $n_0 = \max(n_1, n_2)$ and $b_n = \frac{1}{a_n}$ for all $n \geq n_0$. Then $\lim\limits_{n\to+\infty} b_n = 1$ and $v_n = b_n u_n$ for all $n \geq n_0$. Hence $v_n \underset{+\infty}{\sim} u_n$.

3) If $u_n \underset{+\infty}{\sim} v_n$ and $v_n \underset{+\infty}{\sim} w_n$, then there exist two real sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ and two ranks $n_1, n_2 \in \mathbb{N}$ such that $\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} b_n = 1$ and $u_n = a_n v_n$ (for all $n \geq n_1$) and $v_n = b_n w_n$ (for all $n \geq n_2$). Put $n_0 = \max(n_1, n_2)$ and $c_n = a_n b_n$ for all $n \geq n_0$. Then $\lim\limits_{n\to+\infty} c_n = 1$ and $u_n = c_n w_n$ for all $n \geq n_0$. Hence $u_n \underset{+\infty}{\sim} w_n$.

4) Suppose that there exists $n_1 \in \mathbb{N}$ such that $v_n \neq 0$ for all $n \geq n_1$.
N.C. If $u_n \underset{+\infty}{\sim} v_n$, then there exists a real sequence $(a_n)_{n\in\mathbb{N}}$ and a certain rank $n_2 \in \mathbb{N}$ such that $\lim_{n\to+\infty} a_n = 1$ and $u_n = a_n v_n$ for all $n \geq n_2$. Put $n_0 = \max(n_1, n_2)$, then $\frac{u_n}{v_n} = a_n$ for all $n \geq n_0$. So

$$\lim_{n\to+\infty} \frac{u_n}{v_n} = \lim_{n\to+\infty} a_n = 1.$$

S.C. Put $a_n = \frac{u_n}{v_n}$ for all $n \geq n_1$, then $\lim_{n\to+\infty} a_n = 1$ and $u_n = a_n v_n$ for all $n \geq n_1$. So $u_n \underset{+\infty}{\sim} v_n$.

5) Suppose that $u_n \underset{+\infty}{\sim} v_n$, then there exists a real sequence $(a_n)_{n\in\mathbb{N}}$ and a certain rank $n_1 \in \mathbb{N}$ such that $\lim_{n\to+\infty} a_n = 1$ and $u_n = a_n v_n$ for all $n \geq n_1$. As $\lim_{n\to+\infty} a_n > 0$, then there exists $n_2 \in \mathbb{N}$ such that $a_n > 0$ for all $n \geq n_2$ (by the exercise 46).

   i) Put $n_0 = \max(n_1, n_2)$, then $u_n = a_n v_n$ and $a_n > 0$ for all $n \geq n_0$. So u_n and v_n have the same sign for all $n \geq n_0$.

   ii) Suppose that there exists $n_3 \in \mathbb{N}$ such that $v_n \neq 0$ for all $n \geq n_3$. Put $n_0' = \max(n_0, n_3)$, then $u_n = a_n v_n$, $a_n > 0$ and $v_n \neq 0$ for all $n \geq n_0'$. So $u_n \neq 0$ for all $n \geq n_0'$. □

Example 2.7.1

1) $\frac{n+1}{n} \underset{+\infty}{\sim} 1$ *since* $\lim_{n\to+\infty} \frac{n+1}{n} = 1$.

2) $\frac{n^2+1}{n} \underset{+\infty}{\sim} n$ *since*

$$\lim_{n\to+\infty} \frac{\frac{n^2+1}{n}}{n} = \lim_{n\to+\infty} \frac{n^2+1}{n^2} = 1.$$

Set $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$, and call it the completed real line.

Proposition 2.7.2 *Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two real sequences.*

1) If $\lim_{n\to+\infty} u_n = \ell \in \mathbb{R}^*$, then $u_n \underset{+\infty}{\sim} \ell$.

2) If $u_n \underset{+\infty}{\sim} v_n$, then the two sequences have the same nature, i.e., if one of them converges (resp. diverges), then the other one converges (resp. diverges). Moreover, if $\lim_{n\to+\infty} v_n = \ell \in \overline{\mathbb{R}}$, then $\lim_{n\to+\infty} u_n = \ell$.

Proof

1) Put $v_n = \ell \neq 0$ for all $n \in \mathbb{N}$. Since $\lim\limits_{n\to+\infty} \dfrac{u_n}{v_n} = 1$, then $u_n \underset{+\infty}{\sim} v_n$ by the part (4) of the proposition 2.7.1. Hence $u_n \underset{+\infty}{\sim} \ell$.

2) Suppose that $u_n \underset{+\infty}{\sim} v_n$, then there exists a real sequence $(a_n)_{n\in\mathbb{N}}$ and a certain rank $n_0 \in \mathbb{N}$ such that $\lim\limits_{n\to+\infty} a_n = 1$ and $u_n = a_n v_n$ for all $n \geq n_0$.
If $\lim\limits_{n\to+\infty} v_n = \ell \in \overline{\mathbb{R}}$, then

$$\lim_{n\to+\infty} u_n = \left(\lim_{n\to+\infty} a_n\right)\left(\lim_{n\to+\infty} v_n\right) = \lim_{n\to+\infty} v_n = \ell.$$

If $\lim\limits_{n\to+\infty} u_n = \ell \in \overline{\mathbb{R}}$, then

$$\lim_{n\to+\infty} v_n = 1 \times \left(\lim_{n\to+\infty} v_n\right) = \left(\lim_{n\to+\infty} a_n\right)\left(\lim_{n\to+\infty} v_n\right) = \lim_{n\to+\infty} u_n = \ell.$$

Hence
$$\lim_{n\to+\infty} u_n = \ell \in \overline{\mathbb{R}} \Leftrightarrow \lim_{n\to+\infty} v_n = \ell.$$
Thus the two sequences have the same nature. □

Proposition 2.7.3 *(Operations on the equivalent sequences)*
Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be real sequences.

1) *If $u_n \underset{+\infty}{\sim} v_n$, then $ku_n \underset{+\infty}{\sim} kv_n$ for all $k \in \mathbb{R}$.*

2) *If $u_n \underset{+\infty}{\sim} v_n$, then $|u_n| \underset{+\infty}{\sim} |v_n|$. The converse is not necessary true.*

3) *If $u_n \underset{+\infty}{\sim} x_n$ and $v_n \underset{+\infty}{\sim} x_n$, then*

$$\boxed{u_n + v_n \underset{+\infty}{\sim} 2x_n}$$

4) *Suppose that $u_n \underset{+\infty}{\sim} v_n$ and $x_n \underset{+\infty}{\sim} y_n$. Then:*

   i) $u_n x_n \underset{+\infty}{\sim} v_n y_n$.

   ii) If the sequence $(x_n)_{n\in\mathbb{N}}$ is nonzero after a certain rank, then

$$\boxed{\frac{u_n}{x_n} \underset{+\infty}{\sim} \frac{v_n}{y_n}}$$

5) *Suppose that $u_n \underset{+\infty}{\sim} v_n$.*

   i) $u_n^p \underset{+\infty}{\sim} v_n^p$ for all $p \in \mathbb{N}$.

ii) If $u_n \neq 0$ after a certain rank, then $u_n^p \underset{+\infty}{\sim} v_n^p$ for all $p \in \mathbb{Z}$.

iii) If $u_n > 0$ after a certain rank, then, for all $\alpha \in \mathbb{R}^+$,

$$\boxed{u_n^\alpha \underset{+\infty}{\sim} v_n^\alpha}$$

Proof

1) Let $k \in \mathbb{R}$. If $u_n \underset{+\infty}{\sim} v_n$, then there exists a real sequence $(a_n)_{n\in\mathbb{N}}$ and a certain rank $n_0 \in \mathbb{N}$ such that $\lim\limits_{n\to+\infty} a_n = 1$ and $u_n = a_n v_n$ for all $n \geq n_0$. So $ku_n = a_n(kv_n)$ for all $n \geq n_0$. Hence $ku_n \underset{+\infty}{\sim} kv_n$.

2) If $u_n \underset{+\infty}{\sim} v_n$, then there exists a real sequence $(a_n)_{n\in\mathbb{N}}$ and a certain rank $n_0 \in \mathbb{N}$ such that $\lim\limits_{n\to+\infty} a_n = 1$ and $u_n = a_n v_n$ for all $n \geq n_0$. So $\lim\limits_{n\to+\infty} |a_n| = 1$ and $|u_n| = |a_n||v_n|$ for all $n \geq n_0$. Hence $|u_n| \underset{+\infty}{\sim} |v_n|$. The converse is not necessary true. Indeed, for example, $|(-1)^n| \underset{+\infty}{\sim} |1|$, but $(-1)^n$ and 1 are not equivalent.

3) Suppose that $u_n \underset{+\infty}{\sim} x_n$ and $v_n \underset{+\infty}{\sim} x_n$. Then there exist two real sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ and two ranks $n_1, n_2 \in \mathbb{N}$ such that $\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} b_n = 1$ and $u_n = a_n x_n$ (for all $n \geq n_1$) and $v_n = b_n x_n$ (for all $n \geq n_2$). Put $n_0 = \max(n_1, n_2)$ and $c_n = \frac{1}{2}(a_n + b_n)$ for all $n \geq n_0$. Then $\lim\limits_{n\to+\infty} c_n = \dfrac{1}{2}(1+1) = 1$ and, for all $n \geq n_0$,

$$u_n + v_n = (a_n + b_n)x_n = c_n(2x_n).$$

Hence $u_n + v_n \underset{+\infty}{\sim} 2x_n$.

4) Suppose that $u_n \underset{+\infty}{\sim} v_n$ and $x_n \underset{+\infty}{\sim} y_n$. Then there exist two real sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ and two ranks $n_1, n_2 \in \mathbb{N}$ such that $\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} b_n = 1$ and $u_n = a_n v_n$ (for all $n \geq n_1$) and $x_n = b_n y_n$ (for all $n \geq n_2$).

   i) Put $n_0 = \max(n_1, n_2)$ and $c_n = a_n b_n$ for all $n \geq n_0$. Then $\lim\limits_{n\to+\infty} c_n = 1$ and, for all $n \geq n_0$,

   $$u_n x_n = (a_n b_n)(v_n y_n) = c_n(v_n y_n).$$

   Hence $u_n x_n \underset{+\infty}{\sim} v_n y_n$.

   ii) Suppose that $(x_n)_{n\in\mathbb{N}}$ is nonzero after a certain rank. As $x_n \underset{+\infty}{\sim} y_n$, then $(y_n)_{n\in\mathbb{N}}$ is nonzero after a certain rank by the proposition 2.7.1. As $\lim\limits_{n\to+\infty} b_n \neq 0$, then $(b_n)_{n\in\mathbb{N}}$ is nonzero after a certain rank by the exercise 46. Hence there exists $n_3 \in \mathbb{N}$ such that $x_n \neq 0$, $y_n \neq 0$ and $b_n \neq 0$ for all $n \geq n_3$.

Put $n'_0 = \max(n_0, n_3)$ and $d_n = \frac{a_n}{b_n}$ for all $n \geq n'_0$. Then $\lim_{n\to+\infty} d_n = 1$ and, for all $n \geq n'_0$,

$$\frac{u_n}{x_n} = \left(\frac{a_n}{b_n}\right)\left(\frac{v_n}{y_n}\right) = d_n\left(\frac{v_n}{y_n}\right).$$

Hence

$$\frac{u_n}{x_n} \underset{+\infty}{\sim} \frac{v_n}{y_n}.$$

5) Suppose that $u_n \underset{+\infty}{\sim} v_n$, then there exists a real sequence $(a_n)_{n\in\mathbb{N}}$ and a certain rank $n_1 \in \mathbb{N}$ such that $\lim_{n\to+\infty} a_n = 1$ and $u_n = a_n v_n$ for all $n \geq n_1$.

i) Let $p \in \mathbb{N}$. Put $b_n = a_n^p$ for all $n \geq n_1$. Then $\lim_{n\to+\infty} b_n = 1$ and, for all $n \geq n_1$,

$$u_n^p = a_n^p v_n^p = b_n v_n^p.$$

Hence $u_n^p \underset{+\infty}{\sim} v_n^p$.

ii) Let $p \in \mathbb{Z}$. If there exists $n_2 \in \mathbb{N}$ such that $u_n \neq 0$ for all $n \geq n_2$, then put $n_0 = \max(n_1, n_2)$ and $b_n = a_n^p$ for all $n \geq n_0$, and $\lim_{n\to+\infty} b_n = 1$ and, for all $n \geq n_0$,

$$u_n^p = a_n^p v_n^p = b_n v_n^p.$$

Hence $u_n^p \underset{+\infty}{\sim} v_n^p$.

iii) Let $\alpha \in \mathbb{R}^+$. As $u_n \underset{+\infty}{\sim} v_n$ and $u_n > 0$ after a certain rank, then $v_n > 0$ after a certain rank by the proposition 2.7.1. As $\lim_{n\to+\infty} a_n > 0$, then $a_n > 0$ after a certain rank by the exercise 46. So there exists $n_2 \in \mathbb{N}$ such that $u_n > 0$, $v_n > 0$ and $a_n > 0$ for all $n \geq n_2$. Put $n_0 = \max(n_1, n_2)$ and $b_n = a_n^\alpha$ for all $n \geq n_0$. Then $\lim_{n\to+\infty} b_n = 1$ and, for all $n \geq n_0$,

$$u_n^\alpha = a_n^\alpha v_n^\alpha = b_n v_n^\alpha.$$

Hence $u_n^\alpha \underset{+\infty}{\sim} v_n^\alpha$. □

Example 2.7.2

1) *For any* $a_0, a_1, \ldots, a_s \in \mathbb{R}$ *such that* $a_s \neq 0$,

$$\boxed{a_s n^s + \cdots + a_1 n + a_0 \underset{+\infty}{\sim} a_s n^s}$$

In other words, a polynomial sequence is equivalent to is term of higher degree. Consequently, for any $b_0, b_1, \ldots, b_t \in \mathbb{R}$ *such that* $b_t \neq 0$,

$$\boxed{\frac{a_s n^s + \cdots + a_1 n + a_0}{b_t n^t + \cdots + b_1 n + b_0} \underset{+\infty}{\sim} \frac{a_s n^s}{b_t n^t}}$$

2) The equivalence of sequences is not compatible with the addition. Indeed, for example,

$$u_n = n^2 + n + 1 \underset{+\infty}{\sim} n^2 + n \quad \textit{and} \quad v_n = -n^2 + 4n + 5 \underset{+\infty}{\sim} -n^2.$$

On the other hand, $u_n + v_n = 5n + 6 \underset{+\infty}{\sim} 5n \neq n = (n^2 + n) - n^2$.
Hence, we can not move terms from left to right of the symbol $\underset{+\infty}{\sim}$.

3) If $\lim\limits_{n\to+\infty} u_n = 0$, *then we have the following basic equivalences:*

$$\boxed{\sin u_n \underset{+\infty}{\sim} u_n} \quad \boxed{\tan u_n \underset{+\infty}{\sim} u_n} \quad \boxed{\ln(1+u_n) \underset{+\infty}{\sim} u_n}$$

$$\boxed{e^{u_n} \underset{+\infty}{\sim} 1 + u_n} \quad \boxed{\cos u_n \underset{+\infty}{\sim} 1 - \frac{u_n^2}{2}} \quad \boxed{\sqrt{1+u_n} \underset{+\infty}{\sim} 1 + \frac{u_n}{2}}$$

2.7.2 Negligible sequences

Definition 2.7.2 *Let* $(u_n)_{n\in\mathbb{N}}$ *and* $(v_n)_{n\in\mathbb{N}}$ *be two real sequences. We say that* $(u_n)_{n\in\mathbb{N}}$ *is negligible in front of* $(v_n)_{n\in\mathbb{N}}$, *and we write* $u_n \underset{+\infty}{=} o(v_n)$ *(or simply* $u_n = o(v_n)$ *or* $u_n \ll v_n$*), if there exists a real sequence* $(\varepsilon_n)_{n\in\mathbb{N}}$ *and a certain rank* $n_0 \in \mathbb{N}$ *such that the following two conditions are satisfied:*

i) $\lim\limits_{n\to+\infty} \varepsilon_n = 0$.

ii) $u_n = \varepsilon_n v_n$ *for all* $n \geq n_0$.

Remark 2.7.1

1) The notation $u_n = o(v_n)$ *is called Landau's notation or "small o". The notation* $u_n \ll v_n$ *is called Hardy's notation.*

2) Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ *and* $(w_n)_{n\in\mathbb{N}}$ *be real sequences.*

i) We have:

$$\boxed{u_n \underset{+\infty}{\sim} v_n \Leftrightarrow u_n - v_n = o(v_n)}$$

In other words, $u_n \underset{+\infty}{\sim} v_n$ *if and only if there exists a real sequence* $(\varepsilon_n)_{n\in\mathbb{N}}$ *and a certain rank* $n_0 \in \mathbb{N}$ *such that, for all* $n \geq n_0$,

$$\boxed{u_n = v_n + v_n\varepsilon_n, \quad \textit{with} \quad \lim_{n\to+\infty} \varepsilon_n = 0}$$

ii) If $u_n \underset{+\infty}{\sim} v_n$ *and* $v_n = o(w_n)$, *then* $u_n = o(w_n)$. *Indeed, if* $u_n \underset{+\infty}{\sim} v_n$ *and* $v_n = o(w_n)$, *then there exist two real sequences* $(a_n)_{n\in\mathbb{N}}$ *and* $(\varepsilon_n)_{n\in\mathbb{N}}$ *and* *a certain rank* $n_0 \in \mathbb{N}$ *such that* $\lim\limits_{n\to+\infty} a_n = 1$, $\lim\limits_{n\to+\infty} \varepsilon_n = 0$ *and, for all* $n \geq n_0$,

$$u_n = a_n v_n \quad \text{and} \quad v_n = \varepsilon_n w_n.$$

Put $\varphi_n = a_n \varepsilon_n$ *for all* $n \geq n_0$, *then* $\lim\limits_{n\to+\infty} \varphi_n = 0$ *and* $u_n = \varphi_n w_n$ *for all* $n \geq n_0$. *Hence* $u_n = o(w_n)$.

Theorem 2.7.1 *Let* $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *be real sequences.*

1) *If* $(v_n)_{n\in\mathbb{N}}$ *is nonzero after a certain rank, then*

$$\boxed{u_n = o(v_n) \Leftrightarrow \lim_{n\to+\infty} \frac{u_n}{v_n} = 0}$$

2) *If* $\ell \in \mathbb{R}$, *then*

$$\boxed{\lim_{n\to+\infty} u_n = \ell \Leftrightarrow u_n = \ell + o(1)}$$

In particular,

$$\boxed{\lim_{n\to+\infty} u_n = 0 \Leftrightarrow u_n = o(1)}$$

3) *If* $u_n = o(v_n)$, *then, for all* $p \in \mathbb{N}$,

$$\boxed{u_n^p = o(v_n^p)}$$

4) *If* $u_n = o(v_n)$ *and* $(u_n)_{n\in\mathbb{N}}$ *is nonzero after a certain rank, then* $(v_n)_{n\in\mathbb{N}}$ *is nonzero after a certain rank and*

$$\boxed{\frac{1}{v_n} = o\left(\frac{1}{u_n}\right)}$$

5) *If* $u_n = o(v_n)$ *and* $v_n = o(x_n)$, *then* $u_n = o(x_n)$.

6) *If* $u_n = o(v_n)$, *then* $ku_n = o(v_n)$ *and* $u_n = o(kv_n)$ *for all* $k \in \mathbb{R}^*$.

7) *If* $u_n = o(v_n)$ *and* $x_n = o(v_n)$, *then*

$$\boxed{u_n + x_n = o(v_n)}$$

8) *If* $u_n = o(v_n)$ *and* $x_n = o(y_n)$, *then*

$$\boxed{u_n x_n = o(v_n y_n)}$$

Example 2.7.3 *For any* $a_0, a_1, \ldots, a_s \in \mathbb{R}$ *such that* $a_s \neq 0$,

$$\boxed{a_s n^s + \cdots + a_1 n + a_0 = o(n^{s+1})} \qquad \boxed{n^s = o(n^t) \Leftrightarrow s < t}$$

2.8 Adjacent sequences

Definition 2.8.1 *Two sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are said to be adjacent if one of them is increasing, the other one is decreasing and their difference tends to zero. In other words, for example, if the following conditions are satisfied:*

- $(u_n)_{n\in\mathbb{N}}$ *is increasing and* $(v_n)_{n\in\mathbb{N}}$ *is decreasing.*
- $\lim\limits_{n\to+\infty}(u_n - v_n) = 0$.

Example 2.8.1

1) *The two sequences $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ of the exercise 51 are adjacent.*

2) *For every $n \in \mathbb{N}^*$, put $u_n = \frac{n}{n+1}$ and $v_n = \frac{n+1}{n}$. The sequence $(u_n)_{n\in\mathbb{N}^*}$ is increasing (by the example 2.4.1) and the sequence $(v_n)_{n\in\mathbb{N}^*}$ is decreasing since $v_n = \frac{1}{u_n}$ and $u_n > 0$ for all $n \in \mathbb{N}^*$ (by using the exercise 39). Moreover,*

$$\lim_{n\to+\infty}(u_n - v_n) = \lim_{n\to+\infty}\frac{-2n-1}{n(n+1)} = \lim_{n\to+\infty}\frac{-2}{n} = 0.$$

So $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$ are adjacent.

Theorem 2.8.1 *Two adjacent sequences are convergent to the same limit. More precisely, if $(u_n)_{n\in\mathbb{N}}$ is an increasing sequence and $(v_n)_{n\in\mathbb{N}}$ is a decreasing sequence such that $\lim\limits_{n\to+\infty}(u_n - v_n) = 0$, then $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are convergent with*

$$\lim_{n\to+\infty} u_n = \sup u_n = \inf v_n = \lim_{n\to+\infty} v_n.$$

In particular, if $\ell = \lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} v_n$, then $u_n \le \ell \le v_n$ for all $n \in \mathbb{N}$.

Proof Since $\lim\limits_{n\to+\infty}(u_n - v_n) = 0$, then, for $\varepsilon = 1 > 0$, there exists $n_0 \in \mathbb{N}$ such that $|u_n - v_n| < 1$ for all $n \ge n_0$. So

$$(*) \qquad v_n - 1 < u_n < v_n + 1, \quad \forall n \ge n_0.$$

As $(v_n)_{n\in\mathbb{N}}$ is decreasing, then $v_n \le v_0$ for all $n \ge n_0$. So

$$u_n \le v_0 + 1, \quad \forall n \ge n_0.$$

Therefore the sequence $(u_n)_{n\ge n_0}$ is bounded from above by v_0+1. As $(u_n)_{n\in\mathbb{N}}$ is increasing, then $(u_n)_{n\in\mathbb{N}}$ is also bounded from above by $v_0 + 1$. So $(u_n)_{n\in\mathbb{N}}$ is increasing and bounded from above, hence it is convergent to $\ell = \sup u_n$. In the other side, by $(*)$, we have:

$$v_n > u_n - 1, \quad \forall n \ge n_0.$$

As $(u_n)_{n\in\mathbb{N}}$ is increasing, then $u_n \ge u_0$ for all $n \ge n_0$. So

$$v_n \ge u_0 - 1, \quad \forall n \ge n_0.$$

Therefore the sequence $(v_n)_{n\geq n_0}$ is bounded from below by u_0-1. As $(v_n)_{n\in\mathbb{N}}$ is decreasing, then $(v_n)_{n\in\mathbb{N}}$ is also bounded from below by u_0-1. So $(v_n)_{n\in\mathbb{N}}$ is decreasing and bounded from below, hence it is convergent to $\ell' = \inf v_n$. Moreover,

$$\ell - \ell' = \lim_{n\to+\infty} u_n - \lim_{n\to+\infty} v_n = \lim_{n\to+\infty}(u_n - v_n) = 0.$$

Hence $\ell = \ell'$, and therefore $\sup u_n = \inf v_n$. Consequently, for all $n \in \mathbb{N}$, we have:

$$u_n \leq \sup u_n = \ell = \inf v_n \leq v_n.$$

Remark that we can also prove that $(v_n)_{n\in\mathbb{N}}$ is convergent to ℓ by the following way: for all $n \in \mathbb{N}$, put $w_n = u_n - v_n$, then $v_n = u_n - w_n$. As $(u_n)_{n\in\mathbb{N}}$ is convergent to ℓ and $(w_n)_{n\in\mathbb{N}}$ is convergent to 0, then $(v_n)_{n\in\mathbb{N}}$ is convergent and

$$\lim_{n\to+\infty} v_n = \lim_{n\to+\infty} u_n - \lim_{n\to+\infty} w_n = \ell - 0 = \ell. \quad \square$$

Exercise 53
For every $n \in \mathbb{N}^*$, put

$$u_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \quad \text{and} \quad v_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \frac{1}{n} = u_n + \frac{1}{n}.$$

1) **Prove that the two sequences $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$ are adjacent.**

2) **Let ℓ be their common limit. Find an approximate value of ℓ up to 10^{-1}.**

Solution

1) For any $n \in \mathbb{N}^*$, $u_{n+1} - u_n = \frac{1}{(n+1)^2} \geq 0$ and

$$v_{n+1} - v_n = \frac{1}{(n+1)^2} + \frac{1}{n+1} - \frac{1}{n} = \frac{-1}{n(n+1)^2} \leq 0.$$

So $(u_n)_{n\in\mathbb{N}^*}$ is increasing and $(v_n)_{n\in\mathbb{N}^*}$ is decreasing. Moreover, $\lim_{n\to+\infty}(v_n - u_n) = \lim_{n\to+\infty} \frac{1}{n} = 0$. Hence $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$ are adjacent.

2) We know that $u_n \leq \ell \leq v_n$ for all $n \in \mathbb{N}^*$. If ℓ_0 is an approximate value of ℓ, then $|\ell - \ell_0| \leq |v_n - u_n| = \frac{1}{n}$. If we look for ℓ_0 to be an approximate value of ℓ up to 10^{-1} (i.e., $|\ell - \ell_0| \leq 10^{-1}$), it is sufficient to take n such that $\frac{1}{n} \leq 10^{-1}$, i.e., $n \geq 10$. For $n = 10$, every real number in the interval $[u_{10}, v_{10}]$ is an approximate value of ℓ up to 10^{-1}. For example, $u_{10} = 1,547$ is a such approximate value. $\square$

Exercise 54
Consider the two sequences $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ defined, for every $n \in \mathbb{N}$, by:

$$\begin{cases} u_0 = 3 \\ u_{n+1} = \dfrac{u_n + v_n}{2} \end{cases} \quad \text{and} \quad \begin{cases} v_0 = 4 \\ v_{n+1} = \dfrac{u_{n+1} + v_n}{2} \end{cases}$$

1) Calculate u_1, v_1, u_2 and v_2.

2) Let the sequence $(w_n)_n$ defined, for every $n \in \mathbb{N}$, by $w_n = v_n - u_n$.

i) Show that $(w_n)_n$ is a geometric sequence with ratio $\frac{1}{4}$.

ii) Express w_n in terms of n and deduce the limit of the sequence $(w_n)_n$.

3) Study the monotonicity of the sequences $(u_n)_n$ and $(v_n)_n$ and prove that they are adjacent. What can we deduce ?

4) Consider the sequence $(t_n)_n$ defined, for every $n \in \mathbb{N}$, by $t_n = \dfrac{u_n + 2v_n}{3}$.

a) Prove that $(t_n)_n$ is a constant sequence.

b) Deduce the limits of the sequences $(u_n)_n$ and $(v_n)_n$.

Solution

1) $u_1 = \dfrac{u_0 + v_0}{2} = \dfrac{3+4}{2} = \dfrac{7}{2}$, $\quad v_1 = \dfrac{u_1 + v_0}{2} = \dfrac{\frac{7}{2}+4}{2} = \dfrac{15}{4}$,

$u_2 = \dfrac{u_1 + v_1}{2} = \dfrac{\frac{7}{2}+\frac{15}{4}}{2} = \dfrac{29}{8}$, $\quad v_2 = \dfrac{u_2 + v_1}{2} = \dfrac{\frac{29}{8}+\frac{15}{4}}{2} = \dfrac{59}{16}$.

2) i) For any $n \in \mathbb{N}$,

$$\begin{aligned} w_{n+1} = v_{n+1} - u_{n+1} &= \frac{u_{n+1}+v_n}{2} - u_{n+1} = \frac{v_n - u_{n+1}}{2} \\ &= \frac{v_n - \frac{u_n+v_n}{2}}{2} = \frac{v_n - u_n}{4} = \frac{1}{4} w_n. \end{aligned}$$

So $(w_n)_n$ is a geometric sequence with ratio $q = \frac{1}{4}$.

ii) For any $n \in \mathbb{N}$, $w_n = w_0 q^n$, where $w_0 = v_0 - u_0 = 1$. So $w_n = \dfrac{1}{4^n}$ and therefore $\lim\limits_{n\to+\infty} w_n = 0$.

3) For any $n \in \mathbb{N}$,

$$u_{n+1} - u_n = \frac{u_n+v_n}{2} - u_n = \frac{v_n - u_n}{2} = \frac{w_n}{2}$$

and

$$v_{n+1} - v_n = \frac{u_{n+1}+v_n}{2} - v_n = \frac{\frac{u_n+v_n}{2} - v_n}{2} = \frac{u_n - v_n}{4} = -\frac{w_n}{2}.$$

As $w_n = \dfrac{1}{4^n} > 0$, then $u_{n+1} - u_n > 0$ and $v_{n+1} - v_n < 0$, so $(u_n)_n$ is strictly increasing and $(v_n)_n$ is strictly decreasing. But

$$\lim_{n\to+\infty} (v_n - u_n) = \lim_{n\to+\infty} w_n = 0,$$

then $(u_n)_n$ and $(v_n)_n$ are adjacent. We deduce that the sequences $(u_n)_n$ and $(v_n)_n$ are convergent to the same limit ℓ.

4) a) For any $n \in \mathbb{N}$,

$$\begin{aligned} t_{n+1} &= \frac{u_{n+1}+2v_{n+1}}{3} = \frac{u_{n+1}+u_{n+1}+v_n}{2} \\ &= \frac{2u_{n+1}+v_n}{2} = \frac{u_n+v_n+v_n}{2} = \frac{u_n+2v_n}{2} = t_n. \end{aligned}$$

So $(t_n)_n$ is a constant sequence.

b) For any $n \in \mathbb{N}$,

$$t_n = t_0 = \frac{u_0+2v_0}{3} = \frac{11}{3}.$$

So $\frac{\ell+2\ell}{3} = \frac{11}{3}$ and therefore $\ell = \frac{11}{3}$. □

2.9 Cauchy sequences

Definition 2.9.1 *A sequence $(u_n)_{n\in\mathbb{N}}$ is said to be a Cauchy sequence if it satisfies the following property:*

$$\Big(\forall\, \varepsilon > 0\Big)\Big(\exists\, n_0 \in \mathbb{N}\Big)\Bigg[\Big(\forall\, p, q \in \mathbb{N}\Big)\Big((p \geq n_0) \text{ and } (q \geq n_0) \Rightarrow |u_p - u_q| < \varepsilon\Big)\Bigg].$$

Proposition 2.9.1 *Every convergent real sequence is a Cauchy sequence.*

Proof Let $(u_n)_{n\in\mathbb{N}}$ be a convergent real sequence to a real number ℓ. Let $\varepsilon > 0$. Since $\lim\limits_{n\to+\infty} u_n = \ell$, then there exists $n_0 \in \mathbb{N}$ such that $|u_n - \ell| < \frac{\varepsilon}{2}$ for all $n \geq n_0$. If $p \geq n_0$ and $q \geq n_0$, then $|u_p - \ell| < \frac{\varepsilon}{2}$ and $|u_q - \ell| < \frac{\varepsilon}{2}$, so

$$|u_p - u_q| = |u_p - \ell + \ell - u_q| \leq |u_p - \ell| + |u_q - \ell| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $|u_p - u_q| < \varepsilon$. Hence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. □

Exercise 55

Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence.

1) **Show that $(u_n)_{n\in\mathbb{N}}$ is bounded.**

2) **Suppose that $(u_n)_{n\in\mathbb{N}}$ has a convergent subsequence. Show that $(u_n)_{n\in\mathbb{N}}$ is convergent.**

Solution

1) Let $\varepsilon = 1$. Since $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, then there exists $n_0 \in \mathbb{N}$ such that $|u_p - u_q| < 1$ for all $p \geq n_0$ and $q \geq n_0$. Let's fix $q = n_0$, then $|u_p - u_{n_0}| < 1$ for all $p \geq n_0$. So, for all $p \geq n_0$,

$$|u_p| = |u_p - u_{n_0} + u_{n_0}| \leq |u_p - u_{n_0}| + |u_{n_0}| < 1 + |u_{n_0}|.$$

Hence, for all $n \in \mathbb{N}$,

$$|u_n| \leq \max(|u_0|, \ldots, |u_{n_0-1}|, 1 + |u_{n_0}|).$$

Thus $(u_n)_{n\in\mathbb{N}}$ is bounded.

2) Let $(u_{\varphi(n)})_{n\in\mathbb{N}}$ be a convergent subsequence of $(u_n)_{n\in\mathbb{N}}$ to a limit $\ell \in \mathbb{R}$. Let's show that $(u_n)_{n\in\mathbb{N}}$ converges to ℓ. Indeed, let $\varepsilon > 0$, since $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, then there exists $n_1 \in \mathbb{N}$ such that $|u_p - u_q| < \frac{\varepsilon}{2}$ for all $p \geq n_1$ and $q \geq n_1$. As $\lim\limits_{n\to+\infty} u_{\varphi(n)} = \ell$, then there exists $n_2 \in \mathbb{N}$ such that

$$|u_{\varphi(n)} - \ell| < \frac{\varepsilon}{2} \quad \forall n \geq n_2.$$

Let $n_3 = \max(n_1, n_2)$, then $n_3 \geq n_1$ and $n_3 \geq n_2$. As $\varphi(n_3) \geq n_3$ by the lemma 2.2.1, then $\varphi(n_3) \geq n_1$. Put $n_0 = \varphi(n_3)$. If $n \geq n_0$, then $n \geq n_1$, so

$$|u_n - \ell| = |u_n - u_{\varphi(n_3)} + u_{\varphi(n_3)} - \ell| \leq |u_n - u_{\varphi(n_3)}| + |u_{\varphi(n_3)} - \ell| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $(u_n)_{n\in\mathbb{N}}$ is convergent to ℓ. □

Theorem 2.9.1 *(Completeness of $\mathbb{R}$) or (Cauchy's criterion)*
Every Cauchy real sequence is convergent.

$\boxed{\text{Proof}}$ Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy real sequence. Let $\varepsilon > 0$, then there exists $n_0 \in \mathbb{N}$ such that $|u_p - u_q| < \frac{\varepsilon}{3}$ for all $p \geq n_0$ and $q \geq n_0$. For every $n \in \mathbb{N}$, consider the set

$$A_n = \Big\{u_k,\ k \geq n\Big\}.$$

Let's show that A_n is bounded. Indeed,

- If $n \geq n_0$, then, for any $k \geq n \geq n_0$, $|u_k - u_{n_0}| < \frac{\varepsilon}{3}$, so
$$u_{n_0} - \frac{\varepsilon}{3} < u_k < u_{n_0} + \frac{\varepsilon}{3}.$$
Therefore A_n is bounded from above by $u_{n_0} + \frac{\varepsilon}{3}$ and from below by $u_{n_0} - \frac{\varepsilon}{3}$. Hence A_n is bounded.

- If $n < n_0$, then
$$A_n = \Big\{u_n, \ldots, u_{n_0-1}\Big\} \cup A_{n_0}$$
As A_{n_0} is bounded (by the first case) and $\Big\{u_n, \ldots, u_{n_0-1}\Big\}$ is bounded (finite set), then their union A_n is bounded.

Hence A_n is a nonempty bounded subset of $\mathbb{R}$. By the theorem 1.3.3, A_n has an infimum $a_n = \inf A_n$ and a supremum $b_n = \sup A_n$. As $A_{n+1} \subseteq A_n$, then, by the part (1) of the exercise 10, we obtain:

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n.$$

Hence $(a_n)_{n\in\mathbb{N}}$ is an increasing sequence and $(b_n)_{n\in\mathbb{N}}$ is a decreasing sequence. Let's show that $\lim\limits_{n\to+\infty} (b_n - a_n) = 0$. Indeed, let $n \geq n_0$ fixed,

- Since $a_n = \inf A_n$, then there exists $p \geq n$ (i.e., $u_p \in A_n$) such that $a_n \leq u_p < a_n + \frac{\varepsilon}{3}$ by the infimum criterion.
- Since $b_n = \sup A_n$, then there exists $q \geq n$ (i.e., $u_q \in A_n$) such that $b_n - \frac{\varepsilon}{3} < u_q \leq b_n$ by the supremum criterion.

So

$$|b_n - a_n| \leq |b_n - u_q| + |u_q - u_p| + |u_p - a_n| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Hence $\lim\limits_{n \to +\infty} (b_n - a_n) = 0$. Consequently, $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two adjacent sequences, so they are convergent to the same limit $\ell \in \mathbb{R}$ by the theorem 2.8.1. In the other side, for all $n \in \mathbb{N}$, $u_n \in A_n$, so $a_n \leq u_n \leq b_n$. Hence $\lim\limits_{n\to+\infty} u_n = \ell$ by the Sandwich theorem. □

Exercise 56
Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence.

1) **Show that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if and only if**

$$\Big(\forall\, \varepsilon > 0\Big)\Big(\exists\, n_0 \in \mathbb{N}\Big)\Bigg[\Big(\forall\, n, p \in \mathbb{N}\Big)\Big((n \geq n_0) \Rightarrow |u_{n+p} - u_n| < \varepsilon\Big)\Bigg].$$

2) **Suppose that there exist $a \in [0, 1[$ and $k \in \mathbb{R}$ such that**

$$|u_{n+1} - u_n| \leq k a^n, \quad \forall\, n \in \mathbb{N}.$$

Show that $(u_n)_{n\in\mathbb{N}}$ is convergent.

Solution

1) N.C. Let $\varepsilon > 0$. Since $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, then there exists $n_0 \in \mathbb{N}$ such that

$$(*) \qquad |u_p - u_q| < \varepsilon, \quad \text{for all } p, q \in \mathbb{N} \text{ such that } p \geq n_0 \text{ and } q \geq n_0.$$

If $n \geq n_0$ and $p \in \mathbb{N}$ arbitrary, then $n + p \geq n_0$, so $|u_{n+p} - u_n| < \varepsilon$ by $(*)$.
S.C. Let $\varepsilon > 0$. By the hypothesis, there exists $n_0 \in \mathbb{N}$ such that

$$(**) \qquad |u_{n+p} - u_n| < \varepsilon, \quad \text{for all } n, p \in \mathbb{N} \text{ such that } n \geq n_0.$$

Let $p, q \in \mathbb{N}$ such that $p \geq n_0$ and $q \geq n_0$.

- If $p \leq q$, then put $p' = q - p \in \mathbb{N}$. Since $p \geq n_0$, then, by using $(**)$, we obtain:
$$|u_p - u_q| = |u_q - u_p| = |u_{p+p'} - u_p| < \varepsilon.$$
- If $p > q$, then $q \leq p$, and therefore $|u_q - u_p| < \varepsilon$ by the first case. Hence $|u_p - u_q| < \varepsilon$.

Thus $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

2) Let's show that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence by using the part (1). Indeed, let $\varepsilon > 0$ and $n, p \in \mathbb{N}$, then

$$\begin{aligned} |u_{n+p} - u_n| &= \Big|(u_{n+p} - u_{n+p-1}) + \cdots + (u_{n+1} - u_n)\Big| \\ &\leq |u_{n+p} - u_{n+p-1}| + \cdots + |u_{n+1} - u_n| \\ &\leq ka^{n+p-1} + \cdots + ka^n \;=\; ka^n(a^{p-1} + \cdots + 1) \\ &= ka^n\left(\frac{1-a^p}{1-a}\right) \;\leq\; \frac{ka^n}{1-a} \quad (\text{since } 1 - a^p \leq 1). \end{aligned}$$

Since $a \in [0, 1[$, then $\lim\limits_{n\to+\infty} \dfrac{ka^n}{1-a} = 0$, so there exists $n_0 \in \mathbb{N}$ such that $\frac{ka^n}{1-a} < \varepsilon$ for all $n \geq n_0$. Hence $|u_{n+p} - u_n| < \varepsilon$ for all $n \geq n_0$. Consequently, $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence by the part (1), and therefore it is convergent. $\square$

Exercise 57

In each of the following cases, study the nature of the real sequence $(u_n)_n$. Find its limit if it is convergent.

1) $u_n = \dfrac{3n^3 + n^2 - 4}{2n^4 + 3n + 1}$,

2) $u_n = \dfrac{1 + 2 + 2^2 + \cdots + 2^n}{2^{n+1}}$.

3) $u_n = \sqrt{n^3 + 3n} - \sqrt{n^3 + n^2}$,

4) $u_n = \dfrac{n + \cos n}{n - \sin n}$.

5) $u_n = \dfrac{(-1)^n}{\ln n}$,

6) $u_n = n^{-1+(-1)^n}$.

7) $u_n = \cos(\pi\sqrt{n})$,

8) $u_n = \dfrac{\ln(n^2 + n)}{\ln(n^2 + 2^n)}$.

9) $u_n = \dfrac{1}{n} + 2(-1)^n$,

10) $u_n = \dfrac{n! + 2^n}{(n+1)! + 3^n}$.

Solution

1) $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} \dfrac{3n^3}{2n^4} = \lim\limits_{n\to+\infty} \dfrac{3}{2n} = 0.$

2) We have:

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \frac{2^{n+1} - 1}{2^{n+1}} = \lim_{n\to+\infty} \left(1 - \frac{1}{2^{n+1}}\right) = 1.$$

3) For any $n \in \mathbb{N}^*$,

$$\begin{aligned} u_n &= \frac{(n^3+3n)-(n^3+n^2)}{\sqrt{n^3+3n}+\sqrt{n^3+n^2}} \quad \text{(by multiplying by the conjugate)} \\ &= \frac{-n^2+3n}{\sqrt{n^3+3n}+\sqrt{n^3+n^2}} = \frac{n^2\left(-1+\frac{3}{n}\right)}{\sqrt{n^3}\left(\sqrt{1+\frac{3}{n^2}}+\sqrt{1+\frac{1}{n}}\right)} \\ &= \frac{\sqrt{n}\left(-1+\frac{3}{n}\right)}{\sqrt{1+\frac{3}{n^2}}+\sqrt{1+\frac{1}{n}}} \underset{+\infty}{\sim} \frac{-\sqrt{n}}{2}. \end{aligned}$$

So $\lim_{n\to+\infty} u_n = -\infty$.

4) $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \frac{1+\frac{\cos n}{n}}{1-\frac{\sin n}{n}} = \frac{1+0}{1-0} = 1.$

5) Since $0 \leq |u_n| \leq \frac{1}{\ln n}$ for all $n \geq 2$ and $\lim_{n\to+\infty} \frac{1}{\ln n} = 0$, then $\lim_{n\to+\infty} |u_n| = 0$ by the Sandwich theorem, so $\lim_{n\to+\infty} u_n = 0$.

6) $\lim_{n\to+\infty} u_{2n} = 1$ and $\lim_{n\to+\infty} u_{2n+1} = \lim_{n\to+\infty} \frac{1}{(2n+1)^2} = 0$.
As $\lim_{n\to+\infty} u_{2n} \neq \lim_{n\to+\infty} u_{2n+1}$, then the sequence $(u_n)_n$ is divergent.

7) For any $n \in \mathbb{N}$, $u_{(2n)^2} = \cos(2n\pi) = 1$ and $u_{(2n+1)^2} = \cos((2n+1)\pi) = -1$ As

$$\lim_{n\to+\infty} u_{(2n)^2} = 1 \neq -1 = \lim_{n\to+\infty} u_{(2n+1)^2},$$

then the sequence $(u_n)_n$ is divergent.

8) For any $n \in \mathbb{N}$,

$$\begin{aligned} u_n &= \frac{\ln\left[n^2\left(1+\frac{1}{n}\right)\right]}{\ln\left[2^n\left(1+\frac{n^2}{2^n}\right)\right]} = \frac{2\ln n + \ln\left(1+\frac{1}{n}\right)}{n\ln 2 + \ln\left(1+\frac{n^2}{2^n}\right)} \\ &= \frac{2\ln n}{n\ln 2}\left(\frac{1+\frac{\ln\left(1+\frac{1}{n}\right)}{2\ln n}}{1+\frac{\ln\left(1+\frac{n^2}{2^n}\right)}{n\ln 2}}\right) \underset{+\infty}{\sim} \frac{2\ln n}{n\ln 2}, \end{aligned}$$

by using the fact that

$$\lim_{n\to+\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{2\ln n} = 0 \quad \text{and} \quad \lim_{n\to+\infty} \frac{\ln\left(1+\frac{n^2}{2^n}\right)}{n\ln 2} = 0.$$

So $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \frac{2\ln n}{n\ln 2} = 0$.

9) $\lim_{n\to+\infty} u_{2n} = \lim_{n\to+\infty} \left(\frac{1}{2n}+2\right) = 2$ and $\lim_{n\to+\infty} u_{2n+1} = \lim_{n\to+\infty} \left(\frac{1}{2n+1}-2\right) = -2$.
As $\lim_{n\to+\infty} u_{2n} \neq \lim_{n\to+\infty} u_{2n+1}$, then the sequence $(u_n)_n$ is divergent.

10) For any $n \in \mathbb{N}$,

$$u_n = \frac{n!\left(1+\frac{2^n}{n!}\right)}{(n+1)!\left(1+\frac{3^n}{(n+1)!}\right)} = \frac{1}{n+1}\left(\frac{1+\frac{2^n}{n!}}{1+\frac{3^n}{(n+1)!}}\right) \underset{+\infty}{\sim} \frac{1}{n+1},$$

by using the fact that $\lim\limits_{n\to+\infty} \frac{2^n}{n!} = 0$ and $\lim\limits_{n\to+\infty} \frac{3^n}{(n+1)!} = 0$. So $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} \frac{1}{n+1} = 0$. □

Exercise 58

Consider the recursive sequence $(u_n)_{n\geq 0}$, defined by $u_0 = \frac{1}{2}$ and

$$u_{n+1} = \frac{u_n^2 + u_n}{2} \quad \text{for } n \geq 0.$$

a) Show that $0 < u_n < 1$ for all $n \in \mathbb{N}$.

b) Show that the sequence $(u_n)_n$ is monotone.

c) Deduce that the sequence $(u_n)_n$ is convergent and determine its limit.

Solution

a) By mathematical induction: for $n = 0$, $0 < u_0 = \frac{1}{2} < 1$. Suppose that $0 < u_k < 1$ for a certain $k \in \mathbb{N}$. Then $1 < u_k + 1 < 2$ and therefore $0 < u_k(u_k+1) < 2$. So

$$0 < u_{k+1} = \frac{u_k(u_k+1)}{2} < \frac{2}{2} = 1.$$

b) For any $n \in \mathbb{N}$,

$$u_{n+1} - u_n = \frac{u_n^2 - u_n}{2} = \frac{u_n(u_n-1)}{2} < 0,$$

since $u_n > 0$ and $u_n - 1 < 0$. Then the sequence $(u_n)_n$ is strictly decreasing.

c) The sequence $(u_n)_n$ is decreasing and bounded from below by 0, so it is convergent to $\ell = \inf u_n$. Taking limits in the formula $u_{n+1} = \frac{u_n^2 + u_n}{2}$, we obtain $\ell = \frac{\ell^2 + \ell}{2}$, so $\ell^2 - \ell = 0$, therefore $\ell = 0$ or $\ell = 1$. If $\ell = 1$, then $\inf u_n = 1$ and therefore $u_n \geq 1$ for all $n \in \mathbb{N}$, which is in contradiction with the part (a). Hence $\ell \neq 1$, and therefore $\ell = 0$. □

Exercise 59

Let $(u_n)_{n\geq 0}$ be the sequence defined by:

$$u_0 \geq 2 \quad \text{and} \quad u_{n+1} = u_n^2 - \frac{2n}{n+1}, \quad \forall n \geq 0.$$

1) **Show that $u_n \geq 4$ for all $n \geq 1$.**

2) **Deduce that the sequence $(u_n)_{n\geq 0}$ is divergent.**

3) **Show that $u_{n+1} - u_n \geq (u_n + 1)(u_n - 2)$ for all $n \geq 0$, and deduce that the sequence $(u_n)_{n\geq 0}$ is increasing.**

4) **Is the sequence $(u_n)_{n\geq 0}$ bounded from above ? Justify your answer.**

Solution

1) For $n = 1$, $u_1 = u_0^2 \geq 4$. Suppose that $u_k \geq 4$ for a certain $k \geq 1$, then $u_k^2 \geq 16$. As $\dfrac{2k}{k+1} \leq 2$, then

$$u_{k+1} = u_k^2 - \frac{2k}{k+1} \geq 16 - 2 \geq 4.$$

Hence $u_n \geq 4$ for all $n \geq 1$.

2) Suppose that $(u_n)_{n\geq 0}$ is convergent to a limit ℓ, then $\lim\limits_{n\to+\infty} u_n^2 = \ell^2$. Since $\lim\limits_{n\to+\infty} \dfrac{2n}{n+1} = 2$, then, by taking limits in the recursive formula, we obtain $\ell = \ell^2 - 2$, so $\ell = -1$ or $\ell = 2$. But $u_n \geq 4$ for all $n \geq 1$, then $\ell \geq 4$, which is impossible. Hence the sequence $(u_n)_{n\geq 0}$ is divergent.

3) For any $n \geq 0$,

$$u_{n+1} - u_n = u_n^2 - \frac{2n}{n+1} - u_n \geq u_n^2 - 2 - u_n = (u_n + 1)(u_n - 2).$$

As $u_0 \geq 2$ and $u_n \geq 4 \geq 2$ for all $n \geq 1$, then $u_n + 1 \geq 0$ and $u_n - 2 \geq 0$, so $(u_n + 1)(u_n - 2) \geq 0$, therefore $u_{n+1} - u_n \geq 0$. Hence $(u_n)_{n\geq 0}$ is increasing.

4) Suppose that the sequence $(u_n)_{n\geq 0}$ is bounded from above, and as it is increasing, then it is convergent, which is impossible by the part (2). So $(u_n)_{n\geq 0}$ is not bounded from above. $\square$

Exercise 60

Let $(u_n)_{n\geq 1}$ be the sequence defined by:

$$u_1 = 3 \quad \text{and} \quad u_{n+1} = \frac{(n+2)u_n + 2(n^2 + n - 1)}{(n+1)^2}, \quad \forall n \geq 1.$$

1) **Show that the sequence $(u_n)_{n\geq 1}$ is decreasing.**

2) **Show that $(u_n)_{n\geq 1}$ is convergent to a limit ℓ to be determined.**

3) **For every $n \geq 1$, express u_n in terms of n.**
(Hint: we may use the expression of $u_n - 2$).

Solution

1) For any $n \geq 1$, we have:

$$u_{n+1} - u_n = \frac{(n^2+n-1)(2-u_n)}{(n+1)^2}.$$

As $n^2+n-1 \geq 0$ (since $n \geq 1$), then the sign of $u_{n+1} - u_n$ is the same as that of $2 - u_n$. Let's show, by mathematical induction, that $2 - u_n \leq 0$ for all $n \geq 1$. Indeed, $2 - u_1 = 2 - 3 = -1 \leq 0$. Suppose that $2 - u_k \leq 0$ for a certain $k \geq 1$. Then

$$(*) \qquad 2 - u_{k+1} = \frac{(k+2)(2-u_k)}{(k+1)^2} \leq 0.$$

Hence $2 - u_n \leq 0$ for all $n \geq 1$. Therefore $u_{n+1} - u_n \leq 0$ for all $n \geq 1$. Thus $(u_n)_{n\geq 1}$ is decreasing.

2) Since $u_n \geq 2$ for all $n \geq 1$, then $(u_n)_{n\geq 1}$ is bounded from below by 2, and as it is decreasing, then it is convergent to a limit ℓ. For any $n \geq 1$, we have:

$$u_{n+1} = \frac{n+2}{(n+1)^2}u_n + \frac{2(n^2+n-1)}{(n+1)^2}.$$

By taking limits, we obtain $\ell = (0)(\ell) + 2 = 2$.

3) By $(*)$, for any $n \geq 2$,

$$\begin{aligned} u_n - 2 &= \frac{(n+1)}{n^2}(u_{n-1} - 2) \\ &= \frac{(n+1)}{n^2} \times \frac{n}{(n-1)^2} \times \cdots \times \frac{3}{2^2} \times (u_1 - 2) \\ &= \frac{1}{2} \times \frac{(n+1)!}{(n!)^2} = \frac{n+1}{2n!}. \end{aligned}$$

So $u_n = \frac{n+1}{2n!} + 2$. Moreover, this formula is also true for $n = 1$. So it is true for all $n \geq 1$. $\square$

Exercise 61

Let $(u_n)_{n\geq 0}$ be a real sequence.

1) **Prove that if $\lim_{n\to+\infty} \frac{u_n}{1+u_n} = 0$, then $\lim_{n\to+\infty} u_n = 0$.**

2) **Prove that if $(u_n)_{n\geq 0}$ is bounded and $\lim_{n\to+\infty} \frac{u_n}{1+u_n^2} = 0$, then $\lim_{n\to+\infty} u_n = 0$.**

Solution

1) For every $n \in \mathbb{N}$, put $a_n = \dfrac{u_n}{1+u_n}$, then $u_n = \dfrac{a_n}{1-a_n}$ (it is obvious that $a_n \neq 1$).

   As $\lim_{n\to+\infty} a_n = 0$, then $\lim_{n\to+\infty} u_n = \dfrac{0}{1-0} = 0$.

2) For every $n \in \mathbb{N}$, put $b_n = \dfrac{u_n}{1+u_n^2}$, then $u_n = (1+u_n^2)b_n$. As $(u_n)_{n\geq 0}$ is bounded, then the sequence $(1+u_n^2)_{n\geq 0}$ is bounded. Moreover, as $\lim\limits_{n\to+\infty} b_n = 0$, then $\lim\limits_{n\to+\infty} u_n = 0$ (as the product of a bounded sequence and a convergent sequence to 0). □

Exercise 62
Let $a > 1$ be a given real number. Put

$$u_0 = a \quad \text{and} \quad u_{n+1} = u_n + \frac{1}{u_n}, \quad \forall n \geq 0.$$

1) **Show that the sequence $(u_n)_{n\geq 0}$ is well-defined.**

2) **Study the monotonicity and the convergence of the sequence $(u_n)_{n\geq 0}$ (find its limit if it exists).**

3) **Show that $u_{n+1} \underset{+\infty}{\sim} u_n$.**

4) **Show that, for all $k \geq 0$,**

$$2 < u_{k+1}^2 - u_k^2 < 2 + u_{k+1} - u_k.$$

5) **Deduce that, for all $n \geq 1$,**

$$2n + a^2 < u_n^2 < 2n + u_n + a^2 - a.$$

6) **Show that $u_n \underset{+\infty}{\sim} \sqrt{2n}$.**

Solution

1) Let's show, by mathematical induction, that $u_n \neq 0$ for all $n \in \mathbb{N}$. For $n = 0$, $u_0 = a \neq 0$. Suppose that $u_k \neq 0$ for a certain $k \geq 0$. If $u_{k+1} = 0$, then $u_k + \frac{1}{u_k} = 0$, so $u_k^2 = -1$, which is impossible, hence $u_{k+1} \neq 0$. Thus $u_n \neq 0$ for all $n \in \mathbb{N}$. Consequently, the sequence $(u_n)_{n\geq 0}$ is well-defined.

2) Let's show, by mathematical induction, that $u_n > 1$ for all $n \in \mathbb{N}$. For $n = 0$, $u_0 = a > 1$. Suppose that $u_k > 1$ for a certain $k \geq 0$. Then $u_{k+1} = u_k + \frac{1}{u_k} > 1$ (since $\frac{1}{u_k} > 0$). So $u_n > 1$ for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, $u_{n+1} - u_n = \frac{1}{u_n} > 0$. So the sequence $(u_n)_{n\geq 0}$ is strictly increasing. In the other side, suppose that $(u_n)_{n\geq 0}$ is convergent to a limit ℓ. As $u_n > 1$ for all $n \in \mathbb{N}$, then $\ell \geq 1$, so $\ell \neq 0$. By taking limits in the relation $u_{n+1} = u_n + \frac{1}{u_n}$, we obtain $\ell = \ell + \frac{1}{\ell}$, so $\frac{1}{\ell} = 0$, which is impossible. Hence $(u_n)_{n\geq 0}$ is divergent. If $(u_n)_{n\geq 0}$ is bounded from above, it is convergent (since it is increasing), which is impossible. So $(u_n)_{n\geq 0}$ is an increasing sequence which is not bounded from above, hence $\lim\limits_{n\to+\infty} u_n = +\infty$.

3) For any $n \in \mathbb{N}$,
$$\frac{u_{n+1}}{u_n} = 1 + \frac{1}{u_n^2}.$$
As $\lim\limits_{n\to+\infty} u_n = +\infty$, then $\lim\limits_{n\to+\infty} u_n^2 = +\infty$, so $\lim\limits_{n\to+\infty} \frac{1}{u_n^2} = 0$. Therefore $\lim\limits_{n\to+\infty} \frac{u_{n+1}}{u_n} = 1$. Hence $u_{n+1} \underset{+\infty}{\sim} u_n$.

4) Let $k \geq 0$, then
$$u_{k+1}^2 - u_k^2 = \left(u_k + \frac{1}{u_k}\right)^2 - u_k^2 = 2 + \frac{1}{u_k^2}.$$
As $\frac{1}{u_k^2} > 0$, then $u_{k+1}^2 - u_k^2 > 2$. In the other side, as $u_k > 1$ (by the proof of the part (2)), then $u_k^2 > u_k$, so $\frac{1}{u_k^2} < \frac{1}{u_k}$. But $\frac{1}{u_k} = u_{k+1} - u_k$, then
$$u_{k+1}^2 - u_k^2 = 2 + \frac{1}{u_k^2} < 2 + \frac{1}{u_k} = 2 + u_{k+1} - u_k.$$

5) Let $n \geq 1$. Applying the sum $\sum\limits_{k=0}^{n-1}$ on the double inequality of the part (4), we obtain:
$$\sum_{k=0}^{n-1} 2 < \sum_{k=0}^{n-1}(u_{k+1}^2 - u_k^2) < \sum_{k=0}^{n-1}(2 + u_{k+1} - u_k).$$
But $\sum\limits_{k=0}^{n-1} 2 = 2n$,
$$\sum_{k=0}^{n-1}(u_{k+1}^2 - u_k^2) = (u_1^2 - u_0^2) + (u_2^2 - u_1^2) + \cdots + (u_n^2 - u_{n-1}^2) = u_n^2 - u_0^2 = u_n^2 - a^2,$$
$$\begin{aligned}\sum_{k=0}^{n-1}(2 + u_{k+1} - u_k) &= (2 + u_1 - u_0) + (2 + u_2 - u_1) + \cdots + (2 + u_n - u_{n-1}) \\ &= 2n + u_n - u_0 \;=\; 2n + u_n - a,\end{aligned}$$
then $2n < u_n^2 - a^2 < 2n + u_n - a$, and therefore
$$2n + a^2 < u_n^2 < 2n + u_n + a^2 - a.$$

6) By using the part (5), for all $n \geq 1$,
$$u_n^2 - u_n - a^2 + a < 2n < u_n^2 - a^2.$$
So, dividing by $u_n^2 \neq 0$, we obtain:
$$1 - \frac{1}{u_n} + \frac{a^2 - a}{u_n} < \frac{2n}{u_n^2} < 1 - \frac{a^2}{u_n}.$$

Since $\lim_{n \to +\infty} u_n = +\infty$, then

$$\lim_{n \to +\infty} \frac{1}{u_n} = \lim_{n \to +\infty} \frac{a^2 - a}{u_n} = \lim_{n \to +\infty} \frac{a^2}{u_n} = 0.$$

So $\lim_{n \to +\infty} \frac{2n}{u_n^2} = 1$ by the Sandwich theorem. Therefore $\lim_{n \to +\infty} \frac{\sqrt{2n}}{u_n} = \sqrt{1} = 1$ by the exercise 50 ($u_n > 0$). Hence $u_n \underset{+\infty}{\sim} \sqrt{2n}$. □

Exercise 63
Let $(u_n)_{n \geq 1}$ be the sequence defined by:

$$u_1 = 1 \quad \text{and} \quad u_{n+1} = \frac{n + u_n}{n^2}, \quad \forall\, n \geq 1.$$

1) **Show that $u_n \leq 2$ for all $n \geq 1$. Deduce that the sequence $(u_n)_{n \geq 1}$ is convergent to a limit ℓ to be determined.**

2) **Show that, for all $n \geq 2$,**

$$\frac{1}{n-1} \leq u_n \leq \frac{n+1}{(n-1)^2}.$$

3) **Deduce a simple equivalent sequence to u_n.**

4) **For every $n \geq 2$, put**

$$v_n = \frac{n}{n^2 - 1}.$$

(a) **Show that the sequence $(v_n)_{n \geq 2}$ is decreasing.**
(b) **Show that $u_n \geq v_n$ for all $n \geq 2$.**
(c) **Deduce that the sequence $(u_n)_{n \geq 2}$ is decreasing.**
(d) **Is the sequence $(u_n)_{n \geq 1}$ decreasing ? Justify your answer.**

Solution

1) For $n = 2$, $u_2 = 2 \leq 2$. Suppose that $u_k \leq 2$ for a certain $k \geq 2$. Then

$$u_{k+1} = \frac{k + u_k}{k^2} \leq \frac{k+2}{k^2} = \frac{1}{k} + \frac{2}{k^2} \leq \frac{1}{2} + \frac{2}{2^2} = 1 \leq 2.$$

So $u_n \leq 2$ for all $n \geq 2$. But $u_1 = 1 \leq 2$, then $u_n \leq 2$ for all $n \geq 1$. In the other side, for all $n \geq 2$,

$$u_n = \frac{(n-1) + u_{n-1}}{(n-1)^2} = \frac{1}{n-1} + \frac{u_{n-1}}{(n-1)^2}.$$

We can easily prove, by mathematical induction on n, that $u_n > 0$ for all $n \geq 1$. So the sequence $(u_{n-1})_n$ is bounded, and as $\lim\limits_{n\to+\infty} \frac{1}{(n-1)^2} = 0$, then $\lim\limits_{n\to+\infty} \frac{u_{n-1}}{(n-1)^2} = 0$. Hence

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \frac{1}{n-1} + \lim_{n\to+\infty} \frac{u_{n-1}}{(n-1)^2} = 0.$$

2) Let $n \geq 2$. As $u_{n-1} > 0$ (by the proof of the part (1)), then

$$u_n = \frac{(n-1) + u_{n-1}}{(n-1)^2} \geq \frac{n-1}{(n-1)^2} = \frac{1}{n-1}.$$

In the other side, as $u_{n-1} \leq 2$ (by the part (1)), then

$$u_n = \frac{(n-1) + u_{n-1}}{(n-1)^2} \leq \frac{(n-1)+2}{(n-1)^2} = \frac{n+1}{(n-1)^2}.$$

3) For any $n \geq 2$, multiplying the double inequality of the part (2) by n, we obtain:

$$\frac{n}{n-1} \leq nu_n \leq \frac{n(n+1)}{(n-1)^2}.$$

As $\lim\limits_{n\to+\infty} \frac{n}{n-1} = \lim\limits_{n\to+\infty} \frac{n(n+1)}{(n-1)^2} = 1$, then $\lim\limits_{n\to+\infty} nu_n = 1$ by the Sandwich theorem. Hence $u_n \underset{+\infty}{\sim} \frac{1}{n}$.

4) (a) For any $n \geq 2$,

$$v_{n+1} - v_n = \frac{n+1}{n^2+2n} - \frac{n}{n^2-1} = \frac{-(n^2+n+1)}{(n^2+2n)(n^2-1)} \leq 0.$$

So the sequence $(v_n)_{n\geq 2}$ is decreasing.

(b) For any $n \geq 2$, $u_n \geq \frac{1}{n-1}$ (by the part (2)), so

$$u_n - v_n \geq \frac{1}{n-1} - \frac{n}{n^2-1} = \frac{1}{n^2-1} \geq 0.$$

So $u_n \geq v_n$.

(c) For any $n \geq 2$,

$$u_{n+1} - u_n = \frac{n+u_n}{n^2} - u_n = \frac{n-(n^2-1)u_n}{n^2} = \frac{n - \frac{nu_n}{v_n}}{n^2} = \frac{1 - \frac{u_n}{v_n}}{n} \leq 0,$$

by using the fact that $\frac{u_n}{v_n} \geq 1$ (since $u_n \geq v_n$). Hence the sequence $(u_n)_{n\geq 2}$ is decreasing.

(d) If the sequence $(u_n)_{n\geq 1}$ is decreasing, then $u_1 \geq u_2$, i.e., $1 \geq 2$, which is impossible. So $(u_n)_{n\geq 1}$ is not decreasing. □

Exercise 64
In each of the following cases, discuss, according to the values of the real parameters a, b and c, the nature of the real sequence $(u_n)_n$. Find its limit if it is convergent.

1) $u_n = \dfrac{an^2 + bn + c}{n+1}$, **2)** $u_n = \sqrt{n+a} - \sqrt{n+b}$.

3) $u_n = \dfrac{n^a + a^n}{n^{2a} + a^{2n}}$ **where** $a > 0$.
Hint: we may consider the following three cases:

$$0 < a < 1, \qquad a = 1, \qquad a > 1.$$

4) $u_n = \dfrac{a^n - b^n}{a^n + b^n}$ **where** $a > 0$ **and** $b > 0$, **5)** $u_n = \dfrac{2^{an} + 3n + 1}{2^{an} + 1}$.

Solution

1)
- If $a \neq 0$, then $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \dfrac{an^2}{n} = \lim_{n\to+\infty} an = \pm\infty$ according to the sign of a.
- If $a = 0$ and $b \neq 0$, then $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \dfrac{bn}{n} = \lim_{n\to+\infty} b = b$.
- If $a = 0$ and $b = 0$, then $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \dfrac{c}{n} = 0$.

2) For any $n \geq 1$, multiplying by the conjugate, we obtain:

$$u_n = \frac{a-b}{\sqrt{n+a} + \sqrt{n+b}} = \frac{a-b}{\sqrt{n}\left[\sqrt{1+\frac{a}{n}} + \sqrt{1+\frac{b}{n}}\right]}.$$

- If $a = b$, then $u_n = 0$, so $\lim_{n\to+\infty} u_n = 0$.
- If $a \neq b$, then $u_n \underset{+\infty}{\sim} \dfrac{a-b}{\sqrt{n}}$. So $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \dfrac{a-b}{\sqrt{n}} = 0$.

Hence $\lim_{n\to+\infty} u_n = 0$.

3)
- Suppose that $0 < a < 1$. Then

$$u_n = \frac{n^a(1 + \frac{a^n}{n^a})}{n^{2a}(1 + \frac{a^{2n}}{n^{2a}})} \underset{+\infty}{\sim} \frac{n^a}{n^{2a}} = \frac{1}{n^a},$$

by using the fact that $\lim_{n\to+\infty} \dfrac{a^n}{n^a} = \lim_{n\to+\infty} \dfrac{a^{2n}}{n^{2a}} = 0$. So

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \frac{1}{n^a} = 0.$$

- If $a = 1$, then $u_n = \dfrac{n+1}{n^2+1}$, so $\lim\limits_{n\to+\infty} u_n = 0$.
- Suppose that $a > 1$. Then

$$u_n = \frac{a^n(\frac{n^a}{a^n}+1)}{a^{2n}(\frac{n^{2a}}{a^{2n}}+1)} \underset{+\infty}{\sim} \frac{a^n}{a^{2n}} = \frac{1}{a^n},$$

by using the fact that $\lim\limits_{n\to+\infty} \dfrac{n^a}{a^n} = \lim\limits_{n\to+\infty} \dfrac{n^{2a}}{a^{2n}} = 0$. So

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \frac{1}{a^n} = 0.$$

Hence the sequence $(u_n)_n$ is convergent to 0 in all cases.

4) - Suppose that $a > b$. Then

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \frac{1-\left(\frac{b}{a}\right)^n}{1+\left(\frac{b}{a}\right)^n} = 1,$$

by using the fact that $\lim\limits_{n\to+\infty} \left(\dfrac{b}{a}\right)^n = 0$ since $\frac{b}{a} \in]0,1[$.

- If $a = b$, then $u_n = 0$ for all $n \in \mathbb{N}$, so $\lim\limits_{n\to+\infty} u_n = 0$.
- Suppose that $a < b$. Then

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \frac{\left(\frac{a}{b}\right)^n - 1}{\left(\frac{a}{b}\right)^n + 1} = -1,$$

by using the fact that $\lim\limits_{n\to+\infty} \left(\dfrac{a}{b}\right)^n = 0$ since $\frac{a}{b} \in]0,1[$.

5) - Suppose that $a > 0$. Then

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \frac{1+\frac{3n+1}{2^{an}}}{1+\frac{1}{2^{an}}} = 1.$$

- If $a = 0$, then $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} \dfrac{3n+2}{2} = +\infty$.
- Suppose that $a < 0$. Then

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \frac{1+\frac{2^{an}+1}{3n}}{\frac{1}{3n}+\frac{1}{2^{-an}3n}} = +\infty. \qquad \square$$

Exercise 65
Let $a > 0$ be a given real number. We define the sequence $(u_n)_{n\geq 0}$ by:

$$u_0 > 0 \quad \text{and} \quad u_{n+1} = \frac{1}{2}\left(u_n + \frac{a}{u_n}\right), \quad \forall n \geq 0.$$

1) Show that the sequence $(u_n)_{n\geq 0}$ is well-defined.

2) Verify that, for all $n \geq 0$,

$$u_{n+1}^2 - a = \frac{(u_n^2 - a)^2}{4u_n^2}.$$

3) Deduce that $u_n \geq \sqrt{a}$ for all $n \geq 1$.

4) Deduce that the sequence $(u_n)_{n\geq 0}$ is decreasing.

5) Deduce that $(u_n)_{n\geq 0}$ is convergent, and find its limit.

Solution

1) Let's show that $u_n \neq 0$ for all $n \in \mathbb{N}$. Indeed, we have $u_0 \neq 0$. Suppose that $u_k \neq 0$ for a certain $k \in \mathbb{N}$. If $u_{k+1} = 0$, then $u_k + \frac{a}{u_k} = 0$, and therefore $u_k^2 = -a < 0$, which is impossible. Therefore $u_{k+1} \neq 0$. Hence $u_n \neq 0$ for all $n \in \mathbb{N}$. Thus the sequence $(u_n)_{n\geq 0}$ is well-defined.

2) For any $n \geq 0$,

$$u_{n+1}^2 - a = \frac{1}{4}\left(u_n + \frac{a}{u_n}\right)^2 - a = \frac{(u_n^2 + a)^2}{4u_n^2} - a = \frac{(u_n^2 - a)^2}{4u_n^2}.$$

3) Let's show that $u_n > 0$ for all $n \in \mathbb{N}$. Indeed, $u_0 > 0$ by the hypothesis. Suppose that $u_k > 0$ for a certain $k \in \mathbb{N}$, then $u_{k+1} = \frac{u_k}{2} + \frac{a}{2u_k} > 0$. Hence $u_n > 0$ for all $n \in \mathbb{N}$. In the other side, by the part (2), for all $n \geq 1$, we have:

$$u_n^2 - a = \frac{(u_{n-1}^2 - a)^2}{4u_{n-1}^2} \geq 0.$$

Moreover, as $u_n > 0$, then $u_n \geq \sqrt{a}$.

4) First, $u_n > 0$ for all $n \in \mathbb{N}$. In the other side, for all $n \in \mathbb{N}$,

$$\frac{u_{n+1}}{u_n} = \frac{1}{2}\left(1 + \frac{a}{u_n^2}\right) \leq \frac{1}{2}(1+1) = 1,$$

by using the fact that $u_n^2 \geq a$ (since $u_n \geq \sqrt{a}$). Hence the sequence $(u_n)_{n\geq 0}$ is decreasing.

5) Since the sequence $(u_n)_{n\geq 0}$ is decreasing and bounded from below by $\sqrt{a}$, then it is convergent to a limit $\ell \in \mathbb{R}$. Since $u_n \geq \sqrt{a}$ for all $n \geq 1$, then $\ell \geq \sqrt{a} > 0$, so $\ell > 0$ (in particular $\ell \neq 0$). Taking limits in the relation $u_{n+1} = \frac{1}{2}\left(u_n + \frac{a}{u_n}\right)$, we obtain $\ell = \frac{1}{2}\left(\ell + \frac{a}{\ell}\right)$, so $\ell^2 = a$, hence $\ell = \sqrt{a}$ (since $\ell > 0$). □

Exercise **66 (Manipulating with the definition of the limit).**
Let $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ be two convergent sequences to ℓ and ℓ' respectively. For every $n \geq 0$, put $w_n = \min(u_n, v_n)$.

1) Suppose that $\ell = \ell'$. Show that the sequence $(w_n)_{n\geq 0}$ converges to ℓ.

2) Suppose that $\ell < \ell'$.

(a) Show that there exists $n_0 \in \mathbb{N}$ such that $u_n < v_n$ for all $n \geq n_0$.

(b) Deduce that the sequence $(w_n)_{n\geq 0}$ converges to ℓ.

3) Deduce that the sequence $(w_n)_{n\geq 0}$ is convergent, and determine its limit.

Solution

1) Suppose that $\ell = \ell'$. Let $\varepsilon > 0$, since $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} v_n = \ell$, then there exist $n_1, n_2 \in \mathbb{N}$ such that

$$|u_n - \ell| < \varepsilon,\ \forall\, n \geq n_1 \quad \text{and} \quad |v_n - \ell| < \varepsilon,\ \forall\, n \geq n_2.$$

Put $n_0 = \max(n_1, n_2)$. Let $n \geq n_0$, then $n \geq n_1$ and $n \geq n_2$.

- If $u_n \leq v_n$, then $w_n = u_n$, so $|w_n - \ell| = |u_n - \ell| < \varepsilon$.
- If $v_n \leq u_n$, then $w_n = v_n$, so $|w_n - \ell| = |v_n - \ell| < \varepsilon$.

Hence the sequence $(w_n)_{n\geq 0}$ converges to ℓ.

2) Suppose that $\ell < \ell'$.

(a) Let $\varepsilon = \frac{\ell'-\ell}{2} > 0$. Since $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} v_n = \ell$, then there exist $n_1, n_2 \in \mathbb{N}$ such that

$$(\forall\, n \geq n_1)(\ell - \varepsilon < u_n < \ell + \varepsilon) \quad \text{and} \quad (\forall\, n \geq n_2)(\ell' - \varepsilon < v_n < \ell' + \varepsilon).$$

Put $n_0 = \max(n_1, n_2)$. If $n \geq n_0$, then $n \geq n_1$ and $n \geq n_2$, so

$$u_n < \ell + \varepsilon = \frac{\ell' + \ell}{2} = \ell' - \varepsilon < v_n.$$

(b) Since $u_n < v_n$ for all $n \geq n_0$, then $w_n = u_n$ for all $n \geq n_0$. So $\lim\limits_{n\to+\infty} w_n = \lim\limits_{n\to+\infty} u_n = \ell$.

3)
- If $\ell = \ell'$, then $\lim\limits_{n\to+\infty} w_n = \ell = \min(\ell, \ell')$.
- If $\ell < \ell'$, then $\lim\limits_{n\to+\infty} w_n = \ell = \min(\ell, \ell')$.
- If $\ell' < \ell$, then $\lim\limits_{n\to+\infty} w_n = \ell' = \min(\ell, \ell')$.

Hence, in the three cases, the sequence $(w_n)_{n\geq 0}$ is convergent to $\min(\ell, \ell')$. □

Exercise 67 (Cesaro's theorem: method of arithmetic mean)
Let $(u_n)_{n\geq 1}$ be a real sequence. For every $n \in \mathbb{N}^*$, put:

$$v_n = \frac{u_1 + \cdots + u_n}{n}.$$

1) Show that if $\lim_{n\to+\infty} u_n = 0$, then $\lim_{n\to+\infty} v_n = 0$.

2) Is the converse of the part (1) always true ? Justify your answer.

3) Show that if $\lim_{n\to+\infty} u_n = \ell \in \mathbb{R}$, then $\lim_{n\to+\infty} v_n = \ell$.

Solution

1) Let $\varepsilon > 0$. Since $\lim_{n\to+\infty} u_n = 0$, then there exists $n_1 \in \mathbb{N}^*$ such that $|u_n| < \frac{\varepsilon}{2}$ for all $n \geq n_1$. So, for $n \geq n_1$,

$$\begin{aligned} |v_n| &\leq \frac{|u_1| + \cdots + |u_{n_0-1}| + |u_{n_0}| + \cdots + |u_n|}{n} \\ &< \frac{|u_1| + \cdots + |u_{n_0-1}| + (n - n_0 + 1)\frac{\varepsilon}{2}}{n}. \end{aligned}$$

Put

$$w_n = \frac{|u_1| + \cdots + |u_{n_0-1}| + (n - n_0 + 1)\frac{\varepsilon}{2}}{n}.$$

Since $\lim_{n\to+\infty} w_n = \dfrac{\varepsilon}{2}$, then there exists $n_2 \in \mathbb{N}^*$ such that $|w_n - \frac{\varepsilon}{2}| < \frac{\varepsilon}{2}$ for all $n \geq n_2$, so $w_n < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $n \geq n_2$. Put $n_0 = \max(n_1, n_2) \in \mathbb{N}$. If $n \geq n_0$, then $n \geq n_1$ and $n \geq n_2$, so $|v_n| < w_n < \varepsilon$. Thus $\lim_{n\to+\infty} v_n = 0$.

2) The converse of the part (1) is not necessary true. Indeed, if we take $u_n = (-1)^n$ for all $n \in \mathbb{N}^*$, then $v_{2n} = 0$ and $v_{2n+1} = -\frac{1}{2n+1}$ for all $n \in \mathbb{N}^*$. Since $\lim_{n\to+\infty} v_{2n} = \lim_{n\to+\infty} v_{2n+1} = 0$, then $\lim_{n\to+\infty} v_n = 0$. On the other hand, the sequence $(u_n)_{n\geq 1}$ has no limit.

3) For every $n \in \mathbb{N}^*$, put $u'_n = u_n - \ell$. Then $\lim_{n\to+\infty} u'_n = 0$. For any $n \in \mathbb{N}^*$,

$$v_n - \ell = \frac{(u_1 - \ell) + \cdots + (u_n - \ell)}{n} = \frac{u'_1 + \cdots + u'_n}{n}.$$

By the part (1), $\lim_{n\to+\infty} (v_n - \ell) = 0$, so $\lim_{n\to+\infty} v_n = \ell$. $\square$

Chapter 3

Continuity of a function

3.1 Introduction

Definition 3.1.1 *Let E and F be two sets. A function f from E to F is every correspondence from E to F, such that to each element x of E associates* **at most** *an element y of F, denoted by $y = f(x)$.*

- *The element y (if it exists) is said to be the image of x by f, and x is said to be* **a** *preimage of y by f.*
- *The set elements of E having image by f is called the set of definition (or the domain of definition) of the function f, and it is denoted by D_f.*
- *The subset*
$$G_f = \{(x, y) \in E \times F,\ (x \in D_f) \wedge (y = f(x))\}$$
of the cartesian product $E \times F$ is called the graph (or the curve) of f.
- *If X is a subset of D_f, then the set*
$$\begin{aligned} f(X) &= \Big\{f(x),\ \text{such that } x \in X\Big\} \\ &= \Big\{y \in \mathbb{R},\ \text{such that there exists } x \in X \text{ satisfying } y = f(x)\Big\} \end{aligned}$$
is called the image of X by f.
- *If f is a function from E to F, we adopt the notation:*
$$\begin{array}{rccc} f: & D_f & \longmapsto & F \\ & x & \longmapsto & f(x). \end{array}$$
- *If f is a function from E to $\mathbb{R}$, we say that f is a numerical function (or a real-valued function). We will denote by $\mathcal{F}(E, \mathbb{R})$, the set of numerical functions from E to $\mathbb{R}$. If, in addition, $E \subseteq \mathbb{R}$, we say that f is a numerical function of a real variable.*

Example 3.1.1

1) *Let $f : \mathbb{R} \longmapsto \mathbb{R}$ be the function defined by $f(x) = x^2$. The domain of definition of f is $D_f = \mathbb{R}$. The image of $\mathbb{R}$ by f is $f(\mathbb{R}) = [0, +\infty[$.*

2) *Let* $f : \mathbb{R} \longmapsto \mathbb{R}$ *be the function defined by* $f(x) = \dfrac{\sqrt{x}}{x(x^2-1)}$*. The domain of definition of* f *is* $D_f =]0,1[\cup]1,+\infty[$.

3.1.1 Parity and symmetry of a function

Definition 3.1.2 *We say that a subset* E *of* $\mathbb{R}$ *is symmetric with respect to* $\mathbf{0}$ *(or centered at* $\mathbf{0}$*) if* $-x \in E$ *for all* $x \in E$.

Example 3.1.2

1) *The sets* $\mathbb{R}$, $\mathbb{R}^*$, $[-a,a]$, $]-a,a[$, $]-\infty,-a] \cup [a,+\infty[$, $]-\infty,-a[\cup]a,+\infty[$ *(where* $a > 0$*) and* $[-2,-1] \cup [1,2]$ *are symmetric with respect to* $\mathbf{0}$.

2) *The sets* $\mathbb{R} - \{1\}$, $\mathbb{R}^+$, $\mathbb{R}^-$, $[-a,a[$ *and* $]-a,a]$ *(where* $a > 0$*),* $[a,+\infty[$, $]a,+\infty[$, $]-\infty,a]$, $]-\infty,a[$ *(where* $a \in \mathbb{R}$*) are not symmetric with respect to* $\mathbf{0}$.

Remark 3.1.1

1) *The complementary in* $\mathbb{R}$ *of a symmetric with respect to* $\mathbf{0}$ *set is symmetric with respect to* $\mathbf{0}$.

2) *The intersection and the union of two symmetric with respect to* $\mathbf{0}$ *sets are also symmetric with respect to* $\mathbf{0}$.

Definition 3.1.3 *Let* $f \in \mathcal{F}(\mathbb{R},\mathbb{R})$ *and* E *be a subset of* D_f*. We say that* f *is even (resp. odd) on* E *if the following two conditions are satisfied:*

i) E *is symmetric with respect to* $\mathbf{0}$.

ii) $f(-x) = f(x)$ *(resp.* $f(-x) = -f(x)$*) for all* $x \in E$.

Example 3.1.3

1) *Every constant function is even on* $\mathbb{R}$.

2) *For* $n \in \mathbb{N}^*$*, the* n*-th power function* f *defined by* $f(x) = x^n$ *is even on* $\mathbb{R}$ *if* n *is even and odd on* $\mathbb{R}$ *if* n *is odd. In general, every polynomial function having only even (resp. odd) powers is an even (resp. odd) function.*

3) *The function* f *defined by* $f(x) = x + 1$ *is not even, neither odd on* $\mathbb{R}$.

4) *The function* $\cos$ *(resp.* $\sin$*) is even on* $\mathbb{R}$ *(resp. odd on* $\mathbb{R}$*).*

5) *The function* $\tan$ *is odd on* $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$.

Remark 3.1.2 *Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ and E be a subset of D_f symmetric with respect to 0.*

1) *If f is odd on E and $0 \in E$, then $f(0) = 0$. Indeed, $f(-0) = -f(0)$, so $2f(0) = 0$, and therefore $f(0) = 0$.*

2) *If f is even and odd on E, then, for all $x \in E$, $f(-x) = f(x)$ and $f(-x) = -f(x)$, so $f(x) = -f(x)$, and therefore $2f(x) = 0$, hence $f(x) = 0$. In other words, the only function which is even and odd on E is the zero function on E.*

3) *Let C_f be the representative curve of f on E in an orthonormal system (xoy).*
   - *f is even on E if and only if C_f is symmetric with respect to the y-axis.*
   - *f is odd on E if and only if C_f is symmetric with respect to the origin.*

4) *Let $g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ such that $E \subseteq D_g$.*
   - *If f and g are even (resp. odd) on E, then $af + bg$ is an even (resp. odd) function on E for all $a, b \in \mathbb{R}$.*
   - *If f and g are even (resp. odd) on E, then their product fg is an even function on E. If one of these two functions f and g is even and the other is odd, then fg is odd.*

5) *Let $g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ and F be a subset of D_g symmetric with respect to 0. Suppose that $f(E) \subseteq F$ in a way that the composite $g \circ f$ is well-defined on E.*
   - *If f is even on E, then $g \circ f$ is even on E.*
   - *If f is odd on E and g is even on F, then $g \circ f$ is even on E.*
   - *If f is odd on E and g is odd on F, then $g \circ f$ is odd on E.*

Exercise 68
Prove that every even (resp. odd) polynomial function on $\mathbb{R}$ has only even (resp. odd) powers.

Solution Let P be an even polynomial function on $\mathbb{R}$, let $Q(x)$ be the polynomial obtained by taking only even powers from $P(x)$. Then the function $P - Q$ is even (by the remark 3.1.2). But the polynomial function $P - Q$ is also odd on $\mathbb{R}$ since it has only odd powers, then $P - Q = 0$. (by the remark 3.1.2). Hence $P = Q$, i.e., P has only even powers. We make a similar proof in the case of an odd polynomial function on $\mathbb{R}$. □

Exercise 69
Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ and E be a symmetric with respect to 0 subset of D_f. Consider the two functions $f_e, f_o \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ defined by:

$$f_e(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x) - f(-x)}{2}.$$

1) **Verify that $f = f_e + f_o$ on E.**

2) Show that f_e is even on E and f_o is odd on E.

3) Prove that f can be written in a unique way as sum of an even function and an odd function on E. The function f_e (resp. f_o) is called the even (resp. odd) part of f on E.

Solution

1) For any $x \in E$,

$$f_e(x) + f_o(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = \frac{2f(x)}{2} = f(x).$$

So $f = f_e + f_o$ on E.

2) For any $x \in E$,

$$f_e(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(-x) + f(x)}{2} = f_e(x)$$

and

$$f_o(-x) = \frac{f(-x) - f(-(-x))}{2} = \frac{f(-x) - f(x)}{2} = -f_o(x).$$

So f_e is even on E and f_o is odd on E.

3) Suppose that there exist two functions $g, h \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ such that $f = g + h$ on E with g is even on E and h is odd on E. For any $x \in E$,

$$\begin{aligned} f_e(x) &= \frac{f(x) + f(-x)}{2} = \frac{g(x) + h(x) + g(-x) + h(-x)}{2} \\ &= \frac{g(x) + h(x) + g(x) - h(x)}{2} \quad \text{(since } g \text{ is even and } h \text{ est odd on } E\text{)} \\ &= g(x). \end{aligned}$$

$$\begin{aligned} f_o(x) &= \frac{f(x) - f(-x)}{2} = \frac{g(x) + h(x) - g(-x) - h(-x)}{2} \\ &= \frac{g(x) + h(x) - g(x) + h(x)}{2} \quad \text{(since } g \text{ is even and } h \text{ est odd on } E\text{)} \\ &= h(x). \end{aligned}$$

So $g = f_e$ and $h = f_o$. Hence f can be written in a unique way as sum of an even function (which is f_e) and an odd function (which is f_o) on E. □

Proposition 3.1.1 *Let $a, b \in \mathbb{R}$ and $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$. Let C_f be the representative curve of f on D_f in an orthonormal system (xoy).*

1) The vertical line of equation $x = a$ is an axis of symmetry of C_f if and only if, for any nonzero real number x such that $a + x \in D_f$, we have:

$$a - x \in D_f \quad \textit{and} \quad f(a - x) = f(a + x).$$

This is equivalent to the fact that, for any $x \in D_f$, we have:

$$2a - x \in D_f \quad \textit{and} \quad f(2a - x) = f(x).$$

2) *The point* $\boldsymbol{A(a,b)}$ *is a center of symmetry of* $\boldsymbol{C_f}$ *if and only if, for any nonzero real number* $\boldsymbol{x}$ *such that* $\boldsymbol{a+x \in D_f}$*, we have:*

$$\boldsymbol{a - x \in D_f} \quad \text{and} \quad \boldsymbol{f(a-x) + f(a+x) = 2b.}$$

Example 3.1.4

1) *Let* $\boldsymbol{f(x) = x^2 + 4x + 1 = (x+2)^2 - 3}$. *Since* $\boldsymbol{D_f} = \mathbb{R}$*, then, for any nonzero real number* $\boldsymbol{x}$*, if* $\boldsymbol{-2+x \in D_f}$*, then* $\boldsymbol{-2-x \in D_f}$ *and*

$$\boldsymbol{f(-2-x) = x^2 - 3 = f(-2+x).}$$

So the vertical line of equation $\boldsymbol{x = -2}$ *is an axis of symmetry of the curve* $\boldsymbol{C_f}$ *of* $\boldsymbol{f}$ *on* $\mathbb{R}$.

2) *Let* $\boldsymbol{f(x) = \dfrac{x-4}{x-2}}$. *Since* $\boldsymbol{D_f} = \mathbb{R} - \boldsymbol{\{2\}}$*, then, for any nonzero real number* $\boldsymbol{x}$*, if* $\boldsymbol{2+x \in D_f}$*, then* $\boldsymbol{2-x \in D_f}$ *and*

$$\boldsymbol{f(2-x) + f(2+x) = \frac{-x-2}{-x} + \frac{x-2}{x} = 2 = 2 \times 1.}$$

So the point $\boldsymbol{A(2,1)}$ *is a center of symmetry of the curve* $\boldsymbol{C_f}$ *of* $\boldsymbol{f}$ *on* $\boldsymbol{D_f}$.

3.1.2 Periodic functions

Definition 3.1.4 *Let* $\boldsymbol{f} \in \boldsymbol{\mathcal{F}}(\mathbb{R},\mathbb{R})$ *and* $\boldsymbol{E}$ *be a subset of* $\boldsymbol{D_f}$.

1) *We say that* $\boldsymbol{f}$ *is periodic on* $\boldsymbol{E}$ *if there exists a nonzero real number* $\boldsymbol{T}$ *such that the following two conditions are satisfied:*

   i) *For any* $\boldsymbol{x \in D_f}$*, we have the equivalence:*

   $$\boldsymbol{x \in E \Leftrightarrow x + T \in E}$$

   ii) $\boldsymbol{f(x+T) = f(x)}$ *for all* $\boldsymbol{x \in E}$.

   In this case, $\boldsymbol{f}$ *is said to be* $\boldsymbol{T}$*-periodic on* $\boldsymbol{E}$ *and* $\boldsymbol{T}$ *is said to be a period of* $\boldsymbol{f}$ *on* $\boldsymbol{E}$.

2) *The smallest strictly positive real number* $\boldsymbol{T_0}$ *such that* $\boldsymbol{f}$ *is* $\boldsymbol{T_0}$*-periodic on* $\boldsymbol{E}$*,* **if it exists***, is called the fundamental period of* $\boldsymbol{f}$ *on* $\boldsymbol{E}$ *(or simply* **the period** *of* $\boldsymbol{f}$ *on* $\boldsymbol{E}$*).*

Remark 3.1.3

1) *If* $\boldsymbol{f}$ *is* $\boldsymbol{T}$*-periodic on* $\boldsymbol{E}$*, then* $\boldsymbol{f(x+nT) = f(x)}$ *for all* $\boldsymbol{n} \in \mathbb{Z}$ *and* $\boldsymbol{x \in E}$. *Hence, if* $\boldsymbol{T}$ *is a period of* $\boldsymbol{f}$ *on* $\boldsymbol{E}$*, then* $\boldsymbol{nT}$ *is also a period of* $\boldsymbol{f}$ *on* $\boldsymbol{E}$ *for all* $\boldsymbol{n} \in \mathbb{Z} - \boldsymbol{\{0\}}$.

2) *If* $\boldsymbol{T}$ *and* $\boldsymbol{T'}$ *are two periods of* $\boldsymbol{f}$ *on* $\boldsymbol{E}$*, then* $\boldsymbol{T + T'}$ *is also a period of* $\boldsymbol{f}$ *on* $\boldsymbol{E}$.

Example 3.1.5

1) *Every constant function is periodic on* $\mathbb{R}$ *where every nonzero real number is a period.*

2) *The functions* **cos** *and* **sin** *are periodic on* $\mathbb{R}$ *with period* 2π.

3) *The function* **tan** *is periodic with period* π *on the set*

$$D = \mathbb{R} - \left\{(2k+1)\frac{\pi}{2},\ k \in \mathbb{Z}\right\}.$$

4) *Let* $f(x) = x - E(x)$. *Since* $D_f = \mathbb{R}$, *then* $x + 1 \in D_f$ *for all* $x \in D_f$. *Moreover, for all* $x \in D_f$,

$$f(x+1) = x + 1 - E(x+1) = x + 1 - E(x) - 1 = x - E(x) = f(x).$$

So f *is periodic on* D_f *with period* $T = 1$. *Hence, every nonzero integer is a period of* f *on* D_f.

3.1.3 Bounded functions

Definition 3.1.5 *Let* $f \in \mathcal{F}(\mathbb{R},\mathbb{R})$ *and* E *be a nonempty subset of* D_f.

1) *We say that* f *is bounded from above on* E *if there exists* $M \in \mathbb{R}$ *such that* $f(x) \leq M$ *for all* $x \in E$ *(a such* M *is said to be an upper bound of* f *on* E*).*

2) *We say that* f *is bounded from below on* E *if there exists* $N \in \mathbb{R}$ *such that* $f(x) \geq N$ *for all* $x \in E$ *(a such* N *is said to be a lower bound of* f *on* E*)..*

3) *We say that* f *is bounded on* E *if it is bounded from above and bounded from below on* E. *In other words,* f *is bounded on* E *if and only if there exists* $M > 0$ *such that* $|f(x)| \leq M$ *for all* $x \in E$.

Remark 3.1.4 *Let* $f \in \mathcal{F}(\mathbb{R},\mathbb{R})$ *and* E *be a nonempty subset of* D_f. *Then* f *is bounded from above (resp. bounded from below, bounded) on* E *if and only if the set* $f(E)$ *is bounded from above (resp. bounded from below, bounded) in* $\mathbb{R}$. *Hence, if* f *is bounded on* E, *then* $f(E)$ *is a nonempty bounded subset of* $\mathbb{R}$, *so* $f(E)$ *has supremum* M *and infimum* m *in* $\mathbb{R}$. *We write*

$$m = \inf f(E) = \inf_{x \in E} f(x) \quad \text{and} \quad M = \sup f(E) = \sup_{x \in E} f(x).$$

3.1.4 Monotone functions

Definition 3.1.6 *Let* $f \in \mathcal{F}(\mathbb{R},\mathbb{R})$ *and* I *be a* **non trivial** *interval (i.e., nonempty and not reduced to a single point) of* $\mathbb{R}$ *contained in* D_f.

1) *We say that* f *is increasing (resp. decreasing) on* I *if, for all* $a, b \in I$ *such that* $a < b$, *we have:*

$$f(a) \leq f(b) \quad (\textit{resp. } f(a) \geq f(b)).$$

2) *We say that f is strictly increasing (resp. strictly decreasing) on I if, for all $a, b \in I$ such that $a < b$, we have:*

$$f(a) < f(b) \quad (\text{resp. } f(a) > f(b)).$$

3) *We say that f is monotone (resp. strictly monotone) on I if it is increasing or decreasing (resp. strictly increasing or strictly decreasing) on I.*

Example 3.1.6

1) *The function f defined by $f(x) = \sqrt{x}$ is strictly increasing on $\mathbb{R}^+$ by the exercise 2.*

2) *The function f defined by $f(x) = \frac{1}{x}$ is strictly decreasing on $]0, +\infty[$ and on $]-\infty, 0[$ by the exercise 2. On the other hand, f is not decreasing on $\mathbb{R}$ since, for example, $-2 < 1$ and $f(-2) = -\frac{1}{2} < 1 = f(1)$. Hence the monotonicity of a function on an interval does not depend only on its expression but also on the nature of the interval.*

3) *The integer part function E is increasing (but not strictly increasing) on $\mathbb{R}$. It is constant on each interval $]k, k+1[$ with $k \in \mathbb{Z}$ ($E(x) = k$ for all $x \in]k, k+1[$).*

4) *Let $n \in \mathbb{N}^*$ and f be the function defined by $f(x) = x^n$. By the exercise 33,*
   - *If n is even, then f is strictly decreasing on $\mathbb{R}^-$ and strictly increasing on $\mathbb{R}^+$.*
   - *If n is odd, then f is strictly increasing on $\mathbb{R}$.*

Remark 3.1.5

1) *A function f is increasing (resp. strictly increasing) if and only if the function $-f$ is decreasing (resp. strictly decreasing).*

2) *The composite of two increasing (resp. decreasing) functions is an increasing function. More precisely, let I and J be two non trivial intervals of $\mathbb{R}$. If $f : I \longmapsto J$ is an increasing (resp. decreasing) function on I and $g : J \longmapsto \mathbb{R}$ is an increasing (resp. decreasing) function on J, then $g \circ f : I \longmapsto \mathbb{R}$ is an increasing function on I.*

3) *The composite of an increasing function and of a decreasing function is a decreasing function. More precisely, let I and J be two non trivial intervals of $\mathbb{R}$. If $f : I \longmapsto J$ is an increasing function on I and $g : J \longmapsto \mathbb{R}$ is a decreasing function on J, then $g \circ f : I \longmapsto \mathbb{R}$ is a decreasing function on I.*

4) *The sum of two increasing (resp. decreasing) functions is an increasing (resp. decreasing) function.*

5) *The product of two increasing (resp. strictly increasing) functions with positive values is an increasing (resp. strictly increasing) function.*

6) *If f is an increasing (resp. decreasing) function on a non trivial interval I of $\mathbb{R}$ with values in $]0, +\infty[$ (or in $]-\infty, 0[$), then the function $\frac{1}{f}$ is decreasing (resp. increasing) on I since it is the composite of the decreasing function $x \mapsto \frac{1}{x}$ and f. In general, the quotient of an increasing function on I with positive values by a decreasing function on I with values in $]0, +\infty[$ is an increasing function on I.*

7) f is constant on a non trivial interval I of $\mathbb{R}$ if and only if f is both increasing and decreasing on I.

Exercise 70
Prove that the function f defined by $f(x) = \dfrac{2x+1}{x-1}$ is strictly decreasing on the interval $I = [2, +\infty[$.

Solution Let $a, b \in I$ such that $a < b$. We have:

$$f(b) - f(a) = \frac{2b+1}{b-1} - \frac{2a+1}{a-1} = \frac{3(a-b)}{(a-1)(b-1)} < 0$$

since $a - b < 0$ and $(a-1)(b-1) > 0$ ($a > 1$ and $b > 1$), so $f(a) > f(b)$. Hence f is strictly decreasing on I. □

Remark 3.1.6 *Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ and I be a non trivial interval of $\mathbb{R}$ contained in D_f.*

1) f is increasing (resp. decreasing) on I if and only if, for all $a, b \in I$ such that $a \neq b$, we have:

$$\frac{f(a)-f(b)}{a-b} \geq 0 \quad \left(\text{resp. } \frac{f(a)-f(b)}{a-b} \leq 0\right).$$

2) f is strictly increasing (resp. strictly decreasing) on I if and only if, for all $a, b \in I$ such that $a \neq b$, we have:

$$\frac{f(a)-f(b)}{a-b} > 0 \quad \left(\text{resp. } \frac{f(a)-f(b)}{a-b} < 0\right).$$

Proposition 3.1.2 *If f is a strictly monotone function on a non trivial interval I of $\mathbb{R}$, then f is an injection from I into $\mathbb{R}$.*

Proof Suppose that f is strictly increasing on I (we make a similar proof when f is strictly decreasing on I). Suppose that f is not injective, then there exist $x, x' \in I$ such that $x \neq x'$ and $f(x) = f(x')$.

- If $x < x'$, then $f(x) < f(x')$ (since f is strictly increasing on I), which is impossible.
- If $x' < x$, then $f(x') < f(x)$ (since f is strictly increasing on I), which is impossible.

Hence f is an injection from I into $\mathbb{R}$. □

3.2 Limit of a function

Let $f : \mathbb{R} \longmapsto \mathbb{R}$ be a real function. The aim of this section is to study the behavior of $f(x)$ in each of the following cases:

- When x approaches a point $x_0 \in \mathbb{R}$ (from the two sides, from the right only or from the left only) with points in D_f, distincts from x_0 (f is not necessary defined at x_0).
- When x approaches to $+\infty$ (resp. $-\infty$) with points in D_f.

3.2.1 Neighborhood

Definition 3.2.1 *Let V be a subset of $\mathbb{R}$.*

1) *We say that V is a neighborhood of a point $x_0 \in \mathbb{R}$ if there exists $r > 0$ such that*

$$]x_0 - r, x_0 + r[\subseteq V.$$

2) *We say that V is a left (resp. right) neighborhood of a point $x_0 \in \mathbb{R}$ if there exists $r > 0$ such that*

$$]x_0 - r, x_0] \subseteq V \qquad (\textit{resp. } [x_0, x_0 + r[\subseteq V).$$

3) *We say that V is a* **pointed** *neighborhood of a point $x_0 \in \mathbb{R}$ if $x_0 \notin V$ and if there exists $r > 0$ such that*

$$]x_0 - r, x_0[\cup]x_0, x_0 + r[\subseteq V.$$

4) *We say that V is a left (resp. right)* **pointed** *neighborhood of a point $x_0 \in \mathbb{R}$ if $x_0 \notin V$ and if there exists $r > 0$ such that*

$$]x_0 - r, x_0[\subseteq V \qquad (\textit{resp. }]x_0, x_0 + r[\subseteq V).$$

5) *We say that V is a neighborhood of $+\infty$ if there exists $r > 0$ such that*

$$]r, +\infty[\subseteq V.$$

6) *We say that V is a neighborhood of $-\infty$ if there exists $r > 0$ such that*

$$] - \infty, -r[\subseteq V.$$

Example 3.2.1

1) *$\mathbb{R}$ is a neighborhood for each of its points. Similarly, $\mathbb{R}$ is a neighborhood of $\pm\infty$.*

2) *$\mathbb{R}^*$ is a pointed neighborhood of 0.*

3) *Every interval I of the form $[a, b]$, $]a, b]$, $[a, b[$ or $]a, b[$ (where $a < b$) is a neighborhood of each point x_0 such that $a < x_0 < b$. On the other hand, I is not a neighborhood of a, neither of b, nor of $\pm\infty$.*

4) *Every interval I of the form $[a, +\infty[$ or $]a, +\infty[$ (resp. $] - \infty, b]$ or $] - \infty, b[$) is a neighborhood of each point x_0 such that $x_0 > a$ (resp. $x_0 < b$), I is also neighborhood of $+\infty$ (resp. of $-\infty$). On the other hand, I is not a neighborhood of a (resp. of b).*

5) *Every neighborhood of a point $x_0 \in \mathbb{R}$ is a left and right neighborhood of x_0.*

6) *Every interval of the form $[a, b]$ (where $a < b$) is a left neighborhood of b and a right neighborhood of a.*

Exercise 71
Show that the intersection of two neighborhoods of a point $x_0 \in \mathbb{R}$ (resp. of $\pm\infty$) is a neighborhood of x_0 (resp. of $\pm\infty$).

Solution

- Let V and W be two neighborhoods of a point $x_0 \in \mathbb{R}$, then there exist $r_1, r_2 > 0$ such that
$$]x_0 - r_1, x_0 + r_1[\ \subseteq V \quad \text{and} \quad]x_0 - r_2, x_0 + r_2[\ \subseteq W.$$
Put $r = \min(r_1, r_2) > 0$, then $r \leq r_1$ and $r \leq r_2$. So
$$]x_0 - r, x_0 + r[\ \subseteq\ ]x_0 - r_1, x_0 + r_1[\ \subseteq V$$
and
$$]x_0 - r, x_0 + r[\ \subseteq\ ]x_0 - r_2, x_0 + r_2[\ \subseteq W.$$
Hence $]x_0 - r, x_0 + r[\ \subseteq\ V \cap W$. Thus $V \cap W$ is a neighborhood of x_0.
- Let V and W be two neighborhoods of $+\infty$ (we make a similar proof for $-\infty$), then there exist $r_1, r_2 > 0$ such that
$$]r_1, +\infty[\ \subseteq V \quad \text{and} \quad]r_2, +\infty[\ \subseteq W.$$
Put $r = \max(r_1, r_2) > 0$, then $r \geq r_1$ and $r \geq r_2$. So
$$]r, +\infty[\ \subseteq\ ]r_1, +\infty[\ \subseteq V \quad \text{and} \quad]r, +\infty[\ \subseteq\ ]r_2, +\infty[\ \subseteq W.$$
Hence $]r, +\infty[\ \subseteq\ V \cap W$. Thus $V \cap W$ is a neighborhood of $+\infty$. □

In the sequel of this section, $f : \mathbb{R} \longmapsto \mathbb{R}$ is a real function defined on a pointed neighborhood of a point $x_0 \in \overline{\mathbb{R}}$ (f is not necessary defined at x_0 when $x_0 \in \mathbb{R}$).

3.2.2 Limit as x tends to x_0

Definition 3.2.2

1) *We say that $f(x)$ has limit $\ell \in \mathbb{R}$ as x tends to x_0 if*
$$(\forall\, \varepsilon > 0)(\exists\, \delta > 0)\Big(\forall\, x \in D_f,\ 0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon\Big).$$
This means that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $x \in D_f$, we have the following implication:
$$\Big(x \in\ ]x_0 - \delta, x_0[\ \cup\ ]x_0, x_0 + \delta[\Big) \Rightarrow f(x) \in\ ]\ell - \varepsilon, \ell + \varepsilon[.$$

2) We say that $f(x)$ has limit $+\infty$ as x tends to x_0 if

$$(\forall A > 0)(\exists \delta > 0)\Big(\forall x \in D_f,\ 0 < |x - x_0| < \delta \Rightarrow f(x) > A\Big).$$

This means that, for any $A > 0$, there exists $\delta > 0$ such that, for any $x \in D_f$, we have the following implication:

$$\Big(x \in\,]x_0 - \delta, x_0[\, \cup\,]x_0, x_0 + \delta[\Big) \Rightarrow f(x) > A.$$

We write $\lim\limits_{x \to x_0} f(x) = +\infty$ *or* $\lim\limits_{\substack{x \to x_0 \\ x \neq x_0}} f(x) = +\infty$.

3) We say that $f(x)$ has limit $-\infty$ as x tends to x_0 if

$$(\forall A > 0)(\exists \delta > 0)\Big(\forall x \in D_f,\ 0 < |x - x_0| < \delta \Rightarrow f(x) < -A\Big).$$

This means that, for any $A > 0$, there exists $\delta > 0$ such that, for any $x \in D_f$, we have the following implication:

$$\Big(x \in\,]x_0 - \delta, x_0[\, \cup\,]x_0, x_0 + \delta[\Big) \Rightarrow f(x) < -A.$$

We write $\lim\limits_{x \to x_0} f(x) = -\infty$ *or* $\lim\limits_{\substack{x \to x_0 \\ x \neq x_0}} f(x) = -\infty$.

Remark 3.2.1 *$f(x)$ has limit $\ell \in \overline{\mathbb{R}}$ as x tends to x_0 if and only if, for any neighborhood W of ℓ, there exists a pointed neighborhood V of x_0 contained in D_f such that $f(V) \subseteq W$.*

Proposition 3.2.1 *If $f(x)$ has limit $\ell \in \mathbb{R}$ as x tends to x_0, then ℓ is unique. This limit will be denoted by* $\lim\limits_{x \to x_0} f(x)$ *or* $\lim\limits_{\substack{x \to x_0 \\ x \neq x_0}} f(x)$ *or* $\lim\limits_{x_0} f$.

Proof Suppose that $f(x)$ has limits $\ell \in \mathbb{R}$ and $\ell' \in \mathbb{R}$ as x tends to x_0. Let's show that $\ell = \ell'$. Indeed, let $\varepsilon > 0$, then there exist $\delta_1, \delta_2 > 0$ such that, for any $x \in D_f$,

$$\left(0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - \ell| < \frac{\varepsilon}{2}\right) \quad \text{and} \quad \left(0 < |x - x_0| < \delta_2 \Rightarrow |f(x) - \ell'| < \frac{\varepsilon}{2}\right).$$

Put $\delta = \min(\delta_1, \delta_2)$. If $0 < |x - x_0| < \delta$, then $0 < |x - x_0| < \delta_1$ and $0 < |x - x_0| < \delta_2$, so $|f(x) - \ell| < \frac{\varepsilon}{2}$ and $|f(x) - \ell'| < \frac{\varepsilon}{2}$. Hence

$$|\ell - \ell'| = |\ell - f(x) + f(x) - \ell'| \leq |\ell - f(x)| + |f(x) - \ell'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $|\ell - \ell'| \leq \varepsilon$ for all $\varepsilon > 0$. So $\ell - \ell' = 0$ by the exercise 9. Thus $\ell = \ell'$. □

Example 3.2.2 *Let's show, by using the definition of the limit of a function, that, for all $x_0 \in \mathbb{R}$,*

$$a)\ \lim_{x \to x_0} |x| = |x_0| \qquad\qquad b)\ \lim_{x \to x_0} x^2 = x_0^2.$$

a) Let $\varepsilon > 0$*, let's find* $\delta > 0$ *such that, for any* $x \in \mathbb{R}$*,*

$$0 < |x - x_0| < \delta \Rightarrow \Big||x| - |x_0|\Big| < \varepsilon.$$

For any $x \in \mathbb{R}$*,* $\Big||x| - |x_0|\Big| \leq |x - x_0|$*. So it is sufficient to take* $\delta = \varepsilon$*. Hence* $\lim_{x \to x_0} |x| = |x_0|$*.*

b) Let $\varepsilon > 0$*, let's find* $\delta > 0$ *such that, for any* $x \in \mathbb{R}$*,*

$$0 < |x - x_0| < \delta \Rightarrow |x^2 - x_0^2| < \varepsilon.$$

For any $x \in \mathbb{R}$*,* $|x^2 - x_0^2| \leq |x - x_0||x + x_0|$*. But*

$$|x + x_0| = |x - x_0 + 2x_0| \leq |x - x_0| + 2|x_0|.$$

We take

$$\delta = \min\left(\frac{\varepsilon}{1 + 2|x_0|}, 1\right) > 0.$$

If $0 < |x - x_0| < \delta$*, then* $|x + x_0| \leq \delta + 2|x_0| \leq 1 + 2|x_0|$*, and therefore*

$$|x^2 - x_0^2| \leq |x - x_0||x + x_0| < \delta(1 + 2|x_0|) \leq \frac{\varepsilon}{1 + 2|x_0|}(1 + 2|x_0|) = \varepsilon.$$

Hence $\lim_{x \to x_0} x^2 = x_0^2$*.*

Exercise 72
Show, by using the definition of the limit of a function, that

$$\text{a)}\ \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4 \qquad\qquad \text{b)}\ \lim_{x \to 1} \frac{2x - 1}{x + 2} = \frac{1}{3}.$$

Solution

a) Let $\varepsilon > 0$, let's find $\delta > 0$ such that, for any $x \in \mathbb{R}$,

$$0 < |x - 2| < \delta \Rightarrow \left|\frac{x^2 - 4}{x - 2} - 4\right| < \varepsilon.$$

But $\left|\frac{x^2-4}{x-2} - 4\right| = |x - 2|$, then it is sufficient to take $\delta = \varepsilon$.

b) Let $\varepsilon > 0$, let's find $\delta > 0$ such that, for any $x \in \mathbb{R} \setminus \{-2\}$,

$$0 < |x - 1| < \delta \Rightarrow \left|\frac{2x - 1}{x + 2} - \frac{1}{3}\right| < \varepsilon.$$

<u>1^{st} method</u>: If $x > 0$, then $3(x + 2) > 6 > 1$, so

$$\left|\frac{2x - 1}{x + 2} - \frac{1}{3}\right| = \left|\frac{5(x - 1)}{3(x + 2)}\right| \leq 5|x - 1|.$$

But $5|x-1| < \varepsilon$ if and only if $|x-1| < \frac{\varepsilon}{5}$. We take $\delta = \min\left(\frac{1}{2}, \frac{\varepsilon}{5}\right) > 0$. If $0 < |x-1| < \delta$, then $0 < |x-1| < \frac{1}{2}$, so $\frac{1}{2} < x < \frac{3}{2}$, in particular $x > 0$, and as $|x-1| < \frac{\varepsilon}{5}$, then

$$\left|\frac{2x-1}{x+2} - \frac{1}{3}\right| \leq 5|x-1| < \varepsilon.$$

2nd method: If $|x-1| < \delta$, then $1-\delta < x < 1+\delta$, so $3-\delta < x+2 < 3+\delta$. If $\delta < 3$, then $3-\delta > 0$, so

$$\left|\frac{2x-1}{x+2} - \frac{1}{3}\right| < \frac{5\delta}{3(3-\delta)}.$$

But $\dfrac{5\delta}{3(3-\delta)} = \varepsilon$ if and only if $\delta = \dfrac{9\varepsilon}{3\varepsilon+5}$. It is sufficient to take $\delta = \dfrac{9\varepsilon}{3\varepsilon+5}$ since

$$3-\delta = \frac{15}{3\varepsilon+5} > 0. \quad \square$$

Exercise 73

By using the fact that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$, show, by using the definition of the limit of a function, that, for all $x_0 \in \mathbb{R}$,

1) $\lim\limits_{x \to x_0} \sin x = \sin x_0$.

2) $\lim\limits_{x \to x_0} \cos x = \cos x_0$.

Solution

1) Let $\varepsilon > 0$, let's find $\delta > 0$ such that, for any $x \in \mathbb{R}$,

$$0 < |x - x_0| < \delta \Rightarrow |\sin x - \sin x_0| < \varepsilon.$$

For any $x \in \mathbb{R}$,

$$\begin{aligned} |\sin x - \sin x_0| &= \left|2\sin\left(\frac{x-x_0}{2}\right)\cos\left(\frac{x+x_0}{2}\right)\right| \\ &\leq 2\left|\sin\left(\frac{x-x_0}{2}\right)\right| \leq 2\left|\frac{x-x_0}{2}\right| = |x-x_0|. \end{aligned}$$

So it is sufficient to take $\delta = \varepsilon$. Hence $\lim\limits_{x \to x_0} \cos x = \cos x_0$.

2) Let $\varepsilon > 0$, let's find $\delta > 0$ such that, for any $x \in \mathbb{R}$,

$$0 < |x - x_0| < \delta \Rightarrow |\cos x - \cos x_0| < \varepsilon.$$

For any $x \in \mathbb{R}$,

$$\begin{aligned} |\cos x - \cos x_0| &= \left|-2\sin\left(\frac{x-x_0}{2}\right)\sin\left(\frac{x+x_0}{2}\right)\right| \\ &\leq 2\left|\sin\left(\frac{x-x_0}{2}\right)\right| \leq 2\left|\frac{x-x_0}{2}\right| = |x-x_0|. \end{aligned}$$

So it is sufficient to take $\delta = \varepsilon$. Hence $\lim\limits_{x \to x_0} \cos x = \cos x_0$. $\square$

Proposition 3.2.2 *If* $\lim\limits_{x \to x_0} f(x) = \ell \in \mathbb{R}$*, then* f *is bounded on a pointed neighborhood of* x_0*.*

Proof For $\varepsilon = 1$, there exists $\delta > 0$ such that $f(x) \in]\ell - 1, \ell + 1[$ for all $x \in V =]x_0 - \delta, x_0[\cup]x_0, x_0 + \delta[$. So f is bounded on the pointed neighborhood V of x_0. □

3.2.3 Limit from the left and from the right as x tends to x_0

Definition 3.2.3

1) *We say that* $f(x)$ *has limit* $\ell \in \mathbb{R}$ *as* x *tends to* x_0^+ *(i.e., from the right of* x_0*) if*

$$(\forall\, \varepsilon > 0)(\exists\, \delta > 0)\Big(\forall\, x \in D_f,\ 0 < x - x_0 < \delta \Rightarrow |f(x) - \ell| < \varepsilon\Big).$$

This means that, for any $\varepsilon > 0$*, there exists* $\delta > 0$ *such that, for any* $x \in D_f$*, we have the following implication:*

$$x \in]x_0, x_0 + \delta[\Rightarrow f(x) \in]\ell - \varepsilon, \ell + \varepsilon[.$$

We write $\lim\limits_{x \to x_0^+} f(x) = \ell$ *or* $\lim\limits_{\substack{x \to x_0 \\ x > x_0}} f(x) = \ell$ *or* $f(x_0^+) = \ell$*.*

2) *We say that* $f(x)$ *has limit* $+\infty$ *as* x *tends to* x_0^+ *(i.e., from the right of* x_0*) if*

$$(\forall\, A > 0)(\exists\, \delta > 0)\Big(\forall\, x \in D_f,\ 0 < x - x_0 < \delta \Rightarrow f(x) > A\Big).$$

This means that, for any $A > 0$*, there exists* $\delta > 0$ *such that, for any* $x \in D_f$*, we have the following implication:*

$$x \in]x_0, x_0 + \delta[\Rightarrow f(x) > A.$$

We write $\lim\limits_{x \to x_0^+} f(x) = +\infty$ *or* $\lim\limits_{\substack{x \to x_0 \\ x > x_0}} f(x) = +\infty$*.*

3) *We say that* $f(x)$ *has limit* $-\infty$ *as* x *tends to* x_0^+ *(i.e., from the right of* x_0*) if*

$$(\forall\, A > 0)(\exists\, \delta > 0)\Big(\forall\, x \in D_f,\ 0 < x - x_0 < \delta \Rightarrow f(x) < -A\Big).$$

This means that, for any $A > 0$*, there exists* $\delta > 0$ *such that, for any* $x \in D_f$*, we have the following implication:*

$$x \in]x_0, x_0 + \delta[\Rightarrow f(x) < -A.$$

We write $\lim\limits_{x \to x_0^+} f(x) = -\infty$ *or* $\lim\limits_{\substack{x \to x_0 \\ x > x_0}} f(x) = -\infty$*.*

Definition 3.2.4

1) *We say that* $f(x)$ *has limit* $\ell \in \mathbb{R}$ *as* x *tends to* x_0^- *(i.e., from the left of* x_0*) if*

$$(\forall\, \varepsilon > 0)(\exists\, \delta > 0)\Big(\forall\, x \in D_f,\ 0 < x_0 - x < \delta \Rightarrow |f(x) - \ell| < \varepsilon\Big).$$

This means that, for any $\varepsilon > 0$*, there exists* $\delta > 0$ *such that, for any* $x \in D_f$*, we have the following implication:*

$$x \in \,]x_0 - \delta, x_0[\ \Rightarrow f(x) \in \,]\ell - \varepsilon, \ell + \varepsilon[.$$

We write $\lim\limits_{x \to x_0^-} f(x) = \ell$ *or* $\lim\limits_{\substack{x \to x_0 \\ x < x_0}} f(x) = \ell$ *or* $f(x_0^-) = \ell$*.*

2) *We say that* $f(x)$ *has limit* $+\infty$ *as* x *tends to* x_0^- *(i.e., from the left of* x_0*) if*

$$(\forall\, A > 0)(\exists\, \delta > 0)\Big(\forall\, x \in D_f,\ 0 < x_0 - x < \delta \Rightarrow f(x) > A\Big).$$

This means that, for any $A > 0$*, there exists* $\delta > 0$ *such that, for any* $x \in D_f$*, we have the following implication:*

$$x \in \,]x_0 - \delta, x_0[\ \Rightarrow f(x) > A.$$

We write $\lim\limits_{x \to x_0^-} f(x) = +\infty$ *or* $\lim\limits_{\substack{x \to x_0 \\ x < x_0}} f(x) = +\infty$*.*

3) *We say that* $f(x)$ *has limit* $-\infty$ *as* x *tends to* x_0^- *(i.e., from the left of* x_0*) if*

$$(\forall\, A > 0)(\exists\, \delta > 0)\Big(\forall\, x \in D_f,\ 0 < x_0 - x < \delta \Rightarrow f(x) < -A\Big).$$

This means that, for any $A > 0$*, there exists* $\delta > 0$ *such that, for any* $x \in D_f$*, we have the following implication:*

$$x \in \,]x_0 - \delta, x_0[\ \Rightarrow f(x) < -A.$$

We write $\lim\limits_{x \to x_0^-} f(x) = -\infty$ *or* $\lim\limits_{\substack{x \to x_0 \\ x < x_0}} f(x) = -\infty$*.*

Remark 3.2.2 $f(x)$ *has limit* $\ell \in \overline{\mathbb{R}}$ *as* x *tends to* x_0^+ *(resp.* x_0^-*) if and only if, for any neighborhood* W *of* ℓ*, there exists a right (resp. left) pointed neighborhood* V *of* x_0 *contained in* D_f *such that* $f(V) \subseteq W$*.*

Example 3.2.3 *By using the definition of the limit of a function, we will prove that:*

$$\lim_{x \to x_0^+} \frac{1}{x - x_0} = +\infty \qquad \text{and} \qquad \lim_{x \to x_0^-} \frac{1}{x - x_0} = -\infty.$$

Let $A > 0$*.*

- *For the first limit, we take* $\delta = \frac{1}{A} > 0$*. If* $0 < x - x_0 < \delta$*, then*

$$\frac{1}{x - x_0} > \frac{1}{\delta} = A.$$

- *For the second limit, we take* $\delta = \frac{1}{A} > 0$*. If* $0 < x_0 - x < \delta$*, then*

$$\frac{1}{x - x_0} = -\frac{1}{x_0 - x} < -\frac{1}{\delta} = -A.$$

Proposition 3.2.3 *(Relation between limit, right limit and left limit)*

1) *If* $\lim\limits_{x \to x_0} f(x) = \ell \in \mathbb{R}$ *then* $\lim\limits_{x \to x_0^+} f(x) = \ell$ *and* $\lim\limits_{x \to x_0^-} f(x) = \ell$*.*

2) *If* $\lim\limits_{x \to x_0^+} f(x) = \ell_1 \in \mathbb{R}$ *and* $\lim\limits_{x \to x_0^-} f(x) = \ell_2 \in \mathbb{R}$*, then we have the following two cases:*

   (a) *If* $\ell_1 \neq \ell_2$*, then* $\lim\limits_{x \to x_0} f(x)$ *does not exist.*

   (b) *If* $\ell_1 = \ell_2$*, then* $\lim\limits_{x \to x_0} f(x) = \ell_1 = \ell_2$*.*

Example 3.2.4

1) *Since*

$$\lim_{x \to 1^+} \frac{|x-1|}{x+1} = \lim_{\substack{x \to 1 \\ x > 1}} \frac{x-1}{x+1} = 0 \quad \textit{and} \quad \lim_{x \to 1^-} \frac{|x-1|}{x+1} = \lim_{\substack{x \to 1 \\ x < 1}} \frac{1-x}{x+1} = 0,$$

then $\lim\limits_{x \to 1} \dfrac{|x-1|}{x+1} = 0$*.*

2) *Let* $f : \mathbb{R} \setminus \{0\} \longmapsto \mathbb{R}$ *be the function defined by* $f(x) = \dfrac{|x|}{x}$*. Since*

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x}{x} = 1 \quad \textit{and} \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{-x}{x} = -1,$$

then f *has no limit as* x *tends to* 0*.*

3.2.4 Limit as x tends to $+\infty$ or $-\infty$

Definition 3.2.5

1) *We say that* $f(x)$ *has limit* $\ell \in \mathbb{R}$ *as* x *tends to* $+\infty$ *if*

$$(\forall\, \varepsilon > 0)(\exists\, \delta > 0)\Big(\forall\, x \in D_f,\ x > \delta \Rightarrow |f(x) - \ell| < \varepsilon\Big).$$

This means that, for any $\varepsilon > 0$*, there exists* $\delta > 0$ *such that, for any* $x \in D_f$*, we have the following implication:*

$$x \in \,]\delta, +\infty[\; \Rightarrow f(x) \in \,]\ell - \varepsilon, \ell + \varepsilon[.$$

We write $\lim\limits_{x \to +\infty} f(x) = \ell$*.*

2) We say that $\boldsymbol{f(x)}$ *has limit* $+\infty$ *as* $\boldsymbol{x}$ *tends to* $+\infty$ *if*

$$(\forall \boldsymbol{A} > \mathbf{0})(\exists\, \boldsymbol{\delta} > \mathbf{0})\Big(\forall\, \boldsymbol{x} \in \boldsymbol{D_f},\ \boldsymbol{x} > \boldsymbol{\delta} \Rightarrow \boldsymbol{f(x)} > \boldsymbol{A}\Big).$$

This means that, for any $\boldsymbol{A} > \mathbf{0}$*, there exists* $\boldsymbol{\delta} > \mathbf{0}$ *such that, for any* $\boldsymbol{x} \in \boldsymbol{D_f}$*, we have the following implication:*

$$\boldsymbol{x} \in \,]\boldsymbol{\delta}, +\infty[\; \Rightarrow \boldsymbol{f(x)} > \boldsymbol{A}.$$

We write $\lim\limits_{x \to +\infty} \boldsymbol{f(x)} = +\infty$.

3) We say that $\boldsymbol{f(x)}$ *has limit* $-\infty$ *as* $\boldsymbol{x}$ *tends to* $+\infty$ *if*

$$(\forall \boldsymbol{A} > \mathbf{0})(\exists\, \boldsymbol{\delta} > \mathbf{0})\Big(\forall\, \boldsymbol{x} \in \boldsymbol{D_f},\ \boldsymbol{x} > \boldsymbol{\delta} \Rightarrow \boldsymbol{f(x)} < -\boldsymbol{A}\Big).$$

This means that, for any $\boldsymbol{A} > \mathbf{0}$*, there exists* $\boldsymbol{\delta} > \mathbf{0}$ *such that, for any* $\boldsymbol{x} \in \boldsymbol{D_f}$*, we have the following implication:*

$$\boldsymbol{x} \in \,]\boldsymbol{\delta}, +\infty[\; \Rightarrow \boldsymbol{f(x)} < -\boldsymbol{A}.$$

We write $\lim\limits_{x \to +\infty} \boldsymbol{f(x)} = -\infty$.

Definition 3.2.6

1) We say that $\boldsymbol{f(x)}$ *has limit* $\boldsymbol{\ell} \in \mathbb{R}$ *as* $\boldsymbol{x}$ *tends to* $-\infty$ *if*

$$(\forall \varepsilon > \mathbf{0})(\exists\, \boldsymbol{\delta} > \mathbf{0})\Big(\forall\, \boldsymbol{x} \in \boldsymbol{D_f},\ \boldsymbol{x} < -\boldsymbol{\delta} \Rightarrow |\boldsymbol{f(x)} - \boldsymbol{\ell}| < \varepsilon\Big).$$

This means that, for any $\varepsilon > \mathbf{0}$*, there exists* $\boldsymbol{\delta} > \mathbf{0}$ *such that, for any* $\boldsymbol{x} \in \boldsymbol{D_f}$*, we have the following implication:*

$$\boldsymbol{x} \in \,]-\infty, -\boldsymbol{\delta}[\; \Rightarrow \boldsymbol{f(x)} \in \,]\boldsymbol{\ell} - \varepsilon, \boldsymbol{\ell} + \varepsilon[.$$

We write $\lim\limits_{x \to -\infty} \boldsymbol{f(x)} = \boldsymbol{\ell}$.

2) We say that $\boldsymbol{f(x)}$ *has limit* $+\infty$ *as* $\boldsymbol{x}$ *tends to* $-\infty$ *if*

$$(\forall \boldsymbol{A} > \mathbf{0})(\exists\, \boldsymbol{\delta} > \mathbf{0})\Big(\forall\, \boldsymbol{x} \in \boldsymbol{D_f},\ \boldsymbol{x} < -\boldsymbol{\delta} \Rightarrow \boldsymbol{f(x)} > \boldsymbol{A}\Big).$$

This means that, for any $\boldsymbol{A} > \mathbf{0}$*, there exists* $\boldsymbol{\delta} > \mathbf{0}$ *such that, for any* $\boldsymbol{x} \in \boldsymbol{D_f}$*, we have the following implication:*

$$\boldsymbol{x} \in \,]-\infty, -\boldsymbol{\delta}[\; \Rightarrow \boldsymbol{f(x)} > \boldsymbol{A}.$$

We write $\lim\limits_{x \to -\infty} \boldsymbol{f(x)} = +\infty$.

3) We say that $f(x)$ has limit $-\infty$ as x tends to $-\infty$ if

$$(\forall\, A > 0)(\exists\, \delta > 0)\Big(\forall\, x \in D_f,\ x < -\delta \Rightarrow f(x) < -A\Big).$$

This means that, for any $A > 0$, there exists $\delta > 0$ such that, for any $x \in D_f$, we have the following implication:

$$x \in]-\infty, -\delta[\Rightarrow f(x) < -A.$$

We write $\lim\limits_{x \to -\infty} f(x) = -\infty$.

Remark 3.2.3 *$f(x)$ has limit $\ell \in \overline{\mathbb{R}}$ as x tends to $+\infty$ (resp. $-\infty$) if and only if, for any neighborhood W of ℓ, there exists a neighborhood V of $+\infty$ (resp. $-\infty$) contained in D_f such that $f(V) \subseteq W$.*

Example 3.2.5 *Let's show, by using the definition of the limit of a function, that*

$$\lim_{x \to +\infty} \frac{2x^3 + 1}{x - 3} = +\infty.$$

Indeed, let $A > 0$, let's find $\delta > 0$ such that, for any $x \in \mathbb{R} \setminus \{3\}$,

$$x > \delta \Rightarrow \frac{2x^3 + 1}{x - 3} > A.$$

Let $x > 3$, then $x - 3 > 0$. As $0 < x - 3 < x$, then $\frac{1}{x-3} > \frac{1}{x} > 0$. Multiplying by $2x^3 + 1 > 0$, we obtain:

$$\frac{2x^3 + 1}{x - 3} > \frac{2x^3 + 1}{x} > \frac{2x^3}{x} = 2x^2$$

But $2x^2 > A$ if and only if $x > \sqrt{\frac{A}{2}}$ (since $x > 0$), then we take

$$\delta = \max\left(3, \sqrt{\frac{A}{2}}\right).$$

Hence, if $x > \delta$, then $x > 3$ and $x > \sqrt{\frac{A}{2}}$, so $\dfrac{2x^3 + 1}{x - 3} > 2x^2 > A$.

Exercise 74
Let's show, by using the definition of the limit of a function, that, for all $n \in \mathbb{N}^*$,

$$\lim_{x \to +\infty} x^n = +\infty.$$

Solution Let $n \in \mathbb{N}^*$ and $A > 0$. Let's find $\delta > 0$ such that, for any $x \in \mathbb{R}$,

$$x > \delta \Rightarrow x^n > A.$$

We know that if $x > 1$, then $x^n > x$ (by the exercise 33), so we take $\delta = \max(A, 1) > 0$. If $x > \delta$, then $x > A$ and $x > 1$, so $x^n > x > A$. Hence $\lim\limits_{x \to +\infty} x^n = +\infty$. □

3.2.5 Operations on the limits

Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$, $\ell, \ell', \alpha \in \mathbb{R}$ and $x_0 \in \mathbb{R}$. In the following table, on the first line we have operations on the functions f and g, and on the other lines, under each function, we have its limit as x tends to x_0. The symbol "IF" means "Indeterminate Form", to say that there is no fixed result in general. The table remains true if the limits as x tends to x_0 are replaced by limits as x tends to x_0^+ (resp. x_0^-, $+\infty$, or $-\infty$).

f	g	$f+g$	fg	αf	$\lvert f\rvert$	$\frac{f}{g}$
ℓ	ℓ'	$\ell+\ell'$	$\ell\ell'$	$\alpha\ell$	$\lvert\ell\rvert$	$\frac{\ell}{\ell'}$ if $\ell' \neq 0$
ℓ	$+\infty$	$+\infty$	$+\infty$ if $\ell > 0$ $-\infty$ if $\ell < 0$ IF if $\ell = 0$	$\alpha\ell$	$\lvert\ell\rvert$	0
ℓ	$-\infty$	$-\infty$	$-\infty$ if $\ell > 0$ $+\infty$ if $\ell < 0$ IF if $\ell = 0$	$\alpha\ell$	$\lvert\ell\rvert$	0
$+\infty$	ℓ'	$+\infty$	$+\infty$ if $\ell' > 0$ $-\infty$ if $\ell' < 0$ IF if $\ell' = 0$	$+\infty$ if $\alpha > 0$ $-\infty$ if $\alpha < 0$ 0 if $\alpha = 0$	$+\infty$	$+\infty$ if $\ell' \geq 0$ $-\infty$ if $\ell' < 0$
$-\infty$	ℓ'	$-\infty$	$-\infty$ if $\ell' > 0$ $+\infty$ if $\ell' < 0$ IF if $\ell' = 0$	$-\infty$ if $\alpha > 0$ $+\infty$ if $\alpha < 0$ 0 if $\alpha = 0$	$+\infty$	$-\infty$ if $\ell' \geq 0$ $+\infty$ if $\ell' < 0$
$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$ if $\alpha > 0$ $-\infty$ if $\alpha < 0$ 0 if $\alpha = 0$	$+\infty$	IF
$+\infty$	$-\infty$	IF	$-\infty$	$+\infty$ if $\alpha > 0$ $-\infty$ if $\alpha < 0$ 0 if $\alpha = 0$	$+\infty$	IF
$-\infty$	$-\infty$	$-\infty$	$+\infty$	$-\infty$ if $\alpha > 0$ $+\infty$ if $\alpha < 0$ 0 if $\alpha = 0$	$+\infty$	IF

Proposition 3.2.4 *Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$, $x_0 \in \overline{\mathbb{R}}$ and $\ell \in \mathbb{R}$.*

1) $\lim\limits_{x \to x_0} f(x) = \ell$ *if and only if* $\lim\limits_{x \to x_0} (f(x) - \ell) = 0$.

2) *If* $\lim\limits_{x \to x_0} f(x) = \ell$, *then* $\lim\limits_{x \to x_0} |f(x)| = |\ell|$.

3) $\lim\limits_{x \to x_0} |f(x)| = 0$ *if and only if* $\lim\limits_{x \to x_0} f(x) = 0$.

4) *If* $\lim\limits_{x \to x_0} f(x) = 0$ *and g is bounded on a pointed neighborhood of x_0, then*

$$\lim_{x \to x_0} f(x)g(x) = 0.$$

Proof Suppose that $x_0 \in \mathbb{R}$ (we make a similar proof when $x_0 = \pm\infty$).

1) By the fact that $|f(x) - \ell| = \Big|(f(x) - \ell) - 0\Big|$.

2) Let $\varepsilon > 0$. Since $\lim\limits_{x\to x_0} f(x) = \ell$, then there exists $\delta > 0$ such that, for any $x \in D_f$,
$$0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon.$$
If $0 < |x - x_0| < \delta$, then
$$\Big||f(x)| - |\ell|\Big| \leq |f(x) - \ell| < \varepsilon.$$
Hence $\lim\limits_{x\to x_0} |f(x)| = |\ell|$.

3) If $\lim\limits_{x\to x_0} f(x) = 0$, then $\lim\limits_{x\to x_0} |f(x)| = |0| = 0$ by the part (2). Conversely, let $\varepsilon > 0$. Since $\lim\limits_{x\to x_0} |f(x)| = 0$, then there exists $\delta > 0$ such that, for any $x \in D_f$,
$$0 < |x - x_0| < \delta \Rightarrow |f(x) - 0| = \Big||f(x)| - 0\Big| < \varepsilon.$$
Hence $\lim\limits_{x\to x_0} f(x) = 0$.

4) Let V be a pointed neighborhood of x_0 such that g is bounded on V. Then there exists $r > 0$ such that
$$D =]x_0 - r, x_0 + r[\ \setminus\{x_0\} \subseteq V.$$
Since g is bounded on V, then g is bounded on D, so there exists $M > 0$ such that $|g(x)| \leq M$ for all $x \in D$. Let $\varepsilon > 0$, since $\lim\limits_{x\to x_0} f(x) = 0$, then there exists $\delta > 0$ such that, for any $x \in D_f$,
$$0 < |x - x_0| < \delta \Rightarrow |f(x)| < \frac{\varepsilon}{M}.$$
Let $\delta_0 = \min(r, \delta)$. If $0 < |x - x_0| < \delta_0$, then $0 < |x - x_0| < r$ (i.e., $x \in D$) and $0 < |x - x_0| < \delta$, so
$$|f(x)g(x)| \leq M|f(x)| < M\left(\frac{\varepsilon}{M}\right) = \varepsilon.$$
Hence $\lim\limits_{x\to x_0} f(x)g(x) = 0$. □

Example 3.2.6 *Since* $\lim\limits_{x\to 0} x^2 = 0$ *and* $\left|\sin\frac{1}{x}\right| \leq 1$ *for all* $x \in \mathbb{R}^*$*, then*
$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0.$$

Theorem 3.2.1 *(Change of variable)*
Let $x_0, \ell, \ell' \in \overline{\mathbb{R}}$ *and* $u, f \in \mathcal{F}(\mathbb{R},\mathbb{R})$ *such that* $u(D_u) \subseteq D_f$*. If* $\lim_{x\to x_0} u(x) = \ell$ *and* $\lim_{y\to\ell} f(y) = \ell'$*, then* $\lim_{x\to x_0} f(u(x)) = \ell'$*. In other words,*

$$\lim_{x\to x_0} f(u(x)) = \lim_{u\to\ell} f(u).$$

Proof Suppose that $x_0, \ell, \ell' \in \mathbb{R}$. We make a similar proof for the other cases. Let $\varepsilon > 0$. Since $\lim_{y\to\ell} f(y) = \ell'$, then there exists $\delta_1 > 0$ such that, for any $y \in D_f$,

$$(*) \qquad 0 < |y-\ell| < \delta_1 \Rightarrow |f(y)-\ell'| < \varepsilon.$$

For $\delta_1 > 0$, since $\lim_{x\to x_0} u(x) = \ell$, then there exists $\delta > 0$ such that, for any $x \in D_u$,

$$(**) \qquad 0 < |x-x_0| < \delta \Rightarrow |u(x)-\ell| < \delta_1.$$

Hence, for any $x \in \mathbb{R}$, by using $(*)$ and $(**)$, we obtain:

$$0 < |x-x_0| < \delta \Rightarrow |u(x)-\ell| < \delta_1 \Rightarrow |f(u(x))-\ell'| < \varepsilon.$$

Thus $\lim_{x\to x_0} f(u(x)) = \ell'$. □

Example 3.2.7 *By using the theorem 3.2.1, we obtain:*

$$\lim_{x\to 0} \sin\left(x^2 + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$$

Theorem 3.2.2 *(Comparison of limits)*
Let $f, g \in \mathcal{F}(\mathbb{R},\mathbb{R})$ *and* $x_0 \in \overline{\mathbb{R}}$*. Suppose that there exists a pointed neighborhood* V *of* x_0 *such that*

$$f(x) \leq g(x), \quad \forall x \in V.$$

1) If $\lim_{x\to x_0} f(x) = \ell_1 \in \mathbb{R}$ *and* $\lim_{x\to x_0} g(x) = \ell_2 \in \mathbb{R}$*, then* $\ell_1 \leq \ell_2$*.*

2) If $\lim_{x\to x_0} f(x) = +\infty$*, then* $\lim_{x\to x_0} g(x) = +\infty$*.*

3) If $\lim_{x\to x_0} g(x) = -\infty$*, then* $\lim_{x\to x_0} f(x) = -\infty$*.*

Proof Suppose that $x_0 \in \mathbb{R}$ (we make a similar proof when $x_0 = \pm\infty$).

1) Suppose that $\ell_1 > \ell_2$ and put $\varepsilon = \frac{\ell_1-\ell_2}{2} > 0$. Since $\lim_{x\to x_0} f(x) = \ell_1$ and $\lim_{x\to x_0} g(x) = \ell_2$, then there exist $\delta_1, \delta_2 > 0$ such that,

$$0 < |x-x_0| < \delta_1 \Rightarrow |f(x)-\ell_1| < \varepsilon \quad \text{and} \quad 0 < |x-x_0| < \delta_2 \Rightarrow |g(x)-\ell_2| < \varepsilon.$$

Let $\delta = \min(r, \delta_1, \delta_2)$. If $0 < |x-x_0| < \delta$, then $0 < |x-x_0| < \delta_1$ and $0 < |x-x_0| < \delta_2$, so

$$g(x) < \ell_2 + \varepsilon = \frac{\ell_1+\ell_2}{2} = \ell_1 - \varepsilon < f(x),$$

this contradicts the hypothesis (since $0 < |x-x_0| < r$, i.e., $x \in]x_0-r, x_0+r[\ \setminus\{x_0\}$). Hence $\ell_1 \leq \ell_2$.

2) Let $A > 0$. Since $\lim_{x \to x_0} f(x) = +\infty$, then there exists $\delta > 0$ such that $f(x) > A$ for any $x \in D_f$ satisfying $0 < |x - x_0| < \delta$. So, if $0 < |x - x_0| < \delta$, then $g(x) \geq f(x) > A$, and therefore $g(x) > A$. Hence $\lim_{x \to x_0} g(x) = +\infty$.

3) Ii is sufficient to apply the part (2) on the functions $-f$ and $-g$ taking into account that $-g(x) \leq -f(x)$ for all $x \in V$. $\square$

Example 3.2.8 *For any* $x \in \mathbb{R}$, $-1 \leq \sin x \leq 1$, *so*

$$x - 1 \leq x + \sin x \leq x + 1.$$

By the theorem 3.2.2, since $\lim_{x \to -\infty} (x + 1) = -\infty$ *(resp.* $\lim_{x \to +\infty} (x - 1) = +\infty$*), then* $\lim_{x \to -\infty} (x + \sin x) = -\infty$ *(resp.* $\lim_{x \to +\infty} (x + \sin x) = +\infty$*).*

Theorem 3.2.3 *(Sandwich theorem on the functions)*
Let $f, u, v \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ *and* $x_0 \in \overline{\mathbb{R}}$. *Suppose that there exists a pointed neighborhood* V *of* x_0 *such that*

$$u(x) \leq f(x) \leq v(x), \quad \forall x \in V.$$

If $\lim_{x \to x_0} u(x) = \lim_{x \to x_0} v(x) = \ell \in \mathbb{R}$, *then* $\lim_{x \to x_0} f(x) = \ell$.

Proof Suppose that $x_0 \in \mathbb{R}$ (we make a similar proof when $x_0 = \pm\infty$). Let $\varepsilon > 0$. Since $\lim_{x \to x_0} u(x) = \ell$ and $\lim_{x \to x_0} v(x) = \ell$, then there exist $\delta_1, \delta_2 > 0$ such that,

$$0 < |x - x_0| < \delta_1 \Rightarrow |u(x) - \ell| < \varepsilon \quad \text{and} \quad 0 < |x - x_0| < \delta_2 \Rightarrow |v(x) - \ell| < \varepsilon.$$

Let $\delta = \min(r, \delta_1, \delta_2)$. If $0 < |x - x_0| < \delta$, then

$$0 < |x - x_0| < r, \qquad 0 < |x - x_0| < \delta_1 \qquad \text{and} \qquad 0 < |x - x_0| < \delta_2,$$

so

$$\ell - \varepsilon < u(x) \leq f(x) \leq v(x) < \ell + \varepsilon,$$

i.e., $|f(x) - \ell| < \varepsilon$. Hence $\lim_{x \to x_0} f(x) = \ell$. $\square$

Example 3.2.9 *Let* $f(x) = x \sin \frac{1}{x}$. *For any* $x \in]-1, 1[- \{0\}$, *as* $\left|\sin \frac{1}{x}\right| \leq 1$, *then*

$$0 \leq |f(x)| \leq |x|.$$

Since $\lim_{x \to 0} |x| = 0$, *then* $\lim_{x \to 0} |f(x)| = 0$ *by the Sandwich theorem. Hence* $\lim_{x \to 0} f(x) = 0$ *by the proposition 3.2.4.*

Remark 3.2.4 *In the theorems 3.2.2 and 3.2.3, if the inequalities are strict in the given condition, then taking limits does not necessary give strict inequalities. For example,* $\frac{1}{x+1} < \frac{1}{x}$ *for all* $x > 0$, *but* $\lim_{x \to +\infty} \frac{1}{x + 1} = \lim_{x \to +\infty} \frac{1}{x} = 0$.

Proposition 3.2.5 *(References limits)*

1) For any $n \in \mathbb{N}^*$,

$$\boxed{\lim_{x \to +\infty} x^n = +\infty} \quad \text{and} \quad \boxed{\lim_{x \to -\infty} x^n = (-1)^n \infty = \begin{cases} +\infty \text{ if } n \text{ is even} \\ -\infty \text{ if } n \text{ is odd} \end{cases}}$$

2) For any $n \in \mathbb{N}^*$,

$$\boxed{\lim_{x \to \pm\infty} \frac{1}{x^n} = 0} \quad \boxed{\lim_{x \to 0^+} \frac{1}{x^n} = +\infty} \quad \text{and} \quad \boxed{\lim_{x \to 0^-} \frac{1}{x^n} = (-1)^n \infty}$$

3) Let P *and* Q *be two real polynomials. Then*

$$\boxed{\lim_{x \to x_0} \frac{P(x)}{Q(x)} = \frac{P(x_0)}{Q(x_0)} \quad \text{if } Q(x_0) \neq 0} \quad \text{and} \quad \boxed{\lim_{x \to \pm\infty} \frac{P(x)}{Q(x)} = \lim_{x \to \pm\infty} \frac{Lt(P)}{Lt(Q)}}$$

where $Lt(P)$ *and* $Lt(Q)$ *are the leading terms of* P *and* Q *respectively.*

4) Let $\alpha \in \mathbb{R}^*$. *We have:*

$$\boxed{\lim_{x \to +\infty} x^\alpha = \begin{cases} +\infty & \text{if } \alpha > 0 \\ 0 & \text{if } \alpha < 0 \end{cases}} \quad \text{and} \quad \boxed{\lim_{x \to 0^+} x^\alpha = \begin{cases} 0 & \text{if } \alpha > 0 \\ +\infty & \text{if } \alpha < 0 \end{cases}}$$

5) We have:

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

6) For any real number $a > 0$,

$$\boxed{\lim_{x \to 0^+} \ln x = -\infty} \quad \boxed{\lim_{x \to 0^+} x^a \ln x = 0} \quad \boxed{\lim_{x \to +\infty} \ln x = +\infty} \quad \text{and} \quad \boxed{\lim_{x \to +\infty} \frac{\ln x}{x^a} = 0}$$

7) Let a *be a strictly positive real number such that* $a \neq 1$. *Then*

$$\boxed{\lim_{x \to 0^+} \log_a x = \begin{cases} -\infty & \text{if } a > 1 \\ +\infty & \text{if } a < 1 \end{cases}} \quad \text{and} \quad \boxed{\lim_{x \to +\infty} \log_a x = \begin{cases} +\infty & \text{if } a > 1 \\ -\infty & \text{if } a < 1 \end{cases}}$$

8) For any $n \in \mathbb{N}$,

$$\boxed{\lim_{x \to +\infty} \frac{e^x}{x^n} = +\infty} \quad \text{and} \quad \boxed{\lim_{x \to -\infty} x^n e^x = 0}$$

3.2.6 Asymptotic behavior of a function

Definition 3.2.7 *Let $f \in \mathcal{F}(\mathbb{R},\mathbb{R})$ and C_f be its representative curve on D_f in an orthonormal system (xoy).*

1) *We say that a straight-line (D) is an asymptote to the curve C_f (or of f) if the distance between (D) and C_f approaches zero as they tend to infinity.*

2) *We say that a horizontal straight-line of equation $y = y_0$ is an asymptote to the curve C_f (or of f) at $-\infty$ (resp. at $+\infty$) if*

$$\lim_{x\to-\infty} f(x) = y_0 \quad (\textit{resp.} \lim_{x\to+\infty} f(x) = y_0).$$

3) *We say that a vertical straight-line of equation $x = x_0$ is an asymptote to the curve C_f (or of f) if*

$$\lim_{x\to x_0} f(x) = \pm\infty.$$

4) *We say that an oblique straight-line of equation $y = ax+b$ (where $a \in \mathbb{R}^*$ and $b \in \mathbb{R}$) is an asymptote to the curve C_f (or of f) at $-\infty$ (resp. at $+\infty$) if*

$$\lim_{x\to-\infty} \Big[f(x) - (ax+b)\Big] = 0 \quad \Big(\textit{resp.} \lim_{x\to+\infty} \Big[f(x) - (ax+b)\Big] = 0\Big).$$

Example 3.2.10

1) *Let $f(x) = \frac{1}{x}$. Since $\lim_{x\to\pm\infty} f(x) = 0$ and $\lim_{x\to 0} f(x) = \pm\infty$, then the horizontal line of equation $y = 0$ (i.e., the x-axis) is an asymptote to the curve of f on $\mathbb{R}^*$ at $-\infty$ and at $+\infty$, and the vertical line of equation $x = 0$ (i.e., the y-axis) is an asymptote to the curve of f on $\mathbb{R}^*$.*

2) *Let $f(x) = 2x + 1 + \frac{1}{x^2}$. Since*

$$\lim_{x\to\pm\infty} \Big[f(x) - (2x+1)\Big] = \lim_{x\to\pm\infty} \frac{1}{x^2} = 0,$$

then the oblique line of equation $y = 2x + 1$ is an asymptote at $-\infty$ and at $+\infty$ to the curve of f on $\mathbb{R}^*$.

Proposition 3.2.6 *Let $f \in \mathcal{F}(\mathbb{R},\mathbb{R})$ and C_f be its representative curve on D_f in an orthonormal system (xoy). The straight-line of equation $y = ax + b$ (where $a, b \in \mathbb{R}$) is an asymptote to the curve C_f at $\pm\infty$ if and only if*

$$\boxed{a = \lim_{x\to\pm\infty} \frac{f(x)}{x} \in \mathbb{R}} \quad \textit{and} \quad \boxed{b = \lim_{x\to\pm\infty} \Big(f(x) - ax\Big) \in \mathbb{R}}$$

Proof If the straight-line of equation $y = ax + b$ (where $a, b \in \mathbb{R}$) is an asymptote to the curve C_f at $+\infty$ (we make a similar reasoning at $-\infty$), then $\lim_{x\to+\infty}\Big[f(x) - (ax+b)\Big] = 0$, so

$$\lim_{x\to+\infty}\frac{f(x)}{x} - a = \lim_{x\to+\infty}\frac{f(x)}{x} - \lim_{x\to+\infty}\frac{ax+b}{x} = \lim_{x\to+\infty}\frac{f(x)-(ax+b)}{x} = 0.$$

Hence

$$(*) \qquad a = \lim_{x\to+\infty}\frac{f(x)}{x}.$$

Moreover,

$$(**) \qquad b = \lim_{x\to+\infty}\Big(f(x) - ax\Big).$$

Conversely, it is obvious that if the two limits in $(*)$ and $(**)$ exist in $\mathbb{R}$, then the straight-line of equation $y = ax + b$ is an asymptote to the curve C_f at $+\infty$. □

Remark 3.2.5 *Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ and C_f, C_g be their representative curves on D_f and D_g respectively in an orthonormal system (xoy).*

1)
- *If $a = \lim_{x\to\pm\infty}\frac{f(x)}{x} \in \mathbb{R}$ and $f(x) - ax$ has no real limit at $\pm\infty$, then we say that the curve C_f has as asymptotic direction the straight-line of equation $y = ax$ at $\pm\infty$.*
- *If $a = \lim_{x\to\pm\infty}\frac{f(x)}{x} \in \mathbb{R}$ and $\lim_{x\to+\infty}\Big(f(x) - ax\Big) = \pm\infty$, then we say that the curve C_f has a parabolic branche of direction the straight-line of equation $y = ax$ at $\pm\infty$.*

2) *In general, we define a curve asymptote to another curve in the following way: we say that the curve C_g is asymptote to the curve C_f at $\pm\infty$ if*

$$\lim_{x\to\pm\infty}\Big[f(x) - g(x)\Big] = 0.$$

3) *In order to study the relative position of the curve C_f with respect to a straight-line (D) of equation $y = ax + b$ (where $a, b \in \mathbb{R}$) on a subset E of D_f, it is sufficient to determine the sign of the function h defined by $h(x) = f(x) - (ax + b)$ on E.*

- *If $h(x) \geq 0$ for all $x \in E$, then C_f is above (D) on E.*
- *If $h(x) \leq 0$ for all $x \in E$, then C_f is below (D) on E.*

In general, to study the relative position of the curves C_f and C_g on a subset E of $D_f \cap D_g$, it is sufficient to determine the sign of the function h defined by $h(x) = f(x) - g(x)$ on E.

- *If $h(x) \geq 0$ for all $x \in E$, then C_f is above C_g on E.*
- *If $h(x) \leq 0$ for all $x \in E$, then C_f is below C_g on E.*

Exercise 75
Determine the asymptotes of the function f defined by $f(x) = \sqrt{x^2-1}+2x$, and study their relative positions with respect to the curve C_f of f on D_f.

Solution The domain of definition of f is $D_f =]-\infty, -1] \cup [1, +\infty[$.
Asymptote at $+\infty$: Since $\lim\limits_{x\to+\infty} f(x) = +\infty$, then C_f has no horizontal asymptote at $+\infty$. Let's look for an oblique asymptote at $+\infty$.

$$\begin{aligned}
\lim_{x\to+\infty} \frac{f(x)}{x} &= \lim_{x\to+\infty} \frac{\sqrt{x^2-1}+2x}{x} = \lim_{x\to+\infty} \frac{\sqrt{x^2\left(1-\frac{1}{x^2}\right)}+2x}{x} \\
&= \lim_{x\to+\infty} \frac{x\sqrt{1-\frac{1}{x^2}}+2x}{x} \quad (\text{since } x>0 \text{ at } +\infty) \\
&= \lim_{x\to+\infty} \sqrt{1-\frac{1}{x^2}}+2 = 3 \in \mathbb{R}.
\end{aligned}$$

So, put $a = 3$. Moreover,

$$\begin{aligned}
\lim_{x\to+\infty} \left(f(x)-ax\right) &= \lim_{x\to+\infty} \left(\sqrt{x^2-1}-x\right) = \lim_{x\to+\infty} \frac{(x^2-1)-x^2}{\sqrt{x^2-1}+x} \\
&= \lim_{x\to+\infty} \frac{-1}{\sqrt{x^2-1}+x} = 0 \in \mathbb{R}.
\end{aligned}$$

So, put $b = 0$. Hence, by the proposition 3.2.6, the straight-line (D) of equation $y = 3x$ is an asymptote to the curve C_f at $+\infty$. In order to study the relative position of (D) with respect to C_f in a neighborhood of $+\infty$, consider the difference

$$f(x) - 3x = \frac{-1}{\sqrt{x^2-1}+x} < 0, \quad \forall\, x \in [1, +\infty[.$$

So C_f is below (D) on $[1, +\infty[$ (i.e., in a neighborhood of $+\infty$).
Asymptote at $-\infty$: In a neighborhood of $-\infty$, $x < 0$, then $\sqrt{x^2} = |x| = -x$, therefore

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x\left(-\sqrt{1-\frac{1}{x^2}}+2\right) = -\infty.$$

Hence C_f has no horizontal asymptote at $-\infty$. Let's look for an oblique asymptote at $-\infty$.

$$\lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \left(-\sqrt{1-\frac{1}{x^2}}+2\right) = 1 \in \mathbb{R}.$$

So, put $a = 1$. Moreover,

$$\begin{aligned}
\lim_{x\to-\infty} \left(f(x)-ax\right) &= \lim_{x\to-\infty} \left(\sqrt{x^2-1}+x\right) = \lim_{x\to-\infty} \frac{(x^2-1)-x^2}{\sqrt{x^2-1}-x} \\
&= \lim_{x\to-\infty} \frac{-1}{\sqrt{x^2-1}-x} = 0 \in \mathbb{R}.
\end{aligned}$$

So, put $b = 0$. Hence, by the proposition 3.2.6, the straight-line (D') of equation $y = x$ is an asymptote to the curve C_f at $-\infty$. In order to study the relative position of (D') with respect to C_f in a neighborhood of $-\infty$, consider the difference

$$f(x) - x = \frac{-1}{\sqrt{x^2 - 1} - x} < 0, \quad \forall x \in]-\infty, -1].$$

So C_f is below (D') on $]-\infty, -1]$ (i.e., in a neighborhood of $-\infty$). □

3.2.7 Equivalent functions and negligible functions

In this paragraph, we will generalize the notion of equivalent and negligible sequences (section 2.7) to equivalent and negligible functions (in a neighborhood of a certain $x_0 \in \overline{\mathbb{R}}$).

Definition 3.2.8 *Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be functions defined in a neighborhood of a certain $x_0 \in \overline{\mathbb{R}}$. We say that f is equivalent to g in a neighborhood of x_0, and we write $f \underset{x_0}{\sim} g$ (or $f(x) \underset{x_0}{\sim} g(x)$), if there exists a function $h \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ defined in a neighborhood V of x_0 such that the following two conditions are satisfied:*

i) $\lim\limits_{x \to x_0} h(x) = 1$.

ii) $f(x) = h(x)g(x)$ *for all* $x \in V$.

Remark 3.2.6 *Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ such that $\mathbb{R}^+ \subseteq D_f \cap D_g$. Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be the two sequences defined by $u_n = f(n)$ and $v_n = g(n)$ for all $n \in \mathbb{N}$. Then*

$$\boxed{f \underset{+\infty}{\sim} g \Rightarrow u_n \underset{+\infty}{\sim} v_n}$$

Proposition 3.2.7 *Let $f, g, h \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be functions defined in a neighborhood of a certain $x_0 \in \overline{\mathbb{R}}$.*

1) $f \underset{x_0}{\sim} f$.

2) If $f \underset{x_0}{\sim} g$, then $g \underset{x_0}{\sim} f$. In this case, we say that f and g are equivalent in a neighborhood of x_0.

3) If $f \underset{x_0}{\sim} g$ and $g \underset{x_0}{\sim} h$, then $f \underset{x_0}{\sim} h$.

4) If g is nonzero in a certain neighborhood of x_0, then

$$\boxed{f \underset{x_0}{\sim} g \Leftrightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = 1}$$

5) Suppose that $f \underset{x_0}{\sim} g$. Then *i)* *The two functions* $\boldsymbol{f}$ *and* $\boldsymbol{g}$ *have the same sign in a certain neighborhood of* $\boldsymbol{x_0}$.

ii) *If one of the two functions is nonzero in a certain neighborhood of* $\boldsymbol{x_0}$, *then the other function is also nonzero in a certain neighborhood of* $\boldsymbol{x_0}$.

Proposition 3.2.8 *Let* $\boldsymbol{f, g, h} \in \boldsymbol{\mathcal{F}}(\mathbb{R}, \mathbb{R})$ *be functions defined in a neighborhood of a certain* $\boldsymbol{x_0} \in \overline{\mathbb{R}}$.

1) *If* $\lim\limits_{x \to x_0} f(x) = \ell \in \mathbb{R}^*$, *then* $f \underset{x_0}{\sim} \ell$.

2) *If* $f \underset{x_0}{\sim} g$, *then* f *and* g *have the same limits at* x_0, *i.e., if* $\lim\limits_{x \to x_0} g(x) = \ell \in \overline{\mathbb{R}}$, *then* $\lim\limits_{x \to x_0} f(x) = \ell$.

Proposition 3.2.9 *(Operations on equivalent functions)*
Let $\boldsymbol{f, g, u, v} \in \boldsymbol{\mathcal{F}}(\mathbb{R}, \mathbb{R})$ *be functions defined in a neighborhood of a certain* $\boldsymbol{x_0} \in \overline{\mathbb{R}}$.

1) *If* $f \underset{x_0}{\sim} g$, *then* $kf \underset{x_0}{\sim} kg$ *for all* $k \in \mathbb{R}$.

2) *If* $f \underset{x_0}{\sim} g$, *then* $|f| \underset{x_0}{\sim} |g|$. *The converse is not necessary true.*

3) *If* $f \underset{x_0}{\sim} u$ *and* $g \underset{x_0}{\sim} u$, *then*

$$\boxed{f + g \underset{x_0}{\sim} 2u}$$

4) *Suppose that* $f \underset{x_0}{\sim} g$ *and* $u \underset{x_0}{\sim} v$. *Then:*

i) $fu \underset{x_0}{\sim} gv$.

ii) *If the function* u *is nonzero in a certain neighborhood of* x_0, *then*

$$\boxed{\frac{f}{u} \underset{x_0}{\sim} \frac{g}{v}}$$

5) *Suppose that* $f \underset{x_0}{\sim} g$.

i) $f^p \underset{x_0}{\sim} g^p$ *for all* $p \in \mathbb{N}$.

ii) *If* $f(x) \neq 0$ *in a certain neighborhood of* x_0, *then* $f^p \underset{x_0}{\sim} g^p$ *for all* $p \in \mathbb{Z}$.

iii) *If* $f(x) > 0$ *in a certain neighborhood of* x_0, *then, for any* $\alpha \in \mathbb{R}^+$,

$$\boxed{f^\alpha \underset{x_0}{\sim} g^\alpha}$$

6) *If* $\lim\limits_{x \to x_0} u(x) = y_0 \in \overline{\mathbb{R}}$ *and* $f \underset{y_0}{\sim} g$, *then*

$$\boxed{f \circ u \underset{x_0}{\sim} g \circ u}$$

If $\lim\limits_{n\to+\infty} u_n = y_0 \in \overline{\mathbb{R}}$ *and* $f \underset{y_0}{\sim} g$*, then*

$$\boxed{f(u_n) \underset{+\infty}{\sim} g(u_n)}$$

Example 3.2.11

1) *For any* $a_s, \dots, a_r \in \mathbb{R}$ *such that* $a_s \neq 0$ *and* $a_r \neq 0$ *(with* $s \geq r$*),*

$$\boxed{a_s x^s + \cdots + a_r x^r \underset{\pm\infty}{\sim} a_s x^s} \qquad \boxed{a_s x^s + \cdots + a_r x^r \underset{0}{\sim} a_r x^r}$$

In other words, a polynomial function is equivalent to is leading term (term of higher degree) in a neighborhood of $\pm\infty$*. It is equivalent to its term of lower degree in a neighborhood of* 0*.*
Consequently, for any $b_{s'}, \dots, b_{r'} \in \mathbb{R}$ *such that* $b_{s'} \neq 0$ *and* $b_{r'} \neq 0$ *(with* $s' \geq r'$*),*

$$\boxed{\frac{a_s x^s + \cdots + a_r x^r}{b_{s'} x^{s'} + \cdots + b_{r'} x^{r'}} \underset{\pm\infty}{\sim} \frac{a_s x^s}{b_{s'} x^{s'}}} \qquad \boxed{\frac{a_s x^s + \cdots + a_r x^r}{b_{s'} x^{s'} + \cdots + b_{r'} x^{r'}} \underset{0}{\sim} \frac{a_r x^r}{b_{r'} x^{r'}}}$$

2) *We have the following basic equivalences:*

$$\boxed{\sin x \underset{0}{\sim} x} \qquad \boxed{\tan x \underset{0}{\sim} x} \qquad \boxed{\ln(1+x) \underset{0}{\sim} x}$$

$$\boxed{e^x \underset{0}{\sim} 1+x} \qquad \boxed{\cos x \underset{0}{\sim} 1 - \frac{x^2}{2}} \qquad \boxed{\sqrt{1+x} \underset{0}{\sim} 1 + \frac{x}{2}}$$

Exercise 76
By using the equivalences, calculate the following limits:

$$\ell_1 = \lim_{x\to 0} \frac{1-\cos x}{x} \qquad \text{and} \qquad \ell_2 = \lim_{x\to-\infty} x\ln(1+e^x).$$

Solution Since $1-\cos x \underset{0}{\sim} \frac{x^2}{2}$, then

$$\frac{1-\cos x}{x} \underset{0}{\sim} \frac{\frac{x^2}{2}}{x} \underset{0}{\sim} \frac{x}{2}.$$

So $\ell_1 = \lim\limits_{x\to 0} \frac{x}{2} = 0$. In the other side, since $\lim\limits_{x\to-\infty} e^x = 0$ et $\ln(1+u) \underset{0}{\sim} u$, then, by the part (6) of the proposition 3.2.9,

$$\ln(1+e^x) \underset{-\infty}{\sim} e^x, \quad \text{and therefore} \quad x\ln(1+e^x) \underset{-\infty}{\sim} xe^x.$$

So $\ell_2 = \lim\limits_{x\to-\infty} xe^x = 0$. □

Definition 3.2.9 *Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be functions defined in a neighborhood of a certain $x_0 \in \overline{\mathbb{R}}$. We say that f is negligible in front of g (or that g is preponderant in front of f) in a neighborhood of x_0, and we write $f \underset{x_0}{=} o(g)$ (or simply $f = o(g)$ or $f \ll_{x_0} g$), if there exists a function $\varepsilon \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ defined in a neighborhood V of x_0 such that the following two conditions are satisfied:*

i) $\lim_{x \to x_0} \varepsilon(x) = 0$.

ii) $f(x) = \varepsilon(x) g(x)$ *for all* $x \in V$.

Remark 3.2.7

1) The notation $f = o(g)$ is called Landau's notation or "small o". The notation $f <<_{x_0} g$ is called Hardy's notation.

2) Writing $f = o(g)$ does not mean that $f = g$. For example, $x \neq x^2$, but

$$x \underset{0}{=} o(x^4) \quad \text{and} \quad x^2 \underset{0}{=} o(x^4).$$

Writing $f = o(g)$ means that the function f belongs to the set of negligible functions in front of the function g, and sometimes, we write $f \in o(g)$.

3) Let $f, g, h \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be functions defined in a neighborhood of a certain $x_0 \in \overline{\mathbb{R}}$. Then

i) We have:

$$\boxed{f \underset{x_0}{\sim} g \Leftrightarrow f - g \underset{x_0}{=} o(g)}$$

In other words, $f \underset{x_0}{\sim} g$ if and only if there exists a function $\varepsilon \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ defined in a neighborhood V of x_0 such that, for all $x \in V$,

$$\boxed{f(x) = g(x) + g(x)\varepsilon(x), \quad \text{with} \lim_{x \to x_0} \varepsilon(x) = 0}$$

Hence, by using the example 3.2.11, in a neighborhood of 0, we obtain:

$$\boxed{\sin x = x + o(x)} \qquad \boxed{\tan x = x + o(x)} \qquad \boxed{\ln(1+x) = x + o(x)}$$

$$\boxed{e^x = 1 + x + o(x)} \qquad \boxed{\cos x = 1 - \frac{x^2}{2} + o(x^2)} \qquad \boxed{\sqrt{1+x} = 1 + \frac{x}{2} + o(x)}$$

ii) If $f \underset{x_0}{\sim} g$ and $g \underset{x_0}{=} o(h)$, then $f \underset{x_0}{=} o(h)$.

Theorem 3.2.4 *Let $f, g, h, u, v \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be functions defined in a neighborhood of a certain $x_0 \in \overline{\mathbb{R}}$.*

1) *If the function* g *is nonzero in a certain neighborhood of* x_0, *then*

$$\boxed{f \underset{x_0}{=} o(g) \Leftrightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = 0}$$

2) *If* $\ell \in \mathbb{R}$, *then*

$$\boxed{\lim_{x \to x_0} f(x) = \ell \Leftrightarrow f \underset{x_0}{=} \ell + o(1)}$$

In particular,

$$\boxed{\lim_{x \to x_0} f(x) = 0 \Leftrightarrow f \underset{x_0}{=} o(1)}$$

3) *If* $f \underset{x_0}{=} o(g)$ *and* $g \underset{x_0}{=} o(h)$, *then* $f \underset{x_0}{=} o(h)$.

4) *If* $f \underset{x_0}{=} o(g)$, *then* $kf \underset{x_0}{=} o(g)$ *and* $f \underset{x_0}{=} o(kg)$ *for all* $k \in \mathbb{R}^*$. *More generally, if* $f \underset{x_0}{=} o(g)$ *and* h *is bounded in a neighborhood of* x_0, *then* $hf \underset{x_0}{=} o(g)$.

5) *If* $f \underset{x_0}{=} o(u)$ *and* $g \underset{x_0}{=} o(u)$, *then, for all* $a, b \in \mathbb{R}$,

$$\boxed{af + bg \underset{x_0}{=} o(u)}$$

6) *If* $f \underset{x_0}{=} o(g)$ *and* $u \underset{x_0}{=} o(v)$, *then*

$$\boxed{uf \underset{x_0}{=} o(vg)}$$

7) *If* $f \underset{x_0}{=} o(g)$, *then, for all* $p \in \mathbb{N}$,

$$\boxed{f^p \underset{x_0}{=} o(g^p)}$$

8) *If* $f \underset{x_0}{=} o(g)$ *and* f *is nonzero in a certain neighborhood of* x_0, *then* g *is nonzero in a certain neighborhood of* x_0 *and*

$$\boxed{\frac{1}{g} \underset{x_0}{=} o\left(\frac{1}{f}\right)}$$

Example 3.2.12 *For any* $a_0, a_1, \dots, a_s \in \mathbb{R}$ *such that* $a_s \neq 0$,

$$\boxed{a_s x^s + \cdots + a_1 x + a_0 \underset{+\infty}{=} o(x^{s+1})}$$

Moreover,

$$\boxed{x^s \underset{+\infty}{=} o(x^t) \Leftrightarrow s < t} \qquad \boxed{x^s \underset{0}{=} o(x^t) \Leftrightarrow s > t}$$

3.3 Continuity of a function

In this section, $f : \mathbb{R} \longmapsto \mathbb{R}$ is a real function defined in a neighborhood of a point $x_0 \in \mathbb{R}$. The aim of this section is to study the continuity of f at the point x_0.

Definition 3.3.1

1) *We say that f is continuous at x_0 if $\lim\limits_{x \to x_0} f(x) = f(x_0)$. Else, we say that f is discontinuous at x_0.*

2) *We say that f is right continuous at x_0 if $\lim\limits_{x \to x_0^+} f(x) = f(x_0)$.*

3) *We say that f is left continuous at x_0 if $\lim\limits_{x \to x_0^-} f(x) = f(x_0)$.*

4) *We say that f is continuous on a subset E of D_f if it is continuous at every point of E. We will denote by $\mathcal{C}(E, F)$ the set of continuous functions from E to a subset F of $\mathbb{R}$.*

Remark 3.3.1

1) *By the proposition 3.2.3, the function f is continuous at x_0 if and only if it is right and left continuous at x_0.*

2) *As a particular case of the above definition, we define $\mathcal{C}(\,]a, b[, F)$ to be the set of continuous functions from $]a, b[$ to F. We can extend this definition to the closed interval $[a, b]$ in the following way: f is said to be continuous on $[a, b]$ if it is continuous at every point of $]a, b[$, right continuous at a and left continuous at b. We will denote by $\mathcal{C}([a, b], F)$ the set of continuous functions from $[a, b]$ to F.*

Example 3.3.1

1) *By the example 3.2.2, the absolute value function is continuous on $\mathbb{R}$.*

2) *By the exercise 73, the functions sine and cosine are continuous on $\mathbb{R}$.*

3) *If $x_0 \in \mathbb{R}^+$, then*

$$\lim_{x \to x_0} (\sqrt{x} - \sqrt{x_0}) = \lim_{x \to x_0} \frac{(\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0})}{\sqrt{x} + \sqrt{x_0}} = \lim_{x \to x_0} \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}} = 0.$$

So $\lim\limits_{x \to x_0} \sqrt{x} = \sqrt{x_0}$, and therefore the square root function is continuous at x_0. Hence it is continuous on $\mathbb{R}^+$.

4) *Let $x_0 \in \mathbb{Z}$. We know that $E(x) = x_0$ if $x_0 < x < x_0 + 1$ (as x tends to x_0 from the right) and $E(x) = x_0 - 1$ if $x_0 - 1 < x < x_0$ (as x tends to x_0 from the left). So*

$$\lim_{x \to x_0^+} E(x) = \lim_{x \to x_0^+} x_0 = x_0 = E(x_0)$$

$$\lim_{x \to x_0^-} E(x) = \lim_{x \to x_0^-} (x_0 - 1) = x_0 - 1 \neq E(x_0).$$

So the integer part function E is right continuous at x_0 but it is not left continuous at x_0. Hence it is not continuous at x_0.

5) *Let $x_0 \in \mathbb{R}$ such that $x_0 \neq (2k+1)\frac{\pi}{2}$ for all $k \in \mathbb{Z}$, then $\cos x_0 \neq 0$. So*

$$\lim_{x \to x_0} \tan x = \lim_{x \to x_0} \frac{\sin x}{\cos x} = \frac{\sin x_0}{\cos x_0} = \tan x_0.$$

Hence the tangent function is continuous at x_0.

6) *If P, Q are two real polynomials and $x_0 \in \mathbb{R}$ such that $Q(x_0) \neq 0$, then the function f defined by $f(x) = \dfrac{P(x)}{Q(x)}$ is continuous at x_0.*

7) *Let D be the function defined by*

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

The function D, called Dirichlet's function, is not continuous at any point of $\mathbb{R}$. Indeed, let $x_0 \in \mathbb{R}$, we suppose that D is continuous at x_0, then $\lim_{x \to x_0} D(x) = D(x_0)$. Two cases are possible ($x_0 \in \mathbb{Q}$ or $x_0 \in \mathbb{R} - \mathbb{Q}$).

- *Suppose that $x_0 \in \mathbb{Q}$, then $D(x_0) = 1$. For $\varepsilon = \frac{1}{2} > 0$, there exists $\delta > 0$ such that*

$$(*) \qquad x \in \left]x_0 - \delta, x_0 + \delta\right[\Rightarrow D(x) \in \left]1 - \varepsilon, 1 + \varepsilon\right[= \left]\frac{1}{2}, \frac{3}{2}\right[.$$

  Since $\mathbb{R} - \mathbb{Q}$ is dense in $\mathbb{R}$ (by the theorem 1.6.2), then between the two real numbers $x_0 - \delta$ and $x_0 + \delta$, there exists an irrational number x_1, so $D(x_1) \in \left]\frac{1}{2}, \frac{3}{2}\right[$ by $(*)$. But $x_1 \in \mathbb{R} - \mathbb{Q}$, then $D(x_1) = 0$, so $0 \in \left]\frac{1}{2}, \frac{3}{2}\right[$, which is impossible.

- *Suppose that $x_0 \in \mathbb{R} - \mathbb{Q}$, then $D(x_0) = 0$. For $\varepsilon = \frac{1}{2} > 0$, there exists $\delta > 0$ such that*

$$(**) \qquad x \in \left]x_0 - \delta, x_0 + \delta\right[\Rightarrow D(x) \in \left]0 - \varepsilon, 0 + \varepsilon\right[= \left]-\frac{1}{2}, \frac{1}{2}\right[.$$

  Since $\mathbb{Q}$ is dense in $\mathbb{R}$ (by the theorem 1.6.1), then between the two real numbers $x_0 - \delta$ and $x_0 + \delta$, there exists a rational number x_2, so $D(x_2) \in \left]-\frac{1}{2}, \frac{1}{2}\right[$ by $(**)$. But $x_2 \in \mathbb{Q}$, then $D(x_2) = 1$, so $1 \in \left]-\frac{1}{2}, \frac{1}{2}\right[$, which is impossible.

Hence D is not continuous at x_0.

Proposition 3.3.1 *(Operations on continuous functions)*
Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ and $x_0 \in D_f \cap D_g$. If f and g are continuous at x_0 then

1) *$f + g$ is continuous at x_0.*

2) αf *is continuous at* x_0 *for all* $\alpha \in \mathbb{R}$.

3) fg *is continuous at* x_0.

4) $\dfrac{f}{g}$ *is continuous at* x_0 *if* $g(x_0) \neq 0$.

Proof The proof is done by using the table of operations on the limits. □

Proposition 3.3.2 *Let* $x_0 \in \mathbb{R}$ *and* $u, f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ *such that* $u(D_u) \subseteq D_f$.

1) If $\lim\limits_{x \to x_0} u(x) = \ell \in \mathbb{R}$ *and* f *is continuous at* ℓ, *then* $\lim\limits_{x \to x_0} f(u(x)) = f(\ell)$. *In other words,*

$$\lim_{x \to x_0} f(u(x)) = f\left(\lim_{x \to x_0} u(x)\right).$$

2) If u *is continuous at* x_0 *and* f *is continuous at* $u(x_0)$, *then the composite function* $f \circ u$ *is continuous at* x_0. *In other words, the composite of two continuous functions is a continuous function.*

Proof

1) Since f is continuous at ℓ, then $\lim\limits_{y \to \ell} f(y) = f(\ell)$. Moreover, as $\lim\limits_{x \to x_0} u(x) = \ell$, then $\lim\limits_{x \to x_0} f(u(x)) = f(\ell)$ by the theorem 3.2.1.

2) Since u is continuous at x_0, then $\lim\limits_{x \to x_0} u(x) = u(x_0)$. Moreover, as f is continuous at $u(x_0)$, then, by the part (1),

$$\lim_{x \to x_0} (f \circ u)(x) = \lim_{x \to x_0} f(u(x)) = f(u(x_0)) = (f \circ u)(x_0).$$

Hence $f \circ u$ is continuous at x_0. □

Definition 3.3.2 *Let* $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ *and* $x_0 \in \mathbb{R}$. *We say that* f *is extendable by continuity (resp. continuity from the right, continuity from the left) at* x_0 *if the following two conditions are satisfied:*

1) f *is not defined at* x_0.

2) $\lim\limits_{x \to x_0} f(x) = \ell \in \mathbb{R}$ *(resp.* $\lim\limits_{x \to x_0^+} f(x) = \ell \in \mathbb{R}$, $\lim\limits_{x \to x_0^-} f(x) = \ell \in \mathbb{R}$*).*

In this case, the function $\varphi \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ *defined by:*

$$\varphi(x) = \begin{cases} f(x) & \text{if } x \neq x_0 \\ \ell & \text{if } x = x_0. \end{cases}$$

is said to be the extension by continuity (resp. continuity from the right, continuity from the left) of f *at* x_0.

Example 3.3.2

1) *Let* $f(x) = \frac{|x|}{x}$. *Since* f *is not defined at* 0 *and* $\lim_{x\to 0^+} f(x) = 1 \in \mathbb{R}$ *and* $\lim_{x\to 0^-} f(x) = -1 \in \mathbb{R}$, *then* f *is extendable by continuity from the right and from the left at* 0. *On the other hand,* f *is not extendable by continuity at* 0.

2) *Let* $f(x) = \frac{\sin x}{x}$. *Since* f *is not defined at* 0 *and* $\lim_{x\to 0} f(x) = 1 \in \mathbb{R}$, *then* f *is extendable by continuity at* 0.

3) *Let* $f(x) = x \sin \frac{1}{x}$. *Since* f *is not defined at* 0 *and* $\lim_{x\to 0} f(x) = 0 \in \mathbb{R}$ *(by the example 3.2.9), then* f *is extendable by continuity at* 0.

Remark 3.3.2

1) *If* f *is extendable by continuity (resp. from the right, from the left) at* x_0, *then its extension* φ *by continuity (resp. continuity from the right, continuity from the left) at* x_0 *is continuous (resp. right continuous, left continuous) at* x_0. *Indeed,*

$$\lim_{\substack{x\to x_0 \\ x\neq x_0}} \varphi(x) = \lim_{x\to x_0} f(x) = \ell = \varphi(x_0).$$

2) *By the proposition 3.2.3,* f *is extendable by continuity at* x_0 *if and only if* f *is extendable by continuity from the right and from the left at* x_0 *and* $\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x)$.

3.4 Sequence defined by a function

Theorem 3.4.1 *Let* $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ *and* $x_0, \ell \in \overline{\mathbb{R}}$. *Then* $\lim_{x\to x_0} f(x) = \ell$ *if and only if, for any sequence* $(u_n)_{n\in\mathbb{N}}$ *of elements of* D_f, *the following implication is true:*

$$\lim_{n\to+\infty} u_n = x_0 \Rightarrow \lim_{n\to+\infty} f(u_n) = \ell.$$

Proof Suppose that $x_0, \ell \in \mathbb{R}$. We make a similar proof for the other cases.
N.C. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of elements of D_f such that $\lim_{n\to+\infty} u_n = x_0$. Let $\varepsilon > 0$. Since $\lim_{x\to x_0} f(x) = \ell$, then there exists $\delta > 0$ such that, for any $x \in \mathbb{R}$,

$$(*) \qquad 0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon.$$

For $\delta > 0$, since $\lim_{n\to+\infty} u_n = x_0$, then there exists $n_0 \in \mathbb{N}$ such that, for any $n \in \mathbb{N}$,

$$(**) \qquad n \geq n_0 \Rightarrow |u_n - x_0| < \delta.$$

Hence, for any $n \in \mathbb{N}$, by using $(*)$ and $(**)$, we obtain:

$$n \geq n_0 \Rightarrow |u_n - x_0| < \delta \Rightarrow |f(u_n) - \ell| < \varepsilon.$$

Thus $\lim\limits_{n\to+\infty} f(u_n) = \ell$.
S.C. Suppose that $\lim\limits_{x\to x_0} f(x) \neq \ell$, then there exists $\varepsilon > 0$ such that, for any $\delta > 0$, there exists $x \in D_f$ satisfying:

$$0 < |x - x_0| < \delta \quad \text{and} \quad |f(x) - \ell| \geq \varepsilon.$$

For $\delta_n = \frac{1}{n} > 0$ (with $n \in \mathbb{N}^*$), there exists $x_n \in D_f$ satisfying:

$$0 < |x_n - x_0| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - \ell| \geq \varepsilon.$$

So $\lim\limits_{n\to+\infty} |x_n - x_0| = 0$ (by the Sandwich theorem), and therefore $\lim\limits_{n\to+\infty} x_n = x_0$. Hence $\lim\limits_{n\to+\infty} f(x_n) = \ell$ (by using the hypothesis), so $\lim\limits_{n\to+\infty} |f(x_n) - \ell| = 0$, and therefore $0 \geq \varepsilon$, which is impossible. Hence $\lim\limits_{x\to x_0} f(x) = \ell$. $\square$

Example 3.4.1

1) *By the theorem 3.4.1, in order to prove that $\lim\limits_{x\to x_0} f(x)$ does not exist in $\overline{\mathbb{R}}$, it is sufficient to find a sequence $(u_n)_{n\in\mathbb{N}}$ of elements of D_f such that $\lim\limits_{n\to+\infty} u_n = x_0$ and $\lim\limits_{n\to+\infty} f(u_n)$ does not exist in $\overline{\mathbb{R}}$. For example, if $u_n = \frac{1}{(2n+1)\frac{\pi}{2}}$, then $\lim\limits_{n\to+\infty} u_n = 0$, but*

$$\lim_{n\to+\infty} \sin\frac{1}{u_n} = \lim_{n\to+\infty} \sin\left(\frac{\pi}{2} + n\pi\right) = \lim_{n\to+\infty} (-1)^n$$

does not exist in $\overline{\mathbb{R}}$. Hence $\lim\limits_{x\to 0} \sin\frac{1}{x}$ does not exist in $\overline{\mathbb{R}}$.

2) *By the theorem 3.4.1, in order to prove that $\lim\limits_{x\to x_0} f(x)$ does not exist in $\overline{\mathbb{R}}$, it is sufficient to find two sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ of elements of D_f such that*

$$\lim_{n\to+\infty} u_n = x_0 \quad \textit{and} \quad \lim_{n\to+\infty} v_n = x_0,$$

but

$$\lim_{n\to+\infty} f(u_n) \neq \lim_{n\to+\infty} f(v_n).$$

For example, if $u_n = 2n\pi$ and $v_n = \frac{\pi}{2} + 2n\pi$, then $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} v_n = +\infty$, but

$$\lim_{n\to+\infty} \sin u_n = 0 \neq 1 = \lim_{n\to+\infty} \sin v_n.$$

Hence $\lim\limits_{x\to+\infty} \sin x$ does not exist in $\overline{\mathbb{R}}$. In a similar way, we prove that

$$\lim_{x\to-\infty} \sin x, \quad \lim_{x\to+\infty} \cos x \quad \textit{and} \quad \lim_{x\to-\infty} \cos x$$

do not exist in $\overline{\mathbb{R}}$.

3) *By using the change of variable* $u(x) = \frac{1}{x}$*, the theorem 3.2.1 and the fact that* $\lim_{x\to 0^\pm} u(x) = \pm\infty$*, we deduce that the limits*

$$\lim_{x\to 0^+} \sin\frac{1}{x}, \quad \lim_{x\to 0^-} \sin\frac{1}{x}, \quad \lim_{x\to 0} \sin\frac{1}{x}, \quad \lim_{x\to 0^+} \cos\frac{1}{x}, \quad \lim_{x\to 0^-} \cos\frac{1}{x}, \quad \lim_{x\to 0} \cos\frac{1}{x}$$

do not exist in $\overline{\mathbb{R}}$.

Corollary 3.4.1 *(Heine's theorem)*
Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ *and* $x_0 \in D_f$*. Then* f *is continuous at* x_0 *if and only if, for any sequence* $(u_n)_{n\in\mathbb{N}}$ *of elements of* D_f *which is convergent to* x_0*, the sequence* $(f(u_n))_{n\in\mathbb{N}}$ *converges to* $f(x_0)$.

Proof f is continuous at x_0 if and only if $\lim_{x\to x_0} f(x) = f(x_0)$. This is equivalent, by the theorem 3.4.1, to the fact that, for any sequence $(u_n)_{n\in\mathbb{N}}$ of elements of D_f which is convergent to x_0, the sequence $(f(u_n))_{n\in\mathbb{N}}$ converges to $f(x_0)$. □

Example 3.4.2

1) *Since* $\lim_{n\to+\infty} \frac{\ln n}{n} = 0$ *and the exponential function is continuous at* 0*, then, by Heine's theorem, we obtain:*

$$\lim_{n\to+\infty} (\ln n)^{\frac{1}{n}} = \lim_{n\to+\infty} e^{\frac{\ln n}{n}} = e^0 = 1.$$

2) *Let* $(u_n)_{n\geq 0}$ *be the recursive sequence defined by* $u_0 = 1$ *and* $u_{n+1} = 2u_n^2$ *for all* $n \in \mathbb{N}$*. If* $(u_n)_{n\geq 0}$ *converges to a limit* ℓ*, then* $\ell = 2\ell^2$*, so* $\ell = 0$ *or* $\ell = \frac{1}{2}$.

Exercise 77
Let a and b be two positive real parameters. For every $n \geq 0$, put:

$$u_n = \sqrt{a + bn + \pi^2 n^2} - n\pi \qquad \text{and} \qquad v_n = \sin\sqrt{a + bn + \pi^2 n^2}.$$

1) Calculate $\lim_{n\to+\infty} u_n$.

2) Verify that $v_n = (-1)^n \sin u_n$ for all $n \geq 0$.

3) For which values of a and b, the sequence $(v_n)_{n\geq 0}$ is convergent ? Determine $\lim_{n\to+\infty} v_n$ in these cases.

Solution

1) For any $n \geq 0$, multiplying by the conjugate, we obtain:

$$u_n = \frac{a + bn}{\sqrt{a + bn + \pi^2 n^2} + n\pi} = \frac{\frac{a}{n} + b}{\sqrt{\frac{a}{n^2} + \frac{b}{n} + \pi^2} + \pi}.$$

So $\lim_{n\to+\infty} u_n = \frac{b}{2\pi}$.

2) Let $n \geq 0$, then $v_n = \sin(u_n + n\pi)$.

   - If $n = 2k$ is even (with $k \in \mathbb{N}$), then $(-1)^n = (-1)^{2k} = 1$ and
   $$v_n = \sin(u_n + 2k\pi) = \sin u_n = (-1)^n \sin u_n.$$
   - If $n = 2k+1$ is odd (with $k \in \mathbb{N}$), then $(-1)^n = (-1)^{2k+1} = -1$ and
   $$v_n = \sin(u_n + 2k\pi + \pi) = \sin(u_n + \pi) = -\sin u_n = (-1)^n \sin u_n.$$

3) Since the sine function is continuous at $\frac{b}{2\pi}$ and $\lim\limits_{n\to+\infty} u_{2n} = \lim\limits_{n\to+\infty} u_{2n+1} = \dfrac{b}{2\pi}$, then, by Heine's theorem, we obtain:
$$\lim_{n\to+\infty} v_{2n} = \lim_{n\to+\infty} \sin u_{2n} = \sin\left(\frac{b}{2\pi}\right)$$
$$\lim_{n\to+\infty} v_{2n+1} = -\lim_{n\to+\infty} \sin u_{2n+1} = -\sin\left(\frac{b}{2\pi}\right).$$
We have:
$$\begin{aligned}(v_n)_{n\geq 0} \text{ is convergent} \quad &\Leftrightarrow \quad \lim_{n\to+\infty} v_{2n} = \lim_{n\to+\infty} v_{2n+1}\\ &\Leftrightarrow \quad \sin\left(\frac{b}{2\pi}\right) = 0\\ &\Leftrightarrow \quad \exists\, k \in \mathbb{Z} \text{ such that } \frac{b}{2\pi} = k\pi\\ &\Leftrightarrow \quad \exists\, k \in \mathbb{Z} \text{ such that } b = 2k\pi^2.\end{aligned}$$
Hence if a is arbitrary and if there exists $k \in \mathbb{Z}$ such that $b = 2k\pi^2$, then $\lim\limits_{n\to+\infty} v_n = 0$. $\square$

Proposition 3.4.1 *Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ such that $\mathbb{R}^+ \subseteq D_f$. Let $(x_n)_{n\in\mathbb{N}}$ be the sequence defined by $x_n = f(n)$ for all $n \in \mathbb{N}$.*

1) If f is increasing (resp. decreasing, strictly increasing, strictly decreasing) on $\mathbb{R}^+$, then $(x_n)_{n\in\mathbb{N}}$ is increasing (resp. decreasing, strictly increasing, strictly decreasing).

2) If $\lim\limits_{x\to+\infty} f(x) = \ell \in \overline{\mathbb{R}}$, then $\lim\limits_{n\to+\infty} x_n = \ell$.

Proof

1) Suppose that f is increasing on $\mathbb{R}^+$ (we make a similar proof in the other cases). Let $n \in \mathbb{N}$, since $n < n+1$ and $n, n+1 \in \mathbb{R}^+$, then $f(n) \leq f(n+1)$ (since f is increasing on $\mathbb{R}^+$), so $x_n \leq x_{n+1}$. Hence $(x_n)_{n\in\mathbb{N}}$ is increasing.

2) For every $n \in \mathbb{N}$, put $u_n = n$, then $x_n = f(u_n)$ for all $n \in \mathbb{N}$. Since $\lim\limits_{x\to+\infty} f(x) = \ell$ and $\lim\limits_{n\to+\infty} u_n = +\infty$, then $\lim\limits_{n\to+\infty} x_n = \lim\limits_{n\to+\infty} f(u_n) = \ell$ by the theorem 3.4.1. $\square$

3.5 Theorems on continuous functions

Theorem 3.5.1 *(Extreme value theorem)*
Let $[a, b]$ *be a closed bounded interval of* $\mathbb{R}$ *(where* $a < b$*). If* $f : [a, b] \longmapsto \mathbb{R}$ *is a continuous function on* $[a, b]$*, then* f *is bounded and it reaches its bounds on* $[a, b]$*. Moreover, if* $m = \inf_{x \in [a,b]} f(x)$ *and* $M = \sup_{x \in [a,b]} f(x)$*, then*

$$f([a, b]) = [m, M],$$

i.e., f *is a surjection from* $[a, b]$ *onto* $[m, M]$*.*

Proof Admitted. □

Corollary 3.5.1 *(Intermediate value theorem)*
Let $[a, b]$ *be a closed bounded interval of* $\mathbb{R}$ *(where* $a < b$*). If* $f : [a, b] \longmapsto \mathbb{R}$ *is a continuous function on* $[a, b]$ *and* $f(a)f(b) \leq 0$*, then there exists* $r \in [a, b]$ *such that* $f(r) = 0$*. If, in addition,* f *is strictly monotone on* $[a, b]$*, then* r *is unique.*

Proof Since f is continuous on $[a, b]$, then, by the theorem 3.5.1, there exist real numbers m and M such that $f([a, b]) = [m, M]$.

- Suppose that $f(a) \leq 0$, then $f(b) \geq 0$, so

$$m \leq f(a) \leq 0 \leq f(b) \leq M.$$

  Hence $0 \in [m, M] = f([a, b])$, therefore there exists $r \in [a, b]$ such that $f(r) = 0$.

- We make a similar proof if $f(a) \geq 0$ and $f(b) \leq 0$.

In the other side, we suppose that f is strictly increasing on $[a, b]$ (we make a similar proof when f is strictly decreasing on $[a, b]$). Suppose that r is not unique, i.e., there exists $r' \in [a, b]$ such that $r' \neq r$ and $f(r') = 0$.

- If $r < r'$, then $f(r) < f(r')$ (since f is strictly increasing on $[a, b]$), so $0 < 0$, which is impossible.

- If $r' < r$, then $f(r') < f(r)$ (since f is strictly increasing on $[a, b]$), so $0 < 0$, which is impossible.

Hence r is the unique real number in $[a, b]$ such that $f(r) = 0$. □

Example 3.5.1

1) *Let's show that the equation* $x^3 - x + 1 = 0$ *has a solution in the interval* $]-2, 0[$*. Indeed, let* $f(x) = x^3 - x + 1$*. Since* f *is continuous on* $[-2, 0]$ *(polynomial function) and* $f(-2)f(0) = (-5)(1) = -5 \leq 0$*, then, by the intermediate value theorem, there exists* $r \in [-2, 0]$ *such that* $f(r) = 0$*, i.e.,* r *is a solution of the equation* $x^3 - x + 1 = 0$ *in* $[-2, 0]$*. But* $f(-2) = -5 \neq 0$ *and* $f(0) = 1 \neq 0$*, then* $r \in]-2, 0[$*.*

2) Let $f(x) = x + \cos x$. *Since* f *is continuous on the interval* $[-\pi, 0]$ *and* $f(-\pi)f(0) = -\pi - 1 \leq 0$, *then, by the intermediate value theorem, there exists* $c \in [-\pi, 0]$ *such that* $f(c) = 0$.

Exercise 78 (Fixed point theorem)
Let $f : [a, b] \longmapsto [a, b]$ be a continuous function on an interval $[a, b]$ (where $a < b$). Show that f admits fixed point in $[a, b]$, i.e., there exists $c \in [a, b]$ such that $f(c) = c$.

Solution Consider the function $g : [a, b] \longmapsto \mathbb{R}$ defined by $g(x) = f(x) - x$. Since f is continuous on $[a, b]$, then g is continuous on $[a, b]$. In the other side, as $f(a), f(b) \in [a, b]$, then

$$g(a) = f(a) - a \geq 0 \quad \text{and} \quad g(b) = f(b) - b \leq 0$$

So $g(a)g(b) \leq 0$. Hence, by the intermediate value theorem, there exists $c \in [a, b]$ such that $g(c) = 0$, i.e., $f(c) = c$. □

Exercise 79
Let $f : [0, 2] \longmapsto \mathbb{R}$ be a continuous function on $[0, 2]$ such that $f(0) = f(2)$. Show that there exists $\alpha \in [0, 1]$ such that $f(\alpha + 1) = f(\alpha)$.

Solution Consider the function g defined by $g(x) = f(x + 1) - f(x)$. If $x \in [0, 1]$, then $x + 1 \in [1, 2] \subseteq [0, 2]$, so $f(x + 1)$ exists, and as $x \in [0, 2]$, then $f(x)$ exists. Hence g is defined and continuous on $[0, 1]$. Moreover, as

$$\begin{aligned} g(0)g(1) &= \Big(f(1) - f(0)\Big)\Big(f(2) - f(1)\Big) \\ &= \Big(f(1) - f(0)\Big)\Big(f(0) - f(1)\Big) \quad (\text{since } f(2) = f(0)) \\ &= -\Big(f(1) - f(0)\Big)^2 \leq 0, \end{aligned}$$

then, by the intermediate value theorem, there exists $\alpha \in [0, 1]$ such that $g(\alpha) = 0$, i.e., $f(\alpha + 1) = f(\alpha)$. □

Exercise 80
Let $n \in \mathbb{N}^*$ and f be a continuous function on $\mathbb{R}$. Prove that if $0 \leq f(x) \leq 1$ for all $x \in [0, 1]$, then there exists $c \in [0, 1]$ such that $f(c) = c^n$.

Solution Consider the function g defined by $g(x) = f(x) - x^n$. Since f is continuous on $[0, 1]$, then g is continuous on $[0, 1]$. Moreover, as $f(0) \geq 0$ and $f(1) \leq 1$, then

$$g(0)g(1) = f(0)(f(1) - 1) \leq 0.$$

So, by the intermediate value theorem, there exists $c \in [0, 1]$ such that $g(c) = 0$, i.e., $f(c) = c^n$. □

Exercise 81

Let a, b be two strictly positive real numbers. Show that the equation $\frac{a}{x-1} + \frac{b}{x-3} = 0$ admits at least a solution in $\mathbb{R}$.

Solution Consider the function g defined by $g(x) = a(x-3) + b(x-1)$. We have:

$$g(1)g(3) = (-2a)(2b) = -4ab \leq 0.$$

Moreover, as g is continuous on the interval $[1, 3]$, then, by the intermediate value theorem, there exists $c \in [1, 3]$ such that $g(c) = 0$. But $g(1) = -2a \neq 0$ and $g(3) = 2b \neq 0$, then $c \in]1, 3[$. So

$$\frac{a}{c-1} + \frac{b}{c-3} = \frac{a(c-3) + b(c-1)}{(c-1)(c-3)} = \frac{g(c)}{(c-1)(c-3)} = 0.$$

Hence c is a solution of the equation $\frac{a}{x-1} + \frac{b}{x-3} = 0$ in $\mathbb{R}$. □

Exercise 82

Let $f : [a, b] \longmapsto [a, b]$ be a function such that there exists $k \in]0, 1[$ satisfying the following property:

$$\forall\, x, x' \in [a, b], \quad |f(x) - f(x')| \leq k|x - x'|.$$

1) **Show that f is continuous on $[a, b]$.**

2) **Show that there exists one and only one $c \in [a, b]$ such that $f(c) = c$ (c is called a fixed point of f in $[a, b]$).**

Solution

1) Let $x_0 \in [a, b]$. Let's show that f is continuous at x_0. Indeed, let $\varepsilon > 0$, we take $\delta = \frac{\varepsilon}{k} > 0$. If $x \in [a, b]$ such that $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| \leq k|x - x_0| < k\left(\frac{\varepsilon}{k}\right) = \varepsilon.$$

Hence $\lim\limits_{x \to x_0} f(x) = f(x_0)$, and therefore f is continuous at x_0. Thus f is continuous on $[a, b]$.

2) Since f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that $f(c) = c$ by the exercise 78. In the other side, suppose that there exists $c' \in [a, b]$ such that $c' \neq c$ and $f(c') = c'$, then

$$|c - c'| = |f(c) - f(c')| \leq k|c - c'|.$$

Simplifying by $|c - c'| \neq 0$, we obtain $1 \leq k$, which is impossible. Hence c is the only real number in $[a, b]$ such that $f(c) = c$. □

Corollary 3.5.2 *(Intermediate value theorem: general form)*
Let $[a, b]$ *be a closed bounded interval of* $\mathbb{R}$ *(where* $a < b$*). If* $f : [a, b] \longmapsto \mathbb{R}$ *is a continuous function on* $[a, b]$*, then* f *reaches any value between* $f(a)$ *and* $f(b)$*, i.e., for any real number* β *between* $f(a)$ *and* $f(b)$*, there exists* $\alpha \in [a, b]$ *such that* $f(\alpha) = \beta$*.*

Proof

- The result is obvious if $f(a) = f(b)$.
- Suppose that $f(a) < f(b)$. The result is obvious if $\beta = f(a)$ (take $\alpha = a$) or if $\beta = f(b)$ (take $\alpha = b$). Let $\beta \in]f(a), f(b)[$ and g be the function defined on $[a, b]$ by $g(x) = f(x) - \beta$. Then

$$g(a) = f(a) - \beta < 0 \qquad \text{and} \qquad g(b) = f(b) - \beta > 0.$$

  So $g(a)g(b) < 0$, and as g is continuous on $[a, b]$, then, by the intermediate value theorem, there exists $\alpha \in]a, b[$ such that $g(\alpha) = 0$, so $f(\alpha) = \beta$.
- If $f(a) > f(b)$, then we make a similar proof. □

Exercise 83 (Another demonstration of the corollary 3.5.2)
Let $[a, b]$ be a closed bounded interval of $\mathbb{R}$ (where $a < b$). Let $f : [a, b] \longmapsto \mathbb{R}$ be a continuous function on $[a, b]$ and $\beta \in \mathbb{R}$ such that $f(a) \leq \beta \leq f(b)$. Put:

$$A = \Big\{x \in [a, b], \text{ such that } f(x) \leq \beta\Big\}.$$

1) Show that A has a supremum α such that $\alpha \in [a, b]$.

2) Show that $f(\alpha) = \beta$.

Solution

1) Since $a \in [a, b]$ and $f(a) \leq \beta$, then $a \in A$, so A is a nonempty subset of $\mathbb{R}$. Moreover, as A is bounded from above by b, then A has a supremum $\alpha \leq b$. As $a \in A$ and α is an upper bound of A, then $a \leq \alpha$. So $\alpha \in [a, b]$.

2) As f is continuous on $[a, b]$, then f is continuous at α, therefore $\lim\limits_{x \to \alpha} f(x) = f(\alpha)$. Since $f(x) \leq \beta$ for all $x \in A$, then, by taking limits as x tends to α, we obtain $f(\alpha) \leq \beta$. In the other side,
   - If $\alpha = b$, then $\beta \leq f(b) = f(\alpha)$.
   - Suppose that $\alpha < b$, then the interval $]\alpha, b]$ is nonempty. As α is an upper bound of A, then $A \cap]\alpha, b] = \varnothing$, i.e., $f(x) > \beta$ for all $x \in]\alpha, b]$. By taking limits as x tends to α, we obtain $f(\alpha) \geq \beta$.

   Hence $f(\alpha) = \beta$. □

Lemma 3.5.1 *(Theorem of the monotone limit)*
Let I be an interval of $\mathbb{R}$ with extremities $a, b \in \overline{\mathbb{R}}$ such that $a < b$. If $f : I \longmapsto \mathbb{R}$ is a monotone function on I, then:

1) *f has right and left limits in $\mathbb{R}$ at every point $x_0 \in]a, b[$ with $f(x_0^-) \leq f(x_0) \leq f(x_0^+)$.*

2) *The following limits exist in $\overline{\mathbb{R}}$:*

$$\ell_a = \lim_{x \to a^+} f(x) \quad \text{and} \quad \ell_b = \lim_{x \to b^-} f(x).$$

$\boxed{\text{Proof}}$ Suppose that $I = [a, b[$ with $a, b \in \mathbb{R}$ and that f is increasing on I (we make a similar proof for the other forms of I).

1) Let $x_0 \in]a, b[$.

- For all $x \in [a, x_0[$, $f(x) \leq f(x_0)$ since f is increasing on I. So f is bounded from above on the interval $[a, x_0[$, and therefore the set $f([a, x_0[)$ is a nonempty subset (since $[a, x_0[$ is nonempty) and bounded from above in $\mathbb{R}$. Therefore $f([a, x_0[)$ has a supremum α in $\mathbb{R}$. Let's show that $\lim\limits_{x \to x_0^-} f(x) = \alpha$. Indeed, let $\varepsilon > 0$, as $\alpha = \sup f([a, x_0[)$, then, by the supremum criterion, there exists $x_1 \in [a, x_0[$ such that
$$\alpha - \varepsilon < f(x_1) \leq \alpha.$$
Put $\delta = x_0 - x_1 > 0$. If $x \in I$, then
$$\begin{aligned} 0 < x_0 - x < \delta \quad &\Rightarrow \quad 0 < x_0 - x < x_0 - x_1 \\ &\Rightarrow \quad x > x_1 \\ &\Rightarrow \quad f(x) \geq f(x_1) \quad (\text{since } f \text{ is increasing on } I) \\ &\Rightarrow \quad f(x) > \alpha - \varepsilon \quad (\text{since } f(x_1) > \alpha - \varepsilon) \\ &\Rightarrow \quad \alpha - \varepsilon < f(x) < \alpha + \varepsilon \quad (\text{since } f(x) \leq \alpha < \alpha + \varepsilon) \\ &\Rightarrow \quad |f(x) - \alpha| < \varepsilon. \end{aligned}$$
Hence $\lim\limits_{x \to x_0^-} f(x) = \alpha$, i.e., f has a left limit at x_0 with $f(x_0^-) = \alpha$.

- For all $x \in]x_0, b[$, $f(x) \geq f(x_0)$ since f is increasing on I. So f is bounded from below on the interval $]x_0, b[$, and therefore the set $f(]x_0, b[)$ is a nonempty subset (since $]x_0, b[$ is nonempty) and bounded from below in $\mathbb{R}$. Therefore $f(]x_0, b[)$ has an infimum β in $\mathbb{R}$. Let's show that $\lim\limits_{x \to x_0^+} f(x) = \beta$. Indeed, let $\varepsilon > 0$, as $\beta = \inf f(]x_0, b[)$, then, by the infimum criterion, there exists $x_2 \in]x_0, b[$ such that
$$\beta \leq f(x_2) < \beta + \varepsilon.$$

Put $\delta = x_2 - x_0 > 0$. If $x \in I$, then

$$\begin{aligned} 0 < x - x_0 < \delta &\Rightarrow 0 < x - x_0 < x_2 - x_0 \\ &\Rightarrow x < x_2 \\ &\Rightarrow f(x) \leq f(x_2) \quad (\text{since } f \text{ is increasing on } I) \\ &\Rightarrow f(x) < \beta + \varepsilon \quad (\text{since } f(x_2) < \beta + \varepsilon) \\ &\Rightarrow \beta - \varepsilon < f(x) < \beta + \varepsilon \quad (\text{since } f(x) \geq \beta > \beta - \varepsilon) \\ &\Rightarrow |f(x) - \beta| < \varepsilon. \end{aligned}$$

Hence $\lim\limits_{x \to x_0^+} f(x) = \beta$, i.e., f has right limit at x_0 with $f(x_0^+) = \beta$.

In the other side, since $f(x) \leq f(x_0)$ for all $x \in [a, x_0[$ (since f is increasing on I), then $f(x_0)$ is an upper bound of $f([a, x_0[)$, and therefore $\alpha \leq f(x_0)$ since α is the least upper bound of $f([a, x_0[)$. Similarly, since $f(x_0) \leq f(x)$ for all $x \in]x_0, b[$ (since f is increasing on I), then $f(x_0)$ is a lower bound of $f(]x_0, b[)$, and therefore $f(x_0) \leq \beta$ since β is the greatest lower bound of $f(]x_0, b[)$. Hence

$$f(x_0^-) = \alpha \leq f(x_0) \leq \beta = f(x_0^+).$$

2) Limit at a: For all $x \in I$, $f(x) \geq f(a)$ since f is increasing on I. So f is bounded from below on the interval I, and therefore the set $f(I)$ is a nonempty subset (since I is nonempty) and bounded from below in $\mathbb{R}$. Therefore $f(I)$ has an infimum β in $\mathbb{R}$. Let's show that $\lim\limits_{x \to a^+} f(x) = \beta$. Indeed, let $\varepsilon > 0$, as $\beta = \inf f(I)$, then, by the infimum criterion, there exists $x_0 \in I$ such that

$$\beta \leq f(x_0) < \beta + \varepsilon.$$

Put $\delta = x_0 - a > 0$. If $x \in I$, then

$$\begin{aligned} 0 < x - a < \delta &\Rightarrow 0 < x - a < x_0 - a \\ &\Rightarrow x < x_0 \\ &\Rightarrow f(x) \leq f(x_0) \quad (\text{since } f \text{ is increasing on } I) \\ &\Rightarrow f(x) < \beta + \varepsilon \quad (\text{since } f(x_0) < \beta + \varepsilon) \\ &\Rightarrow \beta - \varepsilon < f(x) < \beta + \varepsilon \quad (\text{since } f(x) \geq \beta > \beta - \varepsilon) \\ &\Rightarrow |f(x) - \beta| < \varepsilon. \end{aligned}$$

Hence $\lim\limits_{x \to a^+} f(x) = \beta$, i.e., $\ell_a = \beta \in \mathbb{R}$.

Limit at b: We have two cases: f is not bounded from above on I or f is bounded from above on I.

- Suppose that f is not bounded from above on I, then the set $f(I)$ is not bounded from above in $\mathbb{R}$. Let's show that $\lim\limits_{x \to b^-} f(x) = +\infty$. Indeed, let $A > 0$, as A is not an upper bound of $f(I)$ (since $f(I)$ is not bounded from above in $\mathbb{R}$), then there exists $x_0 \in I$ such that $f(x_0) > A$. Put $\delta = b - x_0 > 0$, if $x \in I$, then

$$\begin{aligned} 0 < b - x < \delta &\Rightarrow 0 < b - x < b - x_0 \\ &\Rightarrow x > x_0 \\ &\Rightarrow f(x) \geq f(x_0) \quad \text{(since } f \text{ is increasing on } I) \\ &\Rightarrow f(x) > A \quad \text{(since } f(x_0) > A). \end{aligned}$$

Hence $\lim_{x \to b^-} f(x) = +\infty$.

- Suppose that f is bounded from above on I, then the set $f(I)$ is a nonempty subset (since I is nonempty) and bounded from above of $\mathbb{R}$, so $f(I)$ has a supremum α in $\mathbb{R}$. Let's show that $\lim_{x \to b^-} f(x) = \alpha$. Indeed, let $\varepsilon > 0$, as $\alpha = \sup f(I)$, then, by the supremum criterion, there exists $x_0 \in I$ such that

$$\alpha - \varepsilon < f(x_0) \leq \alpha.$$

Put $\delta = b - x_0 > 0$, if $x \in I$, then

$$\begin{aligned} 0 < b - x < \delta &\Rightarrow 0 < b - x < b - x_0 \\ &\Rightarrow x > x_0 \\ &\Rightarrow f(x) \geq f(x_0) \quad \text{(since } f \text{ is increasing on } I) \\ &\Rightarrow f(x) > \alpha - \varepsilon \quad \text{(since } f(x_0) > \alpha - \varepsilon) \\ &\Rightarrow \alpha - \varepsilon < f(x) < \alpha + \varepsilon \quad \text{(since } f(x) \leq \alpha < \alpha + \varepsilon) \\ &\Rightarrow |f(x) - \alpha| < \varepsilon. \end{aligned}$$

Hence $\lim_{x \to b^-} f(x) = \alpha$.

If f is decreasing on $I = [a, b[$, then $g = -f$ is increasing on I, so, by the above discussion, $\lim_{x \to b^-} g(x) \in \overline{\mathbb{R}}$. But $\lim_{x \to b^-} g(x) = -\lim_{x \to b^-} f(x)$, then $\lim_{x \to b^-} f(x) \in \overline{\mathbb{R}}$. □

Theorem 3.5.2 *The image of an interval by a continuous function is an interval, i.e., if f is a continuous function on an interval I of $\mathbb{R}$, then $f(I)$ is an interval. Moreover, if f is strictly monotone on I and if $a, b \in \overline{\mathbb{R}}$ are the extremities of the interval I such that $a < b$, then the extremities of the interval $f(I)$ are*

$$\ell_a = \lim_{x \to a^+} f(x) \in \overline{\mathbb{R}} \quad \text{and} \quad \ell_b = \lim_{x \to b^-} f(x) \in \overline{\mathbb{R}}.$$

More precisely,

- *If f is strictly increasing, then*

$$f([a, b]) = [f(a), f(b)], \qquad f(]a, b[) = \left] \lim_{x \to a^+} f(x), \lim_{x \to b^-} f(x) \right[$$

$$f([a, b[) = \left[f(a), \lim_{x \to b^-} f(x) \right[, \qquad f(]a, b]) = \left] \lim_{x \to a^+} f(x), f(b) \right]$$

$$f([a,+\infty[)=\left[f(a),\lim_{x\to+\infty}f(x)\right[,\qquad f(]a,+\infty[)=\left]\lim_{x\to a^+}f(x),\lim_{x\to+\infty}f(x)\right[$$

$$f(]-\infty,b])=\left]\lim_{x\to-\infty}f(x),f(b)\right],\qquad f(]-\infty,b[)=\left]\lim_{x\to-\infty}f(x),\lim_{x\to b^-}f(x)\right[.$$

- *If f is strictly decreasing, then*

$$f([a,b])=[f(b),f(a)],\qquad f(]a,b[)=\left]\lim_{x\to b^-}f(x),\lim_{x\to a^+}f(x)\right[$$

$$f([a,b[)=\left]\lim_{x\to b^-}f(x),f(a)\right],\qquad f(]a,b])=\left[f(b),\lim_{x\to a^+}f(x)\right[$$

$$f([a,+\infty[)=\left]\lim_{x\to+\infty}f(x),f(a)\right],\qquad f(]a,+\infty[)=\left]\lim_{x\to+\infty}f(x),\lim_{x\to a^+}f(x)\right[$$

$$f(]-\infty,b])=\left[f(b),\lim_{x\to-\infty}f(x)\right[,\qquad f(]-\infty,b[)=\left]\lim_{x\to b^-}f(x),\lim_{x\to-\infty}f(x)\right[.$$

Proof Let $c,d\in f(I)$ such that $c\le d$, then there exist $u,v\in I$ such that $c=f(u)$ and $d=f(v)$. Suppose that $u\le v$ (we make a similar proof if $v<u$). Since I is an interval, then $[u,v]\subseteq I$. As f is continuous on I, then f is continuous on $[u,v]$. If $c\le\beta\le d$, then $f(u)\le\beta\le f(v)$, and therefore there exists $\alpha\in[u,v]$ such that $\beta=f(\alpha)$ by the intermediate value theorem (Corollary 3.5.2). Then $\alpha\in I$, and therefore $\beta=f(\alpha)\in f(I)$. Thus $f(I)$ is an interval.

In the other side, we suppose that $I=[a,b[$ with $a,b\in\mathbb{R}$ and that f is strictly increasing on I (we make a similar proof for the other forms of I). By the proof of the lemma 3.5.1, $\ell_b=+\infty$ if f is not bounded from above on I or $\ell_b=\sup f(I)$ if f is bounded from above on I. Let's show that $f(I)=[f(a),\ell_b[$. Indeed, if $y\in f(I)$, then there exists $x\in I$ such that $y=f(x)$. As $x\ge a$ and f is strictly increasing on I, then $f(x)>f(a)$, so $y=f(x)\ge f(a)$. Moreover, $y<\ell_b$ since $\ell_b=+\infty$ or $\ell_b=\sup f(I)$. So $y\in[f(a),\ell_b[$, and therefore $f(I)\subseteq[f(a),\ell_b[$. Conversely, let $y\in[f(a),\ell_b[$.

- If $\ell_b=+\infty$ (i.e., f is not bounded from above on I), then y is not an upper bound of $f(I)$, so there exists $x_0\in I$ such that $y<f(x_0)<\ell_b$.
- If $\ell_b=\sup f(I)$ (i.e., f is bounded from above on I), then put $\varepsilon=\ell_b-y>0$. By the supremum criterion, there exists $x_0\in I$ such that

$$\ell_b-\varepsilon<f(x_0)<\ell_b\quad\text{i.e.,}\quad y<f(x_0)<\ell_b.$$

In the two cases, $f(a)\le y<f(x_0)$. As f is continuous on the interval $[a,x_0]$ (since f is continuous on I and $[a,x_0]\subseteq I$), then there exists $\alpha\in[a,x_0]\subseteq I$ such that $y=f(\alpha)$ by the intermediate value theorem (Corollary 3.5.2), therefore $y\in f(I)$. Hence $[f(a),\ell_b[\subseteq f(I)$. Thus $f(I)=[f(a),\ell_b[$.

If f is strictly decreasing on $I=[a,b[$, then $g=-f$ is continuous and strictly increasing on I, so, by the above discussion, $g(I)=[g(a),\ell'_b[$, where

$$g(a)=-f(a)\quad\text{and}\quad\ell'_b=\lim_{x\to b^-}g(x)=-\lim_{x\to b^-}f(x)=-\ell_b.$$

So

$$g(I) = \Big[- f(a), -\ell_b \Big[\ = - \Big] \ell_b, f(a) \Big]$$

But $g(I) = -f(I)$, then $f(I) = \Big] \ell_b, f(a) \Big]$. $\square$

Corollary 3.5.3 *(Generalization of the intermediate value theorem)*
Let I be an interval of $\mathbb{R}$ with extremities $a, b \in \overline{\mathbb{R}}$ such that $a < b$. If $f : I \longmapsto \mathbb{R}$ is a continuous and strictly monotone function on I, then

$$\ell_a = \lim_{x \to a^+} f(x) \quad \textit{and} \quad \ell_b = \lim_{x \to b^-} f(x) \quad \textit{exist in } \overline{\mathbb{R}},$$

and, for every real number β strictly between ℓ_a and ℓ_b, there exists one and only one $\alpha \in I$ such that $f(\alpha) = \beta$.

Proof Since f is a continuous and strictly monotone function on I, then $f(I)$ is an interval with extremities ℓ_a and ℓ_b by the theorem 3.5.2. If β is a real number strictly between ℓ_a and ℓ_b, then $\beta \in f(I)$, so there exists $\alpha \in I$ such that $f(\alpha) = \beta$. In the other side, as f is strictly monotone on I, then f is injective on I by the proposition 3.1.2. If there exists $\alpha' \in I$ such that $f(\alpha') = \beta$, then $f(\alpha') = f(\alpha)$, so $\alpha' = \alpha$ (since f is injective on I). Hence α is unique. $\square$

Remark 3.5.1 *The corollary 3.5.3 is also true if the condition on the strict monotonicity of f is replaced by the fact that ℓ_a and ℓ_b exist in $\overline{\mathbb{R}}$ (i.e., without using the lemma 3.5.1). Hence, if I is an interval of $\mathbb{R}$ with extremities $a, b \in \overline{\mathbb{R}}$ such that $a < b$ and if $f : I \longmapsto \mathbb{R}$ is a continuous function on I such that*

$$\ell_a = \lim_{x \to a^+} f(x) \quad \textit{and} \quad \ell_b = \lim_{x \to b^-} f(x) \quad \textit{exist in } \overline{\mathbb{R}},$$

then, for every real number β strictly between ℓ_a and ℓ_b, there exists $\alpha \in I$ such that $f(\alpha) = \beta$ (α is not necessary unique).

Exercise 84 (A result from algebra)
Prove that every real polynomial of odd degree admits at least a real root.

Solution Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a real polynomial of odd degree n ($a_n \neq 0$). Put

$$g(x) = \frac{1}{a_n} f(x) = x^n + \frac{a_{n-1}}{a_n} x^{n-1} \cdots + \frac{a_1}{a_n} x + \frac{a_0}{a_n}.$$

Since f is continuous on $\mathbb{R}$, then g is continuous on the interval $\mathbb{R} =]a, b[=] - \infty, +\infty[$. As n is odd, then

$$\ell_a = \lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} x^n = -\infty, \quad \ell_b = \lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} x^n = +\infty,$$

By the remark 3.5.1, for every $\beta \in] - \infty, +\infty[$, there exists $\alpha \in \mathbb{R}$ such that $g(\alpha) = \beta$. In particular, for $\beta = 0$, there exists $r \in \mathbb{R}$ such that $g(r) = 0$, so $f(r) = 0$, i.e., r is a real root of the polynomial $f(x)$. $\square$

Lemma 3.5.2 *Every surjective monotone function from an interval onto another interval is continuous.*

Proof Let f be a surjective monotone function from an interval I onto another interval J. Suppose that f is increasing on I (we make a similar proof when f is decreasing on I). Suppose that f is not continuous on I, then there exists $c \in I$ such that f is not continuous at c. Since f is increasing on I, then f has at c a left limit $f(c^-)$ and a right limit $f(c^+)$ (or only one of the two limits if c is an extremity of the interval I) by the lemma 3.5.1. As f is not continuous at c, then $f(c^-) < f(c)$ or $f(c) < f(c^+)$.

- Suppose that $f(c^-) < f(c)$. Then, for any $x \in I$, $f(x) \leq f(c^-)$ if $x < c$ and $f(x) \geq f(c)$ if $x \geq c$ (since f is increasing on I). So, for all $x \in I$,
$$f(x) \notin \left]f(c^-), f(c)\right[,$$
i.e., the elements of $]f(c^-), f(c)[$ have no preimages by f in I, this contradicts the hypothesis that f is surjective since $]f(c^-), f(c)[\subseteq J$ (from the fact that J is an interval).

- Suppose that $f(c) < f(c^+)$. Then, for any $x \in I$, $f(x) \leq f(c)$ if $x \leq c$ and $f(x) \geq f(c^+)$ if $x > c$ (since f is increasing on I). So, for all $x \in I$,
$$f(x) \notin \left]f(c), f(c^+)\right[,$$
i.e., the elements of $]f(c), f(c^+)[$ have no preimages by f in I, this contradicts the hypothesis that f is surjective since $]f(c), f(c^+)[\subseteq J$ (from the fact that J is an interval).

Thus f is continuous on I. □

Theorem 3.5.3 *(Inverse functions theorem)*
If f is a **continuous** *and* **strictly monotone** *function on an interval I of $\mathbb{R}$, then f is a bijection from I on the interval $J = f(I)$. Moreover, the inverse mapping f^{-1} of f is a bijective,* **continuous** *and* **strictly monotone** *function from J on I with the same monotonicity as that of f on I. The representative curve of f^{-1} on J is the symmetric with respect to the first bissector of that of f on I.*

Proof It is obvious that f is a surjection from I onto $J = f(I)$. Moreover, since f is continuous on the interval I, then $f(I)$ is an interval by the theorem 3.5.2. In the other side, as f is strictly monotone on I, then f is injective by the proposition 3.1.2. Therefore f is a bijection from I on J. Hence the inverse mapping f^{-1} of f exists and it is a bijective function from J on I.

- Let's show that f^{-1} is strictly monotone on J with the same monotonicity as that of f on I. Indeed, suppose that f is strictly increasing on I (we make a similar proof when f is strictly decreasing on I). Let $y_1, y_2 \in J$ such that $y_1 < y_2$, then there exist $x_1, x_2 \in I$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$, and therefore $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. Suppose, by contradiction, that $f^{-1}(y_1) \geq f^{-1}(y_2)$, then $x_1 \geq x_2$, so $f(x_1) \geq f(x_2)$ (since f is strictly increasing on I), therefore $y_1 \geq y_2$, this contradicts the hypothesis. Hence $f^{-1}(y_1) < f^{-1}(y_2)$. Thus f^{-1} is strictly increasing on J.

- Since f^{-1} is a surjective monotone function from the interval J onto the interval I, then f^{-1} is continuous on J by the lemma 3.5.2.

In the other side, let C_f (resp. $C_{f^{-1}}$) the representative curve of f on I (resp. of f^{-1} on $f(I)$). Let $(x, y) \in \mathbb{R}^2$, then

$$(x, y) \in C_{f^{-1}} \Leftrightarrow y = f^{-1}(x) \Leftrightarrow x = f(y) \Leftrightarrow (y, x) \in C_f.$$

So $C_{f^{-1}}$ is the symmetric of C_f with respect to the first bissector. □

Example 3.5.2

1) *The function f defined by $f(x) = 1 - x^2$ is continuous and strictly decreasing on the interval $\mathbb{R}^+$. Then, by the inverse functions theorem, f admits an inverse function f^{-1} defined, continuous and strictly decreasing on the interval*

$$f(\mathbb{R}^+) = f([0, +\infty[) = \left]\lim_{x\to+\infty} f(x), f(0)\right] =]-\infty, 1].$$

Hence $f^{-1} :]-\infty, 1] \longmapsto \mathbb{R}^+$ defined by, for every $x \in]-\infty, 1]$, by:

$$f^{-1}(x) = y \Leftrightarrow f(y) = x \Leftrightarrow 1 - y^2 = x \Leftrightarrow y = \sqrt{1-x} \quad (\textit{since } y \geq 0).$$

So $f^{-1}(x) = \sqrt{1-x}$.

2) *The function f defined by $f(x) = xe^x$ is continuous and strictly increasing on the interval $\mathbb{R}^+$ (as it is the product of two strictly increasing functions on $\mathbb{R}^+$ with positive values). Then, by the inverse functions theorem, f admits an inverse function f^{-1} defined, continuous and strictly increasing on the interval*

$$f(\mathbb{R}^+) = f([0, +\infty[) = \left[f(0), \lim_{x\to+\infty} f(x)\right[= [0, +\infty[= \mathbb{R}^+.$$

Hence $f^{-1} : \mathbb{R}^+ \longmapsto \mathbb{R}^+$ defined by $f^{-1}(x) = y$ if and only if $x = f(y) = ye^y$. But, from the relation $x = ye^y$, we cannot express y in terms of x by using basic functions, this yields to define a new function f^{-1}, known as Lambert's function W.

Exercise 85

Prove that the function f defined by $f(x) = \dfrac{2x+1}{x-1}$ admits an inverse function on the interval $I = [2, +\infty[$ to be determined.

Solution The function f is continuous on the interval I since its denominator does not vanish at any point of I. Moreover, f is strictly decreasing on I by the exercise 70, so, by the inverse functions theorem, f admits an inverse function f^{-1} defined, continuous and strictly decreasing on the interval

$$f(I) = f([2, +\infty[) = \left]\lim_{x\to+\infty} f(x), f(2)\right] =]2, 5].$$

Hence $f^{-1} :]2, 5] \longmapsto [2, +\infty[$ defined, for every $x \in]2, 5]$, by:

$$f^{-1}(x) = y \Leftrightarrow f(y) = x \Leftrightarrow \frac{2y+1}{y-1} = x \Leftrightarrow y = \frac{x+1}{x-2}$$

So $f^{-1}(x) = \dfrac{x+1}{x-2}$. □

Remark 3.5.2 *Let f be a bijection from an interval I on an interval J of $\mathbb{R}$. Let $x_0 \in \overline{\mathbb{R}}$ ($x_0 \in I$ or x_0 is an extremity of I).*
If $\lim\limits_{x \to x_0} f(x) = \ell \in \overline{\mathbb{R}}$ and $\lim\limits_{y \to \ell} f^{-1}(y) \in \overline{\mathbb{R}}$, then

$$\lim_{y \to \ell} f^{-1}(y) = x_0.$$

Indeed, since $f^{-1} \circ f = id_I$, then $f^{-1}(f(x)) = x$ for all $x \in I$.
Since $\lim\limits_{x \to x_0} f(x) = \ell$ and $\lim\limits_{y \to \ell} f^{-1}(y) \in \overline{\mathbb{R}}$, then, by using the theorem 3.2.1,

$$\lim_{y \to \ell} f^{-1}(y) = \lim_{x \to x_0} f^{-1}(f(x)) = \lim_{x \to x_0} x = x_0.$$

In the exercise 85, we have:

$$\lim_{x \to +\infty} f(x) = 2 \quad \text{and} \quad \lim_{y \to 2} f^{-1}(y) = +\infty.$$

Exercise 86
Let $n \in \mathbb{N}^*$. Consider the function f_n defined on the interval $I = [0, +\infty[$ by:

$$f_n(x) = e^{-x} - \frac{x-n}{x+n}.$$

1) **Show that f_n is a bijection from I on an interval J to be determined.**

2) **Deduce that the equation $f_n(x) = 0$ admits a unique solution in I, we denote it a_n.**

3) **Study the monotonicity of the inverse function f_n^{-1} of f_n.**

4) **Verify that $f_n(n) > 0$, and deduce that $a_n > n$. Deduce the nature of the sequence $(a_n)_{n \geq 1}$.**

Solution

1) For any $x \in I$, $f_n(x) = g(x) + h(x)$ where $g(x) = e^{-x}$ and $h(x) = \dfrac{n-x}{n+x}$.
The function g is continuous and strictly decreasing on I as it is the composite of the exponential function (continuous and strictly increasing on $\mathbb{R}$) and the function $x \mapsto -x$ (continuous and strictly decreasing on I). In the other side, if $a, b \in I$ such that $a < b$, then

$$h(a) - h(b) = \frac{n-a}{n+a} - \frac{n-b}{n+b} = \frac{2n(b-a)}{(n+a)(n+b)} > 0$$

So $h(a) > h(b)$. Therefore h is strictly decreasing on I. Moreover, as the denominator of $h(x)$ does not vanish at any point of I (since $-n \notin I$), then h is continuous on I. Thus f_n is continuous and strictly decreasing on the interval I as it is the sum of two functions g and h which are continuous and strictly decreasing on I. By the inverse functions theorem, f_n is a bijection from I on the interval

$$J = f_n(I) = f_n([0,+\infty[) = \left]\lim_{x\to+\infty} f_n(x), f_n(0)\right] =]-1, 2].$$

2) Since $0 \in J$ and f_n is a bijection from I on J, then there exists one and only one $a_n \in I$ such that $f_n(a_n) = 0$, i.e., the equation $f_n(x) = 0$ admits a unique solution in I.

3) Since f_n is strictly decreasing on I, then f_n^{-1} is strictly decreasing on J by the inverse functions theorem.

4) We have:

$$(*) \qquad f_n(n) = e^{-n} > 0 = f_n(a_n).$$

As f_n^{-1} is strictly decreasing on J, then, applying f_n^{-1} on $(*)$, we obtain $a_n > n$ (since $f_n^{-1} \circ f_n = id_I$). In the other side, since $\lim\limits_{n\to+\infty} n = +\infty$, then $\lim\limits_{n\to+\infty} a_n = +\infty$ (the sequence $(a_n)_{n\geq 1}$ is divergent). □

3.6 Some basic functions and their inverses

3.6.1 The n^{th} power function and the n^{th} root function

Let $n \in \mathbb{N}^*$ and f be the function defined by $f(x) = x^n$.

- Suppose that n is **even**. The function f is continuous and strictly increasing on the interval $\mathbb{R}^+$ (by the example 3.1.6). Then, by the inverse functions theorem, f admits an inverse function f^{-1} defined, continuous and strictly increasing on the interval

$$f(\mathbb{R}^+) = f([0,+\infty[) = \left[f(0), \lim_{x\to+\infty} f(x)\right[= [0,+\infty[= \mathbb{R}^+.$$

Hence $f^{-1} : \mathbb{R}^+ \longmapsto \mathbb{R}^+$ defined by $f^{-1}(x) = y$ if and only if $x = f(y) = y^n$. The function f^{-1} is said to be the positive n^{th} root function. We write

$$f^{-1}(x) = \sqrt[n]{x} = x^{\frac{1}{n}} \quad \text{for all } x \in \mathbb{R}^+.$$

In particular, for $n = 2$, f^{-1} is the square root function $\sqrt{\cdot}$.
For any $x, y \in \mathbb{R}^+$, we have the following properties:

i)

$$\boxed{\sqrt[n]{x} = y \Leftrightarrow y^n = x}$$

In particular, $\sqrt[n]{x} = 0 \Leftrightarrow x = 0$.

ii) As $\boldsymbol{f \circ f^{-1} = f^{-1} \circ f = id_{\mathbb{R}^+}}$, then

$$\boxed{(x^n)^{\frac{1}{n}} = \sqrt[n]{x^n} = x} \quad \text{and} \quad \boxed{(x^{\frac{1}{n}})^n = (\sqrt[n]{x})^n = x}$$

iii) By using the part (ii), we obtain:

$$(\sqrt[n]{x}\ \sqrt[n]{y})^n = (\sqrt[n]{x})^n (\sqrt[n]{y})^n = xy.$$

As $\sqrt[n]{x}\ \sqrt[n]{y} \in \mathbb{R}^+$ and $xy \in \mathbb{R}^+$, then, by the part (i)

$$\boxed{\sqrt[n]{xy} = \sqrt[n]{x}\ \sqrt[n]{y}}$$

iv) Suppose that $y \neq 0$, then put $z = \frac{x}{y} \in \mathbb{R}^+$, so $x = yz$. By the part (iii),

$$\sqrt[n]{x} = \sqrt[n]{yz} = \sqrt[n]{y}\ \sqrt[n]{z}.$$

But, by the part (i), $\sqrt[n]{y} \neq 0$ (since $y \neq 0$), then $\sqrt[n]{z} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$. Hence

$$\boxed{\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}}$$

v) For every $k \in \mathbb{N}$, we define:

$$\boxed{x^{\frac{k}{n}} = \left(x^{\frac{1}{n}}\right)^k = (\sqrt[n]{x})^k}$$

Since

$$\left[x^{\frac{k}{n}}\right]^n = \left[(\sqrt[n]{x})^k\right]^n = (\sqrt[n]{x})^{kn} = \left[(\sqrt[n]{x})^n\right]^k = x^k,$$

and as $x^{\frac{k}{n}} \in \mathbb{R}^+$ and $x^k \in \mathbb{R}^+$, then, by the part (i),

$$\boxed{x^{\frac{k}{n}} = \sqrt[n]{x^k} = \left(x^k\right)^{\frac{1}{n}}}$$

vi) If $x \neq 0$, then $\sqrt[n]{x} \neq 0$, and therefore, for every $k \in \mathbb{N}^*$, we define:

$$\boxed{x^{\frac{-k}{n}} = \left(x^{\frac{1}{n}}\right)^{-k} = (\sqrt[n]{x})^{-k} = \tfrac{1}{(\sqrt[n]{x})^k}}$$

By using the parts (iv) and (v) and the fact that $\sqrt[n]{1} = 1$, we obtain:

$$\frac{1}{(\sqrt[n]{x})^k} = \frac{\sqrt[n]{1}}{\sqrt[n]{x^k}} = \sqrt[n]{\frac{1}{x^k}} = \sqrt[n]{x^{-k}}.$$

So

$$\boxed{x^{\frac{-k}{n}} = \sqrt[n]{x^{-k}} = \left(x^{-k}\right)^{\frac{1}{n}}}$$

vii) If $x \neq 0$, then, by grouping the parts (v) and (vi), we obtain:

$$\boxed{x^{\frac{k}{n}} = (\sqrt[n]{x})^k = \sqrt[n]{x^k}, \quad \forall\, k \in \mathbb{Z}}$$

viii) Let $m \in \mathbb{N}^*$ be an **even** integer. By using the part (ii) and the fact that $\sqrt[n]{x} \in \mathbb{R}^+$, we obtain:

$$\left(\sqrt[m]{\sqrt[n]{x}}\right)^{mn} = \left[\left(\sqrt[m]{\sqrt[n]{x}}\right)^{m}\right]^{n} = (\sqrt[n]{x}\,)^n = x.$$

As mn is even, $\sqrt[m]{\sqrt[n]{x}} \in \mathbb{R}^+$ and $x \in \mathbb{R}^+$, then, by the part (i),

$$\boxed{\sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x}}$$

- Suppose that n is **odd**. The function f is continuous and strictly increasing on the interval $\mathbb{R}$ (by the example 3.1.6). Then, by the inverse functions theorem, f admits an inverse function f^{-1} defined, continuous and strictly increasing on the interval

$$f(\mathbb{R}) = f(\,]-\infty, +\infty[) = \left]\lim_{x\to-\infty} f(x), \lim_{x\to+\infty} f(x)\right[= \,]-\infty, +\infty[\, = \mathbb{R}.$$

Hence $f^{-1} : \mathbb{R} \longmapsto \mathbb{R}$ defined by $f^{-1}(x) = y$ if and only if $x = f(y) = y^n$. The function f^{-1} is said to be the n^{th} root function. We write

$$f^{-1}(x) = \sqrt[n]{x} = x^{\frac{1}{n}} \quad \text{for all } x \in \mathbb{R}.$$

All the above properties from (i) to (viii) are also true for all $x, y \in \mathbb{R}$.

Exercise 87
Let $n \in \mathbb{N}^*$ and $a \in \mathbb{R}$. Solve, in $\mathbb{R}$, the equation $x^n = a$.

Solution

- Suppose that n is even. If $a < 0$, then the equation $x^n = a$ has no solutions in $\mathbb{R}$ since $x^n \geq 0$ for all $x \in \mathbb{R}$. Suppose that $a \geq 0$ (i.e., $a \in \mathbb{R}^+$). If $x_0 \in \mathbb{R}$ is a solution of the equation $x^n = a$, then $x_0^n = a$, and therefore, as n is even,

$$|x_0|^n = x_0^n = a.$$

As $|x_0| \in \mathbb{R}^+$ and $a \in \mathbb{R}^+$, then $|x_0| = \sqrt[n]{a}$, and therefore $x_0 = \pm\sqrt[n]{a}$. Conversely, as n is even, we have:

$$(\pm\sqrt[n]{a})^n = a.$$

Hence the set of real solutions of the equation $x^n = a$ is $S = \{-\sqrt[n]{a}, \sqrt[n]{a}\}$.

- Suppose that n is odd. If $x_0 \in \mathbb{R}$ is a solution of the equation $x^n = a$, then $x_0^n = a$, and therefore $x_0 = \sqrt[n]{a}$ (n is odd). Conversely, $(\sqrt[n]{a})^n = a$. Hence the set of real solutions of the equation $x^n = a$ is $S = \{\sqrt[n]{a}\}$. □

3.6.2 The rational power function

Lemma 3.6.1 *Let $p, p' \in \mathbb{Z}$ and $q, q' \in \mathbb{N}^*$. If $\dfrac{p}{q} = \dfrac{p'}{q'}$, then*

$$\boxed{x^{\frac{p}{q}} = x^{\frac{p'}{q'}}, \quad \forall x \in]0, +\infty[}$$

$\boxed{\text{Proof}}$ Let $x \in]0, +\infty[$. By the part (vii) above (cases even and odd), we have:

$$x^{\frac{p}{q}} = \sqrt[q]{x^p} \quad \text{and} \quad x^{\frac{p'}{q'}} = \sqrt[q']{x^{p'}}.$$

As $\dfrac{p}{q} = \dfrac{p'}{q'}$, then $pq' = p'q$. So

$$\left(\sqrt[q]{x^p}\right)^{q'} = \sqrt[q]{x^{pq'}} = \sqrt[q]{x^{p'q}} = \left(\sqrt[q]{x^q}\right)^{p'} = x^{p'}.$$

Hence $\sqrt[q]{x^p} = \sqrt[q']{x^{p'}}$, i.e., $x^{\frac{p}{q}} = x^{\frac{p'}{q'}}$. □

The lemma 3.6.1 shows that, if $x > 0$, then the power $x^{\frac{p}{q}}$ is independent of the choice of the representative of the rational number $\frac{p}{q}$, this justifies the following definition:

Definition 3.6.1 *Let $r = \frac{p}{q} \in \mathbb{Q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}^*$. For every $x \in]0, +\infty[$, we define the power*

$$x^r = x^{\frac{p}{q}} = \sqrt[q]{x^p} = \left(\sqrt[q]{x}\right)^p.$$

Proposition 3.6.1 *If $r, r' \in \mathbb{Q}$ and $x, y \in]0, +\infty[$, then we have the following properties:*

1) $(xy)^r = x^r y^r$.

2) $\left(\dfrac{x}{y}\right)^r = \dfrac{x^r}{y^r}$.

3) $x^{-r} = \dfrac{1}{x^r}$.

4) $x^{r+r'} = x^r x^{r'}$.

5) $(x^r)^{r'} = x^{rr'}$.

$\boxed{\text{Proof}}$ Since $r, r' \in \mathbb{Q}$, then there exist $p, p' \in \mathbb{Z}$ and $q, q' \in \mathbb{N}^*$ such that $r = \dfrac{p}{q}$ and $r' = \dfrac{p'}{q'}$, then

1) We have:

$$(xy)^r = \sqrt[q]{(xy)^p} = \sqrt[q]{x^p y^p} = \sqrt[q]{x^p}\sqrt[q]{y^p} = x^r y^r.$$

2) We have:

$$\left(\frac{x}{y}\right)^r = \sqrt[q]{\left(\frac{x}{y}\right)^p} = \sqrt[q]{\frac{x^p}{y^p}} = \frac{\sqrt[q]{x^p}}{\sqrt[q]{y^p}} = \frac{x^r}{y^r}.$$

3) We have:

$$x^{-r} = x^{\frac{-p}{q}} = \sqrt[q]{x^{-p}} = \sqrt[q]{\frac{1}{x^p}} = \frac{\sqrt[q]{1}}{\sqrt[q]{x^p}} = \frac{1}{x^r}.$$

4) We have:

$$\begin{aligned} x^{r+r'} &= x^{\frac{p}{q}+\frac{p'}{q'}} = x^{\frac{pq'+p'q}{qq'}} = \sqrt[qq']{x^{pq'+p'q}} = \sqrt[qq']{x^{pq'}x^{p'q}} = \sqrt[qq']{x^{pq'}}\,\sqrt[qq']{x^{p'q}} \\ &= \sqrt[q]{\sqrt[q']{(x^p)^{q'}}}\,\sqrt[q']{\sqrt[q]{(x^{p'})^q}} = \sqrt[q]{x^p}\,\sqrt[q']{x^{p'}} = x^r x^{r'}. \end{aligned}$$

5) We have:

$$\begin{aligned} x^{rr'} &= x^{\left(\frac{p}{q}\right)\left(\frac{p'}{q'}\right)} = x^{\frac{pp'}{qq'}} = \sqrt[qq']{x^{pp'}} \\ &= \sqrt[q']{\sqrt[q]{(x^p)^{p'}}} = \sqrt[q']{\left(\sqrt[q]{x^p}\right)^{p'}} = \left(\sqrt[q]{x^p}\right)^{\frac{p'}{q'}} = (x^r)^{r'}. \quad \square \end{aligned}$$

3.6.3 The logarithm of base a and the exponential of base a

The natural logarithm: We admit that the natural logarithm function $\ln$ is continuous and strictly increasing on the interval $]0,+\infty[$, then it admits, by the inverse functions theorem, an inverse function $\ln^{-1}$ defined, continuous and strictly increasing on the interval

$$\ln\Big(]0,+\infty[\Big) = \Big]\lim_{x\to 0^+}\ln x, \lim_{x\to+\infty}\ln x\Big[= \Big]-\infty,+\infty\Big[= \mathbb{R}.$$

Hence $\ln^{-1} : \mathbb{R} \longmapsto]0,+\infty[$ defined by $\ln^{-1}(x) = y$ if and only if $x = \ln y$. The function $\ln^{-1}$ is called the exponential function. We write

$$\ln^{-1}(x) = e^x = \exp(x) \quad \text{for all } x \in \mathbb{R}.$$

Thus

$$\boxed{\ln(e^x) = x,\ \forall\, x \in \mathbb{R}} \quad \text{and} \quad \boxed{e^{\ln x} = x,\ \forall\, x \in]0,+\infty[}$$

Hence, for all $x \in \mathbb{R}$ and $y \in]0,+\infty[$,

$$\boxed{e^x = y \Leftrightarrow \ln y = x}$$

In particular, $e^0 = 1$ since $\ln 1 = 0$.

The logarithm of base a: Let a be a strictly positive real number such that $a \neq 1$. The function $\log_a$, called logarithm of base a, defined by $\log_a x = \dfrac{\ln x}{\ln a}$, is continuous and strictly monotone (increasing if $a > 1$ and decreasing if $a < 1$) on the interval $]0,+\infty[$, then it admits, by the inverse functions theorem, an inverse function $\log_a^{-1}$ defined, continuous and strictly monotone (increasing if $a > 1$ and decreasing if $a < 1$) on the interval

$$\log_a\Big(\,]0,+\infty[\Big) = \begin{cases} \Big]\lim\limits_{x\to 0^+}\log_a x,\ \lim\limits_{x\to+\infty}\log_a x\Big[& \text{if } a>1 \\ \\ \Big]\lim\limits_{x\to +\infty}\log_a x,\ \lim\limits_{x\to 0^+}\log_a x\Big[& \text{if } a<1 \end{cases}$$
$$= \Big]-\infty,+\infty\Big[= \mathbb{R}.$$

Hence $\log_a^{-1} : \mathbb{R} \longmapsto]0,+\infty[$ defined by $\log_a^{-1}(x) = y$ if and only if $x = \log_a y$. The function $\log_a^{-1}$ is called the exponential function of base a, it will be denoted by $\exp_a$. Remark that if $n \in \mathbb{N}$, then

$$\log_a a^n = n\log_a a = n,$$

so $\exp_a(n) = a^n$, this justifies the notation

$$\boxed{a^x = \exp_a(x), \quad \forall x \in \mathbb{R}}$$

Thus

$$\boxed{\log_a(a^x) = x,\ \forall x \in \mathbb{R}} \quad \text{and} \quad \boxed{a^{\log_a x} = x,\ \forall x \in]0,+\infty[}$$

Hence, for all $x \in \mathbb{R}$ and $y \in]0,+\infty[$,

$$\boxed{a^x = y \Leftrightarrow \log_a y = x}$$

For any $x \in \mathbb{R}$, as

$$\log_a(e^{x\ln a}) = \frac{\ln(e^{x\ln a})}{\ln a} = \frac{x\ln a}{\ln a} = x,$$

then $\boxed{a^x = e^{x\ln a}}$. Moreover, for every $x \in \mathbb{R}$, put $\boxed{1^x = 1}$.
In particular, if $a = e \simeq 2,718$ (called Euler's number or Neper's number), then $\log_e$ is the natural logarithm $\ln$, and therefore $\exp_e$ is the exponential function $\exp$.

In the other side, by using the remark 3.5.2 and the references limits of the function $\log_a$ (proposition 3.2.5), we obtain:

$$\boxed{\lim_{x\to-\infty} a^x = \begin{cases} 0 & \text{if } a>1 \\ +\infty & \text{if } a<1 \end{cases}} \quad \text{and} \quad \boxed{\lim_{x\to+\infty} a^x = \begin{cases} +\infty & \text{if } a>1 \\ 0 & \text{if } a<1 \end{cases}}$$

Proposition 3.6.2 *For any strictly positive real number a, $a^0 = 1$ and, for all $x, y \in \mathbb{R}$,*

$$\boxed{a^{x+y} = a^x a^y} \quad \text{and} \quad \boxed{(a^x)^y = a^{xy}}$$

Proof The proposition is trivially satisfied when $a = 1$ since, by definition, $1^x = 1$ for all $x \in \mathbb{R}$. Suppose that a is a strictly positive real number such that $a \neq 1$. As $\log_a 1 = 0$, then $a^0 = 1$. Let $x, y \in \mathbb{R}$, since

$$\log_a(a^x a^y) = \log_a(a^x) + \log_a(a^y) = x + y,$$

then $a^{x+y} = a^x a^y$. In the other side, put $b = a^x > 0$.

- If $\boldsymbol{x = 0}$, then $\boldsymbol{b = a^0 = 1}$, so

$$\boldsymbol{(a^x)^y = b^y = 1^y = 1 = a^0 = a^{xy}.}$$

- Suppose that $\boldsymbol{x \neq 0}$, then $\boldsymbol{b \neq 1}$. Since

$$\boldsymbol{\log_b(a^{xy}) = \frac{\ln(a^{xy})}{\ln b} = \frac{\ln(a^{xy})}{\ln a} \times \frac{\ln a}{\ln b} = \log_a(a^{xy}) \times \frac{1}{\log_a b} = \frac{xy}{x} = y,}$$

then

$$\boldsymbol{(a^x)^y = b^y = a^{xy}.} \quad \square$$

Chapter 4

Differentiability of a function

4.1 Definitions and properties

Definition 4.1.1 *Let* $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ *and* $x_0 \in D_f$.

1) *We say that* f *is differentiable at* x_0 *if*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{exists in } \mathbb{R}.$$

We denote then this limit $f'(x_0)$ *or* $\frac{df}{dx}(x_0)$ *or* $\left.\frac{df}{dx}\right|_{x=x_0}$, *and we call it the derivative of* f *at the point* x_0 *or the derivative number of* f *at the point* x_0.

2) *We say that* f *is differentiable on a nonempty subset* E *of* D_f *if it is differentiable at every point of* E.

3) *We say that* f *is right differentiable at* x_0 *if*

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{exists in } \mathbb{R}.$$

We denote then $f'_+(x_0)$ *(or* $f'_r(x_0)$*) this limit, and we call it the right derivative of* f *at the point* x_0.

4) *We say that* f *is left differentiable at* x_0 *if*

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{exists in } \mathbb{R}.$$

We denote then $f'_-(x_0)$ *(or* $f'_l(x_0)$*) this limit, and we call it the left derivative of* f *at the point* x_0.

5) *We say that* f *is differentiable on an interval* $[a, b]$ *contained in* D_f *if it is differentiable on* $]a, b[$, *right differentiable at* a *and left differentiable at* b. *In a similar way, we define the differentiability of* f *on the other forms of intervals:*

$$[a, b[, \quad]a, b], \quad [a, +\infty[, \quad]a, +\infty[, \quad]-\infty, b], \quad]-\infty, b[.$$

Remark 4.1.1 *Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ and $x_0 \in D_f$.*

1) *By using the proposition 3.2.3, f is differentiable at x_0 if and only if f is right and left differentiable at x_0 and $f'_+(x_0) = f'_-(x_0)$.*

2) *By using the change of variable $h = x - x_0$ (i.e., $x = x_0 + h$), we obtain:*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Example 4.1.1

1) *Let $f(x) = x$ and $x_0 \in \mathbb{R}$. Since*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{x - x_0}{x - x_0} = 1 \in \mathbb{R},$$

then f is differentiable at x_0 and $f'(x_0) = 1$. Hence f is differentiable on $\mathbb{R}$.

2) *Let $f(x) = x^2 + 3x + 1$ and $x_0 \in \mathbb{R}$. Since*

$$\begin{aligned} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} &= \lim_{x \to x_0} \frac{x^2 + 3x + 1 - x_0^2 - 3x_0 - 1}{x - x_0} \\ &= \lim_{x \to x_0} \frac{(x - x_0)(x + x_0) + 3(x - x_0)}{x - x_0} \\ &= \lim_{x \to x_0} (x + x_0 + 3) = 2x_0 + 3 \in \mathbb{R}, \end{aligned}$$

then f is differentiable at x_0 and $f'(x_0) = 2x_0 + 3$. Hence f is differentiable on $\mathbb{R}$.

3) *Let $f(x) = \sin x$. For any $x_0 \in \mathbb{R}$, we have:*

$$\begin{aligned} \lim_{x \to x_0} \frac{\sin x - \sin x_0}{x - x_0} &= \lim_{x \to x_0} \frac{2 \sin\left(\frac{x - x_0}{2}\right) \cos\left(\frac{x + x_0}{2}\right)}{x - x_0} \\ &= \left(\lim_{x \to x_0} \frac{\sin\left(\frac{x - x_0}{2}\right)}{\frac{x - x_0}{2}} \right) \lim_{x \to x_0} \cos\left(\frac{x + x_0}{2}\right) \\ &= \cos x_0, \end{aligned}$$

by using the fact that $\lim_{u \to 0} \frac{\sin u}{u} = 1$ and the function $\cos$ is continuous at x_0. So f is differentiable at x_0 and $f'(x_0) = \cos x_0$. Hence f is differentiable on $\mathbb{R}$.

4) *Let $f(x) = \cos x$. For any $x_0 \in \mathbb{R}$, we have:*

$$\begin{aligned} \lim_{x \to x_0} \frac{\cos x - \cos x_0}{x - x_0} &= \lim_{x \to x_0} \frac{-2 \sin\left(\frac{x - x_0}{2}\right) \sin\left(\frac{x + x_0}{2}\right)}{x - x_0} \\ &= -\left(\lim_{x \to x_0} \frac{\sin\left(\frac{x - x_0}{2}\right)}{\frac{x - x_0}{2}} \right) \lim_{x \to x_0} \sin\left(\frac{x + x_0}{2}\right) \\ &= -\sin x_0. \end{aligned}$$

So f is differentiable at x_0 and $f'(x_0) = -\sin x_0$. Hence f is differentiable on $\mathbb{R}$.

5) *Let* $f(x) = \sqrt{x}$. *If* $x_0 > 0$, *then*

$$\begin{aligned}\lim_{x\to x_0} \frac{\sqrt{x}-\sqrt{x_0}}{x-x_0} &= \lim_{x\to x_0} \frac{(\sqrt{x}-\sqrt{x_0})(\sqrt{x}+\sqrt{x_0})}{(x-x_0)(\sqrt{x}+\sqrt{x_0})} = \lim_{\substack{x\to x_0\\ x\neq x_0}} \frac{x-x_0}{(x-x_0)(\sqrt{x}+\sqrt{x_0})}\\ &= \lim_{x\to x_0} \frac{1}{\sqrt{x}+\sqrt{x_0}} = \frac{1}{2\sqrt{x_0}} \in \mathbb{R}.\end{aligned}$$

So f *is differentiable at* x_0 *and* $f'(x_0) = \frac{1}{2\sqrt{x_0}}$. *In the other side,*

$$\lim_{x\to 0^+} \frac{\sqrt{x}-\sqrt{0}}{x-0} = \lim_{x\to 0^+} \frac{1}{\sqrt{x}} = +\infty.$$

So f *is not right differentiable at* 0, *and therefore* f *is not differentiable at* 0. *Hence* f *is differentiable only on* $]0, +\infty[$.

6) *Let* $f(x) = |x|$. *Since*

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{\substack{x\to 0\\ x>0}} \frac{|x|-|0|}{x} = \lim_{x\to 0} \frac{x}{x} = 1 \in \mathbb{R}$$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{\substack{x\to 0\\ x<0}} \frac{|x|-|0|}{x} = \lim_{x\to 0} \frac{-x}{x} = -1 \in \mathbb{R},$$

then f *is right and left differentiable at* 0 *with* $f'_+(0) = 1$ *and* $f'_-(0) = -1$. *In the other side, as* $f'_+(0) \neq f'_-(0)$, *then* f *is not differentiable at* 0.

7) *Let* $f(x) = \frac{1}{x}$. *If* $x_0 \neq 0$, *then*

$$\begin{aligned}\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} &= \lim_{x\to x_0} \frac{\frac{1}{x}-\frac{1}{x_0}}{x-x_0} = \lim_{\substack{x\to x_0\\ x\neq x_0}} \frac{x_0-x}{xx_0(x-x_0)}\\ &= \lim_{x\to x_0} \frac{-1}{xx_0} = -\frac{1}{x_0^2} \in \mathbb{R}.\end{aligned}$$

So f *is differentiable at* x_0 *and* $f'(x_0) = -\dfrac{1}{x_0^2}$. *Hence* f *is differentiable on* $\mathbb{R}^*$.

8) *Let* f *be the function defined by:*

$$f(x) = \begin{cases} x+1 & \text{if } x \leq 1\\ x^2+x+1 & \text{if } x > 1.\end{cases}$$

Since

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{\substack{x\to 1\\ x<1}} \frac{x+1-2}{x-1} = 1 \in \mathbb{R},$$

$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{\substack{x\to 1\\ x>1}} \frac{x^2+x+1-2}{x-1} = +\infty,$$

then f *is left differentiable at* 1 *(with* $f'_-(1) = 1$*), but it is not right differentiable at* 1. *Hence* f *is not differentiable at* 1.

Exercise 88
Let f be the function defined by:

$$f(x) = \begin{cases} 0 & \text{if} \quad x \leq 2 \\ a - \frac{b}{x} & \text{if} \quad 2 < x \leq 4 \\ 1 & \text{if} \quad x > 4 \end{cases}$$

1) **Determine a and b for which f is continuous on $\mathbb{R}$.**

2) **For these values of a and b, study the differentiability of f on $\mathbb{R}$.**

Solution

1) The function f is continuous on $]-\infty, 2[\, \cup \,]2, 4[\, \cup \,]4, +\infty[$ since the restriction of f on each of theses three intervals is a continuous function. So f is continuous on $\mathbb{R}$ if and only if f is continuous at 2 and at 4.
   Continuity at 2: We have:

$$f(2^+) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \left(a - \frac{b}{x}\right) = a - \frac{b}{2}$$

$$f(2^-) = \lim_{x \to 2^-} f(x) = 0.$$

   f is continuous at 2 if and only if

$$f(2^+) = f(2^-) = f(2) = 0.$$

   This is equivalent to $a - \frac{b}{2} = 0$.
   Continuity at 4: We have:

$$f(4^+) = \lim_{x \to 4^+} f(x) = 1$$

$$f(4^-) = \lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} \left(a - \frac{b}{x}\right) = a - \frac{b}{4}.$$

   f is continuous at 4 if and only if

$$f(4^+) = f(4^-) = f(4) = a - \frac{b}{4}.$$

   This is equivalent to $a - \frac{b}{4} = 1$.
   Hence f is continuous on $\mathbb{R}$ if and only if $a = 2$ and $b = 4$.

2) For $a = 2$ and $b = 4$,

$$f(x) = \begin{cases} 0 & \text{if} \quad x \leq 2 \\ 2 - \frac{4}{x} & \text{if} \quad 2 < x \leq 4 \\ 1 & \text{if} \quad x > 4 \end{cases}$$

   We have:

$$f'_-(2) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = 0$$

and

$$f'_+(2) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{2 - \frac{4}{x} - 0}{x - 2} = \lim_{x \to 2^+} \frac{2}{x} = 1.$$

As $f'_-(2) \neq f'_+(2)$, then f is not differentiable at 2. Hence f is not differentiable on $\mathbb{R}$. □

Exercise 89
Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be a differentiable function at a certain point $x_0 \in D_f$. Prove that if $f'(x_0) \neq 0$, then

$$f(x) - f(x_0) \underset{x_0}{\sim} f'(x_0)(x - x_0).$$

Solution Since f is differentiable at x_0, then

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \in \mathbb{R}^*.$$

So $\dfrac{f(x) - f(x_0)}{x - x_0} \underset{x_0}{\sim} f'(x_0)$ by the proposition 3.2.8. Therefore, multiplying by $x - x_0$, we obtain the asked equivalence (by the part (4) of the proposition 3.2.9). □

Proposition 4.1.1 *Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ and $x_0 \in D_f$. Then f is differentiable at x_0 if and only if there exist $\lambda \in \mathbb{R}$ and a function ε such that*

$$f(x_0 + h) = f(x_0) + \lambda h + h\varepsilon(h) \quad \textit{with } \lim_{h \to 0} \varepsilon(h) = 0.$$

Proof

N.C. It is sufficient to take $\lambda = f'(x_0) \in \mathbb{R}$ and $\varepsilon(h) = \dfrac{f(x_0 + h) - f(x_0)}{h} - \lambda$.
S.C. Since

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x \to x_0} (\lambda + \varepsilon(h)) = \lambda \in \mathbb{R},$$

then f is differentiable at x_0 and $f'(x_0) = \lambda$. □

Proposition 4.1.2 *Every differentiable function is continuous, i.e., if $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ is differentiable at a point $x_0 \in D_f$, then f is continuous at x_0.*

Proof By the proposition 4.1.1, there exist $\lambda \in \mathbb{R}$ and a function ε such that

$$f(x_0 + h) = f(x_0) + \lambda h + h\varepsilon(h) \quad \text{with } \lim_{h \to 0} \varepsilon(h) = 0.$$

Then, by using the change of variable $h = x - x_0$ (i.e., $x = x_0 + h$), we obtain:

$$\lim_{x \to x_0} f(x) = \lim_{h \to 0} f(x_0 + h) = f(x_0).$$

So f is continuous at x_0. □

Remark 4.1.2

1) *The converse of the proposition 4.1.2 is not always true. Indeed, the absolute value function and the square root function are continuous at* $\mathbf{0}$ *(by the example 3.3.1), but they are not differentiable at* $\mathbf{0}$ *(by the example 4.1.1).*

2) *The extension by continuity of a function at a certain point is not necessary differentiable at this point. Indeed, for example, the function* f *defined by* $f(x) = x \sin \frac{1}{x}$ *is extendable by continuity at* $\mathbf{0}$ *(by the example 3.3.2). But its extension* φ *by continuity at* $\mathbf{0}$ *is not differentiable at* $\mathbf{0}$*. Indeed, we have:*

$$\lim_{\substack{x \to 0 \\ x \neq 0}} \frac{\varphi(x) - \varphi(0)}{x - 0} = \lim_{\substack{x \to 0 \\ x \neq 0}} \frac{x \sin \frac{1}{x} - 0}{x} = \lim_{x \to 0} \sin \frac{1}{x}.$$

This last limit does not exist in $\mathbb{R}$ *by the example 3.4.1. So* φ *is not differentiable at* $\mathbf{0}$*.*

3) *The contrapositive of the proposition 4.1.2 is useful to prove that a function is not differentiable. For example, since the integer part function* E *is not continuous at any point of* $\mathbb{Z}$ *(by the example 3.3.1), then it is not differentiable at any point of* $\mathbb{Z}$*. Similarly, since Dirichlet's function* D *defined in the example 3.3.1 is not continuous at any point of* $\mathbb{R}$*, then it is not differentiable at any point of* $\mathbb{R}$*.*

4) *A little modification of the proof of the proposition 4.1.2 shows that every right (resp. left) differentiable function is right (resp. left) continuous function.*

Remark 4.1.3 *(Geometric interpretation of the derivative at a point).*
Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$*.*

- *Suppose that* f *is differentiable at a point* $x_0 \in D_f$*. Let* $M_0(x_0, f(x_0))$ *be the point on the curve* C_f *of* f *with abscissa* x_0*. For every* $x \in D_f$*, we denote* $M(x, f(x))$ *the point on* C_f *with abscissa* x*. Let* (T) *be the tangent line to* C_f *at* M_0*. Let* α_0 *(resp.* α*) be the angle between the* x*-axis and the straight-line* (T) *(resp.* (M_0M)*). The slope of the straight-line* (T) *(resp.* (M_0M)*) is*

$$m_0 = \tan \alpha_0 \qquad \left(\text{resp. } m = \tan \alpha = \frac{f(x) - f(x_0)}{x - x_0} \right).$$

We remark that when x *tends to* x_0*, the point* M *approaches to* M_0 *staying on the curve* C_f *(since* f *is continuous at* x_0*), and therefore the straight-line* (M_0M) *tends to the same position of the tangent* (T)*. Hence*

$$m_0 = \tan \alpha_0 = \lim_{x \to x_0} \tan \alpha = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0).$$

Consequently, the equation of (T) *is*

$$y - f(x_0) = f'(x_0)(x - x_0).$$

- *Suppose that f is left and right differentiable at a point $x_0 \in D_f$ with $f'_-(x_0) \neq f'_+(x_0)$, then f is not differentiable at x_0. In this case, we say that the curve of f has two half tangents at the point $M_0(x_0, f(x_0))$.*

- *Suppose that f is continuous at x_0 and $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = \pm\infty$, then f is not differentiable at x_0. In this case, we say that the curve of f has a vertical tangent at the point $M_0(x_0, f(x_0))$.*

Example 4.1.2 *The equation of the tangent to the curve of the sine function at the point with abscissa π is:*

$$y - \sin \pi = (\cos \pi)(x - \pi) \quad \textit{i.e.,} \quad y = -x + \pi.$$

Definition 4.1.2 *Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ and $x_0 \in D_f$.*

1) We say that x_0 is a local maximum (resp. minimum) of f if there exists a neighborhood V of x_0 contained in D_f such that, for all $x \in V$, we have:

$$f(x) \leq f(x_0) \quad (\textit{resp. } f(x) \geq f(x_0)).$$

2) We say that x_0 is a local extremum of f if x_0 is a local maximum or a local minimum of f.

3) We say that x_0 is a global maximum (resp. minimum) of f if, for all $x \in D_f$,

$$f(x) \leq f(x_0) \quad (\textit{resp. } f(x) \geq f(x_0)).$$

4) We say that x_0 is a global extremum of f if x_0 is a global maximum or a global minimum of f.

Proposition 4.1.3 *Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ and $x_0 \in D_f$. If f is differentiable at x_0 and if x_0 is a local extremum of f, then $f'(x_0) = 0$.*

Proof Suppose that x_0 is a local minimum of f (we make a similar proof if x_0 is a local maximum of f). Then there exists a neighborhood V of x_0 contained in D_f such that $f(x) \geq f(x_0)$ for all $x \in V$. So $f(x) - f(x_0) \geq 0$ for all $x \in V$. In the other side, since f is differentiable at x_0, then f is right and left differentiable at x_0 and

$$f'_+(x_0) = f'_-(x_0) = f'(x_0).$$

- If $x \in V$ and $x > x_0$, then $\frac{f(x)-f(x_0)}{x-x_0} \geq 0$, and therefore

$$f'(x_0) = f'_+(x_0) = \lim_{\substack{x \to x_0 \\ x > x_0}} \frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

- If $x \in V$ and $x < x_0$, then $\frac{f(x)-f(x_0)}{x-x_0} \leq 0$, and therefore

$$f'(x_0) = f'_-(x_0) = \lim_{\substack{x \to x_0 \\ x < x_0}} \frac{f(x) - f(x_0)}{x - x_0} \leq 0.$$

Hence $f'(x_0) = 0$. □

Remark 4.1.4

1) *A global extremum of f is necessary a local extremum of f.*

2) *The converse of the proposition 4.1.3 is not necessary true. Indeed, if $f(x) = x^3$, then $f'(0) = 0$, but 0 is not a local extremum of f (to see below in the example 4.2.3 or in the example 5.1.1). We will consider the converse in the corollary 5.1.1.*

Proposition 4.1.4 *(Operations on differentiable functions)*
Let $u, v \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ and $x_0 \in D_u \cap D_v$. If u and v are differentiable at x_0, then

1) *$u + v$ is differentiable at x_0 and*

$$(u + v)'(x_0) = u'(x_0) + v'(x_0).$$

In other words, the derivative of the sum of two functions at a point is the sum of their derivatives at this point.

2) *For any $\alpha \in \mathbb{R}$, αu is differentiable at x_0 and*

$$(\alpha u)'(x_0) = \alpha u'(x_0).$$

3) *uv is differentiable at x_0 and*

$$(uv)'(x_0) = u'(x_0)\, v(x_0) + v'(x_0)\, u(x_0).$$

4) *If $v(x_0) \neq 0$, then $\frac{u}{v}$ is differentiable at x_0 and*

$$\left(\frac{u}{v}\right)'(x_0) = \frac{u'(x_0)\, v(x_0) - v'(x_0)\, u(x_0)}{(v(x_0))^2}.$$

Proposition 4.1.5 *(composite of two differentiable functions)*
Let $x_0 \in \mathbb{R}$ and $u, f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ such that $u(D_u) \subseteq D_f$. If u is differentiable at x_0 and f is differentiable at $u(x_0)$, then the composite function $f \circ u$ is differentiable at x_0 and

$$(f \circ u)'(x_0) = u'(x_0)\, f'(u(x_0)).$$

In other words, the composite of two differentiable functions is a differentiable function.

Proof We have:

$$\lim_{x\to x_0}\frac{(f\circ u)(x)-(f\circ u)(x_0)}{x-x_0}=\lim_{x\to x_0}\frac{f(u(x))-f(u(x_0))}{u(x)-u(x_0)}\times\frac{u(x)-u(x_0)}{x-x_0}.$$

Since u is differentiable at x_0, then u is continuous at x_0, so $\lim_{x\to x_0}u(x)=u(x_0)$. Therefore, by the theorem 3.2.1 and the fact that f is differentiable at $u(x_0)$, we get

$$\lim_{x\to x_0}\frac{f(u(x))-f(u(x_0))}{u(x)-u(x_0)}=\lim_{u\to u(x_0)}\frac{f(u)-f(u(x_0))}{u-u(x_0)}=f'(u(x_0)).$$

In the other side, as u is differentiable at x_0, then

$$\lim_{x\to x_0}\frac{u(x)-u(x_0)}{x-x_0}=u'(x_0).$$

Hence

$$\lim_{x\to x_0}\frac{(f\circ u)(x)-(f\circ u)(x_0)}{x-x_0}=f'(u(x_0))u'(x_0).$$

Thus $f\circ u$ is differentiable at x_0 and $(f\circ u)'(x_0)=u'(x_0)f'(u(x_0))$. $\square$

Example 4.1.3

1) *Let $x_0\in\mathbb{R}$ such that $x_0\neq(2k+1)\frac{\pi}{2}$ for all $k\in\mathbb{Z}$ (i.e., $\cos x_0\neq 0$). Since the two functions $\sin$ and $\cos$ are differentiable at x_0, then the function $\tan$ is differentiable at x_0 and*

$$(\tan x)'(x_0)=\frac{(\cos x_0)(\cos x_0)-(-\sin x_0)(\sin x_0)}{\cos^2 x_0}=\frac{1}{\cos^2 x_0}=1+\tan^2 x_0.$$

2) *Let $x_0\in\mathbb{R}$ such that $x_0\neq k\pi$ for all $k\in\mathbb{Z}$ (i.e., $\sin x_0\neq 0$). Since the two functions $\sin$ and $\cos$ are differentiable at x_0, then the function $\cot$ is differentiable at x_0 and*

$$(\cot x)'(x_0)=\frac{(-\sin x_0)(\sin x_0)-(\cos x_0)(\cos x_0)}{\sin^2 x_0}=\frac{-1}{\sin^2 x_0}=-(1+\cot^2 x_0).$$

3) *Remarking that $\cos x=\sin(\frac{\pi}{2}-x)=f(u(x))$ where $f(x)=\sin x$ and $u(x)=\frac{\pi}{2}-x$, we obtain, for all $x\in\mathbb{R}$,*

$$(\cos x)'=u'(x)f'(u(x))=-\cos\left(\frac{\pi}{2}-x\right)=-\sin x.$$

Hence, we found again the derivative of the cosine function.

4) *Let $f(x)=\sin(\sqrt{x})$ and $x_0>0$. Since the function $\sqrt{\cdot}$ is differentiable at x_0 and the function $\sin$ is differentiable at $\sqrt{x_0}$, then f is differentiable at x_0 and*

$$f'(x_0)=\frac{1}{2\sqrt{x_0}}\cos(\sqrt{x_0}).$$

5) *Put* $f_n(x) = x^n$ *where* $n \in \mathbb{N}$ *such that* $n \geq 2$. *Let's show, by mathematical induction on* $n \geq 2$, *that*
$$f_n'(x) = nx^{n-1}, \quad \forall x \in \mathbb{R}.$$
(the case $n = 1$ *is already seen in the example 4.1.1). Indeed,*

- *Suppose that* $n = 2$, *then* $f_2(x) = x^2 = xx$. *So, by using the proposition 4.1.4, we obtain:*
$$f_2'(x) = (1)(x) + (x)(1) = 2x = 2x^{2-1}.$$
- *Suppose that this property is true for a certain integer* $k \geq 2$, *i.e.,* $f_k'(x) = kx^{k-1}$ *for all* $x \in \mathbb{R}$. *Since* $f_{k+1}(x) = x^{k+1} = xx^k = xf_k(x)$, *then, by using the proposition 4.1.4, we obtain:*
$$f_{k+1}'(x) = (1)f_k(x) + xf_k'(x) = x^k + xkx^{k-1} = (k+1)x^k = (k+1)x^{(k+1)-1}.$$

Hence $f_n'(x) = nx^{n-1}$ *for all* $x \in \mathbb{R}$ *and* $n \in \mathbb{N}^*$.

6) *Put* $f_n(x) = x^n$ *where* $n \in \mathbb{Z}$ *such that* $n \leq -1$. *Since* $f_n(x) = x^n = \dfrac{1}{x^{-n}}$, *then, for all* $x \in \mathbb{R}^*$,
$$f_n'(x) = \frac{-(-n)x^{-n-1}}{(x^{-n})^2} = nx^{n-1}.$$

7) *Let* $u \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ *be a differentiable function on an interval* I *of* D_u.

- *For any* $n \in \mathbb{N}^*$, *the power function* u^n *is differentiable on* I *and*
$$(u^n)'(x) = nu'(x)u^{n-1}(x) \quad \text{for all } x \in I.$$
For example, if $f(x) = (3x^2+1)^7$, *then* $f'(x) = 42x(3x^2+1)^6$ *for all* $x \in \mathbb{R}$.
- *If* $u(x) > 0$ *for all* $x \in I$, *then the functions* $\sqrt{u}$ *and* $\ln u$ *are differentiable on* I *and, for all* $x \in I$, *we have:*
$$(\sqrt{u})'(x) = \frac{u'(x)}{2\sqrt{u(x)}} \quad \text{and} \quad (\ln u)'(x) = \frac{u'(x)}{u(x)}.$$

Exercise 90
By using the derivatives, calculate the limit $\ell = \lim\limits_{x \to 0} \dfrac{\sin x}{\cos(\frac{\pi}{2} - x)}$.

Solution The limit ℓ is an indeterminate form $\frac{0}{0}$. Put $f(x) = \sin x$ and $g(x) = \cos(\frac{\pi}{2} - x)$. The functions f and g are differentiable on $\mathbb{R}$ with $f'(x) = \cos x$ and $g'(x) = \sin(\frac{\pi}{2} - x)$ for all $x \in \mathbb{R}$. Since $f(0) = g(0) = 0$, then
$$\ell = \lim_{\substack{x \to 0 \\ x \neq 0}} \frac{f(x) - f(0)}{g(x) - g(0)} = \lim_{x \to 0} \frac{\frac{f(x)-f(0)}{x-0}}{\frac{g(x)-g(0)}{x-0}} = \frac{\lim\limits_{x \to 0} \frac{f(x)-f(0)}{x-0}}{\lim\limits_{x \to 0} \frac{g(x)-g(0)}{x-0}} = \frac{f'(0)}{g'(0)} = \frac{\cos 0}{\sin(\frac{\pi}{2} - 0)} = 1. \quad \square$$

Proposition 4.1.6 *Let f be a bijective and continuous function from an interval I on an interval J of $\mathbb{R}$. Denote by f^{-1} its inverse function. Let $x_0 \in I$ and $y_0 = f(x_0)$, i.e., $x_0 = f^{-1}(y_0)$. Suppose that f is differentiable at x_0.*

1) If $f'(x_0) \neq 0$, then f^{-1} is differentiable at y_0 and

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'\left(f^{-1}(y_0)\right)}.$$

2) If $f'(x_0) = 0$, then f^{-1} is not differentiable at y_0.

In other words, f^{-1} is differentiable only at the points $f(x) \in J$ (with $x \in I$) such that f is differentiable at x and $f'(x) \neq 0$, with

$$\left(f^{-1}\right)'(f(x)) = \frac{1}{f'(x)}.$$

Proof Every real number y in J can be written, in a unique way, in the form $y = f(x)$ where $x \in I$. Since f is continuous at x_0, then $\lim\limits_{x \to x_0} f(x) = f(x_0) = y_0$. So, by using the theorem 3.2.1, we obtain:

$$\begin{aligned}\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} &= \lim_{x \to x_0} \frac{f^{-1}(f(x)) - f^{-1}(f(x_0))}{f(x) - f(x_0)} \\ &= \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \lim_{x \to x_0} \frac{1}{\frac{f(x)-f(x_0)}{x-x_0}}.\end{aligned}$$

1) If $f'(x_0) \neq 0$, i.e., $\lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} \neq 0$, then

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{\lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}} = \frac{1}{f'(x_0)}.$$

So f^{-1} is differentiable at y_0 and $\left(f^{-1}\right)'(y_0) = \dfrac{1}{f'(x_0)}$.

2) If $f'(x_0) = 0$, i.e., $\lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = 0$, then

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \pm\infty.$$

So f^{-1} is not differentiable at y_0.

Example 4.1.4

1) Let $n \in \mathbb{N}^*$ be an **even** *integer and $g(x) = \sqrt[n]{x} = x^{\frac{1}{n}}$ for all $x \in \mathbb{R}^+$. By the example 3.5.2, g is the inverse function of the function f defined by $f(x) = x^n$ on $\mathbb{R}^+$. The function f is differentiable at any point $x \in \mathbb{R}^+$ with $f'(x) = nx^{n-1}$. As*

$f'(x) = 0$ *if and only if* $x = 0$, *then, by the proposition 4.1.6,* g *is differentiable at any point* $x \in \mathbb{R}^+$ *except at the point* $f(0) = 0$. *Hence, for all* $x \in]0, +\infty[$,

$$g'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{n(\sqrt[n]{x})^{n-1}} = \frac{1}{n}x^{\frac{1}{n}-1}.$$

In particular, for $n = 2$, *we found the derivative of the square root function obtained in the example 4.1.1:*

$$(\sqrt{x})' = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}, \quad \forall x \in]0, +\infty[.$$

2) *Let* $n \in \mathbb{N}^*$ *be an* **odd** *integer and* $g(x) = \sqrt[n]{x} = x^{\frac{1}{n}}$ *for all* $x \in \mathbb{R}$. *By the example 3.5.2,* g *is the inverse function of the function* f *defined by* $f(x) = x^n$ *on* $\mathbb{R}$. *The function* f *is differentiable at any point* $x \in \mathbb{R}$ *with* $f'(x) = nx^{n-1}$.

- *If* $n = 1$, *then* $g(x) = x$ *for all* $x \in \mathbb{R}$, *and therefore* $g'(x) = 1$ *for all* $x \in \mathbb{R}$.
- *If* $n \geq 3$, *then* $f'(x) = 0$ *if and only if* $x = 0$, *then, by the proposition 4.1.6,* g *is differentiable at any point* $x \in \mathbb{R}$ *except at the point* $f(0) = 0$. *Hence, for all* $x \in \mathbb{R}^*$,

$$g'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{n(\sqrt[n]{x})^{n-1}} = \frac{1}{n}x^{\frac{1}{n}-1}.$$

3) *We admit that the natural logarithm* $\ln$ *(denoted by* f *here) is differentiable at any point* $x \in]0, +\infty[$ *with* $f'(x) = \frac{1}{x} \neq 0$, *then the exponential function* $\exp$ *is differentiable at any point* $x \in \mathbb{R}$ *with*

$$(e^x)' = \frac{1}{f'(e^x)} = \frac{1}{\frac{1}{e^x}} = e^x.$$

4) *Let* a *be a strictly positive real number such that* $a \neq 1$. *Since the function* $\log_a$ *(denoted by* f *here) is differentiable at any point* $x \in]0, +\infty[$ *with* $f'(x) = \frac{1}{x \ln a} \neq 0$, *then the exponential function* $\exp_a$ *of base* a *is differentiable at any point* $x \in \mathbb{R}$ *with*

$$(a^x)' = \frac{1}{f'(a^x)} = \frac{1}{\frac{1}{a^x \ln a}} = a^x \ln a.$$

Definition 4.1.3 *(Successive derivatives)*
Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$.

1) *The function*

$$\begin{array}{rccc} f': & D_f & \longmapsto & \mathbb{R} \\ & x & \longmapsto & f'(x). \end{array}$$

is called the first order derivative function of f *or the derivative of* f *of order* 1, *denoted by* $f^{(1)}$. *The domain of definition* $D_{f'}$ *of* f' *is the set of real numbers* $x \in D_f$ *where* f *is differentiable, i.e.,*

$$D_{f'} = \Big\{x \in D_f, \quad \text{such that } f \text{ is differentiable at } x\Big\}.$$

2) *We say that* f *is twice-differentiable at a point* $x_0 \in D_f$ *if* f *is differentiable on a neighborhood of* x_0 *and the function* f' *is differentiable at* x_0. *In this case, we write*

$$f''(x_0) = (f')'(x_0),$$

and we call it the second derivative of f *at the point* x_0.

3) *The function*

$$\begin{array}{rccc} f'': & D_f & \longmapsto & \mathbb{R} \\ & x & \longmapsto & f''(x). \end{array}$$

is called the second derivative function of f *or the derivative of* f *of order* 2, *denoted by* $f^{(2)}$. *The domain of definition* $D_{f''}$ *of* f'' *is the set of real numbers* $x \in D_f$ *where where* f *is twice-differentiable, i.e.,*

$$D_{f''} = \Big\{x \in D_f, \quad \text{such that } f \text{ is twice-differentiable at } x\Big\}.$$

4) *Let* $n \in \mathbb{N}^*$. *By induction, we say that* f *is* n*-times differentiable at a point* $x_0 \in D_f$ *if* f *is* $(n-1)$*-times differentiable on a neighborhood of* x_0 *and the derivative function* $f^{(n-1)}$ *of* f *of order* $n-1$ *is differentiable at* x_0. *We denote* $f = f^{(0)}$, $f' = f^{(1)}$, $f'' = f^{(2)}$ *and the function*

$$f^{(n)} = \left(f^{(n-1)}\right)'.$$

and we call it the n*-th derivative function of* f *or the derivative of* f *of order* n.

Definition 4.1.4 *Let* $n \in \mathbb{N}^*$, $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ *and* I *be an interval contained in* D_f.

1) *We say that* f *is* n*-times differentiable on* I *if* f *is* n*-times differentiable at every point of* I.

2) *We say that* f *is of class* $\mathcal{C}^n$ *(or that* f *is* n*-times continuously differentiable) on* I *if the following two conditions suivantes are satisfied:*

   i) f *is* n*-times differentiable on* I.

   ii) $f^{(n)}$ *is continuous on* I.

3) *We say that* f *is of class* $\mathcal{C}^\infty$ *(or that* f *is infinitely differentiable) on* I *if* f *is of class* $\mathcal{C}^n$ *for all* $n \in \mathbb{N}^*$.

4) *We denote by* $\mathcal{C}^n(I, \mathbb{R})$ *the set of functions of class* $\mathcal{C}^n$ *on* I. *By convention, put* $\mathcal{C}^0(I, \mathbb{R}) = \mathcal{C}(I, \mathbb{R})$. *We denote by* $\mathcal{C}^\infty(I, \mathbb{R})$ *the set of functions of class* $\mathcal{C}^\infty$ *on* I, *i.e.,*

$$\mathcal{C}^\infty(I, \mathbb{R}) = \bigcap_{n \in \mathbb{N}^*} \mathcal{C}^n(I, \mathbb{R}).$$

Example 4.1.5

1) *Every polynomial function is of class* $\mathcal{C}^\infty$ *on* $\mathbb{R}$. *More generally, every rational function is of class* $\mathcal{C}^\infty$ *on its domain of definition.*

2) The exponential function is of class $\mathcal{C}^\infty$ *on* $\mathbb{R}$*. For any* $n \in \mathbb{N}$ *and* $x \in \mathbb{R}$*, we have:*

$$\boxed{(\exp)^{(n)}(x) = \exp x}$$

More generally, if a *is a strictly positive real number such that* $a \neq 1$*, the exponential function* $\exp_a$ *of base* a *is of class* $\mathcal{C}^\infty$ *on* $\mathbb{R}$*. For any* $n \in \mathbb{N}$ *and* $x \in \mathbb{R}$*,*

$$\boxed{(\exp_a)^{(n)}(x) = a^x (\ln a)^n}$$

3) The natural logarithm $\ln$ *is of class* $\mathcal{C}^\infty$ *on the interval* $]0, +\infty[$*. For any* $n \in \mathbb{N}^*$ *and* $x \in]0, +\infty[$*, we have:*

$$\boxed{\ln^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{x^n}}$$

Indeed, by mathematical induction on n*,*

- *For* $n = 1$*,* $\ln^{(1)} x = \frac{1}{x} = (-1)^{1+1}\frac{(1-1)!}{x^1}$ *for all* $x \in]0, +\infty[$*.*
- *Suppose that this formula is true for a certain* $k \in \mathbb{N}^*$*. For any* $x \in]0, +\infty[$*,*

$$\begin{aligned}
\ln^{(k+1)} x &= (\ln^{(k)} x)' = \left[(-1)^{k+1}\frac{(k-1)!}{x^k}\right]' \\
&= (-1)^{k+1}(k-1)!\left[\frac{-kx^{k-1}}{x^{2k}}\right] \\
&= (-1)^{k+2}\frac{k(k-1)!}{x^{k+1}} = (-1)^{k+2}\frac{k!}{x^{k+1}}.
\end{aligned}$$

Hence the formula is true for all $n \in \mathbb{N}^*$*. Similarly, the function* $x \longmapsto \ln(1+x)$ *is of class* $\mathcal{C}^\infty$ *on the interval* $]-1, +\infty[$*, and, for all* $n \in \mathbb{N}^*$ *and* $x \in]-1, +\infty[$*, we have:*

$$\boxed{\Big(\ln(1+x)\Big)^{(n)} = (-1)^{n+1}\frac{(n-1)!}{(1+x)^n}}$$

4) The functions $\sin$ *and* $\cos$ *are of class* $\mathcal{C}^\infty$ *on* $\mathbb{R}$*. For any* $n \in \mathbb{N}$ *and* $x \in \mathbb{R}$*, we have:*

$$\boxed{\sin^{(n)} x = \sin\left(x + n\frac{\pi}{2}\right)} \quad \text{and} \quad \boxed{\cos^{(n)} x = \cos\left(x + n\frac{\pi}{2}\right)}$$

Indeed, let's show, for example, the first formula by mathematical induction on n*.*

- *For* $n = 0$*,* $\sin^{(0)} x = \sin x = \sin\left(x + 0\frac{\pi}{2}\right)$ *for all* $x \in \mathbb{R}$*.*
- *Suppose that this formula is true for a certain* $k \in \mathbb{N}$*. For any* $x \in \mathbb{R}$*,*

$$\begin{aligned}
\sin^{(k+1)} x &= (\sin^{(k)} x)' = \left[\sin\left(x + k\frac{\pi}{2}\right)\right]' = \cos\left(x + k\frac{\pi}{2}\right) \\
&= \sin\left(\frac{\pi}{2} + x + k\frac{\pi}{2}\right) = \sin\left(x + (k+1)\frac{\pi}{2}\right)
\end{aligned}$$

Hence the formula is true for all $n \in \mathbb{N}$.

Exercise 91
Let $f(x) = \dfrac{1}{1-x}$. Calculate $f^{(n)}(x)$ for all $n \in \mathbb{N}$ and $x \in]-1,1[$.

Solution Let $x \in]-1,1[$. In order to find the recursive formula which determines $f^{(n)}(x)$, we begin by calculating the first derivatives.

- For $n = 0$, $f^{(0)}(x) = f(x) = (1-x)^{-1}$.
- For $n = 1$, $f^{(1)}(x) = f'(x) = (1-x)^{-2}$.
- For $n = 2$, $f^{(2)}(x) = f''(x) = 2(1-x)^{-3}$.
- For $n = 3$, $f^{(3)}(x) = (2)(3)(1-x)^{-4} = 3!(1-x)^{-4}$.
- For $n = 4$, $f^{(4)}(x) = (2)(3)(4)(1-x)^{-5} = 4!(1-x)^{-5}$.

Let's show, by mathematical induction on $n \in \mathbb{N}$, that, for all $n \in \mathbb{N}$,

$$f^{(n)}(x) = n!(1-x)^{-(n+1)}.$$

Indeed, this formula is true for $n = 0$. Suppose that it is true for a certain $k \in \mathbb{N}$. Then

$$f^{(k+1)}(x) = \left(f^{(k)}(x)\right)' = (-1)\Big(-(k+1)\Big)k!(1-x)^{-(k+1)-1} = (k+1)!(1-x)^{-\Big((k+1)+1\Big)}.$$

Hence the formula is true for all $n \in \mathbb{N}$. $\square$

Exercise 92
Let $\alpha \in \mathbb{R}$ and $f(x) = (1+x)^{\alpha}$. Determine $f^{(n)}(x)$ for all $n \in \mathbb{N}$ and for all $x \in]-1,+\infty[$.

Solution Let $x \in]-1,+\infty[$. In order to find the recursive formula which determines $f^{(n)}(x)$, we begin by calculating the first derivatives.

- For $n = 0$, $f^{(0)}(x) = f(x) = (1+x)^{\alpha}$.
- For $n = 1$, $f^{(1)}(x) = f'(x) = \alpha(1+x)^{\alpha-1}$.
- For $n = 2$, $f^{(2)}(x) = f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$.
- For $n = 3$, $f^{(3)}(x) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}$.

Let's show, by mathematical induction on $n \in \mathbb{N}$, that, for all $n \in \mathbb{N}$,

$$f^{(n)}(x) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)(1+x)^{\alpha-n}.$$

Indeed, this formula is true for $n = 0$. Suppose that it is true for a certain $k \in \mathbb{N}$. Then

$$\begin{aligned} f^{(k+1)}(x) &= \left(f^{(k)}(x)\right)' \\ &= \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)(\alpha-k)(1+x)^{\alpha-k-1} \\ &= \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(k+1)+1)(1+x)^{\alpha-(k+1)} \end{aligned}$$

Hence the formula is true for all $n \in \mathbb{N}$. □

Remark 4.1.5 *Let $n \in \mathbb{N}^*$, $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ and I be an interval contained in $D_f \cap D_g$.*

1) *If f is n-times differentiable on I, then all the derivatives $f^{(0)}, f^{(1)}, \ldots, f^{(n-1)}$ are continuous on I by the proposition 4.1.2.*

2) *Let $m, n \in \mathbb{N}^*$ such that $m \leq n$. If f is of class $\mathcal{C}^n$ on I, then it is of class $\mathcal{C}^m$ on I. In other words, $\mathcal{C}^n(I, \mathbb{R}) \subseteq \mathcal{C}^m(I, \mathbb{R})$.*

3) *f is of class $\mathcal{C}^\infty$ on I if and only if all the successive derivatives of f exist and continuous on I.*

4) *If f and g are of class $\mathcal{C}^n$ (resp. $\mathcal{C}^\infty$) on I, then:*

   - *$f + g$ and fg are of class $\mathcal{C}^n$ (resp. $\mathcal{C}^\infty$) on I.*
   - *f^k is of class $\mathcal{C}^n$ (resp. $\mathcal{C}^\infty$) on I for all $k \in \mathbb{N}$.*
   - *$\frac{f}{g}$ is of class $\mathcal{C}^n$ (resp. $\mathcal{C}^\infty$) on I (if g does not vanish on I).*

5) *If f is of class $\mathcal{C}^n$ (resp. $\mathcal{C}^\infty$) on I and g is of class $\mathcal{C}^n$ (resp. $\mathcal{C}^\infty$) on an interval J such that $f(I) \subseteq J$, then the composite function $g \circ f$ is of class $\mathcal{C}^n$ (resp. $\mathcal{C}^\infty$) on I.*

6) *If f and g are n-times differentiable on I, then their product fg is also n-times differentiable on I, and we have the following Leibniz formula:*

$$(fg)^{(n)} = \sum_{k=0}^{n} C_n^k f^{(k)} g^{(n-k)} \quad \text{where} \quad C_n^k = \frac{n!}{k!(n-k)!}.$$

Exercise 93
Prove that the function f defined by:

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not of class $\mathcal{C}^1$ on $\mathbb{R}$.

Solution The function f is differentiable on $\mathbb{R}^*$ as it is the product of two differentiable functions on $\mathbb{R}^*$. Moreover,

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} x \sin \frac{1}{x} = 0 \in \mathbb{R},$$

by using the fact that $\lim\limits_{x\to 0} x = 0$ and the function $x \mapsto \sin\frac{1}{x}$ is bounded in a neighborhood of 0 (see also the example 3.2.9). So f is differentiable at 0 and $f'(0) = 0$. Thus f is differentiable on $\mathbb{R}$ and

$$f'(x) = \begin{cases} 2x\sin\dfrac{1}{x} - \cos\dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

For any $x \neq 0$,

$$\cos\frac{1}{x} = 2x\sin\frac{1}{x} - f'(x).$$

Suppose that f' is continuous at 0, then $\lim\limits_{x\to 0} f'(x) = f'(0) = 0$. But $\lim\limits_{x\to 0} x\sin\dfrac{1}{x} = 0$ (as above), then

$$\lim_{x\to 0}\cos\frac{1}{x} = 0 - 0 = 0,$$

which is impossible by the example 3.4.1. So f' is not continuous at 0, and therefore f' is not continuous on $\mathbb{R}$. Thus f is not of class $\mathcal{C}^1$ on $\mathbb{R}$. □

Exercise 94
Prove that the function f defined by:

$$f(x) = \begin{cases} x^3\sin\dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not twice-differentiable at 0.

Solution A similar reasoning to that of the exercise 93 shows that f is differentiable on $\mathbb{R}$ and

$$f'(x) = \begin{cases} 3x^2\sin\dfrac{1}{x} - x\cos\dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Moreover, the limit

$$\lim_{x\to 0}\frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0}\left(3x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$$

does not exist in $\mathbb{R}$ since $\lim\limits_{x\to 0} x\sin\dfrac{1}{x} = 0$ and $\lim\limits_{x\to 0}\cos\dfrac{1}{x}$ does not exist in $\mathbb{R}$ (by the example 3.4.1). So f' is not differentiable at 0. Hence the function f is not twice-differentiable at 0. □

Exercise 95
Prove that the function f defined by:

$$f(x) = \begin{cases} \dfrac{1-\cos x}{e^x - 1} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is of class $\mathcal{C}^1$ on $\mathbb{R}$.

Solution

- The function f is differentiable on $\mathbb{R}^*$ as it is the quotient of two differentiable functions on $\mathbb{R}^*$ (where the denominator $e^x - 1$ does not vanish at any point of $\mathbb{R}^*$). Moreover, put:
$$\ell = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{\substack{x\to 0\\ x\neq 0}} \frac{f(x)}{x} = \lim_{x\to 0} \frac{1 - \cos x}{x(e^x - 1)}.$$
But $\cos x = 1 - 2\sin^2(\frac{x}{2})$, then $1 - \cos x = 2\sin^2(\frac{x}{2})$. So
$$\ell = \lim_{x\to 0} \frac{2\sin^2(\frac{x}{2})}{x(e^x - 1)} = \lim_{x\to 0} \frac{\sin(\frac{x}{2})}{\frac{x}{2}} \times \frac{\sin(\frac{x}{2})}{e^x - 1}.$$
Put $u(x) = \frac{x}{2}$, we have $\lim_{x\to 0} u(x) = 0$, and therefore
$$\lim_{x\to 0} \frac{\sin(\frac{x}{2})}{\frac{x}{2}} = \lim_{u\to 0} \frac{\sin u}{u} = 1.$$
In the other side, put $g(x) = \sin(\frac{x}{2})$ and $h(x) = e^x$, then
$$\lim_{x\to 0} \frac{\sin(\frac{x}{2})}{e^x - 1} = \lim_{x\to 0} \frac{g(x) - g(0)}{h(x) - h(0)} = \lim_{\substack{x\to 0\\ x\neq 0}} \frac{\frac{g(x)-g(0)}{x-0}}{\frac{h(x)-h(0)}{x-0}} = \frac{g'(0)}{h'(0)}.$$
But $g'(x) = \frac{1}{2}\cos(\frac{x}{2})$ and $h'(x) = e^x$ for all $x \in \mathbb{R}$, then $g'(0) = \frac{1}{2}$ and $h'(0) = 1$. Hence
$$\lim_{x\to 0} \frac{\sin(\frac{x}{2})}{e^x - 1} = \frac{1}{2}.$$
Consequently, $\ell = \frac{1}{2} \in \mathbb{R}$, and therefore f is differentiable at 0 and $f'(0) = \ell = \frac{1}{2}$. Thus f is differentiable on $\mathbb{R}$.

- The function f' is given by:
$$f'(x) = \begin{cases} \dfrac{(e^x - 1)\sin x - (1 - \cos x)e^x}{(e^x - 1)^2} & \text{if } x \neq 0 \\[2ex] \frac{1}{2} & \text{if } x = 0 \end{cases}$$
The function f' is continuous on $\mathbb{R}^*$ as it is the quotient of two continuous functions on $\mathbb{R}^*$ (where the denominator $(e^x - 1)^2$ does not vanish at any point of $\mathbb{R}^*$). Moreover,
$$\begin{aligned} \lim_{x\to 0} f'(x) &= \lim_{\substack{x\to 0\\ x\neq 0}} \left[\frac{\sin x}{e^x - 1} - e^x \frac{1 - \cos x}{(e^x - 1)^2}\right] \\ &= \lim_{x\to 0} \left[\frac{\sin x}{e^x - 1} - 2e^x \left(\frac{\sin(\frac{x}{2})}{e^x - 1}\right)^2\right]. \end{aligned}$$
But $\lim_{x\to 0} \frac{\sin(\frac{x}{2})}{e^x - 1} = \frac{1}{2}$ by the above discussion, $\lim_{x\to 0} e^x = 1$ and
$$\lim_{x\to 0} \frac{\sin x}{e^x - 1} = \lim_{x\to 0} \frac{\sin x - \sin 0}{e^x - e^0} = \lim_{\substack{x\to 0\\ x\neq 0}} \frac{\frac{\sin x - \sin 0}{x-0}}{\frac{e^x - e^0}{x-0}} = \frac{\cos 0}{e^0} = 1.$$

So
$$\lim_{x\to 0} f'(x) = 1 - 2\left(\frac{1}{2}\right)^2 = \frac{1}{2} = f'(0).$$
Hence f' is continuous at 0. Consequently, f' is continuous on $\mathbb{R}$. Thus f is of class $\mathcal{C}^1$ on $\mathbb{R}$. $\square$

Exercise 96
By using Leibniz formula, calculate the n-th derivative of the function h defined by $h(x) = (x^3 + 2x^2 + 1)e^x$ for all $n \in \mathbb{N}^*$.

Solution Put $f(x) = x^3 + 2x^2 + 1$ and $g(x) = e^x$. Then
$$f'(x) = 3x^2 + 4x, \quad f''(x) = 6x + 4, \quad f'''(x) = 6 \quad \text{and} \quad f^{(k)}(x) = 0, \ \forall\, k \geq 4.$$
$$g^{(k)}(x) = e^x, \ \forall\, k \in \mathbb{N}.$$
Let $n \in \mathbb{N}^*$. By Leibniz formula,
$$h^{(n)}(x) = (fg)^{(n)}(x) = \sum_{k=0}^{n} C_n^k f^{(k)}(x) g^{(n-k)}(x).$$

- If $n = 1$, then
$$\begin{aligned} h'(x) &= C_1^0 f(x)g'(x) + C_1^1 f'(x)g(x) \\ &= (x^3 + 2x^2 + 1)e^x + (3x^2 + 4x)e^x \\ &= (x^3 + 5x^2 + 4x + 1)e^x. \end{aligned}$$

- If $n = 2$, then
$$\begin{aligned} h''(x) &= C_2^0 f(x)g''(x) + C_2^1 f'(x)g'(x) + C_2^2 f''(x)g(x) \\ &= (x^3 + 2x^2 + 1)e^x + 2(3x^2 + 4x)e^x + (6x + 4)e^x \\ &= (x^3 + 8x^2 + 14x + 5)e^x. \end{aligned}$$

- If $n \geq 3$, then
$$\begin{aligned} h^{(n)}(x) &= \sum_{k=0}^{3} C_3^k f^{(k)}(x) g^{(3-k)}(x) \quad (\text{since } f^{(k)}(x) = 0, \ \forall\, k \geq 4) \\ &= C_3^0 f(x)g'''(x) + C_3^1 f'(x)g''(x) + C_3^2 f''(x)g'(x) + C_3^3 f'''(x)g(x) \\ &= (x^3 + 2x^2 + 1)e^x + 3(3x^2 + 4x)e^x + 3(6x + 4)e^x + 6e^x \\ &= (x^3 + 11x^2 + 30x + 19)e^x. \quad \square \end{aligned}$$

4.2 Theorems on differentiable functions

Theorem 4.2.1 *(Darboux theorem)*
Let $[a, b]$ be a closed bounded interval of $\mathbb{R}$ (where $a < b$). If $f : [a, b] \longmapsto \mathbb{R}$ is a **differentiable** *function on $[a, b]$, then the derivative function f' reaches any value between $f'(a)$ and $f'(b)$, i.e., for any real number β between $f'(a)$ and $f'(b)$, there exists $\alpha \in [a, b]$ such that $f'(\alpha) = \beta$.*

Proof Let β be a real number such that $f'(a) < \beta < f'(b)$ (we make a similar proof in the other case). Consider the function $g : [a, b] \longmapsto \mathbb{R}$ defined by $g(x) = f(x) - \beta x$. The function g is differentiable on $[a, b]$ with $g'(x) = f'(x) - \beta$ for all $x \in [a, b]$. In particular,

$$g'(a) = f'(a) - \beta < 0 \quad \text{and} \quad g'(b) = f'(b) - \beta > 0.$$

Since g is differentiable on $[a, b]$, then g is continuous on $[a, b]$, so g is bounded and it reaches its bounds on $[a, b]$ by the theorem 3.5.1.

- If g reaches a minimum at a, then $\dfrac{g(x) - g(a)}{x - a} \geq 0$ for all $x \in [a, b]$, and therefore
$$g'(a) = \lim_{x \to a^+} \frac{g(x) - g(a)}{x - a} \geq 0,$$
which is impossible. Hence a is not a global minimum of g.

- If g reaches a minimum at b, then $\dfrac{g(x) - g(b)}{x - b} \leq 0$ for all $x \in [a, b]$, and therefore
$$g'(b) = \lim_{x \to b^-} \frac{g(x) - g(b)}{x - b} \leq 0,$$
which is impossible. Hence b is not a global minimum of g.

Hence g reaches a minimum at a point $\alpha \in \,]a, b[$. By the remark 4.1.4, α is a local minimum of g, so $g'(\alpha) = 0$ by the proposition 4.1.3 (since g is differentiable at α). Hence $f'(\alpha) = \beta$. □

Corollary 4.2.1 *If $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ is a* **differentiable** *function on an interval I, then $f'(I)$ is an interval.*

Proof Let $c, d \in f'(I)$ such that $c \leq d$, then there exist $a, b \in I$ such that $c = f'(a)$ and $d = f'(b)$. Suppose that $a \leq b$ (we make a similar proof when $b < a$). Since I is an interval, then $[a, b] \subseteq I$. As f is differentiable on I, then f is differentiable on $[a, b]$. If $c \leq \beta \leq d$, then $f'(a) \leq \beta \leq f'(b)$, and therefore there exists $\alpha \in [a, b]$ such that $\beta = f'(\alpha)$ by Darboux theorem (theorem 4.2.1). Hence $\alpha \in I$, and therefore $\beta = f'(\alpha) \in f'(I)$. Thus $f'(I)$ is an interval. □

Theorem 4.2.2 *(Rolle's theeorem)*
Let $[a, b]$ be a closed bounded interval of $\mathbb{R}$ (where $a < b$). If $f : [a, b] \longmapsto \mathbb{R}$ is continuous on $[a, b]$ and differentiable on $]a, b[$ such that $f(a) = f(b)$, then there exists $c \in \,]a, b[$ such that $f'(c) = 0$.

Proof

- If f is constant, then $f'(x) = 0$ for all $x \in]a, b[$, and therefore the conclusion of Rolle's theorem is satisfied.

- Suppose that f is not constant. Since f is continuous on $[a, b]$, then f is bounded and it reaches its bounds on $[a, b]$ by the theorem 3.5.1. So there exist $\alpha, \beta \in [a, b]$ such that α (resp. β) is a global minimum (resp. maximum) of f. Hence $f(\alpha) \leq f(x) \leq f(\beta)$ for all $x \in [a, b]$. If $f(\alpha) = f(\beta)$, then $f(x) = f(\alpha) = f(\beta)$ for all $x \in [a, b]$, i.e., f is constant, which is impossible. So $f(\alpha) \neq f(\beta)$. As $f(a) = f(b)$, then one of the global extremums α and β belongs to $]a, b[$ (denote by $c \in]a, b[$ this global extremum). By the remark 4.1.4, c is a local extremum of f, so $f'(c) = 0$ by the proposition 4.1.3 (since f is differentiable at c). □

Remark 4.2.1

1) *Rolle's theorem ensures the existence of a point in $]a, b[$ where the tangent to the curve of f is horizontal.*

2) *In Rolle's theorem, the point $c \in]a, b[$ such that $f'(c) = 0$ is not necessary unique. For example, if $f(x) = \sin x$, then f is continuous on $[0, 2\pi]$ and differentiable on $]0, 2\pi[$ with $f(0) = f(2\pi) = 0$. As $f'(x) = \cos x$, then $f'(\frac{\pi}{2}) = 0$ and $f'(\frac{3\pi}{2}) = 0$ with $\frac{\pi}{2}, \frac{3\pi}{2} \in]0, 2\pi[$.*

Exercise 97 (A result from algebra)
Prove that if P is a real polynomial having at least n distinct real roots, then its derivative polynomial P' has at least $(n-1)$ distinct real roots.

Solution Let $a_1, \ldots, a_n$ be distinct real roots of P such that $a_1 < \cdots < a_n$. For each $1 \leq i \leq n-1$, the polynomial function P is continuous on $[a_i, a_{i+1}]$ and differentiable on $]a_i, a_{i+1}[$ (since P is differentiable on $\mathbb{R}$) with $P(a_i) = 0 = P(a_{i+1})$. By Rolle's theorem, there exists $c_i \in]a_i, a_{i+1}[$ such that $P'(c_i) = 0$, i.e., c_i is a root of P'. In the other side, for all $1 \leq i \leq n-1$ and $1 \leq j \leq n-1$ such that $i \neq j$, we have $c_i \neq c_j$ since $]a_i, a_{i+1}[\cap]a_j, a_{j+1}[= \emptyset$. Hence $c_1, \ldots, c_{n-1}$ are distinct real roots of P'. □

Corollary 4.2.2 *(Mean-value theorem: Lagrange)*
Let $[a, b]$ be a closed bounded interval of $\mathbb{R}$ (where $a < b$). If $f : [a, b] \longmapsto \mathbb{R}$ is continuous on $[a, b]$ and differentiable on $]a, b[$, then there exists $c \in]a, b[$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof Consider the function h defined by:

$$h(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a}\right) x.$$

Since $\boldsymbol{f}$ is continuous on $[\boldsymbol{a}, \boldsymbol{b}]$ and differentiable on $]\boldsymbol{a}, \boldsymbol{b}[$, then $\boldsymbol{h}$ is continuous on $[\boldsymbol{a}, \boldsymbol{b}]$ and differentiable on $]\boldsymbol{a}, \boldsymbol{b}[$. Moreover, as

$$h(a) = f(a) - \left(\frac{f(b) - f(a)}{b - a}\right) a = bf(a) - af(b)$$

and

$$h(b) = f(b) - \left(\frac{f(b) - f(a)}{b - a}\right) b = -af(b) + bf(a),$$

then $h(a) = h(b)$. By Rolle's theorem, there exists $c \in \,]a, b[$ such that $h'(c) = 0$. But, for all $x \in \,]a, b[$,

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

So $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. □

Remark 4.2.2 *(Geometric interpretation of the mean-value theorem).*
The real number $\alpha = \dfrac{f(b) - f(a)}{b - a}$ *is the slope of the straight-line* (AB) *between the two points* $A(a, f(a))$ *and* $B(b, f(b))$ *of the curve* C_f *of* f *on* $[a, b]$*. The derivative* $f'(c)$ *is the slope of the tangent* (T) *to the curve* C_f *at the point* $C(c, f(c))$*. The equality* $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ *means that* (T) *is parallel to* (AB)*. Hence the mean-value theorem ensures the existence of a point in* $]a, b[$ *where the tangent to the curve of* f *is parallel to* (AB)*.*

Exercise 98 (Mean-value inequality)
Let $[a, b]$ be a closed bounded interval of $\mathbb{R}$ (where $a < b$). Let $f : [a, b] \longmapsto \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $]a, b[$. Suppose that there exists $k \in \mathbb{R}$ such that $|f'(x)| \leq k$ for all $x \in \,]a, b[$. Prove that

$$\left|\frac{f(b) - f(a)}{b - a}\right| \leq k.$$

Solution By the mean-value theorem, there exists $c \in \,]a, b[$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

So

$$\left|\frac{f(b) - f(a)}{b - a}\right| = |f'(c)| \leq k. \quad \square$$

Exercise 99 (Theorem of the limit of the derivative)
Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be a continuous function on an interval I in D_f and $x_0 \in I$. Prove that, if f is differentiable on $I - \{x_0\}$ and $\lim\limits_{x \to x_0} f'(x) = \ell \in \mathbb{R}$, then f is differentiable at x_0 and $f'(x_0) = \ell$.

Solution Let $x \in I$ such that $x \neq x_0$ (suppose, for example, that $x_0 < x$). Since f is continuous on I and $[x_0, x] \subseteq I$, then f is continuous on $[x_0, x]$. Since f is differentiable on $I - \{x_0\}$ and $]x_0, x[\subseteq I - \{x_0\}$, then f is differentiable on $]x_0, x[$. By the mean-value theorem, there exists $c_x \in]x_0, x[$ such that

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(c_x).$$

As $x_0 < c_x < x$, then $\lim\limits_{x \to x_0} c_x = x_0$ (Sandwich theorem), and therefore, by using the theorem 3.2.1, we obtain:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{c_x \to x_0} f'(c_x) = \ell \in \mathbb{R}.$$

Hence f is differentiable at x_0 and $f'(x_0) = \ell$. □

Exercise 100
By applying the mean-value theorem, show that if a and b are two real numbers such that $0 < a < b$, then

$$1 - \frac{a}{b} < \ln \frac{b}{a} < \frac{b}{a} - 1.$$

Solution Let a, b be two real numbers such that $0 < a < b$. Since the function $\ln$ is continuous on the interval $[a, b]$ and differentiable on $]a, b[$ (since $\ln$ is differentiable on the interval $]0, +\infty[$ and $[a, b] \subseteq]0, +\infty[$), then, by the mean-value theorem, there exists $c \in]a, b[$ such that

$$\ln \frac{b}{a} = \ln b - \ln a = (b - a) \ln'(c) = \frac{b - a}{c}.$$

But $a < c < b$, then $\frac{1}{b} < \frac{1}{c} < \frac{1}{a}$. As $b - a > 0$, then

$$\frac{b - a}{b} < \frac{b - a}{c} < \frac{b - a}{a}.$$

Hence

$$1 - \frac{a}{b} < \ln \frac{b}{a} < \frac{b}{a} - 1. \quad \square$$

Exercise 101

1) **Let $x \in \mathbb{R}_+^*$. By applying the mean-value theorem, show that**

$$\frac{x}{1 + x} < \ln(1 + x) < x.$$

2) **Deduce the value of the limit $\lim\limits_{t \to +\infty} (1 + t^2) \ln\left(1 + \frac{1}{t^2}\right)$.**

Solution

1) Let $x \in \mathbb{R}_+^*$. Since the function f defined by $f(t) = \ln(1+t)$ is continuous on the interval $[0, x]$ and differentiable on $]0, x[$ (since f is differentiable on the interval $]-1, +\infty[$ and $[0, x] \subseteq]-1, +\infty[$), then, by the mean-value theorem, there exists $c \in]0, x[$ such that

$$\ln(1+x) - \ln(1) = xf'(c) = \frac{x}{1+c}$$

by using the fact that $f'(t) = \dfrac{1}{1+t}$ for all $t \in]0, x[$. But $0 < c < x$, then $1 < 1 + c < 1 + x$, and therefore $\dfrac{1}{1+x} < \dfrac{1}{1+c} < 1$. Since $x > 0$, then

$$\frac{x}{1+x} < \frac{x}{1+c} < x$$

Hence

$$\frac{x}{1+x} < \ln(1+x) < x. \qquad (*)$$

2) Let $t > 0$. Take $x = \frac{1}{t^2} > 0$ in the double inequality $(*)$, we obtain:

$$\frac{\frac{1}{t^2}}{1+\frac{1}{t^2}} < \ln\left(1+\frac{1}{t^2}\right) < \frac{1}{t^2}.$$

So

$$1 < (1+t^2)\ln(1+\frac{1}{t^2}) < \frac{t^2+1}{t^2}. \qquad (**)$$

By taking limits in $(**)$ and by using the Sandwich theorem, we obtain:

$$\lim_{t\to+\infty}(1+t^2)\ln\left(1+\frac{1}{t^2}\right) = 1. \quad \square$$

Theorem 4.2.3 *(Derivative and monotonicity)*
Let I be a **non trivial** *interval of $\mathbb{R}$ and $f : I \longmapsto \mathbb{R}$ be a* **differentiable** *function on I.*

1) f is increasing on I if and only if $f'(x) \geq 0$ for all $x \in I$.

2) f is decreasing on I if and only if $f'(x) \leq 0$ for all $x \in I$.

3) f is constant on I if and only if $f'(x) = 0$ for all $x \in I$.

Proof

1) N.C. Let $x_0 \in I$. Since f is increasing on I, then, by the remark 3.1.6, for any $x \in I$ such that $x \neq x_0$, we have:

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

As f is differentiable at x_0, then

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

S.C. Let $a, b \in I$ such that $a < b$. Since f is differentiable on I, then f is differentiable on $]a, b[$ (since $]a, b[\subseteq I$) and f is continuous on $[a, b]$ (since f is continuous on I and $[a, b] \subseteq I$). By the mean-value theorem, there exists $c \in]a, b[$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) \geq 0.$$

As $b - a > 0$, then $f(b) - f(a) \geq 0$, so $f(a) \leq f(b)$. Hence f is increasing on I.

2) Put $g(x) = -f(x)$. Since f is differentiable on I, then g is differentiable on I and $g'(x) = -f'(x)$ for all $x \in I$. Hence

$$\begin{aligned} f \text{ is decreasing on } I \quad &\Leftrightarrow \quad g \text{ is increasing on } I \\ &\Leftrightarrow \quad g'(x) \geq 0 \text{ for all } x \in I \quad (\text{by the part (1)}) \\ &\Leftrightarrow \quad f'(x) \leq 0 \text{ for all } x \in I \quad (\text{since } g'(x) = -f'(x)). \end{aligned}$$

3) We have:

$$\begin{aligned} f \text{ is constant on } I \quad &\Leftrightarrow \quad f \text{ is increasing and decreasing on } I \\ &\Leftrightarrow \quad f'(x) \geq 0 \text{ and } f'(x) \leq 0 \text{ for all } x \in I \\ &\qquad (\text{by the parts (1) and (2)}) \\ &\Leftrightarrow \quad f'(x) = 0 \text{ for all } x \in I. \quad \square \end{aligned}$$

Remark 4.2.3 *Let I be a non trivial interval of $\mathbb{R}$ and $f, g : I \longmapsto \mathbb{R}$ be two differentiable functions on I. If $f'(x) = g'(x)$ for all $x \in I$, then there exists a constant $c \in \mathbb{R}$ such that $f(x) = g(x) + c$ for all $x \in I$. Indeed, let $h = f - g$, then h is differentiable on I and $h'(x) = f'(x) - g'(x) = 0$ for all $x \in I$. So h is constant on I by the theorem 4.2.3, i.e., there exists a constant $c \in \mathbb{R}$ such that $h(x) = c$ for all $x \in I$. Hence $f(x) = g(x) + c$ for all $x \in I$.*

Theorem 4.2.4 *(Derivative and strict monotonicity)*
Let I be a **non trivial** *interval of $\mathbb{R}$ and $f : I \longmapsto \mathbb{R}$ be a* **differentiable** *function on I.*

1) *f is strictly increasing on I if and only if $f'(x) \geq 0$ for all $x \in I$ and f' does not vanish on any non trivial sub-interval of I.*

2) *f is strictly decreasing on I if and only if $f'(x) \leq 0$ for all $x \in I$ and f' does not vanish on any non trivial sub-interval of I.*

Proof

1) N.C. Since f is strictly increasing on I, then f is increasing on I, so $f'(x) \geq 0$ for all $x \in I$ by the part (1) of the theorem 4.2.3. In the other side, if f' vanishes on a non trivial sub-interval J of I, then f is constant on J by the part (3) of the theorem 4.2.3. As J is non trivial, then there exist $a, b \in J$ such that $a \neq b$, so $f(a) = f(b)$ (since f is constant on J), which is impossible since f is strictly increasing on I.
S.C. Since $f'(x) \geq 0$ for all $x \in I$, then f is increasing on I by the part (1) of the theorem 4.2.3. Suppose that f is not strictly increasing on I, then there exist $a, b \in I$ such that $a < b$ and $f(a) \geq f(b)$. For any $x \in [a, b]$, as f is increasing on I, then
$$f(a) \leq f(x) \leq f(b).$$
But $f(a) \geq f(b)$, then, for all $x \in [a, b]$,
$$f(b) \leq f(x) \leq f(b).$$
So $f(x) = f(b)$ for all $x \in [a, b]$ (i.e., f is constant on $[a, b]$), and therefore $f'(x) = 0$ for all $x \in [a, b]$, i.e., f' vanishes on the non trivial sub-interval $[a, b]$ of I, which is impossible. Hence f is strictly increasing on I.

2) This part can be deduced from the part (1) by using the function $g = -f$ as in the proof of the part (2) of the theorem 4.2.3.

Remark 4.2.4 *Let I be a* **non trivial** *interval of $\mathbb{R}$ and $f : I \longmapsto \mathbb{R}$ be a* **differentiable** *function on I.*

1) If $f'(x) \geq 0$ (resp. $f'(x) \leq 0$) for all $x \in I$ and f' vanishes only at a finite number of points of I, then f is strictly increasing (resp. strictly decreasing) on I.

2) If $f'(x) > 0$ (resp. $f'(x) < 0$) for all $x \in I$, then f is strictly increasing (resp. strictly decreasing) on I. The converse is not necessary true. For example, the function f defined by $f(x) = x^3$ is strictly increasing on $\mathbb{R}$ (since f is differentiable on $\mathbb{R}$, $f'(x) = 3x^2 \geq 0$ for all $x \in \mathbb{R}$ and f' vanishes only at 0). On the other hand, $f'(x)$ is not strictly positive for all $x \in \mathbb{R}$.

Example 4.2.1

1) Let $f(x) = x^2$, then $f'(x) = 2x$. So $f'(x) = 0$ if and only if $x = 0$. Since $f'(x) \geq 0$ for all $x \in [0, +\infty[$, then f is strictly increasing on $[0, +\infty[$. In the other side, as $f'(x) \leq 0$ for all $x \in]-\infty, 0]$, then f is strictly decreasing on $]-\infty, 0]$.

2) If $f(x) = x^3 + 2x - 1$, then $f'(x) = 3x^2 + 2 > 0$ for all $x \in \mathbb{R}$. So f is strictly increasing on $\mathbb{R}$.

3) If $f(x) = \sin x$, then $f'(x) = \cos x \geq 0$ for all $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and f' vanishes only at the two points $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ of $[-\frac{\pi}{2}, \frac{\pi}{2}]$. So f is strictly increasing on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

4) If $f(x) = \cos x$, then $f'(x) = -\sin x \leq 0$ for all $x \in [0, \pi]$ and f' vanishes only at the two points 0 and π of $[0, \pi]$. So f is strictly decreasing on the interval $[0, \pi]$.

5) If $f(x) = \tan x$, then $f'(x) = \dfrac{1}{\cos^2 x} > 0$ for all $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$. So f is strictly increasing on the interval $]-\frac{\pi}{2}, \frac{\pi}{2}[$.

6) Let $f(x) = \cos^2 x + \sin^2 x$. Since f is differentiable on the interval $\mathbb{R}$ and, for all $x \in \mathbb{R}$,

$$f'(x) = 2(\cos x)(-\sin x) + 2(\sin x)(\cos x) = 0,$$

then f is a constant function on $\mathbb{R}$. So, for all $x \in \mathbb{R}$,

$$f(x) = f(0) = \cos^2 0 + \sin^2 0 = 1. \quad \square$$

Exercise 102
Prove that, for any $x \geq 0$,

$$x - \frac{x^2}{2} \leq \ln(1 + x) \leq x.$$

Solution Let $I = [0, +\infty[$. For every $x \in I$, put:

$$f(x) = x - \ln(1 + x) \quad \text{and} \quad g(x) = \ln(1 + x) - x + \frac{x^2}{2}.$$

- The function f is differentiable on I and, for all $x \in I$,

$$f'(x) = 1 - \frac{1}{1 + x} = \frac{x}{1 + x} \geq 0.$$

  So f is increasing on the interval I. Hence, if $x \geq 0$, then $f(x) \geq f(0) = 0$, so $\ln(1 + x) \leq x$.

- The function g is differentiable on I and, for all $x \in I$,

$$g'(x) = \frac{1}{1 + x} - 1 + x = \frac{x^2}{1 + x} \geq 0.$$

  So g is increasing on the interval I. Hence, if $x \geq 0$, then $g(x) \geq g(0) = 0$, so $x - \frac{x^2}{2} \leq \ln(1 + x)$. $\square$

Exercise 103
Let $n \geq 2$ be a natural number. Consider the function f defined on $\mathbb{R}^+$ by:

$$f(x) = x^n - nx - 1.$$

1) **Study the variations of f on $\mathbb{R}^+$.**

2) **Show that the equation $f(x) = 0$ admits a unique solution in $\mathbb{R}^+$.**

Solution

1) The function f is continuous and differentiable on $\mathbb{R}^+$ with $f'(x) = nx^{n-1} - n$ for all $x \in \mathbb{R}^+$. This derivative vanishes only at 1 in $\mathbb{R}^+$. Moreover, $f'(x) \leq 0$ if $x \leq 1$ and $f'(x) \geq 0$ if $x \geq 1$. The table of variations of f on $\mathbb{R}^+$ is given by:

x	0		1		$+\infty$
$f'(x)$		$-$		$+$	
f	-1	$\searrow$	$-n$	$\nearrow$	$+\infty$

2)
- Since f is continuous and strictly decreasing on the interval $[0, 1]$, then, by the inverse functions theorem, f is a bijection from $[0, 1]$ on the interval $f([0, 1]) = [-n, -1]$. As $0 \notin [-n, -1]$, then the equation $f(x) = 0$ has no solutions in $[0, 1]$.
- Since f is continuous and strictly increasing on the interval $[1, +\infty[$, then, by the inverse functions theorem, f is a bijection from $[1, +\infty[$ on the interval $f([1, +\infty[) = [-n, +\infty[$. As $0 \in [-n, +\infty[$, then there exists one and only one $c \in [1, +\infty[$ such that $f(c) = 0$, i.e., c is the unique solution of the equation $f(x) = 0$ in $[1, +\infty[$.

Hence c is the unique solution of the equation $f(x) = 0$ in $\mathbb{R}^+$. □

Exercise 104
Let f be the function defined by $f(x) = \dfrac{x}{x^2 + x + 1}$.

1) Study the variations of f on its domain of definition.

2) Show that f is a bijection from the interval $[-1, 1]$ on an interval $[a, b]$ to be determined. We denote by f^{-1} its inverse function.

3) Show that f^{-1} is differentiable on $]a, b[$ but it is not differentiable at a, neither at b.

4) Determine the equation of the tangent to the curve of f^{-1} at the point of abscissa $-\frac{2}{3}$.

Solution

1) The discriminant of the trinomial $x^2 + x + 1$ is $\Delta = -3 < 0$, so the denominator of $f(x)$ does not vanish at any point of $\mathbb{R}$, and therefore the domain of definition of f is $D_f = \mathbb{R}$. Moreover, f is differentiable on $\mathbb{R}$ as it is the quotient of two differentiable functions on $\mathbb{R}$ where the denominator does not vanish, and, for all $x \in \mathbb{R}$,

$$(*) \qquad f'(x) = \frac{1 - x^2}{(x^2 + x + 1)^2}.$$

$f'(x)$ vanishes only at ± 1, and $f'(x) \geq 0$ if and only if $x \in [-1, 1]$. The table of variations of f on $\mathbb{R}$ is given by:

x	$-\infty$		-1		1		$+\infty$
$f'(x)$		$-$	0	$+$	0	$-$	
f	0	$\searrow$	-1	$\nearrow$	$\frac{1}{3}$	$\searrow$	0

2) Since $f'(x) \geq 0$ for all $x \in [-1, 1]$ and $f'(x) = 0$ only at $x = \pm 1$, then f is strictly increasing on the interval $[-1, 1]$. Moreover, as f is continuous on $[-1, 1]$, then, by the inverse functions theorem, f is a bijection from $[-1, 1]$ on the interval $[a, b] = f([-1, 1]) = [-1, \frac{1}{3}]$.

3) Since f is differentiable on $]-1, 1[$ and $f'(x) \neq 0$ for all $x \in]-1, 1[$, then the inverse function f^{-1} is differentiable on $f(]-1, 1[) =]-1, \frac{1}{3}[$. In the other side, as f is differentiable at -1 and at 1 with $f'(-1) = 0$ and $f'(1) = 0$, then f^{-1} is not differentiable at $f(-1) = -1$ (from the right), neither at $f(1) = \frac{1}{3}$ (from the left).

4) Let $y_0 = -\frac{2}{3}$ and $x_0 = f^{-1}(y_0) \in [-1, 1]$. Since $y_0 \in]-1, \frac{1}{3}[$, then f^{-1} is differentiable at y_0 by the part (3). Moreover,

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{f'(x_0)}.$$

But $f(x_0) = y_0$, then $\dfrac{x_0}{x_0^2 + x_0 + 1} = -\dfrac{2}{3}$, so $2x_0^2 + 5x_0 + 2 = 0$. The solutions of this last equation are -2 and $-\frac{1}{2}$. As $x_0 \in [-1, 1]$, then $x_0 = -\frac{1}{2}$. So, by using $(*)$, we obtain:

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{3}{4}.$$

The equation of the tangent (T) to the curve of f^{-1} at the point of abscissa y_0 is:

$$(T): \; y - f^{-1}(y_0) = (f^{-1})'(y_0)(x - y_0).$$

So $(T): \; y + \frac{1}{2} = \frac{3}{4}(x + \frac{2}{3})$, i.e., $y = \frac{3}{4}x$. □

Definition 4.2.1 *Let I be an interval of $\mathbb{R}$ and $f : I \longmapsto \mathbb{R}$ be a function defined on I. We denote by C_f the representative curve of f on I.*

1) *We say that f is convex on I (or that C_f is convex) if, for any two points A and B of the curve C_f, the segment $[AB]$ is above C_f. In this case, we say that the curve C_f turns upward its concavity.*

2) *We say that f is concave on I (or that C_f is concave) if the function $-f$ is convex on I, i.e., if, for any two points A and B of the curve C_f, the segment $[AB]$ is below C_f. In this case, we say that the curve C_f turns downward its concavity.*

3) *We say that a point x_0 of I is an inflection point of f (or that the point $(x_0, f(x_0))$ is an inflection point of C_f) if C_f changes its concavity by passing through the point of abscissa x_0.*

Remark 4.2.5 *Let I be an interval of $\mathbb{R}$ and $f : I \longmapsto \mathbb{R}$ be a function defined on I.*

1) *By using the exercise 4, the function f is convex on I if and only if, for any $x, y \in I$ and $t \in [0, 1]$, we have:*

$$f\Big(tx + (1-t)y\Big) \leq tf(x) + (1-t)f(y).$$

2) Let C_f be the representative curve of f on I. If f is differentiable on I, then:

- *f is convex on I if and only if the curve C_f is above each of its tangents.*
- *f is concave on I if and only if the curve C_f is below each of its tangents.*
- *A point x_0 of I is an inflection point of f if and only if the tangent at the point of abscissa x_0 crosses the curve C_f.*

Example 4.2.2

1) Every constant real function is convex and concave on $\mathbb{R}$.

2) The function f defined by $f(x) = |x|$ is convex on $\mathbb{R}$. Indeed, if $x, y \in \mathbb{R}$ and $t \in [0,1]$, then

$$f\Big(tx + (1-t)y\Big) = \Big|tx + (1-t)y\Big| \leq t|x| + (1-t)|y| = tf(x) + (1-t)f(y).$$

Theorem 4.2.5 *(Derivative and concavity)*
Let I be an interval of $\mathbb{R}$ and $f : I \longmapsto \mathbb{R}$ be a differentiable function on I.

1) f is convex on I if and only if the derivative function f' is increasing on I.

2) f is concave on I if and only if the derivative function f' is decreasing on I.

Proof

1) N.C. Suppose that f is convex on I, then the curve C_f of f on I is above each of its tangents. Let $x_1, x_2 \in I$ such that $x_1 \leq x_2$. Since f is differentiable on I, then the equation of the tangent (T_1) (resp. (T_2)) to C_f at the point of abscissa x_1 (resp. x_2) is

$$(T_1):\ y = f(x_1) + f'(x_1)(x - x_1) \quad \Big(\text{resp. } (T_1):\ y = f(x_2) + f'(x_2)(x - x_2)\Big).$$

Since C_f is above (T_1) and (T_2), then $f(x) - y$ for all $x \in I$. So, for all $x \in I$,

$$(*) \qquad f(x) - f(x_1) - f'(x_1)(x - x_1) \geq 0.$$

and

$$(**) \qquad f(x) - f(x_2) - f'(x_2)(x - x_2) \geq 0.$$

For $x = x_2$ in $(*)$ and for $x = x_1$ in $(**)$, we obtain:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq f'(x_1) \qquad \text{and} \qquad \frac{f(x_1) - f(x_2)}{x_1 - x_2} \leq f'(x_2).$$

So

$$f'(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1) - f(x_2)}{x_1 - x_2} \leq f'(x_2).$$

Hence f' is increasing on I.
S.C. Suppose that f' is increasing on I. In order to prove that f is convex on I, it is

sufficient to show that the curve C_f of f on I is above each of its tangents. Let $x_0 \in I$ and (T) be the tangent to C_f at the point of abscissa x_0. Since f is differentiable on I, then the equation of (T) is

$$y = f(x_0) + f'(x_0)(x - x_0).$$

In order to study the position of (T) with respect to C_f, consider the function g defined by:

$$g(x) = f(x) - y = f(x) - f(x_0) - f'(x_0)(x - x_0).$$

Since f is differentiable on I, then g is differentiable on I and, for all $x \in I$,

$$g'(x) = f'(x) - f'(x_0).$$

- If $x \in I$ and $x \leq x_0$, then $f'(x) \leq f'(x_0)$ (since f' is increasing on I), so $g'(x) \leq 0$. Hence g is decreasing on the sub-interval of I from the left of x_0.
- If $x \in I$ and $x \geq x_0$, then $f'(x) \geq f'(x_0)$ (since f' is increasing on I), so $g'(x) \geq 0$. Hence g is increasing on the sub-interval of I from the right of x_0.

The table of variations of g on I is:

x		x_0	
$g'(x)$	$-$	0	$+$
g	$\searrow$	0	$\nearrow$

So x_0 is a global minimum of g, and therefore $g(x) \geq g(x_0)$ for all $x \in I$. But $g(x_0) = 0$, then $g(x) \geq 0$ for all $x \in I$. Hence $f(x) \geq y$ for all $x \in I$, i.e., C_f is above (T). Consequently, f is convex on I.

2) We have:

$$\begin{aligned} f \text{ is concave on } I &\Leftrightarrow -f \text{ is convex on } I \\ &\Leftrightarrow -f' \text{ is increasing on } I \quad \text{(by the part (1))} \\ &\Leftrightarrow f' \text{ is decreasing on } I. \quad \square \end{aligned}$$

Corollary 4.2.3 *Let I be an interval of $\mathbb{R}$ and $f : I \longmapsto \mathbb{R}$ be a twice-differentiable function on I.*

1) *f is convex on I if and only if $f''(x) \geq 0$ for all $x \in I$.*

2) *f is concave on I if and only if $f''(x) \leq 0$ for all $x \in I$.*

3) *A point x_0 of I is an inflection point of f if and only if $f''(x_0) = 0$ and $f''(x)$ changes its sign when x crosses x_0.*

Proof Since f is twice-differentiable on I, then f' is differentiable on I.

1) We have:

$$\begin{aligned} f \text{ is convex on } I &\Leftrightarrow f' \text{ is increasing on } I \quad \text{(by the theorem 4.2.5)} \\ &\Leftrightarrow f''(x) \geq 0 \text{ for all } x \in I \quad \text{(by the theorem 4.2.3)}. \end{aligned}$$

2) We have:

$$\begin{aligned} f \text{ is concave on } I &\Leftrightarrow f' \text{ is decreasing on } I \quad \text{(by the theorem 4.2.5)} \\ &\Leftrightarrow f''(x) \leq 0 \text{ for all } x \in I \quad \text{(by the theorem 4.2.3).} \ \square \end{aligned}$$

3) This is a direct consequence of the definition of an inflection point and the parts (1) and (2) above.

Example 4.2.3

1) *Let $a > 0$ and $f(x) = ax^2$. Since f is twice-differentiable on $\mathbb{R}$ and $f''(x) = 2a \geq 0$ for all $x \in \mathbb{R}$, then f is convex on $\mathbb{R}$.*

2) *If $f(x) = x^3$, then $f''(x) = 6x \geq 0$ if $x \in [0, +\infty[$ and $f''(x) \leq 0$ if $x \in]-\infty, 0]$. So f is convex on $[0, +\infty[$ and concave on $]-\infty, 0]$. As $f''(0) = 0$ and $f''(x)$ changes its sign when x crosses 0, then 0 is an inflection point of f. So the origin is an inflection point for the curve of f on $\mathbb{R}$.*

x	$-\infty$		0		$+\infty$
$f''(x)$		$-$	0	$+$	
Concavity		*downward*	*inflection*	*upward*	

3) *The natural logarithm is concave on $]0, +\infty[$ since $(\ln x)'' = \frac{-1}{x^2} \leq 0$ for all $x \in]0, +\infty[$.*

4) *The exponential function is convex on $\mathbb{R}$ since $(e^x)'' = e^x \geq 0$ for all $x \in \mathbb{R}$.*

Exercise 105
Study the concavity of the function f defined by $f(x) = xe^{-x}$ and show that it admits an inflection point to be determined.

Solution The function f is twice-differentiable on $\mathbb{R}$ and, for all $x \in \mathbb{R}$,

$$f'(x) = (1-x)e^{-x}, \qquad f''(x) = (x-2)e^{-x}$$

Since $f''(x) \geq 0$ if $x \geq 2$ and $f''(x) \leq 0$ if $x \leq 2$, then f is convex on $[2, +\infty[$ and concave on $]-\infty, 2]$. Moreover, $f''(x) = 0$ if and only if $x = 2$. So 2 is an inflection point of f. As $f(2) = 2e^{-2}$, then the point $(2, 2e^{-2})$ is an inflection point for the curve of f. $\square$

4.3 The inverse functions of the trigonometric functions

4.3.1 The arc sine function

Since the sine function is continuous and strictly increasing on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (by the examples 3.3.1 and 4.2.1). Then, by the inverse functions theorem, the function **sin** has an inverse function $\sin^{-1}$ (denoted by **arcsin**) defined, **continuous** and **strictly increasing** on the interval

$$\sin\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right) = \left[\sin\left(-\frac{\pi}{2}\right), \sin\left(\frac{\pi}{2}\right)\right] = [-1, 1].$$

Hence

$$\boxed{\arcsin : \quad [-1,1] \longmapsto \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$$

For any $x \in [-1, 1]$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have:

$$\boxed{\arcsin x = y \text{ if and only if } \sin y = x}$$

Moreover,

$$\boxed{\arcsin(\sin x) = x \text{ for all } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$$

and

$$\boxed{\sin(\arcsin x) = x \text{ for all } x \in [-1, 1]}$$

Proposition 4.3.1

1) *We have the following table of particular values:*

x	-1	0	1	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\arcsin x$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{6}$

2) *The function* **arcsin** *is odd, i.e., if* $x \in [-1, 1]$*, then* $-x \in [-1, 1]$ *and*

$$\boxed{\arcsin(-x) = -\arcsin x}$$

3) *For all* $x \in [-1, 1]$*,*

$$\cos(\arcsin x) = \sqrt{1 - x^2}.$$

Proof

1) This is due to the fact that if we apply **sin** to the second row of the table, we obtain the first row.

2) Let $x \in [-1, 1]$. Since **sin** is an odd function, then

$$\sin(-\arcsin x) = -\sin(\arcsin x) = -x.$$

So $\arcsin(-x) = -\arcsin x$.

3) Let $x \in [-1, 1]$ and $y = \arcsin x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\sin y = x$. But we know that $\cos^2 y + \sin^2 y = 1$, then $\cos^2 y = 1 - \sin^2 y = 1 - x^2$. As $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\cos y \geq 0$. So

$$\cos(\arcsin x) = \cos y = \sqrt{1 - x^2}. \quad \square$$

Proposition 4.3.2 *The* **arcsin** *function is differentiable on* $]-1,1[$ *with*

$$\boxed{(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \quad \textit{for all } x \in]-1,1[}$$

Moreover, **arcsin** *is not right differentiable at* -1*, neither left differentiable at* 1*. Consequently, if* $u : I \longmapsto]-1,1[$ *is a differentiable function on an interval* I *of* $\mathbb{R}$*, then the composite function* $\arcsin \circ u$ *is differentiable on* I *and*

$$(\arcsin u)'(x) = \frac{u'}{\sqrt{1-u^2}} \quad \textit{for all } x \in I.$$

$\boxed{\text{Proof}}$ Since $(\sin x)' = \cos x = 0$ if and only if $x \in \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$, then, by the proposition 4.1.6, the function **arcsin** is differentiable at every point of $[-1,1]$ except at the points $\sin\left(-\frac{\pi}{2}\right) = -1$ and $\sin\left(\frac{\pi}{2}\right) = 1$. Hence **arcsin** is differentiable on $]-1,1[$ with

$$(\arcsin x)' = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}} \quad \text{for all } x \in]-1,1[,$$

by using the proposition 4.3.1. In the other side, the derivative of the composite function $\arcsin \circ u$ is obtained by using the proposition 4.1.5. □

Remark 4.3.1 *Let* $x \in]-1,1[$ *and* $y = \arcsin x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$*, then* $\sin y = x$*. Differentiating this last relation with respect to* x*, and by using the proposition 4.1.5, we obtain* $y' \cos y = 1$*. As* $\cos y \neq 0$*, then*

$$y' = \frac{1}{\cos y} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}},$$

by using the proposition 4.3.1.

4.3.2 The arc cosine function

Since the cosine function is continuous and strictly decreasing on the interval $[0,\pi]$ (by the examples 3.3.1 and 4.2.1). Then, by the inverse functions theorem, the function **cos** has an inverse function $\cos^{-1}$ (denoted by **arccos**) defined, **continuous** and **strictly decreasing** on the interval

$$\cos([0,\pi]) = [\cos(\pi), \cos(0)] = [-1,1].$$

Hence

$$\boxed{\arccos : \quad [-1,1] \longmapsto [0,\pi]}$$

For any $x \in [-1,1]$ and $y \in [0,\pi]$, we have:

$$\boxed{\arccos x = y \quad \text{if and only if} \quad \cos y = x}$$

Moreover,

$$\boxed{\arccos(\cos x) = x \text{ for all } x \in [0, \pi]}$$

and

$$\boxed{\cos(\arccos x) = x \text{ for all } x \in [-1, 1]}$$

Proposition 4.3.3

1) *We have the following table of particular values:*

x	-1	0	1	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\arccos x$	π	$\frac{\pi}{2}$	0	$\frac{\pi}{4}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$

2) *If $x \in [-1, 1]$, then $-x \in [-1, 1]$ and*

$$\boxed{\arccos(-x) = \pi - \arccos x}$$

The function **arccos** *is neither even, nor odd.*

3) *For any $x \in [-1, 1]$,*

$$\sin(\arccos x) = \sqrt{1 - x^2}.$$

$\boxed{\text{Proof}}$

1) This is due to the fact that if we apply **cos** to the second row of the table, we obtain the first row.

2) Let $x \in [-1, 1]$. Since

$$\cos(\pi - \arccos x) = -\cos(\arccos x) = -x,$$

then $\arccos(-x) = \pi - \arccos x$.

3) Let $x \in [-1, 1]$ and $y = \arccos x \in [0, \pi]$, then $\cos y = x$. But we know that $\cos^2 y + \sin^2 y = 1$, then $\sin^2 y = 1 - \cos^2 y = 1 - x^2$. As $y \in [0, \pi]$, then $\sin y \geq 0$. So

$$\sin(\arccos x) = \sin y = \sqrt{1 - x^2}. \quad \square$$

Proposition 4.3.4 *The function* **arccos** *is differentiable on* $]-1, 1[$ *with*

$$\boxed{(\arccos x)' = \frac{-1}{\sqrt{1 - x^2}} \quad \textit{for all } x \in]-1, 1[}$$

Moreover, **arccos** *is not right differentiable at -1, neither left differentiable at 1. Consequently, if $u : I \longmapsto]-1, 1[$ is a differentiable function on an interval I of $\mathbb{R}$, then the composite function $\arccos \circ u$ is differentiable on I and*

$$(\arccos u)'(x) = \frac{-u'}{\sqrt{1 - u^2}} \quad \textit{for all } x \in I.$$

Proof Since $(\cos x)' = -\sin x = 0$ if and only if $x \in \{0, \pi\}$, then, by the proposition 4.1.6, the function **arccos** is differentiable at every point of $[-1, 1]$ except at the points $\cos(0) = 1$ and $\cos(\pi) = -1$. Hence **arccos** is differentiable on $]-1, 1[$ with

$$(\arccos x)' = \frac{1}{-\sin(\arccos x)} = \frac{-1}{\sqrt{1-x^2}} \quad \text{for all } x \in \,]-1, 1[,$$

by using the proposition 4.3.3. In the other side, the derivative of the composite function $\arccos \circ u$ is obtained by using the proposition 4.1.5. □

Remark 4.3.2 *Let $x \in \,]-1,1[$ and $y = \arccos x \in \,]0, \pi[$, then $\cos y = x$. Differentiating this last relation with respect to x, and by using the proposition 4.1.5, we obtain $-y' \sin y = 1$. As $\sin y \neq 0$, then*

$$y' = \frac{-1}{\sin y} = \frac{-1}{\sin(\arccos x)} = \frac{-1}{\sqrt{1-x^2}},$$

by using the proposition 4.3.1.

Exercise 106
Prove that

$$\arcsin(\cos(2x)) = \begin{cases} \frac{\pi}{2} - 2x & \text{if } x \in \left[0, \frac{\pi}{2}\right] \\ 2x - \frac{3\pi}{2} & \text{if } x \in \left[\frac{\pi}{2}, \pi\right] \end{cases}$$

Solution

- Suppose that $x \in \left[0, \frac{\pi}{2}\right]$, then $\frac{\pi}{2} - 2x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. As

  $$\sin\left(\frac{\pi}{2} - 2x\right) = \cos(2x),$$

  then $\arcsin(\cos(2x)) = \frac{\pi}{2} - 2x$.

- Suppose that $x \in \left[\frac{\pi}{2}, \pi\right]$, then $2x - \frac{3\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. As

  $$\sin\left(2x - \frac{3\pi}{2}\right) = \sin\left(2x - \frac{3\pi}{2} + 2\pi\right) = \sin\left(2x + \frac{\pi}{2}\right) = \cos(-2x) = \cos(2x),$$

  then $\arcsin(\cos(2x)) = 2x - \frac{3\pi}{2}$. □

Exercise 107
Prove that, for any $x \in [-1, 1]$,

$$\arcsin x + \arccos x = \frac{\pi}{2}.$$

Solution Consider the function f defined by $f(x) = \arcsin x + \arccos x$. Since **arcsin** and **arccos** are differentiable on $]-1, 1[$, then f is differentiable on $]-1, 1[$. Moreover, for any $x \in]-1, 1[$,

$$f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}} = 0.$$

Hence f is a constant function on the interval $]-1, 1[$. For any $x \in]-1, 1[$,

$$f(x) = f(0) = \arcsin 0 + \arccos 0 = 0 + \frac{\pi}{2} = \frac{\pi}{2}.$$

In the other side,

$$f(1) = \arcsin 1 + \arccos 1 = \frac{\pi}{2} \quad \text{and} \quad f(-1) = \arcsin(-1) + \arccos(-1) = \frac{\pi}{2}.$$

So $f(x) = \frac{\pi}{2}$ for all $x \in [-1, 1]$, □

4.3.3 The arc tangent function

Since the tangent function is continuous and strictly increasing on the interval $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ (by the examples 3.3.1 and 4.2.1). Then, by the inverse functions theorem, the function **tan** has an inverse function $\tan^{-1}$ (denoted by **arctan**) defined, **continuous** and **strictly increasing** on the interval

$$\tan\left(\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right) = \left]\lim_{x\to(-\frac{\pi}{2})^+} \tan x, \lim_{x\to(\frac{\pi}{2})^-} \tan x\right[=]-\infty, +\infty[\ = \mathbb{R}.$$

Hence

$$\boxed{\arctan : \mathbb{R} \longmapsto \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[}$$

For any $x \in \mathbb{R}$ and $y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, we have:

$$\boxed{\arctan x = y \text{ if and only if } \tan y = x}$$

Moreover,

$$\boxed{\arctan(\tan x) = x \quad \text{for all } x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[}$$

and

$$\boxed{\tan(\arctan x) = x \quad \text{for all } x \in \mathbb{R}}$$

Proposition 4.3.5

1) *We have the following table of particular values:*

x	-1	0	1	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$
$\arctan x$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{6}$

2) The function **arctan** *is odd, i.e., for all* $x \in \mathbb{R}$,

$$\boxed{\arctan(-x) = -\arctan x}$$

3) We have:

$$\boxed{\lim_{x \to -\infty} \arctan x = -\frac{\pi}{2}} \quad \text{and} \quad \boxed{\lim_{x \to +\infty} \arctan x = \frac{\pi}{2}}$$

Proof

1) This is due to the fact that if we apply **tan** to the second row of the table, we obtain the first row.

2) Let $x \in \mathbb{R}$. Since **tan** is an odd function, then

$$\tan(-\arctan x) = -\tan(\arctan x) = -x.$$

So $\arctan(-x) = -\arctan x$.

3) This is obtained by using the remark 3.5.2 and the fact that

$$\lim_{x \to (-\frac{\pi}{2})^+} \tan x = -\infty \qquad \text{and} \qquad \lim_{x \to (\frac{\pi}{2})^-} \tan x = +\infty. \quad \square$$

Proposition 4.3.6 *The function* **arctan** *is differentiable on* $\mathbb{R}$ *with*

$$\boxed{(\arctan x)' = \frac{1}{1+x^2} \quad \text{for all } x \in \mathbb{R}}$$

Consequently, if $u : I \longmapsto \mathbb{R}$ *is a differentiable function on an interval* I *of* $\mathbb{R}$*, then the composite function* $\arctan \circ u$ *is differentiable on* I *and*

$$(\arctan u)'(x) = \frac{u'}{1+u^2} \quad \text{for all } x \in I.$$

Proof Since $(\tan x)' = \dfrac{1}{\cos^2 x} = 1 + \tan^2 x \neq 0$ for all $x \in \mathbb{R}$, then, by the proposition 4.1.6, the function **arctan** is differentiable at every point of $\mathbb{R}$. Hence **arctan** is differentiable on $\mathbb{R}$ with

$$(\arctan x)' = \frac{1}{1 + \tan^2(\arctan x)} = \frac{1}{1+x^2} \quad \text{for all } x \in \mathbb{R}.$$

In the other side, the derivative of the composite function $\arctan \circ u$ is obtained by using the proposition 4.1.5. $\square$

Remark 4.3.3

1) *Let* $x \in \mathbb{R}$ *and* $y = \arctan x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, *then* $\tan y = x$. *Differentiating this last relation with respect to* x, *and by using the proposition 4.1.5, we obtain* $y'(1+\tan^2 y) = 1$. *As* $1 + \tan^2 y \neq 0$, *then*

$$y' = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

2) *There exist other intervals* I *of* $\mathbb{R}$ *where the function* $\sin$ *(resp.* $\cos$, $\tan$*) is continuous and strictly monotone. So, by the inverse functions theorem, the function* $\sin$ *(resp.* $\cos$, $\tan$*) has an inverse function on* I. *Such functions are not known by calculators and computers.*

Exercise 108

Determine the domain of definition of the function f in each of the following cases:

1) $f(x) = \arcsin\left(\dfrac{2x}{1+x^2}\right)$.

2) $f(x) = \arccos\left(\dfrac{x-1}{2x+3}\right)$.

3) $f(x) = \dfrac{x}{\arctan\sqrt{x^2-1}}$.

Solution

1) Since $\arcsin$ is defined only on $[-1,1]$, then $f(x)$ is defined if and only if $\dfrac{2x}{1+x^2} \in [-1,1]$. Let $x \in \mathbb{R}$.

   - As $(1-x)^2 \geq 0$, then $1+x^2-2x \geq 0$, so $2x \leq 1+x^2$, and therefore $\frac{2x}{1+x^2} \leq 1$ (since $1+x^2 > 0$).
   - As $(1+x)^2 \geq 0$, then $1+x^2+2x \geq 0$, so $2x \geq -(1+x^2)$, and therefore $\frac{2x}{1+x^2} \geq -1$ (since $1+x^2 > 0$).

   Hence, for all $x \in \mathbb{R}$,

   $$-1 \leq \frac{2x}{1+x^2} \leq 1.$$

   Thus the domain of definition of f is $D_f = \mathbb{R}$.

2) Since $\arccos$ is defined only on $[-1,1]$, then $f(x)$ is defined if and only if $2x+3 \neq 0$ (i.e., $x \neq -\frac{3}{2}$) and

   $$(*) \qquad \frac{x-1}{2x+3} \in [-1,1].$$

   1^{st} method: Let $x \in \mathbb{R}$ such that $x \neq -\frac{3}{2}$. Two cases are possible:

- Suppose that $x > -\frac{3}{2}$, then $2x+3 > 0$. So the double inequality $(*)$ is equivalent to
$$-2x - 3 \leq x - 1 \leq 2x + 3.$$
This is equivalent to $x \geq -\frac{2}{3}$ and $x \geq -4$, and therefore to $x \geq -\frac{2}{3}$.
- Suppose that $x < -\frac{3}{2}$, then $2x+3 < 0$. So the double inequality $(*)$ is equivalent to
$$2x + 3 \leq x - 1 \leq -2x - 3.$$
This is equivalent to $x \leq -4$ and $x \leq -\frac{2}{3}$, and therefore to $x \leq -4$.

Hence the domain of definition of f is
$$D_f = \left] -\infty, -4\right] \cup \left[-\frac{2}{3}, +\infty\right[.$$

2$^{\text{nd}}$ method: Put $u(x) = \dfrac{x-1}{2x+3}$. For any real number $x \neq -\frac{3}{2}$,
$$u'(x) = \frac{5}{(2x+3)^2} > 0.$$

So u is strictly increasing on $\left]-\infty, -\frac{3}{2}\right[$ and on $\left]-\frac{3}{2}, +\infty\right[$.

- Since u is continuous and strictly increasing on the interval $I_1 = \left]-\infty, -\frac{3}{2}\right[$, then, by the inverse functions theorem, u has an inverse function on I_1, we denote it by u_1, which is defined, continuous and strictly increasing on the interval
$$J_1 = u(I_1) = \left] \lim_{x\to-\infty} u(x), \lim_{x\to(-\frac{3}{2})^-} u(x) \right[= \left]\frac{1}{2}, +\infty\right[.$$
- Since u is continuous and strictly increasing on the interval $I_2 = \left]-\frac{3}{2}, +\infty\right[$, then, by the inverse functions theorem, u has an inverse function on I_2, denoted by u_2, which is defined, continuous and strictly increasing on the interval
$$J_2 = u(I_2) = \left] \lim_{x\to(-\frac{3}{2})^+} u(x), \lim_{x\to+\infty} u(x) \right[= \left]-\infty, \frac{1}{2}\right[.$$
- If $u(x) = \frac{1}{2}$, then $2x - 2 = 2x + 3$, so $-2 = 3$, which is impossible. Hence $u(x) \neq \frac{1}{2}$.

We have:
$$\begin{aligned} u(x) \in [-1, 1] \quad &\Leftrightarrow \quad u(x) \in \left[-1, \frac{1}{2}\right[\quad \text{or} \quad u(x) \in \left]\frac{1}{2}, 1\right] \quad \left(\text{since } u(x) \neq \frac{1}{2}\right) \\ &\Leftrightarrow \quad x \in u_2\left(\left[-1, \frac{1}{2}\right[\right) \quad \text{or} \quad x \in u_1\left(\left]\frac{1}{2}, 1\right]\right). \end{aligned}$$

But u_1 and u_2 are continuous and strictly increasing on the intervals $\left]\frac{1}{2}, 1\right]$ and $\left[-1, \frac{1}{2}\right[$ respectively, then

$$u_1\left(\left]\frac{1}{2}, 1\right]\right) = \left]-\infty, u_1(1)\right] \quad \text{and} \quad u_2\left(\left[-1, \frac{1}{2}\right[\right) = \left[u_2(-1), +\infty\right[.$$

If $a = u_1(1)$, then $u(a) = 1$, so $\frac{a-1}{2a+3} = 1$, and therefore $a = -4$.
If $b = u_2(-1)$, then $u(b) = -1$, so $\frac{b-1}{2b+3} = -1$, and therefore $b = -\frac{2}{3}$.
Hence

$$u(x) \in [-1, 1] \Leftrightarrow x \in \left[-\frac{2}{3}, +\infty\right[\quad \text{or} \quad x \in \left]-\infty, -4\right].$$

Thus the domain of definition of f is

$$D_f = \left]-\infty, -4\right] \cup \left[-\frac{2}{3}, +\infty\right[.$$

3) Since **arctan** is defined on $\mathbb{R}$, then $f(x)$ is defined if and only if

$$x^2 - 1 \geq 0 \quad \text{and} \quad \arctan\sqrt{x^2-1} \neq 0.$$

But $\arctan u = 0$ if and only if $u = \tan 0 = 0$, then $f(x)$ is defined if and only if $x^2 - 1 > 0$. So the domain of definition of f is

$$D_f =]-\infty, -1[\cup]1, +\infty[. \quad \square$$

Exercise 109
Prove that:

1) **For all** $x \in]-1, 1[$, $\tan(\arcsin x) = \dfrac{x}{\sqrt{1-x^2}}$.

2) **For all** $x \in [-1, 1] - \{0\}$, $\tan(\arccos x) = \dfrac{\sqrt{1-x^2}}{x}$.

3) **For all** $x \in \mathbb{R}$,

$$\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}} \quad \text{and} \quad \cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}.$$

Solution

1) Let $x \in]-1, 1[$ and $y = \arcsin x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, then $\sin y = x$, and therefore $\cos^2 y = 1 - \sin^2 y = 1 - x^2$. But $y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, then $\cos y > 0$, and therefore $\cos y = \sqrt{1-x^2}$. Hence

$$\tan(\arcsin x) = \tan y = \frac{\sin y}{\cos y} = \frac{x}{\sqrt{1-x^2}}.$$

2) Let $x \in [-1, 1] - \{0\}$ and $y = \arccos x \in [0, \pi] - \{\frac{\pi}{2}\}$, then $\cos y = x$, and therefore $\sin^2 y = 1 - \cos^2 y = 1 - x^2$. But $y \in [0, \pi] - \{\frac{\pi}{2}\}$, then $\sin y \geq 0$, and therefore $\sin y = \sqrt{1 - x^2}$. Hence

$$\tan(\arccos x) = \tan y = \frac{\sin y}{\cos y} = \frac{\sqrt{1-x^2}}{x}.$$

3) Let $x \in \mathbb{R}$ and $y = \arctan x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, then $\tan y = x$. We know that

$$\cos^2 y = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}.$$

But $y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, then $\cos y > 0$, so

$$\cos(\arctan x) = \cos y = \frac{1}{\sqrt{1 + x^2}}.$$

In the other side,

$$\sin^2 y = 1 - \cos^2 y = \frac{\tan^2 y}{1 + \tan^2 y}.$$

But $\sin y$ and $\tan y$ have the same sign since $\cos y > 0$, then

$$\sin(\arctan x) = \sin y = \frac{\tan y}{\sqrt{1 + \tan^2 y}} = \frac{x}{\sqrt{1 + x^2}}. \quad \square$$

Exercise 110
Prove that:

1) **For all real numbers $x > 0$, $\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$.**

2) **For all real numbers $x < 0$, $\arctan x + \arctan\left(\frac{1}{x}\right) = -\frac{\pi}{2}$.**

Solution

1) Consider the function f defined by $f(x) = \arctan x + \arctan\left(\frac{1}{x}\right)$. Since the function $x \mapsto \frac{1}{x}$ is differentiable on $]0, +\infty[$ and $\arctan$ is differentiable on $\mathbb{R}$, then f is differentiable on $]0, +\infty[$. Moreover, for all $x \in]0, +\infty[$,

$$f'(x) = \frac{1}{1 + x^2} + \frac{-\frac{1}{x^2}}{1 + \left(\frac{1}{x}\right)^2} = \frac{1}{1 + x^2} - \frac{1}{x^2 + 1} = 0.$$

Hence f is a constant function on the interval $]0, +\infty[$. For any $x \in]0, +\infty[$,

$$f(x) = f(1) = \arctan 1 + \arctan 1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

2) Let $x < 0$, then $-x > 0$. By the part (1),

$$\arctan(-x) + \arctan\left(\frac{1}{-x}\right) = \frac{\pi}{2}.$$

But $\arctan$ is an odd function, then $-\arctan x - \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$. Hence

$$\arctan x + \arctan\left(\frac{1}{x}\right) = -\frac{\pi}{2}. \quad \square$$

Exercise 111
Prove, by two different methods, that

$$\arctan\left(\frac{2x}{1-x^2}\right) = 2\arctan x, \qquad \text{for all } x \in \,]-1,1[.$$

Solution

<u>1^{st} method</u>: For any $x \in \,]-1,1[$,

$$\tan(2\arctan x) = \frac{2\tan(\arctan x)}{1-\tan^2(\arctan x)} = \frac{2x}{1-x^2}.$$

So $\arctan\left(\frac{2x}{1-x^2}\right) = 2\arctan x$.
<u>2^{nd} method</u>: Let

$$f(x) = \arctan\underbrace{\left(\frac{2x}{1-x^2}\right)}_{u(x)} - 2\arctan x.$$

For any $x \in \,]-1,1[$,

$$\begin{aligned} f'(x) &= \frac{u'}{1+u^2} - \frac{2}{1+x^2} = \frac{2(1+x^2)}{(1-x^2)^2} \times \frac{1}{1+\dfrac{4x^2}{(1-x^2)^2}} - \frac{2}{1+x^2} \\ &= \frac{2(1+x^2)}{(1+x^2)^2} - \frac{2}{1+x^2} = 0. \end{aligned}$$

So there exists a real constant c such that $f(x) = c$ for all $x \in \,]-1,1[$. For $x = 0$, $c = f(0) = 0$, so $f(x) = 0$ for all $x \in \,]-1,1[$. $\square$

4.4 The hyperbolic functions and their inverses

Definition 4.4.1 *Put:*

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2},$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

The function **cosh** *(resp.* **sinh**, **tanh** *and* **coth***) is called the hyperbolic cosine (resp. sine, tangent, cotangent).*

Remark 4.4.1

1) *The common domain of definition ot the functions* $\cosh, \sinh$ *and* $\tanh$ *is* $\mathbb{R}$*. On the other hand, since* $e^x - e^{-x} = 0$ *if and only if* $e^{2x} = 1$ *(i.e., when* $x = 0$*), then the domain of definition of the function* $\coth$ *is* $\mathbb{R}^*$*.*

2) *The function* **cosh** *(resp.* **sinh***) is the even (resp. odd) part of the exponential function on* $\mathbb{R}$ *(see the exercise 69).*

Proposition 4.4.1

1) *For any* $x \in \mathbb{R}$*,*

   (a) $\cosh x \geq 1$.

   (b) $\sinh x = 0$ *if and only if* $x = 0$.

   (c) $\sinh x > 0$ *if and only if* $x > 0$.

   (d) $\cosh x + \sinh x = e^x$ *and* $\cosh x - \sinh x = e^{-x}$.

   (e) $\cosh^2 x - \sinh^2 x = 1$.

2) *The function* **cosh** *(resp.* **sinh**, **tanh***) is even (resp. odd), i.e.,*

$$\cosh(-x) = \cosh x, \quad \sinh(-x) = -\sinh x \quad \text{and} \quad \tanh(-x) = -\tanh x.$$

3) *We have the following limits:*

$$\lim_{x \to -\infty} \cosh x = +\infty \quad \text{and} \quad \lim_{x \to +\infty} \cosh x = +\infty$$

$$\lim_{x \to -\infty} \sinh x = -\infty \quad \text{and} \quad \lim_{x \to +\infty} \sinh x = +\infty$$

$$\lim_{x \to -\infty} \tanh x = -1 \quad \text{and} \quad \lim_{x \to +\infty} \tanh x = 1.$$

4) *The functions* **cosh**, **sinh** *and* **tanh** *are differentiable (and then continuous) on* $\mathbb{R}$*, and, for any* $x \in \mathbb{R}$*,*

$$\boxed{(\cosh x)' = \sinh x} \quad \text{and} \quad \boxed{(\sinh x)' = \cosh x}$$

$$\boxed{(\tanh x)' = \frac{1}{\cosh^2 x} = 1 - \tanh^2 x}$$

The function **coth** *is differentiable (and then continuous) on* $\mathbb{R}^*$*, and, for all* $x \in \mathbb{R}^*$*,*

$$\boxed{(\coth x)' = \frac{-1}{\sinh^2 x} = 1 - \coth^2 x}$$

5) *The functions* $\sinh$ *and* $\tanh$ *are strictly increasing on* $\mathbb{R}$*. The function* $\cosh$ *is strictly decreasing on* $\mathbb{R}^-$ *and strictly increasing on* $\mathbb{R}^+$*. The function* $\coth$ *is strictly decreasing on* $]-\infty, 0[$ *and strictly decreasing on* $]0, +\infty[$.

Proof

1) Let $x \in \mathbb{R}$.

(a) We have:

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x}.$$

But $(e^x - 1)^2 \geq 0$, then $e^{2x} - 2e^x + 1 \geq 0$, so $e^{2x} + 1 \geq 2e^x$. Hence $\cosh x \geq 1$ since $e^x > 0$.

(b) We have:

$$\sinh x = 0 \Leftrightarrow e^x - e^{-x} = 0 \Leftrightarrow e^{2x} = 1 \Leftrightarrow x = 0.$$

(c) We have:

$$\sinh x > 0 \Leftrightarrow e^x - e^{-x} > 0 \Leftrightarrow e^{2x} > 1 \Leftrightarrow x > 0.$$

(d) Obvious.

(e) We have:

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \frac{1}{4}\left[(e^x + e^{-x})^2 - (e^x - e^{-x})^2\right] \\ &= \frac{1}{4}\left[(e^{2x} + e^{-2x} + 2) - (e^{2x} + e^{-2x} - 2)\right] = 1.\end{aligned}$$

2) Obvious.

3) The first four limits are obtained by using the fact that

$$\lim_{x \to -\infty} e^x = 0 \quad \text{and} \quad \lim_{x \to +\infty} e^x = +\infty.$$

In the other side,

$$\lim_{x \to +\infty} \tanh x = \lim_{x \to +\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \to +\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1 - 0}{1 + 0} = 1.$$

and by taking $u = -x$, we obtain:

$$\lim_{x \to -\infty} \tanh x = \lim_{u \to +\infty} \tanh(-u) = -\lim_{u \to +\infty} \tanh u = -1.$$

4) These derivatives are obtained by using the operations on the derivatives.

5) By using the parts (1) and (4), for all $x \in \mathbb{R}$,

$$(\sinh x)' = \cosh x > 0 \quad \text{and} \quad (\tanh x)' = \frac{1}{\cosh^2 x} > 0.$$

So **sinh** and **tanh** are strictly increasing on $\mathbb{R}$. Moreover, $(\cosh x)' = \sinh x \leq 0$ if $x \in \mathbb{R}^-$ and $(\cosh x)' = \sinh x \geq 0$ if $x \in \mathbb{R}^+$ (the derivative vanishes only at the point 0). So **cosh** is strictly decreasing on $\mathbb{R}^-$ and strictly increasing on $\mathbb{R}^+$ by the theorem 4.2.4. In the other side, $(\coth x)' = \frac{-1}{\sinh^2 x} < 0$ for all $x \in \mathbb{R}^*$. So **coth** is strictly decreasing on $]-\infty, 0[$ and strictly decreasing on $]0, +\infty[$. □

Remark 4.4.2

- *Since* $(\sinh x)'' = \sinh x \leq 0$ *if* $x \in \mathbb{R}^-$ *and* $(\sinh x)'' = \sinh x \geq 0$ *if* $x \in \mathbb{R}^+$*, then the function* **sinh** *is concave on* $\mathbb{R}^-$ *and convex* $\mathbb{R}^+$ *by the corollary 4.2.3, and therefore the representative curve of* **sinh** *turns downward its concavity on* $\mathbb{R}^-$ *and upward on* $\mathbb{R}^+$*. Moreover, the origin* $O(0, 0)$ *is an inflection point of this curve.*

- *Since* $(\cosh x)'' = \cosh x \geq 0$ *for all* $x \in \mathbb{R}$*, then the function* **cosh** *is convex on* $\mathbb{R}$ *by the corollary 4.2.3, and therefore the representative curve of* **cosh** *on* $\mathbb{R}$ *turns upward its concavity.*

- *Since* $(\tanh x)'' = \dfrac{-2\sinh x}{\cosh^3 x} \geq 0$ *if* $x \in \mathbb{R}^-$ *and* $(\tanh x)'' \leq 0$ *if* $x \in \mathbb{R}^+$*, then the function* **tanh** *is convex on* $\mathbb{R}^-$ *and concave on* $\mathbb{R}^+$ *by the corollary 4.2.3, and therefore the representative curve of* **tanh** *turns upward its concavity on* $\mathbb{R}^-$ *and downward on* $\mathbb{R}^+$*. Moreover, the origin* $O(0, 0)$ *is an inflection point of this curve.*

By using the proposition 4.4.1, we obtain the following tables of variations:

The function **sinh***:*

x	$-\infty$		0		$+\infty$
$(\sinh x)'$		$+$		$+$	
sinh		$\nearrow$	0	$\nearrow$	
$(\sinh x)''$		$-$	0	$+$	
Concavity		*downward*		*upward*	

The function **cosh***:*

x	$-\infty$		0		$+\infty$
$(\cosh x)'$		$-$		$+$	
cosh		$\searrow$	1	$\nearrow$	
$(\cosh x)''$		$+$		$+$	
Concavity		*upward*		*upward*	

The function **tanh***:*

x	$-\infty$		0		$+\infty$
$(\tanh x)'$		$+$		$+$	
tanh	-1	$\nearrow$	0	$\nearrow$	1
$(\tanh x)''$		$+$	0	$-$	
Concavity		*upward*		*downward*	

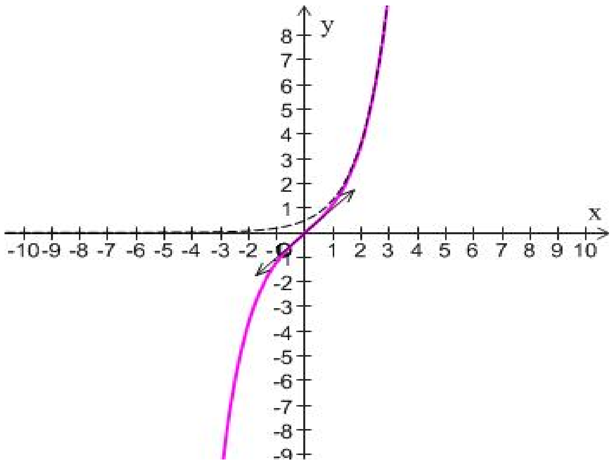

Figure 4.1: The curve of **sinh**

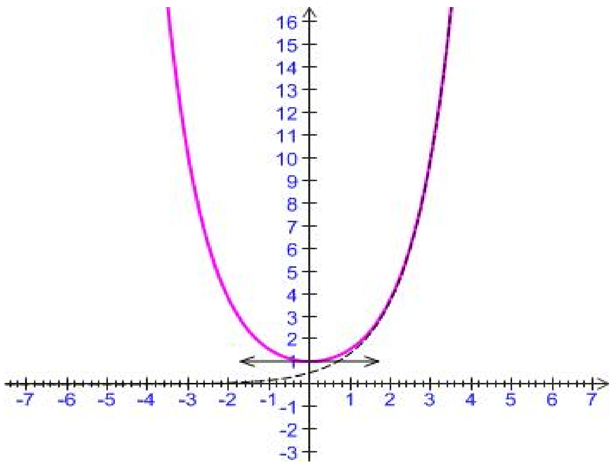

Figure 4.2: The curve of **cosh**

Remark 4.4.3 *By using complex numbers, we have the following formulas:*

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \text{and} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

So

$$\cos x = \cosh(ix), \quad \sin x = -i \sinh(ix) \quad \text{and} \quad \tan x = -i \tanh(ix).$$

$$\cos(ix) = \cosh x, \quad \sin(ix) = i \sinh x \quad \text{and} \quad \tan(ix) = i \tanh x.$$

By using the above relations, we can pass from a trigonometric identity to a hyperbolic identity. For example, we know that $\sin(2x) = 2 \sin x \cos x$*, we replace* x *by* ix*, we obtain* $\sin(2ix) = 2 \sin(ix) \cos(ix)$*, so* $i \sinh(2x) = 2(i \sinh x) \cosh x$*. Hence, simplifying by* i*, we obtain the formula:*

$$\sinh(2x) = 2 \sinh x \cosh x.$$

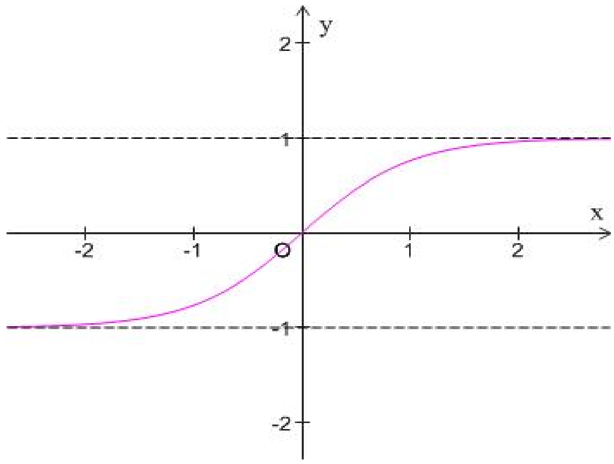

Figure 4.3: The curve of **tanh**

We can then state the following rule: in order to get a hyperbolic identity from a trigonometric identity, it is sufficient to replace $\cos x$ *by* $\cosh x$ *and* $\sin x$ *by* $i \sinh x$*. For example, the relation* $\cos(2x) = 2\cos^2 x - 1$ *becomes* $\cosh(2x) = 2\cosh^2 x - 1$.

4.4.1 The hyperbolic argument sine function

Since the function **sinh** is continuous and strictly increasing on the interval $\mathbb{R}$ (by the proposition 4.4.1). Then, by the inverse functions theorem, the function **sinh** has an inverse function $\sinh^{-1}$ (denoted by **argsinh** and said hyperbolic argument sine) defined, **continuous** and **strictly increasing** on the interval

$$\sinh(\mathbb{R}) = \left] \lim_{x \to -\infty} \sinh x, \lim_{x \to +\infty} \sinh x \right[=]-\infty, +\infty[= \mathbb{R}.$$

Hence

$$\boxed{\operatorname{argsinh} : \mathbb{R} \longmapsto \mathbb{R}}$$

For any $x, y \in \mathbb{R}$, we have:

$$\boxed{\operatorname{argsinh} x = y \text{ if and only if } \sinh y = x}$$

$$\boxed{\operatorname{argsinh}(\sinh x) = x \text{ and } \sinh(\operatorname{argsinh} x) = x}$$

Proposition 4.4.2

1) $\operatorname{argsinh} 0 = 0$.

2) The function **argsinh** *is odd, i.e., if* $x \in \mathbb{R}$*, then*

$$\boxed{\operatorname{argsinh}(-x) = -\operatorname{argsinh} x}$$

3) For all $x \in \mathbb{R}$,

$$\cosh(\operatorname{argsinh} x) = \sqrt{1+x^2}.$$

4) We have:

$$\boxed{\lim_{x \to -\infty} \operatorname{argsinh} x = -\infty} \quad \textit{and} \quad \boxed{\lim_{x \to +\infty} \operatorname{argsinh} x = +\infty}$$

Proof

1) Since $\sinh 0 = 0$, then $\operatorname{argsinh} 0 = 0$.

2) Let $x \in \mathbb{R}$. Since $\sinh$ is an odd function (by the proposition 4.4.1), then

$$\sinh(-\operatorname{argsinh} x) = -\sinh(\operatorname{argsinh} x) = -x.$$

So $\operatorname{argsinh}(-x) = -\operatorname{argsinh} x$.

3) Let $x \in \mathbb{R}$ and $y = \operatorname{argsinh} x \in \mathbb{R}$, then $\sinh y = x$. But $\cosh^2 y - \sinh^2 y = 1$, then $\cosh^2 y = 1 + \sinh^2 y = 1 + x^2$. As $\cosh y > 0$, then

$$\cosh(\operatorname{argsinh} x) = \cosh y = \sqrt{1+x^2}. \quad \square$$

4) This is obtained by using the remark 3.5.2 and the fact that $\lim_{x \to -\infty} \sinh x = -\infty$ and $\lim_{x \to +\infty} \sinh x = +\infty$.

Proposition 4.4.3 *The function* $\operatorname{argsinh}$ *is differentiable on* $\mathbb{R}$ *with*

$$\boxed{(\operatorname{argsinh} x)' = \frac{1}{\sqrt{1+x^2}} \quad \textit{for all } x \in \mathbb{R}}$$

Consequently, if $u : I \longmapsto \mathbb{R}$ *is a differentiable function on an interval* I *of* $\mathbb{R}$*, then the composite function* $\operatorname{argsinh} \circ u$ *is differentiable on* I *and*

$$(\operatorname{argsinh} u)'(x) = \frac{u'}{\sqrt{1+u^2}} \quad \textit{for all } x \in I.$$

Proof Since $(\sinh x)' = \cosh x \neq 0$ for all $x \in \mathbb{R}$, then, by the proposition 4.1.6, the function $\operatorname{argsinh}$ is differentiable at every point of $\mathbb{R}$. Hence $\operatorname{argsinh}$ is differentiable on $\mathbb{R}$ with

$$(\operatorname{argsinh} x)' = \frac{1}{\cosh(\operatorname{argsinh} x)} = \frac{1}{\sqrt{1+x^2}} \quad \text{for all } x \in \mathbb{R},$$

by using the proposition 4.4.2. In the other side, the derivative of the composite function $\operatorname{argsinh} \circ u$ is obtained by using the proposition 4.1.5. $\square$

Remark 4.4.4 *Let* $x \in \mathbb{R}$ *and* $y = \operatorname{argsinh} x \in \mathbb{R}$, *then* $\sinh y = x$. *Differentiating this last relation with respect to* x, *and by using the proposition 4.1.5, we obtain* $y' \cosh y = 1$. *As* $\cosh y \neq 0$, *then*

$$y' = \frac{1}{\cosh y} = \frac{1}{\cosh(\operatorname{argsinh} x)} = \frac{1}{\sqrt{1+x^2}},$$

by using the proposition 4.4.2.

Proposition 4.4.4 *For all* $x \in \mathbb{R}$,

$$\boxed{\operatorname{argsinh} x = \ln\left(x + \sqrt{1+x^2}\right)}$$

Proof Let $x \in \mathbb{R}$ and $y = \operatorname{argsinh} x \in \mathbb{R}$, then $x = \sinh y$.
1^{st} method: Since $\cosh^2 y = \sinh^2 y + 1 = x^2 + 1$ and $\cosh y > 0$, then $\cosh y = \sqrt{x^2+1}$. So

$$e^y = \sinh y + \cosh y = x + \sqrt{x^2+1}.$$

Hence $y = \ln\left(x + \sqrt{x^2+1}\right)$.
2^{nd} method: We have:

$$x = \sinh y = \frac{e^y - e^{-y}}{2} = \frac{e^{2y} - 1}{2e^y}.$$

Put $t = e^y$, then $t^2 - 2xt - 1 = 0$. The discriminant of this trinomial is $\Delta = 4x^2 + 4 > 0$, and therefore its (distinct) real roots are:

$$t_1 = x - \sqrt{1+x^2} < 0 \qquad \text{and} \qquad t_2 = x + \sqrt{1+x^2} > 0.$$

As $t = e^y > 0$, then $t = t_2 = x + \sqrt{1+x^2}$. Hence

$$y = \ln t = \ln\left(x + \sqrt{1+x^2}\right). \quad \square$$

4.4.2 The hyperbolic argument cosine function

Since the function **cosh** is continuous and strictly increasing on the interval $\mathbb{R}^+$ (by the proposition 4.4.1). Then, by the inverse functions theorem, the function **cosh** has an inverse function $\cosh^{-1}$ (denoted by **argcosh** and said hyperbolic argument cosine) defined, **continuous** and **strictly increasing** on the interval

$$\cosh(\mathbb{R}^+) = \cosh\left([0, +\infty[\right) = \left[\cosh 0, \lim_{x\to+\infty} \cosh x\right[= [1, +\infty[.$$

Hence

$$\boxed{\operatorname{argcosh} : [1, +\infty[\longmapsto [0, +\infty[}$$

For any $x \in [1, +\infty[$ and $y \in [0, +\infty[$, we have:

$$\boxed{\operatorname{argcosh} x = y \text{ if and only if } \cosh y = x}$$

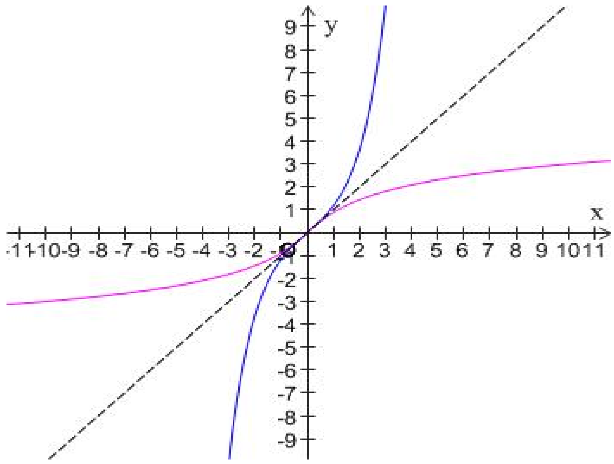

Figure 4.4: The curves of **sinh** and **argsinh**

Moreover,

$$\boxed{\mathbf{argcosh}\,(\cosh x) = x \quad \text{for all} \quad x \in [0, +\infty[}$$

and

$$\boxed{\cosh(\mathbf{argcosh}\, x) = x \quad \text{for all} \quad x \in [1, +\infty[}$$

Proposition 4.4.5

1) $\mathbf{argcosh}\, 1 = 0$.

2) For all $x \in [1, +\infty[$,

$$\sinh(\mathbf{argcosh}\, x) = \sqrt{x^2 - 1}.$$

3) We have:

$$\boxed{\lim_{x \to +\infty} \mathbf{argcosh}\, x = +\infty}$$

Proof

1) Since $\cosh 0 = 1$, then $\mathbf{argcosh}\, 1 = 0$.

2) Let $x \in [1, +\infty[$ and $y = \mathbf{argcosh}\, x \in [0, +\infty[$, then $\cosh y = x$. But we know that $\cosh^2 y - \sinh^2 y = 1$, then $\sinh^2 y = \cosh^2 y - 1 = x^2 - 1$. As $y \in [0, +\infty[$, then $\sinh y \geq 0$, so

$$\sinh(\mathbf{argcosh}\, x) = \sinh y = \sqrt{x^2 - 1}. \quad \square$$

3) This is obtained by using the remark 3.5.2 and the fact that $\lim_{x \to +\infty} \cosh x = +\infty$.

Proposition 4.4.6 *The function* **argcosh** *is differentiable on* $]1, +\infty[$ *with*

$$\boxed{(\operatorname{argcosh} x)' = \frac{1}{\sqrt{x^2 - 1}} \quad \textit{for all } x \in]1, +\infty[}$$

Moreover, **argcosh** *is not right differentiable at* 1.
Consequently, if $u : I \longmapsto]1, +\infty[$ *is a differentiable function on an interval* I *of* $\mathbb{R}$*, then the composite function* $\operatorname{argcosh} \circ u$ *is differentiable on* I *and*

$$(\operatorname{argcosh} u)'(x) = \frac{u'}{\sqrt{u^2 - 1}} \quad \textit{for all } x \in I.$$

$\boxed{\text{Proof}}$ Since $(\cosh x)' = \sinh x = 0$ if and only if $x = 0$, then, by the proposition 4.1.6, the function **argcosh** is differentiable at every point of $[1, +\infty[$ except at the point $\cosh 0 = 1$. Hence **argcosh** is differentiable on $]1, +\infty[$ with

$$(\operatorname{argcosh} x)' = \frac{1}{\sinh(\operatorname{argcosh} x)} = \frac{1}{\sqrt{x^2 - 1}} \quad \text{for all } x \in]1, +\infty[,$$

by using the proposition 4.4.5. In the other side, the derivative of the composite function $\operatorname{argcosh} \circ u$ is obtained by using the proposition 4.1.5. $\square$

Remark 4.4.5

1) *Let* $x \in]1, +\infty[$ *and* $y = \operatorname{argcosh} x \in]0, +\infty[$*, then* $\cosh y = x$. *Differentiating this last relation with respect to* x*, and by using the proposition 4.1.5, we obtain* $y' \sinh y = 1$. *As* $y \neq 0$*, then* $\sinh y \neq 0$*, so*

$$y' = \frac{1}{\sinh y} = \frac{1}{\sinh(\operatorname{argcosh} x)} = \frac{1}{\sqrt{x^2 - 1}},$$

by using the proposition 4.4.5.

2) *There exist other intervals* I *of* $\mathbb{R}$ *where the function* **cosh** *is continuous and strictly monotone (for example,* $I = \mathbb{R}^-$*). So, by the inverse functions theorem, the function* **cosh** *has an inverse function on* I. *Such functions are not known by calculators and computers.*

Proposition 4.4.7 *For all* $x \in [1, +\infty[$,

$$\boxed{\operatorname{argcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right)}$$

$\boxed{\text{Proof}}$ Let $x \in [1, +\infty[$ and $y = \operatorname{argcosh} x \in [0, +\infty[$, then $x = \cosh y$.
1st method: Since $\sinh^2 y = \cosh^2 y - 1 = x^2 - 1$ and $\sinh y \geq 0$ (since $y \geq 0$), then $\sinh y = \sqrt{x^2 - 1}$. So

$$e^y = \cosh y + \sinh y = x + \sqrt{x^2 - 1}.$$

Hence $y = \ln\left(x + \sqrt{x^2 - 1}\right)$.
<u>2^{nd} method</u>: We have:

$$x = \cosh y = \frac{e^y + e^{-y}}{2} = \frac{e^{2y} + 1}{2e^y}.$$

Put $t = e^y$, then $t^2 - 2xt + 1 = 0$. Suppose that $x \in]1, +\infty[$, then $y \in]0, +\infty[$, The discriminant of this trinomial is $\Delta = 4x^2 - 4 > 0$, and therefore its (distinct) real roots are:

$$t_1 = x - \sqrt{x^2 - 1} > 0 \qquad \text{and} \qquad t_2 = x + \sqrt{x^2 - 1} > 0.$$

As $t = e^y$ and $y > 0$, then $t > 1$. But, by multiplying by the conjugate, we obtain:

$$t_1 = \frac{(x - \sqrt{x^2 - 1})(x + \sqrt{x^2 - 1})}{x + \sqrt{x^2 - 1}} = \frac{1}{x + \sqrt{x^2 - 1}}.$$

As $x + \sqrt{x^2 - 1} > x > 1$, then $t_1 < 1$, so $t \neq t_1$. Hence $t = t_2$, and therefore

$$y = \ln t = \ln\left(x + \sqrt{x^2 - 1}\right).$$

In the other side, for $x = 1$, the relation is trivially satisfied. □

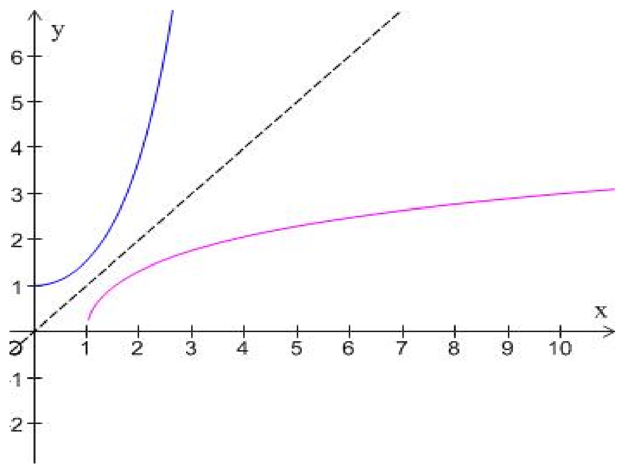

Figure 4.5: The curves of **cosh** and **argcosh**

4.4.3 The hyperbolic argument tangent function

Since the function **tanh** is continuous and strictly increasing on the interval $\mathbb{R}$ (by the proposition 4.4.1). Then, by the inverse functions theorem, the function **tanh** has an inverse function $\tanh^{-1}$ (denoted by **argtanh** and said hyperbolic argument tangent) defined, **continuous** and **strictly increasing** on the interval

$$\tanh(\mathbb{R}) = \left] \lim_{x \to -\infty} \tanh x, \lim_{x \to +\infty} \tanh x \right[= \left] -1, 1 \right[.$$

Hence

$$\boxed{\textbf{argtanh} : \ ]-1,1[\longmapsto \mathbb{R}}$$

For any $x \in]-1,1[$ and $y \in \mathbb{R}$, we have:

$$\boxed{\textbf{argtanh}\, x = y \ \text{ if and only if } \ \tanh y = x}$$

Moreover,

$$\boxed{\textbf{argtanh}\,(\tanh x) = x \ \text{ for all } \ x \in \mathbb{R}}$$

and

$$\boxed{\tanh(\textbf{argtanh}\, x) = x \ \text{ for all } \ x \in]-1,1[}$$

By using the remark 3.5.2 and the fact that $\lim_{x \to -\infty} \tanh x = -1$ and $\lim_{x \to +\infty} \tanh x = 1$, we obtain:

$$\boxed{\lim_{x \to (-1)^+} \textbf{argtanh}\, x = -\infty} \quad \text{and} \quad \boxed{\lim_{x \to 1^-} \textbf{argtanh}\, x = +\infty}$$

Proposition 4.4.8 *The function* **argtanh** *is differentiable on* $]-1,1[$ *with*

$$\boxed{(\textbf{argtanh}\, x)' = \frac{1}{1-x^2} \ \textit{ for all } \ x \in]-1,1[}$$

Consequently, if $u : I \longmapsto]-1,1[$ *is a differentiable function on an interval* I *of* $\mathbb{R}$*, then the composite function* **argtanh** $\circ\, u$ *is differentiable on* I *and*

$$(\textbf{argtanh}\, u)'(x) = \frac{u'}{1-u^2} \ \textit{ for all } \ x \in I.$$

Proof Since $(\tanh x)' = 1 - \tanh^2 x = \dfrac{1}{\cosh^2 x} \neq 0$ for all $x \in \mathbb{R}$, then, by the proposition 4.1.6, the function **argtanh** is differentiable at every point of $]-1,1[$. Hence **argtanh** is differentiable on $]-1,1[$ with

$$(\textbf{argtanh}\, x)' = \frac{1}{1 - \tanh^2(\textbf{argtanh}\, x)} = \frac{1}{1-x^2} \ \text{ for all } \ x \in]-1,1[.$$

In the other side, the derivative of the composite function **argtanh** $\circ\, u$ is obtained by using the proposition 4.1.5. □

Remark 4.4.6 *Let* $x \in]-1,1[$ *and* $y = \textbf{argtanh}\, x \in \mathbb{R}$*, then* $\tanh y = x$*. Differentiating this last relation with respect to* x*, and by using the proposition 4.1.5, we obtain*

$$y'(1 - \tanh^2 y) = 1.$$

As $1 - \tanh^2 y = \dfrac{1}{\cosh^2 y} \neq 0$*, then*

$$y' = \frac{1}{1-\tanh^2 y} = \frac{1}{1-x^2}.$$

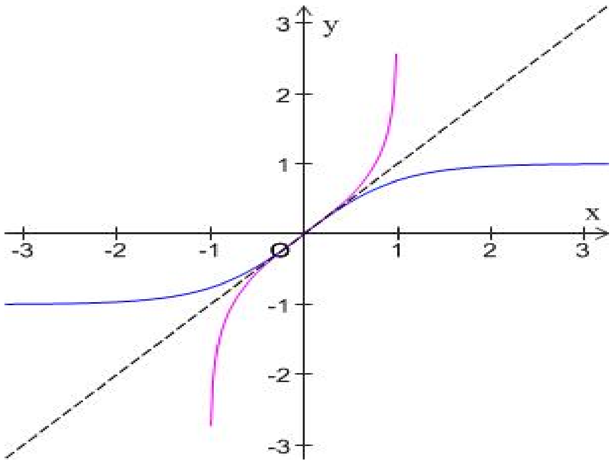

Figure 4.6: The curves of **tanh** and **argtanh**

Proposition 4.4.9 *For all* $x \in]-1,1[$,

$$\boxed{\operatorname{argtanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)}$$

Proof Let $x \in]-1,1[$ and $y = \operatorname{argtanh} x \in \mathbb{R}$, then

$$x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y}-1}{e^{2y}+1}.$$

Put $t = e^y > 0$, then $x = \frac{t^2-1}{t^2+1}$, so $(1-x)t^2 - (1+x) = 0$, and therefore $t^2 = \frac{1+x}{1-x}$. Hence $2\ln t = \ln\left(\frac{1+x}{1-x}\right)$, and then

$$y = \ln t = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right). \quad \square$$

Exercise 112

1) **Show that $|\tanh u| = \tanh |u|$ for all $u \in \mathbb{R}$.**

2) **Prove that, for all $x \in \mathbb{R}$,**

$$\operatorname{argtanh}\sqrt{\frac{\cosh x - 1}{\cosh x + 1}} = \frac{|x|}{2}.$$

Solution

1) Let $u \in \mathbb{R}$,

- If $u \geq 0$, then $|u| = u$ and $\tanh u \geq 0$, and therefore $|\tanh u| = \tanh u = \tanh|u|$.
- If $u < 0$, then $|u| = -u$ and $\tanh u < 0$, and therefore, by using the fact that $\tanh$ is odd, we obtain:

$$|\tanh u| = -\tanh u = \tanh(-u) = \tanh|u|.$$

2) Let $x \in \mathbb{R}$, then

$$\begin{aligned}\frac{\cosh x - 1}{\cosh x + 1} &= \frac{\frac{e^x+e^{-x}}{2} - 1}{\frac{e^x+e^{-x}}{2} + 1} = \frac{e^x + e^{-x} - 2}{e^x + e^{-x} + 2} = \frac{e^x + e^{-x} - 2e^{\frac{x}{2}}e^{-\frac{x}{2}}}{e^x + e^{-x} + 2e^{\frac{x}{2}}e^{-\frac{x}{2}}} \\ &= \frac{\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^2}{\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^2} = \left(\frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}\right)^2 = \tanh^2\left(\frac{x}{2}\right).\end{aligned}$$

So, by using the part (1), we obtain:

$$\sqrt{\frac{\cosh x - 1}{\cosh x + 1}} = \left|\tanh\left(\frac{x}{2}\right)\right| = \tanh\left|\frac{x}{2}\right| = \tanh\frac{|x|}{2}.$$

Hence, as $\frac{|x|}{2} \in \mathbb{R}$, then

$$\operatorname{argtanh}\sqrt{\frac{\cosh x - 1}{\cosh x + 1}} = \frac{|x|}{2}. \quad \square$$

Exercise 113

Let f be the function defined by $f(x) = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$.

1) **Determine the domain of definition D of f and study its parity.**
2) **Calculate the derivative of f on D and deduce that $f(x) = \arcsin x$ for all $x \in D$.**
3) a) **Show that $\frac{t}{1+t^2} < \arctan t < t$ for all $t \in \mathbb{R}_+^*$.**
   b) **Deduce that, for all $x \in]0, 1[$,**

$$x\sqrt{1-x^2} < f(x) < \frac{x}{\sqrt{1-x^2}}.$$

Solution

1) The function $\arctan$ is defined on $\mathbb{R}$. So $f(x)$ is defined if and only if $1 - x^2 > 0$. Hence the domain of definition D of f is $D =]-1, 1[$. In the other side, for any $x \in D$, we have $-x \in D$ and

$$f(-x) = \arctan\left(\frac{-x}{\sqrt{1-(-x)^2}}\right) = -\arctan\left(\frac{x}{\sqrt{1-x^2}}\right) = -f(x),$$

by using the fact that $\arctan$ is an odd function. So f is odd.

2) For every $x \in D$, put $u = \dfrac{x}{\sqrt{1-x^2}}$, then $f(x) = \arctan u$. So

$$f'(x) = \frac{u'}{1+u^2} = \frac{\frac{\sqrt{1-x^2}+\frac{2x^2}{2\sqrt{1-x^2}}}{1-x^2}}{1+\frac{x^2}{1-x^2}} = \frac{1}{\sqrt{1-x^2}} = (\arcsin x)'.$$

As D is an interval, then there exists a real constant c such that

$$f(x) = \arcsin x + c.$$

For $x = 0 \in D$, we have $f(0) = \arcsin 0 + c$, so $c = 0$. Hence $f(x) = \arcsin x$ for all $x \in D$.

3) a) Let $t \in \mathbb{R}_+^*$ and g be the function defined by $g(x) = \arctan x$. Since g is continuous on the interval $[0, t]$ and differentiable on $]0, t[$ (since g is differentiable on $\mathbb{R}$ and $[0, t] \subset \mathbb{R}$), then, by the mean-value theorem, there exists $c \in]0, t[$ such that

$$g(t) - g(0) = (t-0)g'(c) = \frac{t}{1+c^2}$$

But $0 < c < t$, then $\dfrac{1}{1+t^2} < \dfrac{1}{1+c^2} < 1$. As $t > 0$, then

$$\frac{t}{1+t^2} < \frac{t}{1+c^2} < t.$$

In the other side, we have:

$$g(t) - g(0) = \arctan t - \arctan 0 = \arctan t.$$

Hence

$$\frac{t}{1+t^2} < \arctan t < t \qquad (*)$$

b) Let $x \in]0, 1[$. For $t = \frac{x}{\sqrt{1-x^2}} > 0$ in $(*)$, we obtain the double inequality:

$$x\sqrt{1-x^2} < f(x) < \frac{x}{\sqrt{1-x^2}}. \quad \square$$

Exercise 114

Let G be a differentiable function on the interval $I = [0, +\infty[$ such that:

$$G'(x) = \frac{1}{\sqrt{x^4+x^2+1}}, \qquad \forall\, x \in I.$$

Let $F(x) = G(2x) - G(x)$ for all $x \in I$.

1) **Show that $F(x) \geq 0$ for all $x \in I$.**

2) By using the mean-value theorem, show that

$$F(x) \leq \frac{x}{\sqrt{x^4 + x^2 + 1}}, \qquad \forall\, x > 0.$$

3) Deduce the value of the limit $\lim\limits_{x \to +\infty} F(x)$.

4) Study the monotonicity of the function F on I.

Solution

1) Since $G'(x) \geq 0$ for all $x \in I$, then G is increasing on I. Let $x \in I$, then $x \leq 2x$ (since $x \geq 0$), so $G(x) \leq G(2x)$ and therefore $F(x) = G(2x) - G(x) \geq 0$.

2) Let $x > 0$. Since G is differentiable on I and $[x, 2x] \subseteq I$, then G is continuous on $[x, 2x]$ and differentiable on $]x, 2x[$. So, by the mean-value theorem, there exists $c \in]x, 2x[$ such that

$$\begin{aligned} G(2x) - G(x) &= (2x - x)G'(c) \\ &= \frac{x}{\sqrt{c^4 + c^2 + 1}}. \end{aligned}$$

But $c > x > 0$, then $\sqrt{c^4 + c^2 + 1} > \sqrt{x^4 + x^2 + 1}$, and therefore

$$\frac{x}{\sqrt{c^4 + c^2 + 1}} < \frac{x}{\sqrt{x^4 + x^2 + 1}} \quad \text{(since } x > 0\text{)}.$$

Hence

$$F(x) = G(2x) - G(x) \leq \frac{x}{\sqrt{x^4 + x^2 + 1}}.$$

3) By the parts (1) and (2), for any $x > 0$,

$$0 \leq F(x) \leq \frac{x}{\sqrt{x^4 + x^2 + 1}}.$$

As

$$\lim_{x \to +\infty} \frac{x}{\sqrt{x^4 + x^2 + 1}} = \lim_{x \to +\infty} \frac{x}{x^2\sqrt{1 + \frac{1}{x^2} + \frac{1}{x^4}}} = \lim_{x \to +\infty} \frac{1}{x\sqrt{1 + \frac{1}{x^2} + \frac{1}{x^4}}} = 0,$$

then, by the Sandwich theorem, we obtain $\lim\limits_{x \to +\infty} F(x) = 0$.

4) For any $x \in I$,

$$F'(x) = 2G'(2x) - G'(x) = \frac{2}{\sqrt{16x^4 + 4x^2 + 1}} - \frac{1}{\sqrt{x^4 + x^2 + 1}} = \frac{2}{a} - \frac{1}{b}$$

where, to simplify this expression,

$$a = a(x) = \sqrt{16x^4 + 4x^2 + 1} \quad \text{and} \quad b = b(x) = \sqrt{x^4 + x^2 + 1}.$$

$$\begin{aligned} F'(x) &= \frac{2b-a}{ab} = \frac{(2b-a)(2b+a)}{ab(2b+a)} = \frac{4b^2-a^2}{ab(2b+a)} \\ &= \frac{-12x^4+3}{ab(2b+a)} = \frac{3(1-2x^2)(1+2x^2)}{ab(2b+a)}. \end{aligned}$$

But $a > 0$ and $b > 0$, then

$$\begin{cases} F'(x) \geq 0 & \text{if } 0 \leq x \leq \frac{\sqrt{2}}{2} \\ F'(x) \leq 0 & \text{if } x \geq \frac{\sqrt{2}}{2} \end{cases}$$

So F is increasing on the interval $\left[0, \frac{\sqrt{2}}{2}\right]$ and decreasing on the interval $\left[\frac{\sqrt{2}}{2}, +\infty\right[$. On the other hand, F is neither increasing, nor decreasing on I. □

Exercise 115

1) **Establish, for any $x, y \in \mathbb{R}$ of opposite signs such that $xy \neq 1$, the relation:**

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right).$$

2) **Consider the real sequence $(u_n)_{n\in\mathbb{N}}$, defined by $u_n = \arctan\left(\frac{1}{n^2+n+1}\right)$.**

a) **Calculate $\lim\limits_{n\to+\infty} u_n$.**

b) **Show that, for all $n \in \mathbb{N}^*$,**

$$u_n = \arctan\left(\frac{1}{n}\right) - \arctan\left(\frac{1}{n+1}\right).$$

c) **Show that $u_n = \arctan(n+1) - \arctan(n)$ for all $n \in \mathbb{N}$.**

d) **For every $n \in \mathbb{N}^*$, put $S_n = \sum\limits_{k=1}^{n} u_k$.**

i) **Show that, for all $n \in \mathbb{N}^*$,**

$$S_n = \frac{\pi}{4} - \arctan\left(\frac{1}{n+1}\right).$$

ii) **Deduce $\lim\limits_{n\to+\infty} S_n$.**

Solution

1) Let $x, y \in \mathbb{R}$ be two real numbers with opposite signs such that $xy \neq 1$. Suppose that $x \geq 0$ and $y \leq 0$ (we make a similar proof when $x \leq 0$ and $y \geq 0$). Let $a = \arctan x$ and $b = \arctan y$, then $\tan a = x$ and $\tan b = y$, so

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} = \frac{x+y}{1-xy}.$$

As $x \geq 0$ and $y \leq 0$, then $a \in \left[0, \frac{\pi}{2}\right[$ and $b \in \left]-\frac{\pi}{2}, 0\right[$. So $a + b \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. Hence

$$\arctan x + \arctan y = a + b = \arctan\left(\frac{x+y}{1-xy}\right).$$

2) a) Since $\lim\limits_{n\to+\infty} \dfrac{1}{n^2+n+1} = 0$ and $\arctan$ is continuous at 0, then

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \arctan\left(\frac{1}{n^2+n+1}\right) = \arctan 0 = 0.$$

b) Let $n \in \mathbb{N}^*$. By using the part (1) and the fact that $\arctan$ is odd, we obtain:

$$\begin{aligned}
\arctan\left(\frac{1}{n}\right) - \arctan\left(\frac{1}{n+1}\right) &= \arctan\left(\frac{1}{n}\right) + \arctan\left(\frac{-1}{n+1}\right) \\
&= \arctan\left(\frac{\frac{1}{n} - \frac{1}{n+1}}{1 - \left(\frac{1}{n}\right)\left(\frac{-1}{n+1}\right)}\right) \\
&= \arctan\left(\frac{1}{n^2+n+1}\right) = u_n.
\end{aligned}$$

c) Let $n \in \mathbb{N}$. By using the part (1) and the fact that $\arctan$ is odd, we obtain:

$$\begin{aligned}
\arctan(n+1) - \arctan(n) &= \arctan(n+1) + \arctan(-n) \\
&= \arctan\left(\frac{n+1-n}{1-(n+1)(-n)}\right) \\
&= \arctan\left(\frac{1}{n^2+n+1}\right) = u_n.
\end{aligned}$$

d) i) Let $n \in \mathbb{N}^*$. By using the part (b), we obtain:

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} u_k = \sum_{k=1}^{n} \left(\arctan\left(\frac{1}{k}\right) - \arctan\left(\frac{1}{k+1}\right)\right) \\
&= \arctan 1 - \arctan\left(\frac{1}{n+1}\right) = \frac{\pi}{4} - \arctan\left(\frac{1}{n+1}\right).
\end{aligned}$$

ii) Since $\lim\limits_{n\to+\infty} \dfrac{1}{n+1} = 0$ and $\arctan$ is continuous at 0, then

$$\lim_{n\to+\infty} S_n = \frac{\pi}{4} - \arctan 0 = \frac{\pi}{4}. \quad \square$$

Exercise 116

Let f be the function defined on the interval $I =]1, e[$ by $f(x) = \dfrac{x}{\ln x}$.

1) **Show that the function f is a bijection on I and determine the domain of definition of its inverse function f^{-1}.**

2) **Deduce that, for all $n \geq 3$, the equation $e^x = x^n$ admits a unique solution in I, we denote it by a_n.**

3) **Calculate $f(a_n)$ and $f(a_{n+1})$ for all $n \geq 3$. Deduce that the sequence $(a_n)_{n\geq 3}$ is strictly decreasing.**

4) **Deduce that $(a_n)_{n\geq 3}$ is convergent to a limit to be determined.**

Solution

1) The function f is differentiable on I as it is the quotient of two differentiable functions on I where the denominator $\ln x$ does not vanish on I. For all $x \in I$,

$$f'(x) = \frac{\ln x - 1}{(\ln x)^2} < 0.$$

So f is strictly decreasing on I. Moreover, as f is continuous on I, then, by the inverse functions theorem, f is a bijection on I. The domain of definition of its inverse function f^{-1} is

$$f(I) = f(]1, e[) = \left] \lim_{x\to e^-} f(x), \lim_{x\to 1^+} f(x) \right[= \left] e, +\infty \right[.$$

2) Let $n \geq 3$. If $x \in I$, then $\ln x \neq 0$, and therefore

$$e^x = x^n \Leftrightarrow x = n \ln x \Leftrightarrow n = \frac{x}{\ln x} = f(x).$$

Since $n \geq 3$, then $n \in]e, +\infty[$. As f is a bijection from I on $]e, +\infty[$, then there exists one and only one real number a_n in I such that $n = f(a_n)$. Hence the equation $e^x = x^n$ admits a_n as a unique solution in I.

3) Let $n \geq 3$, then $f(a_n) = n$ and $f(a_{n+1}) = n + 1$ by the part (2). Since f is strictly decreasing on I, then f^{-1} is strictly decreasing on $]e, +\infty[$ by the inverse functions theorem. As $n < n + 1$, then $f(a_n) < f(a_{n+1})$, and therefore $f^{-1}(f(a_n)) < f^{-1}(f(a_{n+1}))$, hence $a_n < a_{n+1}$. Thus the sequence $(a_n)_{n\geq 3}$ is strictly decreasing.

4) Since $a_n \in I$ for all $n \geq 3$, then the sequence $(a_n)_{n\geq 3}$ is bounded from below by 1, and as it is strictly decreasing, so it is convergent to a real number ℓ.
<u>1^{st} method</u>: As $1 < a_n < e$ for all $n \geq 3$, then $1 \leq \ell \leq e$. Suppose that $\ell > 1$, then f is continuous at ℓ, so, by Heine's theorem,

$$\lim_{n\to+\infty} f(a_n) = f(\lim_{n\to+\infty} a_n) = f(\ell) \in \mathbb{R}.$$

In the other side, as $f(a_n) = n$ for all $n \geq 3$, then $\lim_{n\to+\infty} f(a_n) = +\infty$, which is impossible. Hence $\ell = 1$.
<u>2^{nd} method</u>: For any $n \geq 3$, as $n = f(a_n) = \frac{a_n}{\ln a_n}$, then $\ln a_n = \frac{a_n}{n}$, and therefore

$$a_n = e^{\frac{a_n}{n}}.$$

But $\lim_{n\to+\infty} a_n = \ell \in \mathbb{R}$, then $\lim_{n\to+\infty} \dfrac{a_n}{n} = 0$. Since the exponential function is continuous at 0, then, by Heine's theorem,

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} e^{\frac{a_n}{n}} = e^0 = 1. \quad \square$$

Exercise 117

Let f be the function defined on the interval $I = \left[-\frac{1}{2}, 1\right]$ by $f(x) = \dfrac{x^3}{3} - \dfrac{x^2}{2}$. Consider the sequence $(u_n)_{n\geq 0}$ defined by:

$$u_0 = \frac{1}{2} \quad \text{and} \quad u_{n+1} = f(u_n), \quad \forall n \geq 0.$$

1) **Study the variations of f on I.**

2) **Deduce that $u_n \in I$ for all $n \geq 0$.**

3) **By studying the variations of the function f' on I, determine a real number $M > 0$ such that $|f'(x)| \leq M$ for all $x \in I$.**

4) **By using the mean-value theorem, show that, for all $n \geq 0$,**
$$|u_{n+1}| \leq M|u_n|.$$

5) **Deduce that the sequence $(u_n)_{n\geq 0}$ is convergent to a limit to be determined.**

Solution

1) The function f is differentiable on I with $f'(x) = x^2 - x$ for all $x \in I$. The table of variations of f on I is:

x	$-\frac{1}{2}$		0		1
$f'(x)$		$+$	0	$-$	0
f	$-\frac{1}{6}$	$\nearrow$	0	$\searrow$	$-\frac{1}{6}$

2) By mathematical induction on n. For $n = 0$, $u_0 = \frac{1}{2} \in I$. Suppose that $u_k \in I$ for a certain $k \geq 0$. By the table of variations of f on I,

$$\begin{aligned} f(I) &= f\left(\left[-\frac{1}{2}, 0\right] \cup \left[0, 1\right]\right) \\ &= f\left(\left[-\frac{1}{2}, 0\right]\right) \cup f\left(\left[0, 1\right]\right) \\ &= \left[-\frac{1}{6}, 0\right] \cup \left[-\frac{1}{6}, 0\right] = \left[-\frac{1}{6}, 0\right] \subseteq I. \end{aligned}$$

As $u_{k+1} = f(u_k)$ and $u_k \in I$, then $u_{k+1} \in f(I) \subseteq I$. So $u_{k+1} \in I$. Hence $u_n \in I$ for all $n \geq 0$.

3) The function f' is differentiable on I with $f''(x) = 2x - 1$ for all $x \in I$. The table of variations of f' on I is:

x	$-\frac{1}{2}$		$\frac{1}{2}$		1
$f''(x)$		$-$	0	$+$	
f'	$\frac{3}{4}$	$\searrow$	$-\frac{1}{4}$	$\nearrow$	0

By the table of variations of f' on I, for all $x \in I$,

$$f'(x) \in \left[-\frac{1}{4}, \frac{3}{4}\right] \subseteq \left[-\frac{3}{4}, \frac{3}{4}\right].$$

So $|f'(x)| \leq \frac{3}{4}$ for all $x \in I$ (we take $M = \frac{3}{4}$).

4) Let $x \in I$ such that $x > 0$. Since f is continuous on $[0, x]$ and differentiable on $]0, x[$, then, by the mean-value theorem, there exists $c \in]0, x[$ such that

$$f(x) - f(0) = f'(c)(x - 0)$$

But $f(0) = 0$, then

$$(*) \qquad |f(x)| = |f'(c)||x| \leq M|x|.$$

If $x \in I$ such that $x < 0$, then we apply the mean-value theorem on the interval $[x, 0]$ to get also the inequality $(*)$. So $(*)$ is satisfied for all $x \in I$. Let $n \geq 0$, since $u_n \in I$ (by the part (2)), then, by $(*)$,

$$|u_{n+1}| = |f(u_n)| \leq M|u_n|.$$

5) Let's show, by mathematical induction on n, that, for all $n \geq 0$,

$$0 \leq |u_n| \leq M^n|u_0|.$$

Indeed, for $n = 0$, $0 \leq |u_0| \leq M^0|u_0|$. Suppose that $0 \leq |u_k| \leq M^k|u_0|$ for a certain $k \geq 0$. Then

$$0 \leq |u_{k+1}| \leq M|u_k| \leq MM^k|u_0| = M^{k+1}|u_0|.$$

Hence, for all $n \geq 0$,

$$0 \leq |u_n| \leq M^n|u_0| \leq \frac{M^n}{2}.$$

As $M = \frac{3}{4} \in]-1, 1[$, then $\lim\limits_{n \to +\infty} \frac{M^n}{2} = 0$. So, by the Sandwich theorem, the sequence $(|u_n|)_{n \geq 0}$ is convergent and $\lim\limits_{n \to +\infty} |u_n| = 0$. Hence the sequence $(u_n)_{n \geq 0}$ is convergent and $\lim\limits_{n \to +\infty} u_n = 0$. □

4.5 L'Hôpital's rule

Theorem 4.5.1 *(Generalized mean-value theorem - Cauchy)*
Let $[a, b]$ be a closed bounded interval of $\mathbb{R}$ (where $a < b$). If $f, g : [a, b] \longmapsto \mathbb{R}$ are continuous on $[a, b]$ and differentiable on $]a, b[$, then there exists $c \in]a, b[$ such that

$$\Big(f(b) - f(a)\Big)g'(c) = \Big(g(b) - g(a)\Big)f'(c).$$

If, in addition, g' does not vanish on $]a, b[$, then $g(a) \neq g(b)$ and

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof Consider the function h defined by:

$$h(x) = \Big(g(b) - g(a)\Big)f(x) - \Big(f(b) - f(a)\Big)g(x).$$

Since f and g are continuous on $[a, b]$ and differentiable on $]a, b[$, then h is continuous on $[a, b]$ and differentiable on $]a, b[$. Moreover, as

$$h(a) = \Big(g(b) - g(a)\Big)f(a) - \Big(f(b) - f(a)\Big)g(a) = g(b)f(a) - f(b)g(a)$$

and

$$h(b) = \Big(g(b) - g(a)\Big)f(b) - \Big(f(b) - f(a)\Big)g(b) = -g(a)f(b) + f(a)g(b),$$

then $h(a) = h(b)$. By Rolle's theorem, there exists $c \in]a, b[$ such that $h'(c) = 0$. But, for all $x \in]a, b[$,

$$h'(x) = \Big(g(b) - g(a)\Big)f'(x) - \Big(f(b) - f(a)\Big)g'(x).$$

So

$$(*) \qquad \Big(f(b) - f(a)\Big)g'(c) = \Big(g(b) - g(a)\Big)f'(c).$$

Moreover, we suppose that g' does not vanish on $]a, b[$. Suppose that $g(a) = g(b)$. Since g is continuous on $[a, b]$ and differentiable on $]a, b[$, then, by Rolle's theorem, there exists $d \in]a, b[$ such that $g'(d) = 0$, which is impossible since g' does not vanish in $]a, b[$. So $g(a) \neq g(b)$ and therefore, dividing the relation $(*)$ by $g(b) - g(a) \neq 0$ and by $g'(c) \neq 0$, we obtain:

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}. \quad \square$$

Remark 4.5.1 *The mean-value theorem (Corollary 4.2.2) is a particular case of the generalized mean-value theorem (theorem 4.5.1) for* $g(x) = x$*.*

Theorem 4.5.2 *(Generalized Rolle's theorem)*
Let I be an interval of $\mathbb{R}$ with extremities $a, b \in \overline{\mathbb{R}}$ such that $a < b$. If $f : I \longmapsto \mathbb{R}$ is a differentiable function on I such that

$$\lim_{x \to a^+} f(x) = \lim_{x \to b^-} f(x) = \ell \in \overline{\mathbb{R}},$$

then there exists $c \in I$ such that $f'(c) = 0$.

Proof Put

$$a' = \lim_{x \to a^+} \arctan x, \qquad b' = \lim_{x \to b^-} \arctan x,$$

then $a', b' \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Consider the function

$$g = \arctan \circ f \circ \tan : \quad]a', b'[\quad \longmapsto \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

Since

$$\lim_{x\to(a')^+} \tan x = a, \quad \lim_{x\to a^+} f(x) = \ell \quad \text{and} \quad \lim_{x\to\ell} \arctan x = \ell' \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

then, by the theorem 3.2.1,

$$\lim_{x\to(a')^+} g(x) = \ell' \in \mathbb{R}.$$

So g is extendable by continuity from the right at a' (put $g(a') = \ell'$).
Similarly, since

$$\lim_{x\to(b')^-} \tan x = b, \quad \lim_{x\to b^-} f(x) = \ell \quad \text{and} \quad \lim_{x\to\ell} \arctan x = \ell' \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

then, by the theorem 3.2.1,

$$\lim_{x\to(b')^-} g(x) = \ell' \in \mathbb{R}.$$

So g is extendable by continuity from the left at b' (put $g(b') = \ell'$).
The function g is continuous on $[a', b']$ and differentiable on $]a', b'[$ (since g is the composite of differentiable functions) and $g(a') = g(b') = \ell'$, then, by Rolle's theorem, there exists $c' \in]a', b'[$ such that $g'(c') = 0$. But, for all $x \in]a', b'[$,

$$g'(x) = (1 + \tan^2 x)\frac{f'(\tan x)}{1 + [f(\tan x)]^2}.$$

Put $c = \tan c' \in I$, then $f'(c) = 0$. □

Theorem 4.5.3 *(Generalization of the generalized mean-value theorem to a non bounded interval)*
Let I be an interval of $\mathbb{R}$ with extremities $a, b \in \overline{\mathbb{R}}$ such that $a < b$. If $f, g : I \longmapsto \mathbb{R}$ are two differentiable functions on I such that the following limits **exist in** *$\mathbb{R}$:*

$$\ell_a = \lim_{x\to a^+} f(x), \quad \ell_b = \lim_{x\to b^-} f(x), \quad L_a = \lim_{x\to a^+} g(x), \quad L_b = \lim_{x\to b^-} g(x).$$

Then there exists $c \in I$ such that

$$(\ell_b - \ell_a)g'(c) = (L_b - L_a)f'(c).$$

If, in addition, g' does not vanish in I, then $L_a \neq L_b$ and

$$\frac{\ell_b - \ell_a}{L_b - L_a} = \frac{f'(c)}{g'(c)}.$$

Proof Consider the function h defined by:

$$h(x) = (L_b - L_a)f(x) - (\ell_b - \ell_a)g(x).$$

Since f and g are differentiable on I, then h is differentiable on I. Moreover, as

$$\lim_{x\to a^+} h(x) = (L_b - L_a)\ell_a - (\ell_b - \ell_a)L_a = L_b\ell_a - \ell_b L_a$$

and

$$\lim_{x \to b^-} h(x) = (L_b - L_a)\ell_b - (\ell_b - \ell_a)L_b = -L_a\ell_b + \ell_a L_b,$$

then $\lim_{x \to a^+} h(x) = \lim_{x \to b^-} h(x)$. By the generalized Rolle's theorem, there exists $c \in I$ such that $h'(c) = 0$. But, for all $x \in I$,

$$h'(x) = (L_b - L_a)f'(x) - (\ell_b - \ell_a)g'(x).$$

So

$$(*) \qquad (\ell_b - \ell_a)g'(c) = (L_b - L_a)f'(c).$$

Moreover, we suppose that g' does not vanish in I. Suppose that $L_a = L_b$. Since g is differentiable on I, then, by the generalized Rolle's theorem, there exists $d \in I$ such that $g'(d) = 0$, which is impossible since g' does not vanish on I. So $L_a \neq L_b$ and therefore, dividing the relation $(*)$ by $L_b - L_a \neq 0$ and by $g'(c) \neq 0$, we obtain:

$$\frac{\ell_b - \ell_a}{L_b - L_a} = \frac{f'(c)}{g'(c)}. \quad \square$$

Theorem 4.5.4 *(L'Hôpital's rule)*
Let $x_0 \in \overline{\mathbb{R}}$. Let f and g be two real functions satisfying the following conditions:

- *f and g are differentiable in a neighborhood V of x_0.*
- $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$.
- *g' does not vanish in V.*
- $\lim_{x \to x_0} \dfrac{f'(x)}{g'(x)}$ *exists in $\overline{\mathbb{R}}$.*

Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

Proof Let $x \in V$ such that $x_0 < x$ (when $x_0 \neq +\infty$). Since f and g are differentiable on V and $]x_0, x[\subseteq V$, then f and g are differentiable on the interval $]x_0, x[$. As f and g are differentiable at x, then f and g are continuous at x, so

$$\lim_{t \to x^-} f(t) = f(x) \in \mathbb{R} \quad \text{and} \quad \lim_{t \to x^-} g(t) = g(x) \in \mathbb{R}.$$

Moreover, by the hypothesis, we have:

$$\lim_{x \to x_0^+} f(x) = 0 \in \mathbb{R} \quad \text{and} \quad \lim_{x \to x_0^+} g(x) = 0 \in \mathbb{R},$$

and as g' does not vanish on $]x_0, x[$, then, by the theorem 4.5.3, there exists $c = c(x) \in]x_0, x[$ such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - 0}{g(x) - 0} = \frac{f'(c(x))}{g'(c(x))}.$$

But $x_0 < c(x) < x$, then $\lim\limits_{x\to x_0^+} c(x) = x_0$ (Sandwich theorem), and as $\lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)} \in \overline{\mathbb{R}}$, then, by using the theorem 3.2.1,

$$\lim_{x\to x_0^+} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(c(x))}{g'(c(x))} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

Similarly, applying the theorem 4.5.3 on the interval $[x, x_0]$ (when $x < x_0$ and $x_0 \neq -\infty$), we obtain:

$$\lim_{x\to x_0^-} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

Hence $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)} = \lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)}$. □

Remark 4.5.2

1) *We admit, without demonstration, that Hôpital's rule is also true when:*

$$x_0 \in \overline{\mathbb{R}} \quad \text{and} \quad \lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \pm\infty.$$

2) *Hôpital's rule is used when we have a limit in one of the indeterminate forms* $\frac{0}{0}$ *and* $\frac{\infty}{\infty}$. *The other indeterminate forms*

$$0 \times \infty, \quad \infty - \infty, \quad 1^\infty, \quad \infty^0 \quad \text{and} \quad 0^0$$

can be transformed by using elementary operations into one of these two forms.

Example 4.5.1

1) *By using Hôpital's rule,*

$$\lim_{x\to 0} \frac{\sin x}{x} \overset{H.R}{=} \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1.$$

$$\lim_{x\to 0^+} \frac{\ln x}{x} \overset{H.R}{=} \lim_{x\to 0^+} \frac{1}{x} = +\infty, \qquad \lim_{x\to +\infty} \frac{\ln x}{x} \overset{H.R}{=} \lim_{x\to +\infty} \frac{1}{x} = 0.$$

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{H.R}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} (-x) = 0.$$

$$\lim_{x\to 0^+} xe^{\frac{1}{x}} = \lim_{x\to 0^+} \frac{e^{\frac{1}{x}}}{\frac{1}{x}} \overset{H.R}{=} \lim_{x\to 0^+} \frac{\frac{-1}{x^2}e^{\frac{1}{x}}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} e^{\frac{1}{x}} = +\infty.$$

2) *We can find again the limit* $\lim\limits_{x\to 0} \frac{1-\cos x}{x(e^x-1)} = \frac{1}{2}$ *(which is already seen in the solution of the exercise 95) by applying twice Hôpital's rule. Indeed,*

$$\lim_{x\to 0} \frac{1-\cos x}{x(e^x-1)} \overset{H.R}{=} \lim_{x\to 0} \frac{\sin x}{(e^x-1)+xe^x} \overset{H.R}{=} \lim_{x\to 0} \frac{\cos x}{(2+x)e^x} = \frac{1}{2}.$$

3) Let $n \in \mathbb{N}^*$. Applying n-times Hôpital's rule, we obtain:

$$\lim_{x\to+\infty} \frac{e^x}{x^n} = \lim_{x\to+\infty} \frac{e^x}{n!} = +\infty.$$

Exercise 118
Let $a \in \mathbb{R}$. Calculate the following limits:

1) $\ell_1 = \lim\limits_{x\to+\infty} \left(1+\dfrac{a}{x}\right)^x$.

2) $\ell_2 = \lim\limits_{x\to0^+} \left(1+\dfrac{a}{x}\right)^x$ **if** $a > 0$.

3) $\ell_3 = \lim\limits_{x\to0^+} x^x$.

Solution

1) This limit is of the indeterminate form 1^∞. To be able to apply Hôpital's rule, one has to transform it to one of the two forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Since $\ell_1 > 0$, then $\ln$ is continuous at ℓ_1. So, by the proposition 3.3.2,

$$\ln \ell_1 = \lim_{x\to+\infty} \ln\left(1+\frac{a}{x}\right)^x = \lim_{x\to+\infty} x\ln\left(1+\frac{a}{x}\right) = \lim_{x\to+\infty} \frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}} \sim \frac{0}{0}.$$

Applying Hôpital's rule, we obtain:

$$\ln \ell_1 = \lim_{x\to+\infty} \frac{\frac{\frac{-a}{x^2}}{1+\frac{a}{x}}}{\frac{-1}{x^2}} = \lim_{x\to+\infty} \frac{a}{1+\frac{a}{x}} = a.$$

Hence $\ell_1 = e^a$.

2) Suppose that $a > 0$. This limit is of the indeterminate form ∞^0. Since $\ell_2 > 0$, then $\ln$ is continuous at ℓ_2. So, by the proposition 3.3.2,

$$\ln \ell_2 = \lim_{x\to0^+} \ln\left(1+\frac{a}{x}\right)^x = \lim_{x\to0^+} x\ln\left(1+\frac{a}{x}\right) = \lim_{x\to0^+} \frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}} \sim \frac{\infty}{\infty}.$$

Applying Hôpital's rule, we obtain:

$$\ln \ell_2 = \lim_{x\to0^+} \frac{\frac{\frac{-a}{x^2}}{1+\frac{a}{x}}}{\frac{-1}{x^2}} = \lim_{x\to0^+} \frac{a}{1+\frac{a}{x}} = 0.$$

Hence $\ell_2 = e^0 = 1$.

3) This limit is of the indeterminate form 0^0. Since the exponential function is continuous at 0 and $\lim\limits_{x\to0^+} x\ln x = 0$ (by the example 4.5.1), then

$$\ell_3 = \lim_{x\to0^+} e^{x\ln x} = e^0 = 1. \quad \square$$

Chapter 5

Finite expansions

5.1 Taylor's formula

Taylor's formula, from the name of the english mathematician *Brook Taylor* who established it in 1712, allows the approximation of a several times differentiable function in a neighborhood of a point by a polynomial where the coefficients depend only on the derivatives of the function at this point.

Theorem 5.1.1 *(Taylor-Lagrange formula)*
Let $n \in \mathbb{N}$, I be an interval of $\mathbb{R}$ and $f : I \longmapsto \mathbb{R}$ be a $(n+1)$-times differentiable function on I. Then, for any $x_0, x \in I$, there exists a real number c between x_0 and x such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1},$$

where

$$\begin{aligned} P_n(x) &= f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \end{aligned}$$

called Taylor's polynomial of f up to order n at the point x_0.

Proof Let $x_0, x \in I$. Suppose that $x_0 < x$ (we make a similar proof if $x < x_0$). The case $n = 0$ corresponds to the mean-value formula on the interval $[x_0, x]$ (Corollary 4.2.2). Suppose that $n \geq 1$. Let $\lambda \in \mathbb{R}$ and, for every $t \in I$, put:

$$g(t) = f(x) - f(t) - \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(x - t)^k - \lambda \frac{(x - t)^{n+1}}{(n+1)!}.$$

We choose λ in a way such that $g(x_0) = 0$. Since g is continuous on the interval $[x_0, x]$, differentiable on $]x_0, x[$ and $g(x_0) = g(x) = 0$, then, by Rolle's theorem, there exists

$c \in]x_0, x[$ such that $g'(c) = 0$. But, for all $t \in I$,

$$\begin{aligned}
g'(t) &= -f'(t) - \sum_{k=1}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} + \lambda \frac{(x-t)^n}{n!} \\
&= -f'(t) - \sum_{k=1}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{s=0}^{n-1} \frac{f^{(s+1)}(t)}{s!}(x-t)^s + \lambda \frac{(x-t)^n}{n!} \\
&\quad \text{(by taking } s = k-1 \text{ in the second sum)} \\
&= -f'(t) - \frac{f^{(n+1)}(t)}{n!}(x-t)^n + f'(t) + \lambda \frac{(x-t)^n}{n!} \quad \text{(after simplifications)} \\
&= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + \lambda \frac{(x-t)^n}{n!}.
\end{aligned}$$

So $\lambda = f^{(n+1)}(c)$. Replacing λ by its value in the expression:

$$0 = g(x_0) = f(x) - f(x_0) - \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(x-x_0)^k - \lambda \frac{(x-x_0)^{n+1}}{(n+1)!},$$

we obtain the requested formula. □

Exercise 119 (Taylor-Lagrange inequality)
Let $n \in \mathbb{N}$, I be an interval of $\mathbb{R}$ and $f : I \longmapsto \mathbb{R}$ be a $(n+1)$-times differentiable function on I.

1) **Show that if there exists a real number $M > 0$ such that $\left|f^{(n+1)}(x)\right| \leq M$ for all $x \in I$, then, for all $x_0, x \in I$,**

$$(*) \qquad \left|f(x) - P_n(x)\right| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}.$$

2) **Show that if f is of class C^{n+1} on I, then, for all $x_0, x \in I$, there exists a real number $M > 0$ such that the inequality $(*)$ is satisfied.**

3) **Deduce that, for all $x \in \mathbb{R}$,**

$$\left|\sin x - x\right| \leq \frac{|x|^3}{3!}.$$

Solution

1) Let $x_0, x \in I$. By Taylor-Lagrange formula, there exists a real number c between x_0 and x such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

As $c \in I$, then $\left|f^{(n+1)}(c)\right| \leq M$, so

$$\left|f(x) - P_n(x)\right| = \frac{\left|f^{(n+1)}(c)\right|}{(n+1)!}|x - x_0|^{n+1} \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}.$$

2) Let $x_0, x \in I$. Suppose that $x_0 < x$ (we make a similar proof if $x < x_0$). Since f is of class C^{n+1} on I, then $f^{(n+1)}$ is continuous on I. As $[x_0, x] \subseteq I$, then $f^{(n+1)}$ is continuous on $[x_0, x]$, and therefore $f^{(n+1)}$ is bounded on $[x_0, x]$ (by the theorem 3.5.1). Hence, there exists a real number $M > 0$ such that $\left|f^{(n+1)}(t)\right| \leq M$ for all $t \in [x_0, x]$. By Taylor-Lagrange formula, there exists $c \in]x_0, x[$ such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

As $\left|f^{(n+1)}(c)\right| \leq M$, then

$$\left|f(x) - P_n(x)\right| = \frac{\left|f^{(n+1)}(c)\right|}{(n+1)!}|x - x_0|^{n+1} \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}.$$

3) Put $f(x) = \sin x$. For $n = 2$, the function f is 3-times differentiable on $\mathbb{R}$ (in fact $\sin$ is of class C^∞ on $\mathbb{R}$ by the example 4.1.5). For $x_0 = 0$, $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, so $P_2(x) = x$. In the other side, as $|f^{(3)}(x)| = |-\cos x| \leq 1$ for all $x \in \mathbb{R}$, then, by the part (1), for all $x \in \mathbb{R}$,

$$\left|\sin x - x\right| = \left|f(x) - P_2(x)\right| \leq \frac{1}{3!}|x - 0|^3 = \frac{|x|^3}{3!}. \quad \square$$

Theorem 5.1.2 *(Taylor-Young formula)*
Let $n \in \mathbb{N}^*$, I be an interval of $\mathbb{R}$ and $f : I \longmapsto \mathbb{R}$ be a n-times differentiable function at a certain point $x_0 \in I$. Then there exists a function $\varepsilon_n : I \longmapsto \mathbb{R}$ such that, for all $x \in I$,

$$f(x) = P_n(x) + (x - x_0)^n \varepsilon_n(x), \qquad \textit{with} \lim_{x \to x_0} \varepsilon_n(x) = 0,$$

and

$$P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Proof For every $x \in I - \{x_0\}$, put:

$$\varepsilon_n(x) = \frac{f(x) - P_n(x)}{(x - x_0)^n}.$$

One has to prove that $\lim_{x \to x_0} \varepsilon_n(x) = 0$. The demonstration is done by mathematical induction on $n \in \mathbb{N}^*$.

- For $n = 1$. Since f is differentiable at x_0, then

$$\begin{aligned}\lim_{x \to x_0} \varepsilon_1(x) &= \lim_{x \to x_0} \frac{f(x) - P_1(x)}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) = f'(x_0) - f'(x_0) = 0.\end{aligned}$$

- Suppose that this property is true up to order $n-1$. If f is n-times differentiable at x_0, then f' is $(n-1)$-times differentiable at x_0. By the induction hypothesis, there exists a function $\varphi_n : I \longmapsto \mathbb{R}$ such that, for all $x \in I$,

$$f'(x) = Q_n(x) + (x-x_0)^{n-1}\varphi_n(x), \quad \text{with } \lim_{x\to x_0} \varphi_n(x) = 0,$$

and

$$Q_n(x) = f'(x_0) + \frac{f''(x_0)}{1!}(x-x_0) + \cdots + \frac{f'^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1}.$$

We remark that $Q_n(x) = P_n'(x)$, so, for all $x \in I - \{x_0\}$,

$$\varphi_n(x) = \frac{f'(x) - P_n'(x)}{(x-x_0)^{n-1}}.$$

Let $\varepsilon > 0$, as $\lim\limits_{x\to x_0} \varphi_n(x) = 0$, then there exists $\delta > 0$ such that, for all $x \in I$,

$$0 < |x - x_0| < \delta \Rightarrow |\varphi_n(x)| < \varepsilon.$$

Put $g(x) = f(x) - P_n(x)$. Let's fix x in the interval $]x_0, x_0+\delta[$. Since g is continuous on the interval $[x_0, x]$ and differentiable on $]x_0, x[$, then, by the mean-value theorem, there exists $c \in\ ]x_0, x[$ such that

$$\frac{g(x) - g(x_0)}{x - x_0} = g'(c).$$

But $g(x_0) = f(x_0) - P_n(x_0) = f(x_0) - f(x_0) = 0$, then

$$\frac{f(x) - P_n(x)}{x - x_0} = f'(c) - P_n'(c).$$

So

$$|\varepsilon_n(x)| = \left|\frac{f(x) - P_n(x)}{(x-x_0)^n}\right| = \left|\frac{f'(c) - P_n'(c)}{(x-x_0)^{n-1}}\right| \leq \left|\frac{f'(c) - P_n'(c)}{(c-x_0)^{n-1}}\right| = |\varphi_n(c)| < \varepsilon.$$

In a similar way, if we fix x in the interval $]x_0 - \delta, x_0[$, we apply the mean-value theorem on the function g and the interval $[x, x_0]$, we obtain $|\varepsilon_n(x)| < \varepsilon$. Hence, for all $x \in I$,

$$0 < |x - x_0| < \delta \Rightarrow |\varepsilon_n(x)| < \varepsilon.$$

So $\lim\limits_{x\to x_0} \varepsilon_n(x) = 0$. □

Remark 5.1.1

1) *Taylor-Lagrange formula is a generalization of the mean-value theorem (Corollary 4.2.2).*

2) *Taylor-Lagrange formula implies the existence of a real number θ (dependent of x) such that $0 < \theta < 1$ and*

$$f(x) = P_n(x) + \frac{f^{(n+1)}\Big(x_0 + \theta(x - x_0)\Big)}{(n+1)!}(x - x_0)^{n+1}.$$

3) *Taylor-Young formula can be written in the form:*

$$f(x) = P_n(x) + o\Big((x - x_0)^n\Big).$$

4) *By taking $h = x - x_0$, Taylor-Lagrange formula can be written in the form:*

$$f(x_0+h) = f(x_0)+\frac{h}{1!}f'(x_0)+\frac{h^2}{2!}f''(x_0)+\cdots+\frac{h^n}{n!}f^{(n)}(x_0)+\frac{h^{n+1}}{(n+1)!}f^{(n+1)}(x_0+\theta h).$$

Taylor-Young formula can be written in the form:

$$f(x_0 + h) = f(x_0) + \frac{h}{1!}f'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots + \frac{h^n}{n!}f^{(n)}(x_0) + h^n\varepsilon_n(h),$$

with $\lim\limits_{h\to 0} \varepsilon_n(h) = 0$,

5) *When $x_0 = 0$ (i.e., $0 \in I$), Taylor-Young and Taylor-Lagrange formulas are called* **Mac-Laurin** *formula.*

Corollary 5.1.1 *Let $n \in \mathbb{N}^*$, I be an open interval of $\mathbb{R}$ and $f : I \longmapsto \mathbb{R}$ be a n-times differentiable function at a certain point $x_0 \in I$. Suppose that*

$$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0 \quad \text{and} \quad f^{(n)}(x_0) \neq 0.$$

1) *Suppose that n is even, then x_0 is a local extremum of f. More precisely,*
   - *If $f^{(n)}(x_0) > 0$, then x_0 is a local minimum of f and f is convex in a neighborhood of x_0.*
   - *If $f^{(n)}(x_0) < 0$, then x_0 is a local maximum of f and f is concave in a neighborhood of x_0.*

2) *If n is odd, then the point $M_0(x_0, f(x_0))$ is an inflection point of f.*

Proof Since f is n-times differentiable at x_0, then, by Taylor-Young formula, there exists a function $\varepsilon : I \longmapsto \mathbb{R}$ such that, for all $x \in I$,

$$f(x) = f(x_0) + \sum_{k=1}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + (x - x_0)^n\varepsilon(x)$$

with $\lim\limits_{x\to x_0} \varepsilon(x) = 0$. As $f^{(k)}(x_0) = 0$ for all $1 \le k \le n - 1$, then

$$f(x) - f(x_0) = \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + (x - x_0)^n\varepsilon(x).$$

So
$$f(x) - f(x_0) \underset{x_0}{\sim} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$
By the proposition 3.2.7, $f(x) - f(x_0)$ and $\frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ have the same sign in a certain neighborhood V of x_0.

1) Suppose that n is even, then $(x - x_0)^n \geq 0$ for all $x \in V$.
   - If $f^{(n)}(x_0) > 0$, then $f(x) - f(x_0) \geq 0$ for all $x \in V$, and therefore x_0 is a local minimum of f and f is convex on V.
   - If $f^{(n)}(x_0) < 0$, then $f(x) - f(x_0) \leq 0$ for all $x \in V$, and therefore x_0 is a local maximum of f and f is concave on V.
2) Suppose that n is odd, then $(x - x_0)^n$ $\left(\text{and then also } \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n\right)$ changes its sign when $x \in V$ passes from left to right of x_0. Hence, $f(x) - f(x_0)$ changes its sign when $x \in V$ passes from left to right of x_0. Consequently, the curve of f on V changes its concavity by passing from left to right of the point $M_0(x_0, f(x_0))$, i.e., M_0 is an inflection point of f. □

Example 5.1.1

1) *Let $f(x) = x^3$. The function f is 3-times differentiable at 0 with*
$$f'(0) = f''(0) = 0 \quad \text{and} \quad f^{(3)}(0) = 6 \neq 0.$$
As $n = 3$ is odd, then, by the corollary 5.1.1, the origin O is an inflection point of f.

2) *If $n = 2$ (so even) in the corollary 5.1.1, i.e., $f'(x_0) = 0$ and $f''(x_0) \neq 0$, then x_0 is a local extremum of f and the sign of $f''(x_0)$ determines the nature of this local extremum. For example, if $f(x) = \cos x$, then $f'(0) = 0$ and $f''(0) = -1 \neq 0$. As $f''(0) < 0$, then 0 is a local maximum of f and f is concave in a neighborhood of 0.*

5.2 Finite expansions

5.2.1 Definitions and properties

Definition 5.2.1 *Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be a function defined in a pointed neighborhood V of a certain point $x_0 \in \mathbb{R}$ (f is not necessary defined at x_0). We say that f has a finite expansion at x_0 (or in a neighborhood of x_0) up to certain order $n \in \mathbb{N}$ (and we write $FE_n(x_0)$ for abbreviation) if there exist $a_0, a_1, \ldots, a_n \in \mathbb{R}$ and a function $\varepsilon \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ defined in V such that, for all $x \in V$,*
$$f(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + (x - x_0)^n \varepsilon(x),$$
with $\lim\limits_{x \to x_0} \varepsilon(x) = 0$. The expression
$$P_n(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n$$

is called the regular part and $R_n(x) = (x - x_0)^n \varepsilon(x)$ *is called the remainder of the finite expansion of* f *in a neighborhood of* x_0 *up to order* n.

Example 5.2.1

1) *The function* $f :]-1, 1[\longmapsto \mathbb{R}$ *defined by* $f(x) = \dfrac{1}{1-x}$, *has a finite expansion at* 0 *at any order* $n \in \mathbb{N}$. *Indeed, for any* $n \in \mathbb{N}$ *and* $x \in]-1, 1[$, *we have the following geometric sum:*

$$1 + x + x^2 + x^3 + \cdots + x^n = \frac{1 - x^{n+1}}{1-x} = \frac{1}{1-x} - x^n \frac{x}{1-x}.$$

So

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + x^n \varepsilon(x)$$

with

$$\varepsilon(x) = \frac{x}{1-x} \quad \text{and} \quad \lim_{x \to 0} \varepsilon(x) = 0.$$

2) *The function* $f : \mathbb{R}^* \longmapsto \mathbb{R}$ *defined by* $f(x) = x^3 \sin \dfrac{1}{x}$, *has a finite expansion at* 0 *up to order* 2. *Indeed, for all* $x \in \mathbb{R}^*$,

$$f(x) = 0 + 0x + 0x^2 + x^2 \varepsilon(x),$$

where $\varepsilon(x) = x \sin \dfrac{1}{x}$ *and* $\lim\limits_{x \to 0} \varepsilon(x) = 0$ *(by the example 3.2.9).*

Exercise 120

1) Let $x_0 \in \mathbb{R}$. Show that every real polynomial $P(x)$ can be written in the form:

$$P(x) = b_0 + b_1(x - x_0) + \cdots + b_m(x - x_0)^m,$$

where $m \in \mathbb{N}$, $b_0, b_1, \ldots, b_m \in \mathbb{R}$ with $b_m \neq 0$.

2) Show that every polynomial function has a finite expansion at any point $x_0 \in \mathbb{R}$ and at any order $n \in \mathbb{N}$.

Solution

1) Let $x_0 \in \mathbb{R}$ and $P(x)$ be a real polynomial, then there exist $m \in \mathbb{N}$ and $a_0, a_1, \ldots, a_m \in \mathbb{R}$ such that $P(x) = a_0 + a_1 x + \cdots + a_m x^m$ with $a_m \neq 0$. Put $t = x - x_0$, i.e., $x = t + x_0$, then

$$P(x) = P(t + x_0) = a_0 + a_1(t + x_0) + \cdots + a_m(t + x_0)^m.$$

Expanding this last expression, we obtain:

$$P(x) = b_0 + b_1 t + \cdots + b_m t^m = b_0 + b_1(x - x_0) + \cdots + b_m(x - x_0)^m,$$

with $b_0, b_1, \ldots, b_m \in \mathbb{R}$ and $b_m = a_m \neq 0$.

2) Let $\boldsymbol{x_0} \in \mathbb{R}$ and $\boldsymbol{n} \in \mathbb{N}$. Let $\boldsymbol{f}$ be a polynomial function, then, by the part (1), there exist $\boldsymbol{m} \in \mathbb{N}$, $\boldsymbol{b_0, b_1, \ldots, b_m} \in \mathbb{R}$ with $\boldsymbol{b_m \neq 0}$ such that

$$\boldsymbol{f(x) = b_0 + b_1(x - x_0) + \cdots + b_m(x - x_0)^m.}$$

- If $\boldsymbol{n < m}$, then

$$\begin{aligned} \boldsymbol{f(x)} &= \boldsymbol{b_0 + b_1(x - x_0) + \cdots + b_n(x - x_0)^n + b_{n+1}(x - x_0)^{n+1} +} \\ &\quad \boldsymbol{\cdots + b_m(x - x_0)^m} \\ &= \boldsymbol{b_0 + b_1(x - x_0) + \cdots + b_n(x - x_0)^n +} \\ &\quad \boldsymbol{(x - x_0)^n \Big[b_{n+1}(x - x_0) + \cdots + b_m(x - x_0)^{m-n} \Big]} \\ &= \boldsymbol{b_0 + b_1(x - x_0) + \cdots + b_n(x - x_0)^n + (x - x_0)^n \varepsilon(x),} \end{aligned}$$

  where

$$\boldsymbol{\varepsilon(x) = b_{n+1}(x - x_0) + \cdots + b_m(x - x_0)^{m-n}} \quad \text{and} \quad \lim_{\boldsymbol{x \to x_0}} \boldsymbol{\varepsilon(x) = 0.}$$

- If $\boldsymbol{n = m}$, then

$$\boldsymbol{f(x) = b_0 + b_1(x - x_0) + \cdots + b_n(x - x_0)^n + (x - x_0)^n \varepsilon(x)}$$

  where $\boldsymbol{\varepsilon(x) = 0}$ and $\lim_{\boldsymbol{x \to x_0}} \boldsymbol{\varepsilon(x) = 0}$.

- If $\boldsymbol{n > m}$, then

$$\begin{aligned} \boldsymbol{f(x)} &= \boldsymbol{b_0 + b_1(x - x_0) + \cdots + b_m(x - x_0)^m + b_{m+1}(x - x_0)^{m+1} +} \\ &\quad \boldsymbol{\cdots + b_n(x - x_0)^n + (x - x_0)^n \varepsilon(x),} \end{aligned}$$

  where $\boldsymbol{b_{m+1} = \cdots = b_n = 0}$, $\boldsymbol{\varepsilon(x) = 0}$ and $\lim_{\boldsymbol{x \to x_0}} \boldsymbol{\varepsilon(x) = 0}$.

Hence $\boldsymbol{f}$ has a finite expansion at $\boldsymbol{x_0}$ up to order $\boldsymbol{n}$. □

Remark 5.2.1 *Let $\boldsymbol{f} \in \boldsymbol{\mathcal{F}}(\mathbb{R}, \mathbb{R})$ be a function defined in a pointed neighborhood $\boldsymbol{V}$ of a certain point $\boldsymbol{x_0} \in \mathbb{R}$.*

1) If $\boldsymbol{f}$ has a finite expansion at $\boldsymbol{x_0}$ up to a certain order $\boldsymbol{n} \in \mathbb{N}$, then, for any $\boldsymbol{m} \in \mathbb{N}$ such that $\boldsymbol{m < n}$, $\boldsymbol{f}$ has a finite expansion at $\boldsymbol{x_0}$ up to order $\boldsymbol{m}$, obtained by truncation. Indeed, suppose that there exist $\boldsymbol{a_0, a_1, \ldots, a_n} \in \mathbb{R}$ and a function $\boldsymbol{\varepsilon} \in \boldsymbol{\mathcal{F}}(\mathbb{R}, \mathbb{R})$ defined in $\boldsymbol{V}$ such that, for all $\boldsymbol{x \in V}$,

$$\boldsymbol{f(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + (x - x_0)^n \varepsilon(x),}$$

with $\lim_{\boldsymbol{x \to x_0}} \boldsymbol{\varepsilon(x) = 0}$. Then, we can write

$$\boldsymbol{f(x) = a_0 + a_1(x - x_0) + \cdots + a_m(x - x_0)^m + (x - x_0)^m \varepsilon_1(x),}$$

where

$$\boldsymbol{\varepsilon_1(x) = a_{m+1}(x - x_0) + \cdots + a_n(x - x_0)^{n-m} + (x - x_0)^{n-m} \varepsilon(x)}$$

and $\lim_{\boldsymbol{x \to x_0}} \boldsymbol{\varepsilon_1(x) = 0}$.

2) *If f has a finite expansion at x_0 up to a certain order $n \in \mathbb{N}$, then f has a finite limit (in $\mathbb{R}$) at x_0. More precisely, if, for all $x \in V$,*

$$f(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + (x - x_0)^n \varepsilon(x),$$

with $\lim\limits_{x \to x_0} \varepsilon(x) = 0$, then

$$\lim_{x \to x_0} f(x) = a_0 = P_n(x_0).$$

If, in addition, f is not defined at x_0, then f is extendable by continuity at x_0 and its extension φ by continuity at x_0 has the same finite expansion at x_0 up to order n as that of f, i.e., for all $x \in V$,

$$\varphi(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + (x - x_0)^n \varepsilon(x).$$

If, also in addition, $n \geq 1$, then φ is differentiable at x_0 and $\varphi'(x_0) = a_1$. Indeed, as $\varphi(x_0) = a_0$, then

$$\begin{aligned}
\lim_{x \to x_0} \frac{\varphi(x) - \varphi(x_0)}{x - x_0} &= \lim_{x \to x_0} \frac{\varphi(x) - a_0}{x - x_0} \\
&= \lim_{x \to x_0} \Big[a_1 + a_2(x - x_0) + \cdots + a_n(x - x_0)^{n-1} \\
&\qquad + (x - x_0)^{n-1} \varepsilon(x)\Big] = a_1.
\end{aligned}$$

3) *Consequences of the part (2):*

- *If f has no finite limit at x_0, then f has no finite expansion at x_0. For example, since $\lim\limits_{x \to 0} \sin \dfrac{1}{x}$ does not exist in $\mathbb{R}$ (by the example 3.4.1), then the function f defined by $f(x) = \sin \dfrac{1}{x}$ has no finite expansion at 0. Similarly, the function $\ln$ has no finite expansion at 0.*
- *If f is continuous at x_0, but not differentiable at x_0, then f has no finite expansion at x_0 up to order 1 (and therefore, by the part (1), f has no finite expansion at x_0 up to order n (for all $n \geq 1$)). For example, for all $n \geq 1$, the function $x \longmapsto \sqrt{x}$ has no finite expansion at 0 up to order n.*

Proposition 5.2.1 *(Uniqueness of the finite expansion)*
Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be a function defined in a pointed neighborhood V of a certain point $x_0 \in \mathbb{R}$. If f has a finite expansion at x_0 up to a certain order $n \in \mathbb{N}$, then this finite expansion is unique. In other words, for each $n \in \mathbb{N}$, the function f has at most a finite expansion at x_0 up to order n.

Proof Let $n \in \mathbb{N}$. Suppose that f has two finite expansions at x_0 up to order n, then, there exist $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n \in \mathbb{R}$ and two functions $\varepsilon_1, \varepsilon_2 \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ defined in V such that, for all $x \in V$,

$$f(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + (x - x_0)^n \varepsilon_1(x),$$

$$f(x) = b_0 + b_1(x - x_0) + \cdots + b_n(x - x_0)^n + (x - x_0)^n \varepsilon_2(x),$$

with $\lim\limits_{x \to x_0} \varepsilon_1(x) = \lim\limits_{x \to x_0} \varepsilon_2(x) = 0$. So, for any $x \in V$, taking the difference, we obtain:

$$(A) \qquad \sum_{k=0}^{n} (a_k - b_k)(x - x_0)^k + (x - x_0)^n (\varepsilon_1(x) - \varepsilon_2(x)) = 0.$$

Taking limits as x tends to x_0 in (A), we obtain $a_0 - b_0 = 0$, and therefore $a_0 = b_0$. So (A) becomes

$$(B) \qquad \sum_{k=1}^{n} (a_k - b_k)(x - x_0)^k + (x - x_0)^n (\varepsilon_1(x) - \varepsilon_2(x)) = 0.$$

For any $x \in V$, simplifying by $x - x_0 \neq 0$ in (B), we obtain:

$$(C) \qquad \sum_{k=1}^{n} (a_k - b_k)(x - x_0)^{k-1} + (x - x_0)^{n-1} (\varepsilon_1(x) - \varepsilon_2(x)) = 0.$$

Taking limits as x tends to x_0 in (C), we obtain $a_1 - b_1 = 0$, and therefore $a_1 = b_1$. Repeating this process $(n + 1)$-times, we obtain $a_k = b_k$ for all $0 \leq k \leq n$ and $\varepsilon_1(x) = \varepsilon_2(x)$ for all $x \in V$. Hence the function f has at most a finite expansion at x_0 up to order n. □

Proposition 5.2.2 *(Parity and finite expansions)*
Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be a function defined in a pointed neighborhood V of a certain point $x_0 \in \mathbb{R}$. Suppose that f has a finite expansion at x_0 up to a certain order $n \in \mathbb{N}$.

1) *If the vertical line of equation $x = x_0$ is an axis of symmetry for the curve of f on V, i.e., if, for any nonzero real number x such that $x_0 + x \in V$, we have:*

$$x_0 - x \in V \quad \text{and} \quad f(x_0 - x) = f(x_0 + x),$$

then the coefficients with **odd** *indexes of the finite expansion of f are zeros.*

2) *If the point $A(x_0, 0)$ is a center of symmetry for the curve of f on V, i.e., if, for any nonzero real number x such that $x_0 + x \in V$, we have:*

$$x_0 - x \in V \quad \text{and} \quad f(x_0 - x) = -f(x_0 + x),$$

then the coefficients with **even** *indexes of the finite expansion of f are zeros.*

Proof Since f has a finite expansion at x_0 up to a certain order $n \in \mathbb{N}$, then there exist $a_0, a_1, \ldots, a_n \in \mathbb{R}$ and a function $\varepsilon \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ defined in V such that, for all $x \in V$,

$$f(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + (x - x_0)^n \varepsilon(x),$$

with $\lim\limits_{x \to x_0} \varepsilon(x) = 0$.

1) For every nonzero real number x such that $x_0 + x \in V$, put $g(x) = f(x_0 + x) = f(x_0 - x)$. Then

$$g(x) = f(x_0 + x) = a_0 + a_1 x + \cdots + a_n x^n + x^n \varepsilon(x_0 + x)$$

and

$$g(x) = f(x_0 - x) = a_0 - a_1 x + \cdots + (-1)^n a_n x^n + (-1)^n x^n \varepsilon(x_0 - x)$$

with $\lim_{x \to 0} \varepsilon(x_0 + x) = 0$ and $\lim_{x \to 0} (-1)^n \varepsilon(x_0 - x) = 0$. By the uniqueness of the finite expansion of g at 0 up to order n (proposition 5.2.1), we obtain $a_k = -a_k$ (and therefore $a_k = 0$) for all odd integers $k \leq n$.

2) For every nonzero real number x such that $x_0 + x \in V$, put $g(x) = f(x_0 + x) = -f(x_0 - x)$. Then

$$g(x) = f(x_0 + x) = a_0 + a_1 x + \cdots + a_n x^n + x^n \varepsilon(x_0 + x)$$

and

$$g(x) = -f(x_0 - x) = -a_0 + a_1 x + \cdots + (-1)^{n+1} a_n x^n + (-1)^{n+1} x^n \varepsilon(x_0 - x)$$

with $\lim_{x \to 0} \varepsilon(x_0 + x) = 0$ and $\lim_{x \to 0} (-1)^{n+1} \varepsilon(x_0 - x) = 0$. By the uniqueness of the finite expansion of g at 0 up to order n (proposition 5.2.1), we obtain $a_k = -a_k$ (and therefore $a_k = 0$) for all even integers $k \leq n$. □

Remark 5.2.2 *If $x_0 = 0$ in the proposition 5.2.2, then its result can be stated in the following way:*

- *If f is* **even** *on V, then the coefficients with* **odd** *indexes of the finite expansion of f are zeros, i.e., the regular part of the finite expansion of f is an* **even** *function on V.*
- *If f is* **odd** *on V, then the coefficients with* **even** *indexes of the finite expansion of f are zeros, i.e., the regular part of the finite expansion of f is an* **odd** *function on V.*

5.2.2 Finite expansions of some basic functions

Let $n \in \mathbb{N}^*$, I be an interval of $\mathbb{R}$ and $f : I \longmapsto \mathbb{R}$ be a function. If f is n-times differentiable at a certain point $x_0 \in I$, then, by Taylor-Young formula (theorem 5.1.2), the function f has the following finite expansion at x_0 up to order n:

$$f(x) = P_n(x) + (x - x_0)^n \varepsilon(x),$$

where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \quad \text{and} \quad \lim_{x \to x_0} \varepsilon(x) = 0.$$

The converse is not true in general. Indeed, for example, the function f defined by

$$f(x) = \begin{cases} x^3 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

has a finite expansion at 0 up to order 2 (by the example 5.2.1). On the other hand, f is not twice-differentiable at 0 (by the exercise 94).

Recall that, by the part (2) of the remark 5.2.1, this converse is true if f is continuous at x_0 and $n = 1$. In other words, if f is continuous at x_0 and it has a finite expansion at x_0 up to order 1, then f is differentiable at x_0 with $f'(x_0) = a_1$ is the coefficient of $(x - x_0)$ in this finite expansion.

In the other side, the uniqueness of the finite expansion (if it exists) and Taylor-Young formula show that if we know the finite expansion of f at a certain point $x_0 \in \mathbb{R}$ up to order n and if f is n-times differentiable at x_0, then we can calculate the successive derivatives of f at x_0 up to order n from the regular part of this finite expansion by the formulas:

$$f^{(k)}(x_0) = a_k k! \quad \text{for all } 0 \leq k \leq n.$$

But in the majority of cases, as in the remainder of this section, we will do the inverse, i.e., we will find the finite expansion from the derivatives.

In the remainder of this section, we consider finite expansions in a neighborhood of 0. The transition to a neighborhood of a point $x_0 \neq 0$ is done by the change of variable $u = x - x_0$ (see the remark 5.2.4 below).

Since the functions $\exp$, $\sin$, $\cos$, $x \longmapsto \ln(1+x)$, $x \longmapsto \dfrac{1}{1-x}$ and $x \longmapsto (1+x)^{\alpha}$ (where $\alpha \in \mathbb{R}$) are n-times differentiable at 0, and their successive derivatives at 0 are given in the example 4.1.5 and the exercises 91 and 92, then, by Taylor-Young formula, the finite expansions of these functions at 0 up to order n are given by:
For all $x \in \mathbb{R}$,

$$\boxed{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + x^n \varepsilon(x) = \sum_{k=0}^{n} \frac{x^k}{k!} + x^n \varepsilon(x)}$$

If $n = 2m + 1$ is odd (where $m \in \mathbb{N}$), then, for all $x \in \mathbb{R}$,

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^m x^{2m+1}}{(2m+1)!} + x^{2m+1} \varepsilon(x)}$$

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^m x^{2m}}{(2m)!} + x^{2m+1} \varepsilon(x)}$$

If $n = 2m$ is even (where $m \in \mathbb{N}^*$), then, for all $x \in \mathbb{R}$,

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^{m+1} x^{2m-1}}{(2m-1)!} + x^{2m} \varepsilon(x)}$$

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^m x^{2m}}{(2m)!} + x^{2m}\varepsilon(x)}$$

We remark that, since the function **sin** (resp. **cos**) is odd (resp. even) on $\mathbb{R}$, then its finite expansion at $\mathbf{0}$ up to order n has only terms with odd (resp. even) exponents, this agrees with the proposition 5.2.2.

For all $x \in]-1, +\infty[$,

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{(-1)^{n+1}x^n}{n} + x^n\varepsilon(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}x^k}{k} + x^n\varepsilon(x)}$$

Consequently, for all $x \in]-\infty, 1[$,

$$\boxed{\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} + x^n\varepsilon(x) = -\sum_{k=1}^{n} \frac{x^k}{k} + x^n\varepsilon(x)}$$

For all $x \in]-1, 1[$,

$$\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + x^n\varepsilon(x) = \sum_{k=0}^{n} x^k + x^n\varepsilon(x)}$$

For any $\alpha \in \mathbb{R}$ and $x \in]-1, +\infty[$,

$$\boxed{(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + x^n\varepsilon(x)}$$

So

$$\boxed{(1+x)^\alpha = 1 + \sum_{k=1}^{n} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}x^k + x^n\varepsilon(x)}$$

In particular, for $\alpha = -1$, for all $x \in]-1, +\infty[$,

$$\boxed{\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + x^n\varepsilon(x) = \sum_{k=0}^{n}(-1)^k x^k + x^n\varepsilon(x)}$$

For $\alpha = \frac{1}{2}$, for all $x \in]-1, +\infty[$,

$$\boxed{\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots + (-1)^{n-1}\frac{1 \times 3 \times 5 \times \cdots \times (2n-3)}{2 \times 4 \times 6 \times \cdots \times (2n)}x^n + x^n\varepsilon(x)}$$

with $\lim\limits_{x \to 0} \varepsilon(x) = 0$.

5.2.3 Operations on finite expansions

Proposition 5.2.3 *Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be two functions defined in a pointed neighborhood V of a certain point $x_0 \in \mathbb{R}$. Suppose that f and g has finite expansions at x_0 up to a certain order $n \in \mathbb{N}$. We denote by $P_n(x)$ and $Q_n(x)$ the regular parts of the finite expansions of f and g at x_0 respectively.*

1) *Linear combination: For any $\alpha, \beta \in \mathbb{R}$, the function $\alpha f + \beta g$ has a finite expansion at x_0 up to order n whose regular part $\alpha P_n(x) + \beta Q_n(x)$.*

2) *Product: The product function fg has a finite expansion at x_0 up to order n whose regular part $R_n(X)$ the sum of terms of degree $\leq n$ obtained in the product $P_n(x)Q_n(x)$. In other words, $R_n(X)$ is the regular part of the finite expansion of the polynomial P_nQ_n at x_0 up to order n.*

3) *Quotient: If $Q_n(x_0) \neq 0$, then the quotient function $\dfrac{f}{g}$ has a finite expansion at x_0 up to order n whose regular part is obtained by dividing $P_n(x)$ by $Q_n(x)$ according to the increasing powers, up to order n (i.e., we stop the division when the degree of the remainder becomes $\geq n+1$).*

Example 5.2.2

1) *For any $n \in \mathbb{N}^*$ and $x \in \mathbb{R}$, as $\cosh x = \dfrac{1}{2}e^x + \dfrac{1}{2}e^{-x}$, $\sinh x = \dfrac{1}{2}e^x - \dfrac{1}{2}e^{-x}$ and*

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \cdots + (-1)^n \frac{x^n}{n!} + x^n \varepsilon(x),$$

then, if $n = 2m+1$ is odd (where $m \in \mathbb{N}$),

$$\boxed{\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2m+1}}{(2m+1)!} + x^{2m+1}\varepsilon(x)}$$

and

$$\boxed{\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2m}}{(2m)!} + x^{2m+1}\varepsilon(x)}$$

and if $n = 2m$ is even (where $m \in \mathbb{N}^*$),

$$\boxed{\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2m-1}}{(2m-1)!} + x^{2m}\varepsilon(x)}$$

and

$$\boxed{\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2m}}{(2m)!} + x^{2m}\varepsilon(x)}$$

with $\lim\limits_{x \to 0} \varepsilon(x) = 0$.

2) *The finite expansions of* $\sin x$ *and* $\cos x$ *at* 0 *up to order* 5 *are given by:*

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + x^5\varepsilon(x) \quad \text{and} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + x^5\varepsilon(x).$$

- *The finite expansion of* $\sin^2 x$ *at* 0 *up to order* 5 *is given by:*

$$\begin{aligned}\sin^2 x &= \sin x \sin x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right) + x^5\varepsilon(x)\\ &= x^2 - \frac{x^4}{6} - \frac{x^4}{6} + x^5\varepsilon(x) = x^2 - \frac{x^4}{3} + x^5\varepsilon(x).\end{aligned}$$

- *The finite expansion of* $\sin(2x)$ *at* 0 *up to order* 5 *is given by:*

$$\begin{aligned}\sin(2x) &= 2\sin x\cos x = 2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right) + x^5\varepsilon(x)\\ &= 2\left(x - \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^3}{6} + \frac{x^5}{12} + \frac{x^5}{120}\right) + x^5\varepsilon(x)\\ &= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 + x^5\varepsilon(x),\end{aligned}$$

with $\lim_{x\to 0}\varepsilon(x) = 0$.

- *For any* $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, $\tan x = \dfrac{\sin x}{\cos x}$. *Dividing* $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ *by* $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ *according to the increasing powers, up to order* 5*:*

		$1 - \frac{x^2}{2} + \frac{x^4}{24}$
	$x - \frac{x^3}{6} + \frac{x^5}{120}$	$x + \frac{x^3}{3} + \frac{2}{15}x^5$
-	$x - \frac{x^3}{2} + \frac{x^5}{24}$	
	$\frac{x^3}{3} - \frac{x^5}{30}$	
-	$\frac{x^3}{3} - \frac{x^5}{6} + \frac{x^7}{72}$	
	$\frac{2}{15}x^5 - \frac{x^7}{72}$	

So the finite expansion of $\tan x$ *at* 0 *up to order* 5 *is:*

$$\boxed{\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + x^5\varepsilon(x)}$$

with $\lim_{x\to 0}\varepsilon(x) = 0$. *We remark that, since the function* $\tan$ *is odd on* $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, *then its finite expansion at* 0 *up to order* 5 *has only terms with odd exponents.*

3) *For all* $x \in \mathbb{R}$, $\tanh x = \dfrac{\sinh x}{\cosh x}$. *The finite expansions of* $\sinh x$ *and* $\cosh x$ *at* 0 *up to order* 5 *are given by:*

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + x^5\varepsilon(x) \quad \text{and} \quad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + x^5\varepsilon(x).$$

Dividing $x + \frac{x^3}{3!} + \frac{x^5}{5!}$ *by* $1 + \frac{x^2}{2!} + \frac{x^4}{4!}$ *according to the increasing powers, up to order* 5*:*

		$1 + \frac{x^2}{2} + \frac{x^4}{24}$
	$x + \frac{x^3}{6} + \frac{x^5}{120}$	$x - \frac{x^3}{3} + \frac{2}{15}x^5$
-	$x + \frac{x^3}{2} + \frac{x^5}{24}$	
	$-\frac{x^3}{3} - \frac{x^5}{30}$	
-	$-\frac{x^3}{3} - \frac{x^5}{6} - \frac{x^7}{72}$	
	$\frac{2}{15}x^5 + \frac{x^7}{72}$	

So the finite expansion of $\tanh x$ *at* 0 *up to order* 5 *is:*

$$\boxed{\tanh x = x - \frac{x^3}{3} + \frac{2}{15}x^5 + x^5\varepsilon(x)}$$

with $\lim_{x\to 0} \varepsilon(x) = 0$*.*

Remark 5.2.3 *Suppose that* $Q_n(x_0) = 0$*. The proposition 5.2.3 says nothing on the existence of the finite expansion of* $\frac{f}{g}$*. In this case,* P_n *and* Q_n *can be written in the forms:*

$$P_n(x) = a_p(x - x_0)^p + a_{p+1}(x - x_0)^{p+1} + \cdots + a_n(x - x_0)^n$$

$$Q_n(x) = b_q(x - x_0)^q + b_{q+1}(x - x_0)^{q+1} + \cdots + b_n(x - x_0)^n,$$

where $p \geq 0$ *and* $q \geq 1$ *(since* $b_0 = Q_n(x_0) = 0$*),* $a_p \neq 0$ *and* $b_q \neq 0$*. Two cases are possible:*

- $\underline{1^{st}\text{ case}}$*: Suppose that* $p < q$*, then* $q - p > 0$*. So*

$$\frac{f(x)}{g(x)} \underset{x_0}{\sim} \frac{a_p(x - x_0)^p}{b_q(x - x_0)^q} = \frac{a_p}{b_q(x - x_0)^{q-p}}.$$

  Hence

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{a_p}{b_q(x - x_0)^{q-p}} = \pm\infty.$$

  Consequently, $\frac{f}{g}$ *has no finite expansion.*

- $\underline{2^{nd}\text{ case}}$*: Suppose that* $p \geq q$*, then* $p - q \geq 0$*. So, after simplification by* $(x - x_0)^q$*, we obtain:*

$$\frac{f(x)}{g(x)} = \frac{a_p(x - x_0)^{p-q} + \cdots + a_n(x - x_0)^{n-q} + (x - x_0)^{n-q}\varepsilon(x)}{b_q + b_{q+1}(x - x_0) + \cdots + b_n(x - x_0)^{n-q} + (x - x_0)^{n-q}\varepsilon(x)}.$$

  It is returned to find the finite expansion of a quotient where the regular part of the denominator does not vanish at x_0 *(case of the proposition 5.2.3, i.e., by a division* *according to the increasing powers). But, be careful to order of this finite expansion, i.e., if we look for a finite expansion of* $\frac{f}{g}$ *at* x_0 *up to order* n*, one has to calculate the finite expansions of* f *and* g *at* x_0 *up to order* $n+q$*, and then, after simplification by* $(x-x_0)^q$*, we divide according to the increasing powers up to order* n*.*

Example 5.2.3 *In order to determine the finite expansion of* $h(x) = \dfrac{\sin x - x}{e^x - 1 - x}$ *at* 0 *up to order* 3*, we begin by looking the beginnings of the finite expansions of its numerator* A *and of its denominator* B*. The finite expansion of* A *at* 0 *begins with* $-\frac{x^3}{3!}$ *(so* $p=3$*), that of* B *begins with* $\frac{x^2}{2!}$ *(so* $q=2$*). As* $p \geq q$*, then we have to simplify by* x^2*, and therefore we have to calculate the finite expansions of* A *and* B *at* 0 *up to order* 5 *to get a finite expansion of* h *at* 0 *up to order* 3*:*

$$A = \sin x - x = -\frac{x^3}{3!} + \frac{x^5}{5!} + x^5\varepsilon(x)$$

$$B = e^x - 1 - x = \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + x^5\varepsilon(x).$$

So, after simplification by x^2*, we obtain:*

$$h(x) = \frac{-\frac{x}{6} + \frac{x^3}{120} + x^3\varepsilon(x)}{\frac{1}{2} + \frac{x}{6} + \frac{x^2}{24} + \frac{x^3}{120} + x^3\varepsilon(x)}.$$

By a division according to the increasing powers up to order 3*, we obtain:*

$$h(x) = -\frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{135} + x^3\varepsilon(x).$$

Proposition 5.2.4 *(Composite of finite expansions)*
Let $u \in \mathcal{F}(\mathbb{R},\mathbb{R})$ *be a function defined in a pointed neighborhood* V *of a certain point* $x_0 \in \mathbb{R}$ *such that* $\lim_{x\to x_0} u(x) = y_0 \in \mathbb{R}$*. Let* $f \in \mathcal{F}(\mathbb{R},\mathbb{R})$ *be a function defined in a pointed neighborhood* W *of* y_0 *such that* $u(V) \subseteq W$*. Suppose* u *(resp.* f*) has a finite expansion at* x_0 *(resp. at* y_0*) up to a certain order* $n \in \mathbb{N}$ *with regular part* $P_n(x)$ *(resp.* $Q_n(x)$*). Then the composite function* $f \circ u$ *has a finite expansion at* x_0 *up to order* n *with regular part obtained by the terms of degrees* $\leq n$ *of the composite polynomial* $Q_n(P_n(x))$*.*

Example 5.2.4

1) *To get the finite expansion of* $\sin(2x)$ *at* 0 *up to order* 5*, we put* $u = 2x$*, then* $\lim_{x\to 0} u = 0$ *and*

$$\begin{aligned}\sin(2x) = \sin u &= u - \frac{u^3}{3!} + \frac{u^5}{5!} + u^5\varepsilon(x)\\ &= 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + (2x)^5\varepsilon(2x)\\ &= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 + x^5\varepsilon(x).\end{aligned}$$

Hence we found the same result as in the example 5.2.4.

2) In a neighborhood of $\mathbf{0}$ *and up to order* $\mathbf{3}$*, we have:*

$$\begin{aligned}
(1+x)^x &= e^{x\ln(1+x)} = e^{x\left(x-\frac{x^2}{2}+x^2\varepsilon(x)\right)} = e^{x^2-\frac{x^3}{2}+x^3\varepsilon(x)} \\
&= e^u \quad \text{where } u = x^2-\frac{x^3}{2}+x^3\varepsilon(x) \text{ and } \lim_{x\to 0} u = 0 \\
&= 1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+u^3\varepsilon(u) \\
&= 1+\left(x^2-\frac{x^3}{2}\right)+\frac{1}{2}\left(x^2-\frac{x^3}{2}\right)^2+\frac{1}{6}\left(x^2-\frac{x^3}{2}\right)^3+x^3\varepsilon(x) \\
&= 1+x^2-\frac{x^3}{2}+x^3\varepsilon(x).
\end{aligned}$$

3) In a neighborhood of $\mathbf{0}$ *and up to order* $\mathbf{3}$*, we have:*

$$\begin{aligned}
\ln(\cos x) &= \ln\left(1-\frac{x^2}{2!}+x^3\varepsilon(x)\right) \\
&= \ln(1+u) \quad \text{where } u = -\frac{x^2}{2}+x^3\varepsilon(x) \text{ and } \lim_{x\to 0} u = 0 \\
&= u-\frac{u^2}{2}+\frac{u^3}{3}+u^3\varepsilon(u) \\
&= -\frac{x^2}{2}-\frac{1}{2}\left(-\frac{x^2}{2}\right)^2+\frac{1}{3}\left(-\frac{x^2}{2}\right)^3+x^3\varepsilon(x) \\
&= -\frac{x^2}{2}+x^3\varepsilon(x).
\end{aligned}$$

Exercise 121

Determine, by four different methods, the finite expansion of $\frac{1}{1-x^2}$ at 0 up to order 4.

Solution

<u>1^{st} method</u>: For any $x \in]-1,1[$, by a decomposition into simple fractions, we obtain:

$$\begin{aligned}
\frac{1}{1-x^2} &= \frac{1}{2}\left(\frac{1}{1-x}\right)+\frac{1}{2}\left(\frac{1}{1+x}\right) \\
&= \frac{1}{2}\left(1+x+x^2+x^3+x^4\right)+\frac{1}{2}\left(1-x+x^2-x^3+x^4\right)+x^4\varepsilon(x) \\
&= 1+x^2+x^4+x^4\varepsilon(x).
\end{aligned}$$

<u>2^{nd} method</u>: For all $x \in]-1,1[$, we have:

$$\begin{aligned}
\frac{1}{1-x^2} &= \left(\frac{1}{1-x}\right)\left(\frac{1}{1+x}\right) \\
&= \left(1+x+x^2+x^3+x^4\right)\left(1-x+x^2-x^3+x^4\right)+x^4\varepsilon(x) \\
&= 1+x^2+x^4+x^4\varepsilon(x).
\end{aligned}$$

$\mathbf{3}^{\text{rd}}$ method: By the division of $\mathbf{1}$ by $\mathbf{1}-x^2$ according to the increasing powers, up to order $\mathbf{4}$, we obtain:

		$1-x^2$
	1	$1+x^2+x^4$
-	$1-x^2$	
	x^2	
-	x^2-x^4	
	x^4	
-	x^4-x^6	
	x^6	

So the finite expansion of $\dfrac{1}{1-x^2}$ at $\mathbf{0}$ up to order $\mathbf{4}$ is:

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^4\varepsilon(x).$$

$\mathbf{4}^{\text{th}}$ method: Put $u = x^2$, then $\lim\limits_{x\to 0} u(x) = 0$. For all $x \in \,]-1, 1[$,

$$\begin{aligned}
\frac{1}{1-x^2} &= \frac{1}{1-u} = 1 + u + u^2 + u^3 + u^4 + u^4\varepsilon(u) \\
&= 1 + x^2 + (x^2)^2 + (x^2)^3 + (x^2)^4 + (x^2)^4\varepsilon(x^2) \\
&= 1 + x^2 + x^4 + x^4\varepsilon(x). \quad \square
\end{aligned}$$

Remark 5.2.4

1) *In the proposition 5.2.4, it is possible that f has a finite expansion at y_0 and that u has no finite expansion at x_0, but $f \circ u$ has a finite expansion at x_0. For example, in a right neighborhood of $\mathbf{0}$, we have:*

$$\begin{aligned}
\cos\sqrt{x} &= 1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} + (\sqrt{x})^4\varepsilon(x) \\
&= 1 - \frac{x}{2!} + \frac{x^2}{4!} + x^2\varepsilon(x).
\end{aligned}$$

On the other hand, the function $x \longmapsto \sqrt{x}$ has no finite expansion at $\mathbf{0}$ up to order $\mathbf{2}$ (by the part (3) of the remark 5.2.1).

2) *In order to determine the finite expansion of a function f in a pointed neighborhood of a point $x_0 \neq 0$ (if there exists), we consider the change of variable $u = x - x_0$ (i.e., $x = u + x_0$) to move into a neighborhood of $\mathbf{0}$, then put:*

$$g(u) = f(x) = f(u + x_0).$$

If the function g has a finite expansion at 0 up to a certain order $n \in \mathbb{N}$ of the form:

$$g(u) = a_0 + a_1 u + a_2 u^2 + \cdots + a_n u^n + u^n \varepsilon(u),$$

then f has a finite expansion at x_0 up to order n, which is:

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + (x - x_0)^n \varepsilon(x),$$

with $\lim_{x \to x_0} \varepsilon(x) = 0$.

Exercise 122
Determine the finite expansion in a neighborhood of $x_0 = 1$ up to order 3 of each of the functions defined by the following expressions:

$$e^x, \quad \sin x \quad \text{and} \quad \ln x.$$

Solution Let $x \in \mathbb{R}$. Put $u = x - x_0 = x - 1$, then $x = u + 1$. So

$$\begin{aligned} e^x = e^{u+1} = e e^u &= e\left(1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + u^3 \varepsilon(u)\right) \\ &= e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + (x-1)^3 \varepsilon(x). \end{aligned}$$

$$\begin{aligned} \sin x = \sin(u+1) &= \sin 1 \cos u + \cos 1 \sin u \\ &= (\sin 1)\left[1 - \frac{u^2}{2!} + u^3 \varepsilon(u)\right] + (\cos 1)\left[u - \frac{u^3}{3!} + u^3 \varepsilon(u)\right] \\ &= \sin 1 + (\cos 1) u - \left(\frac{\sin 1}{2}\right) u^2 - \left(\frac{\cos 1}{6}\right) u^3 + u^3 \varepsilon(u) \\ &= \sin 1 + (\cos 1)(x-1) - \left(\frac{\sin 1}{2}\right)(x-1)^2 - \left(\frac{\cos 1}{6}\right)(x-1)^3 \\ &\quad + (x-1)^3 \varepsilon(x). \end{aligned}$$

$$\begin{aligned} \ln x = \ln(1+u) &= u - \frac{u^2}{2} + \frac{u^3}{3} + u^3 \varepsilon(u) \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + (x-1)^3 \varepsilon(x), \end{aligned}$$

with $\lim_{x \to 1} \varepsilon(x) = 0$. □

Proposition 5.2.5 *(Integration of a finite expansion)*
Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be a differentiable function in a neighborhood of a certain point $x_0 \in D_f$. If the derivative function f' has a finite expansion at x_0 up to a certain order $n \in \mathbb{N}$ with regular part

$$a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n,$$

then f has a finite expansion at x_0 up to order $n + 1$ with regular part

$$P_n(x) = f(x_0) + a_0(x - x_0) + \frac{a_1}{2}(x - x_0)^2 + \cdots + \frac{a_n}{n+1}(x - x_0)^{n+1}.$$

Example 5.2.5

1) *We can find again the finite expansion of* $\ln(1+x)$ *at* 0 *up to a certain order* $n \in \mathbb{N}^*$ *by integrating that of* $\frac{1}{1+x}$ *up to order* $n-1$*. Indeed, since*

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^{n-1}x^{n-1} + x^{n-1}\varepsilon(x),$$

then

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + x^n\varepsilon(x).$$

2) *In a neighborhood of* 0 *and up to order* 4*, dividing* 1 *by* $1+x^2$ *according to the increasing powers, up to order* 4*, we obtain:*

$$(\arctan x)' = \frac{1}{1+x^2} = 1 - x^2 + x^4 + x^4\varepsilon(x).$$

Integrating and taking into account the fact that $\arctan 0 = 0$*, we obtain the finite expansion of* $\arctan x$ *at* 0 *up to order* 5*:*

$$\boxed{\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + x^5\varepsilon(x)}$$

3) *In a neighborhood of* 0 *and up to order* 4*, we have:*

$$\begin{aligned}
(\arcsin x)' &= \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} \\
&= (1+u)^{\alpha} \quad \text{where } \alpha = -\frac{1}{2}, \quad u = -x^2 \text{ and } \lim_{x\to 0} u = 0 \\
&= 1 - \frac{1}{2}u + \frac{3}{8}u^2 - \frac{5}{16}u^3 + \frac{35}{64}u^4 + u^4\varepsilon(u) \\
&= 1 - \frac{1}{2}(-x^2) + \frac{3}{8}(-x^2)^2 - \frac{5}{16}(-x^2)^3 + \frac{35}{64}(-x^2)^4 + (-x^2)^4\varepsilon(-x^2) \\
&= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + x^4\varepsilon(x).
\end{aligned}$$

Integrating and taking into account the fact that $\arcsin 0 = 0$*, we obtain the finite expansion of* $\arcsin x$ *at* 0 *up to order* 5*:*

$$\boxed{\arcsin x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + x^5\varepsilon(x)}$$

4) *By the exercise 107, for all* $x \in [-1, 1]$*,*

$$\arccos x = \frac{\pi}{2} - \arcsin x.$$

So, by the part (3), in a neighborhood of 0 *and up to order* 5*, we obtain:*

$$\boxed{\arccos x = \frac{\pi}{2} - x - \frac{1}{6}x^3 - \frac{3}{40}x^5 + x^5\varepsilon(x)}$$

5) In a neighborhood of $\mathbf{0}$ *and up to order* $\mathbf{4}$*, by using the exercise 121, we obtain:*

$$(\operatorname{argtanh} x)' = \frac{1}{1-x^2} = 1 + x^2 + x^4 + x^4\varepsilon(x).$$

Integrating and taking into account the fact that $\operatorname{argtanh} 0 = 0$*, we obtain the finite expansion of* $\operatorname{argtanh} x$ *at* $\mathbf{0}$ *up to order* $\mathbf{5}$*:*

$$\boxed{\operatorname{argtanh} x = x + \frac{x^3}{3} + \frac{x^5}{5} + x^5\varepsilon(x)}$$

6) In a neighborhood of $\mathbf{0}$ *and up to order* $\mathbf{4}$*, we have:*

$$\begin{aligned}
(\operatorname{argsinh} x)' &= \frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-\frac{1}{2}} \\
&= (1+u)^{\alpha} \quad \text{where } \alpha = -\frac{1}{2}, \quad u = x^2 \text{ and } \lim_{x\to 0} u = 0 \\
&= 1 - \frac{1}{2}u + \frac{3}{8}u^2 - \frac{5}{16}u^3 + \frac{35}{64}u^4 + u^4\varepsilon(u) \\
&= 1 - \frac{1}{2}(x^2) + \frac{3}{8}(x^2)^2 - \frac{5}{16}(x^2)^3 + \frac{35}{64}(x^2)^4 + (x^2)^4\varepsilon(x^2) \\
&= 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 + x^4\varepsilon(x).
\end{aligned}$$

Integrating and taking into account the fact that $\operatorname{argsinh} 0 = 0$*, we obtain the finite expansion of* $\operatorname{argsinh} x$ *at* $\mathbf{0}$ *up to order* $\mathbf{5}$*:*

$$\boxed{\operatorname{argsinh} x = x - \frac{1}{6}x^3 + \frac{3}{40}x^5 + x^5\varepsilon(x)}$$

7) Since the function **argcosh** *is not defined in a neighborhood of* $\mathbf{0}$*, so it has no finite expansion at* $\mathbf{0}$*.*

Exercise 123
Determine the finite expansion of $\arctan(1+x)$ at 0 up to order 3.

Solution In a neighborhood of $\mathbf{0}$ and up to order $\mathbf{2}$, we have:

$$\Big(\arctan(1+x)\Big)' = \frac{1}{1+(1+x)^2} = \frac{1}{2+2x+x^2} = \frac{1}{2} - \frac{1}{2}x + \frac{1}{4}x^2 + x^2\varepsilon(x).$$

The last equality is obtained by a division according to the increasing powers up to order $\mathbf{2}$. Integrating and taking into account the fact that $\arctan(1+0) = \frac{\pi}{4}$, we obtain the finite expansion of $\arctan(1+x)$ at $\mathbf{0}$ up to order $\mathbf{3}$:

$$\arctan(1+x) = \frac{\pi}{4} + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{12}x^3 + x^3\varepsilon(x). \quad \square$$

Corollary 5.2.1 *(Derivation of a finite expansion)*
Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be a differentiable function in a neighborhood of a certain point $x_0 \in D_f$. Suppose that f has a finite expansion at x_0 up to order $n+1$ (where $n \in \mathbb{N}$) with regular part

$$P_{n+1}(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + a_{n+1}(x - x_0)^{n+1}.$$

If the derivative function f' has a finite expansion at x_0 up to order n, then its regular part is

$$P'_{n+1}(x) = a_1 + 2a_2(x - x_0) + \cdots + na_n(x - x_0)^{n-1} + (n+1)a_{n+1}(x - x_0)^n.$$

Example 5.2.6

1) *We can find again the finite expansion of $\cos x$ at 0 up to order $2m$ (where $m \in \mathbb{N}$) by derivation of that of $\sin x$ at 0 up to order $2m+1$.*

2) *Since*

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^4\varepsilon(x)$$

and $f'(x) = \dfrac{1}{(1-x)^2}$ has a finite expansion at 0 up to order 3 (as it is the product of $\frac{1}{1-x}$ by itself), then

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + x^3\varepsilon(x).$$

3) *The condition "f' has a finite expansion at x_0 up to order n" in the corollary 5.2.1 is essential , i.e., there exist differentiable functions in a neighborhood of a certain point $x_0 \in \mathbb{R}$ which have finite expansions at x_0 up to order $n+1$, but their derivatives have no finite expansions at x_0 up to order n. Indeed, put*

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

It is easy to show that f is differentiable on $\mathbb{R}$ with

$$f'(x) = \begin{cases} 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Since $\lim\limits_{x \to 0} x \sin \dfrac{1}{x} = 0$ and $\lim\limits_{x \to 0} \cos \dfrac{1}{x}$ does not exist in $\mathbb{R}$ (by the example 3.4.1), then f' has no finite limit at 0. Hence f' has no finite expansion at 0 of any order. On the other hand, f has a finite expansion at 0 up to order 1 since

$$f(x) = 0 + 0x + x\varepsilon(x), \quad \text{where } \varepsilon(x) = x \sin \frac{1}{x} \quad \text{and} \quad \lim_{x \to 0} \varepsilon(x) = 0. \quad \square$$

5.2.4 Finite expansions in a neighborhood of infinity

Definition 5.2.2 *Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be a function defined in a neighborhood V of $+\infty$ (resp. $-\infty$). We say that f has a finite expansion at $+\infty$ (resp. at $-\infty$) (or in a neighborhood of $+\infty$ (resp. $-\infty$)) up to a certain order $n \in \mathbb{N}$ if there exist $a_0, a_1, \dots, a_n \in \mathbb{R}$ and a function $\varepsilon \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ defined in V such that, for all $x \in V$,*

$$f(x) = a_0 + \frac{a_1}{x} + \cdots + \frac{a_n}{x^n} + \frac{1}{x^n}\varepsilon\left(\frac{1}{x}\right),$$

with $\lim_{x \to +\infty} \varepsilon\left(\frac{1}{x}\right) = 0$. $\left(\text{resp. } \lim_{x \to -\infty} \varepsilon\left(\frac{1}{x}\right) = 0\right)$. *The expression*

$$P_n\left(\frac{1}{x}\right) = a_0 + a_1\left(\frac{1}{x}\right) + \cdots + a_n\left(\frac{1}{x}\right)^n$$

is called the regular part and $R_n\left(\frac{1}{x}\right) = \frac{1}{x^n}\varepsilon\left(\frac{1}{x}\right)$ *is called the remainder of order n of this finite expansion.*

Remark 5.2.5 *Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be a function defined in a neighborhood V of $+\infty$ (resp. $-\infty$).*

1) If f has a finite expansion at $+\infty$ (resp. at $-\infty$) up to a certain order $n \in \mathbb{N}$, then f has a finite limit (in $\mathbb{R}$) at $+\infty$ (resp. at $-\infty$). More precisely, if, for all $x \in V$,

$$f(x) = a_0 + \frac{a_1}{x} + \cdots + \frac{a_n}{x^n} + \frac{1}{x^n}\varepsilon\left(\frac{1}{x}\right),$$

with $\lim_{x \to +\infty} \varepsilon\left(\frac{1}{x}\right) = 0$. $\left(\text{resp. } \lim_{x \to -\infty} \varepsilon\left(\frac{1}{x}\right) = 0\right)$, *then*

$$\lim_{x \to +\infty} f(x) = a_0 \qquad \left(\text{resp. } \lim_{x \to -\infty} f(x) = a_0\right).$$

Hence, the horizontal line of equation $y = a_0$ is an asymptote to the curve of f at $+\infty$ (resp. at $-\infty$).
Consequently, if f has no finite limit at $+\infty$ (resp. at $-\infty$), then f has no finite expansion at $+\infty$ (resp. at $-\infty$). For example, since $\lim_{x \to \pm\infty} \sin x$ does not exist in $\mathbb{R}$ (by the example 3.4.1), then the function $\sin$ has no finite expansion at $+\infty$, neither at $-\infty$. Similarly, as $\lim_{x \to +\infty} \ln x = +\infty$, then the function $\ln$ has no finite expansion at $+\infty$ (neither at $-\infty$ since $\ln$ is not defined in a neighborhood of $-\infty$).

2) As in the proposition 5.2.1, if f has a finite expansion at $+\infty$ (resp. at $-\infty$) up to a certain order $n \in \mathbb{N}$, then this finite expansion is unique. In other words, for any $n \in \mathbb{N}$, the function f has at most a finite expansion at $+\infty$ (resp. at $-\infty$) up to order n.

3) *In practrice, to determine the finite expansion of* f *at* $+\infty$ *(resp. at* $-\infty$*) (if there exists), we consider the change of variable* $u = \frac{1}{x}$ *(i.e.,* $x = \frac{1}{u}$*) to move into a neighborhood of* 0*, then put:*

$$g(u) = f(x) = f\left(\frac{1}{u}\right).$$

If the function g *has a finite expansion at* 0 *up to a certain order* $n \in \mathbb{N}$ *of the form:*

$$g(u) = a_0 + a_1 u + a_2 u^2 + \cdots + a_n u^n + u^n \varepsilon(u),$$

then f *has a finite expansion at* $+\infty$ *(resp. at* $-\infty$*) up to order* n*, which is :*

$$f(x) = a_0 + \frac{a_1}{x} + \cdots + \frac{a_n}{x^n} + \frac{1}{x^n}\varepsilon\left(\frac{1}{x}\right),$$

with $\lim\limits_{x\to+\infty} \varepsilon\left(\frac{1}{x}\right) = 0$. $\left(\text{resp. } \lim\limits_{x\to-\infty} \varepsilon\left(\frac{1}{x}\right) = 0\right)$.

Example 5.2.7

1) *To determine the finite expansion of* $\dfrac{1}{1-x}$ *at* $+\infty$ *up to order* 3*, put* $u = \frac{1}{x}$*, then* $x = \frac{1}{u}$*, and therefore*

$$\begin{aligned} \frac{1}{1-x} &= \frac{1}{1-\frac{1}{u}} = \frac{u}{-1+u} = -u - u^2 - u^3 + u^3\varepsilon(u) \\ &= -\frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} + \frac{1}{x^3}\varepsilon\left(\frac{1}{x}\right) \quad \text{with } \lim_{x\to+\infty} \varepsilon\left(\frac{1}{x}\right) = 0. \end{aligned}$$

2) *We will determine the finite expansion of* $f(x) = \sqrt{x^2+1} - \sqrt{x^2-1}$ *at* $\pm\infty$ *up to order* 2*. We have:*

$$\begin{aligned} f(x) &= \sqrt{x^2+1} - \sqrt{x^2-1} = \sqrt{x^2\left(1+\frac{1}{x^2}\right)} - \sqrt{x^2\left(1-\frac{1}{x^2}\right)} \\ &= |x|\sqrt{1+\frac{1}{x^2}} - |x|\sqrt{1-\frac{1}{x^2}} = |x|\left[\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}} - \left(1-\frac{1}{x^2}\right)^{\frac{1}{2}}\right] \end{aligned}$$

Put $u = \frac{1}{x}$*, then* $x = \frac{1}{u}$*.*
<u>In a neighborhood of</u> $\underline{+\infty}$*: Since* $x > 0$*, then*

$$\begin{aligned} f(x) &= x\left[\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}} - \left(1-\frac{1}{x^2}\right)^{\frac{1}{2}}\right] = \frac{1}{u}\left[(1+u^2)^{\frac{1}{2}} - (1-u^2)^{\frac{1}{2}}\right] \\ &= \frac{1}{u}\Bigg[\left(1+\frac{1}{2}(u^2) - \frac{1}{8}(u^2)^2 + \frac{1}{16}(u^2)^3\right) \\ &\qquad - \left(1+\frac{1}{2}(-u^2) - \frac{1}{8}(-u^2)^2 + \frac{1}{16}(-u^2)^3\right) + u^3\varepsilon(u)\Bigg] \\ &= \frac{1}{u}\left(u^2 + u^3\varepsilon(u)\right) = u + u^2\varepsilon(u) = \frac{1}{x} + \frac{1}{x^2}\varepsilon\left(\frac{1}{x}\right). \end{aligned}$$

In a neighborhood of $-\infty$: Since $x<0$, then

$$\begin{aligned} f(x) &= -x\left[\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}}-\left(1-\frac{1}{x^2}\right)^{\frac{1}{2}}\right] = -\frac{1}{u}\left[(1+u^2)^{\frac{1}{2}}-(1-u^2)^{\frac{1}{2}}\right] \\ &= -\frac{1}{u}\Big(u^2+u^3\varepsilon(u)\Big) = -u+u^2\varepsilon(u) = -\frac{1}{x}+\frac{1}{x^2}\varepsilon\left(\frac{1}{x}\right). \end{aligned}$$

Remark 5.2.6 *If $f(x)=\dfrac{P(x)}{Q(x)}$ is a rational fraction (where P and Q are two real polynomials such that $\deg P<\deg Q$), then the finite expansion of f at $\pm\infty$ up to a certain order $n\in\mathbb{N}$ can be obtained by a division of P by Q according to the* **decreasing** *powers by introducing negative powers if necessary to get the order n in the quotient.*

Example 5.2.8 *Dividing 1 by $-x+1$ according to the decreasing powers, we obtain:*

		$-x+1$
	1	$-\frac{1}{x}-\frac{1}{x^2}-\frac{1}{x^3}$
-	$1-\frac{1}{x}$	
	$\frac{1}{x}$	
-	$\frac{1}{x}-\frac{1}{x^2}$	
	$\frac{1}{x^2}$	
-	$\frac{1}{x^2}-\frac{1}{x^3}$	
	$\frac{1}{x^3}$	

Hence, we find again the finite expansion of $\dfrac{1}{1-x}$ at $+\infty$ up to order 3 given in the example 5.2.7.

5.2.5 Generalized finite expansions

Definition 5.2.3 *Let $f\in\mathcal{F}(\mathbb{R},\mathbb{R})$ be a function defined in a pointed neighborhood V of a certain point $x_0\in\mathbb{R}$ (f is not necessary defined at x_0). Suppose that f has no finite expansion at x_0. We say that f has a* **generalized** *finite expansion at x_0 (or in a neighborhood of x_0) up to a certain order $n\in\mathbb{N}$ if there exists $k\in\mathbb{N}^*$ such that the function g defined by $g(x)=(x-x_0)^k f(x)$, has a finite expansion at x_0 up to order n, i.e., if there exist $a_0,a_1,\ldots,a_n\in\mathbb{R}$ and a function $\varepsilon\in\mathcal{F}(\mathbb{R},\mathbb{R})$ defined on V such that, for all $x\in V$,*

$$g(x)=a_0+a_1(x-x_0)+\cdots+a_n(x-x_0)^n+(x-x_0)^n\varepsilon(x),$$

with $\lim\limits_{x\to x_0}\varepsilon(x)=0$. Writing (if $k<n$)

$$\begin{aligned} f(x) &= \frac{a_0}{(x-x_0)^k}+\frac{a_1}{(x-x_0)^{k-1}}+\cdots \\ &\quad +a_k+a_{k+1}(x-x_0)+\cdots+a_n(x-x_0)^{n-k}+(x-x_0)^{n-k}\varepsilon(x) \end{aligned}$$

is called the generalized finite expansion of f *at* x_0 *of order* $n-k$.

Example 5.2.9 *Let* $f(x) = \dfrac{1}{1-\cos x}$. *The function* f *has no finite expansion at* 0 *since it has no finite limit at* 0. *On the other hand, in a neighborhood of* 0, *we have:*

$$\begin{aligned} x^2 f(x) &= \frac{x^2}{1-\cos x} = \frac{x^2}{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+x^4\varepsilon(x)\right)} \\ &= \frac{24}{12-x^2+x^2\varepsilon(x)} = 2+\frac{1}{6}x^2+x^2\varepsilon(x). \end{aligned}$$

The last equality is obtained by a division according to the increasing powers up to order 2. *Hence the generalized finite expansion of* f *at* 0 *of order* 0 *(here* $n=2$ *and* $k=2$*) is:*

$$f(x) = \frac{2}{x^2}+\frac{1}{6}+\varepsilon(x), \text{ with } \lim_{x\to 0}\varepsilon(x)=0.$$

Definition 5.2.4 *Let* $f \in \mathcal{F}(\mathbb{R},\mathbb{R})$ *be a function defined in a neighborhood* V *of* $+\infty$ *(resp.* $-\infty$*). Suppose that* f *has no finite expansion at* $+\infty$ *(resp. at* $-\infty$*). We say that* f *has a* **generalized** *finite expansion at* $+\infty$ *(resp. at* $-\infty$*) up to a certain order* $n \in \mathbb{N}$ *if there exists* $k \in \mathbb{N}^*$ *such that the function* g *defined by* $g(x) = \dfrac{f(x)}{x^k}$, *has a finite expansion at* $+\infty$ *(resp. at* $-\infty$*) up to order* n, *i.e., if there exist* $a_0, a_1, \dots, a_n \in \mathbb{R}$ *and a function* $\varepsilon \in \mathcal{F}(\mathbb{R},\mathbb{R})$ *defined on* V *such that, for all* $x \in V$,

$$g(x) = a_0 + \frac{a_1}{x} + \cdots + \frac{a_n}{x^n} + \frac{1}{x^n}\varepsilon\left(\frac{1}{x}\right),$$

with $\displaystyle\lim_{x\to+\infty}\varepsilon\left(\frac{1}{x}\right)=0$. $\left(\text{resp. } \displaystyle\lim_{x\to-\infty}\varepsilon\left(\frac{1}{x}\right)=0\right)$. *Writing (if* $k<n$*)*

$$f(x) = a_0x^k + a_1x^{k-1} + \cdots + a_k + \frac{a_{k+1}}{x} + \cdots + \frac{a_n}{x^{n-k}} + \frac{1}{x^{n-k}}\varepsilon\left(\frac{1}{x}\right)$$

is called the generalized finite expansion of f *at* $+\infty$ *(resp. at* $-\infty$*) of order* $n-k$.

Example 5.2.10 *Let* $f(x) = \dfrac{x^2-2}{x-1}$. *The function* f *has no finite expansion at* $+\infty$ *since it has no finite limit at* $+\infty$. *On the other hand, by a division according to the decreasing powers, in a neighborhood of* $+\infty$, *we obtain:*

$$\frac{f(x)}{x} = \frac{x^2-2}{x^2-x} = 1+\frac{1}{x}-\frac{1}{x^2}-\frac{1}{x^3}+\frac{1}{x^3}\varepsilon\left(\frac{1}{x}\right).$$

Hence the generalized finite expansion of f *at* $+\infty$ *of order* 2 *(ici* $n=3$ *and* $k=1$*) is:*

$$f(x) = x+1-\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^2}\varepsilon\left(\frac{1}{x}\right), \text{ with } \lim_{x\to+\infty}\varepsilon\left(\frac{1}{x}\right)=0.$$

5.2.6 Applications

I - Local study of a curve: tangent and concavity

Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be a function defined in a neighborhood of a certain point x_0. Suppose that f is continuous at x_0 and that f has the following finite expansion at x_0 up to a certain order $p \geq 2$:

$$f(x) = a_0 + a_1(x - x_0) + a_p(x - x_0)^p + (x - x_0)^p \varepsilon(x),$$

with $a_p \neq 0$. As f is continuous at x_0, then

$$f(x_0) = \lim_{x \to x_0} f(x) = a_0.$$

By the remark 5.2.1, f is differentiable at x_0 and $f'(x_0) = a_1$. So the equation of the tangent (T) to the curve of f at the point $M_0(x_0, f(x_0))$ is:

$$(T): \quad y = f(x_0) + f'(x_0)(x - x_0) = a_0 + a_1(x - x_0).$$

In order to study the relative position of the tangent (T) with respect to the curve of f in a neighborhood of x_0, we consider the difference:

$$f(x) - y = a_p(x - x_0)^p + (x - x_0)^p \varepsilon(x) \underset{x_0}{\sim} a_p(x - x_0)^p.$$

By the part (5) of the proposition 3.2.7, $f(x) - y$ and $a_p(x - x_0)^p$ have the same sign in a certain neighborhood V of x_0.
1^{st} case: Suppose that p is even, then $(x - x_0)^p \geq 0$ for all $x \in V$.

- If $a_p > 0$, then $f(x) - y \geq 0$ for all $x \in V$, and therefore the curve of f on V is above (T).
- If $a_p < 0$, then $f(x) - y \leq 0$ for all $x \in V$, and therefore the curve of f on V is below (T).

If, in addition, $a_1 = 0$, then $f(x) - f(x_0)$ and $a_p(x - x_0)^p$ have the same sign on V.

- If $a_p > 0$, then $f(x) - f(x_0) \geq 0$ for all $x \in V$, and therefore x_0 is a local minimum of f and f is convex on V.
- If $a_p < 0$, then $f(x) - f(x_0) \leq 0$ for all $x \in V$, and therefore x_0 is a local maximum of f and f is concave on V.

$2^{\text{ème}}$ cas: Suppose that p is odd.

- If $a_p > 0$ and $x \in V$ such that $x \geq x_0$, then $f(x) - y \geq 0$, and therefore the curve of f is above (T) from the right of x_0.
- If $a_p > 0$ and $x \in V$ such that $x \leq x_0$, then $f(x) - y \leq 0$, and therefore the curve of f is below (T) from the left of x_0.
- If $a_p < 0$ and $x \in V$ such that $x \geq x_0$, then $f(x) - y \leq 0$, and therefore the curve of f is below (T) from the right of x_0.

- If $a_p < 0$ and $x \in V$ such that $x \leq x_0$, then $f(x) - y \geq 0$, and therefore the curve of f is above (T) from the left of x_0.

In this case, the curve of f on V crosses the tangent (T) at M_0, so this curve changes its concavity by passing from left to right of M_0, i.e., M_0 is an inflection point of f.

Example 5.2.11

1) *Let $f(x) = (1+x)^x$. By the example 5.2.4, in a neighborhood of 0 and up to order 2, we have:*
$$f(x) = 1 + x^2 + x^2\varepsilon(x).$$
So the tangent (T) to the curve of f at the point $M_0(0,1)$ has equation $y = 1$. Moreover,
$$f(x) - y \underset{0}{\sim} x^2 \geq 0.$$
So the curve of f is above (T) in a neighborhood of 0. In the other side, as $f(x) - f(0) \underset{0}{\sim} x^2 \geq 0$, then 0 is a local minimum of f and f is convex in a neighborhood of 0.

2) *By the example 5.2.5, in a neighborhood of 0 and up to order 3, we have:*
$$\arccos x = \frac{\pi}{2} - x - \frac{1}{6}x^3 + x^3\varepsilon(x).$$
So the tangent (T) to the curve of $\arccos$ at the point $M_0(0, \frac{\pi}{2})$ has equation $y = \frac{\pi}{2} - x$. Moreover,
$$\arccos x - y \underset{0}{\sim} -\frac{1}{6}x^3.$$
   - *If $x > 0$, then $-\frac{1}{6}x^3 < 0$, and therefore $\arccos x - y < 0$. Hence the curve of $\arccos$ is below (T) from the right of 0.*
   - *If $x < 0$, then $-\frac{1}{6}x^3 > 0$, and therefore $\arccos x - y > 0$. Hence the curve of $\arccos$ is above (T) from the left of 0.*

   Thus M_0 is an inflection point of the function $\arccos$.

II - Study of infinite branches at $\pm\infty$

Let $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be a function defined in a neighborhood of $+\infty$ (resp. $-\infty$). Suppose that f has the following **generalized** finite expansion at $+\infty$ (resp. at $-\infty$) of order $p-1$ (where $p \geq 2$):
$$f(x) = ax + b + \frac{a_p}{x^{p-1}} + \frac{1}{x^{p-1}}\varepsilon\left(\frac{1}{x}\right)$$
with $a_p \neq 0$. Let (D) be the straight-line of equation
$$(D): \quad y = ax + b.$$

Since

$$\lim_{x\to+\infty}\Big[f(x)-(ax+b)\Big]=0\quad\Big(\text{resp. }\lim_{x\to-\infty}\Big[f(x)-(ax+b)\Big]=0\Big),$$

then the straight-line (D) is an asymptote to the curve of f at $+\infty$ (resp. at $-\infty$) (horizontal if $a=0$ and oblique if $a\neq 0$). In order to study the relative position of the asymptote (D) with respect to the curve of f in a neighborhood of $+\infty$ (resp. $-\infty$), we consider the difference:

$$f(x)-y=\frac{a_p}{x^{p-1}}+\frac{1}{x^{p-1}}\varepsilon\left(\frac{1}{x}\right)\underset{\pm\infty}{\sim}\frac{a_p}{x^{p-1}}.$$

By the part (5) of the proposition 3.2.7, $f(x)-y$ and $\dfrac{a_p}{x^{p-1}}$ have the same sign in a certain neighborhood V of $+\infty$ (resp. $-\infty$).

<u>1^{st} case</u>: Suppose that p is odd, then $p-1$ is even, and therefore $x^{p-1}>0$ for all $x\in V$.

- If $a_p>0$, then $f(x)-y>0$ for all $x\in V$, and therefore the curve of f on V is above (D).
- If $a_p<0$, then $f(x)-y<0$ for all $x\in V$, and therefore the curve of f on V is below (D).

<u>2^{nd} case</u>: Suppose that p is even, then $p-1$ is odd.

- <u>In a neighborhood of $+\infty$</u>: $x>0$ (and therefore $x^{p-1}>0$) for all $x\in V$.
  - If $a_p>0$, then $f(x)-y>0$, and therefore the curve of f on V is above (D).
  - If $a_p<0$, then $f(x)-y<0$, and therefore the curve of f on V is below (D).
- <u>In a neighborhood of $-\infty$</u>: $x<0$ (and therefore $x^{p-1}<0$) for all $x\in V$.
  - If $a_p>0$, then $f(x)-y<0$, and therefore the curve of f on V is below (D).
  - If $a_p<0$, then $f(x)-y>0$, and therefore the curve of f on V is above (D).

If the curve of f on V is above (resp. below) of (D), then f is convex (resp. concave) on V.

Example 5.2.12 *Let $f(x)=\dfrac{x^2-2}{x-1}$. By the example 5.2.10, the generalized finite expansion of f at $+\infty$ of order 1 is:*

$$f(x)=x+1-\frac{1}{x}+\frac{1}{x}\varepsilon\left(\frac{1}{x}\right),\ \textit{with}\ \lim_{x\to+\infty}\varepsilon\left(\frac{1}{x}\right)=0.$$

So the straight-line (D) of equation $y=x+1$ is an oblique asymptote to the curve of f at $+\infty$. Moreover, as

$$f(x)-y=-\frac{1}{x}+\frac{1}{x}\varepsilon\left(\frac{1}{x}\right)\underset{+\infty}{\sim}-\frac{1}{x}<0,$$

then the curve of f is below (D) in a neighborhood of $+\infty$.
In the other side, the generalized finite expansion of f at $-\infty$ of order 1 is:

$$f(x)=x+1-\frac{1}{x}+\frac{1}{x}\varepsilon\left(\frac{1}{x}\right),\ \textit{with}\ \lim_{x\to-\infty}\varepsilon\left(\frac{1}{x}\right)=0.$$

So (D) is also an oblique asymptote to the curve of f at $-\infty$. On the other hand, as

$$f(x) - y \underset{-\infty}{\sim} -\frac{1}{x} > 0,$$

then the curve of f is above (D) in a neighborhood of $-\infty$.

Exercise 124

Consider the function f, defined on $I = \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[- \{0\}$ by:

$$f(x) = (\cos x)^{\frac{1}{x}}.$$

1) **Show that f is extendable by continuity at 0. Let g be its extension.**

2) **Show that g is differentiable at 0 and find $g'(0)$.**

3) **Determine the equation of the tangent to the curve of g at the point of abscissa 0, and study their relative positions.**

Solution

1) For all $x \in I$,
$$f(x) = (\cos x)^{\frac{1}{x}} = e^{\frac{1}{x}\ln(\cos x)}.$$
By the part (3) of the example 5.2.4, in a neighborhood of 0 and up to order 3, we have:
$$\ln(\cos x) = -\frac{x^2}{2} + x^3\varepsilon(x).$$
So
$$\begin{aligned} f(x) &= e^{-\frac{x}{2}+x^2\varepsilon(x)} = e^u \quad \text{where } u = -\frac{x}{2} + x^2\varepsilon(x) \text{ and } \lim_{x\to 0} u = 0 \\ &= 1 + \frac{u}{1!} + \frac{u^2}{2!} + u^2\varepsilon(u) = 1 - \frac{x}{2} + \frac{x^2}{8} + x^2\varepsilon(x) \end{aligned}$$
with $\lim_{x\to 0}\varepsilon(x) = 0$. So $\lim_{x\to 0} f(x) = 1 \in \mathbb{R}$ and therefore f is extendable by continuity at 0. Let g be its extension, then $g(0) = 1$.

2) Since
$$\lim_{x\to 0}\frac{g(x)-g(0)}{x-0} = \lim_{\substack{x\to 0\\ x\neq 0}}\frac{f(x)-1}{x} = \lim_{x\to 0}\Big(-\frac{1}{2} + \frac{x}{8} + x\varepsilon(x)\Big) = -\frac{1}{2},$$
then g is differentiable at 0 and $g'(0) = -\frac{1}{2}$.

3) For all $x \in I$,
$$g(x) = f(x) = 1 - \frac{x}{2} + \frac{x^2}{8} + x^2\varepsilon(x).$$

So the equation of the tangent (T) to the curve (C) of g at the point of abscissa 0 is $y = 1 - \frac{x}{2}$. Moreover,

$$g(x) - y \underset{0}{\sim} \frac{x^2}{8} \geq 0.$$

So (C) is above (T) in a neighborhood of 0. □

Exercise 125
Let a be a strictly positive real number. By using the finite expansions, study, according to the values of a, the following limit:

$$\lim_{x \to +\infty} \left(\sqrt{1 + a^2x^2} + x \ln \left(\frac{x}{2x - 1} \right) \right).$$

Solution Let

$$h(x) = \sqrt{1 + a^2x^2} + x \ln \left(\frac{x}{2x - 1} \right).$$

Put $u = \frac{1}{x} \to 0^+$ as $x \to +\infty$.

$$\begin{aligned} h(x) &= \frac{1}{u} \left[\sqrt{a^2 + u^2} + \ln \left(\frac{1}{2 - u} \right) \right] = \frac{1}{u} \left[\sqrt{a^2 + u^2} - \ln(2 - u) \right] \\ &= \frac{1}{u} \left[a\sqrt{1 + \frac{u^2}{a^2}} - \ln 2 - \ln \left(1 - \frac{u}{2} \right) \right] \\ &= \frac{1}{u} \left[a\left(1 + \frac{u^2}{2a^2} + u^2\varepsilon(u^2)\right) - \ln 2 + \frac{u}{2} + \frac{u^2}{8} + u^2\varepsilon(u^2) \right] \\ &= \frac{1}{u} \left[a - \ln 2 + \frac{u}{2} + \left(\frac{1}{2a} + \frac{1}{8} \right) u^2 + u^2\varepsilon(u^2) \right] \\ &= (a - \ln 2)x + \frac{1}{2} + \left(\frac{1}{2a} + \frac{1}{8} \right) \frac{1}{x} + \frac{1}{x}\varepsilon\left(\frac{1}{x}\right) \end{aligned}$$

with $\lim\limits_{x \to +\infty} \varepsilon\left(\frac{1}{x}\right) = 0$. So

$$\lim_{x \to +\infty} h(x) = \begin{cases} +\infty & \text{if } a > \ln 2 \\ -\infty & \text{if } a < \ln 2 \\ \frac{1}{2} & \text{if } a = \ln 2 \end{cases} \quad \square$$

Exercise 126

1) **Determine the finite expansions at 0 up to order 3 of $e^{\sqrt{1+x}}$ and $e^{\sqrt{1-x}}$. Deduce the finite expansion at 0 up to order 4 of**

$$f(x) = x\left(e^{\sqrt{1+x}} - e^{\sqrt{1-x}}\right).$$

2) Find the finite expansion at 0 up to order 4 of

$$g(x) = \ln\left(1 + 2\arctan^2 x\right).$$

3) Deduce that the function h defined by $h(x) = \dfrac{f(x)}{g(x)}$ is extendable by continuity at 0.

Solution

1) In a neighborhood of 0 and up to order 3, we have:

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + x^3\varepsilon(x)$$
$$\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + x^3\varepsilon(x)$$

So

$$\begin{aligned}
e^{\sqrt{1+x}} &= ee^u \quad \text{where } u = \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + x^3\varepsilon(x) \text{ and } \lim_{x\to 0} u = 0 \\
&= e\left[1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + u^3\varepsilon(u)\right] \\
&= e\left[1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \frac{1}{2}\left(\frac{1}{4}x^2 - \frac{1}{8}x^3\right) + \frac{1}{48}x^3 + x^3\varepsilon(x)\right] \\
&= e + \frac{e}{2}x + \frac{e}{48}x^3 + x^3\varepsilon(x).
\end{aligned}$$

$$\begin{aligned}
e^{\sqrt{1-x}} &= ee^u \quad \text{where } u = -\frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + x^3\varepsilon(x) \text{ and } \lim_{x\to 0} u = 0 \\
&= e\left[1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + u^3\varepsilon(u)\right] \\
&= e\left[1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \frac{1}{2}\left(\frac{1}{4}x^2 + \frac{1}{8}x^3\right) - \frac{1}{48}x^3 + x^3\varepsilon(x)\right] \\
&= e - \frac{e}{2}x - \frac{e}{48}x^3 + x^3\varepsilon(x).
\end{aligned}$$

Hence

$$f(x) = x(e^{\sqrt{1+x}} - e^{\sqrt{1-x}}) = x\left(ex + \frac{e}{24}x^3 + x^3\varepsilon(x)\right) = ex^2 + \frac{e}{24}x^4 + x^4\varepsilon(x).$$

with $\lim_{x\to 0}\varepsilon(x) = 0$.

2) In a neighborhood of 0 and up to order 4, $\arctan x = x - \dfrac{1}{3}x^3 + x^4\varepsilon(x)$, and therefore

$$\arctan^2 x = x^2 - \frac{2}{3}x^4 + x^4\varepsilon(x).$$

Hence

$$\begin{aligned}
g(x) &= \ln\left(1+2\arctan^2 x\right) = \ln\left(1+2x^2-\frac{4}{3}x^4+x^4\varepsilon(x)\right) \\
&= \ln(1+u) \quad \text{where } u = 2x^2-\frac{4}{3}x^4+x^4\varepsilon(x) \text{ and } \lim_{x\to 0} u = 0 \\
&= u-\frac{u^2}{2}+\frac{u^3}{3}-\frac{u^4}{4}+u^4\varepsilon(u) \\
&= \left(2x^2-\frac{4}{3}x^4\right)-\frac{1}{2}(4x^4)+x^4\varepsilon(x) \;=\; 2x^2-\frac{10}{3}x^4+x^4\varepsilon(x).
\end{aligned}$$

3) By the parts (1) and (2), $f(x) \underset{0}{\sim} ex^2$ and $g(x) \underset{0}{\sim} 2x^2$. So

$$h(x) = \frac{f(x)}{g(x)} \underset{0}{\sim} \frac{ex^2}{2x^2} = \frac{e}{2}.$$

Hence $\lim_{x\to 0} h(x) = \frac{e}{2} \in \mathbb{R}$. Hence h is extendable by continuity at 0. □

Exercise 127

1) Find the finite expansion up to order 2 at 0 of the function f defined by:

$$f(x) = \ln(e+x) - \cos x.$$

2) Determine the real numbers a and b for which the limit

$$\lim_{x\to 0} \frac{f(x)-a-bx}{x\sin x}$$

is finite. What is the value of this limit ?

Solution

1) In a neighborhood of 0, we have:

$$f(x) = \ln(e+x)-\cos x = \ln\left(e\left(1+\frac{x}{e}\right)\right)-\cos x = 1+\ln\left(1+\frac{x}{e}\right)-\cos x.$$

But

$$1+\ln\left(1+\frac{x}{e}\right) = 1+\frac{x}{e}-\frac{1}{2e^2}x^2+x^2\varepsilon(x)$$

and

$$\cos x = 1-\frac{1}{2}x^2+x^2\varepsilon(x),$$

then

$$f(x) = \frac{x}{e}-\frac{1-e^2}{2e^2}x^2+x^2\varepsilon(x)$$

with $\lim_{x\to 0}\varepsilon(x) = 0$.

2) Since $x \sin x = x^2 + x^2\varepsilon(x)$, then

$$\begin{aligned}\frac{f(x)-a-bx}{x\sin x} &= \frac{-a+\left(\frac{1}{e}-b\right)x - \dfrac{1-e^2}{2e^2}x^2 + x^2\varepsilon(x)}{x^2+x^2\varepsilon(x)}\\ &= -\frac{a}{x^2}+\left(\frac{1}{e}-b\right)\frac{1}{x}-\frac{1-e^2}{2e^2}+\varepsilon(x).\end{aligned}$$

So $\lim\limits_{x\to 0}\dfrac{f(x)-a-bx}{x\sin x}$ is finite if $a=0$ and $b=\dfrac{1}{e}$. In this case,

$$\lim_{x\to 0}\frac{f(x)-a-bx}{x\sin x}=\frac{e^2-1}{2e^2}. \quad \square$$

Exercise 128
Let f be the function defined on $\mathbb{R}$ by:

$$f(x)=\sqrt{\frac{x^4}{x^2+1}}$$

By using finite expansions, find the equation of the asymptote to the curve (C) of f in a neighborhood of $+\infty$, and then study its position with respect to (C).

Solution For any $x>0$,

$$\frac{f(x)}{x}=\frac{1}{x}\sqrt{\frac{x^4}{x^2+1}}=\sqrt{\frac{x^4}{x^4+x^2}}=\sqrt{\frac{x^4}{x^4\left(1+\dfrac{1}{x^2}\right)}}=\left(1+\frac{1}{x^2}\right)^{-\frac{1}{2}}.$$

Put $u=\frac{1}{x}$, then $u\to 0^+$ as $x\to+\infty$. So

$$\frac{f(x)}{x}=(1+u^2)^{-\frac{1}{2}}=1-\frac{1}{2}u^2+u^2\varepsilon(u)=1-\frac{1}{2x^2}+\frac{1}{x^2}\varepsilon\left(\frac{1}{x}\right).$$

Hence the generalized finite expansion of f at $+\infty$ of order 1 is:

$$f(x)=x-\frac{1}{2x}+\frac{1}{x}\varepsilon\left(\frac{1}{x}\right), \quad \text{with} \lim_{x\to+\infty}\varepsilon\left(\frac{1}{x}\right)=0.$$

So the equation of the asymptote (D) to the curve (C) of f in a neighborhood of $+\infty$ is $y=x$. Moreover, as

$$f(x)-y \underset{+\infty}{\sim} -\frac{1}{2x}<0,$$

then (C) is below the asymptote (D). $\square$

Bibliography

[1] M. Comenetz. *Calculus: The Elements*, World Scientific, 2002.

[2] C. Deschamps and A. Warusfel, *J'intègre : Mathématiques première année*, Dunod, 1999.

[3] J. Lelong-Ferrand and J.-M. Arnaudiès, *Cours de mathématiques*, T2: Analysis, Bordas, 1977.

[4] F. Liret and D. Martinais, *Analyse* **1**$^{\grave{e}re}$ *année*, **2**$^{\text{nd}}$ edition, Science Sup. Dunod, 2003.

[5] D. Prochasson, *Analyse* **1**$^{\grave{e}re}$ *année, Exercices corrigés*, **2**$^{\text{nd}}$ edition, Science Sup. Dunod, 2003.

[6] Wikipedia and google in general are good ressources, but we have to be careful on the accuracy of the informations.