RSA Attacks

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Outline

Simple Attacks

Small Difference

Insufficient Randomness

Hastad's Broadcast Attack

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- Works with any small set of possible messages

Solution

- To solve this common problem, use randomness
- For example, use the first 128 bits for the message and append 128 more random bits before encryption
- Bob will be able to read the first 128 bits, and this simple attack won't work: more than 2^{128} possible messages

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- One typical solution is to generate random primes for the secret key uniformly among very large, 2048-bit numbers

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- What can Eve do?

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- Try all integers between $\sqrt{n} - r$ and $\sqrt{n}$ as divisors of n
- Factorize n and decrypt the same way as Bob does

Even Faster

- p and q are both odd, so $\frac{p+q}{2}$ and $\frac{p-q}{2}$ are integers
- $n = pq = (\frac{p+q}{2} + \frac{p-q}{2})(\frac{p+q}{2} - \frac{p-q}{2}) = (\frac{p+q}{2})^2 - (\frac{p-q}{2})^2$
- So n is a difference of squares, and one of the squares is small, because $|p - q|$ is small
- We can try adding increasing squares of integers to n until we get an exact square of integer

Solution

- Generate p and q
- If $|p-q|$ is small, regenerate
- Repeat until $|p-q|$ is sufficiently large

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- Attack by Heninger et al. and Lenstra et al.
- Use public keys from different devices!
- Experiment resulted in 0.4% factored HTTPS keys!

OpenSSL RSA key generation:

```
rng = RandomNumberGenerator()
rng.seed(seed)
p = rng.big_random_prime()
rng.add_randomness(bits)
q = rng.big_random_prime()
n = p * q
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What if the `seed` is not random enough?

Example: keys are generated by the router immediately after startup, no incoming network packets to get randomness from yet.

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Sometimes the same p will be generated, with different q

Combine Public Keys

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- Take keys from many routers and try to combine all pairs
- Make sure the random number generator is properly seeded
- Some computer programs ask the user to move mouse for some time to get randomness

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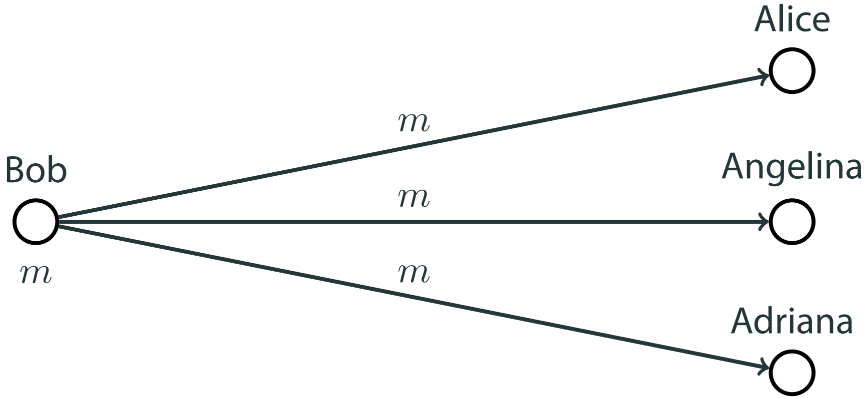
- Hastad came up with an attack in case Bob sends the same message m to several recipients using their public keys
- Uses the fact that the same message m is sent using different keys
- We will consider a very simplified case as an example


Alice
Bob
Angelina
Adriana


Alice
Bob
Angelina
m
Adriana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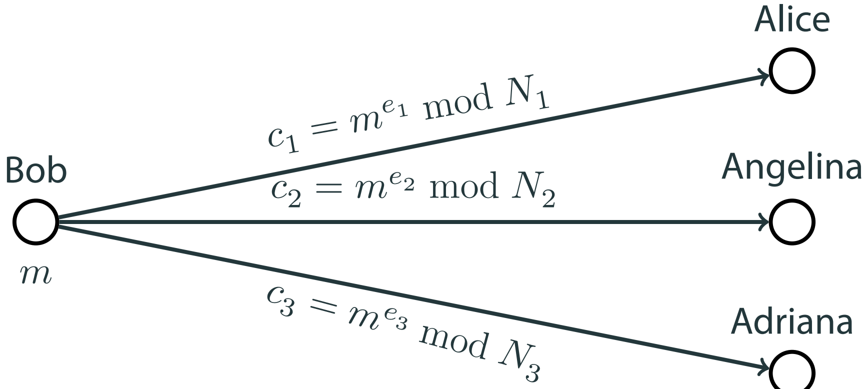
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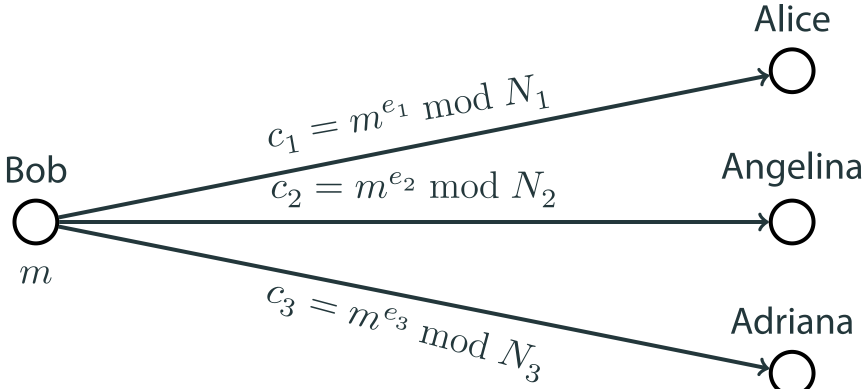
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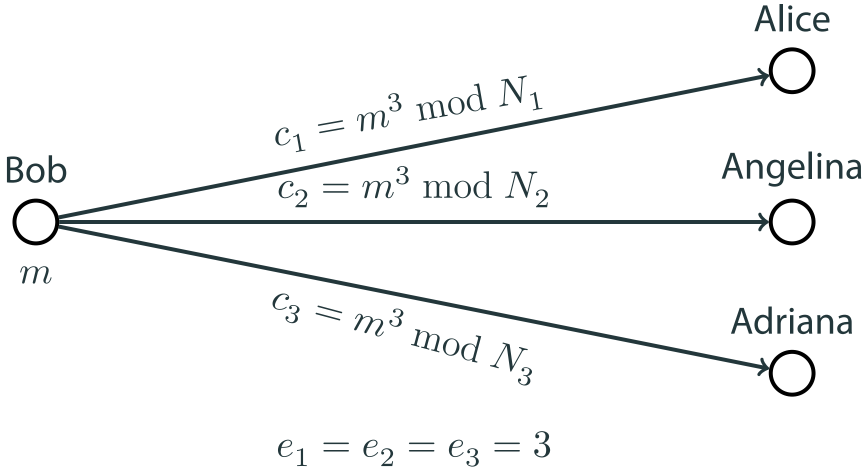
$$e_1 = e_2 = e_3 = 3$$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Again by Chinese Remainder Theorem,

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So $c = m^3$

Eve can decode m as $m = \sqrt[3]{c}$

- Broadcasting the same fixed message is a problem
- Hastad's original attack works even with bigger and different e_i
- Solution — add random padding to m before encryption
- Then it is impossible to compute m using all c_i, because each c_i includes some randomness apart from m

More Attacks

- Time to compute $c^d \bmod n$ can expose d — if one can send ciphertexts to the server which decrypts them and sends some response
- Error return code in case of incorrect ciphertext can expose the message in the same case
- Power consumption while computing $c^d \bmod n$ can expose d — if one tries to decrypt an encrypted hard drive on a stolen computer, or withdraw cash from a stolen card using an ATM

Conclusion

- RSA is a powerful method which is used everywhere
- Hard to implement correctly, although the algorithm itself is relatively simple
- Attacks from unexpected angles
- Deeper dive in dedicated cryptography courses
- Have fun with the problems: let’s break some ciphers!