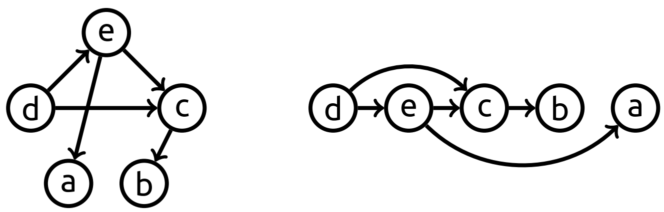
Week 1

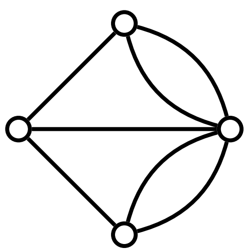
- A graph $G = (V, E)$ consists of the set of vertices V and the set of edges E.
- For an edge $e = \{u, v\}$, we say:
  - e connects u and v;
  - u and v are end points of e;
  - u and e are incident (v and e are incident);
  - u and v are adjacent or neighbors.
- The degree $\deg(v)$ of a vertex v is the number of edges incident to it. A vertex of degree 0 is called isolated.
- In a directed graph, the indegree (outdegree) of a vertex v is the number of edges ending at v (leaving v).
- The degree of a graph is the maximum degree of its vertex. A k-regular graph is a graph where each vertex has degree k.
- The complement of a graph $G = (V, E)$ is a graph $\overline{G} = (V, \overline{E})$ s.t. $(u, v) \in \overline{E}$ if and only if $(u, v) \notin E$.
- A walk in a graph is a sequence of edges, where each edge (except for the 1st one) starts with a vertex where the previous edge ended. The length of a walk is the number of edges in it.
- A path is a walk where all edges are distinct.
- A simple path is a walk where all vertices are distinct.
- A cycle in a graph is a path whose 1st vertex is the same as the last one.
- A simple cycle is a cycle where all vertices except for the 1st one are distinct. (And there 1st vertex is taken twice.)
- A graph is called connected if there is a path between every pair of its vertices.
- A connected component of a graph G is a maximal connected subgraph of G.
- The path graph P_n consists of n vertices $v_1, \ldots, v_n$ and $n-1$ edges $\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}$.
- The cycle graph C_n consists of n vertices $v_1, \ldots, v_n$ and n edges $\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.
- The complete graph (clique) K_n contains n vertices $v_1, \ldots, v_n$ and all $n(n-1)/2$ edges between them.
- Three equivalent definitions of a tree:
  - a connected graph without cycles;
  - a connected graph on n vertices with $n-1$ edges;
  - a graph with a unique simple path between any pair of its vertices.
- A graph G is bipartite if its vertices can be partitioned into two disjoint sets L and R s.t. every edge of G connects a vertex in L with a vertex in R.

Week 2

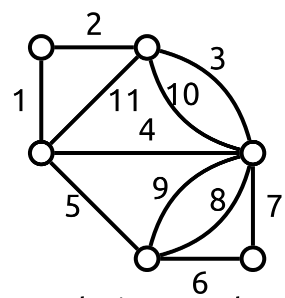
- Degree sum formula: for any undirected graph $G(V, E)$, the sum of degrees of all its nodes is twice the number of edges: $\sum_{v \in V} \text{degree}(v) = 2 \cdot |E|$
- Lower bound on the number of connected components: an undirected graph $G(V, E)$ has at least $|V| - |E|$ connected components.
- A directed acyclic graph (DAG) is a directed graph without cycles.
- A topological ordering of a directed graph is an ordering of its vertices such that, for each edge (u, v), u comes before v. Such an ordering exists, if and only if the graph is acyclic.



- An Eulerian cycle (or path) visits every edge exactly once.



non-Eulerian graph

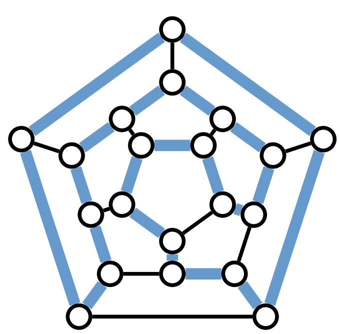


Eulerian graph

Criteria:

  - A connected *undirected* graph contains an Eulerian cycle, if and only if the degree of every node is even.
  - A strongly connected *directed* graph contains an Eulerian cycle, if and only if, for every node, its in-degree is equal to its out-degree.

- A Hamiltonian cycle visits every node exactly once.



Week 3

- A spanning tree of a graph G, is a subgraph of G which is a tree and contains all vertices of G.
- A minimum spanning tree of a weighted graph is a spanning tree of the smallest weight.
- Kruskal's minimum spanning tree algorithm:
  - Start with an empty graph T.
  - Repeat $n-1$ times:
    - Add to T an edge of the smallest weight which doesn't create a cycle in T.
- A graph is bipartite if and only if it has no cycles of odd length.
- A matching in a graph is a set of edges without common vertices.
- A maximal matching is a matching which cannot be extended to a larger matching.
- A maximum matching is a matching of the largest size.
- If $G = (V, E)$ is a graph, and $S \subseteq V$ is its subset of vertices, then the neighborhood $N(S)$ of S is the set of all vertices connected to at least one vertex in S.
- Hall's theorem: In a bipartite graph $G = (L \cup R, E)$, there is a matching which covers all vertices from L if and only if for every subset of vertices $S \subseteq L$, $|S| \leq |N(S)|$.
- A graph is planar if it can be drawn in the plane such that its edges do not meet except at their end points.
- A face of a planar drawing of a graph is a region bounded by the edges of the graph. (There is always one infinitely large outer face.)
- Euler's formula: for a planar drawing of a connected planar graph: $v - e + f = 2$.
- Every planar graph has a vertex of degree ≤ 5.
- In every planar graph on $n \geq 3$ vertices: $e \leq 3v - 6$.
- In every bipartite planar graph on $n \geq 4$ vertices: $e \leq 2v - 4$.