DUALITY IN OPTION PRICING

In a financial market, investors buy and sell options whose value is derived from two stocks. Right now, stock A is worth $20/share, and stock B is worth $15/share. One particular call option has a "strike price" of $25 and matures one month from today. To calculate the payoff of this option at maturity, you first take the average of the two stock prices on that day. If that average is *above* the strike price of $25, the payoff of the option is equal to the difference. If the average of the stock prices is *below* $25, the payoff of the option is $0.

You have already sold one such option to a buyer, but you do not know what the future stock prices (and thus the payoff that you will need to pay to the buyer) will be. However, you have obtained forecasts from three different sources. Each source gives you predictions for both stock prices. Each set of predictions (if correct) will lead to a different payoff for the option, as shown below:

Source	Stock A	Stock B	Average	Option Payoff
1	$34	$18	$26	$1
2	$15	$9	$12	$0
3	$18	$40	$29	$4

You do not know which source will be correct. So, you have decided to invest in a portfolio of cash and stocks whose value one month from now will **always** be enough to pay off the buyer, regardless of which source turns out to be right. To find the cheapest such portfolio, you solve the LP:

x_0 -- amount of cash (in dollars) to set aside

x_i -- number of shares of stock $i = A, B$ to buy (or borrow) now

minimize $x_0 + 20x_A + 15x_B$

subject to

$$x_0 + 34x_A + 18x_B \geq 1$$
$$x_0 + 15x_A + 9x_B \geq 0$$
$$x_0 + 18x_A + 40x_B \geq 4$$
$$x_0, x_A, x_B \quad \textit{free}$$

Note that the decision variables are free, since you can borrow cash and stocks. If the decision variables are positive, that means you are saving cash and buying stocks; if they are negative, you are borrowing cash and borrowing (or selling short) stocks.

You solve this problem and obtain the following sensitivity report:

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
J18	Cash	-1.035587189	0	1	0.312785388	0.343629344
K18	Stock A	-0.008896797	0	20	7.666666667	4.419354839
L18	Stock B	0.129893238	0	15	22.25	3.631578947

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
M20	Source 1	1	0.243772242	1	1E+30	1E+30
M21	Source 2	0	0.633451957	0	1E+30	1E+30
M22	Source 3	4	0.122775801	4	1E+30	1E+30

According to the optimal solution, you should borrow approximately $1.0356 in cash. You should also borrow (or sell short) 0.0089 shares of stock A, and purchase 0.1299 shares of stock B.

Answer the following questions:

1) Explain the meaning of the constraints in the primal LP on the previous page.

2) Write down the dual of the LP. What is the optimal *solution* to the dual?

3) Give an economic interpretation of the dual LP. (Hint: An economist would say that, *on average*, a stock price one month from now should be the same as today's price, otherwise the market would not be fair.)