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Date

CY - A Homework 06.

Q1. If there is a basis B such that $[T]_B$ is diagonal, then A is similar to a diagonal matrix. This means A would have n linearly independent eigen vectors. However, in this case that is not necessary as A only has 2 distinct eigen values.
So, there will exist basis B such that $[T]_B$ is diagonal if and only if geometric multiplicity of eigenvalue 5 is 2.

Q2.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{bmatrix} \qquad \det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 0-\lambda & 1 & 1 \\ 2 & 1-\lambda & 2 \\ 3 & 3 & 2-\lambda \end{bmatrix} = 0.$$

$$(0-\lambda)[(1-\lambda)(2-\lambda) - 6] - [2(2-\lambda) - 6] + [6 - 3(1-\lambda)].$$

$$-\lambda[2 - \lambda - 2\lambda - \lambda^2 - 6] - [4 - 2\lambda - 6] + [6 - 3 + 3\lambda]$$

$$-\lambda[+\lambda^2 - 3\lambda - 4] - [-2\lambda - 2] + [3 + 3\lambda].$$

$$-\lambda^3 + 3\lambda^2 + 4\lambda + 2\lambda + 2 + 3 + 3\lambda$$

$$-\lambda^3 + 3\lambda^2 + 9\lambda + 5 = 0.$$

$$\lambda_1 = 5, \quad \lambda_2 = -1, \quad \lambda_3 = -1$$

For $\lambda_1 = 5$

$$A = \begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \overset{R_2 \leftrightarrow R_1}{\sim} \begin{bmatrix} 2 & -4 & 2 \\ -5 & 1 & 1 \\ 3 & 3 & -3 \end{bmatrix}$$

$$\overset{R_1/2}{\sim} \begin{bmatrix} 1 & -2 & 1 \\ -5 & 1 & 1 \\ 3 & 3 & -3 \end{bmatrix} \xrightarrow{R_2 + 5R_1} \begin{bmatrix} 1 & -2 & 1 \\ 0 & -9 & 6 \\ 3 & 3 & -3 \end{bmatrix}$$

$$\xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & -2 & 1 \\ 0 & -9 & 6 \\ 0 & 9 & -6 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & -2 & 1 \\ 0 & -9 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -x_3 - 2\frac{x_3}{9} = \frac{-11}{9}$$

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$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -9 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ -9x_2 + x_3 &= 0. \end{aligned}$$

$$x_2 = +x_3/+9 = \frac{x_3}{9}.$$

$$x_1 - 2\left[\frac{x_3}{9}\right] + x_3 = 0.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-11}{9}x_3 \\ \frac{x_3}{9} \\ x_3 \end{bmatrix} = \underset{\Downarrow\, V_1}{x_3} \begin{bmatrix} -11/9 \\ 1/9 \\ 1 \end{bmatrix}$$

For $\lambda = -1$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 3 & 3 & 3 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + x_2 + x_3 = 0. \quad , x_1 = -x_2 - x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = \underset{\Downarrow\, V_2}{x_2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \underset{\Downarrow\, V_3}{x_3} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -11/9 & -1 & -1 \\ 1/9 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

as geometric multiplicity $\leq$ algebraic multiplicity.
A is diagonalizable.
It has exactly n i.e 3 n linearly independent vectors.

Q3. $x_0 = 0$, $x_1 = 5$, $x_n = 3x_{n-1} + 4_{n-2}$
for $n \geq 2$.
Here, $x_{n-1} = 1\, x_{n-1} + 0\, x_{n-2}$.

$$\text{So } A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} x_{n-1} \\ x_{n-2} \end{bmatrix}}_{\vec{x}_{n-1}} = \underbrace{\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}}_{\vec{x}_n}$$

$$\begin{aligned} \vec{x}_n &= A\, \vec{x}_{n-1}. \\ &= A(A\vec{x}_{n-2}) \\ &= A(A(\vec{x}_{n-3}) \end{aligned}$$

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$\vec{x_0} = \begin{bmatrix} x_0 \\ x_{-1} \end{bmatrix}$, $\vec{x_1} = \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$x_2 = 3x_1 + 4x_0$

$x_2 = 3(5) + 4(0) = 15$

$\therefore x_2 = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix}$

$A^{n-2} = PD^{n-2}P^{-1}$

$x_n = \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = A^{n-2} x_2$

$= PD^{n-2}P^{-1} \begin{bmatrix} 15 \\ 5 \end{bmatrix}$

Q. 4. $x'(t) = A x(t)$

$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

$\det(A - \lambda I) = 0.$

$\lambda^2 - 3\lambda - 4 = 0.$

$(\lambda - 4)(\lambda + 1) = 0$, $\lambda_1 = 4$, $\lambda_2 = -1$.

for $\lambda_1 = 4$.

$A = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix} \xrightarrow{R/2} \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix}$

$\xrightarrow{R_2 + 3R_1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $x_1 - x_2 = 0.$

$x_1 = x_2$ so $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

V_1.

for $\lambda_2 = -1$. $2x_1 + 3x_2 = 0$ $x_1 = -3/2 \, x_2$

$A = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \Rightarrow x_2 \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$

or $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$x(t) = c_1 e^{4t} - 3c_2 e^{-t}$

$y(t) = c_1 e^{4t} + 2c_2 e^{-t}$

If we consider as one so $X(t) = c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

when $x(0) = 0$ and $y(0) = 5$.

$x(0) = c_1 + 3c_2 = 0.$ } solving these

$y(0) = -(c_1 + 2c_2 = 5.)$ } we get $c_1 = 3$ and $c_2 = -1$.

$c_1 = -5$

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So $x(t) = 3e^{4t} - 3e^{-t}$.

$y(t) = 3e^{4t} + 2e^{-t}$

$x_{k+1} = A x_k$.

$x_k = c_1 \lambda_1^k \vec{v_1} + c_2 \lambda_2^k \vec{v_2} + c_3 \cdot (-1)^k \vec{v_3}$

General solution.

$$x_k = c_1 5^k \begin{bmatrix} -11/9 \\ 1/9 \\ 1 \end{bmatrix} + c_2 (-1) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \cdot (-1)^k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$