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C4 - A

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Homework # 9.

Q. Least square method.

$$y = \beta_0 + \beta_1 x$$

X

$$\rightarrow 3 = \beta_0 + 2\beta_1 x$$
$$\rightarrow 1 = \beta_0 - 4\beta_1 x$$
$$\rightarrow 5 = \beta_0 + 2\beta_1 x$$

$$y = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 2 \\ 1 & -4 \\ 1 & 2 \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

$$\langle x, y \rangle = w_1^2 x_1 y_1 + \dots w_n^2 x_n y_n.$$

Line : $y = ax + b$.

a = slope , b = intercept.

$n = 3$ (no. points)

Σx_i sum of x coord. $= 2 - 4 + 2 = 0$.

Σy_i " y coord $= 3 + 5 + 1 = 9$.

Σx_i^2 sum of square of x coord $= 2^2 + (-4)^2 + 2^2 = 4 + 16 + 4 = 24$

$\Sigma x_i y_i$ Sum of $(x, y) = (2)(3) + (-4)(1) + (2)(5) = 12$.

$$a = \frac{n \times \text{sum}(x.y) - (\text{sum x coord} \times \text{sum y coord})}{n \times \text{sum of square of x-coord} - \text{square of sum of x coord}}$$

$$a = \frac{3 \times 12 - 0}{3 \times 24 - 0} = 1/2$$

$$b = \frac{\text{sum of y coord} - a \times \text{sum of x coord}}{n} = \frac{9 - 1/2(0)}{3}$$

$$= 9/3 = 3.$$

$$y = 1/2\, x + 3.$$

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error $= E = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$

$x = 2, \quad \hat{y} = 1/2(2) + 3 = 4.$

$x = -4, \quad \hat{y} = 1/2(-4) + 3 = 1.$

$x = 2, \quad \hat{y} = 1/2(2) + 3 = 4$

squared errors:

for $(2,3)$:

$(y - \hat{y})^2 = (3-4)^2 = 1.$

$(-4, 1)$:

$(y - \hat{y})^2 = (1-1)^2 = 0.$

$(2, 5)$:

$(y - \hat{y})^2 = 1^2 = 1.$

$E = 1 + 0 + 1 = 2.$

Q 3. a). $u = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$ $\quad u\,u^T = \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \end{bmatrix} \times \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$

$u^T u$

orthonormal basis iff $UU^T = I$

$= 1/4 + 1/4 + 1/4 + 1/4$

$= 1$

$= \begin{bmatrix} 1/4 & 1/4 & -1/4 & -1/4 \\ 1/4 & 1/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 1/4 & 1/4 \\ -1/4 & -1/4 & 1/4 & 1/4 \end{bmatrix}$

$P = \dfrac{uu^T}{u.u = 1} = uu^T$

$P = \begin{bmatrix} 1/4 & 1/4 & -1/4 & -1/4 \\ 1/4 & 1/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 1/4 & 1/4 \\ -1/4 & -1/4 & 1/4 & 1/4 \end{bmatrix}$

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Q1.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}_{3\times 2} \qquad A^T = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{matrix} 1+1 & 1-1 \\ 1-1 & 1+1+1 \end{matrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}_{2\times 2}$$

$$\lambda_1 = 3, \quad \lambda_2 = 2$$

$$\sigma_1 = \sqrt{3} \qquad \sigma_2 = \sqrt{2}$$

$\lambda_1 = 3$: $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 2$: $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{2\times 1}$ standard basis

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad V^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

compute U: $u_i = \frac{A v_i}{\sigma_i}$

$$u_1 = \frac{\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\sqrt{3}} = \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ -1/\sqrt{3} \end{bmatrix}$$

$$u_2 = \frac{A v_2}{\sigma_2} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$U = [u_1 \; u_2] = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \\ -1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$A = U \Sigma V^T.$$

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Q 2. $Q(x_1, x_2, x_3) = 3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_2x_3.$

a)

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

b)

$$\begin{bmatrix} 3-\lambda & 2 & 0 \\ 2 & 4-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{bmatrix}$$

$$(3-\lambda)[(4-\lambda)(5-\lambda) - 4] \quad - 2\,[2(5-\lambda) - 0] \quad + 0.$$

$$(3-\lambda)[(20 - 9\lambda + \lambda^2) - 4] \quad - 2\,[10 - 2\lambda]$$

$$(3-\lambda)(\lambda^2 - 9\lambda + 16) \quad - 20 + 4\lambda.$$

$$= -\lambda^3 + 12\lambda^2 - 43\lambda + 48.$$

$\lambda_1 = 1, \quad \lambda_2 = 4, \quad \lambda_3 = 7$

$\lambda_1 = 1:$ $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 3 & -2 \\ 0 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad v_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$

$\lambda_2 = 4:$ $\begin{bmatrix} -1 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad V_2 = \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$

$x_1 - x_3 = 0$

$x_2 - \frac{1}{2}x_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 1/2\, x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$$

$\lambda_3 = 7:$ $\begin{bmatrix} -4 & 2 & 0 \\ 2 & -3 & -2 \\ 0 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad V_3 = \begin{bmatrix} -1/2 \\ -1 \\ 1 \end{bmatrix}$

$x_1 + 1/2\, x_3 = 0$

$x_2 + x_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1/2 \\ -1 \\ 1 \end{bmatrix}$$

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$$P = \begin{bmatrix} 2 & 1 & -1/2 \\ 2 & 1/2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$Q(x) = x^T A x = y^T D y \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad y^T = [y_1\ y_2\ y_3]$

$$\underset{1\times 3}{[y_1 \quad y_2 \quad y_3]} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_{3\times 3}$$

$$\underset{1\times 3}{[y_1 \quad 4y_2 \quad 7y_3]} \underset{3\times 1}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}} = y_1^2 + 4y_2^2 + 7y_3^2$$

c) $y_1^2 + 4y_2^2 + 7y_3^2 = 1$ surface is positive ~~definite~~ representing ~~...~~ sphere.

Q 6. $P_2(x) = \text{span}\{1, x\}$ apply G.S process.

$v_1 = 1. \quad \|v_1\| = 1$

so $u_1 = 1.$ (normalised vector)

Projection of x onto $u_1(x) = 1.$

$$\text{proj}_{u_1} x = \frac{\langle x, u_1\rangle}{\langle u_1, u_1\rangle} u_1$$

$$\langle x, u_1\rangle = \int_0^1 x \cdot 1\, dx = \left[\frac{x^2}{2}\right]_0^1 = 1/2.$$

$$\langle u_1, u_1\rangle = \int_0^1 1 \cdot 1 \Rightarrow 1.$$

so $\text{proj}_{u_1} x = 1/2$

$v_2 = x - \hat{x} = x - 1/2$

$\|v_2\| = \sqrt{\langle v_2, v_2\rangle}$

$$\int_0^1 \left(x - \frac{1}{2}\right)^2 dx.$$

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$$\int_0^1 \left(x^2 - x + \frac{1}{4}\right) dx$$

$$\int_0^1 x^2\,dx = \frac{1}{2}, \quad \int_0^1 x\,dx = \frac{1}{2}, \quad \int_0^1 \frac{1}{4}\,dx = \frac{1}{4}$$

$$\langle v_2, v_2\rangle = 1/12$$

$$\|v_2\| = \sqrt{3}/6 = 1/\sqrt{12}$$

$$\text{So } u_2 = \frac{6\left(x - \frac{1}{2}\right)}{\sqrt{3}}$$

Orthonormal basis $\{u_1, u_2\}$.

Projection of e^x onto $P_1(x)$ is

$$\text{proj}_{P_1} e^x = \langle e^x, u_1\rangle u_1 + \langle e^x, u_2\rangle u_2$$

orthonormal $\cancel{\langle u_1, u_1\rangle}$ $\cancel{\langle u_2, u_2\rangle}$

$$\langle e^x, u_1\rangle = \int_0^1 e^x \cdot 1\,dx = [e^x]_0^1 = e - 1.$$

$$\langle e^x, u_2\rangle = \int_0^1 e^x \cdot \frac{6(x - 1/2)}{\sqrt{3}}\,dx$$

using integration by parts $u = (x - 1/2)$, $dv = e^x dx$

$$= \frac{6}{\sqrt{3}} \int_0^1 e^x \cdot (x - 1/2)\,dx$$

$$\frac{6}{\sqrt{3}} \int_0^1 u\,dv = uv - \int v\,du.$$

$$\cancel{\text{so } \frac{6}{\sqrt{3}} \int_0^1 e^x (x-1/2) = xe^x - \int e^x \times 1.}$$

$$\underbrace{\frac{6}{\sqrt{3}} \int_0^1 x \cdot e^x\,dx}\; - \;\underbrace{1/2 \int_0^1 e^x\,dx}$$

$du = dx$

$u = x$

$dv = e^x dx$

$$\int u\,dv = uv - \int v\,du$$

$$= x \cdot e^x - \int e^x\,dx$$

$$= xe^x - e^x + C.$$

$$= \frac{6}{\sqrt{3}}\left[e^x(x-1)\right]_0^1$$

$$1/2\,[e^x]_0^1$$

$$\frac{e}{2} - \frac{1}{2}$$

$+1$

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so $\frac{6}{\sqrt{3}}\left(1 - e/2 + 1/2\right)$

$\langle e^x, u_2\rangle = \frac{6}{\sqrt{3}}\left(\frac{3}{2} - e/2\right)$

$proj_{p_1} e^x = \frac{(e-1)\cdot 1}{1} + \frac{6}{\sqrt{3}}\left(\frac{3}{2} - \frac{e}{2}\right) \cdot \frac{6(x-1/2)}{\sqrt{3}}$

$= (e-1) + 12\left(\frac{3}{2} - \frac{e}{2}\right)\left(x - \frac{1}{2}\right)^{1/12}.$

Q 8. a) $p(x) = x^2$ and $P_1 = \text{span}\{1, x\}$

$proj_{P'}(p(x)) = \langle p(x), u_1\rangle u_1 + \langle p(x), u_2\rangle u_2$

where $\{u_1, u_2\}$ is orthonormal basis for P_1.

$v_1 = 1$; $\|v_1\| = \sqrt{\langle v_1, v_1\rangle}$ &

where $\langle v_1, v_1\rangle = v_1(x_0)v_1(x_0) + v_1(x_1)v_1(x_1) + v_1(x_2)v_1(x_2) + v_1(x_3)v_1(x_3)$

$= 1^2 + 1^2 + 1^2 + 1^2 = 4$.

$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{4}} = 1/2$

$v_2 = x - \hat{x}$.

$\hat{x} = proj_{u_1} x = \frac{\langle x, u_1\rangle}{\langle u_1, u_1\rangle} u_1$

$\langle x, u_1\rangle = (-3)\left(\frac{1}{2}\right) + (-1)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right) + (3)\left(\frac{1}{2}\right)$

$= 0$.

So $v_2 = x$.

$\|v_2\|^2 = \langle v_2, v_2\rangle = (-3)^2 + (-1)^2 + (1)^2 + (3)^2$

$= 20$.

$u_2 = \frac{v_2}{\|v_2\|} = \frac{x}{2\sqrt{5}}$.

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Proj of $p(x) = x^2$ onto P_1

$\text{proj}_{P_1} p = \langle p, u_1\rangle u_1 + \langle p, u_2\rangle u_2$

$\langle p, u_1\rangle = p(-3)(1/2) + p(-1)(1/2) + p(1)(1/2) + p(3)(1/2)$

$p(x) = x^2$ $\quad p(-3) = (-3)^2 = 9$, $p(-1) = 1$, $p(1) = 1$, $p(3) = 9$.

$\langle p, u_1\rangle = \frac{1}{2}(9 + 1 + 1 + 9) = 10.$

$\langle p, u_2\rangle = \frac{1}{2\sqrt{5}}\left[p(-3)(-3) + p(-1)(-1) + p(1)(1) + p(3)(3)\right]$

$= \frac{1}{2\sqrt{5}}(\cancel{-27} \cancel{-1} \cancel{+1} \cancel{+27}) = 0.$

So $\text{proj}_{P_1} p = 10 \times \frac{1}{2} + 0.$
$= 5.$

Q.5 $\quad x + 2y + z = 0.$

Normal vector to plane $n = \begin{bmatrix}1\\2\\1\end{bmatrix}$

reflection vector $v = \begin{bmatrix}x\\y\\z\end{bmatrix}$

Projection of v onto n.

$\text{proj}_n v = \frac{v \cdot n}{n \cdot n} n.$

$v \cdot n = x \cdot 1 + y \cdot 2 + z \cdot 1 = x + 2y + z.$

$n \cdot n = 1^2 + 2^2 + 1^2 = 6.$

$\text{proj}_n v = \frac{x + 2y + z}{6}\begin{bmatrix}1\\2\\1\end{bmatrix}$

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$$v_{reflected} = v - 2 \cdot proj_n v$$

$$= \begin{bmatrix} x \\ y \\ z \end{bmatrix} - 2 \begin{bmatrix} x + 2y + z/6 \\ x + 2y + z/3 \\ x + 2y + z/6 \end{bmatrix}$$

$$= \begin{bmatrix} x - \frac{x + 2y + z}{3} \\ y - \frac{2(x + 2y + z)}{3} \\ z - \frac{x + 2y\, z}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 2x - 2y - 2z/3 \\ -2x + 3y - 2z/3 \\ -x - 2y + 2z/3 \end{bmatrix}$$

reflection matrix →

$$R = \begin{bmatrix} 2/3 & -2/3 & -2/3 \\ -2/3 & 1 & -2/3 \\ -1/3 & -2/3 & 2/3 \end{bmatrix}$$