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CY - A.

Linear Algebra. Homework 04.

Q1. $L : \mathbb{R}^3 \to \mathbb{R}^3$

$$L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\0\\1\end{bmatrix}, \quad L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

a) natural basis of $\mathbb{R}^3$:

$$e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$A = [L] = \begin{bmatrix} L(e_1) & L(e_2) & L(e_3) \end{bmatrix}$$

$$L(e_1) = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad L(e_2) = \begin{bmatrix}2\\0\\1\end{bmatrix}, \quad L(e_3) = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$\text{SO, } A = [L] = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) e.g $\vec{u} = \begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}, \quad \vec{v} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$

$$\vec{u} = \vec{e_1}u_1 + \vec{e_2}u_2 + \vec{e_3}u_3.$$

$$L(u) = u_1\begin{bmatrix}1\\1\\0\end{bmatrix} + u_2\begin{bmatrix}2\\1\\0\end{bmatrix} + u_3\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$L(u) = \begin{bmatrix} u_1 + 2u_2 + u_3 \\ u_1 + u_2 \\ u_3 \end{bmatrix}$$

$$L(v) = \begin{bmatrix} v_1 + 2v_2 + v_3 \\ v_1 + v_2 \\ v_3 \end{bmatrix}$$

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$$

$$L(\vec{u} + \vec{v}) = \begin{bmatrix} (u_1 + 2u_2 + u_3) + (v_1 + 2v_2 + v_3) \\ (u_1 + u_2) + (v_1 + v_2) \\ u_3 + v_3. \end{bmatrix}$$

$$= \begin{bmatrix} u_1 + 2u_2 + u_3 + v_1 + 2v_2 + v_3 \\ u_1 + u_2 + v_1 + v_2 \\ u_3 + v_3 \end{bmatrix}$$

Date

$$L(\vec{u}) + L(\vec{v}) = \begin{bmatrix} u_1 + 2u_2 + u_3 \\ u_1 + u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 + 2v_2 + v_3 \\ v_1 + v_2 \\ v_3 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 + 2u_2 + u_3 + v_1 + 2v_2 + v_3 \\ u_1 + u_2 + v_1 + v_2 \\ u_3 + v_3 \end{bmatrix}$$

$\therefore$ $T(u+v) = T(u) + T(v)$ holds

$$c\vec{u} = \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} \qquad L(c\vec{u}) = \begin{bmatrix} cu_1 + 2cu_2 + u_3 \\ cu_1 + cu_2 \\ cu_3 \end{bmatrix}$$

$$cL(u) = c \begin{bmatrix} u_1 + 2u_2 + u_3 \\ u_1 + u_2 \\ u_3 \end{bmatrix}$$

Since $L(c\vec{u}) = cL(\vec{u})$, $L$ is a linear transformation

c) $L\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = A\vec{x}$ and $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\downarrow$
$x$

$$= \underset{3\times 3}{\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

$$= \begin{bmatrix} 1\times 1 + 2\times 2 + 1\times 3 \\ 1\times 1 + 1\times 2 + 0\times 3 \\ 0\times 1 + 0\times 2 + 1\times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 3 \\ 3 \end{bmatrix}$$

d) Ker (L) is set of all vectors $\vec{u}$ such that $L(u) = 0$.

$$L(\vec{u}) = \begin{bmatrix} u_1 + 2u_2 + u_3 \\ u_1 + u_2 \\ u_3 \end{bmatrix}$$

$$\begin{bmatrix} u_1 + 2u_2 + u_3 \\ u_1 + u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Date

$u_1 + 2u_2 + u_3 = 0$ — ①

$u_1 + u_2 = 0$ — ②

$u_3 = 0$ — ③ (put in eq 1)

$u_1 + 2u_2 = 0$

$u_1 + u_2 = 0$. $u_1 = -u_2$.

$u_2 = 0$ and $u_1 = 0$.

Trivial solution. So $\ker(L) = \{\vec{0}\}$.

Basis for $\ker(L) = \{\ \}$.

There are no linearly independent vectors that can span the zero subspace.

e) Range (L) is column space of A.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Basis of Col A = Basis of Range L = pivot columns of A

i.e

$$\text{Basis for Range}(L) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

f). L is one-to-one as there are no free variables so each L (u) is an image of only one $\vec{u}$. L is also one to one because ker of L only contains the zero vector.

g) A has a pivot in every row, so for each $\vec{u}$ in $\mathbb{R}^3$, $A\vec{u} = L(u)$ is consistent. So L is onto.

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~~h) Linear if 1. $T(u+v) = T(u) + T(v)$ and $T(c\vec{u}) = cT(\vec{u})$~~

~~is not linear $\leftarrow$ $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$~~

~~because, $T(\vec{u}+\vec{v}) = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ 1 \end{bmatrix}$ and $T(\vec{u}) + T(\vec{v}) = \begin{bmatrix} u_1 \\ u_2 \\ 1 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ 1 \end{bmatrix}$~~

~~$\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ 1 \end{bmatrix} \neq \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ 2 \end{bmatrix}$~~

Date

h) linear if $T(u+v) = T(u) + T(v)$

$c\,T(\vec{u}) = T(c\vec{u})$.

$$T\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1^2 \\ v_2 \\ v_3 \end{bmatrix}$$ is not linear as

the 1st component is the following:

$(u_1 + v_1)^2 \neq u_1^2 + v_1^2$

and $(cu_1)^2 = c^2u_1^2 \neq c\,T(u)$ that is $c\,(u_1^2)$

$$T\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} = \begin{bmatrix} (cu_1)^2 \\ cu_2 \\ cu_3 \end{bmatrix}$$

$$c\,T\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = c\begin{bmatrix} u_1^2 \\ u_2 \\ u_3 \end{bmatrix}$$