R. M.K. COLLEGE OF ENGINEERING AND TECHNOLOGY
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22MA401 - PROBABILITY AND STATISTICS

UNIT I

ONE DIMENSIONAL RANDOM VARIABLES
PART A

1. Let X be a continuous random variable having the probability density function $f(x) = \begin{cases} \frac{2}{x^3}, & x \geq 1 \\ 0, & otherwise \end{cases}$, Find the distribution function of x.

Solution:

$$F(x) = \int_1^x f(x)\, dx = \int_1^x \frac{2}{x^3} dx = \left[-\frac{1}{x^2}\right]_1^x = 1 - \frac{1}{x^2}$$

2. A random variable X has the probability density function *f(x)* given by $f(x) = \begin{cases} cxe^{-x}, & x \geq 0 \\ 0, & otherwise \end{cases}$. Find the value of c.

Solution:

We know that

$$\int_0^\infty f(x)\, dx = 1 \Rightarrow \int_0^\infty cxe^{-x}\, dx = 1$$

$$c\left[-xe^{-x} - e^{-x}\right]_0^\infty = 1$$

$$c(1) = 1$$

$$c = 1$$

3. If a random variable has the probability density $f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & otherwise \end{cases}$. Find the probability that it will take on a value between 1 and 3. Also, find the probability that it will take on value greater than 0.5.

Solution:

$$P(1 < X < 3) = \int_1^3 f(x)\, dx = \int_1^3 2e^{-2x} dx = \left[-e^{-2x}\right]_1^3 = e^{-2} - e^{-6}$$

$$P(X > 0.5) = \int_{0.5}^\infty f(x)\, dx = \int_{0.5}^\infty 2e^{-2x} dx = \left[-e^{-2x}\right]_{0.5}^\infty = e^{-1}$$

4. Is the function defined as follows a density function?

$$f(x) = \begin{cases} 0, & x < 2 \\ \frac{1}{18}(3+2x), & 2 \leq x \leq 4 \\ 0, & x > 4 \end{cases}$$

Solution:

$$\int_2^4 f(x)\, dx = \int_2^4 \frac{1}{18}(3+2x)\, dx = \left[\frac{(3+2x)^2}{72}\right]_2^4 = 1.$$ Hence it is density function.

5. The cumulative distribution function (CDF) of a random variable X is $F(X) = 1-(1+x)e^{-x}$, $x>0$. Find the probability density function of X.

Solution:

$$f(x) = F'(x)$$
$$= 0 - \left[(1+x)\left(-e^{-x}\right)+(1)\left(e^{-x}\right)\right]$$
$$= xe^{-x}, \quad x>0$$

6. The number of hardware failures of a computer system in a week of operations has the following probability mass function:

No of failures:	0	1	2	3	4	5	6
Probability :	0.18	0.28	0.25	0.18	0.06	0.04	0.01

Find the mean of the number of failures in a week.

Solution:

$$E(X) = \sum x \ P(x) \ = (0)(0.18)+(1)(0.28)+(2)(0.25)+(3)(0.18)+$$
$$(4)(0.06)+(5)(0.04)+(6)(0.01)$$
$$= 1.82$$

7. Given the p.d.f of a continuous R.V X as follows: $f(x) = \begin{cases} 12.5x-1.25 & 0.1 \le x \le 0.5 \\ 0, & elsewhere \end{cases}$

Find P(0.2 < X < 0.3)

Solution:

$$P(0.2 < X < 0.3) \ = \int_{0.2}^{0.3} (12.5x - 1.25) \ dx$$
$$= \left[12.5\frac{x^2}{2} - 1.25x\right]_{0.2}^{0.3}$$
$$= 1.25\left[5(0.3)^2 - 0.3 - 5(0.2)^2 + 0.2\right]$$
$$= 0.1875$$

8. If the MGF of a continuous R.V X is given by $M_X(t) = \frac{3}{3-t}$. Find the mean and variance of X.

Solution:

$$M_X(t) = \frac{3}{3-t} = \frac{1}{1-\frac{t}{3}} = \left(1-\frac{t}{3}\right)^{-1} = 1+\frac{t}{3}+\left(\frac{t}{3}\right)^2+\left(\frac{t}{3}\right)^3+\ldots$$

$$E(X) = (coefficient \ \ of \ \ t) \ 1! = \frac{1}{3} \ \ is \ \ the \ \ mean$$

$$E(X^2) = \left(coefficient \ of \ t^2\right)2! = \frac{1}{9}2! = \frac{2}{9}$$

$$Variance \ = E(X^2) - E(X)^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

9. If the MGF of a discrete R.V X is given by $M_X(t)=\frac{1}{81}\left(1+2e^t\right)^4$, find the distribution of X.

Solution:

$$M_X(t)=\frac{1}{81}\left(1+2e^t\right)^4=\frac{1}{81}\left(1+4C_1\left(2e^t\right)+4C_2\left(2e^t\right)^2+4C_3\left(2e^t\right)^3+4C_4\left(2e^t\right)^4\right)$$

$$=\frac{1}{81}+\frac{8}{81}e^t+\frac{24}{81}e^{2t}+\frac{32}{81}e^{3t}+\frac{16}{81}e^{4t}$$

By definition of MGF ,

$$M_X(t)=\sum e^{tx}p(x)=p(0)+p(1)e^t+p(2)e^{2t}+p(3)e^{3t}+p(4)e^{4t}$$

On comparison with above expansion the probability distribution is

X	0	1	2	3	4
$p(x)$	$\frac{1}{81}$	$\frac{8}{81}$	$\frac{24}{81}$	$\frac{32}{81}$	$\frac{16}{81}$

10. Find the MGF of the R.V X whose p.d.f is $f(x)=\begin{cases}\frac{1}{10}, & 0<x<10\\ 0, & elsewhere\end{cases}$.Hence find its mean.

Solution:

$$M_X(t)=\int_0^{10}\frac{1}{10}e^{tx}\,dx$$

$$=\frac{1}{10}\left(\frac{e^{tx}}{t}\right)_0^{10}$$

$$=\frac{1}{10}\left(\frac{e^{10t}-1}{t}\right)$$

$$=\frac{1}{10t}\left(1+10t+\frac{100t^2}{2!}+\frac{1000t^3}{3!}+....-1\right)$$

$$=1+5t+\frac{50}{3}t^2+.....$$

$$Mean\ =\ coefficient\ of\ \ t\ =5$$

11. Given the p.d.f of a continuous r.v X as follows: $f(x)=\begin{cases}6x(1-x), & 0\le x\le 1\\ 0, & elsewhere\end{cases}$. Find the CDF of X.

Solution:

$$F(x)=\int_0^x f(x)\,dx=\int_0^x 6x(1-x)\,dx=\int_0^x 6x-6x^2\,dx=\left[3x^2-2x^3\right]_0^x=3x^2-2x^3$$

12. Given the probability density function $f(x) = \dfrac{k}{1+x^2}, -\infty < x < \infty$, Find k and C.D.F.

Solution:

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \qquad\qquad F(x) = \int_{-\infty}^{x} f(x)\, dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{k}{1+x^2}\, dx = 1 \qquad\qquad = \int_{-\infty}^{\infty} \frac{k}{1+x^2}\, dx$$

$$\Rightarrow k\left[\tan^{-1} x\right]_{-\infty}^{\infty} = 1 \qquad\qquad = \frac{1}{\pi}\left[\tan^{-1} x\right]_{-\infty}^{x}$$

$$\Rightarrow k\left[\left[\tan^{-1}\infty\right] - \left[\tan^{-1}(-\infty)\right]\right] = 1 \qquad\qquad = \frac{1}{\pi}\left[\left[\tan^{-1} x\right] - \left[\tan^{-1}(-\infty)\right]\right]$$

$$\Rightarrow k\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 1 \qquad\qquad = \frac{1}{\pi}\left(\frac{\pi}{2} - \tan^{-1} x\right) = \frac{1}{\pi}\cot^{-1} x$$

$$\Rightarrow k = \frac{1}{\pi}$$

13. It has been claimed that in 60 % of all solar heat installation the utility bill is Reduced by atleast one-third . Accordingly what are the probabilities that the utility bill will be reduced by atleast one-third in atleast four of five installation.

Solution:

Given n=5 , p=60 % =0.6 and q=1-p=0.4

$$p(x \ge 4) = p[x=4] + p[x=5]$$

$$= 5c_4 (0.6)^4 (0.4)^{5-4} + 5c_5 (0.6)^5 (0.4)^{5-5}$$

$$= 0.337.$$

14. The number of monthly breakdowns of a computer is a r.v. having poisson distribution with mean 1.8. Find the probability that this computer will function for a month with only one breakdown.

Solution:

$$p(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad given\ \lambda = 1.8$$

$$p(x=1) = \frac{e^{-1.8}(1.8)^1}{1!} = 0.2975.$$

15. State the memoryless property of Exponential distribution.

Solution:

If X follows the Exponential distribution then for any two real numbers s,t>0

P(X> s+t / X>s)= P(X > t)

PART B

1. The density function of a random variable X is given by $f(x) = kx(2-x),\ 0 \le x \le 2$. Find *k*, mean , variance and r^{th} moment.

2. The monthly demand for Allwyn watches is known to have the following probability distribution

Demand:	1	2	3	4	5	6	7	8
Probability:	0.08	0.3*k*	0.19	0.24	k^2	0.1	0.07	0.04

Determine the expected demand for watches. Also, compute the variance.

3. The distribution of a random variable X is given by $F(X) = 1-(1+x)e^{-x}$, $x > 0$. Find the r^{th} moment, mean and variance.
4. Suppose that the duration 'X' in minutes of long distance calls from your home, follows exponential law with p.d.f $f(x) = \frac{1}{5} e^{-\frac{x}{5}}$, $x > 0$. Find p(X > 5), p(3≤X≤6),mean and variance.

5. A random variable X has the following probability distribution.

X:	0	1	2	3	4	5	6	7
f(x):	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find (i) the value of k (ii) p(1.5 < X < 4.5 | X > 2) and

(iii) the smallest value of λ such that $p(X \le \lambda) > \frac{1}{2}$.

6. Find the MGF of triangular distribution whose density function is given by

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & elsehwere \end{cases}$$

Hence its mean and variance.

7. Find the MGF of the RV X, whose pdf is given by $f(x) = \frac{1}{2} e^{-|x|}$, $-\infty < x < \infty$.

Hence its mean and variance.

8. The p.m.f of a RV X, is given by $p(X = j) = \frac{1}{2^j}$, $j = 1,2,3\ldots$ Find MGF, mean and variance.

9. Find MGF of the RV X, whose pdf is given by $f(x) = \lambda e^{-\lambda x}$, $x > 0$ and hence find the first four central moments.

10. If the MGF of a (discrete) RV X is $\frac{1}{5-4e^t}$ find the distribution of X and p (X = 5 or 6).
11. Find the moment generating function of Uniform distribution. Hence find its mean and Variance.
12. Find the moment generating function of exponential distribution. Hence find its mean and Variance.
13. Determine the moment generating function of poisson distribution. Hence find its mean and Variance.
14. Determine the moment generating function of Binomial distribution. Hence find its mean and Variance.
15. Find the moment generating function of Gamma distribution. Hence find its mean and Variance.
16. Find the moment generating function of Normal distribution. Hence find its mean and Variance.

17. State and prove the memoryless property of Exponential distribution.
18. State and prove the memoryless property of Geometric distribution.
19. A box contains 100 cell phones. 20 of which are defective.10 cell phones are selected for inspection. Find the probability that (i)atleast one is defective (ii)atmost three are defective (iii)all ten are defective (iv)none of the ten are defective.
20. A manufacturer of pins knows that 2% of his products are defective. If he sells pins in boxes of 100 and guarantees that not more than 4 pins will be defective. What is the probability hat a box will fail to meet the guaranteed quality?
21. If the probability is 0.05 that a certain kind of measuring device will show excessive drift, what is the probability that the sixth of these devices tested will be the first to show excessive drift?
22. Trains arrive at a station at 15 minutes intervals starting at 4 a.m. If a passenger arrive at a station at a time that is uniformly distributed between 9.00 and 9.30 , find the probability that he has to wait for the train for (1)less than 6 minutes (2) more than 10 minutes.
23. Suppose the duration X in mins of long distance calls from your home follows exponential law with pdf $f(x) = \begin{cases} \frac{1}{5} e^{-\frac{x}{5}}, x > 0 \\ 0 \quad , otherwise \end{cases}$ find (i)P(X>5) (ii)P(3≤X≤6) (iii)Mean and VarX.
24. In a certain city, the daily consumption of electric power in millions of kilowatt-hours can be treated as a rv having an Erlang dist. with parameters λ=1/2 and k=3. If the power plant of this city has a daily capacity of 12 millions kilowatt-hours, what is the probability that this power supply will be inadequate on any given day.
25. The standard deviation of a certain group of 1000 high school grades was 11% and the mean grade 78%.Assuming the distribution to be normal find (i)how many grades were above 90% (ii) how many grades were below 60% (iii)How many grades were between 75% and 85% (iv)what was the highest grade of the lowest 10?