R.M.K GROUP OF ENGINEERING INSTITUTIONS

R.M.K
GROUP OF
INSTITUTIONS

R.M.K
GROUP OF INSTITUTIONS

R.M.K GROUP OF INSTITUTIONS

Please read this disclaimer before proceeding:

22MA401

PROBABILITY AND STATISTICS

DEPARTMENT	**Artificial Intelligence and Data Science**
BATCH/YEAR	**2022-2026/ II**
CREATED BY	**Dr. V. BANU PRIYA , Dr. S. VIDHYA**
DATE	**09.01.2024**

Table of content

Course Objectives

S. No.	Course Objectives
1	To Provide the necessary basic concepts of random variables and to introduce some standard distributions.
2	To introduce the basic concepts of two dimensional random variables
3	To test the hypothesis for small and large samples.
4	To introduce the concepts of Analysis of Variances.
5	To understand the concept of statistical quality control.

PREREQUISITES

S.No.	TOPICS	COURSE NAME WITH CODE
1	Differentiation & Integration	Higher Secondary level
2	Basic Probabilities	
3	Knowledge in set theory	

Syllabus

22MA401	**PROBABILITY AND STATISTICS** **(Theory Course with Laboratory Component)**	**L T P C** **3 2 0 4**

UNIT I ONE DIMENSIONAL RANDOM VARIABLES **15**

Basic probability definitions- Independent events- Conditional probability (revisit) - Random variable - Discrete and continuous random variables – Moments – Moment generating functions – Binomial, Poisson, Geometric, Uniform, Exponential and Normal distributions.

Experiments using R Programming:

1. Finding conditional probability.
2. Finding mean, variance and standard deviation.

UNIT II TWO DIMENSIONAL RANDOM VARIABLES **15**

Joint distributions – Marginal and conditional distributions – Covariance – Correlation and linear regression – Transformation of random variables.

Experiments using R Programming:

1. Finding marginal density functions for discrete random variables
2. Calculating correlation and regression

UNIT III TESTING OF HYPOTHESIS **15**

Sampling distributions - Estimation of parameters - Statistical hypothesis - Large sample tests based on Normal distribution for single mean and difference of means -Tests based on t and F distributions for mean and variance – Chisquare - Contingency table (test for independence) - Goodness of fit.

Experiments using R Programming:

1. Testing of hypothesis for given data using Z – test.
2. Testing of hypothesis for given data using t – test.

UNIT IV DESIGN OF EXPERIMENTS **15**

One way and Two way classifications - Completely randomized design – Randomized block design – Latin square design.

Experiments using R Programming:

1. Perform one- way ANOVA test for the given data.
2. Perform two-way ANOVA test for the given data.

UNIT V STATISTICAL QUALITY CONTROL **15**

Control charts for measurements (X and R charts) – Control charts for attributes (p, c and np charts) – Tolerance limits.

Experiments using R Programming:

1. Interpret the results for $\overline{X}$-Chart for variable data
2. Interpret the results for R-Chart for variable data

TOTAL: 75 PERIODS

Course Outcomes

CO's	Course Outcomes	Highest Cognitive Level
CO1	Calculate the statistical measures of standard distributions.	K2
CO2	Compute the correlation and regression for two dimensional random variables.	K2
CO3	Apply the concept of testing the hypothesis.	K2
CO4	Implement the concept of analysis of variance for various experimental designs.	K3
CO5	Demonstrate the control charts for variables and attributes.	K2

CO-PO/CO-PSO Mapping

CO's	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
CO1	3	2	-	-	-	-	-	-	-	-	-	-
CO2	3	2	-	-	-	-	-	-	-	-	-	-
CO3	3	2	-	-	-	-	-	-	-	-	-	-
CO4	3	2	-	-	-	-	-	-	-	-	-	-
CO5	3	2	-	-	-	-	-	-	-	-	-	-

CO's	PSO1	PSO2	PSO3
CO1	-	-	-
CO2	-	-	-
CO3	-	-	-
CO4	-	-	-
CO5	-	-	-

Lecture Plan

S.No	Topics to be covered	No. of periods	Proposed Date	Actual Date	CO	Taxonomy Level	Mode of Delivery
1	Probability	1	3.1.2024		CO1	K3	PPT,Black Board & Chalk
2	Discrete and continuous random variables	1	4.1.2024		CO1	K3	PPT, Black Board & Chalk
3	Problem Solving	1	5.1.2024		CO1	K3	PPT, Black Board & Chalk
4	Moments	1	6.1.2024		CO1	K3	PPT ,Black Board & Chalk
5	Moment Generating Function	1	8.1.2024		CO1	K3	PPT, Black Board & Chalk
6	Binomial Distributions	1	9.1.2024		CO1	K3	PPT ,Black Board & Chalk
7	Binomial Distributions	1	10.1.2024		CO1	K3	PPT, Black Board & Chalk
8	Poisson Distributions	1	11.1.2024		CO1	K3	PPT, Black Board & Chalk
9	Geometric Distributions	1	12.1.2024		CO1	K3	PPT, Black Board & Chalk
10	Uniform Distributions	1	22.1.2024		CO1	K3	PPT, Black Board & Chalk
11	Exponential Distributions	1	22.1.2024		CO1	K3	PPT, Black Board & Chalk
12	Normal Distributions	1	23.1.2024		CO1	K3	PPT, Black Board & Chalk
13	Normal Distributions	1	23.1.2024		CO1	K3	PPT, Black Board & Chalk
14	Lab	1	24.1.2024		CO1	K3	PPT, Black Board & Chalk
15	Lab	1	24.1.2024		CO1	K3	PPT, Black Board & Chalk

ACTIVITY BASED LEARNING

Activity based learning enhances students' critical thinking and collaborative skills. Experiential learning being the core, various activities such as quiz competitions, group discussion, etc. are conducted for all the five units to enhance the learning abilities of students. The students are the center of the activities, where student's opinions are valued, questions are encouraged, and discussions are done. These activities empower the students to explore and learn by themselves.

S.No.	TOPICS	Activity	Link
1	Fundamentals of Probability	Digital Probability Activities	https://alyssateaches.com/probability-activities/
2	Moments	Practice quiz in moments	https://study.com/academy/practice/quiz-worksheet-what-is-the-moment-generating-function.html
3	Binomial Distribution	Self evaluation practice quiz in Binomial Distribution	https://www.sanfoundry.com/probability-statistics-questions-answers-binomial-distribution/
4	Poisson distribution	Use the Poisson distribution calculator to verify the solved problems	https://stattrek.com/online-calculator/poisson.aspx
5	Probability	Self-evaluation practice quiz in probability	https://www.mathopolis.com/questions/q.html?id=700&t=mif&qs=700_7%20%20%2001_702_1475_1476_1477_2175_2176_2177_2178&site=1&ref=2f646174612f70726f626162696c6974792e68746d6c&title=50726f626162696c697479

UNIT I

ONE DIMENSIONAL RANDOM VARIABLES

INTRODUCTION:

Probability is a universally accepted tool for expressing degrees of confidence or doubt about some proposition in the presence of incomplete information or uncertainty. By convention, probabilities are calibrated on a scale of 0 to 1; assigning something a zero probability amounts to expressing the belief that we consider it impossible, whereas assigning a probability of one amounts to considering it a certainty. Most propositions fall somewhere in between.

Probability statements that we make can be based on our past experience, or on our personal judgments. Whether our probability statements are based on past experience or subjective personal judgments, they obey a common set of rules, which we can use to treat probabilities in a mathematical framework, and also for making decisions on predictions, for understanding complex systems, or as intellectual experiments and for entertainment.

Probability theory is one of the most applicable branches of mathematics. It is used as the primary tool for analyzing statistical methodologies; it is used routinely in nearly every branch of science, such as biology, astronomy and physics, medicine, economics, chemistry, sociology, ecology, finance, and many others. A background in the theory, models, and applications of probability is almost a part of basic education.

1.1 Concept of experiments, sample space, event

A repeatable procedure with a set of possible results is called an experiment.

Random Experiment:

Random experiment is one whose results depend on chance, that is the result cannot be predicted.

Example 1:

Throwing dice.

We can throw the dice again and again, so it is repeatable.

The set of possible results from any single throw is {1, 2, 3, 4, 5, 6}.

Example 2:

A fair coin is tossed.

Sample Space:

Set of all the possible outcomes of an random experiment is called sample space.

Example 1:

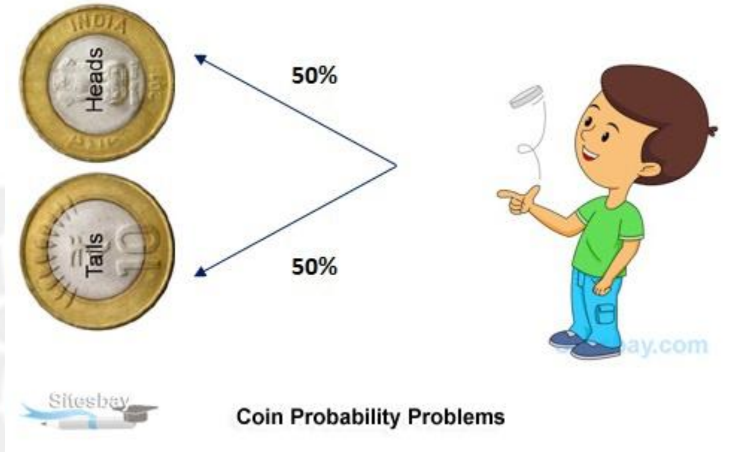
When a die is thrown, its sample space is given by S = $\{1,2,3,4,5,6\}$.

Example 2:

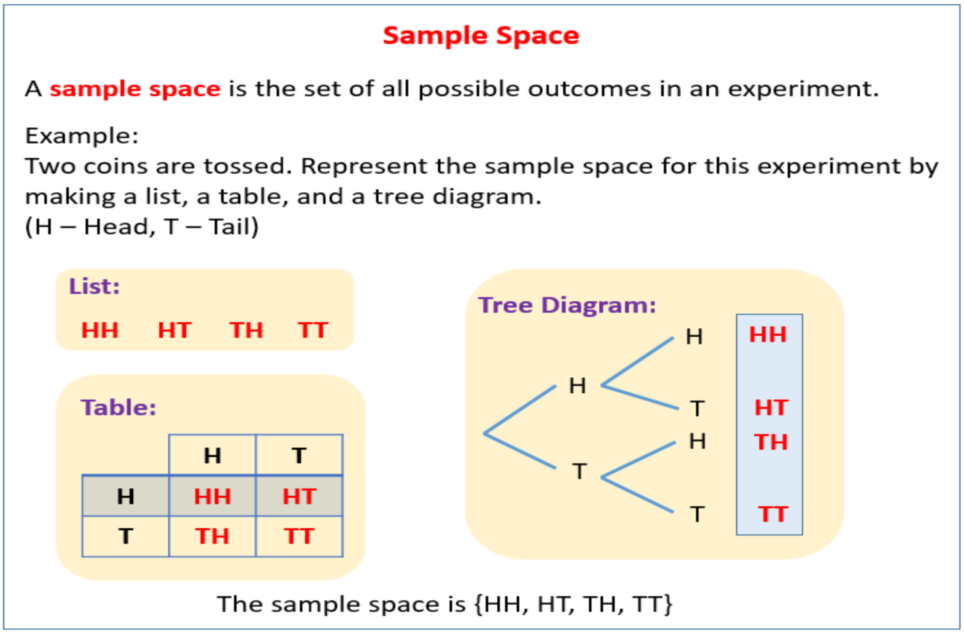
Two coins are tossed. Represent the sample space for the random experiment by making a list, a table and tree diagram (H-Head, T-Tail).

Sample space of some events:

1. When a coin is tossed, S = {H, T}, n(S) = 2 = 2^1.

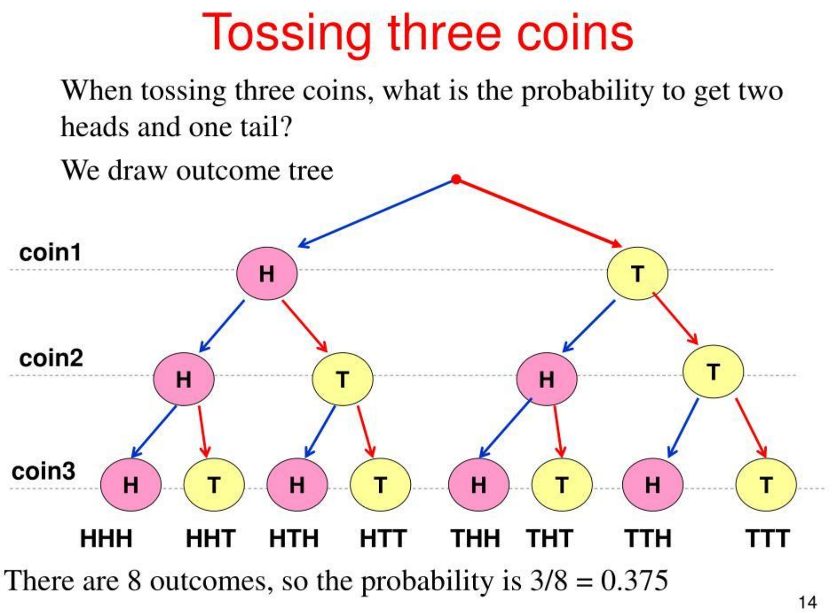


2. When two coins are tossed, S = {HH, HT, TH, TT}, n(S) = 4 = 2^2.

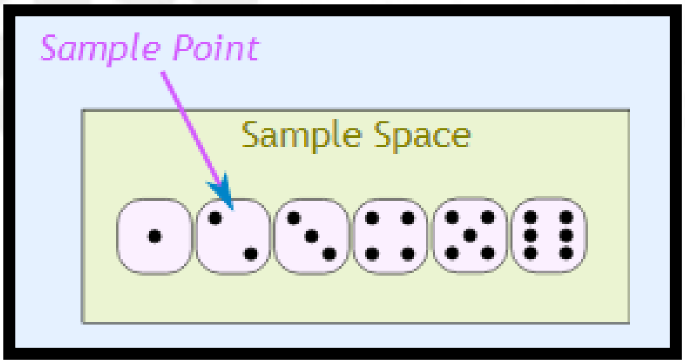


3. When three coins are tossed,

   S = {HHH, HHT, HTH, THH, TTT, TTH, THT, HTT}, n(S) = 8 = 2^3.



4. When a die is thrown, S = {1,2,3,4,5,6}, n(S) = 6 = 6^1.



5. When two dice are thrown S = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)},

   n(S) = 36 = 6^2.

Outcome:

Outcome is a possible result of an random experiment.

Example: 1. Head, Tail are possible outcomes when a coin is tossed.
2. 1,2,3,4,5,6 are possible outcomes when a number cube is rolled.

Event:

One or more outcomes of an random experiment is called an event. That is every non empty subset of the sample space is known as event.

Example: An event can be just one outcome:

- Getting a Tail when tossing a coin.
- Rolling a die getting "5".

An event can include more than one outcome:

- Choosing a "King" from a deck of cards (any of the 4 Kings).
- Rolling a die an "even number" (2, 4 or 6) or E={2, 4, 6}.

Any subset *E* of the sample space *S* is known as an *event*. Some examples of sample space and events are the following.

- If the experiment consists of the flipping of a coin, then S = {H, T},where H means that the outcome of the toss is a head and T that it is a tail.
  If E = {H}, then E is the event that a head appears on the flip of the coin.
  Similarly, if E = {T}, then E would be the event that a tail appears.

- If the experiment consists of rolling a die, then the sample space is
  S = {1, 2, 3, 4, 5, 6}, where the outcome i means that i appeared on the die, i = 1, 2, 3, 4, 5, 6.
  If E = {1}, then E is the event that one appears on the roll of the die.
  If E = {2, 4, 6}, then E would be the event that an even number appears on the roll.

- If the experiments consist of flipping two coins, then the sample space consists of the following four points:

$$S = \{(H,H), (H,T), (T,H), (T,T)\}$$

The outcome will be (H,H) if both coins come up heads; it will be (H,T) if the first coin comes up heads and the second comes up tails; it will be (T,H) if the first comes up tails and the second heads; and it will be (T,T) if both coins come up tails.

- If the experiment consists of rolling two dice, then the sample space consists of the following 36 points:

$$\begin{aligned} S =\{ & (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6) \\ & (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6) \\ & (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6) \\ & (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6) \\ & (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6) \\ & (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\} \end{aligned}$$

where the outcome (i, j) is said to occur if i appears on the first die and j on the second die.

If E = {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}, then E is the event that the sum of the dice equals seven.

If E = (2, 6), then E is the event that the car lasts between two and six years.

- If the experiment consists of measuring the lifetime of a car, then the sample Space consists of all nonnegative real numbers. That is, $S = [0, \infty)$.

Type of events:

- **Simple event:** It has only one sample point of sample space.
- **Compound event**: It has more than one sample point of sample space.
- **Certain event or Sure event:** It is sure to occur in an experiment. The sample space S is called sure event or certain event.
- **Impossible event**: It never occurs in an experiment. The empty set in S is called impossible event.

- **Equally Likely Events:** When the events are equally likely to happen in an experiment. The events having the same chance of occurrences are called equally likely events.

  Example: when a coin is tossed, the events {H} and {T} are equally likely events.

- **Complementary Events:** For an event E non-occurrence of event E is complementary event.

- **Mutually Exclusive Events (Disjoint Events):** When two events do not occur at the same time. Two or more events are said to be mutually exclusive if their intersection is the null set.

  Example: S = {1,2,3,4,5,6}
  A = {1,2} B = {4,5,6} Here $A \cap B = \{\,\}$
  A and B are mutually exclusive.

- **Exhaustive Events:** A set of events is said to be exhaustive events if their union is a sample space.

  Example: S = {1,2,3,4,5,6}
  A = {1,2}, B = {3,4,5,6} Here $A \cup B$ = {1,2,3,4,5,6} = S
  A set of events is said to be exhaustive events if their union is a sample space.

- **Independent events:** Events are said to be independent if the occurrence of the one event does not affect the other.

  Example. Consider a pack of 52 cards. When cards are replaced, it is an example for independent events.

- **Dependent events:** Two events are said to be dependent events if the happening of the second event is affected by the happening of the first event.

  Example. Consider a pack of 52 cards. When cards are not replaced, it is an example for dependent events.

Algebra of an event:

- **Event A or B:**
  Collection of sample points in either A or B (or both), denoted by $A \cup B$ or $A + B$

- **Event A and B:**
  Collection of sample points in both A and B, denoted by $A \cap B$ or AB

- **Event A but not B:**
  Collection of sample points in A but not in B, denoted by $A - B$.

Probability

Probability means possibility. It is a measure of the likelihood of an event to occur. Many events cannot be predicted with total certainty. We can predict only the chance of an event to occur. i.e. how likely they are to happen, using it. Probability can range in from 0 to 1, where 0 means the event to be an impossible one and 1 indicates a certain event. The probability of all the events in a sample space adds up to 1.

Definition of Probability:

The probability for the occurrences of an event A is defined as the ratio between the number of favorable outcomes for the occurrence of the event and the total number of possible outcomes.

$$Probability\ of\ an\ event = \frac{\text{number of favorable outcomes}}{\textbf{Total number of outcomes}}$$

$$(i.e.)\quad \mathrm{P}(A) = \frac{n(A)}{n(S)}$$

Example 1: Tossing a Coin

When a coin is tossed, there are two possible outcomes:

- heads (H) or
- tails (T)

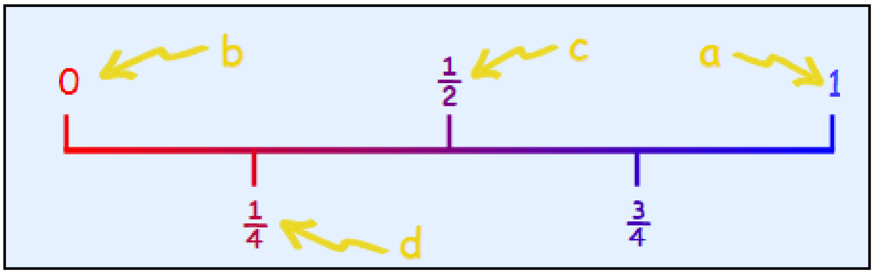
we say that the probability of the coin landing **H** is ½ and the probability of the coin landing **T** is ½.

Example 2: Throwing Dice

When a single die is thrown, there are six possible outcomes:

1, 2, 3, 4, 5, 6. The probability of any one of them is $\frac{1}{6}$.

Example 3: Here we show the probability that



a) The sun will rise tomorrow
b) I will not have to learn mathematics at school
c) If flip a coin it will land heads up
d) Choosing a red ball from a bag with 1 red ball and 3 green balls

Axioms of Probability:

Consider an experiment whose sample space is S. For each event A of the sample space S, we assume that a number P(A)is defined and satisfies the following conditions:

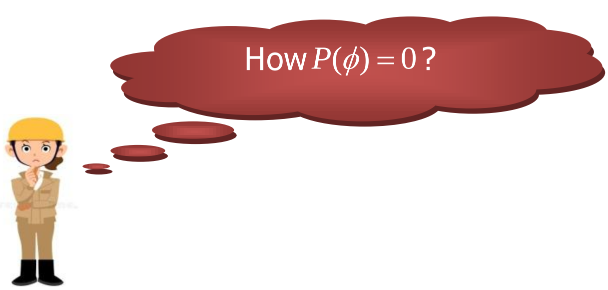
Axioms of Probability:

- For every A in S, $0 \le P(A) \le 1$
- The entire sample space S has the probability $P(S) = 1$

Note:

$P(A \cup A^c) = 1$ and $P(A \cap A^c) = 0$.

The probability of an impossible event is zero, $P(\phi) = 0$.



Addition rule for mutually exclusive events:

For mutually exclusive events A and B

Basic Theorems of Probability

We shall see that the axioms of probability will enable us to build up probability theory and its application to statistics. We begin with three basic theorems.

Complementation Rule

For an event A and its complement in a sample space S,

$$P(A^c) = 1 - P(A)$$

Complementation Rule:

For an event A and its complement in a sample space S,

$$P(A^c) = 1 - P(A)$$

Addition rule for arbitrary events

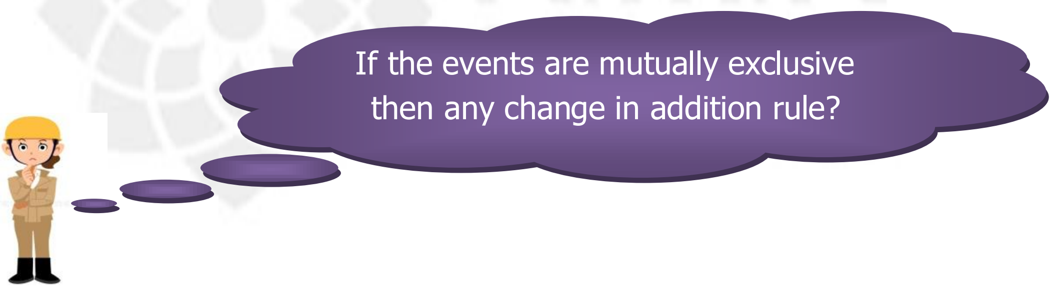
For events A and B in a sample space, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Addition rule for arbitrary events:

For events A and B in a sample space, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Addition rule for mutually exclusive events

Addition rule for mutually exclusive events:

For mutually exclusive events A and B $(A \cap B = \varnothing)$

$$P(A \cup B) = P(A) + P(B)$$

Example 1

A coin is to be tossed until a head appears twice in a row.

i) What is the sample space for this experiment?

ii) If the coin is fair, then what is the probability that it will be tossed exactly four times?.

Solution:

i) The sample space is
S = {HH, THH, TTHH, HTHH, TTTHH, THTHH, HTTHH,…}.
$n(S) = \infty$.

ii) If the coin is fair and it will be tossed exactly four times then the numberof elements in the sample space is $n(S) = 2^4 = 16$.

The only outcomes that satisfy your condition areTTHH and HTHH.

Therefore, A = { TTHH, HTHH }

n(A) = 2

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{16} = \frac{1}{8}$$

Example 2

Pair of fair dice arerolled, what is the probability that the second die lands on a higher value then the first?

Solution:

Let A, B be the event that the first die is higher value and the second die is higher value respectively and C be the event that both die as same value.

Here A,B and C are Exhaustive Events.

Let A_i be the event that the sum is i .

Therefore,

$$A_2 = \{(1,1)\} \Rightarrow n(A_2) = 1$$

$$\Rightarrow P(A_2) = \frac{n(A_2)}{n(S)} = \frac{1}{36},$$

$$A_3 = \{(1,2),(2,1)\} \Rightarrow n(A_3) = 2$$

$$\Rightarrow P(A_3) = \frac{n(A_3)}{n(S)} = \frac{2}{36}$$

$$A_4 = \{(1,3),(2,2),(3,1)\} \Rightarrow n(A_4) = 3$$

$$\Rightarrow P(A_4) = \frac{n(A_4)}{n(S)} = \frac{3}{36}$$

$$A_5 = \{(1,4),(2,3),(3,2),(4,1)\} \Rightarrow n(A_5) = 4$$

$$\Rightarrow P(A_5) = \frac{n(A_5)}{n(S)} = \frac{4}{36}$$

$$A_6=\{(1,5),(2,4),(3,3),(4,2),(5,1)\} \;\Rightarrow\; n(A_6)=5$$

$$\Rightarrow P(A_6)=\frac{n(A_6)}{n(S)}=\frac{5}{36}$$

$$A_7=\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} \;\Rightarrow\; n(A_7)=6$$

$$\Rightarrow P(A_7)=\frac{n(A_7)}{n(S)}=\frac{6}{36}$$

$$A_8=\{(2,6),(3,5),(4,4),(5,3),(6,2)\} \;\Rightarrow\; n(A_8)=5$$

$$\Rightarrow P(A_8)=\frac{n(A_8)}{n(S)}=\frac{5}{36}$$

$$A_9=\{(3,6),(4,5),(5,4),(6,3)\} \;\Rightarrow\; n(A_9)=4$$

$$\Rightarrow P(A_9)=\frac{n(A_9)}{n(S)}=\frac{4}{36}$$

$$A_{10}=\{(4,6),(5,5),(6,4)\} \;\Rightarrow\; n(A_{10})=3$$

$$\Rightarrow P(A_{10})=\frac{n(A_{10})}{n(S)}=\frac{3}{36}$$

$$A_{11}=\{(5,6),(6,5)\} \;\Rightarrow\; n(A_{11})=2$$

$$\Rightarrow P(A_{11})=\frac{n(A_{11})}{n(S)}=\frac{2}{36}$$

$$A_{12}=\{(6,6)\} \;\Rightarrow\; n(A_{12})=1$$

$$\Rightarrow P(A_{12})=\frac{n(A_{12})}{n(S)}=\frac{1}{36}$$

Exercise:

1. A box contains three marbles: one red, one green, and one blue. Consider an experiment that consists of taking one marble from the box then replacing it in the box and drawing a second marble from the box. What is the sample space? If, at all times, each marble in the box is equally likely to be selected, what is the probability of each point in the sample space?

2. An individual uses the following gambling system at Las Vegas. He bets \$1 that the roulette wheel will come up red. If he wins, he quits. If he loses then he makes the same bet a second time only this time he bets \$2; and then regardless of the outcome, quits. Assuming that he has a probability of ½ of winning each bet, what is the probability that he goes home a winner? Why is this system not used by everyone?

3. If P(E) = 0.9 and P(F) = 0.8, show that $P(EF)\geq 0.7$. In general, show that $P(EF)\geq P(E)+P(F)-1$.

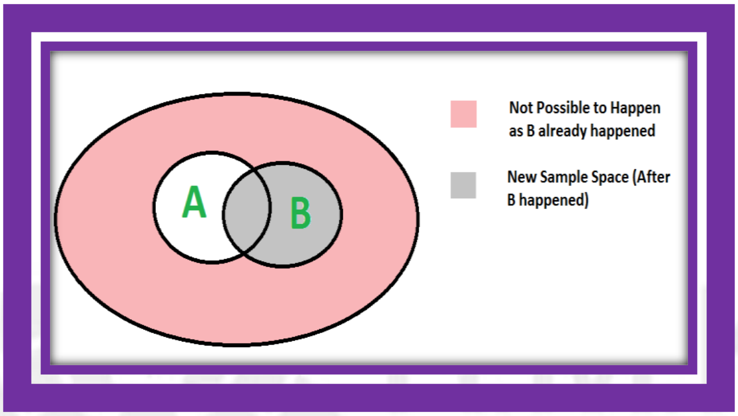
1.2 Conditional Probability:

Why is conditional probability being important?

Conditional probability is needed to understand relative probabilities, which is more often the case in the real-world scenarios instead of looking into the absolute probability of events in isolation.

Conditional Probability:

The probability of an event A under the condition that an event B occurs. This probability is called the conditional probability of A given B and is



denoted by $P(A/B)$ In this case B serves as a new (reduced) sample space, and that probability is the fraction of $P(B)$ which corresponds to $P(A \cap B)$. Thus

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \quad if\ P(B) \neq 0 \qquad \rightarrow (1)$$

Similarly, the conditional probability of B given A is

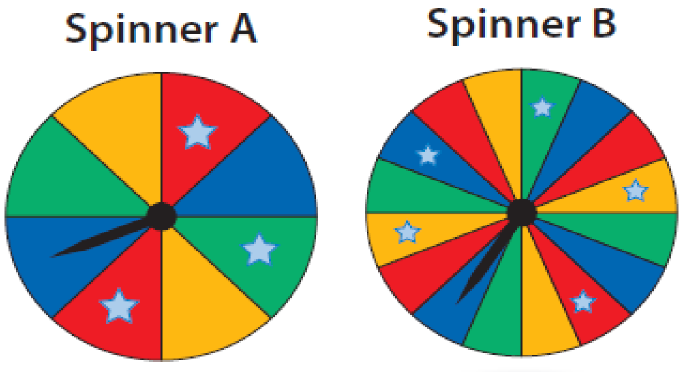
$$P(B/A) = \frac{P(A \cap B)}{P(A)} \quad if\ P(A) \neq 0 \quad \rightarrow (2)$$

From (1) and (2), we conclude that

$$P(A \cap B) = P(B)P(A/B) = P(A)P(B/A) \quad if\ P(B) \neq 0,\ \ P(A) \neq 0$$

Independent Events:

If events A and B are such that $P(A \cap B) = P(A)P(B)$. They are called independent events.



In this case (1) and (2) becomes $P(A/B) = P(A)\, and\ P(B/A) = P(B)$
Similarly,

For m events $A_1, A_2, A_3, \ldots, A_m$ are independent if

$$P\left(A_1 \cap A_2 \cap \ldots \cap A_m\right) = P(A_1)P(A_2)\ldots P(A_m)$$

Problems on conditional probability:

Example:1

Susan took two tests. The probability of her passing both tests is 0.6. The probability of her passing the first test is 0.8. What is the probability of her passing the second test given that she has passed the first test?

Solution:

Let A be the susan passing the second test and B be the susan passing the first test which implies $P(B) = 0.8$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{0.6}{0.8} = 0.75$$

Example:2

What is the probability that the total of two dice will be greater than 9, given that the first die is a 5?

Solution:

Let A = first die is 5 which implies $P(A)=\frac{1}{6}$

Let B = total of two dice is greater than 9

Possible outcomes for A and B: (5, 5), (5, 6)

$$P\left(A \text{ and } B\right)=P(A\cap B)=\frac{2}{36}=\frac{1}{18}$$

$$\therefore P(B/A)=\frac{P(A\cap B)}{P(A)}$$

$$=\frac{1/18}{1/6}$$

$$=\frac{1}{3}$$

Example 3

Suppose cards numbered one through ten are placed in a hat, mixed up, and then one of the cards is drawn. If we are told that the number on the drawn card is at least five, then what is the conditional probability that it is ten?

Solution:

Let E denote the event that the number of the drawn card is ten, and let F be the event that it is at least five. Hence

$$P\left(E\right)=\frac{1}{10} \text{ and } P\left(F\right)=\frac{6}{10}$$

The desired probability is $P\left(E/F\right)$.

However, $E\cap F=E$ since the number of the card will be both ten and at least five if and only if it is number ten. Hence, $P\left(E\cap F\right)=P\left(E\right)=\frac{1}{10}$

Therefore

$$P\left(E|F\right)=\frac{P\left(E\cap F\right)}{P\left(F\right)}=\frac{P\left(E\right)}{P\left(F\right)}=\frac{\frac{1}{10}}{\frac{6}{10}}=\frac{1}{6}.$$

Example 4

A family has two children. What is the conditional probability that both are boys given that at least one of them is a boy? Assume that the sample space S is given by S = [(b, b), (b, g), (g, b), (g, g)], and all outcomes are equally likely. [(b, g) means for instance that the older child is a boy and the younger child a girl.]

Solution:

Letting E denote the event that both children are boys, and F the event that at least one of them is a boy, Hence

$$P(E)=\frac{1}{4} \text{ and } P(F)=\frac{3}{4}$$

Then the desired probability is given by $P(E/F)$.

However, $E\cap F=E$ since the event that both children are boys and one of them is a boy if and only if the event that both children are boys. Hence, $P(E\cap F)=P(E)=\frac{1}{4}$

Therefore

$$P(E|F)=\frac{P(E\cap F)}{P(F)}=\frac{P(E)}{P(F)}$$
$$=\frac{\frac{1}{4}}{\frac{3}{4}}=\frac{1}{3}.$$

Example 5

Suppose an urn contains seven black balls and five white balls. We draw two balls from the urn without replacement. Assuming that each ball in the urn is equally likely to be drawn, what is the probability that both drawn balls are black?

Solution: Let F and E denote respectively the events that the first and second balls drawn are black.

Now, given that the first ball selected is black, there are six remaining black balls and five white balls, and so

$$P(E|F)=\frac{6}{11}.$$

As $P(F)$ is clearly $\frac{6}{11}$&, our desired probability is $P(E\cap F)$.

Therefore

$$P(E\cap F)=P(E|F)P(F)$$
$$=\frac{7}{12}\frac{6}{11}=\frac{42}{132}.$$

Example 6

From a pack of 52 cards, two cards are drawn, the first being replaced before the second is drawn. Find the probability that the first one is diamond and second is king.

Solution: $n(S)=52$. Here the two events are independent.

Let A be the event of getting diamond $n(A)=13$

$$P(A)=\frac{n(A)}{n(S)}=\frac{13}{52}=\frac{1}{4}$$

Let B be an event of getting king card. $n(B)=4$

$$P(B)=\frac{n(B)}{n(S)}=\frac{4}{52}=\frac{1}{13}$$

Our desired probability is $P(A\cap B)$. That is

$$P(A\cap B)=P(A).P(B)=\left(\frac{1}{4}\right).\left(\frac{1}{13}\right)$$

$$=\frac{1}{52}$$

Example 7

A bag contains 10 gold, 8 silver coins. 2 successive drawings of 4 coins are made such that (i) coins are replaced before second trial. (ii) coins are not replaced before second trial. Find the probability that drawing first 4 gold and second 4 silver coins.

Solution. Bag = 10G + 8S = 18 coins.

Let A be the event of getting gold coin

Let B be the event of getting silver coin.

i) **With replacement:**

$$P(A\cap B)=P(A)\times P(B)(Independent)$$

$$=\frac{n(A)}{n(S)}\times\frac{n(B)}{n(S)}$$

$$=\frac{10C_4}{18C_4}\times\frac{8C_4}{18C_4}$$

$$=\left(\frac{10\times 9\times 8\times 7}{18\times 17\times 16\times 15}\right)\times\left(\frac{8\times 7\times 6\times 5}{18\times 17\times 16\times 15}\right)$$

$$=\frac{7}{102}\times\frac{7}{306}$$

$$=\frac{49}{(102\times 306)}$$

ii) **Without replacement:**

$$P(A\cap B)=P(A)\times P\left(B/_{A}\right)(Dependent)$$

$$=\frac{n(A)}{n(S)}\times\frac{n\left(B/_{A}\right)}{n(S)}$$

$$=\frac{10C_4}{18C_4}\times\frac{8C_4}{14C_4}$$

$$=\left(\frac{10\times9\times8\times7}{18\times17\times16\times15}\right)\times\left(\frac{8\times7\times6\times5}{14\times13\times12\times11`}\right)$$

$$=\frac{7}{102}\times\frac{10}{143}=\frac{70}{(102\times143)}$$

Problems for practices:

1. How many ways are there to select a first-prize winner, a second-prize winner and a third-prize winner from 100 different people who have entered a contest?

2. There are 5 red, 4 white and 3 blue marbles in the bag. They are drawn one by one and arranged in a row. Find the number of different arrangements.

3. Mr. Jones has 10 books that he is going to put on his bookshelf. Of these, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. Jones wants to arrange his books so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?

4. Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

5. A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?

5. A basketball team consists of 6 black and 6 white players. The players are to be paired in groups of two for the purpose of determining roommates. If the pairings are done at random, what is the probability that none of the black players will have a white roommate?

6. Bev can either take a course in computers or in chemistry. If Bev takes the computer course, then she will receive an A grade with probability $\frac{1}{2}$ while if she takes the chemistry course then she will receive an A grade with probability $\frac{1}{3}$. Bev decides to base her decision on the flip of a fair coin. What is the probability that Bev will get an A in chemistry?

7. Suppose that each of three men at a party throws his hat into the center of the room. The hats are first mixed up and then each man randomly selects a hat. What is the probability that none of the three men selects his own hat

Answers:

1.970,200	2. 27720 ways	3. 6,912
4.5040 ways	5. 593,775 ways	6. $\frac{5}{231} = 0.0216$
7. $\frac{1}{6}$	8. $\frac{1}{3}$	

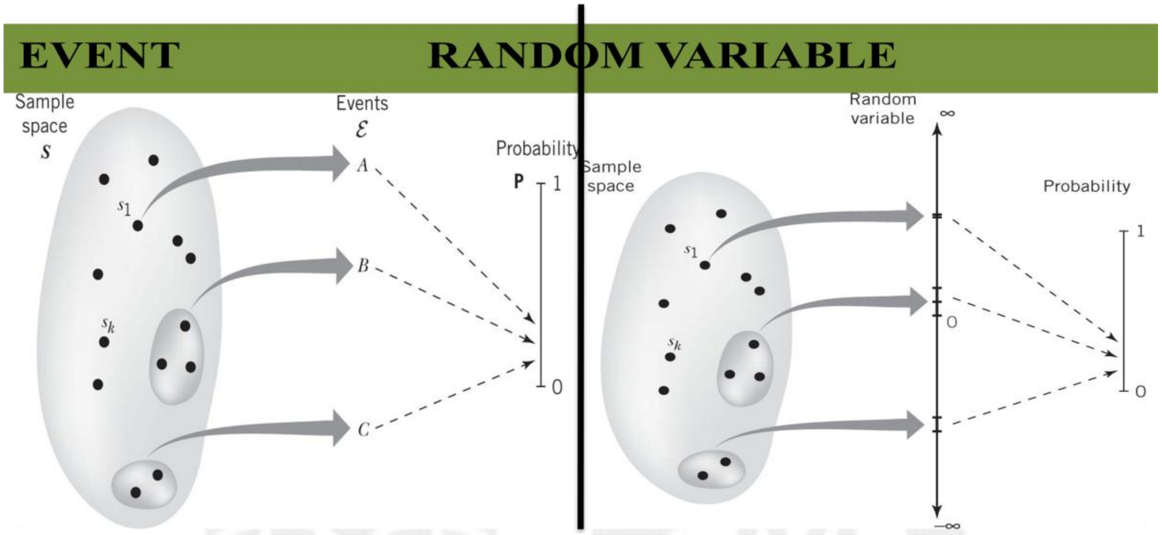
Reference:

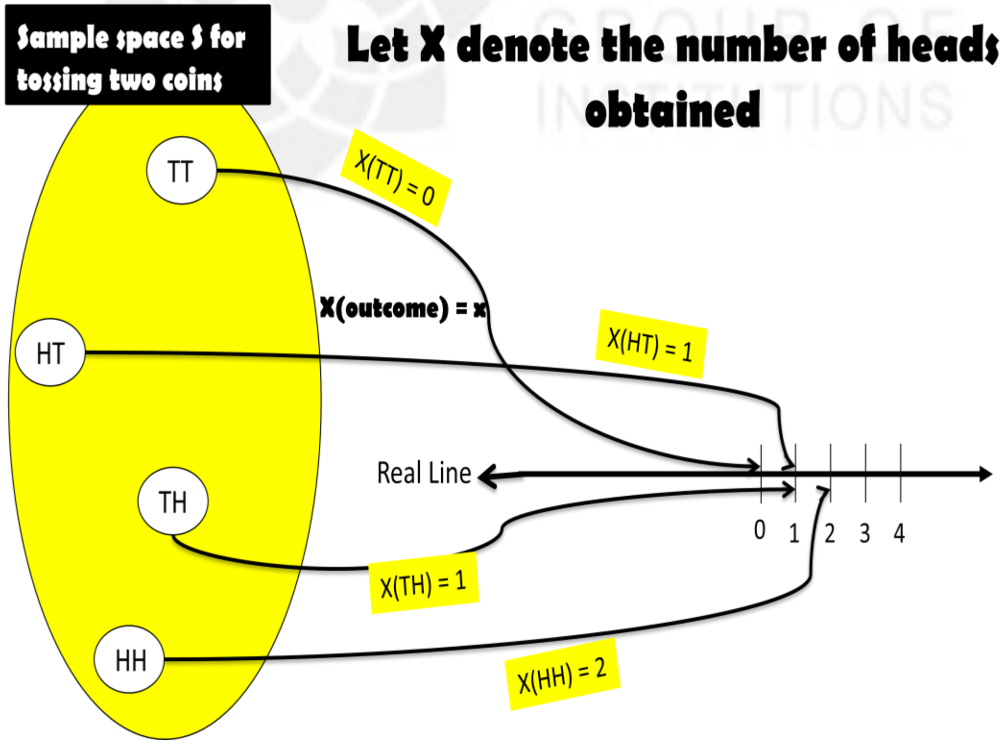
To know more about Sample Spaces, Events, and Their Probabilities click the links

- https://mathigon.org/world/Combinatorics#:~:text=Combinatorics%20has%20many%20applications%20in%20probability%20theory.,total%20number%20of%20possible%20outcomes

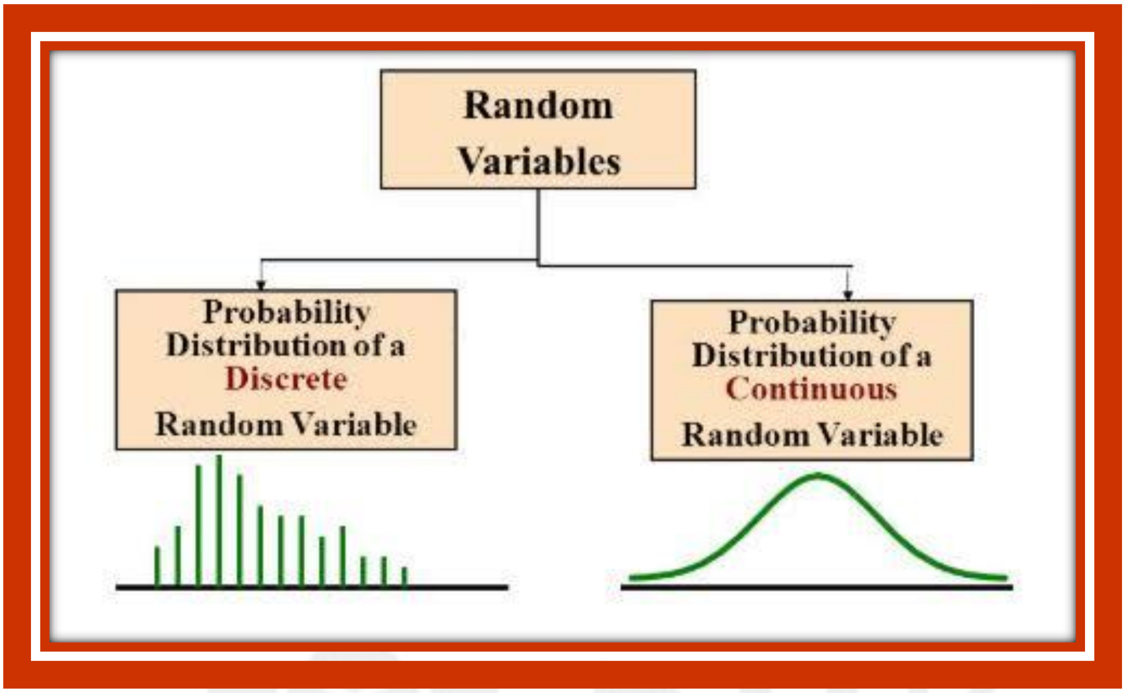
1.4. Discrete and continuous random variables Random Variable

Suppose that to each point of a sample space we assign a number. We then have a function defined on the sample space. This function is called a random variable (or stochastic variable) or more precisely a random function (stochastic function). It is usually denoted by a capital letter such as X or Y. In general, a random variable has some specified physical, geometrical, or other significance.





There are two types of random variables, discrete and continuous.



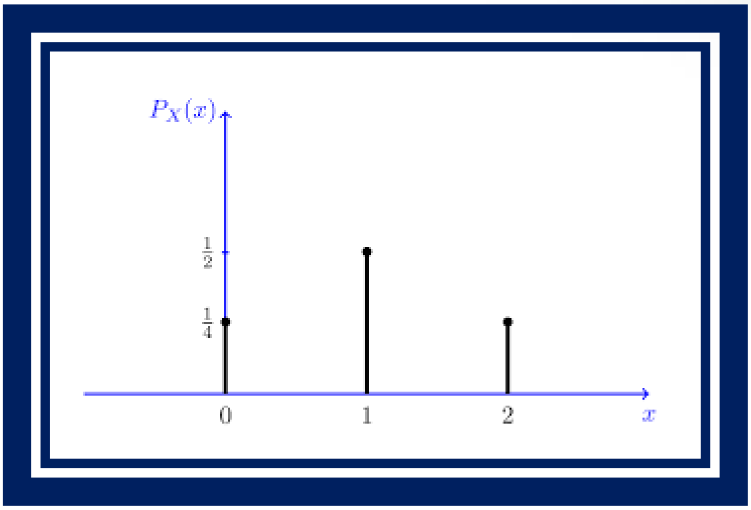
Discrete Random Variable:

A discrete random variable is a random variable 'X' whose possible values constitute finite set of values countably infinite set of values.

Probability mass function or probability function:

Let 'X' be a discrete random variable which takes the values $x_1, x_2, x_3, \ldots$. Let $P(X = x_i) = p(x_i)$ be the probability of x_i. Then the function p is called the **probability mass function** of 'X' if the numbers $p(x_i)$ satisfies the following conditions.

$$i)\quad p(x_i) \geq 0, \quad \forall i \qquad ii)\quad \sum_{i=1}^{\infty} p(x_i) = 1$$



Distribution function or cumulative distribution function of a discrete random variable:

The **cumulative distribution function of a discrete random variable** $'X'$ defined in $(-\infty,\infty)$ is given by

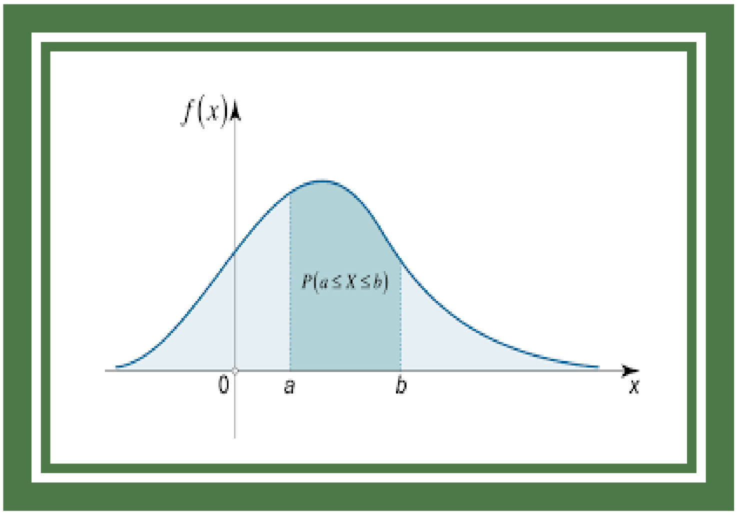
$$F(x) = P(X \le x) = \sum_{\text{for all } x_i \le x} p(x_i)$$

Note:

$$P(a < x \le b) = F(b) - F(a) \text{ where } F(x) = P(X \le x)$$

Continuous Random Variable:

A random variable $'X'$ which takes all possible values in a given interval is called a continuous random variable.



Properties of probability density function of a continuous random variable:

The **probability density function of a continuous random variable** X is a function $f(x)$ it satisfies,

$$i)\ f(x) \geq 0\ x \in (-\infty, \infty) \qquad ii)\ \int_{-\infty}^{\infty} f(x)dx = 1$$

Note: Let 'X' be a continuous random variable with probability density function (p.d.f) $f(x)$. Then

$$P(a < x \leq b) = P(a \leq x < b) = P(a < x < b) = P(a \leq x \leq b) = \int_a^b f(x)dx$$

Distribution function of a continuous random variable:

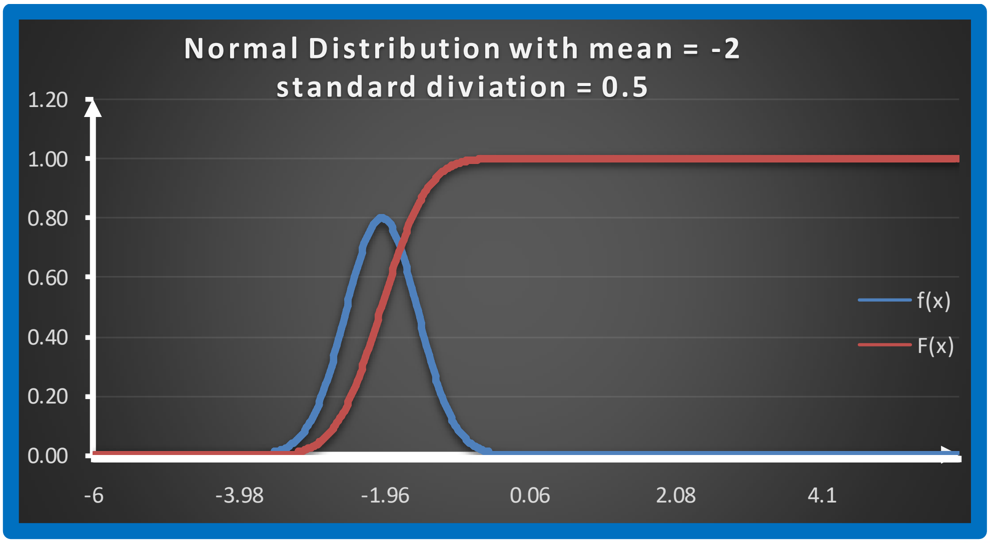
If $f(x)$ is a p.d.f of a continuous random variable X then the function

$$F_X(x) = F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx, \ -\infty < x < \infty$$

is called the **distribution function or cumulative distribution function** of the random variable X.

Relation between distribution function and probability density function:

$$F'(x) = \frac{dF(x)}{dx} = f(x) \quad \text{(or)} \quad dF(x) = f(x)dx$$



Expectation:

The averaging process, when applied to a random variable is called expectation. It is denoted by and is read as the expected value of X or the mean value of X.

Formula: The expectation of a random variable X is denoted by $E(X)$ and is defined by

$$E[X]=\begin{cases}\sum_i x_i p(x_i) \ \ if \ \ X \text{ is discrete and } p(x_i) \text{ is the probability mass function} \\ \int_{-\infty}^{\infty} x f_X(x)dx \ \ if \ \ X \text{ is continuous } and \int_{-\infty}^{\infty} |x| f_X(x)dx<\infty, \ f_X(x) \text{ is the p.d.f}\end{cases}$$

Note:

Mathematical expectation of some real function random variable then $g(X)$ of a discrete random variable X is given by

$$E[g(X)]=\sum_{i=1}^{n} g(x_i)P(x_i)$$

Similarly, if X is a continuous random variable then

$$E[g(X)]=\int_{-\infty}^{\infty} g(x_i)f_X(x)dx$$

Properties of Mathematical Expectation

If X and Y are random variables and a, b are constants then:

(i) $E[a]=a$

(ii) $E(aX)=aE(X)$

(iii) $E(aX+b)=aE(X)+b$

(iv) $E(X+Y)=E(X)+E(Y)$

(v) $E(XY)=E(X).E(Y)$, if X,Y are independent random variables

(vi) $E(X-\bar{X})=0$

(vii) $E(ag(X))=aE(g(X))$

(viii) $E(g(X)+a)=E(g(X))+a$

(ix) $E(g(X))=g[E(X)]$, if $g(X)$ is linear

(x) $E(X)\geq 0$, if $X\geq 0$

(xi) $|E(X)|\leq E(|X|)$

(xii) $|E(XY)^2|\leq E(X^2).E(Y^2)$

(xiii) $P(X\geq a)\leq \dfrac{E(X)}{a}, a>0$

Note:

1. Mean of $X = \bar{X} = E(X)$
2. Standard deviation, $\sigma = \sqrt{\text{Variance}}$
3. Variance of $X = \sigma_x^{\ 2} = E(X^2) - (E(X))^2$ or $\sigma_x^{\ 2} = E((X - E(X))^2) = E((X - \bar{X})^2)$

   Where $\bar{X}$ is the mean value or expected value of the random variable X.

   Case (i):

   If X is a discrete random variable, then

$$\sigma_X^2 = \sum_{i=1}^{n} (x_i - \bar{X})^2 . P(X = x_i)$$

   Case (ii):

   If X is a continuous random variable, then

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x_i - \bar{X})^2 f_X(x) dx$$

Properties of Variance:

(i) $\text{Var}(X) \geq 0$

(ii) $\text{Var}(a)=0$, 'a' is any constant

(iii) If X is a random variable, then $V(aX) = a^2 V(X)$, where a is constant.

Proof :

Let $Y = aX$

$E(Y) = E(aX) = aE(X)$

$\therefore E(Y^2) = E(a^2 X^2) = a^2 E(X^2)$

$$\begin{aligned}\therefore \text{Var(Y)} &= \text{E}(\text{Y}^2) - [\text{E(Y)}]^2 \\ &= a^2 E[X^2] - [aE(X)]^2 \\ &= a^2[E[X^2] - [E(X)]^2] \\ &= a^2 \text{Var}(X)\end{aligned}$$

$\therefore \text{Var}(aX) = a^2 \text{Var}(X)$

(iv) $\text{Var}(aX \pm bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y)$, If X and Y are independent.

Expectation of a Linear Combination of Random Variables

Let $X_1, X_2, \ldots\ldots, X_n$ be any 'n' random variables and if $a_1, a_2, \ldots. a_n$ are constants, then

$$E[a_1 X_1 + a_2 X_2 + \ldots. + a_n X_n] = a_1 E(X_1) + a_2 E(X_2) + \ldots. + a_n E(X_n)$$

Result: If X is a random variable, then

$$\text{Var(aX+b)} = \text{a}^2\text{Var(X)} \text{ where 'a' and 'b' are constants}$$

Example 1

When a die is thrown, X denotes the number that turns up. Find $E(X), E(X^2), \mathrm{Var}(X)$ and standard deviation.

Solution:

$$p=\frac{1}{6}, X=1,2,3,4,5,6$$

Here X is a discrete random variable.

$$\mathrm{E}(X)=\sum \mathrm{x_i p_i}=\left(1\times\frac{1}{6}\right)+\left(2\times\frac{1}{6}\right)+\left(3\times\frac{1}{6}\right)+\left(4\times\frac{1}{6}\right)+\left(5\times\frac{1}{6}\right)+\left(6\times\frac{1}{6}\right)=3.5$$

$$\mathrm{E}(X^2)=\sum_{\mathrm{i=1}}^{6}\mathrm{x_i^2 p_i}=\left(1^2\times\frac{1}{6}\right)+\left(2^2\times\frac{1}{6}\right)+\left(3^2\times\frac{1}{6}\right)+\left(4^2\times\frac{1}{6}\right)+\left(5^2\times\frac{1}{6}\right)+\left(6^2\times\frac{1}{6}\right)=\frac{91}{6}=15.1666$$

$$\mathrm{Var}(X)=\sigma^2{}_{\mathrm{x}}=\mathrm{E}(X^2)\text{-}[\mathrm{E}(X)]^2=15.1666-(3.5)^2=2.9166$$

$$\mathrm{S.D}=\sigma_{\mathrm{x}}=\sqrt{2.9166}=1.7078$$

Example 2

A coin is tossed until a head appears. What is the expectation of the tossed required?

Solution: Let X denote the number of tosses required to get the first head. The first head may appear in the first toss or second toss…and so on.

$\therefore$ The events are H,TH,TTH,TTTH,....

$\therefore$ The corresponding probabilities are: $\frac{1}{2},\frac{1}{2^2},\frac{1}{2^3},....$

$X=x$	1	2	3	4	5	....
$p(x)$	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$	....

$$E(X)=\sum x_i p_i=\frac{1}{2}\left[1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+\ldots\right]$$

$$=\frac{1}{2}\left[1-\frac{1}{2}\right]^{-2} \quad \left[\because (1-x)^{-2}=1+2x+3x^2+\ldots\right]$$

$$E[X]=2$$

Example 3

Find the expectation of the number of failures preceding the first success in an infinite series of independent trials with constant probability of success in each trial.

Solution:

Let X denote the number of failures preceding the first success.

$\therefore$ Events can happen as $S, FS, FFS, FFFS....$

Hence X takes values: 0,1,2,3.....and corresponding probabilities are $p, qp, qqp.....$

$X = x$	0	1	2	3	....
$p(x)$	p	qp	qqp	$qqqp$	....

$$\therefore E(X) = \sum x_i p_i = 0.p + 1.qp + 2.q^2 p + 3.q^3 p + ...$$

$$= qp[1 + 2q + 3q^2 + ...]$$

$$= qp(1-q)^{-2}$$

$$= qpp^{-2}$$

$$= q.p^{-1}$$

$$E(X) = \frac{q}{p}$$

Example 4

A man draws 3 balls from an urn containing 5 white and 7 black balls. He gets Rs.10 for each white ball and Rs.5 for each ball black ball. Find his expectations.

Solution:

Let X be the amount that the man expects to receive. Then X takes values as follows

Balls	3B	1W2B	2W1B	3W
X	15	20	25	30

$$\text{P}(X = 15) = \text{P(selecting 3 black balls)} = \frac{7C_3}{12C_3} = \frac{7}{44}$$

$$P(X = 20) = \text{P(selecting 2B and 1W)} = \frac{5C_1 \times 7C_2}{12C_3} = \frac{21}{44}$$

$$P(X = 25) = \text{P(selecting 2W and 1B)} = \frac{5C_2 \times 7C_1}{12C_3} = \frac{14}{44}$$

$$P(X = 30) = \text{P(selecting 3W)} = \frac{5C_3}{12C_3} = \frac{2}{44}$$

$$E(X) = \left(15 \times \frac{7}{44}\right) + \left(20 \times \frac{21}{44}\right) + \left(25 \times \frac{14}{44}\right) + \left(30 \times \frac{2}{44}\right)$$

$$= \text{Rs.}21.25$$

Example 5

Given the following probability distribution

X	-3	-2	-1	0	1	2	3
P(X)	0.05	0.10	0.30	0	0.30	0.15	0.10

of X compute

(i) $E(X)$

(ii) $E(X^2)$

(iii) $E[2X \pm 3]$

(iv) $\mathrm{Var}(2X \pm 3)$.

Solution:

We know that for a discrete random variable X,

$$(i)\ E(X) = \sum_{i=1}^{7} x_i p(x_i)$$

$$= x_1 p(x_1) + x_2 p(x_2) + x_3 p(x_3) + x_4 p(x_4) + x_5 p(x_5) + x_6 p(x_6) + x_7 p(x_7)$$

$$= (-3)(0.05) - 2(0.1) - 1(0.30) + 0 + 1(0.30) + 2(0.15) + 3(0.10)$$

$$E(X) = 0.25$$

$$(ii)\ E(X^2) = \sum_{i=1}^{7} x^2{}_i\, p(x_i)$$

$$= x_1^2 p(x_1) + x_2^2 p(x_2) + x_3^2 p(x_3) + x_4^2 p(x_4) + x_5^2 p(x_5) + x_6^2 p(x_6) + x_7^2 p(x_7)$$

$$= (-3)^2(0.05) + (-2)^2(0.1) + (-1)^2(0.30) + 0 + 1^2(0.30) + 2^2(0.15) + 3^2(0.10)$$

$$E(X^2) = 2.95$$

$$(iii)\ E(2X \pm 3) = 2E(X) \pm 3 \quad [\because E[aX \pm b] = aE(X) \pm b]$$

$$= 2(0.25) \pm 3$$

$$E(2X \pm 3) = 0.5 \pm 3$$

$$(iv)\ \mathrm{Var}(2X \pm 3) = 2^2\,\mathrm{Var}(X) \quad [\because \mathrm{Var}[aX \pm b] = a^2\,\mathrm{Var}(X)]$$

$$\text{But } \mathrm{Var}(X) = E(X^2) - [E(X)]^2$$

$$= 2.95 - (0.25)^2$$

$$= 2.8875$$

$$\therefore \mathrm{Var}(2X \pm 3) = 4(2.8875) = 11.55$$

Example6

Suppose that the random variable X is equal to the number of hits obtained by a Certain baseball player in his next 3 bats. If $P(X=1)=0.3, P(X=2)=0.2$ and $\text{P}(X=0)\text{=3P}(X=3)$. Find E(X).

Solution: We know that

$$P(X=0)+P(X=1)+P(X=2)+P(X=3)=1$$

Given P(X=0)=3P(X=3)

Substituting (2) in (1) we get

$$3\text{P}(X=3)+0.3+0.2+\text{P}(X=3)=1$$

i.e., $4\text{P}(X=3)=1-0.5$

$$\text{P}(X=3)=\frac{0.5}{4}=0.125$$

Given $\text{P(X=0)=3P(X=3)}=3\times 0.125=0.375$

We know that

$$\text{E}(X)=\sum x_i p(x_i)$$

$$=1\text{P}(X=1)+2\text{P}(X=2)+3\text{P}(X=3)$$

$$=1(0.3)+2(0.2)+3(0.125)$$

$$\text{E}(X)=1.075$$

Example 7

Find the mean and variance of continuous random variable 'X' if it has the density function $f(x)=\begin{cases} 2(x-1), 1<x<2 \\ 0 \quad\quad , \text{otherwise} \end{cases}$.

Solution:

$$f(x)=\begin{cases} 2(x-1), 1<x<2 \\ 0 \quad\quad , \text{otherwise} \end{cases} \qquad (1)$$

$$\text{Mean=E}(X)=\int_{-\infty}^{\infty} xf(x)dx$$

$$=\int_1^2 x.2(x-1)dx \qquad [\text{using (1)}][\because 1<x<2]$$

$$=2\left[\int_1^2 (x^2-x)dx\right]$$

$$= 2\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2$$

$$= 2\left[\left(\frac{8}{3} - \frac{4}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right]$$

$$= 2\left[\left(\frac{16-12}{6}\right) - \left(\frac{2-3}{6}\right)\right]$$

$$= 2\left[\frac{4}{6} + \frac{1}{6}\right]$$

$$= 2\left[\frac{4+1}{6}\right]$$

$$= 2 \times \frac{5}{6}$$

$$\therefore \text{Mean} = E(X) = \frac{5}{3}$$

$$\text{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$$

$$= \int_1^2 x^2 \, 2(x-1)dx \qquad [\text{using } (1)][\because 1 < x < 2]$$

$$= 2\left[\int_1^2 (x^3 - x^2)dx\right]$$

$$= 2\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_1^2$$

$$= 2\left[\left(4 - \frac{8}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)\right]$$

$$= 2\left[\left(\frac{12-8}{3}\right) - \left(\frac{3-4}{12}\right)\right]$$

$$= 2\left[\frac{4}{3} + \frac{1}{12}\right]$$

$$= 2\left[\frac{16+1}{12}\right]$$

$$E(X^2) = \frac{17}{6}$$

$$\text{Var(X)} = E(X^2) - \left(E(X)\right)^2$$

$$= \frac{17}{6} - \left(\frac{5}{3}\right)^2$$

$$= \frac{17}{6} - \frac{25}{9} = \frac{153-150}{54}$$

$$\text{Var(X)} = \frac{1}{18}$$

Example 8

If the p.d.f of 'X' is given by

$$f(x) = \begin{cases} 2(1-x), 0 < x < 1 \\ 0 \qquad \text{Otherwise} \end{cases}$$

(a) Show that $E(X^r) = \dfrac{2}{(r+1)(r+2)}$

(b) Using this result to evaluate $E[(2X+1)^2]$

Solution: Given

$$f(x) = 2(1-x), 0 < x < 1$$

$$E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx$$

$$= \int_0^1 x^r 2(1-x) dx \ [\text{using}(1)][\because 0 < x < 1]$$

$$= 2\left[\int_0^1 (x^r - x^{r+1}) dx\right]$$

$$= 2\left[\frac{x^{r+1}}{r+1} - \frac{x^{r+2}}{r+2}\right]_0^1$$

$$= 2\left[\left(\frac{1}{r+1} - \frac{1}{r+2}\right) - (0-0)\right]$$

$$= 2\left[\frac{1}{r+1} - \frac{1}{r+2}\right]$$

$$= 2\left[\frac{(r+2)-(r+1)}{(r+1)(r+2)}\right]$$

$$= 2\left[\frac{r+2-r-1}{(r+1)(r+2)}\right]$$

$$E(X^r) = \frac{2}{(r+1)(r+2)}$$

$$\therefore \text{When r=1, } E(X) = \frac{2}{(1+1)(1+2)} = \frac{1}{3} \qquad (2)$$

$$\text{When r=2, } E(X^2) = \frac{2}{(2+1)(2+2)} = \frac{1}{6} \qquad (3)$$

$$\therefore\ E[(2X+1)^2] = E[4X^2+1+4X]$$

$$= E(4X^2)+E(1)+E(4X)$$

$$= 4E(X^2)+4E(X)+1\ [\because E(C)=C, E(CX)=CE(X)]$$

$$= \frac{4}{6}+\frac{4}{3}+1 [\text{using (2) and (3)}]$$

$$= \frac{4+8+6}{6}$$

$$E[(2X+1)^2] = 3$$

Example 9

The cumulative distribution function of a random variable 'X' is $F(x)=1-(1+x)e^{-x}, x>0$. Find the probability density function of 'X', mean and variance.

Solution: Given $F(x)=1-(1+x)e^{-x}, 0<x<\infty$

$$\therefore \text{P.d.f is } f(x) = \frac{d}{dx}(F(x))$$

$$= \frac{d}{dx}(1-(1+x)e^{-x})$$

$$= \frac{d}{dx}\left[1-e^{-x}-xe^{-x}\right]$$

$$= e^{-x}-(-xe^{-x}+e^{-x})$$

$$\because f(x) = xe^{-x}, x>0 \qquad (1)$$

$$\text{Mean} = E[X] = \int_0^\infty xf(x)dx$$

$$= \int_0^\infty x.xe^{-x}dx$$

$$= \int_0^\infty x^2e^{-x}dx$$

$$= [x^2(-e^{-x})-(2x)(e^{-x})+(2)(-e^{-x})]_0^\infty \text{ [using Bernoulli's formula]}$$

$$= 0+2$$

$$= 2[\because e^{-\infty}=0] \qquad (2)$$

$$E[X^2]=\int_0^\infty x^2 f(x)dx=\int_0^\infty x^2.xe^{-x}dx \qquad [\text{using}(1)]$$

$$=\int_0^\infty x^3e^{-x}dx$$

$$=[x^3(-e^x)-(3x^2)(e^{-x})+(6x)(-e^{-x})-(6)(e^{-x})]_0^\infty$$

$$=6 \qquad [\text{using Bernoulli's formula}]\ (3)$$

$$\therefore \text{Var}(X)=\text{E}[X^2]\text{-}[\text{E}(X)]^2$$

$$=6-4$$

$$\text{Var}(X)=2$$

Exercise Problems:

1. Let X be a random variable with $\text{E}(X)=10$ and $\text{Var}(X)=25$. Find the positive Values of a and b such that $Y=aX-b$ has expectation 0 and variance1.
2. Find the value of (i) c (ii) mean of the following distribution given $f(x)=\begin{cases} c(x-x^2), 0\le x\le 1 \\ 0, \text{otherwise} \end{cases}$
3. When a die is thrown, 'X' denotes the number that turns up. Find $\text{E(X)},\text{E}(\text{X}^2)$ and Var(X).
4. Let X be a random variable with distribution F given by $F(x)=\begin{cases} 1-e^{-\lambda x}, 0\le x<\infty \\ 0, \text{otherwise} \end{cases}$.Find the p.d.f of X.Determine the mean and variance of the distribution.

Answers:

1. $a=\frac{1}{5}, b=2$

2. $c=6, \text{Mean}=\frac{1}{2}$

3. $E(X)=\frac{7}{2}, E(X^2)=\frac{91}{6}, \text{Var}(X)=\frac{35}{12}$

4. $\text{Mean}=\frac{1}{\lambda}, \quad \text{Variance}=\sigma$

1.5. Moments

Definition

The r^{th} moment about the origin of a random variable X is defined as the expected value of the nth power of X.

	Moments about Origin (Or) Raw moments	Moments about the mean (Or) Central moments
Discrete R.V. X.	$E\left(X^r\right)=\sum_{i=1}^{r} x_i^r p_i=\mu_r' \ ,r\geq 1.$	$E\left[\left(x-\bar{X}\right)^r\right]=\sum_{i=1}^{r}\left(x_i-\bar{X}\right)^r p_i$ $=\mu_r \ , \quad r\geq 1.$
Continuous R.V. X.	$E\left(X^r\right)=\int_{-\infty}^{\infty} x^r f_X\left(x\right)dx$ $=\mu_r' \ , \quad r\geq 1.$	$E\left[\left(x-\bar{X}\right)^r\right]=\int_{-\infty}^{\infty}\left(x-\bar{X}\right)^r f_X\left(x\right)dx$ $=\mu_r \ , \quad r\geq 1.$

The first four moments about origin

Put $r=1$, we get

Mean$=\mu_1' = \sum xp(x)$

Put $r=2$, we get

$$\mu_2' = E[X^2]=\sum_x x^2 p(x)$$

$$\therefore \text{Variance}=\mu_2 = \mu_2' - (\mu_1')^2 = E(X^2)-\{E(X)\}^2$$

The rth moment about mean

$$\mu_r = E[\{X-E(X)\}^r]$$

$$=\sum_x (x-\bar{X})^r p(x), E(X)=\bar{X}$$

Put $r=2$, we get

$$\text{Variance}=\mu_2 = \sum_x (x-\bar{X})^2 p(x)$$

The first four moments about mean

1) $\mu_1 = E\left(x-\bar{X}\right)=E\left(x\right)\text{-E}\left(\bar{X}\right)=\mu-\mu=0.$

   For every distribution discrete or continuous, the first moment about its mean is always zero.

2) $\mu_2 = E\left[\left(X-\bar{X}\right)^2\right]=\text{Variance of } X.$

3) $\mu_3 = E\left[\left(X-\bar{X}\right)^3\right].$

Relation between raw moments and central moments.

The general expression is

$$\mu_r = \mu_r' - {}^rC_1\,\mu_{r-1}'\mu_1' + {}^rC_2\,\mu_{r-2}'\left(\mu_1'\right)^2 - {}^rC_3\,\mu_{r-3}'\left(\mu_1'\right)^3 + {}^rC_4\,\mu_{r-4}'\left(\mu_1'\right)^4 - \ldots(-1)^r\left(\mu_1'\right)^r.$$

1) $\mu_1 = \mu_1' - \mu_0'\mu_1'$

   $= \mu_1' - \mu_1' = 0. \qquad (\because\ \mu_0' = 1)$

2) $\mu_2 = \mu_2' - {}^2C_1\,\mu_1'\mu_1' + {}^2C_2\,\mu_0'\left(\mu_1'\right)^2$

   $= \mu_2' - \left(\mu_1'\right)^2.$

3) $\mu_3 = \mu_3' - {}^3C_1\,\mu_2'\mu_1' + {}^3C_2\,\mu_1'\left(\mu_1'\right)^2 - {}^3C_3\,\mu_0'\left(\mu_1'\right)^3$

   $= \mu_3' - 3\mu_2'\mu_1' + 2\left(\mu_1'\right)^3.$

4) $\mu_4 = \mu_4' - {}^4C_1\,\mu_3'\mu_1' + {}^4C_2\,\mu_2'\left(\mu_1'\right)^2 - {}^4C_3\,\mu_1'\left(\mu_1'\right)^3 + {}^4C_4\,\mu_0'\left(\mu_1'\right)^4$

   $= \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\left(\mu_1'\right)^2 - 3\left(\mu_1'\right)^4.$

EXPECTATIONS TABLE	
Discrete random variable	**Continuous random variable**
1. $E(X^r) = \mu_r' = \sum_x x^r p(x)$	1. $E(X^r) = \mu_r' = \int_{-\infty}^{\infty} x^r f(x)dx$
2. $E(X) = \mu_1' =$ Mean $= \sum xp(x)$	2. $E(X) = \mu_1' = mean = \int_{-\infty}^{\infty} xf(x)dx$
3. $\mu_2' = \sum x^2 p(x)$	3. $\mu_2' = \int_{-\infty}^{\infty} x^2 f(x)dx$
4. Variance $= \mu_2' - \mu_1'^2 = E(X^2) - \{E(X)\}^2$	4. Variance $= \mu_2' - \mu_1'^2 = E(X^2) - \{E(X)\}^2$

Example:1

The first four moments of a distribution about $x=4$ are 1, 4, 10,45. Show that the mean is 5, variance is 3, $\mu_3=0, \mu_4=26$.

Solution: Let μ_1', μ_2', μ_3', μ_4' be the first four moments about $x=4$.

Given $\mu_1'=1$, $\mu_2'=4$, $\mu_3'=10$, $\mu_4'=45$ about $x=4$. (Here $a=4$)

$\mu_1'=E(x-4) \Rightarrow E(x-4)=1 \Rightarrow E(x)-4=1 \Rightarrow E(x)=5.$

Mean=5.

$\mu_2=\mu_2'-(\mu_1')^2 \Rightarrow \mu_2=4-1=3.$

$Var(x)=\mu_2=3.$

$\mu_3=\mu_3'-3\mu_2'\mu_1'+2(\mu_1')^3$

$=10-3(4)(1)+2(1)^3=0.$

$\mu_4=\mu_4'-4\mu_3'\mu_1'+6\mu_2'(\mu_1')^2-3(\mu_1')^4$

$=45-4(10)(1)+6(4)(1)^2-3(1)^4=26.$

Example:2

The first three moments of a distribution about the origin are 5, 26, 78. Show that the three moments about the value $x=3$ are 2, 5, -48.

Solution: Given $E(X)=5$, $E(X^2)=26$ & $E(X^3)=78$.

Let μ_1', μ_2', μ_3', μ_4' be the first four moments about $x=3$.

$\mu_1'=E(X-3)=E(X)-3=5-3=2.$

$\mu_2'=E(X-3)^2=E(X^2-6X+9)=E(X^2)-6E(X)+9=26-6(5)+9=5.$

$\mu_3'=E(X-3)^3=E(X^3-9X^2+27X-27)=E(X^3)-9E(X^2)+27E(X)-27$

$=78-9(26)+27(5)-27=-48.$

Example:3

A continuous random variable X has p.d.f. $f(x)=Kx^2e^{-x}$, $x\geq 0$. Find the r^{th} moment of X about the origin. Hence find the mean and variance.

Solution: Givenp.d.f. of X is $f(x)=Kx^2e^{-x}$, $x\geq 0$.

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx=1$$

R.M.K GROUP OF INSTITUTIONS

$$\int_0^\infty Kx^2e^{-x}\,dx=1\Rightarrow K\int_0^\infty x^2e^{-x}\,dx=1\Rightarrow K\int_0^\infty e^{-x}x^{3-1}\,dx=1$$

$$\text{WKT } \Gamma n=\int_0^\infty e^{-x}x^{n-1}dx$$

$$K\,\Gamma 3=1\Rightarrow K\,2!=1\Rightarrow K=\frac{1}{2}.$$

$\mu_r' = r^{\text{th}}$ moment about the origin

$$=E\left(X^r\right)=\int_{-\infty}^{\infty} x^r f\left(x\right)dx.$$

$$=\int_0^\infty x^r\,\frac{1}{2}x^2e^{-x}dx=\frac{1}{2}\int_0^\infty e^{-x}x^{(r+3)-1}dx=\frac{1}{2}\Gamma(r+3).$$

$$\mu_r'=\frac{1}{2}(r+2)!$$

$$\mu_1'=\frac{1}{2}\times 3!=3.$$

$$\mu_2'=\frac{1}{2}\times 4!=12.$$

Mean of $X=\mu_1'=3.$

Variance of $X=\mu_2=\mu_2'-\left(\mu_1'\right)^2=12-(3)^2=3.$

Example:4

If 'X' has p.d.f $f(x)=\begin{cases}\dfrac{x+1}{2}, -1\le x\le 1\\ 0, \text{otherwise}\end{cases}$.Find the first four central moments.

Solution:

W.K.T $\mu_r'=E(X^r)=\int_{-\infty}^{\infty} x^r f(x)dx$

$$r=1:\mu_1'=E(X)=\int_{-\infty}^{\infty} xf(x)dx$$

$$\mu_1'=\int_{-1}^{1} x\left(\frac{x+1}{2}\right)dx=\frac{1}{3}$$

$$r=2:\mu_2'=\int_{-1}^{1} x^2\left(\frac{x+1}{2}\right)dx=\frac{1}{3}$$

$$r=3:\mu_3'=\int_{-1}^{1} x^3\left(\frac{x+1}{2}\right)dx=\frac{1}{5}$$

$$r=4:\mu_4'=\int_{-1}^{1} x^4\left(\frac{x+1}{2}\right)dx=\frac{1}{5}$$

Then the central moments are:

The general expression is

$$\mu_r = \mu_r^{'} - {}^rC_1\mu_1^{'}\mu_{r-1}^{'} + {}^rC_2\mu_2^{'}\mu_{r-2}^{'} - {}^rC_3\mu_3^{'}\mu_{r-3}^{'} + ...$$

$$r = 1, \mu_1 = 0$$

$$r = 2, \mu_2 = \mu^{'}{}_2 - (\mu_1^{'})^2 = \frac{2}{9}$$

$$r = 3, \mu_3 = \mu^{'}{}_3 - 3\mu^{'}{}_2\mu_1^{'} + 2(\mu_1^{'})^3 = \frac{-8}{135}$$

$$r = 4, \mu_4 = \mu^{'}{}_4 - 4\mu^{'}{}_3\mu_1^{'} + 6\mu^{'}{}_2(\mu_1^{'})^2 - 3(\mu_1^{'})^4 = \frac{48}{405}$$

Example:5

The density function of a random variable ' X ' is given by $f(x) = Kx(2-x), 0 \le x \le 2.$ Find K, mean, variance and r^{th} moment.

Solution:

Given $f(x) = Kx(2-x), 0 \le x \le 2.$ which is a p.d.f

$$\therefore \int_{-\infty}^{\infty} f(x)dx = 1$$

$$i.e., \int_0^2 Kx(2-x)dx = 1[\text{using (1)}][\because 0 < x < 2]$$

$$K\int_0^2 (2x - x^2)dx = 1$$

$$K\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1$$

$$K\left[\left(4 - \frac{8}{3}\right) - (0-0)\right] = 1$$

$$K\left[4 - \frac{8}{3}\right] = 1$$

$$K\left[\frac{12-8}{3}\right] = 1$$

$$K\left(\frac{4}{3}\right) = 1$$

$$K = \frac{3}{4}$$

$$f(x) = \frac{3}{4}x(2-x) \text{ in } 0 \le \text{x} \le 2 \qquad (2)$$

$$\text{Now Mean}=E(X)=\int_{-\infty}^{\infty} xf(x)dx$$

$$=\int_0^2 x.\frac{3}{4}x(2-x)dx \qquad \text{[Using(2)]}$$

$$=\frac{3}{4}\int_0^2 (2x^2-x^3)$$

$$=\frac{3}{4}\left[\frac{2x^3}{3}-\frac{x^4}{4}\right]_0^2$$

$$=\frac{3}{4}\left[\left(\frac{16}{3}-\frac{16}{4}\right)-(0-0)\right]$$

$$=\frac{3}{4}\left[\left(\frac{16}{3}-\frac{16}{4}\right)\right]$$

$$=\frac{3}{4}\times 16\left[\frac{1}{3}-\frac{1}{4}\right]$$

$$=12\left[\frac{4-3}{12}\right]$$

$$=12\times\frac{1}{12}$$

$$\text{Mean}=E(X)=1 \qquad (3)$$

$$E(X^2)=\int_{-\infty}^{\infty} x^2 f(x)dx$$

$$=\int_0^2 x^2.\frac{3}{4}x(2-x)dx \qquad \text{[using(2)]}$$

$$=\frac{3}{4}\int_0^2 (2x^3-x^4)dx$$

$$=\frac{3}{4}\left[\frac{2x^4}{4}-\frac{x^5}{5}\right]_0^2$$

$$=\frac{3}{4}\left[\left(\frac{2\times 16}{4}-\frac{32}{5}\right)-(0-0)\right]$$

$$=\frac{3}{4}\left[\frac{2\times 16}{4}-\frac{32}{5}\right]$$

$$=\frac{3\times 32}{4}\left[\frac{1}{4}-\frac{1}{5}\right]$$

$$=\frac{24}{20}$$

$$\therefore E[X^2] = \frac{6}{5} \qquad (4)$$

$$\therefore \text{Var}(X) = \text{E}[X^2] - [\text{E}(X)]^2 \qquad [\text{using}(3) \text{ and } (4)]$$

$$= \frac{6}{5} - 1$$

$$\text{Var}(X) = \frac{1}{5}$$

The r^{th} moment $= \text{E}(X^r)$

$$= \int_0^2 x^r . \frac{3}{4} x(2-x)dx \quad [\text{using}(2)]$$

$$= \frac{3}{4} \int_0^2 (2x^{r+1} - x^{r+2})dx$$

$$= \frac{3}{4} \left[\frac{2x^{r+2}}{r+2} - \frac{2x^{r+3}}{r+3} \right]_0^2$$

$$= \frac{3}{4} \left[\left(2\frac{2^{r+2}}{r+2} - \frac{2^{r+3}}{r+3} \right) - (0-0) \right]$$

$$= \frac{3}{4} \left[\frac{2.2^r.2^2}{r+2} - \frac{2^r.2^3}{r+3} \right]$$

$$= \frac{2^r \times 3 \times 8}{4} \left[\frac{1}{r+2} - \frac{1}{r+3} \right]$$

$$= 6.2^r \left[\frac{r+3-r-2}{(r+2)(r+3)} \right]$$

$$= \frac{6 \times 2^r}{(r+2)(r+3)}$$

$$E(X^r) = 6.\frac{2^r}{(r+2)(r+3)}$$

The mean and variance can also be got from $E(X^r)$.

Mean $= E(X) = [E(X^r)]_{r=1}$

$$= \left[6.\frac{2^r}{(r+2)(r+3)} \right]_{r=1}$$

$$= \frac{6 \times 2}{3 \times 4} = 1$$

$$E(X^2)=[E(X^r)]_{r=2}$$
$$=\left[6.\frac{2^r}{(r+2)(r+3)}\right]_{r=2}$$
$$=\frac{6\times 4}{4\times 5}=\frac{6}{5}$$
$$\therefore \text{Var}(X)=\text{E}(X^2)\text{-}[\text{E}(X)]^2$$
$$=\frac{6}{5}-1$$
$$\text{Var}(X)=\frac{1}{5}$$

Practice Problems:

1. Find the first three moments of X if X has the following distribution:

$x:$	-2	1	3
$p(x):$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

Hence find the mean and variance.

2. A continuous random variable X has p.d.f. $f(x)=C(1-x),\ 0<x<1$. Find the r^{th} moment of X about the origin. Hence find the mean and variance.

3. Find the first four moments about the mean for $f(x)=\begin{cases} e^{-x} & ,x\geq 0 \\ 0, & \text{otherwise} \end{cases}$.

Answers:

1. $\mu_1'=0,\ \mu_2'=\frac{9}{2},\ \mu_3'=3$ and mean $=0$, $\text{var}(X)=\frac{9}{2}$.

2. C=2, $\mu_r'=\frac{2}{(r+1)(r+2)}$, Mean $=\frac{1}{3}$ & $\text{var}(X)=\frac{1}{18}$.

3. $\mu_1=0,\ \mu_2=1,\ \mu_3=2,\ \mu_4=11.$

Reference:

To know more about expectation

- https://medium.com/@praveenprashant/the-four-moments-of-a-probability-distribution-6b900a25d0d8

Activity:

Use the below link and try out Quiz

- https://study.com/academy/practice/quiz-worksheet-what-is-the-moment-generating-function.html

Videos:

To know more view the following videos

- https://www.youtube.com/watch?v=aankeAV-kC8
- https://www.youtube.com/watch?v=ucbIw-oC20w

1.6. Moment Generating Function

Definition: The moment generating function (m.g.f.) of a random variable X is the function M_X defined by $M_X(t)=E\left(e^{tX}\right)$ for those real t at which the expectation is well defined.

$$M_X(t)=E\left(e^{tX}\right)=\begin{cases}\sum e^{tx}p(x); & X \text{ is discrete}\\ \int e^{tx}f(x)\,dx; & X \text{ is continuous}\end{cases}$$

Theorem: Prove that the r^{th} moment of the random variable 'X' about origin is $M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r'$

Proof: We know that

$$M_X(t)=E\left[e^{tX}\right]$$

$$=E\left[1+\frac{tX}{1!}+\frac{(tX)^2}{2!}+\frac{(tX)^3}{3!}+\ldots\frac{(tX)^r}{r!}\ldots+\ldots\right]$$

$$=1+E\left[\frac{tX}{1!}\right]+E\left[\frac{t^2X^2}{2!}\right]+\ldots+E\left[\frac{t^rX^r}{r!}\right]\ldots+\ldots$$

$$=1+tE(X)+\frac{t^2}{2!}E(X^2)+\ldots+\frac{t^r}{r!}E\left[X^r\right]\ldots+\ldots$$

$$=1+t\mu_1'+\frac{t^2}{2!}\mu_2'+\frac{t^3}{3!}\mu_3'+\ldots+\frac{t^r}{r!}\mu_r'+\ldots$$

(i.e)., $M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r'$

Note: r^{th} moment=co efficient of $\frac{t^r}{r!}$

Example:1

Find μ_1' and μ_2' from $M_X(t)$.

Proof: We know that

$$M_X(t)=E\left[e^{tX}\right]$$

$$M_X(t)=\mu_0'+t\mu_1'+\frac{t^2}{2!}\mu_2'+\frac{t^3}{3!}\mu_3'+\ldots+\frac{t^r}{r!}\mu_r'+\ldots\rightarrow(1)$$

Differentiating (1) with respect to 't' we get

$$M_X{}'(t) = \mu_1' + \frac{2t}{2!}\mu_2' + \frac{3t^2}{3!}\mu_3' + \dots$$

Put t=0,we get

$$M_X{}'(0) = \mu_1' = Mean$$

$$\text{Thus Mean} = M_X{}'(0) \text{ (or) } \left[\frac{d}{dt}(M_X(t))\right]_{t=0}$$

Also,we know that

$$M_X{}''(t) = \mu_2' + t\mu_3' + \dots$$

Put t=0,we get

$$M_X{}''(0) = \mu_2' \text{ (or) } \left[\frac{d^2}{dt^2}(M_X(t))\right]_{t=0}$$

In general $\mu_r' = \left[\frac{d^r}{dt^r}(M_X(t))\right]_{t=0}$

Example:2 Find the m.g.f for the distribution where $f(x) = \begin{cases} \frac{2}{3} & at\ x=1 \\ \frac{1}{3} & at\ x=2 \\ 0 & Otherwise \end{cases}$

Solution: $f(1) = \frac{2}{3}$, $f(2) = \frac{1}{3}$

$$f(3) = f(4) = f(5) = ------- = 0$$

M.g.f of a R.V $'X'$

$$M_X(t) = E\left[e^{tX}\right] = \sum_{x=0}^{\infty} e^{tx} f(x)$$

$$= e^o f(0) + e^t f(1) + e^{2t} f(2) + ---$$

$$= 0 + e^t \frac{2}{3} + e^{2t}\frac{1}{3} + ---$$

$$= \frac{2}{3}e^t + \frac{1}{3}e^{2t}$$

$$M_X(t) = \frac{e^t}{3}\left[2 + e^t\right]$$

Example:3

Find the m.g.f of the random variable whose moments are $\mu_r' = (r+1)!2^r$

Solution: We know that the M.g.fin terms of moments is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

$$= \sum_{r=0}^{\infty} (r+1)!2^r \frac{t^r}{r!}\left[\because \mu_r' = (r+1)!2^r\right]$$

$$= \sum_{r=0}^{\infty} \frac{(2t)^r (r+1)!}{r!}$$

$$= \frac{(2t)^0 1!}{0!} + \frac{(2t)^1 (2)!}{1!} + \frac{(2t)^2 (3)!}{2!} + ---$$

$$= 1 + 2(2t) + 3(2t)^2 + ---$$

$$= (1-2t)^{-2}$$

Example:4

If a random variable $'X'$ has the m.g.f $M_X(t) = \frac{3}{3-t}$, find the S.D of $'X'$

Solution: $M_X(t) = \frac{3}{3-t} = 3(3-t)^{-1}$

$$M_X'(t) = -3(3-t)^{-2}(-1)$$

$$M_X''(t) = -6(3-t)^{-3}(-1)$$

$$E[X] = M_X'(0) = \frac{3}{9} = \frac{1}{3}$$

$$E\left[X^2\right] = M_X''(0) = \frac{6}{27} = \frac{2}{9}$$

$$Var(X) = E\left[X^2\right] - \left[E[X]\right]^2$$

$$= \frac{2}{9} - \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

$$S.D = \sqrt{Var(X)}$$

$$= \sqrt{\frac{1}{9}} = \frac{1}{3}$$

Example:5

If 'X' represents the outcome, when a fair die is tossed, find the M.g.f of 'X' and hence find $E(X)$ and $Var(X)$

Solution: $P(X=x)=\frac{1}{6}, x=1,2,3,4,5,6$

$$M_X(t)=\sum_{x=1}^{6} e^{tx}P(X=x) \quad (\because X \text{ is discrete r.v})$$

$$=\sum_{x=1}^{6} e^{tx}\frac{1}{6}$$

$$=\frac{1}{6}\left[e^{t}+e^{2t}+e^{3t}+e^{4t}+e^{5t}+e^{6t}\right]$$

$$M_X'(t)=\frac{1}{6}\left[e^{t}+2e^{2t}+3e^{3t}+4e^{4t}+5e^{5t}+6e^{6t}\right]$$

$$M_X''(t)=\frac{1}{6}\left[e^{t}+4e^{2t}+9e^{3t}+16e^{4t}+25e^{5t}+36e^{6t}\right]$$

$$E[X]=M_X'(0)=\frac{1}{6}[1+2+3+4+5+6]=\frac{21}{6}=\frac{7}{2}$$

$$E\left[X^2\right]=M_X''(0)=\frac{1}{6}[1+4+9+16+25+36]=\frac{91}{6}$$

$$Var(X)=E\left[X^2\right]-\left[E[X]\right]^2$$

$$=\frac{91}{6}-\left(\frac{7}{2}\right)^2=\frac{91}{6}-\frac{49}{4}$$

$$=\frac{35}{12}$$

Example:6

If the density function of a random variable 'X' is $f(x)=\begin{cases}\frac{1}{3}, -1<x<2\\ 0, otherwise\end{cases}$. Find its Moment Generating function.

Solution: Given that , $f(x)=\begin{cases}\frac{1}{3}, -1<x<2\\ 0, otherwise\end{cases}$

The MGF is $M_X(t)=E\left(e^{Xt}\right)$

$$M_X(t)=E\left(e^{Xt}\right)=\int_{-1}^{2} e^{xt}\frac{1}{3}dx$$

$$=\frac{1}{3}\left[\frac{e^{xt}}{t}\right]_{-1}^{2}$$

$$=\frac{1}{3}\left[\frac{e^{2t}}{t}-\frac{e^{-1t}}{t}\right]$$

Example:7

Find the M.g.f of the random variable X having the density function $f(x)=\begin{cases}\frac{x}{2}, 0\le x\le 2\\ o,\ otherwise\end{cases}$

Solution:

$$f(x)=\frac{x}{2}, 0\le x\le 2$$

$$M_x(t)=\int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$=\int_{0}^{2} e^{tx}\frac{x}{2}dx$$

$$=\frac{1}{2}\int_{0}^{2} e^{tx}x\,dx$$

$$=\frac{1}{2}\left[\frac{xe^{tx}}{t}-\frac{e^{tx}}{t^2}\right]_0^2$$

$$=\frac{1}{2}\left[\frac{2e^{t2}}{t}-\frac{e^{t2}}{t^2}+\frac{1}{t^2}\right]$$

$$=\frac{1}{2}\left[\frac{2te^{2t}-e^{2t}+1}{t^2}\right]$$

Example:8

Find the Moment generating function of the random variable X having p.d.f $f(x)=\begin{cases}x,\ for\ 0<x<1\\ 2-x,\ for\ 1<x<2\\ 0, otherwise\end{cases}$

Solution: We know that

$$M_x(t)=\int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^1 e^{tx} f(x)\,dx + \int_1^2 e^{tx} f(x)\,dx$$

$$= \int_0^1 e^{tx} x\,dx + \int_1^2 e^{tx} (2-x)\,dx$$

$$= \left[x\left(\frac{e^{tx}}{t}\right) - \left(\frac{e^{tx}}{t^2}\right) \right]_0^1 + \left[(2-x)\left(\frac{e^{tx}}{t}\right) - (-1)\left(\frac{e^{tx}}{t^2}\right) \right]_1^2$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}$$

$$= \frac{e^{2t} + 1 - 2e^t}{t^2}$$

$$= \frac{\left(e^t - 1\right)^2}{t^2}$$

Problems for practices

1. Find the Moment generating function of the random variable with the probability law $P(X = x) = q^{x-1} p, x = 1, 2, 3, \ldots$ find the mean and variance.
2. Find the Moment generating function of the random variable with the probability law $P(X = x) = \frac{1}{2^x}, x = 1, 2, 3, ..$ find the mean and variance.
3. If a random variable $'X'$ has the m.g.f $M_X(t) = \frac{2}{2-t}$, find the mean and variance.
4. Find the Moment generating function of the random variable X having p.d.f
$$f(x) = \begin{cases} \frac{1}{k}, \; for\; 0 < x < k \\ 0, \; otherwise \end{cases}$$
5. Find the Moment generating function of the random variable X having p.d.f
$$f(x) = \begin{cases} \lambda e^{-\lambda x}, x > 0 \\ 0, \; otherwise \end{cases}$$

Answers:

1.Mean $=\frac{1}{p}$, Variance= $\frac{q}{p^2}$

2. Mean= 2 , Variance= 2

3. Mean= $\frac{1}{2}$, variance= $\frac{1}{4}$

4. $\frac{(e^{kt}-1)}{kt}$

5. $\frac{\lambda}{\lambda - t}$

Reference:

To know more about expectation

- https://www.probabilitycourse.com/chapter6/6_1_3_moment_functions.php
- https://www.statlect.com/fundamentals-of-probability/moment-generating-function

Activity:

Use the below link and try out Quiz

- https://study.com/academy/practice/quiz-worksheet-moment-generating-functions-for-continuous-random-variables.html

Videos:

To know more view the following videos

- https://www.youtube.com/watch?v=cbmfYoepHPk
- https://www.youtube.com/watch?v=wjwLTNYOuI4

Introduction

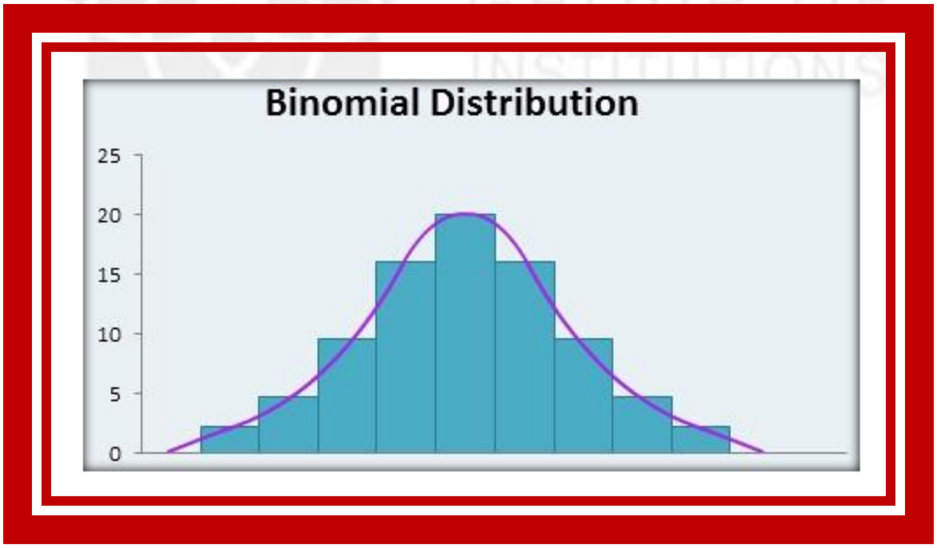
In the previous section we studied the definitions of some probability distributions. In engineering studies we come across many phenomena, which are random in nature. The probability distributions describing these phenomena resemble very frequently to certain distributions, which are known as the standard distributions. The study of these standard distributions both discrete and continuous is of great importance in engineering applications. We shall discuss a number of as well as continuous distributions.

Discrete Distributions:

Let 'X' be a discrete random variable, the probability mass function $p(x)$ categorized into the following types

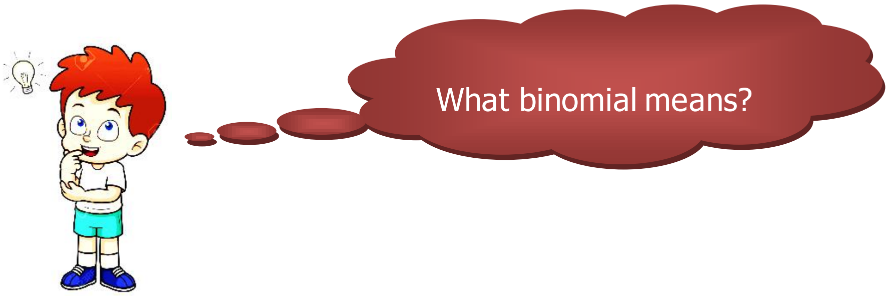
- Binomial distribution
- Poisson distribution
- Geometric distribution

1.7 Binomial Distribution



Starting with an example, if someone tosses the coin then there is an equal chance of outcome it can be heads or tails. There is a 50% chance of the outcomes. Likewise, if you are appearing in an exam then there is also an equal possibility of getting passed or fail.

The binomial distribution summarized the number of trials, survey or experiment conducted. It is very useful when each outcome has the equal chance of attaining a particular value. The binomial distribution have some assumptions which show that there is only one outcome and this outcome have an equal chance of occurrence.



Binomial Distribution Examples

Let's take some real-life instances where you can use the binomial distribution.

- If the WHO introduced a new cure for a disease then there is an equal chance of success and failure. It can either cure the diseases or not.
- If you are purchasing a lottery then either you are going to win money or you are not. In other words, anywhere the outcome could be a success or a failure that can be proved through binomial distribution.

Definition:

Consider an experiment which gives two possible outcomes success and failure. This experiment is repeated n independent times. Let p be the probability of successes and q be the probability of failures. Let us assume that x outcomes are success and the remaining $n-x$ outcomes are failures. The probability of this event is $p^x q^{n-x}$ out of n trials. The x success can happen in any one of nC_x different ways. Therefore, the probability of getting x successes in n trials is $nC_x p^x q^{n-x}$.

Let $'X'$ be a discrete random variable which takes the values $x = 0,1,2,\ldots n$ such that

$$P(X = x) = nC_x p^x q^{n-x} \text{ where } p + q = 1.$$

Then $'X'$ is said to follow the **Binomial distribution** with parameters n and p.

Here,

$$\begin{aligned} X &= \text{Binomial random variable} \\ p &= \text{Probability of succes in a single trial} \\ q &= 1-p = \text{Probability of failrue in a single trial} \\ n &= \text{Number of trials} \\ x &= \text{Number of success in } n \text{ trials} \end{aligned}$$

Assumptions:

- Each trial has only two possible outcomes, success or failure.
- The number of trials n is finite and independent.
- The probability of success is the same for each trial.
- In binomial distribution, mean>variance.

Note:

1. $P(X=x) = nC_x p^x q^{n-x}$
   Here $nC_x p^x q^{n-x}$ is the $(x+1)^{th}$ in the expansion of $(q+p)^n$.
   $\left[\because (q+p)^n = q^n + nC_1 q^{n-1} p^1 + \ldots + nC_x q^{n-x} p^n\right]$
   Which is a Binomial series.
2. Let an experiment constitutes n trials. If this experiment is repeated N times, the frequency function of the binomial distribution is given by
   $$\begin{aligned} f(x) &= Np(x) \\ &= N.nC_x p^x q^{n-x}; \ X = 0,1,2,\ldots n \end{aligned}$$

Mean and variance of a Binomial distribution:

$$\begin{aligned} Mean = \mu_1' &= \sum_{x=0}^{n} xp(x) \\ &= \sum_{x=0}^{n} xnC_x p^x q^{n-x} \\ &= \sum_{x=0}^{n} x\frac{n}{x} n-1C_{x-1} p^x q^{n-x} \end{aligned}$$

$$= n\sum_{x=0}^{n} n-1C_{x-1}\, p^{(x-1)+1} q^{(n-1)-(x-1)}$$

$$= np\sum_{x=0}^{n} n-1C_{x-1}\, p^{x-1} q^{(n-1)-(x-1)}$$

$$= np\left(p+q\right)^{n-1}$$

$$\mu_1^{'} = np \qquad (\because\ p+q=1)$$

Again, we have

$$E(X^2) = \mu_2^{'} = \sum_{x=0}^{n} x^2 p(x)$$

$$= \sum_{x=0}^{n} x^2 nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \left[\left(x^2 - x\right) + x\right] nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \left(x^2 - x\right) nC_x p^x q^{n-x} + \sum_{x=0}^{n} x\, nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)\frac{n(n-1)}{x(x-1)} n-2C_{x-2}\, p^x q^{n-x} + np$$

$$= n(n-1)\sum_{x=0}^{n} n-2C_{x-2}\, p^{x-2+2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2\sum_{x=0}^{n} n-2C_{x-2}\, p^{x-2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2\left(p+q\right)^{n-2} + np$$

$$\mu_2^{'} = n(n-1)p^2 + np \qquad (\because\ p+q=1)$$

We know that

$$Var(X) = E(X^2) - \left(E(X)\right)^2$$

$$= n(n-1)p^2 + np - n^2p^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= -np^2 + np$$

$$= np(1-p)$$

$$Var(X) = npq$$

For Binomial distribution

- Mean = $E(X) = np$
- Variance = $Var(X) = npq$

Moment Generating Function of a Binomial Distribution:

We know that $M_X(t) = E(e^{tX})$

$$= \sum_{x=0}^{n} e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx} nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \left(pe^t\right)^x nC_x q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x \left(pe^t\right)^x q^{n-x}$$

$$\therefore M_X(t) = \left(pe^t + q\right)^n$$

Moment Generating Function of Binomial Distribution is

$$M_X(t) = \left(pe^t + q\right)^n$$

The first three moments of the binomial distribution from M.G.F:

We know that $M_X(t) = \left(pe^t + q\right)^n$

$$M_X'(t) = n\left(pe^t + q\right)^{n-1} pe^t \qquad \rightarrow (1)$$

$$M_X''(t) = np\left[(n-1)\left(pe^t + q\right)^{n-2} pe^t e^t + \left(pe^t + q\right)^{n-1} .e^t\right]$$

$$M_X''(t) = np\left[(n-1)\left(pe^t + q\right)^{n-2} pe^{2t} + \left(pe^t + q\right)^{n-1} .e^t\right] \qquad \rightarrow (2)$$

$$M_X'''(t) = np\left[(n-1)(n-2)\left(pe^t + q\right)^{n-3} pe^t\, pe^{2t} + (n-1)\left(pe^t + q\right)^{n-2} 2pe^{2t}\right]$$

$$+ np\left[\left(pe^t + q\right)^{n-1} .e^t + e^t (n-1)\left(pe^t + q\right)^{n-2} pe^t\right] \qquad \rightarrow (3)$$

put $t = 0$ in (1), we get

$$\mu_1' = M_X'(0) = n(p+q)^{n-1}.p$$

First moment $\left(\mu_1'\right) = np = \text{Mean} \qquad (\because p + q = 1)$

put $t = 0$ in (2), we get

$$\mu_2' = M_X''(0) = np\left[(p+q)^{n-1} + (n-1)(p+q)^{n-2}.p\right]$$

$$= np\left[1 + (n-1)p\right]$$

$$= np\left[1 + np - p\right]$$

$$= np + n^2p^2 - np^2$$

$$= n^2p^2 + np(1-p)$$

$$\mu_2' = n^2p^2 + npq = \text{second moment}$$

$$\text{put } t=0 \text{ in (3), we get}$$

$$\mu_3' = M_X'''(0) = np\left[(n-1)(n-2)p^2 + 2p(n-1) + 1 + p(n-1)\right]$$

$$= np\left[n^2p^2 - 3np^2 + 2p^2 + 2pn - 2p + 1 + pn - p\right]$$

$$\mu_3' = np\left[n^2p^2 - 3np^2 + 2p^2 + 3pn - 3p + 1\right] = \text{third moment}$$

The first three moments of the binomial distribution are

$$\text{First moment, } \mu_1' = np = \text{Mean}$$

$$\text{Second moment, } \mu_2' = n^2p^2 + npq$$

$$\text{Third moment, } \mu_3' = np\left[n^2p^2 - 3np^2 + 2p^2 + 3pn - 3p + 1\right]$$

State and prove additive property of Binomial distribution:

Statement: Sum of two independent Binomial variates is not Binomial variate.

Proof:

Let X and Y be two independent binomial variates with parameter (n_1, p_1) and (n_2, p_2) respectively.

$$\text{Then } M_X(t) = (p_1e^t + q_1)^{n_1}$$

$$M_Y(t) = (p_2e^t + q_2)^{n_2}$$

$$M_{X+Y}(t) = M_X(t).M_Y(t) \quad (\because X \text{ and } Y \text{ are independent})$$

$$= (p_1e^t + q_1)^{n_1}.(p_2e^t + q_2)^{n_2}$$

The R.H.S cannot be expressed in the form of $(pe^t + q)^n$. Hence by uniqueness theorem of M.G.F $X+Y$ is not a binomial variate.

Hence in general, the sum of two binomial variates is not a binomial variate.

Example:1

Check whether the following data follow a binomial distribution or not.

Mean=3, variance=4.

Solution: Given

$$\text{Mean} = np = 3 \qquad \rightarrow (1)$$

$$\text{Variance} = npq = 4 \qquad \rightarrow (2)$$

$$\frac{(2)}{(1)} \Rightarrow \frac{npq}{np} = \frac{4}{3} = 1\frac{1}{3}$$

$$q = 1\frac{1}{3} > 1$$

Since $q > 1$ which is not possible $(0 < q < 1)$. The given data does not follow Binomial distribution.

Example:2

The mean and standard deviation of a Binomial distribution are 5 and 2. Determine the distribution.

Solution: Given

$$\text{Mean} = np = 5 \qquad \rightarrow (1)$$

$$S.D = \sqrt{\text{Variance}} = \sqrt{npq} = 2$$

$$npq = 4 \qquad \rightarrow (2)$$

$$\frac{(2)}{(1)} \Rightarrow \frac{npq}{np} = \frac{4}{5}$$

$$q = 5$$

$$\therefore p = 1 - q = 1 - \frac{4}{5} = \frac{1}{5}$$

$$(i.e)., p = \frac{1}{5} \qquad \rightarrow (3)$$

Substitute (3) in (1), we get

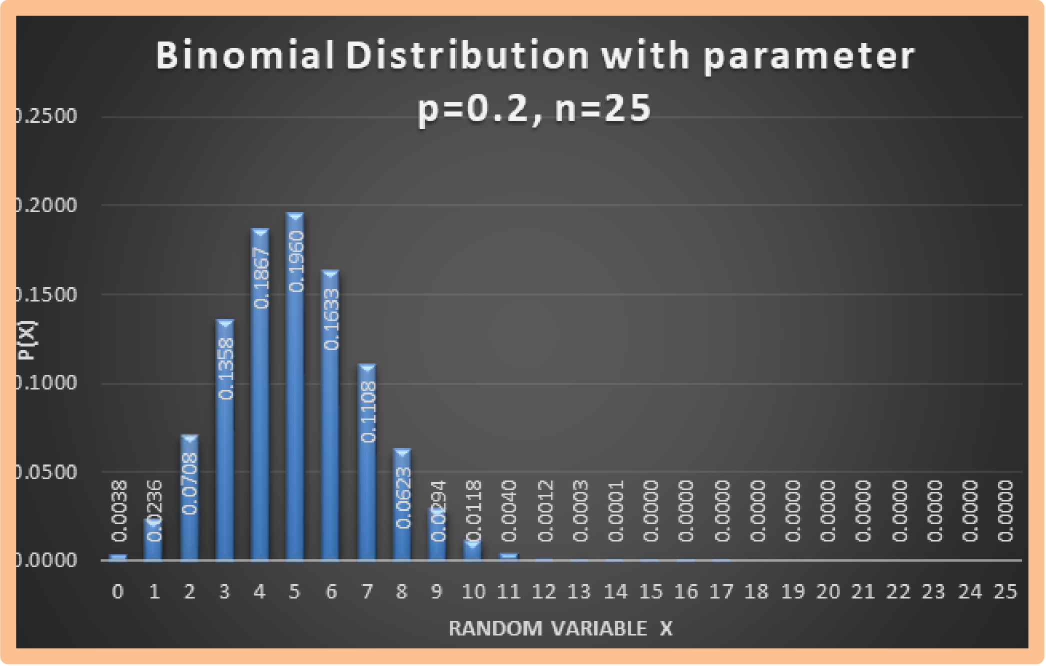
$$n \times \frac{1}{5} = 5$$

$$\therefore n = 25$$

The Binomial Distribution is

$$P(X = x) = nC_x p^x q^{n-x}$$

$$= 25C_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{25-x}$$



Example:3

With the usual notation find $'p'$ for a Binomial random variate $'X'$ if $n=6$ and if $9P(X=4)=P(X=2)$.

Solution: We know that for binomial random variate $'X'$,

$$P(X=x)=nC_x p^x q^{n-x}$$

$$\text{Given } 9P(X=4)=P(X=2)$$

$$9\times 6C_4 p^4 q^{n-4}=6C_2 p^2 q^{n-2}$$

$$9p^2=q^2$$

$$=(1-p)^2$$

$$9p^2=1+p^2-2p$$

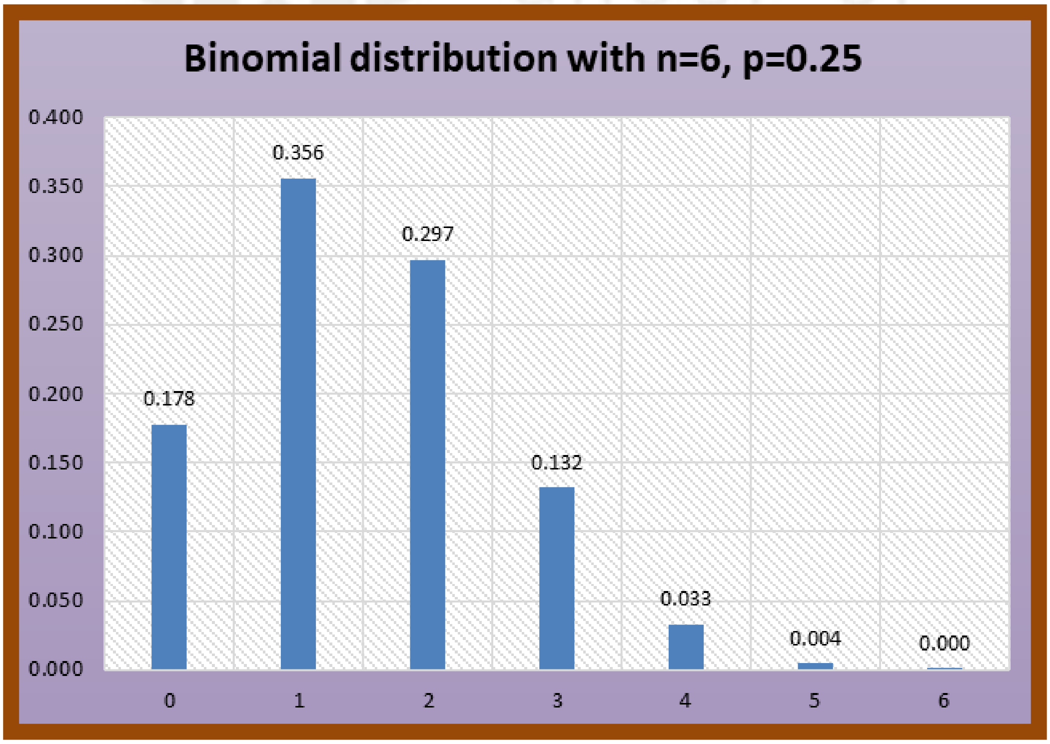
$$8p^2+2p-1=0$$

$$p=\frac{-2\pm\sqrt{4+32}}{16}=\frac{-2\pm 6}{16}=\frac{1}{4} \; or \; \frac{-1}{2}$$

$$p=\frac{-1}{2} \quad \text{(not possible)}$$

$$(i.e)., \; p=\frac{1}{4}$$

$$\therefore q=\frac{3}{4}$$



Example:4

A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability of 2 successes.

Solution: Let 'X' be the random variable of getting doublet.

Doublet means getting the same number in both the dice.

In throwing a pair of dice, we get the following doublets

$$(1,1),(2,2),(3,3),(4,4),(5,5) \text{ and } (6,6).$$

$$\text{Now, } P(\text{getting a doublet}) = \frac{6}{36} = \frac{1}{6}$$

$$p = \frac{1}{6}$$

$$\Rightarrow q = \frac{5}{6} \qquad (\because p + q = 1)$$

$$\text{So, } p = \frac{1}{6},\ q = \frac{5}{6} \quad \text{and } n = 4$$

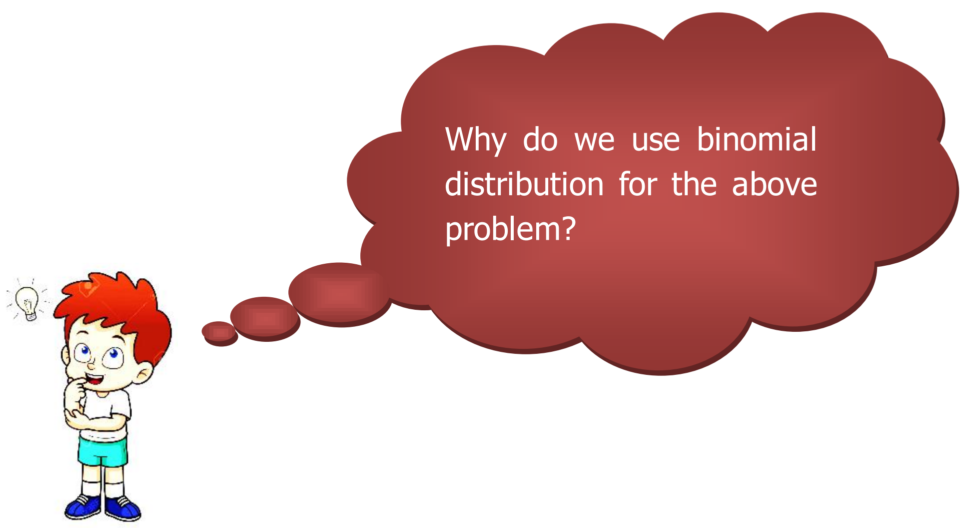
$$\text{We know that } P(X = x) = nC_x p^x q^{n-x}$$

$$P(\text{getting 2 successes}) = P(X = 2)$$

$$= 4C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{4-2}$$

$$= 6\left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2$$

$$= \frac{25}{216} = 0.1157$$



Example:5

Four fair coins are flipped. If the outcomes are assumed independent, what is the probability that two heads and two tails are obtained?

Solution: Let $'X'$ be the random variable denoting the number of successes in a coin.

$$\therefore\ p=\frac{1}{2},\ q=\frac{1}{2}\ \text{ and also given }\ n=4$$

$$\text{We know that } P(X=x)=nC_x p^x q^{n-x}$$

$$P(\text{getting 2 heads and 2 tails}) = P(X=2)$$

$$=4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{4-2}$$

$$=\frac{4\times3}{1\times2}.\frac{1}{4}.\frac{1}{4}=\frac{3}{8}$$

Example:6

Find the probability that in tossing a fair coin 5 times, there are will appear

i) 3 heads

ii) 3 tails and 2 heads

iii) at least one head

iv) not more than one tail.

Solution: Let $'X'$ be the random variable denoting the number of successes in a coin.

$$\therefore\ p=\frac{1}{2},\ q=\frac{1}{2}\ \text{ and also given }\ n=5$$

$$\text{We know that } P(X=x)=nC_x p^x q^{n-x}$$

$$i)\quad P(\text{getting 3 heads}) = P(X=3)$$

$$=5C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{5-3}$$

$$=\frac{5\times4\times3}{1\times2\times3}.\frac{1}{8}.\frac{1}{4}=\frac{5}{16}$$

We note that getting 3 tails and 2 heads is equivalent to getting 3 tails or 2 heads.

$$ii)\quad P(\text{getting 2 heads}) = P(X=2)$$
$$= 5C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{5-2}$$
$$= 10\times\frac{1}{32} = \frac{5}{16}$$

$$iii)\, P(\text{getting atleast 1 head}) = P(X\geq 1)$$
$$= 1 - P(X<1)$$
$$= 1 - P(X=0)$$
$$= 1 - 5C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{5-0}$$
$$= 1 - \frac{1}{32} = \frac{31}{32}$$

$$iv)\, P(\text{not more than 1 tail}) = P(\text{getting 0 tail}) + P(\text{getting 1 tail})$$
$$= P(\text{getting all heads}) + P(\text{getting 4 heads})$$
$$= P(X=5) + P(X=4)$$
$$= 5C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^{5-5} + 5C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{5-4}$$
$$= \frac{1}{32} + \frac{5}{32} = \frac{6}{32} = \frac{3}{16}$$

Example:7

A machine manufacturing screws is known to produce 5% defective. In a random sample of 15 screws, what is the probability that there are

i) exactly three defectives

ii) not more than three defectives.

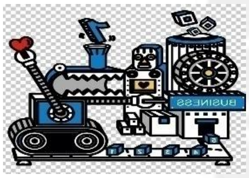
Solution: Let 'X' be the random variable denoting the number of defective screws.

$$\text{Given } p = 5\% = \frac{5}{100} = 0.05$$
$$q = 1 - p = 1 - 0.05 = 0.95$$
$$n = 15$$
$$\text{We know that } P(X=x) = nC_x\, p^x q^{n-x}$$
$$= 15C_x (0.05)^x (0.95)^{15-x}$$

$$\text{i)}P(\text{exactly 3 defectives}) = P(X=3)$$
$$= 15C_3(0.05)^3(0.95)^{15-3}$$
$$= 455(0.000125)(0.54036)$$
$$= 0.0307$$

$$\text{ii) } P(\text{not more than 3 defectives}) = P(X \le 3)$$
$$= P(X=0)+P(X=1)+P(X=2)+P(X=3)$$
$$= 15C_0(0.05)^0(0.95)^{15-0} + 15C_1(0.05)^1(0.95)^{15-1}$$
$$+ 15C_2(0.05)^2(0.95)^{15-2} + 15C_3(0.05)^3(0.95)^{15-3}$$
$$= 0.4632 + 0.3657 + 0.1347 + 0.030733$$
$$= 0.994$$

Example: 8



It is known that all items produced by a certain machine will be defective with probability 0.1, independently of each other. What is the probability that in a sample of three items, at most one will be defective?

Solution: Let $'X'$ be the random variable denoting the number of defective items in the sample.

$$\text{Given } p = 0.1$$
$$q = 1-p = 1-0.1 = 0.9$$
$$n = 3$$

$$\text{We know that } P(X=x) = nC_x p^x q^{n-x}$$
$$= 3C_x(0.1)^x(0.9)^{3-x}$$
$$P(\text{atmost one defectives}) = P(X \le 1)$$
$$= P(X=0)+P(X=1)$$
$$= 3C_0(0.1)^0(0.9)^{3-0} + 3C_1(0.1)^1(0.9)^{3-1}$$
$$= (0.9)^3 + 3(0.1)(0.9)^2$$
$$= 0.972$$

Example:9

The probability of a bomb hitting a target is $\frac{1}{5}$. Two bombs are enough to destroy a bridge. If six bombs are aimed at the bridge, find the probability that the bridge is destroyed?

Solution: Let $'X'$ be the random of hitting the target.

$$\text{Given } p(\text{hitting the target}) = \frac{1}{5}$$

$$q = 1 - p = 1 - \frac{1}{5} = \frac{4}{5}$$

$$n = 6$$

$$\text{We know that } P(X = x) = nC_x p^x q^{6-x}$$

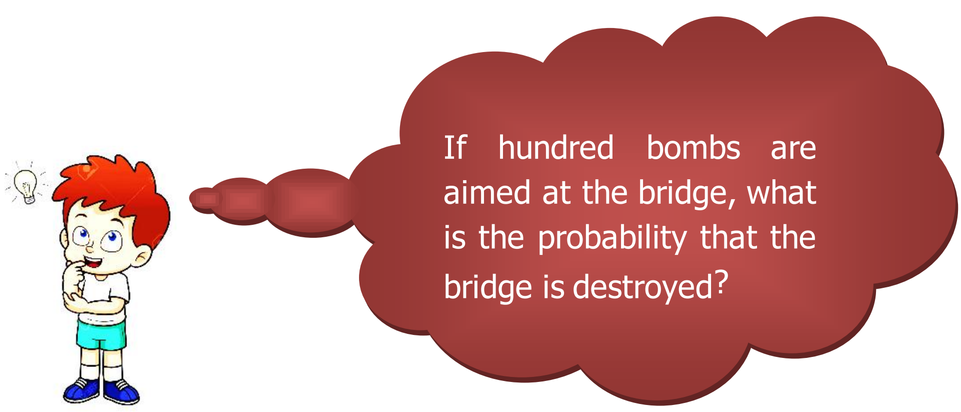
$$= 6C_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{6-x}$$

$$P(\text{the bridge is destroyed}) = P(X = 2)$$

$$= 6C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^4$$

$$= \frac{6\times5}{2\times1} \cdot \frac{1}{25} \cdot \frac{256}{625}$$

$$= \frac{768}{3125} = 0.25$$



Example:9

In certain town, 20% samples of the population are literate and assume that 200 investigators take sample of ten individuals to see whether they are literate. How many investigators would you expect to report that 3 people or less are literates in the samples?

Solution:

Let 'X' be the random variable denoting the number of literate persons.

$$\text{Given } p(\text{an individual is literate}) = 20\% = \frac{20}{100} = 0.2$$

$$q = 1 - p = 1 - 0.2 = 0.8$$

$$n = 10 \text{ (sample size)}$$

$$\text{We know that } P(X = x) = nC_x p^x q^{n-x}$$

$$= 10C_x (0.2)^x (0.8)^{10-x}$$

$$\text{i) } P\begin{Bmatrix}\text{an investigator reporting} \\ 3 \text{ or less as literate}\end{Bmatrix} = P(X \le 3)$$

$$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$= 10C_0 (0.2)^0 (0.8)^{10-0} + 10C_1 (0.2)^1 (0.8)^{10-1}$$

$$+ 10C_2 (0.2)^2 (0.8)^{10-2} + 10C_3 (0.2)^3 (0.8)^{10-3}$$

$$= (0.8)^7 \left[(0.8)^3 + 10(0.2)(0.8)^2 + 45(0.2)^2(0.8) + 120(0.2)^3 \right]$$

$$= 0.2097 [0.512 + 1.28 + 1.44 + 0.96]$$

$$= 0.8790$$

$$\therefore P(200 \text{ investigator reporting } 3 \text{ or less as literate}) = 200 \times 0.8790 = 175.72$$

$$176 \text{ investigators}$$

Example:10

An irregular six faced die is thrown and the expectation that in 10 throws it will give five even numbers is twice the expectation that it will give four even numbers. How many times in 10,000 sets of 10 throws each, would you expect it to give no even number?

Solution:

Let 'X' be the random variable getting the number of even numbers.

$$n = 10 \text{ (sample size)}$$

We know that $P(\text{getting 'x' even number}) = P(X = x) = 10C_x p^x q^{10-x}$

Given that $P(\text{getting 5 even number}) = 2P(\text{getting 4 even number})$

$$P(X=5) = 2P(X=4)$$

$$10C_5 p^5 q^5 = 2 \times 10C_4 p^4 q^6$$

$$\frac{p}{5} = \frac{q}{3}$$

$$3p = 5q = 5(1-p)$$

$$8p = 5 \Rightarrow p = \frac{5}{8}$$

$$\therefore q = \frac{3}{8}$$

$$P(\text{getting 'x' even number}) = P(X = x) = 10C_x \left(\frac{5}{8}\right)^x \left(\frac{3}{8}\right)^{10-x}$$

Therefore, the required number of times that in 10,000 sets of 10 throws each we get no even number

$$= 10,000 \times P(X=0)$$

$$= 10,000 \times 10C_0 \left(\frac{5}{8}\right)^0 \left(\frac{3}{8}\right)^{10-0}$$

$$= 10,000 \times \left(\frac{3}{8}\right)^{10}$$

□1

Example:11

It is known that screws produced by a certain company Wii be defective with probability 0.01 independently of each other. The company sells the screws in packages of 10 and offers a moneyback guarantee that at most 1 of the 10 screws is defective. What proportion of packages sold must the company replace?

Solution:

Let $'X'$ be the random variable denoting the number of defective screws.

$$\text{Given } p = 0.01$$
$$q = 1 - p = 1 - 0.01 = 0.99$$
$$n = 10$$

$$\text{We know that } P(X = x) = nC_x p^x q^{n-x}$$
$$= 10C_x (0.01)^x (0.99)^{10-x}$$

$$P(\text{atmost 1 screw is defective}) = P(X \le 1)$$
$$= P(X = 0) + P(X = 1)$$
$$= 10C_0 (0.01)^0 (0.99)^{10-0} + 10C_1 (0.01)^1 (0.99)^{10-1}$$
$$= (0.99)^{10} + 10(0.01)(0.99)^9 = 0.9957$$

$$\therefore P(\text{a package will have to replace}) = 1 - P(X \le 1)$$
$$= 1 - 0.9957$$
$$= 0.0043$$

$$\therefore 0.43 \text{ percent of the packages will have to replace.}$$

EXERCISE PROBLEMS:

1. If $'X'$ is a binomial random variable with expected value 6 and variance2.4, find $P(X = 5)$.
2. Find the binomial distribution for which the mean is 4 and variance is 3.
3. Five fair coins are flipped. If the outcomes are assumed independent, find the probability mass function of the number of heads obtained.
4. 6 dice are thrown 729 times. How many times do you expect at least three dice to show a five or a six?
5. If the probability of hitting a target is 10% and 10 shots are fired independently, what is the probability that the target will be hit at least once?

Answers:

1. 0.2
2. $P(X=x)=16C_x\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{n-x}$
3. $\frac{1}{32},\frac{5}{32},\frac{5}{16},\frac{5}{16},\frac{5}{32},\frac{1}{32}$
4. 233
5. 0.65

Reference:

To know more about Binomial distribution

- https://www.mathsisfun.com/data/binomial-distribution.html
- https://www.investopedia.com/terms/b/binomialdistribution.asp

Activity:

By Using the Binomial Probability calculator verify the problems you solved.

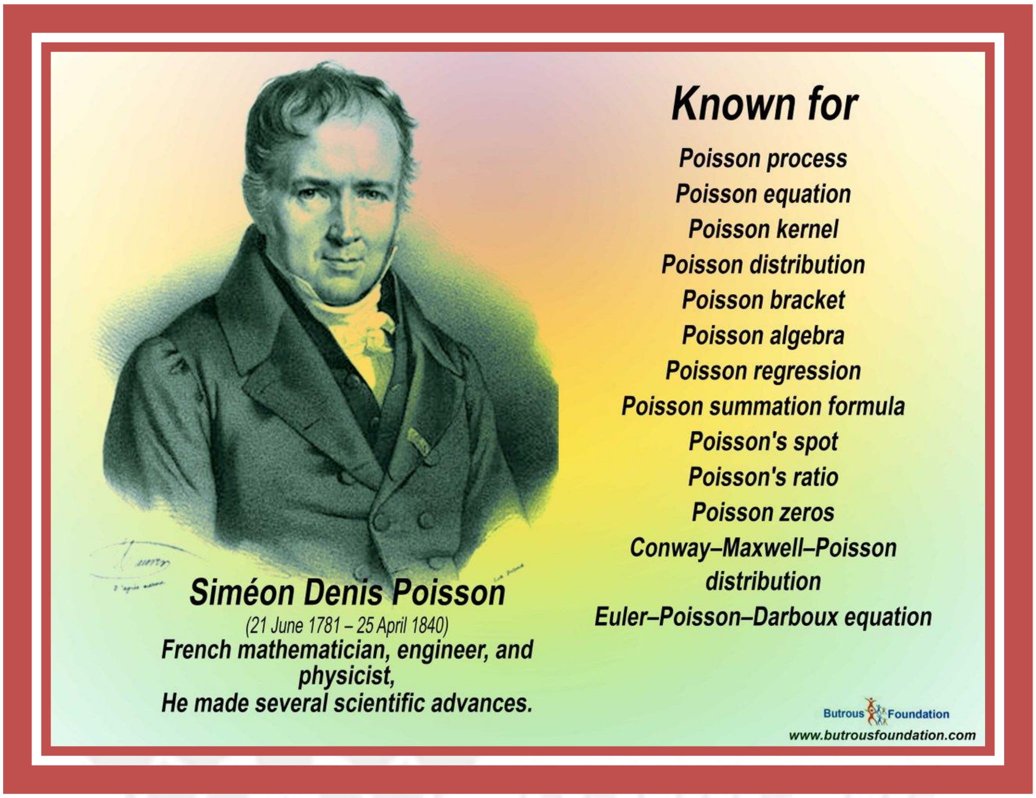
- https://stattrek.com/online-calculator/binomial.aspx

videos:

To know more view the following videos

- https://youtu.be/c06FZ2Yq9rk
- https://youtu.be/J8jNoF-K8E8

1.8 Poisson Distribution



Poisson distribution is a limiting case of Binomial distribution under the following assumptions

- The number of trials 'n' should be indefinitely large i.e., $n \to \infty$
- The probability of successes 'p' for each trial is indefinitely small.
- $np = \lambda$, should be finite where λ is a constant.

Poisson Distribution

The random variable $'X'$ is said to follow a Poisson distribution if it assumes only non-negative values and its probability mass function is given by

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \qquad for \;\; x = 0,1,2,....,\infty$$

Here λ is known as the parameter of the distribution.

Probability mass function of Poisson distribution as a limited case of binomial distribution:

We know that for binomial distribution,

$$P(X=x)=nC_x p^x q^{n-x}$$

$$=\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$$

$$=\frac{1.2.3....(n-x)....n}{1.2.3....(n-x)x!}\left(\frac{p}{1-p}\right)^x(1-p)^n$$

$$=\frac{n(n-1)(n-2)....(n-x+1)}{x!}\left(\frac{\frac{\lambda}{n}}{1-\frac{\lambda}{n}}\right)^x\left(1-\frac{\lambda}{n}\right)^n$$

$$=\frac{n(n-1)(n-2)....(n-x+1)}{x!}\frac{\lambda^x}{n^x}\frac{1}{\left(1-\frac{\lambda}{n}\right)^x}\left(1-\frac{\lambda}{n}\right)^n$$

$$=\frac{n(n-1)(n-2)....(n-x+1)}{x!}\frac{\lambda^x}{n^x}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$=\frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)....\left(1-\frac{(x-1)}{n}\right)}{x!}\lambda^x\left(1-\frac{\lambda}{n}\right)^{n-x}\text{-----------------}(1)$$

We know that $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n-x}=e^{-\lambda}$

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)=\lim_{n\to\infty}\left(1-\frac{2}{n}\right)=\text{------}=\lim_{n\to\infty}\left(1-\left(\frac{x-1}{n}\right)\right)$$

$$=1$$

When $n\to\infty$, the R.H.S of (1) gives $\frac{\lambda^x e^{-\lambda}}{x!}\text{-----------}(2)$

Substituting $(2)\, in\, (1)$, we get

$$P(X=x)=\frac{\lambda^x e^{-\lambda}}{x!}, x=0,1,2,....,\infty$$

The probability mass function is given by

$$P(X=x)=\begin{cases}\frac{e^{-\lambda}\lambda^x}{x!}, x=0,1,2,....,\infty\\ 0, otherwise\end{cases}$$

Moment generating function of the Poisson distribution

$$M_x(t)=\sum_{x=0}^{\infty}e^{tx}P(x)$$

$$=\sum_{x=0}^{\infty}e^{tx}\left(\frac{e^{-\lambda}\lambda^x}{x!}\right)$$

$$=\sum_{x=0}^{\infty}e^{-\lambda}\frac{\left(\lambda e^{tx}\right)^x}{x!}$$

$$=e^{-\lambda}\sum_{x=0}^{\infty}\frac{\left(\lambda e^{tx}\right)^x}{x!}$$

$$=e^{-\lambda}\left\{1+\lambda e^t+\left(\frac{\lambda e^t}{2}\right)^2+----\right\}$$

$$=e^{-\lambda}e^{\lambda e^t}$$

$$=e^{\lambda\left(e^t-1\right)}$$

Moment Generating Function of Poisson Distribution is

$$M_X(t)=e^{\lambda\left(e^t-1\right)}$$

Mean and Variance of Poisson Distribution

Mean = $E(X)=\sum_{x=0}^{\infty}xP(x)$

$$=\sum_{x=0}^{\infty}x\frac{\lambda^x e^{-\lambda}}{x!}=\sum_{x=0}^{\infty}\frac{xe^{-\lambda}\lambda\lambda^{x-1}}{x!}$$

$$=e^{-\lambda}\lambda\sum_{x=1}^{\infty}\frac{x\lambda^{x-1}}{x!}$$

$$=\lambda e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}$$

$$=\lambda e^{-\lambda}\left[1+\lambda+\frac{\lambda^2}{2!}+-----\right]$$

$$=\lambda e^{-\lambda}e^{\lambda}$$

$$=\lambda$$

Now,

$$E(X^2)=\sum_{x=0}^{\infty} x^2 P(x)$$

$$=\sum_{x=0}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!}$$

$$=\sum_{x=0}^{\infty}\left[x(x-1)+x\right]\frac{e^{-\lambda}\lambda^x}{x!}$$

$$=\sum_{x=0}^{\infty}\left[x(x-1)\right]\frac{e^{-\lambda}\lambda^x}{x!}+\sum_{x=0}^{\infty} x\frac{e^{-\lambda}\lambda^x}{x!}$$

$$=\sum_{x=0}^{\infty}\frac{x(x-1)e^{-\lambda}\lambda^2\lambda^{x-2}}{x(x-1)(x-2)\ldots\ldots\ldots\ldots 1}+\lambda$$

$$=e^{-\lambda}\lambda^2\sum_{x=1}^{\infty}\frac{\lambda^{x-2}}{(x-2)!}+\lambda$$

$$=e^{-\lambda}\lambda^2\left[1+\lambda+\frac{\lambda^2}{2!}+------\right]+\lambda$$

$$=e^{-\lambda}\lambda^2 e^{\lambda}+\lambda$$

$$=\lambda^2+\lambda$$

Variance = $E(X^2)-\left[E(X)\right]^2$

$$=\lambda^2+\lambda-(\lambda)^2$$

$$=\lambda$$

For Poisson distribution

- Mean = $E(X)=\lambda$
- Variance = $Var(X)=\lambda$

Aliter: -

Using Moment Generating Function

$$E(X)=M_x{}'(0)$$

$$E(X^2)=M_x{}''(0)$$

$$M_x(t)=e^{\lambda(e^t-1)}=e^{-\lambda}e^{\lambda e^t}$$

$$M_x{}'(t)=\frac{d}{dt}(M_x(t))$$

$$=\lambda e^t e^{-\lambda}e^{\lambda e^t}$$

$$M_x''(t) = \frac{d^2}{dt^2}\left(M_x(t)\right)$$
$$= \frac{d}{dt}\left(M_x'(t)\right)$$
$$= \lambda e^t e^{-\lambda} e^{\lambda e^t} + \lambda e^t e^{-\lambda} e^{\lambda e^t} \lambda e^t$$
$$= \lambda e^t e^{-\lambda} e^{\lambda e^t} + \left(\lambda e^t\right)^2 e^{-\lambda} e^{\lambda e^t}$$

$$Mean = E(X) = M_x'(0) = \lambda e^{-\lambda} e^{\lambda} = \lambda$$
$$E\left(X^2\right) = M_x''(0) = \lambda e^{-\lambda} e^{\lambda} + \lambda^2 e^{-\lambda} e^{\lambda}$$
$$= \lambda^2 + \lambda$$

$$Variance = E\left(X^2\right) - \left[E(X)\right]^2 \; E\left(X^2\right) - \left[E(X)\right]^2$$
$$= \lambda^2 + \lambda - (\lambda)^2$$
$$= \lambda$$

Additive property of independent Poisson variates.

Sum of independent Poisson variates is also a Poisson variate

i.e., If $X_i \,(i = 1, 2, ..., n)$ are n independent Poisson variate with parameters $\lambda_i \,(i = 1, 2, ..., n)$ then $\sum_{i=1}^{n} X_i$ is also a Poisson variate with parameter $\sum_{i=1}^{n} \lambda_i$

Proof:

We know that the M.G.F of the Poisson variate X_i is given by

$$M_{X_i}(t) = e^{\lambda_i\left(e^t - 1\right)};\ i = 1, 2, ..., n$$
$$M_{X_1+X_2+X_3+.....+X_n}(t) = M_{X_1}(t).M_{X_2}(t)........M_{Xn}(t)$$
$$= e^{\lambda_1\left(e^t - 1\right)}.e^{\lambda_2\left(e^t - 1\right)}.e^{\lambda_3\left(e^t - 1\right)}........e^{\lambda_n\left(e^t - 1\right)}$$
$$= e^{\left(\lambda_1 + \lambda_2 + \lambda_3 + ... + \lambda_n\right)\left(e^t - 1\right)}$$

Additive property of independent Poisson variates:

Sum of independent Poisson variates is also a Poisson variate.

Recurrence relation for Poisson distribution

The probability law for the Poisson distribution is

$$P(X=x)=P(x)$$

$$P(X=x)=\frac{e^{-\lambda}\lambda^{x}}{x!}, x=0,1,2,....,\infty$$

$$P(X=x+1)=P(x+1)=\frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!}$$

$$\frac{P(X=x+1)}{P(X=x)}=\frac{\frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!}}{\frac{e^{-\lambda}\lambda^{x}}{x!}}=\frac{\lambda^{x+1}x!}{\lambda^{x}(x+1)!}$$

$$=\frac{\lambda}{x+1}$$

$$P(X=x+1)=\left(\frac{\lambda}{x+1}\right)P(X=x)$$

Recurrence relation for Poisson distribution:

$$P(X=x+1)=\left(\frac{\lambda}{x+1}\right)P(X=x)$$

By using this relation, we can find the frequencies of any Poisson variate starting with

$$P(X=0)=e^{-\lambda}$$

Example:1

If X is a Poisson variate such that $P(X=1)=\frac{3}{10}, P(X=2)=\frac{1}{5}$, find $P(X=0)$, $P(X=3)$

Solution: $P(X=x)=\frac{e^{-\lambda}\lambda^{x}}{x!}$

$$P(X=1)=\frac{e^{-\lambda}\lambda}{1!}=\frac{3}{10}$$

$$P(X=2)=\frac{e^{-\lambda}\lambda^{2}}{2!}=\frac{1}{5}$$

$$e^{-\lambda}\lambda=\frac{3}{10}--------(1)$$

$$e^{-\lambda}\lambda^2 = \frac{2}{5} \text{-------} (2)$$

$$\frac{(2)}{(1)} \Rightarrow \frac{1}{\lambda} = \frac{3}{4}$$

$$\lambda = \frac{4}{3}$$

$$P(X=0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-4/3}$$

$$P(X=3) = \frac{e^{-\lambda}\lambda^3}{3!} = \frac{e^{-4/3}\left(-4/3\right)^3}{3!}$$

Example:2

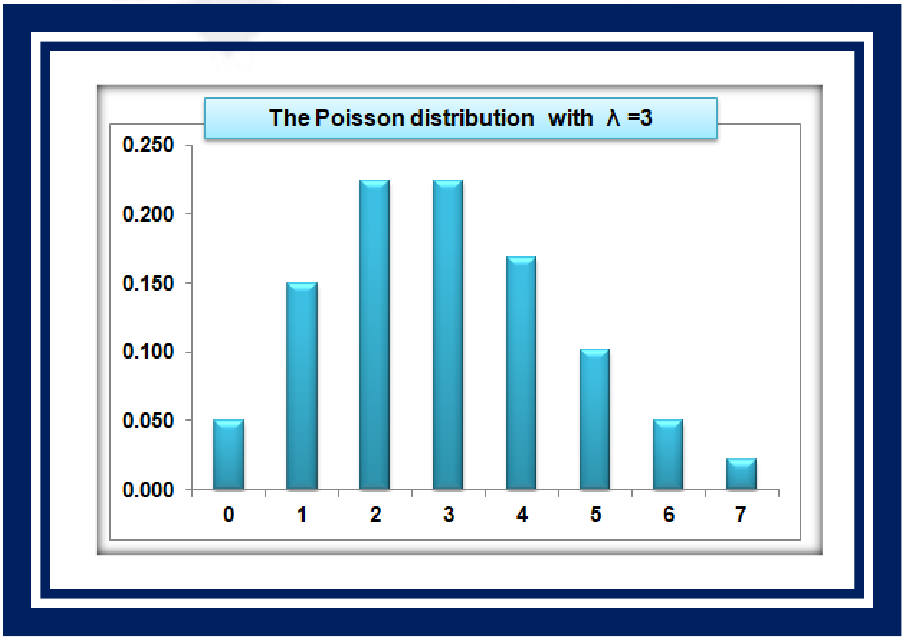
If the number of accidents occurring on a highway each day is a Poisson random variable with parameter $\lambda = 3$, what is the probability that no accidents occur today?

Solution: Given $\lambda = 3$

$$\text{We know that } P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$P(\text{No accident occur today}) = P(X=0)$$

$$= \frac{e^{-3}3^0}{0!} = e^{-3} = 0.05$$



Example:3

If X is a Poisson variate $P(X=2)=9P(X=4)+90P(X=6)$ find

(i) Mean of X

(ii) Variance of X.

Solution: $P(X=x)=\frac{e^{-\lambda}\lambda^{x}}{x!}, x=0,1,2,....,\infty$

$$P(X=2)=9P(X=4)+90P(X=6)$$

$$\frac{e^{-\lambda}\lambda^{2}}{2!}=9\frac{e^{-\lambda}\lambda^{4}}{4!}+90\frac{e^{-\lambda}\lambda^{6}}{6!}$$

$$=e^{-\lambda}\lambda^{2}\left(\frac{9\lambda^{2}}{4!}+\frac{90\lambda^{4}}{6!}\right)$$

$$\Rightarrow \frac{1}{2}=\frac{9\lambda^{2}}{4!}+\frac{90\lambda^{4}}{6!}$$

$$\frac{1}{2}=\frac{3\lambda^{2}}{8}+\frac{\lambda^{4}}{8}$$

$$\lambda^{4}+3\lambda^{2}-4=0$$

$$\lambda^{2}=1 \ or \ \lambda^{2}=-4$$

Therefore, $\lambda=\pm 1 \ or \ \lambda=\pm 2i$

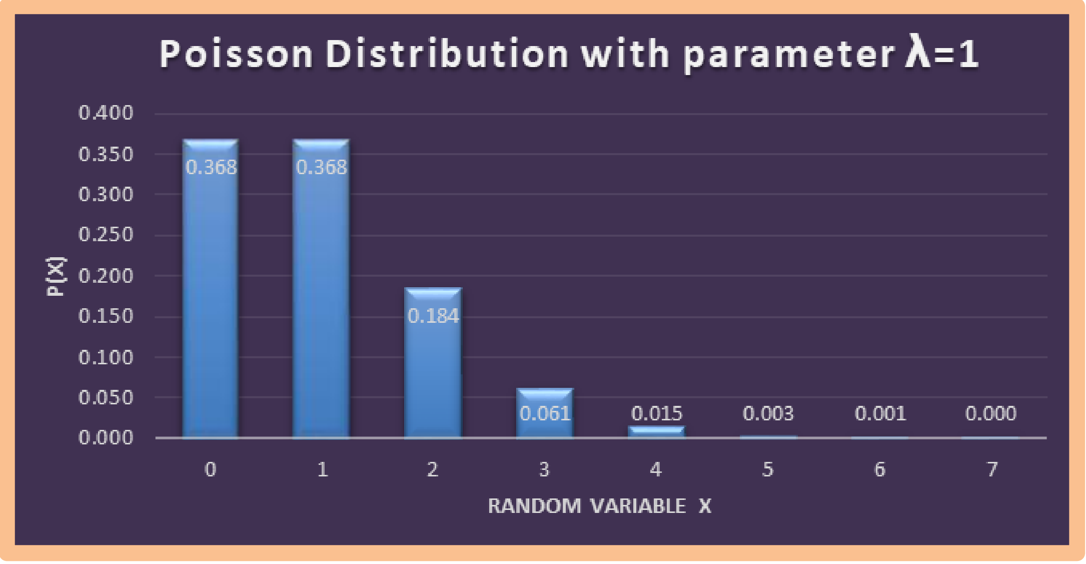
The Parameter λ is always positive real value.

Hence, the possible value is $\lambda=1$ only.

Therefore, Mean = $\lambda=1$

Variance = $\lambda=1$

Standard deviation =1



Example:4

If 3% of the electric bulbs manufactured by a company is defective, find the probability that in a sample of 100 bulbs exactly 5 bulbs are defective $\left(e^{-3} = 0.0498\right)$

Solution:

Let X be the RANDOM VARIABLE denoting the number of defective bulbs

Given $p = \dfrac{3}{100} = 0.03$

$n = 100$

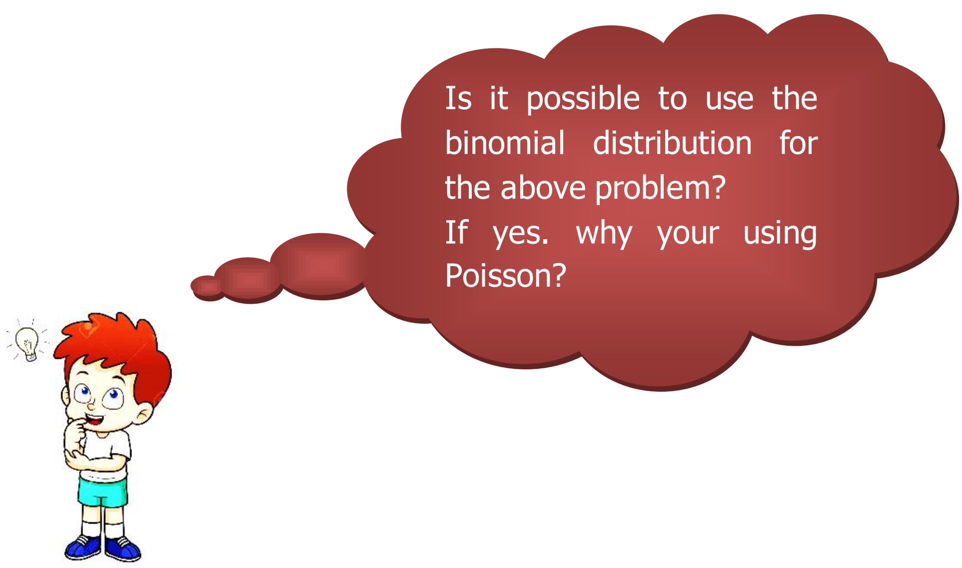
$\lambda = np = 100 \times 0.03 = 3$

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, x = 0,1,2,....,\infty$$

$$\text{P}(\textit{exactly 5 bulbs are defective}) = P(X = 5)$$

$$= \frac{e^{-3}3^{5}}{5!}$$

$$= 0.1008$$



Example:5

A manufacturer of nuts knows that 2% of his products are defective. If he sells the nuts in boxes of 100 and guarantees that not more than 4 nuts will be defective.

What is the probability that a box will fail to meet the guaranteed quality? $\left(e^{-2}=0.13534\right)$

Solution: Let X be the random variable denoting the defective nuts.

$$p=\frac{2}{100}=0.02$$

$$n=100$$

$$\lambda=np=100\times 0.02=2$$

$$P(X=x)=\frac{e^{-\lambda}\lambda^{x}}{x!},x=0,1,2,....,\infty$$

P(Not more than 4 nuts will be defective) = $P(X\le 4)$

$$P(X\le 4)=P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)$$

$$=e^{-2}\left[\frac{2^0}{0!}+\frac{2^1}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}\right]$$

$$=e^{-2}\left[1+2+2+\frac{4}{3}+\frac{2}{3}\right]$$

$$=e^{-2}.7$$

$$=0.9473$$

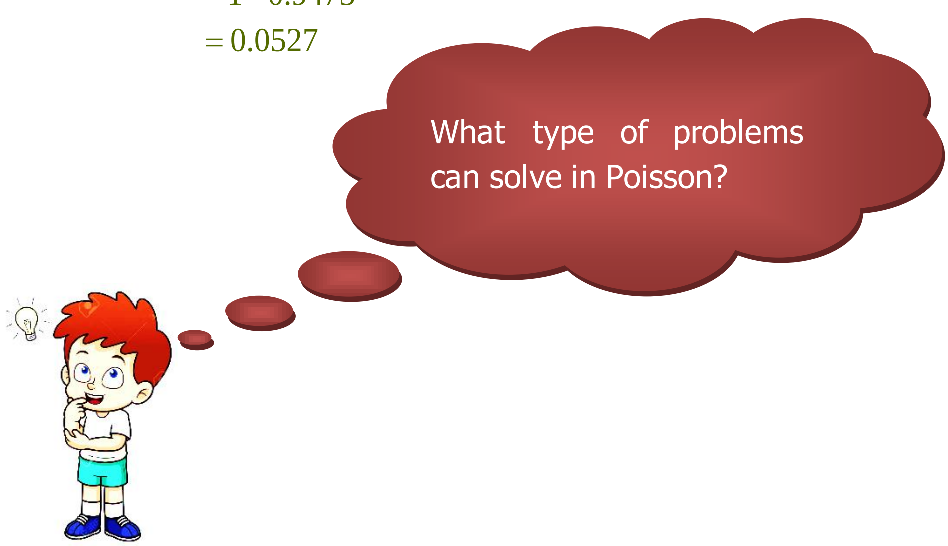
P (A box will fail to meet the guarantee quality)

=P(More than 4 nuts will be defective)

$$P(X>4)=1-P(X\le 4)$$

$$=1-0.9473$$

$$=0.0527$$



Example:6

Six coins are tossed 6400 times. Using the Poisson distribution, what is the approximate probability of getting six heads in 10 times?

Solution:

Let X be the RANDOM VARIABLE. denoting the number of heads.

Given $n = 6400$

Probability of getting one head with one coin= $\frac{1}{2}$

The probability of getting six heads with six coins $= \left(\frac{1}{2}\right)^6 = \frac{1}{64}$

Mean $\lambda = np = 6400 \times \frac{1}{64} = 100$

The Poisson distribution is $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0,1,2,....,\infty$

$$P(X = x) = \frac{e^{-100}(100)^x}{x!}$$

Probability of getting 6 heads in 10 times

$$P(X = 10) = \frac{e^{-100}(100)^{10}}{10!}$$

Example:7

Suppose that the number of typographical errors on a single page of this book has a Poisson distribution with parameter $\lambda = 1$. Calculate the probability that there is at least one error on this page.

Solution: Given $\lambda = 1$

$$\text{We know that } P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$P(X \geq 1) = 1 - P(X < 0)$$

$$= 1 - \frac{e^{-1}1^0}{0!} = 1 - e^{-1} = 0.633$$

Practice Problems:

1. Write down the probability mass function of the Poisson distribution which is approximately equivalent to $B(100, 0.02)$

2. In turning pot certain toys in a manufacturing process in a factory, the average number of defectives is 10%. What is the probability of getting exactly three defective toys in a sample of 10 toys chosen at random by using Poisson distribution?

3. At a busy traffic junction the probability of an individual having an accident is p=0.01. However during a certain part of the day 100 cars passing through the junction. What is the probability that two or more accidents occur during that period?

4. If the probability of a bad reaction from a certain infection is 0.001, what is the chance that out of 2000 individuals more than two will get a bad reaction?

5. A manufacturer knows that the condenser he makes contains on an average of 1% of defective. He packs them in boxes of 100. What is the probability that a box picked at random will contain 3 or more faulty condensers?

Answers:

1. $\dfrac{e^{-2}(2)^x}{x!}, x = 0, 1, 2, ...$
2. 0.61313
3. $\lambda = 10,\ P(X \geq 2) = 0.9995$
4. 0.323
5. 0.08025

Reference:

To know more about Poisson distribution

- https://www.intmath.com/counting-probability/13-poisson-probability-distribution.php
- https://mathworld.wolfram.com/PoissonDistribution.html
- https://stattrek.com/probability-distributions/poisson.aspx
- https://www.onlinemathlearning.com/poisson-distribution.html

Activity:

By Using the Poisson distribution calculator verify the problems you solved.

- https://stattrek.com/online-calculator/poisson.aspx

Play the quiz and gain the idea to solve Poisson distribution problem

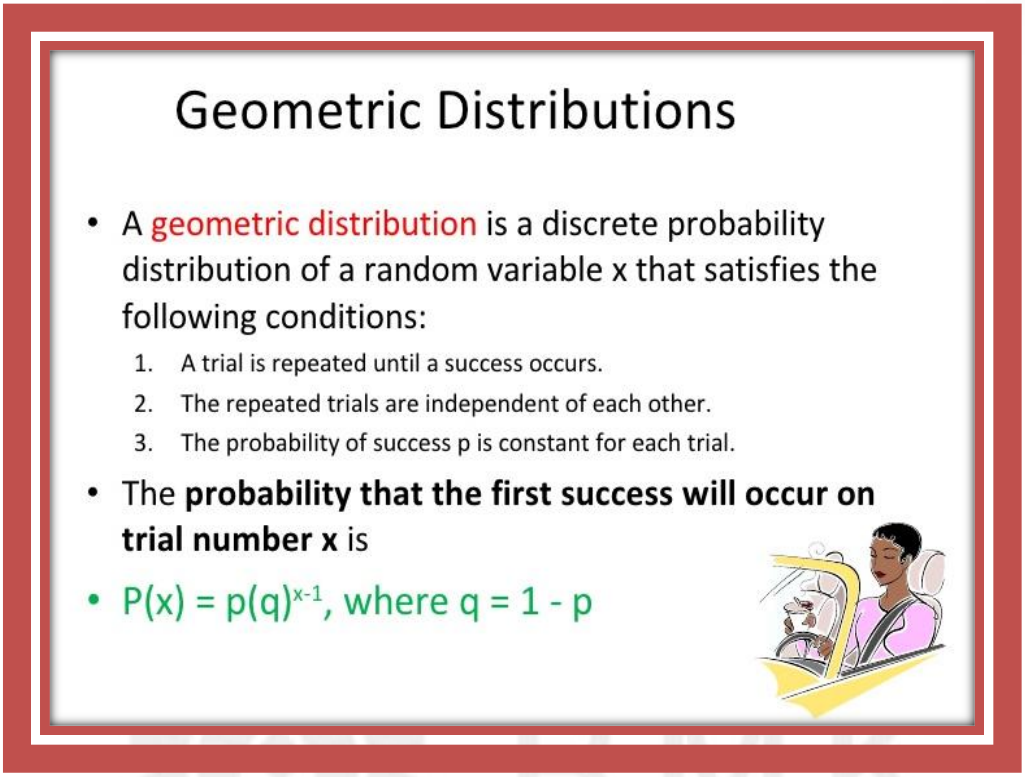
- https://testbook.com/objective-questions/mcq-on-poisson-distribution--5eea6a0a39140f30f369dc67

videos:

To know more view the following videos

- https://www.youtube.com/watch?v=jmqZG6roVqU
- https://www.youtube.com/watch?v=8MpgZJHcB8w
- https://www.youtube.com/watch?v=zA7fp2s7FlM

1.9 Geometric distribution



Suppose we have a series of independent trials or repetitions and on each repetition or trial the probability of success 'p' remains the same. Then the probability that there are 'x' failures preceding the first success is given by $q^x p = (1-p)^x p$, here 'q' denotes the probability of failure.

Geometric distribution

A random variable X is said to follow geometric distribution if it assumes only non-negative values and its probability mass function is given by

$$P(X=x) = p(x) = (1-p)^{x-1} p = q^{x-1} p,$$

where $x = 1, 2, 3, \ldots$, $0 < p \leq 1$ and $q = 1-p$.

Note: Another form of the above definition is $P(X=x) = p(x) = (1-p)^x p$ where $x = 0, 1, 2, \ldots$ *and* $0 < p \leq 1$.

Why the name is geometric distribution?

Since the various probabilities for $x = 0, 1, 2....$, are the various terms of geometric progression, hence the name geometric distribution.

Remarks: 1 $p(n)$ is a probability mass function of geometric distribution, then $\sum_{n=1}^{\infty} P(n) = 1$

Proof:

$$\sum_{n=0}^{\infty} P(n) = p(1) + p(2) + p(3) + \cdots\cdot$$

$$= (1-p)^0 p + (1-p)^1 p + (1-p)^2 p + (1-p)^3 p + \cdots$$

$$= p\left(1 + (1-p)^1 + (1-p)^2 + (1-p)^3 + \cdots\right)$$

$$= p\left(1 + (1-p)^1 + (1-p)^2 + (1-p)^3 + \cdots\right)$$

$$= p\left(1-(1-p)\right)^{-1}$$

$$= p\left(\frac{1}{1-(1-p)}\right)$$

$$= p\left(\frac{1}{p}\right)$$

$$= 1$$

$p(x)$ is a probability mass function of geometric distribution, then $\sum_{n=1}^{\infty} P(x) = 1$.

Moment generating function of geometric distribution

Let $p(x)$ is a probability mass function of geometric distribution with parameter 'p' then $p(x)=q^{x-1}p$ where $x=1,2,3,\ldots$, $0<p\le 1$ and $q=1-p$.

The moment generating function is

$$M_X(t)=E\left[e^{tx}\right]=\sum_{x=1}^{\infty}e^{tx}p(x)$$

$$=\sum_{x=1}^{\infty}e^{tx}\cdot e^{-t}\cdot e^{t}\cdot q^{x-1}p$$

$$=\sum_{x=1}^{\infty}e^{t(x-1)}\cdot e^{t}\cdot q^{x-1}p$$

$$=e^{t}p\sum_{x=1}^{\infty}\left(qe^{t}\right)^{x-1}$$

$$=e^{t}p\left(\left(qe^{t}\right)^{1-1}+\left(qe^{t}\right)^{2-1}+\left(qe^{t}\right)^{3-1}+\left(qe^{t}\right)^{4-1}+\cdots\right)$$

$$=e^{t}p\left(1+\left(qe^{t}\right)^{1}+\left(qe^{t}\right)^{2}+\left(qe^{t}\right)^{3}+\cdots\right)$$

$$=e^{t}p\left(1-qe^{t}\right)^{-1}$$

$$=e^{t}p\frac{1}{1-qe^{t}}$$

$$M_X(t)=\frac{pe^{t}}{1-qe^{t}}.$$

Moment Generating Function of geometric distribution is

$$M_X(t)=\frac{pe^{t}}{1-qe^{t}}$$

Mean and variance of geometric distribution

We know that the *m.g.f.* of geometric distribution is

$$M_X(t)=\frac{pe^{t}}{1-qe^{t}}$$

$$mean=E[X]=\mu_1'=\frac{d}{dt}\left[M_X(t)\right]_{t=0}$$

$$\text{var}\,ience=\mu_2'-(\mu_1')^2 \text{ where } \mu_2'=\frac{d^2}{dt^2}\left[M_X(t)\right]_{t=0}$$

Now,

$$\mu_1' = \frac{d}{dt}\left[\frac{pe^t}{1-qe^t}\right]_{t=0}$$

$$= \left[\frac{\left(1-qe^t\right)pe^t - pe^t\left(-qe^t\right)}{\left(1-qe^t\right)^2}\right]_{t=0}$$

$$= p\left[\frac{\left(1-qe^t\right)e^t - e^t\left(-qe^t\right)}{\left(1-qe^t\right)^2}\right]_{t=0}$$

$$= p\left[\frac{e^t - qe^{2t} + qe^{2t}}{\left(1-qe^t\right)^2}\right]_{t=0}$$

$$= p\left[\frac{e^t}{\left(1-qe^t\right)^2}\right]_{t=0} \qquad \text{-------------->}\ (1)$$

$$= p\left[\frac{e^0}{\left(1-qe^0\right)^2}\right]$$

$$= p\left[\frac{1}{\left(1-q\right)^2}\right]$$

$$= p\left[\frac{1}{\left(1-\left(1-p\right)\right)^2}\right]$$

$$= p\left[\frac{1}{p^2}\right]$$

$$= \frac{1}{p}$$

$$mean = \mu_1' = \frac{1}{p}.$$

$$\mu_2' = \frac{d^2}{dt^2}\left[M_X\left(t\right)\right]_{t=0}$$

$$= \frac{d}{dt}\left[\frac{d}{dt}\left[M_X\left(t\right)\right]\right]_{t=0}$$

$$= \frac{d}{dt}\left[\frac{pe^t}{\left(1-qe^t\right)^2}\right]_{t=0}$$

$$= p\left[\frac{\left(1-qe^t\right)^2 e^t - e^t 2\left(1-qe^t\right)\left(-qe^t\right)}{\left(1-qe^t\right)^4}\right]_{t=0}$$

$$= p\left[\frac{\left(1-qe^0\right)^2 e^0 - e^0 2\left(1-qe^0\right)\left(-qe^0\right)}{\left(1-qe^0\right)^4}\right]$$

$$= p\left[\frac{(1-q)^2 - 2(1-q)(-q)}{(1-q)^4}\right]$$

$$= p\left[\frac{1-2q+q^2+2q-2q^2}{(1-q)^4}\right]$$

$$= p\left[\frac{1-q^2}{(1-q)^4}\right]$$

$$= p\left[\frac{(1-q)(1+q)}{(1-q)^4}\right]$$

$$= p\left[\frac{1+q}{(1-q)^3}\right]$$

$$= p\left[\frac{1+q}{(1-(1-p))^3}\right]$$

$$= p\left[\frac{1+q}{p^3}\right]$$

$$= \frac{1+q}{p^2}$$

$$\mu_2' = \frac{1+q}{p^2}$$

$$\text{var}\,ience = \frac{1+q}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{q}{p^2}$$

For geometric distribution

$$mean = E[X] = \mu_1' = \frac{1}{p} \qquad \text{var}\,ience = \text{var}[X] = \frac{q}{p^2}$$

Memoryless property of geometric distribution:

Statement: If X is a random variable with geometric distribution then X lacks memory in the sense that $P(X > m+n / X > m) = P(X > n)$ for any value $m, n > 0$

Proof: We know that the geometric distribution is $P(X = x) = p(x) = q^{x-1}p$, here $x = 1, 2, 3, \ldots$, $0 < p \le 1$ and $q = 1 - p$.

Now, $P(X > m+n) = \sum_{x=m+n+1}^{\infty} p(x)$

$$= \sum_{x=m+n+1}^{\infty} q^{x-1} p$$

$$= q^{(m+n+1)-1} p + q^{(m+n+2)-1} p + q^{(m+n+3)-1} p + \cdots$$

$$= \frac{p}{q}\left(q^{(m+n+1)} + q^{(m+n+2)} + q^{(m+n+3)} + \cdots\right)$$

$$= \frac{pq^{(m+n+1)}}{q}\left(1 + q + q^2 + \cdots\right)$$

$$= pq^{m+n}\left(1 + q + q^2 + \cdots\right)$$

$$= pq^{m+n}(1-q)^{-1}$$

$$= pq^{m+n}\frac{1}{(1-q)}$$

$$= pq^{m+n}\frac{1}{p}$$

$$= q^{m+n}$$

$$P(X > m+n) = q^{m+n}$$

Similarly,

$$P(X > m) = q^m \text{ and } P(X > n) = q^n$$

Therefore

$$P(X > m+n / X > m) = \frac{P\left((X > m+n) \cap (X > m)\right)}{P(X > m)}$$

$$= \frac{P(X > m+n)}{P(X > m)}$$

$$= \frac{q^{m+n}}{q^m}$$

$$= q^n$$

$$= P(X > n)$$

$$P(X > m+n / X > m) = P(X > n).$$

This means that the first m trials have no success, the conditional probability that the first success will appear after an additional n trials depends only on n and not on the past m trials.

Memoryless property of geometric distribution

If X is a random variable with geometric distribution then X lacks memory in the sense that $P(X > m+n / X > m) = P(X > n)$ for any value $m, n > 0$.

Example 1

A die is tossed until 6 appears. What is the probability that it must be tossed more than 5 times?

Solution: Let X be the number of tasks required to get the first 6.

The probability of getting 6 = $\frac{1}{6}$

$$p = \frac{1}{6} \Rightarrow q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

$$P(X = x) = q^{x-1} p = \left(\frac{5}{6}\right)^{x-1}\left(\frac{1}{6}\right), \ x = 1, 2, 3, \ldots$$

$$\begin{aligned}
P(X > 5) &= 1 - P(X \le 5) \\
&= 1 - \left[P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)\right] \\
&= 1 - \left[q^{1-1}p + q^{2-1}p + q^{3-1}p + q^{4-1}p + q^{5-1}p\right] \\
&= 1 - p\left[1 + q^1 + q^2 + q^3 + q^4\right] \\
&= 1 - \frac{1}{6}\left[1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \left(\frac{5}{6}\right)^4\right] \\
&= 1 - \frac{1}{6}[1 + 0.833 + 0.6944 + 0.578 + 0.4823] \\
&= 1 - 0.5981
\end{aligned}$$

$$P(X > 5) = 0.4019$$

Example 2

A box contains **M** white and **N** Block toys. Toys are randomly selected one at a time until one block is obtained. If we assume that the selected toy is replaced before the next draw, find the probability that it

(i) exactly n draws are needed,

(ii) at least k draws are needed.

Solution:

Let X denote the number of draws required to select a Black toy.

Then

$$P(X=x)=q^{x-1}p \text{, where } x=1,2,3,\ldots$$

Here $p=\dfrac{M}{M+N}$ and $q=\dfrac{N}{M+N}$

(i) $$P(X=n)=q^{n-1}p$$

$$=\left(\frac{N}{M+N}\right)^{n-1}\frac{M}{M+N}$$

$$=\frac{MN^{n-1}}{(M+N)^{n-1}}$$

(ii) $$P(X\geq k)=\sum_{x=k}^{\infty}q^{x-1}p$$

$$=p\left(q^{k-1}+q^{k+1-1}+q^{k+2-1}+q^{k+3-1}+\cdots\right)$$

$$=pq^{k-1}\left(1+q+q+q^{2}+\cdots\right)$$

$$=pq^{k-1}(1-q)^{-1}$$

$$=pq^{k-1}\frac{1}{(1-q)}$$

$$=pq^{k-1}\frac{1}{p}$$

$$=q^{k-1}$$

$$=\left(\frac{N}{M+N}\right)^{k-1}$$

Example 3

A soldier shoots a target in an independent fashion. If the probability that the target is shot on any one of shot is 0.8.

(i) What is the probability that a target would be first hit at the 6th attempt?

(ii) What is the property that it takes less than 5 shots?

Solution: Here P(hitting the target target)=0.8

$$p = 0.8 \Rightarrow q = 1 - 0.8 = 0.2$$

Also

$$P(X = x) = q^{x-1}p, \text{ where } x = 1, 2, 3, \ldots$$

(i) P(target is hit on 6th attempt)

$$P(X = 6) = q^{6-1}p$$

$$= (0.2)^5(0.8)$$

$$= 0.00026$$

(ii) P(hitting in less than 5 shots)

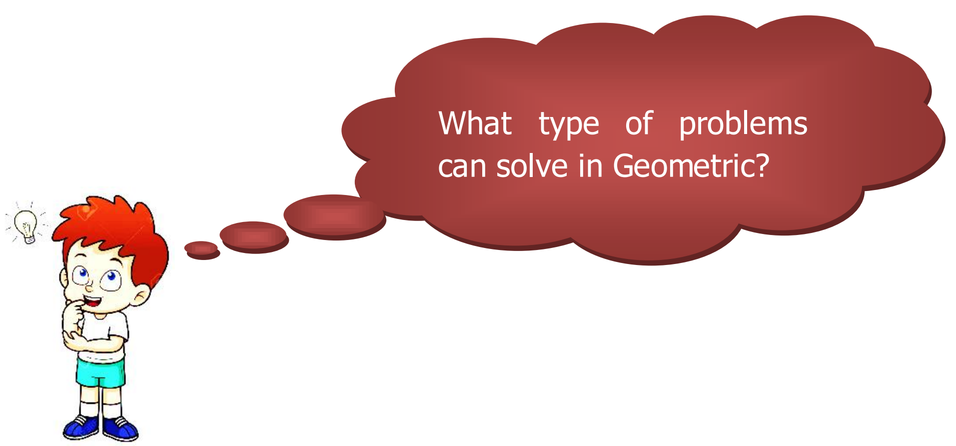
$$P(X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$

$$= q^{1-1}p + q^{2-1}p + q^{3-1}p + q^{4-1}p$$

$$= (0.2)^0(0.8) + (0.2)^1(0.8) + (0.2)^2(0.8) + (0.2)^3(0.8)$$

$$= 0.8\left[(0.2)^0 + (0.2)^1 + (0.2)^2 + (0.2)^3\right]$$

$$= 0.9984$$



Example 4

If the probability of success on each trial is $\frac{1}{3}$. What is the expected number of trials required for the first success?

Solution: Let X be the number of trials required for the first success. Then X follows geometric distribution with

$P(X = x) = q^{x-1}p$, where $x = 1, 2, 3, \ldots$

Here $p = \frac{1}{3}$ and $q = 1 - \frac{1}{3} = \frac{2}{3}$.

For geometric distribution

$$E[X] = \frac{1}{p} = \frac{1}{1/3} = 3.$$

Example 5

A man with n keys wants to open his door and tries the keys independently at random. Find the mean and variance of the number of trials required to open the door,

(i) if unsuccessful keys are not eliminated from further selection, and

(ii) if they are eliminated from further section.

Solution:

(i) Let the first success be got at x^th the trial and not before. Then the random variable X (denoting number of trials required to open the door) follows geometric distribution with

$$P(X = x) = q^{x-1}p, \text{ where } x = 1, 2, 3, \ldots$$

Here $p = \frac{1}{n}$ and $q = 1 - \frac{1}{n}$.

For geometric distribution

Mean= $E[X] = \frac{1}{p} = \frac{1}{1/n} = n$.

Variance = $\frac{q}{p^2} = \frac{\left(1 - \frac{1}{n}\right)}{\left(\frac{1}{n}\right)^2} = \left(\frac{n-1}{n}\right)n^2 = n(n-1)$

(ii) Let X be the number of trials required to open the door. If unsuccessful keys are eliminated, then X will take the values $1,2,3,\ldots,n$

$$P(\text{success in the 1st trial}) = \frac{1}{n}.$$

$$P(\text{first success in the 2nd trial}) = P(\text{failure in the 1st trial}) \text{ x } P(\text{success in the 2nd trial})$$

$$= \left(1-\frac{1}{n}\right)\left(\frac{1}{n-1}\right) = \frac{1}{n}.$$

Similarly,

$$P(\text{first success in the } 3^{rd} \text{ trial}) = \left(1\text{-}\frac{1}{\text{n}}\right)\left(1-\frac{1}{n-1}\right)\left(\frac{1}{n-2}\right)$$

$$= \frac{1}{n}$$

and so on (since in r^{th} trial there are n-r+1 keys)

$$P(X = x) = \text{probability of first success in the } x^{th} \text{ trial} = \frac{1}{n}$$

Example 6

If the probability the applicant for a driver's license will pass the road test on any given trial is 0.8, what is the probability that he will finally pass the test

(i) on the 4^{th} trial?

(ii) in fewer than 4 trial?

Solution:

Here $P(X = x) = q^{x-1}p$, where $x = 1,2,3,\ldots$, and $p = 0.8$ and $q = 1-0.8 = 0.2$.

$$P(X = x) = q^{x-1}p = (0.2)^{x-1}(0.8)$$

(i) P(on the 4^{th} trial)

$$P(X = 4) = (0.2)^{4-1}(0.8) = 0.0064$$

(ii) P(in fewer than 4 trial)

$$P(X < 4) = P(X = 1) + P(X = 2) + P(X = 3)$$

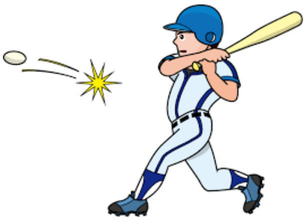
$$= q^{1-1}p + q^{2-1}p + q^{3-1}p$$

$$= (0.2)^0(0.8) + (0.2)^1(0.8) + (0.2)^2(0.8)$$

$$= 0.8\left[(0.2)^0 + (0.2)^1 + (0.2)^2\right]$$

$$= 0.992$$

Example 7



A baseball player has a 30% chance of getting a hit on any given pitch. Ignoring balls, what is the probability that the player earns a hit before he strikes out (which requires three strikes)?

Solution:

In this instance, a success is a hit and a failure is a strike. The player needs to have either 0, 1, or 2 failures in order to get a hit before striking out, so the probability of a hit is

Here $P(X=x)=q^{x-1}p$, where $x=1,2,3,\ldots$, and $p=0.3$ and $q=1-0.3=0.7$.

$$P(X=x)=q^{x-1}p=(0.7)^{x-1}(0.3)$$

$$\begin{aligned}P(X\le 3)&=P(X=1)+P(X=2)+P(X=3)\\&=q^{1-1}p+q^{2-1}p+q^{3-1}p\\&=(0.7)^0(0.3)+(0.7)^1(0.3)+(0.7)^2(0.3)\\&=0.657\end{aligned}$$

Problems for practice:

1. If the probability that the target is destroyed any one shot is 0.5. What is the probability that it would destroy on seventh attempt?

2. Suppose that a trainee soldier shoots a target in an independent fashion. If the probability that the target is shot on any one shot is 0.7,
   (i) what is the probability that the target would be hit on 10th attempt?
   (ii) what is the probability that it takes him less than 4 shoots?
   (iii) what is the probability that it takes him and even number of shots?

3. A test of weld strength involves loading welded joints until a fracture occurs. For a certain type of weld, 80% of the fractures occur in the weld itself, while the other 20% occur in the beam. A number of welds are tested and the tests are independent. Find the probability that the first beam fracture happens on the third trial or later.

4. You play a game of chance that you can either win or lose (there are no other possibilities) until you lose. Your probability of losing is p=0.57. What is the probability that it takes five games until you lose?

Answers:

1. 0.0078
2. (i)0.0000138 (ii)0.973 (iii)0.2308
3. 0.64
4. 0.9853

Reference:

To know more about geometric distribution

- https://online.stat.psu.edu/stat414/lesson/11/11.1

Activity:

By Using the geometric distribution calculator verify the problems you solved

- https://homepage.divms.uiowa.edu/~mbognar/applets/geo1.html
- https://keisan.casio.com/exec/system/1180573193

Enjoy the quiz by clicking the link

- https://study.com/academy/practice/quiz-worksheet-what-is-the-geometric-distribution.html

videos:

To know more view the following videos

- https://www.youtube.com/watch?v=zq9Oz82iHf0
- https://www.youtube.com/watch?v=x29A26UrPkA

Continuous Distribution

Continuous distributions are characterized by an infinite number of possible outcomes, together with the probability of observing a range of these outcomes.

A probability density function lists each range of values along with the probability that an observed value will fall within that range.

1.10. Uniform distribution (or) Rectangular distribution

Definition:

A continuous random variable X is said to follow a uniform distribution over an interval (a,b) if its probability density function is given by $f(x)=\begin{cases} k, a<x<b \\ 0, \text{ otherwise} \end{cases}$, where k is constant. a & b are called the parameters of this distribution.

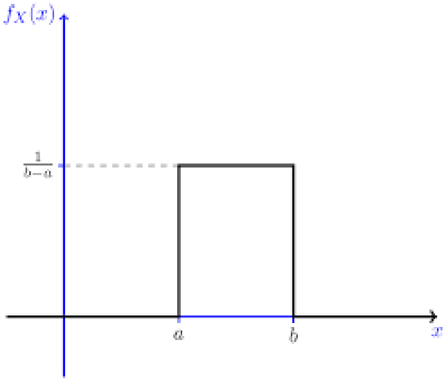
Since $f(x)$ is a p.d.f. $\int_a^b f(x)dx=1$

$$\Rightarrow k(b-a)=1$$

$$\Rightarrow k=\frac{1}{b-a}$$

Therefore the distribution on an interval (a,b) has pdf $f(x)=\begin{cases} \frac{1}{b-a}, a<x<b \\ 0, \text{ otherwise} \end{cases}$

Uniform distribution on an interval (a,b) has pdf $f(x)=\begin{cases} \frac{1}{b-a}, a<x<b \\ 0, \text{ otherwise} \end{cases}$



Moment Generating Function

$$M_X(t)=E\left(e^{tX}\right)=\int_a^b e^{tx}f(x)dx$$

$$=\int_a^b e^{tx}\frac{1}{b-a}dx$$

$$=\frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b$$

$$=\frac{1}{t(b-a)}\left(e^{tb}-e^{ta}\right), \qquad t\neq 0$$

Mean and Variance of uniform distribution

$$E\left(X\right)=\int_a^b xf(x)dx$$

$$=\int_a^b x\frac{1}{b-a}dx$$

$$=\frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$=\frac{1}{2(b-a)}\left(b^2-a^2\right)$$

$$=\frac{b+a}{2}$$

$$E\left(X^2\right)=\int_a^b x^2f(x)dx$$

$$=\int_a^b x^2\frac{1}{b-a}dx$$

$$=\frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$=\frac{1}{3(b-a)}\left(b^3-a^3\right)$$

$$=\frac{b^2+ab+a^2}{3}$$

Therefore

$$Var(X)=E\left(X^2\right)-\left[E\left(X\right)\right]^2$$

$$=\frac{a^2+ab+b^2}{3}-\frac{\left(a+b\right)^2}{4}$$

$$=\frac{1}{12}\left(4a^2+4ab+4b^2-3a^2-3ab-3b^2\right)$$

$$=\frac{1}{12}\left(a-b\right)^2$$

Uniform distribution	
MGF	$\frac{1}{t(b-a)}\left(e^{tb}-e^{ta}\right)$
Mean	$\frac{b+a}{2}$
Variance	$\frac{(a-b)^2}{12}$
Standard deviation	$\sqrt{\frac{(a-b)^2}{12}}$

Distribution function Of Uniform distribution:

We know that

$$F(x)=P(X \le x)$$

$$=\int_{-\infty}^{x} f(x)dx$$

$$=\int_{a}^{x}\frac{1}{b-a}dx$$

$$=\frac{x-a}{b-a}, \quad a<x<b$$

$$F(x) = \begin{cases} 0 \text{ if } x<a \\ \frac{x-a}{b-a} \text{ if } a<x<b \\ 1 \quad \text{if } x \ge b \end{cases}$$

Example:1

If X is uniformly distributed over $(0,10)$ find (i)

$(i) P(X<4)$, $(ii) P(X>6)$, $(iii) P(2<X<5)$

Solution:

X is uniformly distributed over $(0,10)$

Therefore, p.d.f. is $f(x)=\begin{cases}\frac{1}{10}, 0<x<10\\ 0,\ otherwise\end{cases}$

$(i)\ P(X<4)=\int_0^4 \frac{1}{10}dx=\frac{4}{10}=\frac{2}{5}$

$(ii)\ P(X>6)=\int_6^{10} \frac{1}{10}dx=\frac{4}{10}=\frac{2}{5}$

$(iii)\ P(2<X<5)=\int_2^5 \frac{1}{10}dx=\frac{3}{10}$

Example:2

A random variable X has uniform distribution over $(-3,3)$, compute

$(i) P(X<2)$, $(ii) P(|X-2|<2)$, (iii) find K for which $P(X>K)=\frac{1}{3}$

Solution:

X is uniformly distributed over $(-3,3)$

Therefore, p.d.f. is $f(x)=\begin{cases}\frac{1}{6}, -3<x<3\\ 0,\ otherwise\end{cases}$

$(i)\ P(X<2)=\int_{-3}^{2} \frac{1}{6}dx=\frac{5}{10}$

$$(ii)\ P(|X-2|<2)=P(-2<X-2<2)$$
$$=P(0<X<4)$$
$$=\int_0^3 \frac{1}{6}dx \qquad [\because -3<x<3]$$
$$=\frac{1}{6}[x]_0^3=\frac{3}{6}=\frac{1}{2}$$

$$(iii)\ P(X>3)=\frac{1}{3}$$
$$\int_k^3 f(x)dx=\frac{1}{3} \Rightarrow \quad \int_k^3 \frac{1}{6}dx=\frac{1}{3}$$
$$\frac{1}{6}[x]_k^3=\frac{1}{3} \Rightarrow \ 3-k=2$$
$$\Rightarrow k=1$$

Example:3

Subway trains on a certain line run every half hour between mid-night and six in the morning. What is the probability that a man entering the station at random time during this period will have to wait at least twenty minutes.

Solution:

Let X be the random variable which denote the waiting time for the next train. Assume that man arrives at the station at random, a random variable X is distributed uniformly in the interval $(0,30)$ with p.d.f. $f(x)=\begin{cases} \frac{1}{30}, 0<x<30 \\ 0, \ otherwise \end{cases}$

Therefore, the probability that a man entering has to wait at least 20 minutes is

$$P(X\geq 20)=\int_{20}^{30}\frac{1}{30}dx$$
$$=\frac{1}{30}[x]_{20}^{30}$$
$$=\frac{1}{3}$$

Example:4

Let X be uniformly distributed in(0,1). Calculate $E(X^3)$.

Solution: Given interval is (0,1).

The p.d.f. is defined by $f(x)=\frac{1}{b-a}=\frac{1}{1-0}=1$

$$\therefore f(x)=1$$

$$E(X^3)=\int_0^1 x^3 f(x)dx$$

$$=\int_0^1 x^3.1dx$$

$$=\left[\frac{x^4}{4}\right]_0^1=\frac{1}{4}$$

Example:5

Buses arrive at a specified stop at 15 minutes intervals starting at 7 a.m, that is, they arrive at 7,7.15,7.30,7.45 and so on. If passenger arrives at a stop at a random time that is uniformly distributed between 7 and 7.30, find the probability that he waits for

i)Less than 5 minutes for a bus

ii)More than 10 minutes for a bus

Solution:

Let X denote the number of minutes past7a.m that the passenger arrives at the stop

$$f(x)=\frac{1}{30},\ 0<x<30$$

i) Since X is uniformly distributed between 7 and 7.30, the passenger may arrive after

7.10 or 7.25. Hence

$$P[(10<X<15)\cup(25<X<30)] = P(10<X<15)+P(25<X<30)$$

$$= \int_{10}^{15}\frac{1}{30}dx + \int_{25}^{30}\frac{1}{30}dx$$

$$= \frac{1}{30}\left[(x)_{10}^{15}+(x)_{25}^{30}\right]$$

$$= \frac{1}{30}\left[(15-10)+(30-25)\right]$$

$$= \frac{1}{30}(10) = \frac{1}{3}$$

ii)Here the passenger has to arrive between 7and 7.05 or 7.15 and 7.20

$$P[(0<X<5)\cup(15<X<20)] = P(0<X<5)+P(15<X<20)$$

$$= \int_{0}^{5}\frac{1}{30}dx + \int_{15}^{20}\frac{1}{30}dx$$

$$= \frac{1}{30}\left[(x)_{0}^{5}+(x)_{15}^{20}\right]$$

$$= \frac{1}{30}\left[(5-0)+(20-15)\right]$$

$$= \frac{1}{30}(10) = \frac{1}{3}$$

Example:6

If X is uniformly distributed random variable with mean 1 and variance $\frac{4}{3}$, find $P(X<0)$.

Solution: From uniform distribution, we know that

$$\text{Mean} = \frac{a+b}{2} = 1 \Rightarrow a+b=2 \qquad \rightarrow (1)$$

$$\text{variance} = \frac{(b-a)^2}{12} = \frac{4}{3} \Rightarrow 3(b-a)^2 = 48$$

$$\Rightarrow (b-a)^2 = 16$$

$$\Rightarrow b-a = \pm 4 \qquad \rightarrow (2)$$

since, a<x<b, b-a cannot be -4. Hence, b-a=4

Solving (1) and (2), we get

a=-1, b=3

We know that the probability density function is

$$f(x)=\begin{cases}\frac{1}{b-a}, a<x<b \\ 0 \quad \text{otherwise}\end{cases} \Rightarrow \qquad =\begin{cases}\frac{1}{3-(-1)}, -1<x<3 \\ 0 \quad \text{otherwise}\end{cases}$$

$$=\begin{cases}\frac{1}{4}, -1<x<3 \\ 0 \quad \text{otherwise}\end{cases}$$

$$\therefore P(X<0)=\int_{-1}^{0} f(x)dx=\int_{-1}^{0}\frac{1}{4}dx=\frac{1}{4}[x]_{-1}^{0}$$

$$P(X<0)=\frac{1}{4}$$

Example:7

Let X be a uniformly distributed random variable in the interval $(a,9)$ and $P(3<X<5)=\frac{2}{7}$. i)Find 'a' ii)Compute $P(|X-5|<2)$.

Solution: Given X is a uniformly distributed random variable in $(a,9)$

$$\text{Hence } f(x)=\frac{1}{9-a}, \ a<x<9$$

$$i)\text{Given} \int_{3}^{5} f(x)dx=\frac{2}{7}\Rightarrow \qquad \int_{3}^{5}\frac{1}{9-a}dx=\frac{2}{7}$$

$$\Rightarrow \frac{1}{9-a}(x)_{3}^{5}=\frac{2}{7}$$

$$\frac{2}{9-a}=\frac{2}{7}\Rightarrow 9-a=7\Rightarrow \therefore a=2$$

$$\text{Hence } f(x)=\frac{1}{7}, \ 2<x<9$$

$$ii)P\left(|X-5|<2\right)=P\left(-2\le X-5\le 2\right)$$

$$=P\left(3\le X\le 7\right)=\int_{3}^{7}\frac{1}{7}dx \ =\frac{1}{7}(x)_{3}^{7}=\frac{4}{7}$$

Example:8

The amount of time, in minutes, that a person must wait for a bus is uniformly distributed between zero and 15 minutes, inclusive.

1. What is the probability that a person waits fewer than 12.5 minutes?
2. On the average, how long must a person wait? Find the mean, μ, and the standard deviation, σ.

Solution: Given $X \sim U(0,15)$.

The probability density function is

$$\text{Hence } f(x) = \frac{1}{b-a}, \ a < x < b$$

$$f(x) = \frac{1}{15}, \ 0 < x < 15$$

1. Let X= the number of minutes a person must wait for a bus.

$$P(X < 12.5) = \int_0^{12.5} f(x)dx$$

$$= \int_0^{12.5} \frac{1}{15} dx$$

$$= \frac{1}{15}[x]_0^{12.5} \Rightarrow = \frac{12.5}{15} = 0.8333$$

$$2. \ \mu = \frac{a+b}{2} = \frac{15}{2} = 7.5$$

On the average a person must wait 7.5 minutes.

$$\sigma = \sqrt{\frac{(b-a)^2}{12}} = \sqrt{\frac{225}{12}} = 4.3$$

Therefore, the standard deviation is 4.3 minutes.

Exercise:

1. Show that for the uniform distribution $f(x)=\begin{cases}\frac{1}{2a}, -a<x<a\\ 0, \text{ } otherwise\end{cases}$ the m.g.f. about origin is $\frac{\sinh at}{at}$
2. If X is uniformly distributed random variable with mean 1 and variance $\frac{4}{3}$, find $P(X<0)$
3. The number of personal computers sold daily at a compuworld is uniformly distributed with a minimum of 2000 and maximum of 5000 PC. Find
   (i) The probability that daily sales will fall between 2500 and 3000 PC.
   (ii) What is the probability that the compuworld will sell at least 4000 PC.
   (iii) What is the probability that the compuworld will exactly sell 2500 PC.
4. A statistics professor plans classes so carefully that the lengths of her classes are uniformly distributed between 49.0 and 54.0 minutes. Find the probability of selecting a class that runs less than 50.5 minutes.

Answers:

2. $\frac{1}{4}$ 3.(i) $\frac{1}{6}$,(ii) $\frac{1}{3}$, (iii) 0.4. 0.3

Reference:
To know more about Uniform distribution
- https://www.mathsisfun.com/data/random-variables-continuous.html

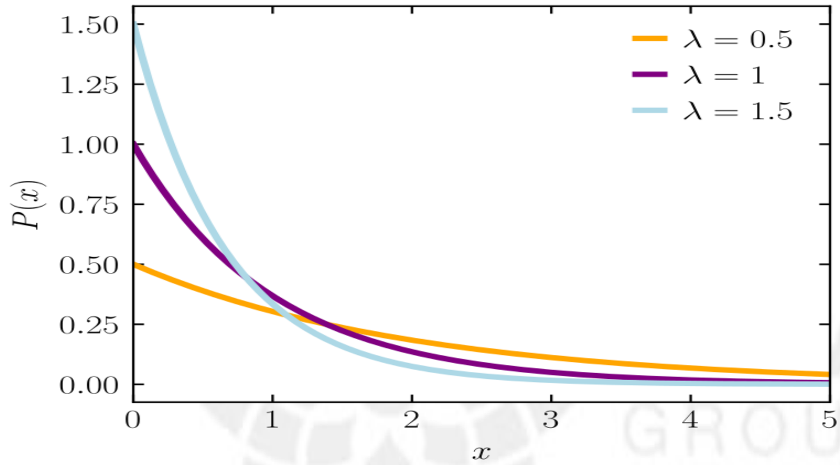
Activity:
Play the quiz to gain the knowledge
- https://wps.prenhall.com/bp_groebner_busstats_6/21/5380/1377488.cw/-/1377513/index.html
- https://wps.pearsoned.com/bajpai_businessstatistics_e/128/32983/8443746.cw/index.html

Videos:
To know more view the following videos
- https://www.youtube.com/watch?v=c5st4uze_ys
- https://www.youtube.com/watch?v=ojoXHILWcEM
- https://www.youtube.com/watch?v=hzb7u3lEtHA

1.11. Exponential distribution

Definition:

A continuous random variable X is said to follow an exponential distribution with parameter $\lambda > 0$ if its probability density function is given by $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & otherwise \end{cases}$, where λ is called the parameter of this distribution.



Moment Generating Function

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^{\infty} e^{-(\lambda - t)x} dx$$

$$= \lambda \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_0^{\infty}$$

$$= \frac{-\lambda}{\lambda - t}(0 - 1)$$

$$= \frac{\lambda}{\lambda - t}, \quad \lambda > t$$

Mean and variance of exponential distribution:

Mean = μ_1' = Coefficient of $\frac{t}{1!}$ in $M_X(t)$

$$M_X(t) = \frac{1}{1-\frac{t}{\lambda}} = \left(1-\frac{t}{\lambda}\right)^{-1}$$

$$= 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \ldots\ldots\ldots$$

Therefore, Mean = $\frac{1}{\lambda}$

Variance = $\mu_2' - \left(\mu_1'\right)^2$

μ_2' = Coefficient of $\frac{t^2}{2!}$ in $M_X(t)$

$$= \frac{2}{\lambda^2}$$

Therefore, Var(X) $= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

Memoryless property of exponential distribution

If X is exponentially distributed, then

$P\left(X > s+t / X > t\right) = P\left(X > s\right)$ for all $s, t > 0$.

Proof:

The exponential distribution is $f\left(x\right) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & otherwise \end{cases}$

Now,

$$P\left(X > m\right) = \int_m^\infty f\left(x\right)dx = \int_m^\infty \lambda e^{-\lambda x} dx = \lambda\left[\frac{e^{-\lambda x}}{-\lambda}\right]_m^\infty$$

$$= -\left(0 - e^{-\lambda m}\right) = e^{-\lambda m} \qquad \rightarrow (1)$$

Solution:

We know that for an exponential distribution the p.d.f. is given by $f(x)=\begin{cases}\lambda e^{-\lambda x}, & x\geq 0\\ 0, & otherwise\end{cases}$

Mean = $\frac{1}{\lambda}$ and Variance = $\frac{1}{\lambda^2}$.

In the given problem $\lambda=\frac{1}{5}$.

Therefore, Mean = 5, Variance(X) = 25.

$$P(X>5)=\int_5^\infty f(x)dx$$
$$=\frac{1}{5}\int_5^\infty e^{-\frac{x}{5}}dx$$
$$=\frac{1}{5}\left[\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right]_5^\infty$$
$$=-\left(0-e^{-1}\right)$$
$$=0.36788$$

$$P(3\leq X\leq 6)=\int_3^6 f(x)dx$$
$$=\frac{1}{5}\int_3^6 e^{-\frac{x}{5}}dx$$
$$=\frac{1}{5}\left[\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right]_3^6$$
$$=-\left(e^{\frac{-6}{5}}-e^{\frac{-3}{5}}\right)$$
$$=0.5488-0.3012$$
$$=0.2476.$$

R.M.K GROUP OF INSTITUTIONS

Example :2

Suppose that the amount of waiting time a customer spends at a restaurant has an exponential distribution with a mean value of 6 minutes. Find the probability that a customer will spend more than 12 minutes in the restaurant.

Solution:

Let X represent the waiting time in the restaurant.

Given X follows exponential distribution with parameter λ, mean = 6 = $\frac{1}{\lambda}$

$$\Rightarrow \lambda = \frac{1}{6}$$

Therefore, the exponential distribution is given by $f(x) = \begin{cases} \frac{1}{6} e^{-\frac{x}{6}}, & x \geq 0 \\ 0, & otherwise \end{cases}$

Required probability is $P(X > 12) = \int_{12}^{\infty} f(x) dx$

$$= \frac{1}{6} \int_{12}^{\infty} e^{-\frac{x}{6}} dx$$

$$= \frac{1}{6} \left[\frac{e^{-\frac{x}{6}}}{-\frac{1}{6}} \right]_{12}^{\infty}$$

$$= -\left(0 - e^{-2}\right)$$

$$= 0.1353$$

Example :3

Suppose that the number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 12000 kms. If a person desires to go on a tour covering a distance of 3000kms, what is the probability that the person will be able to complete the tour without replacing the battery.

Solution:

Let X denote the length of life of the battery.

Given X follows exponential distribution. Therefore $f(x)=\begin{cases}\lambda e^{-\lambda x}, & x\geq 0\\ 0, & otherwise\end{cases}$

Here $\frac{1}{\lambda}=12,000 \Rightarrow \lambda=\frac{1}{12000}$

Therefore, the exponential distribution is given by $f(x)=\begin{cases}\frac{1}{12000}e^{-\frac{x}{12000}}, & x\geq 0\\ 0, & otherwise\end{cases}$

If the battery has been in use for t kms before the tour, by memoryless property of the probability that it will last for another 3000 kms.

$$P(X>t+3000/X>t)=P(X>3000)$$

$$=\int_{3000}^{\infty} f(x)dx$$

$$=\frac{1}{12000}\int_{3000}^{\infty} e^{-\frac{x}{12000}}dx$$

$$=\frac{1}{12000}\left[\frac{e^{-\frac{x}{12000}}}{-\frac{1}{12000}}\right]_{3000}^{\infty}$$

$$=-\left(0-e^{-\frac{1}{4}}\right)$$

$$=0.7788$$

Example :4

The daily consumption of milk in excess of 20000 gallons is approximately exponentially distributed with $\lambda=\frac{1}{3000}$. The city has a daily stock of 35000 gallons. What is the probability that of two days selected at random, the stock is insufficient for both days?

Solution:

Let X denote the daily consumption of milk.

Let $Y=X-20000$, the excess.

Given Y is exponentially distributed with $\lambda = \frac{1}{3000}$

Probability density function of Y is $f(y) = \begin{cases} \frac{1}{3000} e^{-\frac{y}{3000}}, & y \geq 0 \\ 0, & otherwise \end{cases}$

Therefore, probability of stock is insufficient for a day

$$= P(X > 35000)$$
$$= P(Y > 15000)$$

$$= \frac{1}{3000} \int_{15000}^{\infty} e^{-\frac{y}{3000}} dy = \frac{1}{3000} \left[\frac{e^{-\frac{y}{3000}}}{-\frac{1}{3000}} \right]_{15000}^{\infty} = -\left(0 - e^{-5}\right) = e^{-5}$$

Therefore, the probability for insufficient stock for two days = $e^{-5}.e^{-5} = e^{-10} = 0.000045$

Example:5

The mileage that car owners get with certain kind of radial tyre is a random variable having an exponential distribution with mean 4000 km. Find the probabilities that one of these tyres will last i) at least 2000 km ii)at most 3000km.

Solution: L et X be the random variable which denote the mileage obtained with the tyre.

Here $\lambda = \frac{1}{4000}$

Then $f(x) = \frac{1}{4000} e^{\frac{-x}{4000}}, \ x > 0$

$$i)\, P(X > 2000) = \int_{2000}^{\infty} f(x) dx$$

$$= \int_{2000}^{\infty} \frac{1}{4000} e^{\frac{-x}{4000}} dx = \frac{1}{4000} \int_{2000}^{\infty} e^{\frac{-x}{4000}} dx$$

$$= \frac{1}{4000} \left[\frac{e^{\frac{-x}{4000}}}{\frac{-1}{4000}} \right]_{2000}^{\infty} = 0 + e^{\frac{-1}{2}} = 0.6065$$

$$ii)\ P(X \le 3000) = 1 - P(X > 3000)$$

$$= 1 - \int_{3000}^{\infty} f(x)dx$$

$$= 1 - \int_{3000}^{\infty} \frac{1}{4000} e^{\frac{-x}{4000}} dx$$

$$= 1 - \frac{1}{4000} \int_{3000}^{\infty} e^{\frac{-x}{4000}} dx$$

$$= 1 - \frac{1}{4000} \left[\frac{e^{\frac{-x}{4000}}}{\frac{-1}{4000}} \right]_{3000}^{\infty} = 1 - e^{\frac{-3}{4}} = 0.5276$$

Example:6

If a continuous random variable X follows uniform distribution in the interval (0,2) and a continuous random variable Y follows exponential distribution with parameter λ. Find λ such that $P(X < 1) = P(Y < 1)$.

Solution: Since X follows uniform distribution in the interval (0,2), we get

$$f(x) = \begin{cases} \frac{1}{2-0}, & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

and Y follows exponential distribution $f(y) = \lambda e^{-\lambda y}$

Given $P(X < 1) = P(Y < 1)$

$$\int_0^1 f(x)dx = \int_0^1 f(y)dy$$

$$\int_0^1 \frac{1}{2} dx = \int_0^1 \lambda e^{-\lambda y} dy$$

$$\frac{1}{2}(x)_0^1 = \lambda \left(\frac{e^{-\lambda y}}{-\lambda} \right)_0^1$$

$$\frac{1}{2} = -\left[e^{-\lambda} - 1\right] \Rightarrow \quad \frac{1}{2} = 1 - e^{-\lambda}$$

$$e^{-\lambda} = \frac{1}{2} \Rightarrow \quad e^{\lambda} = 2$$

$$\therefore \lambda = \log_e 2$$

Example:7

The time (in hours) required to repair a machine is exponentially distributed with parameter . What is the probability that a repair takes atleast 10 hrs given that its duration exceeds 9 hours.

Solution: Let X be the random variable which represents the time to repair the machine.

$$\therefore P\left[(X > 10) / P(X > 9)\right] = P(X > 1)$$

$$= P(X > 9 + 1 / X > 9)$$

$$= e^{\frac{-1}{2}} \quad \left(\because \text{by memoryless property } P(X > k) = e^{-\lambda k}\right)$$

$$= 0.6065$$

Example:8

If X is a random variable which follows an exponential distribution with parameter λ with $P(X \le 1) = P(X > 1)$, find $\operatorname{var}(X)$.

Solution:

Since X follows an exponential distribution with parameter λ, we have

$$f(x) = \lambda e^{-\lambda x}, \quad x \ge 0$$

$$\text{Given} \quad P(X \le 1) = P(X > 1)$$

$$1 - P(X > 1) = P\left(X > 1\right)$$

$$2P(X > 1) = 1$$

$$P(X > 1) = \frac{1}{2} \qquad \rightarrow (1)$$

Since X is an exponential distribution, we have

$$P(X > 1) = e^{-\lambda(1)} \qquad \rightarrow (2) \quad (\because P(X > K) = e^{-\lambda K})$$

From (1) and (2), we get

$$e^{-\lambda} = \frac{1}{2}$$

$$\frac{1}{e^{\lambda}} = \frac{1}{2}$$

$$e^{\lambda} = 2$$

$$\lambda = \log_e 2$$

Therefore, $\operatorname{var}(X) = \frac{1}{\lambda^2} = \frac{1}{(\log_e 2)^2}$

Exercise:

1. Find the moment generating function of the exponential distribution $f(x) = \frac{1}{c} e^{\frac{-x}{c}}, x \geq 0, c > 0$. Hence find its mean and standard deviation.

2. The time in hours required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$. (i) What is the probability that the repair time exceeds 2 hrs.? (ii) What is the probability that a repair takes at least 10 hrs. given that its duration exceeds 9 hours.

3. The amount of time that a watch will run without having to be reset is a R.V. having an exponential distribution with mean 120 days. Find the probability that such a watch will (i) have to be set in less than 24 days and (ii) not have to be reset in at least 180 days.

Answers:

1. Mean = c, Standard deviation = c.
2. **(i)** $e^{-1} = 0.3679$**,** **(ii)** $e^{-0.5} = 0.6065$
3. (i) 0.1813 (ii) 0.2231

Reference:

To know more about Exponential distribution

- https://homepage.divms.uiowa.edu/~mbognar/applets/exp-like.html

Activity:

Play the quiz to gain the knowledge

- https://wps.pearsoned.com/bajpai_businessstatistics_e/128/32983/8443746.cw/index.html

Videos:

To know more view the following videos

- https://youtu.be/THHZy7AE6wo
- https://www.youtube.com/watch?v=bKkLYSi5XNE
- https://www.youtube.com/watch?v=bM6nFDjvEns

1.12 Normal Distribution

Definition:

A continuous random variable X is said to follow a normal distribution with parameters μ and σ^2 if its probability density function is given by

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}},-\infty<x<\infty,\sigma>0,$$

X is called a normal random variable.

NOTE:

Since the normal distribution is used frequently in statistics, a special notation is used for it. The notation $X \sim N(\mu,\sigma^2)$ means X is normally distributed with mean μ and variance σ^2.

Moment generating function of normal distribution:

Let $X \sim N(\mu,\sigma^2)$

$$M_X(t)=E(e^{tx})=\int_{-\infty}^{\infty}e^{tx}f(x)dx$$

$$=\int_{-\infty}^{\infty}e^{tx}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$$

$$=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{tx-\frac{(x-\mu)^2}{2\sigma^2}}dx$$

Put $z=\frac{x-\mu}{\sigma}$ $\Rightarrow x=\mu+\sigma z\Rightarrow dx=\sigma dz$

$x:-\infty\to\infty\Rightarrow z:-\infty\to\infty$

$$\therefore M_X(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{t(\mu+\sigma z)-\frac{z^2}{2}}dz$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mu t}.e^{t(\sigma z)-\frac{z^2}{2}}dz$$

$$=\frac{1}{\sqrt{2\pi}}e^{\mu t}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2+\frac{1}{2}t^2\sigma^2}dz$$

$$=\frac{e^{\mu t}.e^{\frac{1}{2}t^2\sigma^2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2}dz$$

Now put $u=z-t\sigma \quad \therefore du=dz$

$$\therefore M_X(t)=\frac{e^{\mu t}.e^{\frac{1}{2}t^2\sigma^2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}du$$

$$=\frac{e^{\mu t+\frac{1}{2}t^2\sigma^2}}{\sqrt{2\pi}}.\sqrt{2\pi} \qquad \left[\because \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}du=\sqrt{2\pi}\right]$$

$$\therefore M_X(t)=e^{\mu t+\frac{1}{2}t^2\sigma^2}$$

Mean and variance of normal distribution:

We know that $\mu_r'=M_X^{(r)}(0)$

Mean = $\mu_1'=M_X'(0)$

$$M_X'(t)=e^{\mu t+\frac{\sigma^2t^2}{2}}.\left(\mu+t\sigma^2\right)$$

$$M_X'(0)=\mu$$

$\therefore$ Mean = μ

Variance = $\mu_2'-\left(\mu_1'\right)^2$

$$\mu_2'=M_X''(0)=\left[e^{\mu t+\frac{\sigma^2t^2}{2}}.\left(\mu+t\sigma^2\right)^2+e^{\mu t+\frac{\sigma^2t^2}{2}}.\left(\sigma^2\right)\right]_{at\,t=0}$$

$$=\mu^2+\sigma^2$$

$\therefore$ Var(X) = $\mu^2+\sigma^2-\mu^2$ = σ^2

Normal distribution	
MGF	$e^{\mu t+\frac{1}{2}t^2\sigma^2}$
Mean	μ
Variance	σ^2
Standard deviation	σ

Additive property of normal Distribution:

If $X_1 \sim N\left(\mu_1,\sigma_1^2\right)$ and $X_2 \sim N\left(\mu_2,\sigma_2^2\right)$ be two independent normal random variables.

Then $M_{X_1}\left(t\right)=e^{\mu_1 t+\frac{1}{2}t^2\sigma_1^2}$; $M_{X_2}\left(t\right)=e^{\mu_2 t+\frac{1}{2}t^2\sigma_2^2}$ $\because X_1$ *and* X_2 are independent

$$M_{X_1+X_2}(t)= M_{X_1}(t)M_{X_2}(t)$$
$$=e^{\mu_1 t+\frac{1}{2}t^2\sigma_1^2}.e^{\mu_2 t+\frac{1}{2}t^2\sigma_2^2}$$
$$=e^{(\mu_1+\mu_2)t+\frac{t^2}{2}(\sigma_1^2+\sigma_2^2)}$$

Which is the moment generating function of a random variable with mean $\mu=\mu_1+\mu_2$ and variance $\sigma^2=\sigma_1^2+\sigma_2^2$. Hence $X_1+X_2 \sim N\left(\mu_1+\mu_2,\sigma_1^2+\sigma_2^2\right)$

This property can be extended to any n independent normal random variables.

Properties of normal distribution (or normal curve):

1. The mean, median and mode coincide at $x=\mu$.
2. It is symmetric about $x=\mu$
3. The maximum value of $f\left(x\right)$ is at $x=\mu$ and the maximum value is $\frac{1}{\sigma\sqrt{2\pi}}$.
4. The curve approaches the horizontal axis asymptotically on either side of $x=\mu$
5. The total area under the curve is 1.

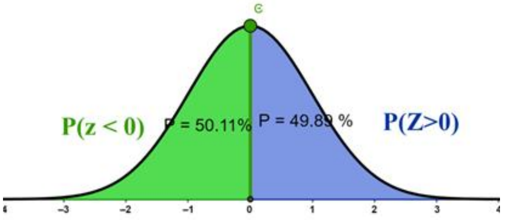
Standard Normal distribution:

Let $X \sim N\left(\mu, \sigma^2\right)$. By transforming X into $Z = \dfrac{X-\mu}{\sigma}$. Then

$$E(z) = \frac{1}{\sigma}\left(E(X) - \mu\right) = \frac{1}{\sigma}(\mu - \mu) = 0$$

$$\text{Var}(z) = \frac{1}{\sigma^2} Var(X) = \frac{1}{\sigma^2}\sigma^2 = 1$$

Z is a normal random variable with mean 0 and variance 1. Any random variable whose mean is 0 and variance is 1 is called as a standard rand normal random variable.



Its p.d.f. is given by $\phi(z) = \dfrac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}, \quad -\infty < z < \infty$

Standard Normal Curve:

$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} f(x)dx$$

$$= \int_{x_1}^{x_2} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

Put $Z = \dfrac{X-\mu}{\sigma}$ when $x = x_1, z_1 = \dfrac{x_1-\mu}{\sigma}$

$\sigma dz = dx$ when $x = x_2, z_2 = \dfrac{x_2-\mu}{\sigma}$

$$\therefore P(z_1 < Z < z_2) = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{\frac{-z^2}{2}} dz = \phi(z)$$

The curve given by $\phi(z) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{z_1}^{z_2} e^{\frac{-z^2}{2}} dz$ is called the standard normal curve and it is bell shaped and symmetrical about the line $z = 0$.

The integral $\frac{1}{\sqrt{2\pi}}\int_{z_1}^{z_2} e^{\frac{-z^2}{2}}\,dz$ is called the probability integral. The values of these integral for different values of z are given in the table.

Example :1

A normal distribution has mean $\mu = 20$ and S.D. $\sigma = 10$. Find $P(15 \le X \le 40)$.

Solution:

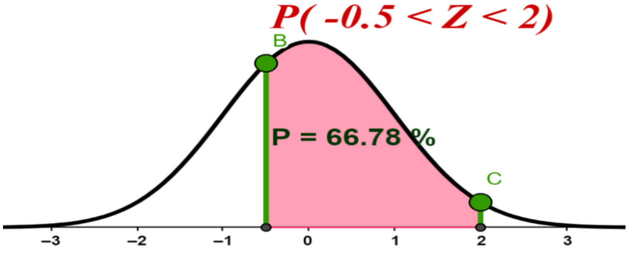
Given $\mu = 20$, $\sigma = 10$.

The normal variate $Z = \frac{X-\mu}{\sigma} = \frac{X-20}{10}$

When $X = 15,\ z = \frac{X-20}{10} = \frac{15-20}{10} = -0.5$

$X = 40,\ z = \frac{X-20}{10} = \frac{40-20}{10} = 2$

$$\begin{aligned}
\therefore P(15 \le X \le 40) &= P(-0.5 \le z \le 2) \\
&= P(-0.5 \le z \le 0) + P(0 \le z \le 2) \\
&= P(0 \le z \le 0.5) + P(0 \le z \le 2) \\
&= 0.1915 + 0.4772 \\
&= 0.6687
\end{aligned}$$



Example :2

If X follows a normal distribution with mean 16 and standard deviation 3.

Find (i) $P(X \ge 19)$ (ii) $P(10 < X < 25)$ (iii) K if $P(X > k) = 0.24$

Solution:

Given $\mu = 16$, $\sigma = 3$

The normal variate $Z = \frac{X-\mu}{\sigma} = \frac{X-16}{3}$

When $X = 19,\ z = \frac{X-16}{3} = \frac{19-16}{3} = 1$

$$
\begin{aligned}
(i) P(X \geq 19) &= P(z \geq 1) \\
&= 0.5 - P(0 < z < 1) \\
&= 0.5 - 0.3413 \\
&= 0.1587
\end{aligned}
$$

$(ii)\ P(10 < X < 25)$

When $X = 10,\ z = \dfrac{X-16}{3} = \dfrac{10-16}{3} = -2$

When $X = 25,\ z = \dfrac{X-16}{3} = \dfrac{25-16}{3} = 3$

$$
\begin{aligned}
P(10 < X < 25) &= P(-2 < z < 3) \\
&= P(-2 < z < 0) + P(0 < z < 3) \\
&= P(0 < z < 2) + P(0 < z < 3) \\
&= 0.4772 + 0.4987 \\
&= 0.9759
\end{aligned}
$$

(iii) Given $P(X > k) = 0.24$

When $X = k,\ z = \dfrac{k-16}{3} = z_1\ say$

$$
\begin{aligned}
& P(Z > z_1) = 0.24 \\
& \Rightarrow P(0 < Z < z_1) = 0.5 - 0.24 = 0.26
\end{aligned}
$$

$\therefore z_1$ is the value of Z corresponding to the area 0.26, and $z_1 = 0.7$ (from table)

$$
\begin{aligned}
& \therefore \frac{k-16}{3} = 0.7 \\
& \Rightarrow k = 16 + 3(0.7) = 18.1
\end{aligned}
$$

Example: 3

The life of a certain kind of electronic device has a mean of 300 hours and standard deviation 25 hours. Assuming that the life times of the devices follow normal distribution.

1. Find the probability that anyone of these devices will have a life time more than 350 hours.

2. What percentage will have life time between 220 and 260 hours?

Solution:

Let $X \sim N(\mu, \sigma^2)$. Given $\mu = 300\ hrs$, $\sigma = 25\,hrs$

The normal variate $Z = \dfrac{X-\mu}{\sigma} = \dfrac{X-300}{25}$

1. Required probability is $P(X \geq 350)$

When $X = 350,\ z = \dfrac{X-300}{25} = \dfrac{350-300}{25} = 2$

$$\begin{aligned}\therefore P(X > 350) &= P(z > 2)\\ &= 0.5 - P(0 < z < 2)\\ &= 0.5 - 0.4772\\ &= 0.0228\end{aligned}$$

$(ii)\ P(220 < X < 260)$

When $X = 220,\ z = \dfrac{X-300}{25} = \dfrac{220-300}{25} = -3.2$

When $X = 260,\ z = \dfrac{X-300}{25} = \dfrac{260-300}{25} = -1.6$

$$\begin{aligned}(ii)\ P(220 < X < 260) &= P(-3.2 < z < -1.6)\\ &= P(-3.2 < z < 0) - P(-1.6 < z < 0)\\ &= P(0 < z < 3.2) + P(0 < z < 1.6)\\ &= 0.4993 - 0.4452\\ &= 0.0541\end{aligned}$$

Percentage of devices = $100 \times 0.0541 = 5.41$

Therefore 5.41 percent of electronic devices will have life time between 220 and 260 hours.

Example :4

In a distribution exactly normal 7% of the items are under 35 and 89% are under 63. What are the mean and standard deviation of the distribution?

Solution:

Let $X \sim N\left(\mu,\sigma^2\right)$. Given

$$P\left(X<35\right)=7\%=0.07$$
$$P\left(X<63\right)=89\%=0.89$$

$$Z=\frac{X-\mu}{\sigma}$$

When $X=35,\ z=\dfrac{35-\mu}{\sigma}=-z_1$, since area below $X=35$ is 0.07 z_1 is negative.

$$\Rightarrow 35-\mu=-\sigma z_1 \qquad \rightarrow (1)$$
$$P\left(Z<-z_1\right)=0.07$$

$\Rightarrow z_1$ is the value corresponding to the area $\int_0^{z_1}\phi\left(z\right)dz=0.43$

When $X=63,\ z=\dfrac{63-\mu}{\sigma}=z_2$, since area below $X=63$ is 89 z_2 is positive.

$$\therefore 63-\mu=\sigma z_2$$
$$P\left(Z<z_2\right)=0.89=0.5+0.39$$

z_2 is the value corresponding to the area = 0.39

From the tables $z_2=1.23$

$$\therefore 63-\mu=1.23\sigma \qquad \rightarrow (2)$$

Solving (1) and (2)

$$\mu=50.3,\sigma=10.33$$

Example:5

In a newly constructed township, 2000 electric lamps are installed with an average life of 1000 burning hours and standard deviation of 200 hours. Assuming the life of the lamps follows normal distribution, find

i)The number of lamps expected to fail during the first 700 hours.

ii) In what period of burning hours 10% of the lamps fail.

Solution:

Given $\mu = 1000,\ \sigma = 200$

i)Let us find the probability of one lamp failing during the first 700 hours

$$\begin{aligned} P(X \le 700) &= P\left(\frac{X-1000}{200} < \frac{700-1000}{200}\right) \\ &= P(Z < -1.5) \\ &= P(Z > 1.5) \\ &= 0.5 - 0.4332 = 0.0668 \end{aligned}$$

The number of lamps that fail to burn in the first 700 hours= $2000 \times 0.0668 = 133.6 \approx 134$

ii) Let t_0 be the period at which 10 of lamps fail.

$$\text{Then} \quad P(X \le t_0) = 0.1$$

$$P\left(\frac{X-1000}{200} \le \frac{t_0-1000}{200}\right) = 0.1$$

$$P\left(Z \le \frac{t_0-1000}{200}\right) = 0.1$$

$$P\left(Z \ge \frac{1000-t_0}{200}\right) = 0.1$$

$$\therefore P\left(0 \le Z \le \frac{1000-t_0}{200}\right) = 0.5 - 0.1 = 0.4$$

From the table this situation arises when $\dfrac{1000-t_0}{200} = 1.28 \Rightarrow t_0 = 744$

By 744 hours, 10% of lamps will fail.

Example:6

The marks obtained by a number of students in a certain subject are assumed to be approximately normally distributed with mean value 65 and with standard deviation 5.

If 3 students are taken at random from this set, what is the probability that exactly 2 of them will have marks over 70.

Solution:

Let X be the random variable denoting the marks obtained by students in the given subject.

Given $\mu = 65, \ \sigma = 5$

$$\begin{aligned} P(X > 70) &= P\left(\frac{X-65}{5} < \frac{70-65}{5}\right) \\ &= P(Z > 1) \\ &= 0.5 - P(0 < Z < 1) \\ &= 0.5 - 0.3413 = 0.1587 \end{aligned}$$

Let Y be the number of students(out of 3) getting marks more than 70. Then Y follows binomial distribution with $n = 3, \ p = 0.1587$

$$\begin{aligned} \therefore P(Y = y) &= nC_y p^y q^{n-y} \\ &= 3C_y (0.1587)^y (0.8413)^{3-y} \end{aligned}$$

Required probability

$$\begin{aligned} P(Y = 2) &= 3C_2 (0.1587)^2 (0.8413)^{3-2} \\ &= 0.06357 \end{aligned}$$

Example:7

Gauges are used to reject a component which is not within the specification $1.50 \pm d$. It is known that this measurement is normally distributed with $\mu = 1.50, \ \sigma = 0.2.$ Determine the value of $'d'$ such that the specification cover 95% of measurement.

Solution:

X is normally distributed.

Given $\mu = 1.50, \ \sigma = 0.2$

$$\text{Now, } P[1.5-d < X < 1.5+d] = P\left[\frac{1.5-d-1.5}{0.2} < \frac{X-1.5}{0.2} < \frac{1.5+d-1.5}{0.2}\right]$$
$$= P\left[\frac{-d}{0.2} < Z < \frac{d}{0.2}\right]$$

Given that the specification covers 95%

$$(i.e)., \quad P\left[\frac{-d}{0.2} < Z < \frac{d}{0.2}\right] = 0.95$$
$$2P\left[0 < Z < \frac{d}{0.2}\right] = 0.95$$
$$P\left[0 < Z < \frac{d}{0.2}\right] = 0.475$$
$$\frac{d}{0.2} = 1.96 \qquad \text{(From normal table)}$$
$$\therefore d = 0.392$$

Example:8

The weekly wages of 1000 workmen are normally distributed around a mean of Rs.70 with a S.D of Rs.5. Estimate the number of workers whose weekly wages will be i) between Rs.69 and Rs.72, ii)less than Rs.69, iii)more than Rs.72.

Solution: Let X be the random variable denoting the weekly wages of a worker.

$$\text{Given } \mu = 70$$
$$\sigma = 5$$

$$\text{The normal variate } z = \frac{X-\mu}{\sigma} = \frac{X-70}{5}$$

$i)$ To find $P(69 < X < 72)$

$$\text{when } X = 69, \ z = \frac{69-70}{5} = -0.2$$

$$\text{when } X = 72, \ z = \frac{72-70}{5} = 0.4$$

$$\therefore P(69 < X < 72) = P(-0.2 < z < 0.4)$$
$$= P(-0.2 < z < 0) + P(0 < z < 0.4)$$
$$= P(0 < z < 0.2) + P(0 < z < 0.4)$$
$$= 0.0793 + 0.1554$$
$$= 0.2347$$

Out of 1000 workmen, the number of workers whose wages lies between Rs.69 and Rs.72

$$= 1000 \times P(69 < X < 72)$$
$$= 1000 \times 0.2347 = 234.7 \approx 235$$

$ii)$ To find $P(X < 69)$

when $X = 69,\ z = \dfrac{69-70}{5} = -0.2$

$$\therefore P(X < 69) = P(z < -0.2)$$
$$= 0.5 - P(0 < z < 0.2)$$
$$= 0.5 - 0.0793$$
$$= 0.4207$$

Out of 1000 workmen, the number of workers whose wages are less than Rs.69

$$= 1000 \times P(X < 69)$$
$$= 1000 \times 0.4207 = 420.7 \approx 421$$

$iii)$ To find $P(X > 72)$

when $X = 72,\ z = \dfrac{72-70}{5} = 0.4$

$$\therefore P(X > 72) = P(z > 0.4)$$
$$= 0.5 - P(0 < z < 0.4)$$
$$= 0.5 - 0.1554$$
$$= 0.3446$$

Out of 1000 workmen, the number of workers whose wages are more than Rs.72

$$= 1000 \times P(X > 72)$$
$$= 1000 \times 0.3446$$
$$= 344.6$$
$$\approx 345$$

Example:9

Assume that mean height of soldiers to be 68.22 inches with a variance of 10.8 inches. How many soldiers in a regiment of 1000 would you expect to be over 6 feet tall?

Solution: Let X be a random variable denoting the heights of soldiers.

Given $\mu = 68.22$

$\sigma^2 = 10.8$

$\Rightarrow \sigma = 3.286$

The normal variate $z = \dfrac{X-\mu}{\sigma} = \dfrac{X-68.22}{3.286}$

Now, 6 feet=72 inches

$$\therefore P(\text{height of a soldier is over 6 feet fall}) = P(X > 6)$$
$$= P(X > 72 \text{ inches})$$

when $X = 72,\ z = \dfrac{72-68.22}{3.286} = 1.1503$

$$\begin{aligned}\therefore P(X > 72) &= P(z > 1.1503)\\ &= 0.5 - P(0 < z < 1.1503)\\ &= 0.5 - 0.3749\\ &= 0.1251\end{aligned}$$

For 1000 soldiers, the number of soldiers greater than 6 feet (72 inches)

$$\begin{aligned}&= 1000 \times P(X > 72)\\ &= 1000 \times 0.1251\\ &= 125 \text{ soldiers}\end{aligned}$$

Example:10

Assuming that the diameters of 1000 brass pilugs taken consecutively from a machine form a normal distribution with mean 0.7515 cm and standard deviation 0.0020 cm, how many of the plugs are likely to be rejected if the approved diameter is 0.752 ± 0.004 cm.

Solution: Let X be the random variable which denotes the diameter of brass plugs.

The approved diameter s ranges from 0.752+0.004 and 0.752-0.004

(i.e), 0.752+0.004=0.756

& 0.752-0.004=0.748

Given $\mu = 0.7515$

$\sigma = 0.0020$

The normal variate $z = \dfrac{X-\mu}{\sigma} = \dfrac{X-0.7515}{0.0020}$

when $X = 0.748,\ z = \dfrac{0.748-0.7515}{0.0020} = -1.75$

when $X = 0.756,\ z = \dfrac{0.756-0.7515}{0.0020} = 2.25$

$$\begin{aligned}
\therefore P(0.748 < X < 0.756) &= P(-1.75 < z < 2.25) \\
&= P(-1.75 < z < 0) + P(0 < z < 2.25) \\
&= P(0 < z < 1.75) + P(0 < z < 2.25) \\
&= 0.4599 + 0.4878 \\
&= 0.9477
\end{aligned}$$

Out of 1000, the number of plugs of approved diameters

$$\begin{aligned}
&= 1000 \times P(0.748 < X < 0.756) \\
&= 1000 \times 0.9477 \\
&= 947.7 \\
&\approx 948
\end{aligned}$$

Exercise:

1. The moment generating function of a normal R.V. X is e^{3t+8t^2}. Find $P(-1 < X < 9)$
2. The average seasonal rainfall in a place is 16inches with a S.D. of 4 inches. What is the probability that in a year the rainfall in the place will be between 20 and 24 inches?
3. The savings bank account of a customer showed an average balance of Rs.150 and a S.D Rs.50. Assuming that the account balances are normally distributed.
   a) What percentage of account is over Rs.200?
   b) What percentage of account is between Rs.120 and Rs.170?
   c) What percentage of account is less than Rs.75?
4. In a normal distribution 31% of the items are under 45 and 8% are over 64. Find the mean and variance of the distribution.

Answers:

1. 0.7745 2. 0.1359 3. a) 15.87 b)38.11 c)6.68 4. $\mu = 50, \sigma = 10$

Reference:

To know more about Normal distribution

- https://www.mathsisfun.com/data/standard-normal-distribution.html
- https://homepage.divms.uiowa.edu/~mbognar/applets/normal.html
- https://stattrek.com/online-calculator/normal.aspx

Activity:

Play the quiz to gain the knowledge

- https://www.maths.usyd.edu.au/u/UG/JM/MATH1015/Quizzes/quiz6.html
- https://www.cliffsnotes.com/study-guides/statistics/sampling/quiz-properties-of-the-normal-curve

Videos:

To know more view the following videos

- https://www.youtube.com/watch?v=778LOolLu8s
- https://www.youtube.com/watch?v=72QjzHnYvL0

Practice Quiz

1.	When you roll a die, what is the sample space? a) {1, 2, 3, 4, 5, 6} b) {2, 4, 6} c) {1, 3, 5} d) {1, 2, 3}
2.	When you tossing a coin P(H)=? a) 1 b) 1/3 c) ¼ d) ½
3.	What is the total number of combinations of $'n'$ different things taken atleast one at a time? a)$2^n - 1$ b)$2n$ c)2^n d)n^2
4.	Find the value of 'r' if $5P_r = 60$. a) -1 b)0 c)3 d)1
5.	Find the value of 'n' if $nP_3 = 5nP_2$. a)3 b)7 c)2 d)0
6.	If A and B are independent events then $a) P(A \cap B) = P(A) + P(B) - P(A \cup B)$ $b) P(A \cap B) = P(A) - P(B)$ $c) P(A \cap B) = P(A) + P(B)$ $d) P(A \cap B) = P(A)P(B)$

7.	Previous probabilities in Bayes Theorem that are changed with help of new available information are classified as __________________ a) independent probabilities b) posterior probabilities c) interior probabilities d) dependent probabilities
8.	Three companies A, B and C supply 25%, 35% and 40% of the notebooks to a school. Past experience shows that 5%, 4% and 2% of the notebooks produced by these companies are defective. If a notebook was found to be defective, what is the probability that the notebook was supplied by A? a) $^{44}/_{69}$ b) $^{25}/_{69}$ c) $^{13}/_{24}$ d) $^{11}/_{24}$
9.	If 'X' is a random variable having density function $f(x)=\begin{cases}\frac{x}{6}, \text{for } x=1,2,3\\ 0, \text{otherwise}\end{cases}$, then the mean is (a) 2 (b) 2.333 (c) 2.755 (d) 3
10.	The random variable 'X' can only take the values 2 and 5.Given that the value 5 is twice as likely the value 2, determine the expectation of X. (a) 2 (b) 3 (c) 4 (d) 5

11.	If Var(X)=4, find Var($4X+5$) where, X is a random variable. (a)54 (b)44 (c)74 (d)64
12.	Find the mean of the random variable X whose pdf is $f(x)=\frac{1}{b-a}, a<x<b$. (a) $a+b$ (b) $a-b$ (c) $\frac{a-b}{2}$ (d) $\frac{a+b}{2}$
13.	A coin is tossed until a tail appears. What is the expectation of the number of tosses. (a) 2 (b)20 (c)5 (d)7
14.	Let X be a random variable. If mean=5.4 and variance=0.2. What is the second moment? a)23.69 b)29.36 c)39.26 d)32.96

15.	The moments about mean are called (a) Raw moments (b) Central moments (c) Moments about origin (d) All of the above
16.	The moments about origin are called: (a) Moments about zero (b) Raw moments (c) Both (a) and (b) (d) Neither (a) nor (b)
17.	The first and second moments about arbitrary constant are -2 and 13 respectively, the standard deviation will be: (a) -2 (b) 3 (c) 9 (d)13
18.	If $f(x) = e^{-x},\ x \geq 0$ What is the moment generating function of X?. $a)\dfrac{1}{1-t}$ $b)\dfrac{1}{t-1}$ $c)\dfrac{1}{1+t}$ $d)\dfrac{1}{1-2t}$

1.	Given the following moment generating function, find the mean of the continuous random variable X: $M_X(t) = e^{-4t}$ $a)0$ $b)-4$ $c)8$ $d)-8$
2.	If X and Y are independent random variables, then $M_{X+Y}(t)$ $a)\dfrac{M_X(t)}{M_Y(t)}$ $b)M_X(t)+M_Y(t)$ $c)M_X(t).M_Y(t)$ $d)M_Y(t)-M_X(t)$

Answers

1. (a)	2. (d)
3.(a)	4.(c)
5.(b)	6.(d)
7.(b)	8. (b)
9.(b)	10.(c)
11.(d)	12.(d)
13.(a)	14. (b)
15.(b)	16. (c)
17.(b)	18.(a)
19.(b)	20.(c)

Practice Quiz

1.	If 'X' is a random variate following binomial distribution with mean 2.4 and variance 1.44, find $P(X \geq 5)$. a)0.4096 b)0.004096 c)0.04096 d)0.0004096
2.	In a binomial distribution consisting of 5 independent trials probability of 1 and 2 successes are 0.4096 and 0.2048 respectively. What is the value of 'p'? a)0.2 b)0.5 c)0.1 d)0.001
3.	10 coins are thrown simultaneously. What is the probability of getting at least 7 heads? a)0.21719 b)0.01719 c)0.1729 d)0.1719
4.	A surgery has a success rate of 75%. Suppose that the surgery is performed on three patients. a. What is the probability that the surgery is successful on exactly 2 patients? a)0.422 b)0.244 c)0.0242 d)0.224
5.	In a Binomial Distribution, if p = q, then P(X = x) is given by? a) $nC_x(0.5)^n$ b) $nC_n(0.5)^n$ c) $nC_x p^{n-x}$ d) $nC_n p^{n-x}$
6.	The mean and variance of a binomial distribution is 3 and 2. What is the probability function? a) $3C_x\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^{3-x}$ b) $9C_x\left(\frac{1}{3}\right)^9\left(\frac{2}{3}\right)^{9-x}$ c) $9C_x\left(\frac{2}{3}\right)^9\left(\frac{1}{3}\right)^{9-x}$ d) $9C_x\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{3-x}$

7.	Six dice are thrown 1000 times. What is the probability of getting 5 or 6 in 3 dice? a)0.912 b)0.291 c)0.921 d)0.219
8.	The moment generating function of Poisson distribution is (a) $e^{-\lambda(e^t-1)}$ (b) $e^{\lambda(e^t-1)}$ (c) $e^{-\lambda(e^t-1)}$ (d) $e^{\lambda(1-e^t)}$
9.	Write down the probability mass function of the Poisson distribution which is approximately equivalent to $B(100, 0.03)$ (a) $\frac{e^{-3}(3)^x}{x!}$ (b) $\frac{e^{3}(-3)^x}{x!}$ (c) $\frac{e^{-3}(-3)^x}{x!}$ (d) $\frac{e^{3}(3)^x}{x!}$
10.	What does Poisson distribution describe? (a) Future events (b)Total number of events (c) Rare events (d) Non occurrence of events
11.	A cafe receives an average of 4 orders of coffee every 5 minutes. What is the probability of receiving exactly 14 orders in a 15 minute period? What is the value of λ? (a) 21 (b) 12 (c) 31 (d) 13
12.	The Poisson distribution was named after a mathematician, engineer, and physicist, Siméon Denis Poisson. In which country is he from? (a) Philippines (b) USA (c) Portugal (d) France

13.	A programmer has a 90% chance of finding a bug every time he compiles his code, and it takes him two hours to rewrite his code every time he discovers a bug. What is the probability that he will finish his program by the end of his workday? a) 0.344 b) 0.355 c)0.244 d)0.433
14.	If your probability of success is 0.2, what is the probability you meet an independent voter on your third try? a)0.121 b)0.128 c)0.218 d)0.124
15.	The expected value of a geometric random variable is a) $\frac{1}{q}$ b) $\frac{1}{p}$ c) $\frac{q}{p}$ d) $\frac{1}{p^2}$

Answers

1. c)	2. a)	3.(d)
4.(a)	5.(a)	6. (b)
7.(d)	8. (b)	9.(a)
10.(c)	11. (b)	12. (d)
13.(a)	14. (b)	15.(b)

Assignment-I

1. An individual uses the following gambling system at Las Vegas. He bets \$1 that the roulette wheel will come up red. If he wins, he quits. If he loses then he makes the same bet a second time only this time he bets \$2; and then regardless of the outcome, quits. Assuming that he has a probability of ½ of winning each bet, what is the probability that he goes home a winner? Why is this system not used by everyone

2. A basketball team consists of 6 black and 6 white players. The players are to be paired in groups of two for the purpose of determining roommates. If the pairings are done at random, what is the probability that none of the black players will have a white roommate?

3. Bev can either take a course in computers or in chemistry. If Bev takes the computer course, then she will receive an A grade with probability $\frac{1}{2}$ while if she takes the chemistry course then she will receive an A grade with probability $\frac{1}{3}$. Bev decides to base her decision on the flip of a fair coin. What is the probability that Bev will get an A in chemistry?

4. Suppose that each of three men at a party throws his hat into the center of the room. The hats are first mixed up and then each man randomly selects a hat. What is the probability that none of the three men selects his own hat?

5. Suppose an urn contains seven black balls and five white balls. We draw two balls from the urn without replacement. Assuming that each ball in the urn is equally likely to be drawn, what is the probability that both drawn balls are black?

6. A certain communication system consists of a transmitter that sends one of the two possible symbols 1 or 0 over a channel to a receiver. The channel occasionally causes errors. The probability that a transmitted 0 is received as0 is0.93 and the probability that transmitted 1 is received as 1 is 0.92. If the probability that a '0' is transmitted is 0.6, find the probability that (i) a '1' is received and (ii) a '1' was transmitted given that a '1' is received.

Answers

1. 3/4. If he wins, he only wins \$1, while if he loses, he loses \$3.

2. $\frac{5}{231} = 0.0216$

3. $\frac{1}{6}$

4. $\frac{1}{3}$

5. $\frac{42}{132}$

6. i) 0.410 ii) 0.897

Assignment-II

1. The monthly demand for Allwyn watches is known to have the following probability distribution:

x_i	1	2	3	4	5	6	7	8
$p(x_i)$	0.08	0.12	0.19	0.24	0.16	0.10	0.07	0.04

Find the expected demand for watches and also compute the variance.

2. Find the mean and variance of the random variable 'X' whose pdf is $f(x)=\frac{1}{b-a},\ a<x<b$

3. From an urn containing 3 red and 2 black balls a man is to draw 2 balls without replacement. He gets Rs.20 for each red ball and Rs.10 for each black ball. Find his expectation.

4. Let X have the density function $f(x)=\begin{cases}\frac{1}{b-a}, & a\le x\le b\\ 0, & \text{otherwise}\end{cases}$. Find the k^{th} moment about
   i) the origin
   ii) the mean.

5. Find the mean and variance of the following distribution:

x:	-3	-2	-1	0	1	2	3
$p(x)$:	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$

6. Find the moment generating function of a random variable X whose density function is given by $f(x)=\lambda e^{-\lambda(x-a)},\ x\ge a$. Hence find its mean and variance.

7. Consider a discrete random variable X with probability function $P(X=x)=\begin{cases}\frac{1}{x(x+1)}, & x=1,2,\ldots \\ 0, & otherwise\end{cases}$. Show that $E(X)$ does not exist even though M.g.f. exist.

8. Find the Moment generating function of the random variable with the probability law $P(X=x)=q^{x-1}p, x=1,2,3,\ldots$ find the mean and variance.

Answers

1. $E(X)=4.06,\ Var(X)=3.21$

2. $Mean=\frac{a+b}{2},\ Variance=\frac{(b-a)^2}{12}$

3. 32

4. $i)\ \frac{b^{k+1}-a^{k+1}}{(k+1)(b-a)}$ $\quad ii)\frac{\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)}$

5. $\text{Mean}=0,\ \text{var}(X)=4.$

6. $\text{mgf }=\frac{\lambda e^{at}}{\lambda-t},\ \text{mean}=\frac{a\lambda+1}{\lambda},\ \text{variance}=\frac{1}{\lambda^2}$

8. $\text{Mean }=\frac{1}{p},\ \text{Variance}=\frac{q}{p^2}$

Assignments

ASSIGNMENT-III

1. The mean and variance of a binomial variate are 8 and 6, find $P(X \geq 2)$.

2. Let $'X'$ follows a binomial distribution. Suppose $P(X=0)=1-P(X=1)$. If $E(X)=3.Var(X)$, find $P(X=0)$.

3. A perfect cubic die is thrown a large number of times in sets of 8. The presence of a 5 or 6 is treated as a success. In what percentage of the sets can we expect 3 success.

4. In 256 sets of twelve tosses of a coin, how many cases may one expect eight heads and four tails?

5. An irregular six faced die is thrown such that the probability that it gives 3 even numbers in 5 throws is twice the probability that it gives 2 even numbers in 5 throws. How many sets of exactly 5 trials can be expected to give no even number out of 2500 sets?

6. Assume that half of the population is vegetarian so that the chance of an individual being a vegetarian is $\frac{1}{2}$. Assuming that 100 investigators take samples of 10 individual each to see whether they are vegetarian, how many investigators would you expect to report that three people or less were vegetarians?

7. The M.G.F of a random variable X is $\left(\frac{2}{3}+\frac{1}{3}e^t\right)^9$. Show that

$$P(\mu-2\sigma < x < \mu+2\sigma)=\sum_{x=1}^{5} 9C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}$$

Answers

1) 0.9988	2)0.33
3) 27.31%	4)31
5)10	6)17

ASSIGNMENT-IV

1. If X and Y are independent Poisson variate $X = 2Y$ and $P(Y=2) = P(Y=3)$. Find the Variance of $X = 2Y$.

2. The probability that a man aged 35 years will die before reaching the age of 40 years may be taken as 0.018. Out of a group of 400 men aged 35 years, what is the probability that 2 men will die within next 5 years?

3. Out of 1000 balls to are red and the rest are white. If 60 balls are picked at random, what is the probability of picking up (i) 3 red balls (ii) Not more than 3 red balls in the sample.

4. Let one copy of a magazine out of 10 copies bears a special prize following geometric distribution. Determine its mean and variance.

5. If the probability is 0.05 that certain kind of measuring device will show excessive drift, what is the probability that the sixth of these measuring devices tested will be the first to show excessive drift?

6. If 'X' is a geometric variate taking values $1, 2, \ldots \infty$. Prove that $P(X = \text{odd}) = \dfrac{1}{1+q}$

Answers

1) Variance =14

2) 0.01935

3) $P(X=3) = 0.2241$, $P(X \le 3) = 0.6474$

4) 90

5) 0.0387

Part A Questions & Answers

Q. No.	Questions &Answers	K Level	CO
1.	Define experiment, sample space and outcome. **Solution:** **Experiment:** An experiment whose all possible outcomes are known, but it is not possible to predict the outcome. **Sample Space:** Set of all the possible outcomes of an experiment. **Outcome:** A possible result of an experiment.	**K1**	**CO2**
2.	Define probability. **Solution:** Probability of the event A is defined as Probability of an event happening $= \frac{\text{Number of ways it can happen}}{\text{Total number of outcomes}} = \frac{n(A)}{n(S)}$	**K1**	**CO2**
3.	If X_1 *and* X_2 are independent Poisson variate, show that $X_1 - X_2$ is not a Poisson variate. **Solution:** Let X_1 *and* X_2 be independent Poisson variate having parameters λ_1 *and* λ_2 respectively. $M_{x_1 - x_2}(t) = M_{x_1}(t).M_{-x_2}(t)$ $= M_{x_1}(t)M_{x_2}(-t)$ $= e^{\lambda_1(e^t - 1)}e^{\lambda_2(e^{-t}-1)}$ The above cannot be expressed in the form of $= e^{\lambda(e^t-1)}$. $\therefore X_1 - X_2$ is not a Poisson variate.	**K2**	**CO2**
4.	If $M_x(t) = (5-4e^t)^{-1}$, *find* $P(X = 5 \, or \, 6)$ **Solution:** M.G.F of Geometric Distribution is $M_x(t) = p(1-qe^t)^{-1}$ Given $M_x(t) = (5-4e^t)^{-1} = \frac{1}{5}\left(1 - \frac{4}{5}e^t\right)^{-1}$ $\therefore p = \frac{1}{5}, q = \frac{4}{5}$ $P[X = 5 \, or \, 6] = P(X=5) + P(X=6)$ $= \left(\frac{4}{5}\right)^5 \frac{1}{5} + \left(\frac{4}{5}\right)^6 \frac{1}{5}$ $= 0.1179.$	**K1**	**CO2**

5.	Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities? **Solution:** The number of possible paths between the cities is the number of permutations of seven elements, because the first city is determined, but the remaining seven can be ordered arbitrarily. Consequently, there are $7!=7\times6\times5\times4\times3\times2\times1=5040$ ways Forthesaleswoman to choose her tour.	**K2**	**CO2**
6.	State Bayes Theorem. **Solution:** Let $B_1, B_{2,}.......B_n$ be mutually exclusive and exhaustive events in a sample space S with $P(B_i)>0$ for each i, and A be any event in S with $P(A)>0$, then $P(B_r / A)=\dfrac{P(A/B_r).P(B_r)}{\sum_{i=1}^{n}P(A/B_i).P(B_i)}.$	**K1**	**CO2**
7.	The cumulative distribution function of a random variable X is $F(x)=1-(1+x)e^{-x}$, $x>0$. Find the probability density function of X. **Solution:** Given : $F(x)=1-(1+x)e^{-x}$, $x>0$. $=1-e^{-x}-xe^{-x}$, $x>0$ We know that pdf $f(x)=\dfrac{d}{dx}F(X)$ $=e^{-x}+xe^{-x}-e^{-x}$ $\therefore\ f(x)=xe^{-x}$, $x>0$	**K2**	**CO2**

8.

Check whether the function given by

$f(x)=\dfrac{x+2}{25}$ *for* $x=1,2,3,4,5$ is the probability density function.

Solution:

x	1	2	3	4	5
f(x)	3/25	4/25	5/25	6/25	7/25

$$\sum_i p_i=\frac{3}{25}+\frac{4}{25}+\frac{5}{25}+\frac{6}{25}+\frac{7}{25}=1$$

f(x) is a p.d.f

K1 | **CO2**

9.

A continuous random variable X has the following probability density function $f(x)=k\,xe^{-x/2}, k\text{ is constant}, x>0$. Find the value of k for $f(x)$ to be a valid probability density function.

Solution:

For a function $f(x)$ to be a probability density function, we must have

$$\int_{-\infty}^{\infty} f(x)\,dx=1$$

$$\therefore\; k\int_{-\infty}^{\infty} x\,e^{-x/2}dx=1 \Rightarrow k\left[\frac{xe^{-x/2}}{-1/2}-\frac{e^{-x/2}}{1/4}\right]_0^{\infty}=1$$

$$\Rightarrow k\left[(0-0)-\left(0-\frac{1}{1/4}\right)\right]=1$$

$$\Rightarrow 4k=1,\;\Rightarrow k=\frac{1}{4}.$$

K2 | **CO2**

10. Find the value of C and mean of the following distribution.

$$f(x)=\begin{cases} C(x-x^2) & for\ 0<x<1 \\ 0 & otherwise \end{cases}$$

Solution: Since f(x) ispdf $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\therefore \int_0^1 C(x-x^2)dx = 1$$

$$\Rightarrow C\left[\frac{1}{6}\right]=1 \qquad \therefore C=6$$

Mean = E[X] $= \int_{-\infty}^{\infty} x\, f(x)dx$

$$= \int_{-\infty}^{\infty} x\, 6(x-x^2)dx$$

$$= 6\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1 = \frac{1}{2}$$

K2 **CO2**

11. By throwing a fair die, a game is played. A player wins Rs 20 if 2 turns up, Rs 40 if 4 turns up and losses Rs 30 if 6 turns up. And he never loses or gains if any other number turns up. Find the expected value of money won.

Solution:

No. on the top face	1	2	3	4	5	6
$X=x$	0	20	0	40	0	-30
P(x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Let X denote the amount that the player gets on any throw

$\therefore$ Expected given $E(X)=\sum x_i\, p_i$

$$=20\left(\frac{1}{6}\right)+40\left(\frac{1}{6}\right)+(-30)\left(\frac{1}{6}\right)=\frac{30}{6}=\text{Rs.}5$$

K1 **CO2**

12.	If X and Y are independent RVs with variance 2 and 3, then find Var(3X + 4Y). **Solution:** If X and Y are independent, then $\operatorname{var}(aX+bY)=a^2\operatorname{var}(X)+b^2\operatorname{var}(Y)$ Given $\operatorname{var}(X)=2,\ \operatorname{var}(Y)=3$ $\therefore\therefore\operatorname{var}(3X+4Y)=9\operatorname{var}(X)+16\operatorname{var}(Y)$ $=9(2)+16(3)=66$.	**K2**	**CO2**
13.	A continuous random variable X has p.d.f. $f(x)=Kx^2e^{-x},\ x\geq 0$. Find the r^{th} moment of X about the origin. **Solution:** Givenp.d.f. of X is $f(x)=Kx^2e^{-x},\ x\geq 0$. $\therefore\int_{-\infty}^{\infty}f(x)dx=1$ $\int_0^{\infty}Kx^2e^{-x}\,dx=1\Rightarrow K\int_0^{\infty}x^2e^{-x}\,dx=1\Rightarrow K\int_0^{\infty}e^{-x}\,x^{3-1}\,dx=1$ WKT $\Gamma n=\int_0^{\infty}e^{-x}x^{n-1}dx$ $K\,\Gamma 3=1\Rightarrow K\,2!=1\Rightarrow K=\frac{1}{2}$. $\mu_r'=r^{\text{th}}\text{moment about the origin}$ $=E(X^r)=\int_{-\infty}^{\infty}x^r f(x)dx$. $=\int_0^{\infty}x^r\frac{1}{2}x^2e^{-x}dx=\frac{1}{2}\int_0^{\infty}e^{-x}x^{(r+3)-1}dx=\frac{1}{2}\Gamma(r+3)$. $\mu_r'=\frac{1}{2}(r+2)!$	**K2**	**CO2**

14.	Find the moment generating function for the distribution where $f(x)=\begin{cases}\frac{2}{3} & at\ x=1\\ \frac{1}{3} & at\ x=2\\ 0 & otherwise\end{cases}$ Solution: $M_X(t)=E\left[e^{tX}\right]=\sum_{0}^{\infty}e^{tx}f(x)$ $=0+e^{t}\frac{2}{3}+e^{2t}\frac{1}{3}+0+...$ $=\frac{e^{t}}{3}\left[2+e^{t}\right]$	K2	CO2
15.	If a RV X has the MGF $M(t)=\frac{3}{3-t}$, obtain the standard deviation of X. **Solution:** Given $M(t)=\frac{3}{3-t}$ $Var(X)=E(X^2)-\left(E(X)\right)^2=\mu_2^1-\mu_1^{1^2}\Rightarrow S.D.=\sqrt{Var(X)}$ $\mu_1^1=\left(M'(t)\right)_{t=0}=\left(\frac{3}{(3-t)^2}\right)_{t=0}=\frac{1}{3}$ and $\mu_2^1=\left(M''(t)\right)_{t=0}=\left(\frac{6}{(3-t)^3}\right)_{t=0}$ Therefore, variance= $\frac{2}{9}-\frac{1}{9}=\frac{1}{9}$ S.D. = $\frac{1}{3}$	K1	CO2
16.	Prove that the MGF of the sum of n -independent random variables is equal to the product of their respective MGFs. **Solution:** To prove $M_{X_1+X_2+.......+X_n}(t)=M_{X_1}(t).M_{X_2}(t)...M_{X_n}(t)$ $M_{X_1+X_2+.......+X_n}(t)=E\left[e^{(X_1+X_2+...X_n)t}\right]$ $=E\left[e^{X_1t}.e^{X_2t}...e^{X_nt}\right]$ $=E\left[e^{X_1t}\right].E\left[e^{X_2t}\right]....E\left[e^{X_nt}\right]$ ($\because X_1,X_2...X_n$ are independent) $=M_{X_1}(t).M_{X_2}(t)...M_{X_n}(t)$	K1	CO2

PART A QUESTIONS & ANSWERS

Q. No.	Questions &Answers	K Level	CO
17	For a Binomial Distribution mean is 6 and S.D. is $\sqrt{2}$. Find the first two terms of the distribution. **Solution:** For a B.D., mean = np = 6 --------(1) Variance = npq = 2 --------(2) $\Rightarrow \frac{npq}{np} = \frac{2}{6} = \frac{1}{3}$ $q = \frac{1}{3} \Rightarrow p = 1 - q = \frac{2}{3}$ $np = 6 \Rightarrow n = 9$ $P(X = x) = nC_x p^x q^{n-x} = 9C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{9-x}; x = 0,1,...9$ (i) $P(X = 0) = \left(\frac{1}{3}\right)^9$ (ii) $P(X = 1) = 9\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^8$	K1	CO2
18	It is known that 5% of the books bound at a certain binding unit have defective bindings. Find the probability that 2 out of 100 books bound by this binding unit will have defective bindings. **Solution:** $x = 2,\ n = 100,\ p = 0.05$ $\lambda = np = 5$ $P(X = 2) = \frac{e^{-\lambda}\lambda^2}{x!} = \frac{e^{-5}5^2}{2} = 0.084$	K1	CO2
19	The life length (in months) X of an electronic component follows an exponential distribution with parameter $\lambda = \frac{1}{2}$ .What is the probability that the component survives at least 10 months, given that already it had survived for more than 9 months. **Solution:** $f(x) = \lambda e^{-\lambda x}$ $mean = \lambda = \frac{1}{2}$ To find $P(X \geq 10 / X \geq 9) = P(X > 1)$ (By memory less property) $= \int_1^\infty \frac{1}{2} e^{-x/2} dx = 0.6065.$	K2	CO2

Q. No.	Questions &Answers	K Level	CO
20	If X is a Poisson variate shows that $P(X=2)=9P(X=4)+90P(X=6)$, find the variance. **Solution:** $P(X=2)=9P(X=4)+90P(X=6)$ $\frac{e^{-\lambda}\lambda^2}{2!}=9\frac{e^{-\lambda}\lambda^4}{4!}+90\frac{e^{-\lambda}\lambda^6}{6!}$ $\frac{1}{2}=\frac{9}{24}\lambda^2+\frac{90\lambda^4}{720}$ $\frac{1}{2}=\frac{3\lambda^2+\lambda^4}{8}\Rightarrow 3\lambda^2+\lambda^4=4$ $\lambda^4+3\lambda^2=4$ Put $\lambda^2=t$ $t^2+3t-4=0$ $(t+4)(t-1)=0$ $t=-4,1$ $\therefore \lambda^2=-4\Rightarrow\lambda=\pm 2i$ $\lambda^2=1\Rightarrow\lambda=\pm 1$ $\therefore \lambda$ is always +ve $\therefore \lambda=1$ $\text{variance}=1$	K1	CO2
21	Find the M.G.F. of a Poisson variate. **Solution:** $M_x(t)=E\left[e^{tx}\right]=\sum_{x=0}^{\infty}e^{tx}p(x)$ $=\sum_{x=0}^{\infty}e^{tx}\frac{e^{-\lambda}\lambda^x}{x}$ $=e^{-\lambda}\left[1+\frac{\lambda e^t}{1!}+\frac{(\lambda e^t)^2}{2!}+\frac{(\lambda e^t)^3}{3!}+\dots\right]$ $=e^{-\lambda}e^{\lambda e^t}$ $=e^{-\lambda(1-e^t)}$	K1	CO2

22	Ten coins thrown simultaneously. Find the probability of getting at least seven heads. **Solution:** $n=10,\ \ p=\frac{1}{2}=9$ $P(X\geq 7)=P(X=7)+P(X=8)+P(X=9)P(X=10)$ $=10C_7\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^3+10C_7\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^3+10C_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2$ $+10C_9\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^1+10C_{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^0$ $P(X\geq 7)=\frac{176}{1024}$	**K2**	**CO2**
23	Assuming that the probability of a fatal accident in a factory during the year is $\frac{1}{1200}$. Calculate the probability that in a factory employing 300 workers, there will be at least two fatal accidents in a year. **Solution:** $p=\frac{1}{1200},\ \ n=300,$ $\lambda=\frac{1}{1200}\times 300=\frac{1}{4}=0.25$ $\Rightarrow P(X\geq 2)=\frac{e^{-1/4}(1/4)^2}{2}=0.0265$.	**K2**	**CO2**
24	State and prove the additive property of Poisson distribution. **Solution:** If X and Y be independent Poisson variate with parameter λ_1 and λ_2 respectively then X+Y is also a Poisson variate with parameter $\lambda_1+\lambda_2$. X is a Poisson variate with parameter λ_1 $\Rightarrow M_x(t)=e^{\lambda_1(e^t-1)}$ Y is a Poisson variate with parameter λ_2 $\Rightarrow M_y(t)=e^{\lambda_2(e^t-1)}$ Now $M_{x+y}(t)=M_x(t).M_y(t)$ $=e^{\lambda_1(e^t-1)}e^{\lambda_2(e^t-1)}$ $=e^{(\lambda_1+\lambda_2)(e^t-1)}$ $\therefore X+Y$ is also a Poisson variate with parameter $(\lambda_1+\lambda_2)$.	**K2**	**CO2**

PART B QUESTIONS

Q. No.	Questions	K Level	CO
1.	If 10% of the screws produced by an automatic machine are defective, find the probability that out of 20 screws selected at random, there are i) exactly 2 defective ii) at most 3 defective iii) at least 2 defectives and iv) between 1 and 3 defectives(inclusive). **Ans:** $i)P(\text{exactly 2 defective}) = 190\times\frac{9^{18}}{10^{20}}$ $ii)P(\text{atmost 3 defective}) = 5199\times\frac{9^{17}}{10^{20}}$ $iii)P(\text{atleast 2 defective}) = 1-29.\frac{9^{19}}{10^{20}}$ $iv)P(\text{between 1 and 3}) = 4470\times\frac{9^{17}}{10^{20}}$	**K1**	**CO2**
2.	Find M.G.F, Mean and variance of a Binomial distribution. **Ans:** $\text{M.G.F} = (q+pe^t)^n$, $\text{Mean} = np$, $\text{Variance} = npq$	**K1**	**CO2**
3.	In a large consignment of electric bulbs 10% are defective. A random sample of 20 is taken for inspection. Find the probability that i) All are good bulbs ii) Atmost there are 3 defective bulbs iii) Exactly there are three defective bulbs. **Ans:** i) 0.1216 ii) 0.8666 iii) 0.19	**K2**	**CO2**
4.	If the probability that a man aged 60 will live to be 70 is 0.65, what is the probability that out of 10 men, now 60, at least 7 will live to be 70? **Ans:**0.509	**K2**	**CO2**
5.	In certain town, 20% samples of the population are literate and assume that 200 investigators take sample of ten individuals to see whether they are literate. How many investigators would you expect to report that 3 people or less are literates in the samples? **Ans:**176	**K2**	**CO2**
6.	The number of monthly breakdowns of a computer is RANDOM VARIABLE having a Poisson distribution with mean equal to 1-8. Find the probability that this computer will function for a month with only one breakdown. **Ans :** 0.2975	**K2**	**CO2**

Q. No.	Questions	K Level	CO
7.	An insurance company found that only 0.01% of the population is involved in a certain type of accident each year. If its 1000 policy holders were randomly selected from the population, what is the probability that not more than two of its clients are involved in such an accident next year? **Ans :** 0.9998	K1	CO2
8.	In a certain factory turning razor blades there is a small chance of 1/500 for any blade to be defective. The blades are in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing (i) No defective, (ii) One defective , (iii) 2 defective blades respectively in a consignment of 10000 packets. **Ans :**(i) 9802 packets (ii) 196 packets (iii) 2	K1	CO2
9.	Wireless sets are manufactured with 25 soldered joints each on the average 1 joint in 500 defective. How many sets can be expected to be free from defective joints in a consignment of 10000 sets. **Ans :** 9512	K2	CO2
10.	The atoms of a radioactive element are randomly disintegrating. If every gram of this element, on average, emits 3-9 alpha particles per second, what is the probability that during the next second the number of alpha particles emitted from 1 gm is (a) at most 6 (b) at least 2 (c) at least 3 and at most 6 **Ans :** (a) 0.899 (b) 0.901 (c) 0.646	K2	CO2
11.	Find Moment generating function , mean and variance of the Poisson distribution **Ans:** $\text{M.G.F} = e^{-\lambda} e^{\lambda(e^t - 1)}$, $\text{Mean} = \lambda$, $\text{Variance} = \lambda$	K2	CO2
12.	If the mgf of X is $(5-4e^t)^{-1}$, find the distribution of X and $P(X=5)$. **Ans:** $\frac{256}{3125}$.	K2	CO2

13.	State and prove memoryless property of Geometric distribution.	**K2**	**CO2**
14.	A soldier shoots a target in an independent fashion. If the probability that the target is shot on any one of shot is 0.8. (i) What is the probability that a target would be first hit at the 6^{th} attempt? (ii) What is the property that it takes less than 5 shots? Ans: i)0.00026 iii)0.9984	**K2**	**CO2**
15.	Find mean and variance of the geometric distribution without using the concept of m.g.f.	**K2**	**CO2**

Part B Questions

Q. No.	Questions	K Level	CO
1.	How many bit strings of length 10 contain i) Exactly four 1's ii) Atmost four 1's iii) Atleast four 1's iv) An equal number of 0's and 1's. **Ans**: i)210 ii)386 iii)848 iv)252	K1	CO2
2.	From a club consisting of six men and seven women. In how many ways we select a committee of i) 3 men and 4 women ii) 4 person which has atleast 1 women iii) 4 person which has atlmost 1 man iv) 4 person that has children of both sexes. **Ans**: i)700 ii)700 iii)245 iv)665	K1	CO2
3.	A random variable X has density function given by a. $f(x)=\begin{cases}\frac{1}{k}, & for\ 0<x<k\\ 0, & otherwise\end{cases}$ Find (1) m.g.f (2) r^{th} moment (3) mean (4) variance. **Ans:** $\frac{(e^{kt}-1)}{kt}, \frac{k^r}{(r+1)!}, \frac{k}{2}, \frac{k^2}{12}$	K2	CO2
4.	The density function of a RV is given by $f(x)=k\,x(2-x)\,, 0\le x\le 2.$ Find k, Mean, Variance and r^{th}moment. **Ans:** k = ¾, 1, 1/5, $\frac{3.2^{r+1}}{(r+2)(r+3)}$	K2	CO2
5.	X is a discrete random variable with probability function $p(x)=\frac{1}{K^x}\ , x=1,2,\ldots$ where k is a constant. Find its (i) M.G.F. (ii) mean (iii) Variance. **Ans:** $\frac{e^t}{k-e^t}, \frac{k}{(k-1)^2}, \frac{k^3-k^2-k}{(k-1)^4}$	K2	CO2

6.	Find the M.G.F. of the random variable with p.d.f. $f(x)=\begin{cases} x & 0\le x<1 \\ 2-x & 1\le x<2 \\ 0 & otherwise. \end{cases}$. Also Find its m.g.f., mean and variance. **Ans:** $\frac{(1-e^t)^2}{t^2}$, 1, 1/6	**K2**	**CO2**
7.	If X represents the outcome, when a fair die is tossed, find the M.G.F. of X and hence find E(X) and Var(X). **Ans:** $\frac{e^t(1-e^{6t})}{6(1-e^t)}$, 7/2, 5/4	**K1**	**CO2**
8.	A coin is tossed until a head appears. What is the expectation of the number of tosses required? **Ans:** 2	**K1**	**CO2**
9.	Let X be a random variable with p.d.f $f(x)=\begin{cases} \frac{1}{3}e^{-x/3} , & x>0 \\ 0 , & otherwise \end{cases}$ Find (i) M.G.F of X (ii) $E[X]$ (iii) $\mathrm{var}(X)$ **Ans:** $i)m.g.f=\frac{1}{1-3t}$ $ii)E[X]=3$ $iii)Var[X]=9$	**K2**	**CO2**

Supportive Online Certification Courses

- Online Course: NPTEL
  Course Name: NOC:Introduction to probability and Statistics
  Course Instructor: Prof. G. Srinivasan, IIT Madras
  Duration : 4 weeks
  https://nptel.ac.in/courses/111/106/111106112/

- Online Course: NPTEL
  Course Name: NOC: Introduction to Probability Theory and Stochastic Processes
  Course Instructor: Dr. S. Dharmaraja, IIT Delhi
  Duration : 12 weeks
  https://nptel.ac.in/courses/111/102/111102111/

- Online Course: NPTEL
  Course Name: NOC:Probability and Statistics
  Course Instructor: Prof. Somesh Kumar
  Department of MathematicsIIT Kharagpur
  Duration : 10 weeks
  https://nptel.ac.in/courses/111/105/111105090/

- Online Course: Coursera
  Course Name: Probability Theory : Foundation for Data Science
  Course Instructor: 1. Anne Dougherty, Senior Instructor and Teaching Professor, University of Colorado Boulder.
  2. Jem Corcoran, Associate Professor, University of Colorado Boulder
  Duration : 1-3 months

https://www.coursera.org/learn/probability-theory-foundation-for-data-science

R.M.K GROUP OF INSTITUTIONS

Real Time Applications

- Concept of Probability - Basics- Why Learn Probability?
  https://www.youtube.com/watch?v=74zR4OQ-ByY

- Concepts of Empirical Probability - Basics - Understanding Empirical Probability
  https://www.youtube.com/watch?v=oliMtbLoDTE

- Concept of Probability - Basics - Application Question
  https://www.youtube.com/watch?v=C0dslb8h5yQ
  https://www.youtube.com/watch?v=GSQWPHd3-CM
  https://www.youtube.com/watch?v=Ru1s7K4wed4
  https://www.youtube.com/watch?v=74zR4OQ-ByY
  https://www.youtube.com/watch?v=YoM87Td5_os
  https://www.youtube.com/watch?v=f2kc88oqUOg
  https://www.youtube.com/watch?v=vNhDhkMTqcI
  https://www.youtube.com/watch?v=9t9WnQErTbk

- Applications of Empirical Probability - Real Life Applications - Application
  https://www.youtube.com/watch?time_continue=30&v=_rmswvrmZ8w&feature=emb_logo
  https://www.youtube.com/watch?v=2rR-vFT2r7M
  https://www.youtube.com/watch?v=PO5wG5_x3Oc
  https://www.youtube.com/watch?v=-6kdwzYgl_c
  https://www.youtube.com/watch?v=HM9-HUVGl2c
  https://www.youtube.com/watch?v=1uvJuBc9l_8
  https://www.youtube.com/watch?v=J7QCWreTio4
  https://www.youtube.com/watch?v=9qkII19r1Ac

- Understanding the applications of Probability in Machine Learning
  https://www.datasciencecentral.com/profiles/blogs/understanding-the-applications-of-probability-in-machine-learning

- probability and applications videos
  https://www.bing.com/videos/search?q=probability+and+applications&qpvt=probability+and+applications&FORM=VDRE

MINI PROJECT

Find Probability distributions using any Programming Language

https://www.tutorialspoint.com/python_data_science/python_binomial_distribution.htm

http://landofprogramming.blogspot.com/2014/08/c-program-for-finding-binomial.html

https://vitalflux.com/poisson-distribution-explained-with-python-examples/

https://reference.wolfram.com/language/ref/PoissonDistribution.html

https://vitalflux.com/normal-distribution-explained-python-examples/

Prescribed Text Books & Reference Books

PROBABILITY AND STATISTICS	
S. No.	**TEXT BOOKS**
1.	Miller and Freund's Probability and Statistics for Engineers, Johnson, R.A., Miller, I and Freund J., Pearson Education, Asia, 8th Edition, 2015.
2.	Introduction to Probability and Statistics, Milton. J. S. and Arnold. J.C., Tata McGraw Hill,4 th Edition, 2017.
REFERENCES :	
1.	Devore. J.L., "Probability and Statistics for Engineering and the Sciences‖,Cengage Learning, New Delhi, 9th Edition, 2016.
2.	Ross, S.M., "Introduction to Probability and Statistics for Engineers and Scientists", 6thEdition, Elsevier, 2020.
3.	Spiegel. M.R., Schiller. J. and Srinivasan, R.A., "Schaum's Outline of Theory and Problems of Probability and Statistics", Tata McGraw Hill Edition, 2004.
4.	Walpole. R.E., Myers. R.H., Myers. S.L. and Ye. K., "Probability and Statistics for Engineers and Scientists", Pearson Education, Asia, 9th Edition, 2012.

Thank you

Disclaimer: