

Linear Algebra

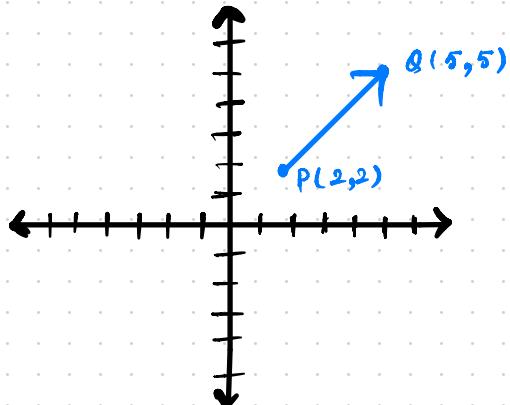
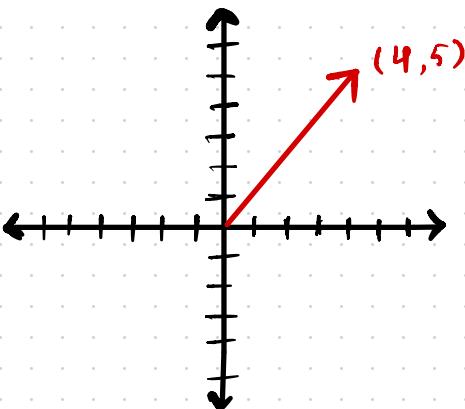
What is Linear Algebra?

- Field of mathematics that is concerned with study of lines in high-dimensional space.
- It helps in solving unknowns within system of linear equations.

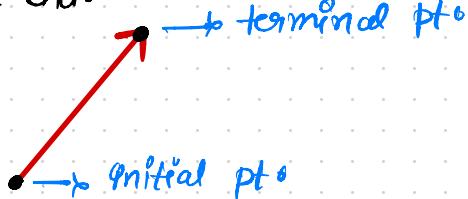
Example:- → The sum of two nos. is 25. One of them exceeds the other by 9. find the nos.

↑
"Unknowns"

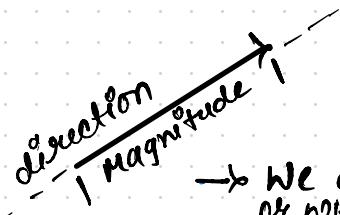
Vectors :-



→ A vector is a specific quantity as line seg. with an arrowhead at one end.



→ It is defined by its Magnitude or the length of a line and its direction



A vector is directed line segment.

→ We denote vectors as lower case, boldfaced with or without an arrow like $\vec{a}, \vec{B}, \mathbf{C}, d$

→ Given initial pt. P and terminal pt. Q, a vector can be expressed as \overrightarrow{PQ} .

→ Given initial pt. (0, 0), a vector can be represented as $\begin{bmatrix} a \\ b \end{bmatrix}$.

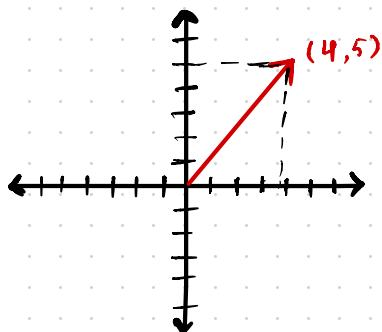
$\begin{bmatrix} a \\ b \end{bmatrix}$ is vector's Standard position. The pos. vector has initial pt. as (0, 0) and terminal pt. as (a, b).

Most of the time we will only deal with position vectors.

Eg:- Given: P(2,3) and Q(6,4); What's Pos. vector?

$$\begin{aligned} r &= \langle 6-2, 4-3 \rangle \\ &= \langle 4, 1 \rangle \\ &= \begin{bmatrix} 4 \\ 1 \end{bmatrix} \rightarrow x\text{- component} \\ &\quad \rightarrow y\text{- component} \end{aligned}$$

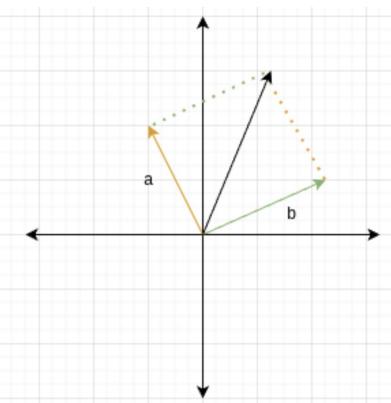
Magnitude of a vector :-



$$v = \begin{bmatrix} 4 \\ 5 \end{bmatrix}; |v| = \sqrt{x^2 + y^2}$$

$|v| = \sqrt{4^2 + 5^2}$
So, $|v|$ can be found using pythagoras theorem.

Addition of Vectors :-



- Geometric view of this operation

Vector-Vector Product

$$\vec{a} + \vec{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

Transpose of vector :-

$$\vec{x}^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}^T = [x_1, x_2, x_3 \dots x_m]$$

make rows from columns

$$\vec{x}^T = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}^T = [2, 3, 4]$$

$$\vec{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\vec{a} + \vec{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

* Scalar-Vector Product :-

Scalar \rightarrow It's just an integer.

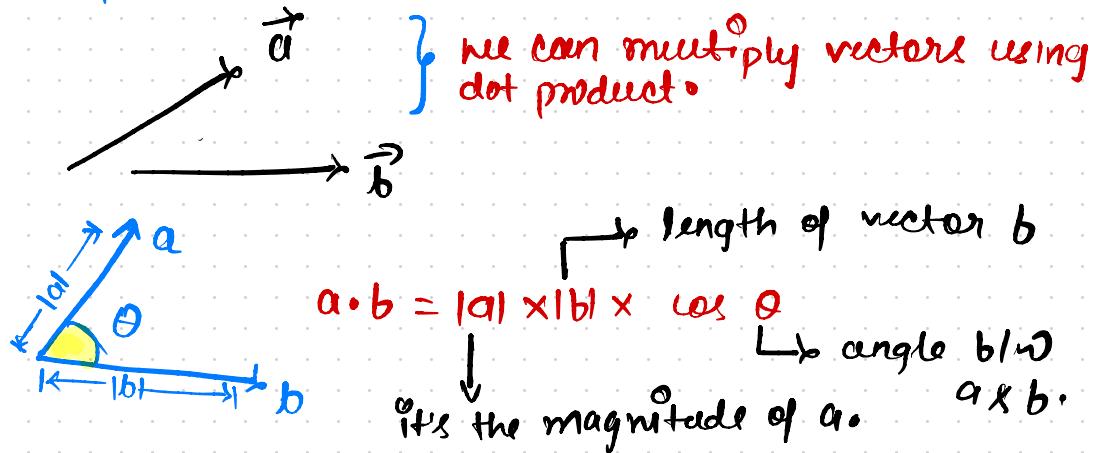
$$c \cdot \vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$$

$$2 \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

it scales your vector by some c.

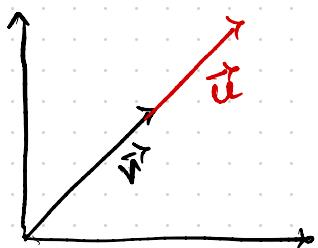
Strategies for Multiplying two vectors :-

* Dot product :-



→ It's about combining two vectors into a single no. It tells us about much two vectors point in the same direction.

case :- When $\theta = 0$

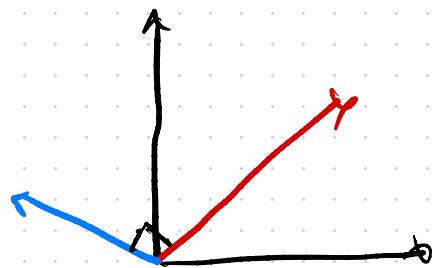


(here dot product is largest as it points in same direction)

$$\vec{v} \cdot \vec{u} = |\vec{v}| \times |\vec{u}| \times \cos(0)$$

$$\downarrow \cos(0) = 1$$

The more two vector points in same direction, the larger dot product will be!



$$\cos(90^\circ) = 0; a \cdot b = |\vec{a}| \times |\vec{b}| \times 0 = 0$$

* These type of vector are called orthogonal vectors.

A better way to calculate dot product :-

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \vec{a} \cdot \vec{b} = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3 \\ = \sum_{i=0}^n a_i \cdot b_i$$

Cross Product :-

$$\vec{a} \times \vec{b} = \vec{c}$$

resulting vector

Properties :-

$$|\vec{c}| = |\vec{a}| |\vec{b}| \sin(\theta)$$

case : When $\theta = 90^\circ$

$$\sin(90^\circ) = 1$$

$$|\vec{c}| = |\vec{a}| |\vec{b}| 1$$

in this case cross product is the longest.

$\rightarrow \vec{c}$ is perpendicular to both $\vec{a} \times \vec{b}$.
 $\rightarrow |\vec{c}|$ is a measure of how far apart \vec{a} & \vec{b} are pointing augmented by their magnitudes.

Linear Combination :-

* Given any no. of vectors, linear combination when we multiply each vector by a scalar and sum it all up!

$$u = \begin{bmatrix} -5 \\ 0 \end{bmatrix} \text{ & } v = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \text{ when we multiply with some scalars.}$$

$$(1) \begin{bmatrix} -5 \\ 0 \end{bmatrix} + (2) \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}; \begin{bmatrix} -5 \\ 2 \end{bmatrix} \text{ is the lin. combination of } u \text{ & } v.$$

\rightarrow A vector w is said to be a lin. combination of a, b, c etc. if there exists scalars $x, y, z, \dots, \text{etc.}$ s.t. $w = x a + y b + z c + \dots \text{etc.}$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = (-1)u + 2v \quad \rightarrow \quad \begin{bmatrix} 4 \\ 2 \end{bmatrix} \text{ is a lin. combination of } u \text{ & } v.$$

Linear dependence & Independence :-

- * A set of vectors is said to be lin. independent if no vector can be represented as lin. combination of the remaining vectors.
- * If a vector is said to be lin. dependent if a vector can be represented as lin. combination of the vectors.

Span :-

- * The set of all the possible lin. combination of given group of vectors is called the Span of those vectors.

The Span of vectors v_1, v_2, v_3 is written as Span of { v_1, v_2, v_3 }.

Eg:- $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ is the Span of $\begin{bmatrix} -5 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

Norms :-

- * L2 Norm :- Euclidean Norm, calculates the dist. from the origin;

$$\|x\|_2 = \sqrt{\sum_i x_i^2}$$

- * L1 Norm :- Manhattan distance, calculates the sum of absolute values.

$$\|x\|_1 = \sum_{r=1}^n |x_r|$$

- * Max Norm :- distance by taking out the max element.

$$\|x\|_\infty = \max_i |x_i| \quad \left[\begin{array}{c} 2 \\ 5 \\ 9 \end{array} \right] \quad \text{max} \rightarrow 9$$

Orthogonal vectors :-

- * Two vectors (assume a & b) are orthogonal, if they are perpendicular to each other.

$$a \cdot b = 0$$

↳ dot product

Matrices :-

- It is a rectangular array or table of numbers, symbols arranged in rows and columns.
- data which have n , no. of rows and m , no. of cols.

$$\begin{matrix} 1 & 2 & \dots & n \\ 1 & a_{11} & a_{12} & \dots & a_{1n} \\ 2 & a_{21} & a_{22} & \dots & a_{2n} \\ 3 & a_{31} & a_{32} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m & a_{m1} & a_{m2} & \dots & a_{mn} \end{matrix}$$

Operations :-

- * Scalar-Matrix Product :-

$$2 \cdot \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 4 & 8 \end{bmatrix}$$

- * Matrix Addition :-

$$A + B = C ; \begin{bmatrix} 2 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 4 \\ 4 & 8 & 6 \end{bmatrix}$$

Properties :-

↑ ↓ ↓ ↓
A n B C

- * Commutative Property :-

$$A + B = B + A$$

- * Associative Property :- $A + (B + C) = (A + B) + C$

- * Dimensions should be same.

Operations on matrices

Matrix-Matrix Multiplication

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{pmatrix}$$

Number of columns in the first matrix must be equal to the number of rows in the second matrix.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 2 \cdot 2 + 3 \cdot 5 & 2 \cdot 3 + 3 \cdot 6 & 2 \cdot 4 + 3 \cdot 7 \\ - & - & - \end{bmatrix}$$

$$C = \begin{pmatrix} a_{11}b_{11} + \dots + a_{1n}b_{n1} & a_{11}b_{12} + \dots + a_{1n}b_{n2} & \dots & a_{11}b_{1p} + \dots + a_{1n}b_{np} \\ a_{21}b_{11} + \dots + a_{2n}b_{n1} & a_{21}b_{12} + \dots + a_{2n}b_{n2} & \dots & a_{21}b_{1p} + \dots + a_{2n}b_{np} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{11} + \dots + a_{mn}b_{n1} & a_{m1}b_{12} + \dots + a_{mn}b_{n2} & \dots & a_{m1}b_{1p} + \dots + a_{mn}b_{np} \end{pmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}$$

$$\begin{aligned} A + (B + C) &= \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} + \left\{ \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \right\} \\ &= \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 6 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 3 \\ 9 & 3 \end{bmatrix}_{2 \times 2} \end{aligned}$$

$$(A + B) + C$$

$$\begin{bmatrix} 5 & 2 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 9 & 3 \end{bmatrix}$$

Operations on matrices

Matrix-Vector Product

$$A = \begin{bmatrix} 2 & 6 \\ 3 & 4 \\ 2 & 2 \end{bmatrix} \cdot v = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ so, } \vec{s} = \begin{bmatrix} 4+18 \\ 6+12 \\ 4+6 \end{bmatrix} = \begin{bmatrix} 22 \\ 18 \\ 10 \end{bmatrix}$$

Note: You will write out the findings from this operation which is given in your problem set.

Operations on matrices

Transpose of a Matrix

Transpose of a matrix is an operator which flips over its diagonal.

- Reflects A over it's main diagonal to obtain A^T .
- Write the rows of A as the cols of A^T .
- Write the cols of A as the rows of A^T .

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Special Types of Matrices

A Matrix is called symmetric matrix if it is equal to its transpose.

$$B = \begin{pmatrix} 0 & 4 & 7 \\ -4 & 0 & -3 \\ -7 & 3 & 0 \end{pmatrix}$$

$$B^T = \begin{pmatrix} 0 & -4 & -7 \\ 4 & 0 & 3 \\ 7 & -3 & 0 \end{pmatrix}$$

Identity matrix is a matrix where all diagonal elements are 1 and remaining elements are 0.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determinant of a Matrix

The determinant is a scalar value that is a function of the entries of a square matrix. It gives you the area under the parallelogram.

The determinant of a matrix A is denoted $\det(A)$, $\det A$, or $|A|$. In the case of a 2×2 matrix the determinant can be defined as:-

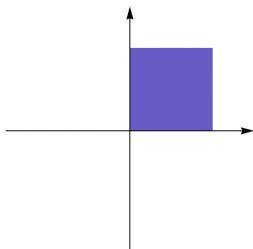
$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Similarly,

$$\begin{aligned} |A| &= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= aei + bfg + cdh - ceg - bdi - afh. \end{aligned}$$

Minor of Matrix A

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \end{vmatrix} = 1.$$



Co-Factor, Minor of a Matrix

A minor of matrix A is the determinant of some smaller square matrix, cut down from A by removing one or more of its rows and columns.

We obtain minors by removing just one row and one column from a square matrices and this is required for calculating matrix co-factor.

It is useful in calculation of determinant and Inverse!

$$\begin{bmatrix} 1 & 4 & 7 \\ 3 & 0 & 5 \\ -1 & 9 & 11 \end{bmatrix} \quad \text{Minor of a Matrix:- } M_{2,3} = \det \begin{bmatrix} 1 & 4 \\ -1 & 9 \end{bmatrix} = 9 - (-4) = 13$$

The Co-Factor of (2,3) entry is, $C_{2,3} = (-1^{2+3}) * M_{2,3} = -13$

$$\det(A) = \sum_{i=1}^n a_{ij} (-1)^{i+1} M_{ij}$$

Adjugate of a Matrix

The adjugate of a square matrix is just the transpose of the it's Co-Factor Matrix.

Inverse Matrix

Let's start with a simple example, $A^{-1} = \frac{1}{A}$ But when we talk about Inverse of matrix, it's just high-dimensional generalization.

- Calculate the minors of all elements of matrix A.
- Compute the Co-Factors of all elements of matrix A.
- Take out the adj A by taking out transposing of matrix A.
- Multiply the adj A by reciprocal of $\det(A)$.

Note:- We will look into uses cases as we go in depth of M2.