Introduction to Kinetic theory

- Continuum description

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = -\nabla p + \nabla \cdot \left(\mu \left[\nabla u + \nabla u^T \right] \right) + \vec{F}_{ext}$$

Continuum assumes that macroscopic variables (density, pressure, velocity) vary smoothly across all length scales. Equations ① and ② are valid for compressible fluids.

In other words, size of molecules, typical distance before molecule collide with another molecule (mean free-path), type of interactions between the molecules does not enter in above equations.

For incompressible flow, density ρ is Constant

$$\nabla \cdot \vec{u} = 0, \quad \rho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = -\nabla p + \mu \nabla^2 \vec{u} + \vec{F}_{ext}$$

At any point $\vec{x}$, number of unknowns are $\vec{u}$ (three velocity components) and pressure p. we have four equations and four unknowns.

For Compressible flow, pressure and density are related by equation of state.

$$p = f(\rho, T) \quad \text{— equation of state}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0, \quad \rho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = -\nabla p + \mu \nabla \cdot \left[\nabla u + \nabla u^T \right] + \vec{F}_{ext}$$

In this case, there are five unknowns and five equations.

for ideal gas $p = \rho RT$.

Consider a classroom of size $10 \times 7 \times 5\,m$ at room temperature $T = 298K$ and atmospheric pressure $P = 10^5\,Pa$.

Number of moles is $n = \frac{PV}{RT} = \frac{10^5 \times 350}{8.31 \times 298} \sim 10^4$

Number of air molecules $N \sim 10^{23} \times 10^4 \sim 10^{27}$

$$\vec{u}_K^{\,t+\Delta t} = \vec{u}_K^{\,t} + \frac{\vec{F}_K \Delta t}{m_K}$$

$$\vec{x}_K^{\,t+\Delta t} = \vec{x}_K^{\,t} + \left(\vec{u}_K^{\,t} + \vec{u}_K^{\,t+\Delta t}\right)\frac{\Delta t}{2}$$

subscript K denotes particle index, Δt is time increment, $\vec{u}_K$ is particle velocity of K^{th} particle, $\vec{F}_K$ is force on K^{th} particle.

Number of floating point operations required to updat position of all particles is $N_{FLO} = 6N \sim 10^{27}$

Even worlds top supercomputers (with millions of cores), Can perform $N_{FLOPS} \sim 10^{18}$ (floating point operations per second). To update position of all particles it will take them ~ 30 years. You can also estimate space required to store all the position and velocity data.

Molecular dynamics Infact solves Newton's equations of motion explicitly.. It is easy to see why it is limited in terms of physical space explored. If we solve everything at molecular level, macroscopic properties such as density, pressure, Can we make progress by talking about collection of particles, instead of talking about individual particles?

Note that continuum approximation in fact already does that. In this approximation, there is well defined macroscopic velocity $\vec{u}$ along with pressure p and density ρ.

To make progress, the aim is to work on length scale that is in between particle dynamics length scale and macroscopic length scale.

To this end, particle distribution function is defined $f(\vec{x}, \vec{\xi}, t)$, where $\vec{\xi}$ is particle velocity and denotes mass density at position $\vec{x}$ moving with velocity $\vec{\xi}$ at time t.

$$\rho(\vec{x}, t) = \int f(\vec{x}, \vec{\xi}, t)\, d^3\xi = \sum f(\vec{x}, \vec{\xi}, t)\, d^3\xi$$

$$\rho(\vec{x}, t)\, \vec{u}(x, t) = \sum f(\vec{x}, \vec{\xi}, t)\, \vec{\xi}\, d^3\xi = \int f(\vec{x}, \vec{\xi}, t)\, \vec{\xi}\, d^3\xi$$

Similarly,

$$\rho E = \frac{1}{2} \int \xi^2 f(\vec{x}, \vec{\xi}, t)\, d^3\xi$$

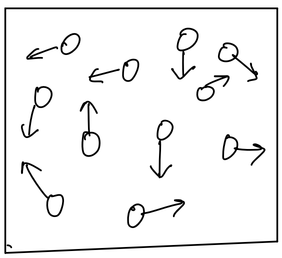
particle distribution function provides information on length-scale that is intermediate between particle size and macroscopic size.

If $f(\vec{x}, \vec{\xi}, t)$ is known, we can find out the macroscopic information of interest $p, \rho, \vec{u}$. We have changed the question of finding $p, \rho, \vec{u}$ to finding $f(\vec{x}, \vec{\xi}, t)$ but not have answered the question yet.

Infact, we have made the question even more Complex. At any point $\vec{x}$, on large (macroscopic) scale, we have only 5 variables. (P, ρ, u_x, u_y, u_z). particle distribution function at any point $\vec{x}$ has infinite variables $f(\vec{x}, \vec{\xi}, t)$ as particle velocity (each Component) Can Continuously vary from $-\infty$ to $+\infty$. (As we will see, the trick is to limit possible directions by allowing only Certain discrete ones)

We proceed anyway and look for evolution rule for particle distribution function $f(\vec{x}, \vec{\xi}, t)$. The first Step is to make use of equilibrium distribution function f^{eq}

Fig. (1)



Consider a Control volume which has large number of particles moving in different directions such that they have an average velocity $\vec{u}$

Boundary Conditions are periodic on all sides. Collision between particles are elastic. momentum and energy of the system is Conserved. If let Collisions take place for some time particles will exchange momentum and energy such that probability of particles having speed different than mean Speed $|\vec{u}|$ will decrease. It makes sense to define relative particle velocity

$$\vec{v} = \vec{\xi} - \vec{u}$$

we expect $f^{eq}(\vec{x},\vec{v},t)$ to be Symmetric around $\vec{v}=0$ that is if $f^{eq}(\vec{x},\vec{v},t)$ is expanded as a series in $\vec{v}$

$$f^{eq}(\vec{x},\vec{v},t) = C_0 + \vec{C_1}\cdot\vec{v} + C_2\,\vec{v}\cdot\vec{v} + (\vec{C_3}\cdot\vec{v})\,v_3^2 + \cdots$$

the odd coefficients $\vec{C_1}, \vec{C_3}$ to be zero to ensure

$f^{eq}(\vec{x},\vec{v},t) = f^{eq}(\vec{x},-\vec{v},t)$. (that is change of sign for $\vec{v}$)

thus,

$$f^{eq}(\vec{x},\vec{v},t) = C_0 + C_2\,\vec{v}\cdot\vec{v} + C_4\,(\vec{v}\cdot\vec{v})^2 + \cdots\cdots$$

We also expect that most of the particles will have a velocity close to the mean velocity $\vec{u}$ of the Control volume. That is $f^{eq}(\vec{x},\vec{v},t)$ will have maximum for $\vec{v} = \vec{\xi} - \vec{u} = 0$.

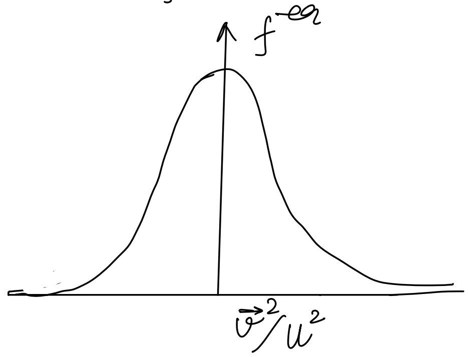


Fig (2)

qualitative expectation for f^{eq}. as $\left|\frac{\vec{v}}{\vec{u}}\right|$ increases, f^{eq} should decrease, finally vanishing for $\left|\frac{\vec{v}}{\vec{u}}\right| \gg 1$. x coordinate is square of scaled relative velocity

For simplicity, consider a two dimensional case so that $v_x = \xi_x - u_x$ and $v_y = \xi_y - u_y$. , subscripts x, y stand for velocity Components in x and y direction.

if one of the scaled Components of the relative velocity $\vec{v}$, say v_y, deviates significantly from zero,

$$\frac{v_y^2}{u^2} \gg 1$$

it follows that $$\frac{v_x^2 + v_y^2}{u^2} = \frac{\vec{v}\cdot\vec{v}}{\vec{u}\cdot\vec{u}} \gg 1$$

from our qualitative Fig (2), then we conclude that probability of finding such particles is very small.

A simplest model that can reproduce the qualitative behaviour is a multiplicative distribution function

$$f^{eq}(\vec{x}, \vec{v}^2, t) = f^{eq}(\vec{x}, v_x^2, t)\, f^{eq}(\vec{x}, v_y^2, t)$$

The assumption behind the construction being, relative velocity components [velocity deviation from mean velocity] are not correlated. The assumption is popularly known as molecular chaos and is applicable for binary collision model.

This multiplicative distribution function fits well with expectation that if one of velocity deviation is large, overall distribution function will decrease, independently of other components of velocity deviation.

At any given position $\vec{x}$ and time $\vec{t}$, f^{eq} depends upon v^2 alone. Thus, assuming molecular chaos,

$$f^{eq}(\vec{v}^2) = f^{eq}(v_x^2 + v_y^2 + v_z^2) = f^{eq}(v_x^2)\, f^{eq}(v_y^2)\, f^{eq}(v_z^2)$$

if $f(x+y) = f(x)f(y) \Rightarrow f(x) = Ke^{Gx}$

Thus $f^{eq}(\vec{v}^2) = K_1 e^{-K_2 \vec{v}^2}$, ($K_1$, K_2 are constants such that $K_1 > 0$, $K_2 > 0$)

$$- \int f^{eq}\, d^3\xi = \int K_1 e^{-K_2 v^2}\, d^3\xi = K_1 \int e^{-K_2 v^2}\, d\xi_x\, d\xi_y\, d\xi_z$$

$$\rho = K_1 \int_0^\infty e^{-K_2 U^2} (4\pi U^2)\, dU = \frac{K_1 \pi^{3/2}}{K_2^{3/2}} \quad -(1)$$

$$\rho e = \frac{K_1}{2} \int e^{-K_2 U^2} (4\pi U^2)\, U^2 dU = \frac{3}{4} \frac{K_1 \pi^{3/2}}{K_2^{5/2}}$$

$$f^{eq}(x, \vec{v}, t) = \rho \left(\frac{3}{4\pi e}\right)^{3/2} e^{-\frac{3}{4e} U^2}$$

the equilibrium particle distribution f^{eq} next is used in the evolution of particle distribution function f.

$\Rightarrow$ Evolution of particle distribution function $f(\vec{x}, \vec{\xi}, t)$

Consider total differential change in f

$$\Delta f = \frac{\partial f}{\partial t} \Delta t + \nabla f \cdot \Delta \vec{x} + \nabla_\xi f \cdot \Delta \vec{\xi}$$

$$\frac{\Delta f}{\Delta t} = \frac{\partial f}{\partial t} + \nabla f \cdot \vec{\xi} + \nabla_\xi f \cdot \frac{\vec{F_\xi}}{\rho}$$

where $\frac{\Delta \vec{x}}{\Delta t} = \vec{\xi}$ and $\frac{\Delta \vec{\xi}}{\Delta t} = \frac{\vec{F_\xi}}{\rho}$ force per unit volume

For clarity, consider a 2D case

$f(x, y, \xi_x, \xi_y, t)$

$$\Delta f = \frac{\partial f}{\partial t} \Delta t + \Delta x \frac{\partial f}{\partial x} + \Delta y \frac{\partial f}{\partial y} + \Delta \xi_x \frac{\partial f}{\partial \xi_x} + \Delta \xi_y \frac{\partial f}{\partial \xi_y}$$

which is also written as

$$= \frac{\partial f}{\partial t} \Delta t + \Delta \vec{x} \cdot \nabla f + \Delta \vec{\xi} \cdot \nabla_\xi f$$

∇_ξ stands for gradient with respect to particle velocity.

$$\nabla_\xi = \left(\frac{\partial}{\partial \xi_x}, \frac{\partial}{\partial \xi_y}, \frac{\partial}{\partial \xi_z}\right)$$

$$\frac{\Delta f}{\Delta t} = \frac{\partial f}{\partial t} + \frac{\Delta \vec{x}}{\Delta t} \cdot \nabla f + \frac{\Delta \vec{\xi}}{\Delta t} \cdot \nabla_{\xi} f = \frac{Df}{Dt}$$

$\frac{Df}{Dt}$ is total or material derivative. $\frac{\partial \vec{\xi}}{\partial t}$ represents rate of particle velocity change and can be written as

$\frac{\partial \vec{\xi}}{\partial t} = \frac{\vec{F}_{\xi}}{\rho}$ where $\vec{F}_{\xi}$ is force in $\vec{\xi}$ direction with dimensions of force per unit volume.

The displacement $\Delta \vec{x} = \vec{\xi} \Delta t$, $\vec{\xi}$ is particle velocity.

$$\frac{Df_{\xi}}{Dt} = \frac{\partial f_{\xi}}{\partial t} + \vec{\xi} \cdot \nabla f_{\xi} + \frac{\vec{F}_{\xi}}{\rho} \cdot \nabla_{\xi} f_{\xi}$$

for clarity subscript ξ is included in the above equation. One can solve for all possible directions $\vec{\xi}$ to compute total differential change Δf_{ξ} in a particular direction.

The total change of f needs to be equated to physical reasons f_{ξ} can change. $\frac{Df}{Dt}$ includes changes due to change in time, position and force. Other reason f_{ξ} can change is due to particle collisions.

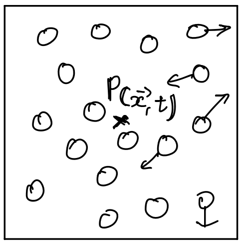
$$\frac{Df}{Dt} = \Omega(f) \quad ; \quad \Omega(f) \text{ is collision operator.}$$

In general, $\Omega(f)$ should take account of all binary collisions. The integral, with explicit inclusion of all the possible collision, becomes expensive to evaluate computationally.

instead of explicit form, we look for a simpler collision operator that essentially captures physics of binary collisions.

The physical ingradients are (i) Collisions do not change local mass, momentum and energy (ii) binary Collisions lead to equilibrium distribution function

When searching for f^{eq}, a closed Control Volume was Considered,



Control Volume

$P(\vec{x})$ is the Center of Control volume. If the Control volume is closed with respect mass, momentum and energy, we assume that we end up with an equilibrium distribution function.

In other words, whatever be the initial Conditions on individual particles, given the initial mass, momentum and energy of the Control volume, all different particle distribution function will tend towards the equilibrium distribution.

One possible approach is

$$\frac{Df}{Dt} = -\frac{(f-f^{eq})}{\tau} = \Omega(f) \Bigg] \rightarrow \text{BGK Collision operator}$$

τ has dimensions of time and Controls the timescale in which equilibrium is reached.

It is easy to check that BGK Collision operator Conserves local mass, momentum and energy.

mass Conservation : $\int \Omega(f)\, d^3\xi = \int (f-f^{eq})\, d^3\xi = \rho - \rho = 0$

momentum : $\int \Omega(f)\, \vec{\xi}\, d^3\xi = \int (f-f^{eq})\, \vec{\xi}\, d^3\xi = \rho\vec{u} - \rho\vec{u} = 0$

Questions

(1) Given $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \vec{u}) = 0$; Show that

$$\frac{\partial}{\partial t}(\rho \vec{u}) + \nabla\cdot(\rho \vec{u}\vec{u}) = \rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right)$$

– you must seen the identity in previous Courses (fluid mechanics, transport phenomena)! We use index (Einstein Convention) notation to prove the identity. Index notation will be used extensively in the Course.

Let $\vec{a}, \vec{b}$ be two vectors.

$$\nabla\cdot(\vec{a}\vec{b}) = \hat{\delta}_\alpha \frac{\partial}{\partial x_\alpha} \cdot \left(a_\beta \hat{\delta}_\beta b_\nu \hat{\delta}_\nu\right)$$

$$= \left[\frac{\partial}{\partial x_\alpha}(a_\beta b_\nu)\right] \delta_{\alpha\beta} \hat{\delta}_\nu$$

$$= \frac{\partial}{\partial x_\alpha}(a_\alpha b_\nu)\hat{\delta}_\nu$$

$$= a_\alpha \frac{\partial b_\nu}{\partial x_\alpha}\hat{\delta}_\nu + b_\nu \frac{\partial a_\alpha}{\partial x_\alpha}\hat{\delta}_\nu$$

$$= \vec{a}\cdot\nabla\vec{b} + \vec{b}(\nabla\cdot\vec{a})$$

Use $\vec{a} = \rho\vec{u}$, $\vec{b} = \vec{u}$

$$\nabla\cdot(\rho\vec{u}\vec{u}) = \rho\vec{u}\cdot\nabla\vec{u} + \vec{u}\left(\nabla\cdot(\rho\vec{u})\right) \quad -(1)$$

$$\frac{\partial}{\partial t}(\rho\vec{u}) = \frac{\partial\rho}{\partial t}\vec{u} + \rho\frac{\partial\vec{u}}{\partial t} \quad -(2)$$

Add (1) and (2)

$$\frac{\partial}{\partial t}(\rho\vec{u}) + \nabla\cdot(\rho\vec{u}\vec{u}) = \vec{u}\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u})\right] + \rho\left[\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right]$$

$$= \rho\left[\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right]$$

also check Exercise 1.1 along similar lines

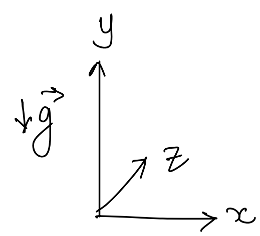
(2) Exercise 1.2 and 1.3 from the textbook

→ You must have done in earlier courses.

(3) Assume that the air follows ideal gas relation $p = \rho R T$, find pressure distribution as a function of height under gravity. Assume gravity strength does not change with height and temperature is constant.

For a static fluid,

$$\rho\left(\cancel{\frac{\partial \vec{u}}{\partial t}} + \cancel{\vec{u}\cdot\nabla\vec{u}}\right) = -\nabla p + \cancel{\mu\nabla^2\vec{u}} + \rho\vec{g}$$



$$0 = -\nabla p + \rho\vec{g}$$

assuming gravity acts in $-y$ direction

$$\vec{g} = (0, -g, 0)$$

$$0 = -\frac{\partial p}{\partial y} - \rho g \quad \Rightarrow \quad \frac{\partial}{\partial y}(\rho R T) = -\rho g$$

$$\ln \rho \Big|_{\rho_0}^{\rho} = -\frac{g y}{RT}\Big|_0^y \quad \Rightarrow \quad \rho = \rho_0 e^{-\left(\frac{gy}{RT}\right)}$$

We will verify the relation using simulation in couple of weeks.

(4) Exercise 1.5 (textbook)

$f(\vec{x}, \vec{\xi}, t) = \rho\, \delta(\vec{\xi} - \vec{u})$ is given find $\int f\, d^3\xi$, $\int f \vec{\xi}\, d^3\xi$

The delta distribution $\delta(\vec{\xi} - \vec{u})$ given in the problem in fact is multiplication of three one dimensional delta distributions.

$\delta(\vec{\xi}-\vec{u}) = \delta(\xi_x - u_x)\,\delta(\xi_y - u_y)\,\delta(\xi_z - u_z)$ and has di

$$I_1 = \int f\, d^3\xi = \int \rho\, \delta(\xi_x - u_x)\,\delta(\xi_y - u_y)\,\delta(\xi_z - u_z)\, d\xi_x\, d\xi_y\, d\xi_z$$

$$= \rho \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\xi_x - u_x)\,\delta(\xi_y - u_y)\,\delta(\xi_z - u_z)\, d\xi_x\, d\xi_y\, d\xi_z$$

$$= \rho$$

$$I_2 = \int f\, \vec{\xi}\, d^3\xi = \rho \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\xi_x - u_x)\,\delta(\xi_y - u_y)\,\delta(\xi_z - u_z)\left(\xi_x \hat{i} + \xi_y \hat{j} + \xi_z \hat{k}\right) d^3\xi$$

$$= \rho \vec{u}$$

$$I_3 = \int f\, \vec{\xi}^2\, d^3\xi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho\, \delta(\xi_x - u_x)\,\delta(\xi_y - u_y)\,\delta(\xi_z - u_z)\left(\xi_x^2 + \xi_y^2 + \xi_z^2\right) d^3\xi$$

$$= \rho \vec{u}^2$$

$$I_4 = \int f\, \vec{v}^2\, d^3\xi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho\, \delta(\xi_x - u_x)\,\delta(\xi_y - u_y)\,\delta(\xi_z - u_z)\left[(\xi_x - u_x)^2 + (\xi_y - u_y)^2 + (\xi_z - u_z)^2\right] d^3\xi$$

$$= 0$$

(5) Exercise 1.6 textbook

(a) $$\int \vec{v} f(x, \vec{\xi}, t)\, d^3\xi = \int \vec{\xi} f(x, \vec{\xi}, t)\, d^3\xi - \int \vec{u} f(x, \vec{\xi}, t)\, d^3\xi$$

$$= \rho\vec{u} - \rho\vec{u} = 0$$

(b) $I = \frac{1}{2}\int \vec{v}^2 f(x,\vec{\xi},t)\, d^3\xi = \frac{1}{2}\int \left(\xi^2 + u^2 - 2\vec{u}\cdot\vec{\xi}\right) f(\vec{x},\vec{\xi},t)\, d^3\xi$

$$= \frac{1}{2}\int \xi^2 f(x,\vec{\xi},t)\, d^3\xi + \frac{u^2}{2}\int f(\vec{x},\vec{\xi},t)\, d^3\xi$$

$$- \int \left(u_x\xi_x + u_y\xi_y + u_z\xi_z\right) f(\vec{x},\vec{\xi},t)\, d^3\xi$$

$$\rho e = \rho E + \frac{\rho u^2}{2} - \rho u^2 = \rho E - \frac{\rho u^2}{2}$$

(6) find equilibrium distribution in two dimensions; that is find K_1, K_2

$$f^{eq}(\vec{x},\vec{\xi},t) = K_1 e^{-\left[(u_x - \xi_x)^2 + (u_y - \xi_y)^2\right]K_2}$$
$$= K_1 e^{-v^2 K_2}$$

Integral over velocity space will give density.

$$\rho = \int f^{eq}(\vec{x},\vec{\xi},t)\, d\xi_x\, d\xi_y$$
$$= K_1 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left[(u_x-\xi_x)^2 + (u_y-\xi_y)^2\right]K_2}\, d\xi_x\, d\xi_y$$

change variable $v_x = \xi_x - u_x$, $v_y = \xi_y - u_y$

$dv_x = d\xi_x$, $dv_y = d\xi_y$

$$\rho = K_1 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(v_x^2 + v_y^2)K_2}\, dv_x\, dv_y$$

Let $v_x = z\cos\theta$, $v_y = z\sin\theta$ (again change of variable)

$$dv_x\, dv_y = z\, dz\, d\theta$$

$$\rho = K_1 \int_0^{2\pi} \int_0^{\infty} e^{-z^2 K_2}\, z\, dz\, d\theta$$

$$= 2\pi K_1 \left. \frac{e^{-z^2 K_2}}{-2K_2} \right|_0^{\infty} = \frac{2\pi K_1}{2K_2}(1) = \pi \frac{K_1}{K_2}$$

$$\rho e = \frac{1}{2}\int f^{eq}\, v^2\, d\xi_x\, d\xi_y$$

$$= \frac{K_1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-K_2\left[(\xi_x - u_x)^2 + (\xi_y - u_y)^2\right]} \left[(\xi_x - u_x)^2 + (\xi_y - u_y)^2\right] d\xi_x\, d\xi_y$$

change variable $\xi_x - u_x = z\cos\theta$, $\xi_y - u_y = z\sin\theta$.

$$= \frac{K_1}{2} \int_0^{2\pi} \int_0^{\infty} e^{-K_2 (z^2)} \left(z^2\, z\, dz\, d\theta\right)$$

$$= -\frac{K_1 (2\pi)}{2} \left. \frac{e^{-z^2 K_2}\left(1 + K_2 z^2\right)}{2K_2^2} \right|_0^{\infty}$$

$$\rho e = \frac{\pi K_1}{2K_2^2} \Rightarrow e = \frac{1}{2K_2}, \quad K_1 = \frac{\rho e}{\pi} \times 2 \times \frac{1}{4e^2} = \frac{\rho}{2\pi} \frac{1}{e}$$

$$f^{eq} = \frac{\rho}{2\pi e} e^{-\frac{u^2}{2e}}$$

note the difference from 3D form of the f^{eq}

(7) take 2D equilibrium distribution function $f^{eq} = K_1 e^{-K_2 U^2}$

Constants K_1, K_2 were determined by demanding that zeroth and second moment of f^{eq} with respect to relative particle velocity are ρ and ρe respectively.

Show that first moment of f^{eq} with respect to particle velocity $\vec{\xi}$ is $\rho\vec{u}$.

$$\int f^{eq} \vec{\xi}\, d\xi^2 = \int \left(\frac{\rho}{2\pi e}\right) e^{-\frac{U^2}{2e}} \vec{\xi}\, d\xi^2$$

$$I_1 = \frac{\rho}{2\pi e} \int e^{-\frac{U^2}{2e}} \xi_x \, d\xi_x \, d\xi_y$$

$$= \frac{\rho}{2\pi e} \int e^{-\left[(\xi_x - u_x)^2 + (\xi_y - u_y)^2\right]/2e} \xi_x \, d\xi_x \, d\xi_y$$

Change variable $\quad z\cos\theta = \xi_x - u_x$

$\qquad\qquad\qquad\quad z\sin\theta = \xi_y - u_y$

$$= \frac{\rho}{2\pi e} \int_0^{2\pi}\int_0^{\infty} e^{-z^2/2e} (z\cos\theta + u_x)\, z\, dz\, d\theta$$

$$= \frac{\rho}{2\pi e} \int_0^{2\pi}\int_0^{\infty} e^{-(z^2/2e)} z\cos\theta \, z\, dz\, d\theta + \frac{\rho}{2\pi e}\int_0^{2\pi}\int_0^{\infty} e^{-\frac{z^2}{2e}} u_x\, z\, dz\, d\theta$$

$$= \frac{\rho}{2\pi e}(0) + \frac{\rho}{2\pi e}(2\pi)\, u_x \left(e^{-\frac{z^2}{2e}}\right)(-e)\Big|_0^{\infty}$$

$$I_1 = \rho u_x$$

similarly $\quad I_2 = \rho u_y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^{eq} \xi_y \, d\xi_x \, d\xi_y$

due to symmetry, as soon as $\int f^{eq} d^2\xi = \rho$ is ensured, $\int f^{eq} \vec{\xi}\, d^2\xi = \rho\vec{u}$ follows. Also applies in 3 dimensions.

(8) exercise 1.9 textbook

Show that $\int v^2 \Omega(f)\, d^3\xi$ given $\int \xi^2 \Omega(f)\, d^3\xi = 0$

$$I = \int v^2 \Omega(f)\, d^3\xi$$
$$= \int (\vec{\xi} - \vec{u})^2 \Omega(f)\, d^3\xi$$
$$= \int \xi^2 \Omega(f)\, d^3\xi + u^2 \int \Omega(f) d^3\xi - 2\int (\vec{u}\cdot\vec{\xi}) \Omega(f) d^3\xi$$
$$= \int \xi^2 \Omega(f)\, d^3\xi + u^2(0) - 2\vec{u}\cdot \int \vec{\xi}\, \Omega(f) d^3\xi$$
$$= \int \xi^2 \Omega(f) d^3\xi + 0 + 0$$
$$= 0$$

(9) show that $\Omega(f) = -\frac{(f - f^{eq})}{\tau}$ conservers mass, momentum and energy.

Mass Conservation

$$I_1 = \int \Omega(f)\, d^3\xi = -\int \frac{(f - f^{eq})}{\tau} d^3\xi = -\frac{\int f d^3\xi}{\tau} + \frac{\int f^{eq} d^3\xi}{\tau}$$
$$= -\frac{\rho}{\tau} + \frac{\rho}{\tau} = 0$$

Momentum Conservation

$$I_2 = \int \Omega(f) \vec{\xi}\, d^3\xi = \int -\frac{(f - f^{eq})}{\tau} \vec{\xi}\, d^3\xi = -\frac{1}{\tau}\left[\rho\vec{u} - \rho\vec{u}\right] = 0$$

Energy Conservation

$$I_3 = \int \Omega(f)\, \xi^2 d^3\xi = -\frac{1}{\tau}\int (f - f^{eq})\, \vec{\xi}^2 d^3\xi = -\frac{1}{\tau}(\rho E - \rho E)$$
$$= 0$$

(10) Show that collision operator of the type

$$\Omega(f) = \frac{(f-f^{eq})^2}{\tau f^{eq}}$$ does not follow mass conservation

Mass Conservation Condition

$$I = \int \Omega(f)\, d^3\xi = 0$$

For the proposed collision operator

$$I = \int \frac{(f-f^{eq})^2}{\tau f^{eq}}\, d^3\xi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(f-f^{eq})^2}{\tau f^{eq}}\, d\xi_x\, d\xi_y\, d\xi_z$$

Given the integrand is always positive, $I > 0$. the only case when it is zero is when $f \equiv f^{eq}$ identically. Thus, in general, the given Collision operator does not Conserve mass.

(11) exercise 1.10 textbook

a force-free spatially homogeneous particle distribution $f(\vec{x}, \vec{\xi}, t)$ is given.

$$\frac{\partial f}{\partial t} + \cancel{\vec{\xi} \cdot \nabla f} + \cancel{\frac{\vec{F}}{\rho} \cdot \nabla_\xi f} = \Omega(f)$$

$$\nabla f = 0, \quad \vec{F} = 0$$

$$\frac{\partial f}{\partial t} = \Omega(f) = -\frac{(f-f^{eq})}{\tau}$$

$$f = f^{eq} + (f(t=0) - f^{eq})^{-\frac{t}{\tau}}$$