EE2703 : ASSIGNMENT 8

Sarthak Vora

May 7, 2021

1 Abstract

In this assignment, we analyse and use **DFT** to find the Fourier transform of periodic and non-periodic signals using fast fourier transform algorithms which are implemented using `fft` and `fftshift` functions in **numpy's** `fft` module.

2 Introduction

The **Discrete Fourier transform** (DFT) converts a finite sequence of equally-spaced samples of a function into a same-length sequence of equally-spaced samples of the discrete-time Fourier transform (**DTFT**), which is a complex-valued function of frequency.
Let's suppose f[n] are the samples of some continous function $f(t)$ then we define the Z transform as

$$F(z) = \sum_{n=-\infty}^{n=\infty} f(n) z^{-n} \tag{1}$$

Replacing z with $e^{j\omega}$ we get DTFT of the sampled function

$$F(e^{j\omega}) = \sum_{n=-\infty}^{\infty} f(n) e^{-j\omega n} \tag{2}$$

$F(e^{j\omega})$ is continuous and periodic. $f[n]$ is discrete and aperiodic. Suppose, $f[n]$ is itself periodic with a period N,

$$f[n+N] = f[n]$$

Then, it should have samples for its DTFT. This is true, and leads to the Discrete Fourier Transform or the DFT. The DTFT of the sequence is also a periodic sequence $F[k]$ with the same period N.

$$F[k] = \sum_{n=0}^{N-1} f[n] e^{-j\frac{2\pi nk}{N}} = \sum_{n=0}^{N-1} f[n] W^{nk} \tag{3}$$

$$f[n] = \frac{1}{N} \sum_{k=0}^{N-1} F[k] W^{-nk} \tag{4}$$

Here $W = e^{-j\frac{2\pi}{N}}$. k is sampled values of continuous variable ω at multiples of $\frac{2\pi}{N}$.
This means that **DFT is a sampled version of the DTFT, which is the digital version of the analog Fourier Transform**.

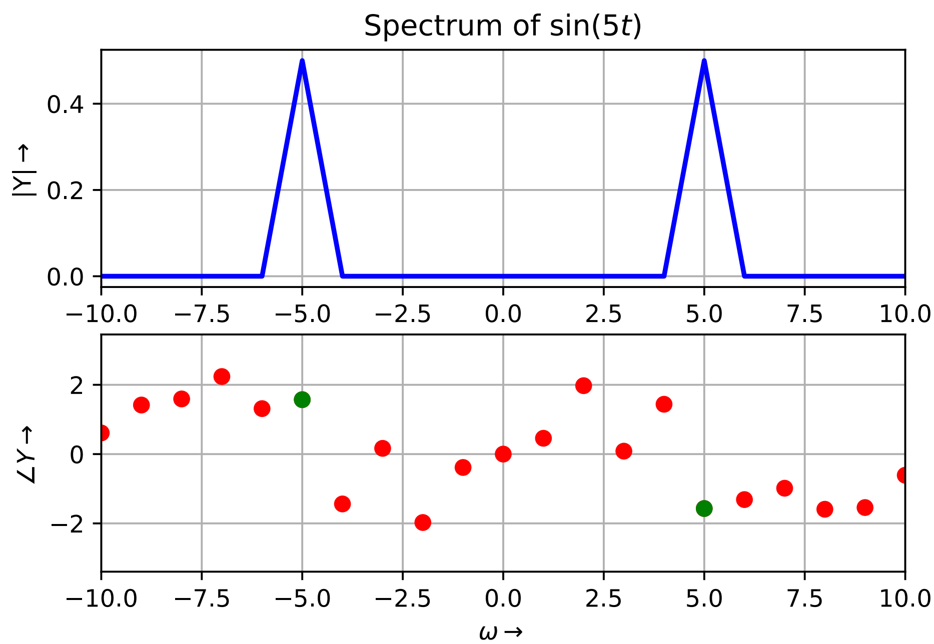
3 Spectrum of $\sin(5t)$

- We calculate the DFT of $f(t) = \sin(t)$ by the method mentioned above using `fft` and `fftshift` functions.
- To compare the spectrum obtained for $\sin(5t)$, we use

$$\sin(5t) = \frac{1}{2j}e^{j5} - \frac{1}{2j}e^{-j5} \tag{5}$$

- We then plot the **magnitude** and **phase** plot of the **DFT** and analyse them .The following is obtained for $\sin(5t)$ -

3.1 Plots



3.2 Observation

- The Fourier transform of $\sin(5t)$ using above relation is

$$F(\sin(5t)) \rightarrow \frac{1}{2j}(\delta(\omega - 5) - \delta(\omega + 5)) \tag{6}$$

- As expected from the expression of **Fourier Transform** of $\sin(5t)$, we get **two peaks** at $\omega = 5$ and $\omega = -5$ with a height of **0.5**.
- Since, the amplitude is 0 everywhere else except at these two peaks, all the **energy** is stored in the frequency component $|w| = 5$ in the signal $\sin(5t)$.
- The phase angle at $\omega = 5$ and $\omega = -5$ is essentially $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ as can be seen from the graph and expression for the **Fourier transform**.

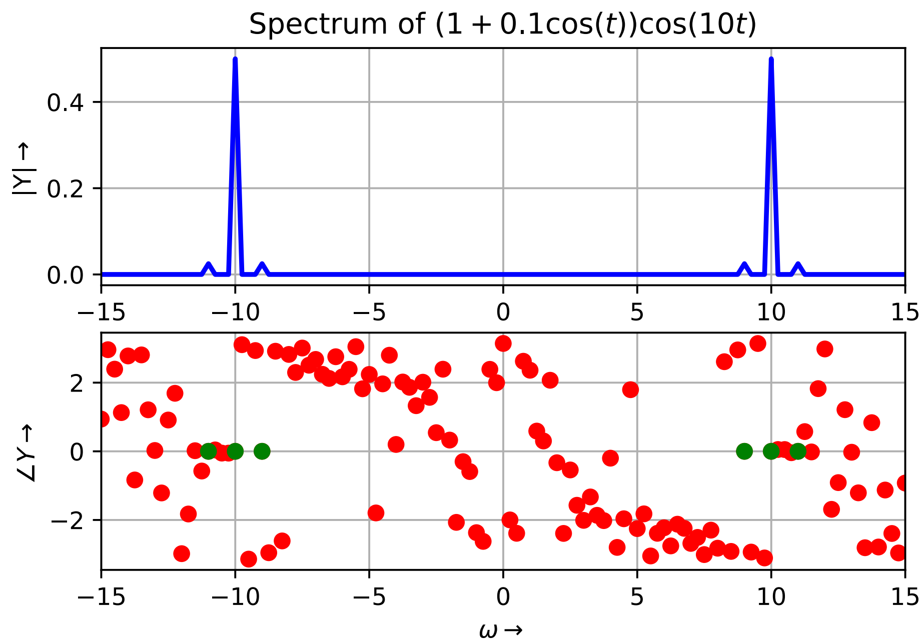
4 Spectrum of Amplitude modulated wave $(1 + 0.1\cos(t))\cos(10t)$

- We calculate the Fourier transform of $(1 + 0.1\cos(t))\cos(10t)$ by the method mentioned above using `fft` and `fftshift` functions.
- To compare the spectrum obtained for $(1 + 0.1\cos(t))\cos(10t)$, we use

$$(1 + 0.1cos(t))cos(10t) = \frac{1}{2}(e^{10jt} + e^{-10jt}) + 0.025(e^{11jt} + e^{-11jt} + e^{9jt} + e^{-9jt}) \tag{7}$$

- We then plot the **magnitude** and **phase** plot of the **DFT** and analyse them .The following is obtained for $(1 + 0.1\cos(t))\cos(10t)$ -

4.1 Plots



4.2 Observation

- As we observe from the plot, the magnitude variation has a **maximum peak** of 0.5 at center frequencies of $\omega = 10$ and $\omega = -10$ from the carrier signal $\cos(10t)$.
- We also get a set of **peaks** of value 0.025 as expected, due to **side-band frequencies** of $\omega = 9$, $\omega = -9$, $\omega = 11$ $\omega = -11$ inside the actual signal $(1 + 0.1\cos(t))$
- Since, the amplitude of the actual message signal is changed by the carrier wave $\cos(10t)$, it is called an **amplitude modulated wave**.
- The phase angle at the point of peaks $\omega = 10$, $\omega = -10$, $\omega = 9$, $\omega = -9$, $\omega = 11$ and $\omega = -11$ is essentially 0 as phase spectra consists of `cosine` terms which has **zero phase**.

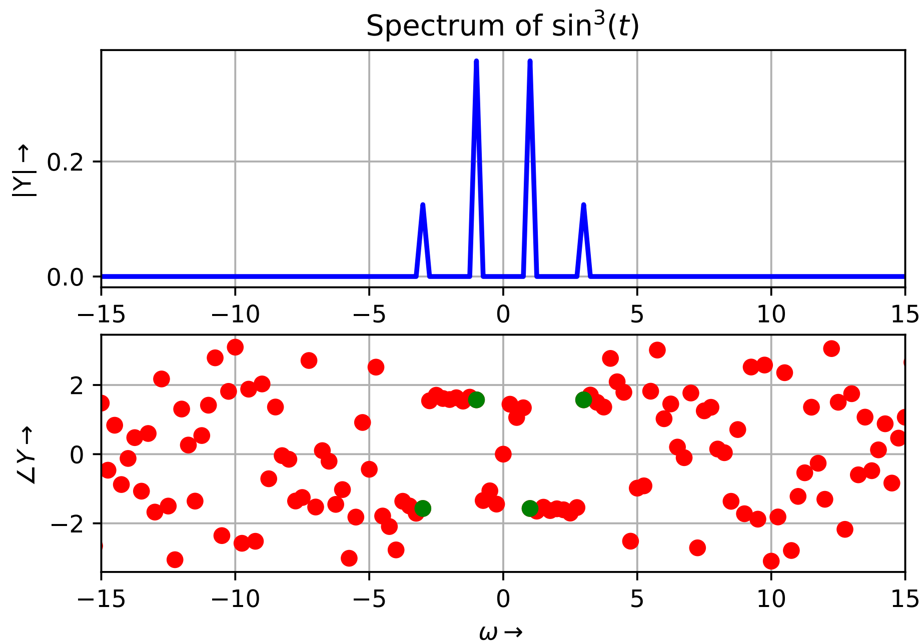
5 Spectrum of $\sin^3(t)$

- We calculate the Fourier transform of $f(t) = \sin^3(t)$ by the method mentioned above using `fft` and `fftshift` functions.
- To compare the spectrum obtained for $\sin^3(t)$, we use

$$sin^3(t) = \frac{3}{4}sin(t) - \frac{1}{4}sin(3t) \tag{8}$$

- We then plot the **magnitude** and **phase** plot of the **DFT** and analyse them .The following is obtained for $\sin^3(t)$ -

5.1 Plots



5.2 Observation

- The Fourier transform of $\sin^3(t)$ using above relation is

$$F(\sin^3(t)) \rightarrow \frac{3}{8j}(\ \delta(\omega-1) - \delta(\omega+1)\ ) - \frac{1}{8j}(\ \delta(\omega-3) - \delta(\omega+3)\ ) \tag{9}$$

- As expected from the expression of **Fourier Transform** of $\sin^3(t)$, we get **4 peaks** at $\omega = 1$, $\omega = -1$ of height **0.375** and at $\omega = 3$, $\omega = -3$ with a height of **0.125**. These amplitudes are in the ratio **3:1**.
- Since, the amplitude is 0 everywhere else except at these 4 peaks, all the **energy** is stored in the frequency components $|w| = 1$ and $|w| = 3$ in the signal $\sin^3(t)$.
- The phase angle at $\omega = -3$, $\omega = -1$, $\omega = 1$, $\omega = 3$ is essentially $\frac{-\pi}{2}$, $\frac{\pi}{2}$, $\frac{-\pi}{2}$, $\frac{\pi}{2}$ as can be seen from the graph and expression for the **Fourier transform**.

6 Spectrum of $\cos^3(t)$

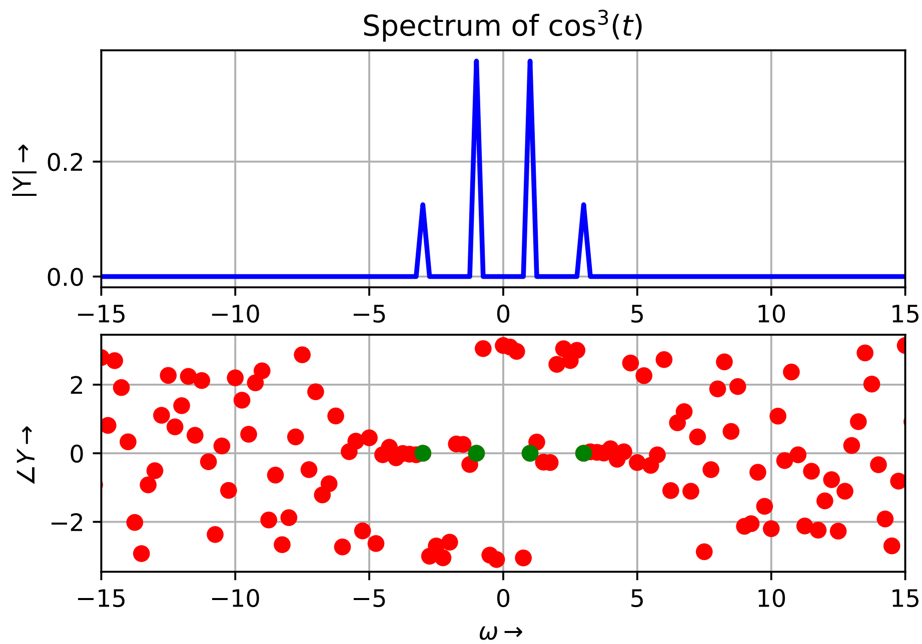
- We calculate the Fourier transform of $f(t) = \cos^3(t)$ by the method mentioned above using `fft` and `fftshift` functions.

- To compare the spectrum obtained for $\cos^3(t)$, we use

$$\cos^3(t) = \frac{3}{4}\cos(t) + \frac{1}{4}\cos(3t) \tag{10}$$

- We then plot the **magnitude** and **phase** plot of the **DFT** and analyse them .The following is obtained for $\cos^3(t)$ -

6.1 Plots



6.2 Observation

- The Fourier transform of $\cos^3(t)$ using above relation is

$$F(\cos^3(t)) \rightarrow \frac{3}{8}(\ \delta(\omega - 1) + \delta(\omega + 1)\ ) + \frac{1}{8}(\ \delta(\omega - 3) + \delta(\omega + 3)\ ) \tag{11}$$

- As expected from the expression of **Fourier Transform** of $\cos^3(t)$, we get **4 peaks** at $\omega = 1$, $\omega = -1$ of height **0.375** and at $\omega = 3$, $\omega = -3$ with a height of **0.125**. These amplitudes are in the ratio **3:1**.

- Since, the amplitude is 0 everywhere else except at these 4 peaks, all the **energy** is stored in the frequency components $|w| = 1$ and $|w| = 3$ in the signal $\cos^3(t)$.

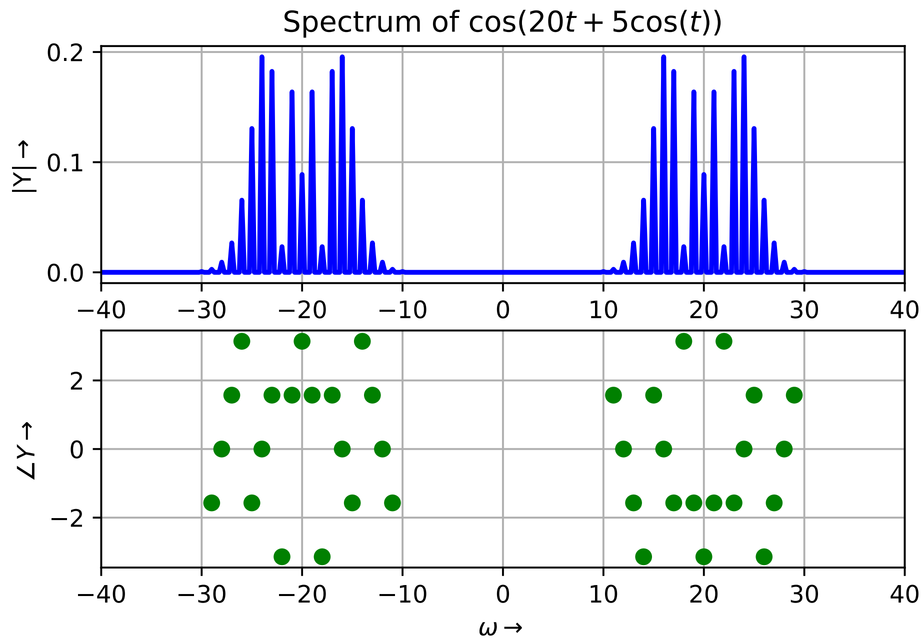
- The phase angle at $\omega = -3$, $\omega = -1$, $\omega = 1$, $\omega = 3$ is essentially 0 everywhere as the expression of the signal is purely `cosine`, hence the phase is **0** at the peaks.

- However, due to lack of infinite computing power, we can see that eventhough the magnitude is approximately 0 everywhere, the phase is not 0. **Zero amplitude doesn't imply zero phase**.

7 Spectrum of Frequency modulated signal $\cos(20t + 5\cos(t))$

- We calculate the Fourier transform of $f(t) = \cos(20t + 5\cos(t))$ by the method mentioned above using `fft` and `fftshift` functions.
- Here, we plot the phase diagram only where magnitude is significant (10^{-3}).
- We then plot the **magnitude** and phase plot of the **DFT** and analyse them .The following is obtained for $\cos(20t + 5\cos(t))$ -

7.1 Plots



7.2 Observation

- In this example of frequency modulated wave, the amplitude corresponding to each frequency, follows the amplitude curve of a **bessel** function from -30, with an initial rise at around -24, **minima** at around -22 reaching the **median** value at the **centre** frequency of -20 following a mirror image path to -10. Since, the actual function is still a **cosine** wave, the LHP and RHP have some graphs.
- In the phase plot, instead of getting the same phase diagrams due to cosine waves in 2 halves, the one on the right is the **negative** of the **reflection** along -20 line of the one on the left.
- This is because of the $5\cos(t)$ term in the **phase** of the original expression. This term can go to both negative and positive and the phase will around the **central frequency** of 20 rad/s, as given in the first term in the phase `20t`.
- Also, when the complex representation of the equation is taken, the sign of $5\cos(5t)$ would be opposite for both the terms.
- The reflection around phase $= 0$ line for the RHP diagram, as compared to the LHP diagram, can be attributed to the fact that, when the initial function is written in it's complex form, the frequency of the first term would be $20t + 5\cos(t)$, whereas that of the second would be negative of this.
- Thus, the entire phase is **negative** in the RHP, and therefore, the phase graph is as seen above.

8 Spectrum of the Gaussian $e^{-t^2/2}$

- The Gaussian $f(t) = e^{-t^2/2}$ is not '**bandlimited**' in frequency, in the sense has frequency spectrum has non-zero values even for very large frequencies.
- The **Continuous Time Fourier Transform** for the Gaussian is given by -

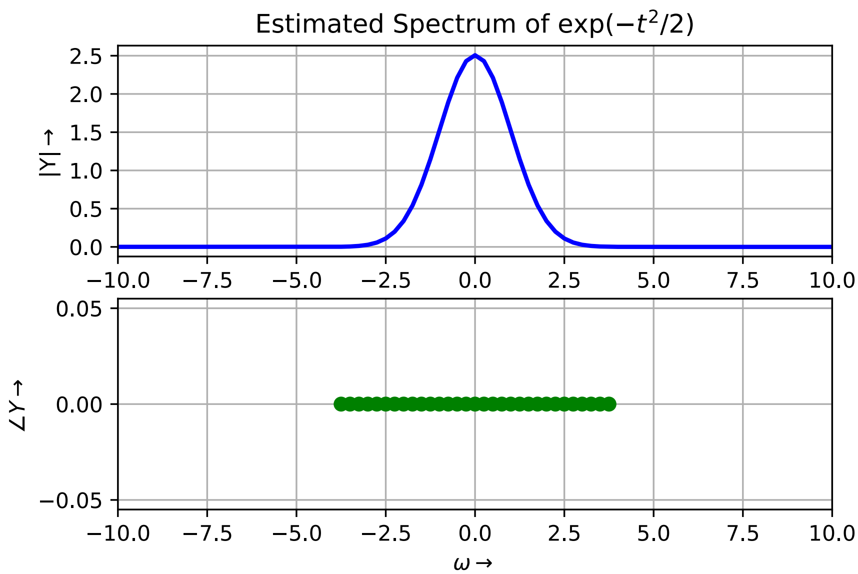
$$F(\omega) = \sqrt{2\pi} e^{-\omega^2/2} \tag{12}$$

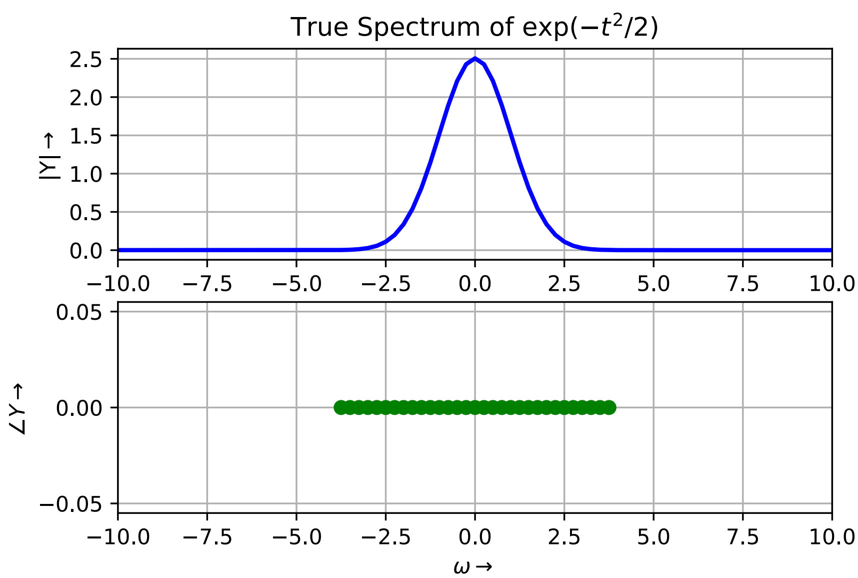
- We know that, the Fourier Transform for Gaussian is Gaussian itself. i.e -

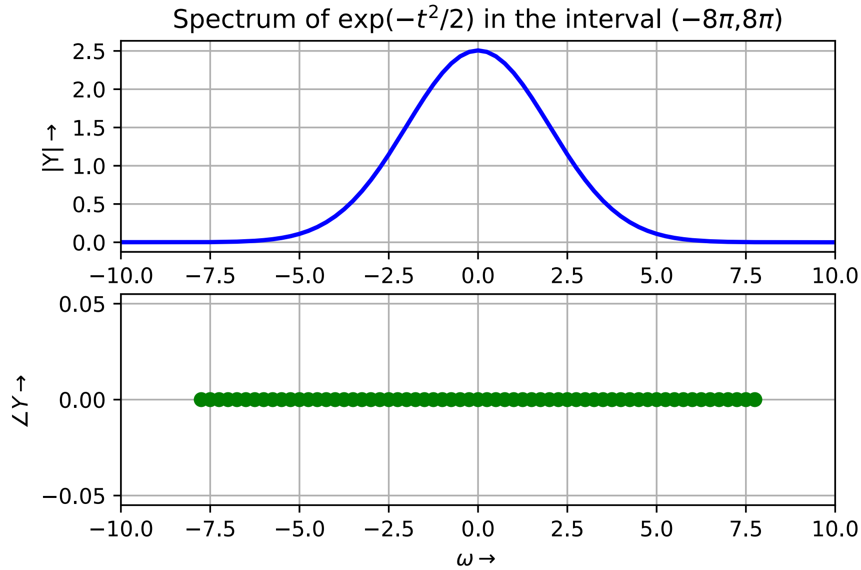
$$e^{-t^2/2} \xrightarrow{\mathcal{F}} \sqrt{2\pi} e^{-\omega^2/2} \tag{13}$$

- In the below plots, we analyse the **expected** and **true** plot of the Gaussian spectrum by comparing the input Gaussian with it's FFT.
- For plotting the spectrum, we consider two different time ranges : $(-4\pi, 4\pi)$ and $(-8\pi, 8\pi)$.

8.1 Plots







8.2 Observation

- The Gaussian function is **aperiodic**, it is not band-limited in the frequency domain. Hence, the spectrum is **different** for different values of time limit we consider and also the spectrum might be slightly **inaccurate**.
- As we observe, the magnitude spectrum of $e^{-t^2/2}$ is almost the same as the Fourier transform (true spectrum) plotted above.
- The phase at all the points will be **zero** since the Fourier transform is purely real.
- The Fourier transform for the time period $(-4\pi, 4\pi)$ almost exactly matches with the **expected** magnitude spectrum plot for the Gaussian function.

---

9 Conclusion

- Hence we analysed, how to find **DFT** for various types of signals and how to estimate normalising factors for Gaussian functions and hence recover the analog Fourier transform using DFT.
- The `fft` library in Python provides a useful kit for analysis of DFT signals. The Discrete Fourier transforms of **sinusoids**, **amplitude** modulated signals were analysed.
- In the case of pure **sinusoids**, the DFT contained impulses at sinusoid **frequencies**.
- The amplitude modulated wave had a frequency spectrum with impulses at the **carrier** and **sideband** frequencies.
- The frequency modulated wave, having an infinite number of **sideband** frequencies, gave rise to a **DFT** with non-zero values for a **broader** range of frequencies.
- The DFT of a Gaussian is also a Gaussian and the spectrum was found to **broaden** for greater time ranges.
- FFT works well for signals with samples in 2^k, as it divides the sample into even and odd and goes dividing further to compute the DFT.