# EE2703: Assignment 6

Sarthak Vora (EE19B140)

April 17, 2021

## 1 Introduction

We analyse **LTI systems** in continuous time using **Laplace Transforms** to find the output of system to a given input with the help of python library, namely `scipy.signal` toolbox.

## 2 Time response of a spring oscillator system

Our goal is to find the response of a **spring oscillator**, governed by the equation:

$$\ddot{x} + 2.25x = f(t) \tag{1}$$

where,

x(t) = Displacement of spring
f(t) = Force applied on the spring

We consider that the force applied on the spring is given by:

$$f(t) = e^{-at}\cos(wt)u(t) \tag{2}$$

We shall do a case-by-case analysis for the following values of a and $\omega$:

| a ($sec^{-1}$) | w (rad $sec^{-1}$) |
|---|---|
| 0.5 | 1.5 |
| 0.05 | 1.5 |
| 0.05 | 1.4 |
| 0.05 | 1.45 |
| 0.05 | 1.5 |
| 0.05 | 1.55 |
| 0.05 | 1.6 |

### 2.1 Laplace Transforms

The **Laplace transform** of $f(t) = e^{-at}\cos(wt)u(t)$ is given as:

$$\mathcal{L}\{f(t)\} = \frac{s+a}{(s+a)^2+\omega^2}$$

From the property of Laplace transforms, we know:

$$x(t) \longleftrightarrow \mathcal{X}(s)$$
$$\implies \dot{x}(t) \longleftrightarrow \ s\mathcal{X}(s) - x(0^-)$$
$$\implies \ddot{x}(t) \longleftrightarrow \ s^2\mathcal{X}(s) - sx(0^-) - \dot{x}(0^-)$$

From the above equations, we get, for $a = 0.5$ and $\omega = 1.5$:

$$\mathcal{F}(s) = \mathcal{L}\{f(t)\} = \frac{s+0.5}{(s+0.5)^2+2.25}$$

So, the equation of the **spring oscillator** can be written as:

$$s^2\mathcal{X}(s) - sx(0^-) - \dot{x}(0^-) + 2.25\mathcal{X}(s) = \frac{s+0.5}{(s+0.5)^2+2.25}$$

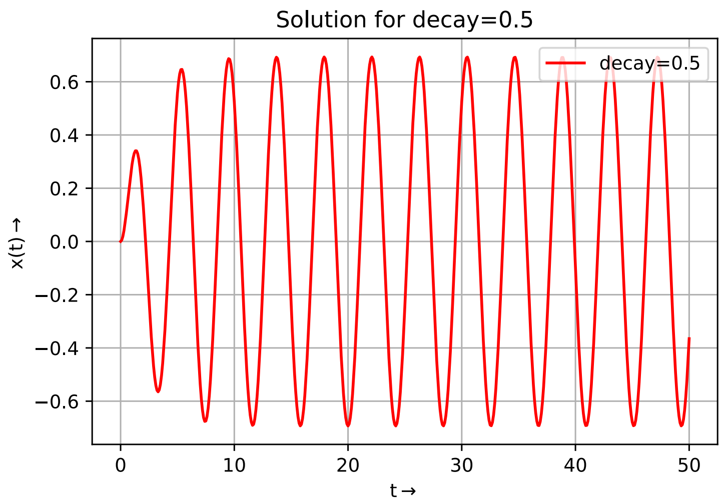
Given that $x(0)$ and $\dot{x}(0)$ are 0, we get:

$$s^2\mathcal{X}(s) + 2.25\mathcal{X}(s) = \frac{s+0.5}{(s+0.5)^2+2.25}$$

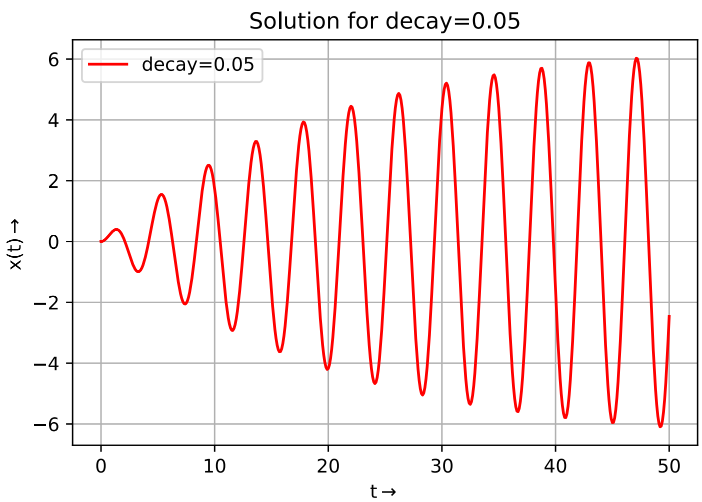
or,

$$\mathcal{X}(s) = \frac{s+0.5}{((s+0.5)^2+2.25)(s^2+2.25)}$$

Using `scipy.signal.impulse` to find $x(t)$, plotting it (for $0 < t < 50s$), we get:

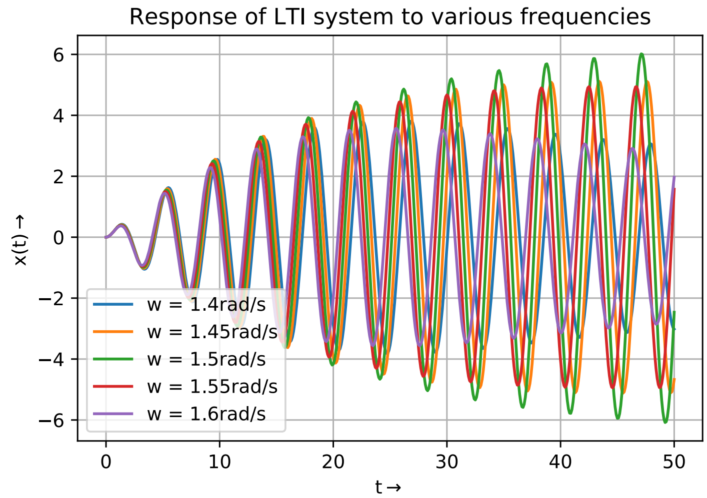



If we use a smaller decay of $a = 0.05$, then we get:



## 2.2 Response for different frequencies

Modelling the system as an **LTI system** and computing the response for various **frequencies** $w$ (and $a = 0.05$), we get:




### 2.2.1 Observation

- From the given equation, we can see that the **natural response** has the frequency $\omega = 1.5\ rad/s$.
- Thus, as expected, the **maximum amplitude** of oscillation is obtained when the frequency of $f(t)$ is $1.5\ rad/s$, as a case of resonance.

# 3 Coupled Spring Problem

The coupled equations we are interested in solving are:

$$\ddot{x} + (x - y) = 0$$
$$\ddot{y} + 2(y - x) = 0$$

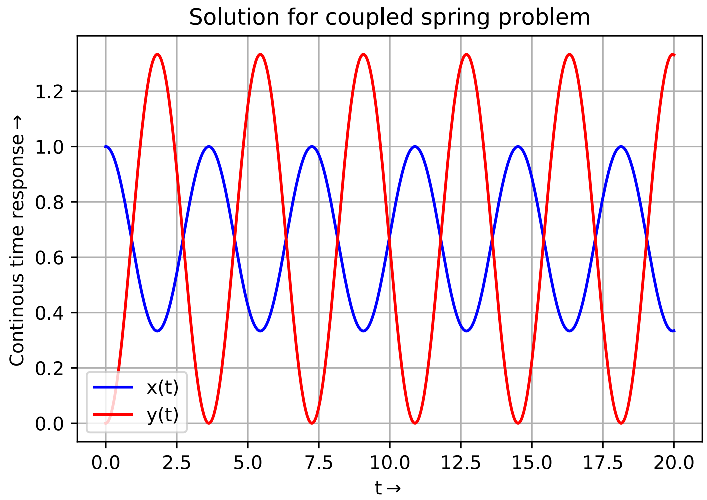
Substituting for $y$ from the $1^{st}$ equation, into the $2^{nd}$, we get a $4^{th}$ order differential equation in $x$:

$$\ddot{\ddot{x}} + 3\ddot{x} = 0$$

Given the conditions $x(0) = 1$ and $\dot{x}(0) = y(0) = \dot{y}(0) = 0$, we can write the above differential equation in the Laplace domain as:

$$s^4\mathcal{X}(s) - s^3 + 3(s^2\mathcal{X}(s) - s) = 0$$
$$\implies \mathcal{X}(s) = \frac{s^2 + 3}{s^3 + 3s}$$
$$\implies \mathcal{Y}(s) = \frac{2}{s^3 + 3s}$$

We can easily solve for $x(t)$ and $y(t)$ using `scipy.signal.impulse` attribute with the above $\mathcal{X}(s)$ and $\mathcal{Y}(s)$. We get the following graph for $x(t)$ and $y(t)$ -




### 3.1 Observation

- We can see that, $x(t)$ and $y(t)$ are **sinusoidal signals** of the **same frequency**.
- However, both the signals have **different amplitude** and **phase** as can be seen in the above plot.

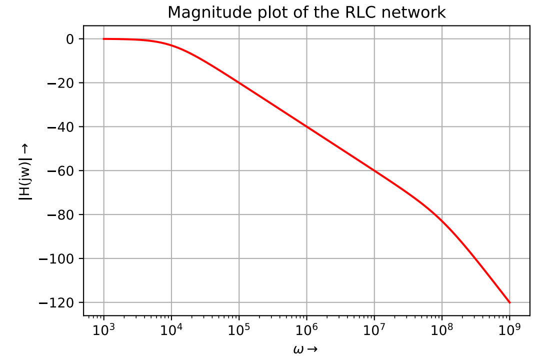
---

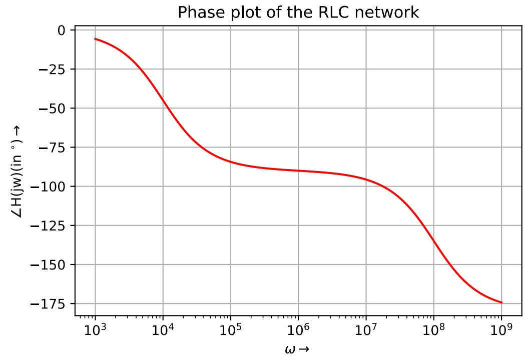
## 4 Two-port Network

The steady-state **Transfer function** of the given **RLC two-port network** can be written as:

$$\frac{V_o(s)}{V_i(s)} = \mathcal{H}(s) = \frac{10^6}{s^2 + 100s + 10^6}$$

The **Bode magnitude** and **phase plots** can be found using the `scipy.signal.bode()` attribute. The plots are shown below:



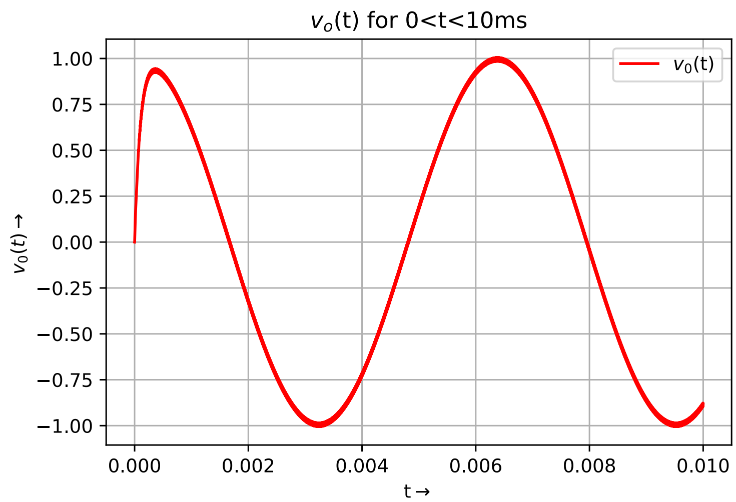
Now, when the input to this system is set as -

$$v_i(t) = (\cos(10^3 t) - \cos(10^6 t))u(t) \tag{3}$$

The output in Laplace domain can be expressed as -

$$V_o(s) = V_i(s)\mathcal{H}(s)$$

Since we have already found $\mathcal{H}(s)$ above, $V_o(t)$ can be easily found out by inputting $\mathcal{H}(s)$ and $V_i(t)$ in the `scipy.signal.lsim` attribute. The plot for the obtained $V_o(t)$ in the time domain $0 < t < 10ms$ is shown below:

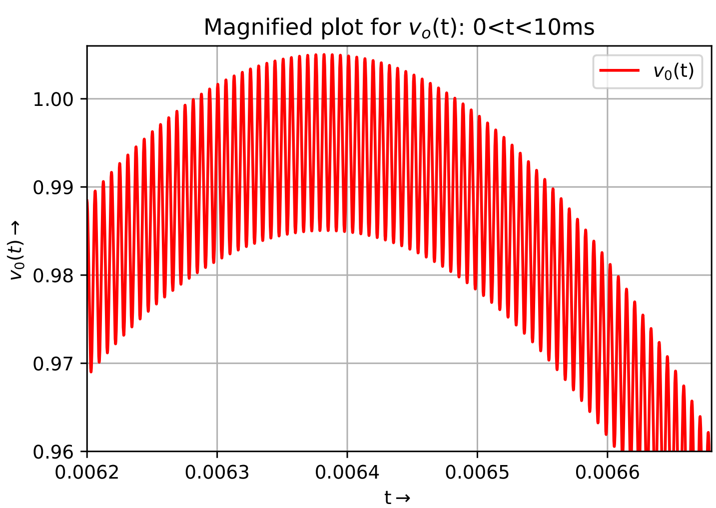



## 4.1 Observation

- The above plot is a **sinusoidal curve** of frequency approximately being **160 Hz**.
- The RLC network acts as a **low pass filter -** it allows low frequencies to pass through unchanged, while **damping high frequencies** to huge extent.

---

However, if we **zoom** in at the above plot for $v_o(t)$ and take a look at the magnified plot, we can actually see the **high frequency signal**. The magnified plot is shown below:

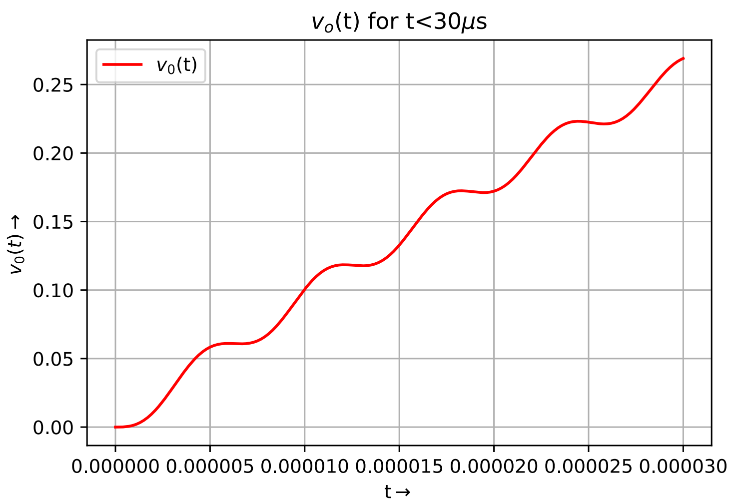


### 4.2 Observation

- We observe that the **peak-to-peak variation** of this high frequency variation is very less, approximately **0.02V**, as compared to the initial **2V**.
- This is expected as we get a gain of $|H(s)|_{dB} \approx -40$ at $\omega = 10^6\ rad/sec$ from the Bode plot, which corresponds to **gain factor** of **0.01**.
- The system provides **unity gain** for a **low frequency** of $10^3 rad/sec$. Thus, low frequency components are more or less preserved in the output.
- The circuit inherently acts as a **low-pass filter**, thus dampening higher frequencies.

---

Another peculiarity of the above $V_o(t)$ plot is the **initial variation**. When zoomed in, for $0 < t < 30\ us$, we get the following plot:




### 4.3 Observation

- Due to the application of a **step input**, the output shows an **irregular behaviour** in the initial phase.
- The variations due to the step input are determined by the **high frequency component** of $v_i(t)$.

# 5 Conclusion

- The `scipy.signal` library was used for circuit analysis of **LTI systems** in Laplace domain.
- Thus, the **forced response** of a simple spring body system was obtained over various **frequencies** of the applied force and the highest amplitude was obtained at **resonant frequency.**
- The coupled spring problem was solved similarly using `scipy.signal.impulse` function attribute in Laplace domain to obtain two **sinusoids** of the **same frequency**.
- A two-port network, functioning as a **low-pass filter** was analysed and a filtered output was obtained for a **mixed frequency input**.

---