EE2703 : ASSIGNMENT 9

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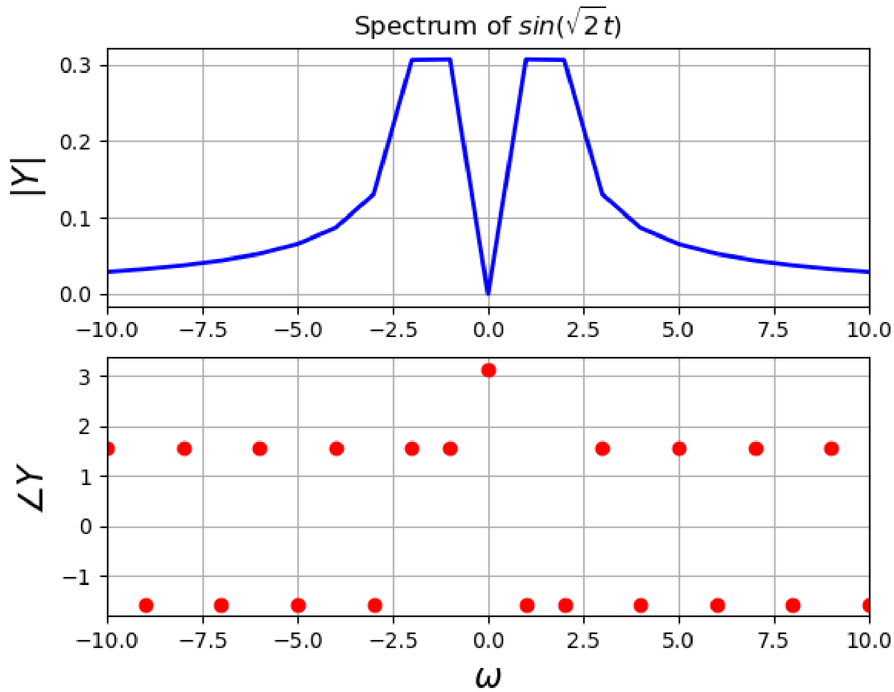
June 3, 2021

1 Introduction

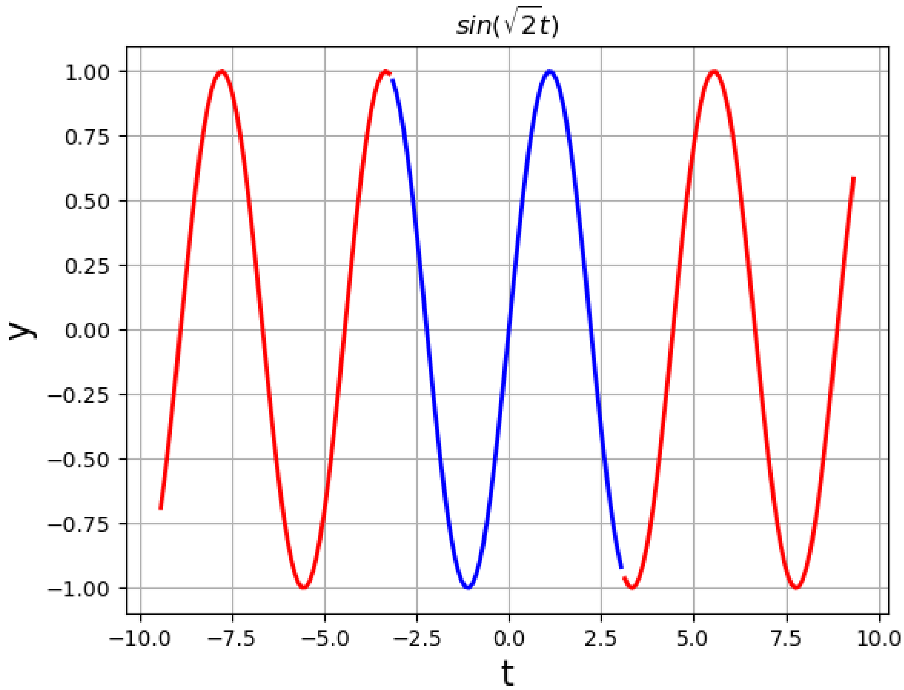
In this week's assignment, we continue our explorations on the DFT of a finite-length sequence with the FFT algorithm, using the `numpy.fft` module. We shall look at finding the DFTs of non-periodic functions, and problems associated with them, namely the *Gibbs Phenomenon* and how to overcome them, using *windowing*.

2 Spectrum of $\sin(\sqrt{2}t)$

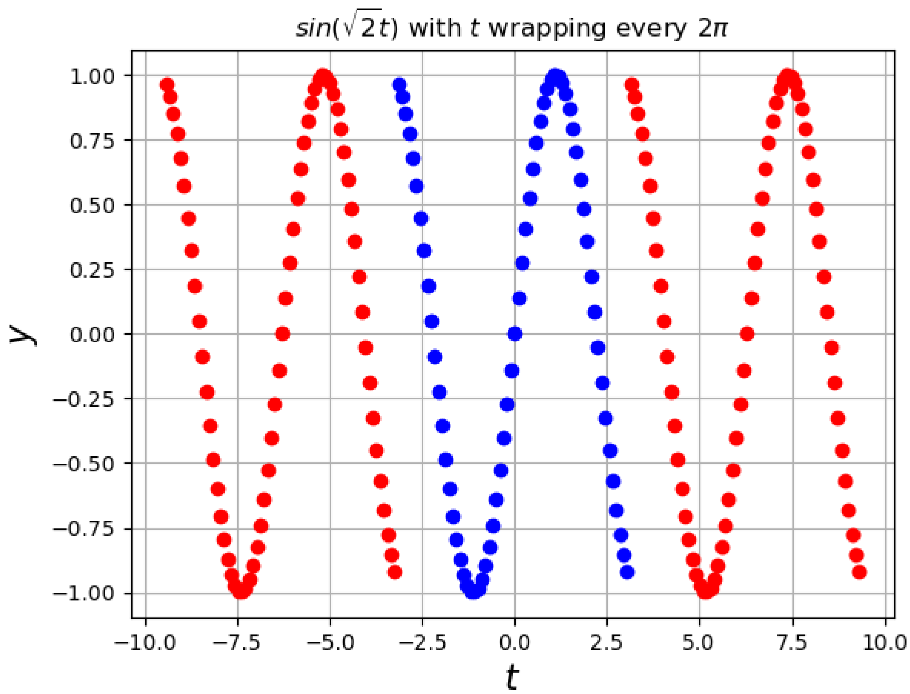
- We begin by looking into the function $\sin(\sqrt{2}t)$. Initially, we obtain the DFT and plot the spectrum of $\sin(\sqrt{2}t)$, by considering 64 samples over the range $(0,\ 2\pi)$. We have used the function `fftshift()` for obtaining the DFT.



- We observe that the peaks are not at the expected frequencies. We got two peaks each with two values and a gradually decaying magnitude. The phase plot obtained is correct though. This happened because the part of the signal $\sin(\sqrt{2}t)$ in the range of $(-\pi, \pi)$ is not periodically repeated. We plot this function for several periods to observe this, as shown in the below figure.

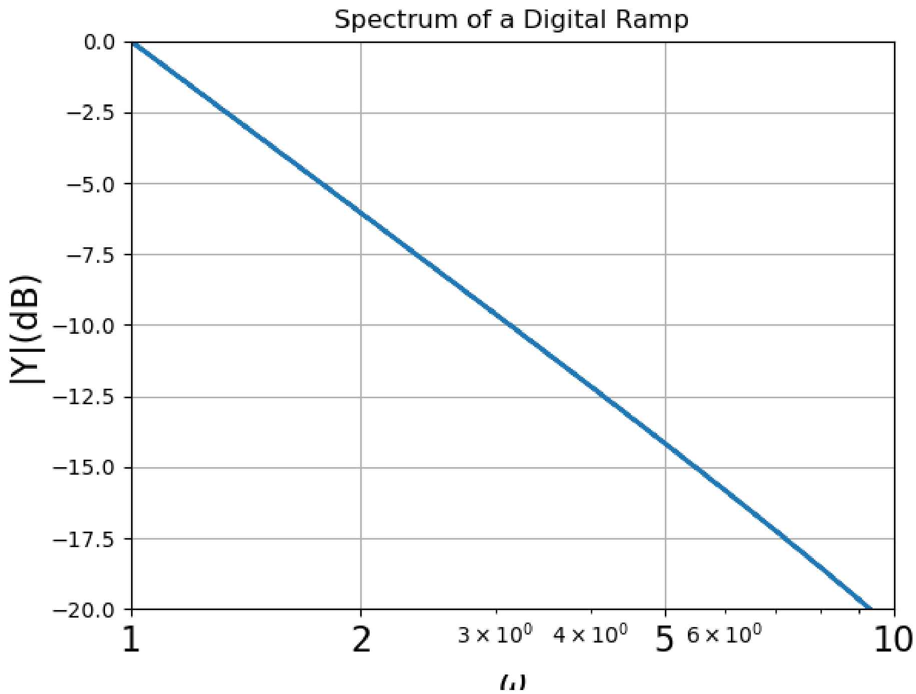


- Therefore, the DFT is not analysing the function $\sin(\sqrt{2}t)$, but is rather analysing the function which is a periodic repetition of the blue part highlighted in the above figure.



3 Spectrum of Digital Ramp

- The ramp function is also discontinuous in time, i.e., it is a non-periodic function and hence it is similar to the above discussed sinusoid. The DFT samples decay as $1/\omega$. We plot the spectrum of the digital ramp as shown below:



- In order to remove discontinuities, we introduce a hamming window function given by -

$$x[n] = 0.54 + 0.46\frac{2\pi n}{N-1} \tag{1}$$

- Hence, the non-periodic signal is multiplied with the hamming window to remove the discontinuities.

4 Windowing property: Hamming Window

- We observe that the spikes happen at the end of the periodic interval. So, we damp the function there, i.e., we multiply our function sequence f[n] by a window sequence w[n]

$$g[n] = f[n] \times w[n] \tag{2}$$

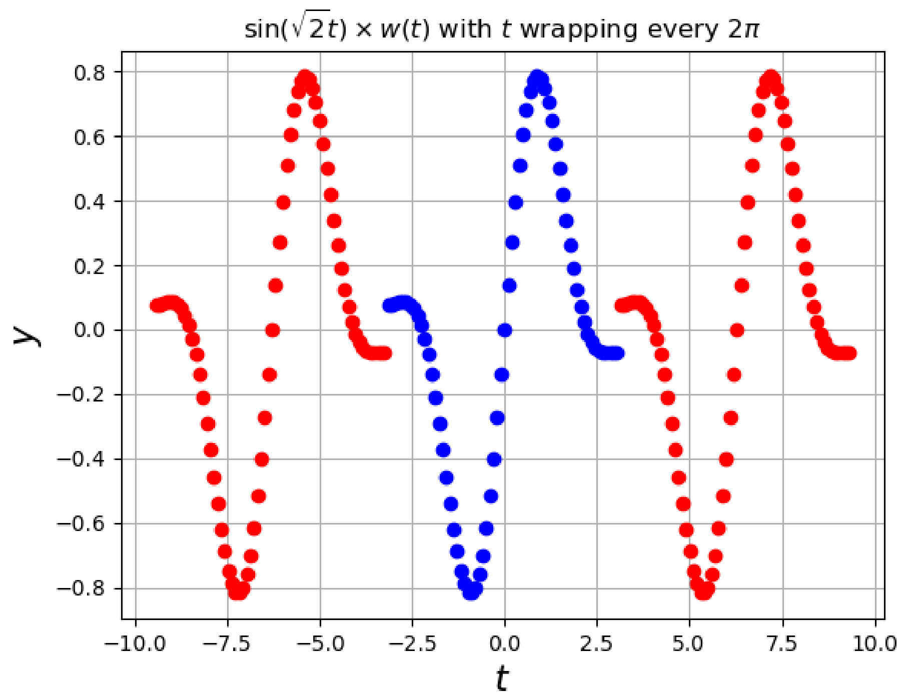
- The new spectrum is obtained by convolving the two fourier transforms:

$$G_k = \sum_{n=0}^{N-1} F_n W_{k-n} \tag{3}$$

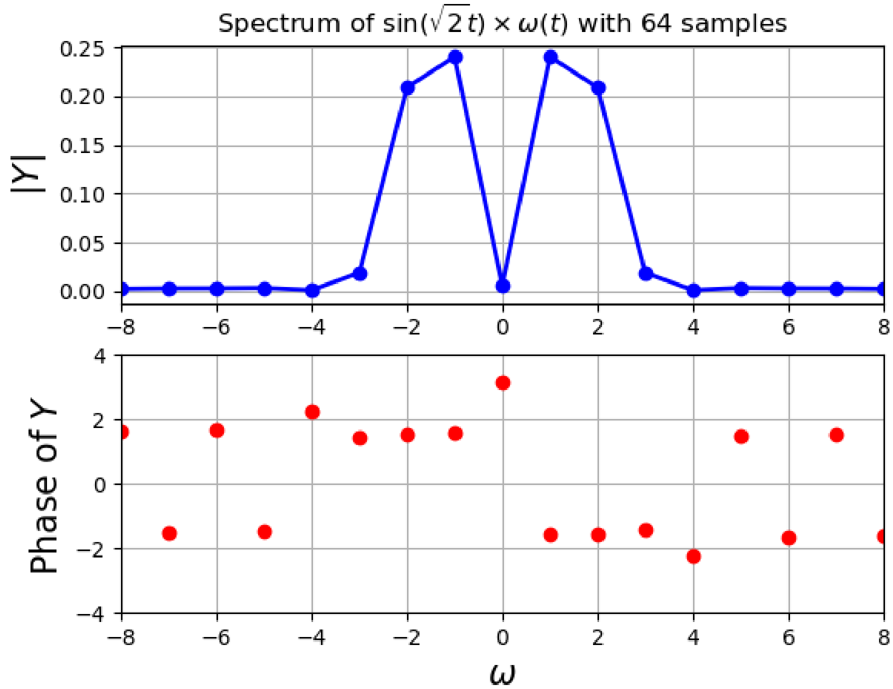
- As mentioned before, the window we use is called the `Hamming Window` and is given by -

$$x[n] = 0.54 + 0.46 \frac{2\pi n}{N-1} \tag{4}$$

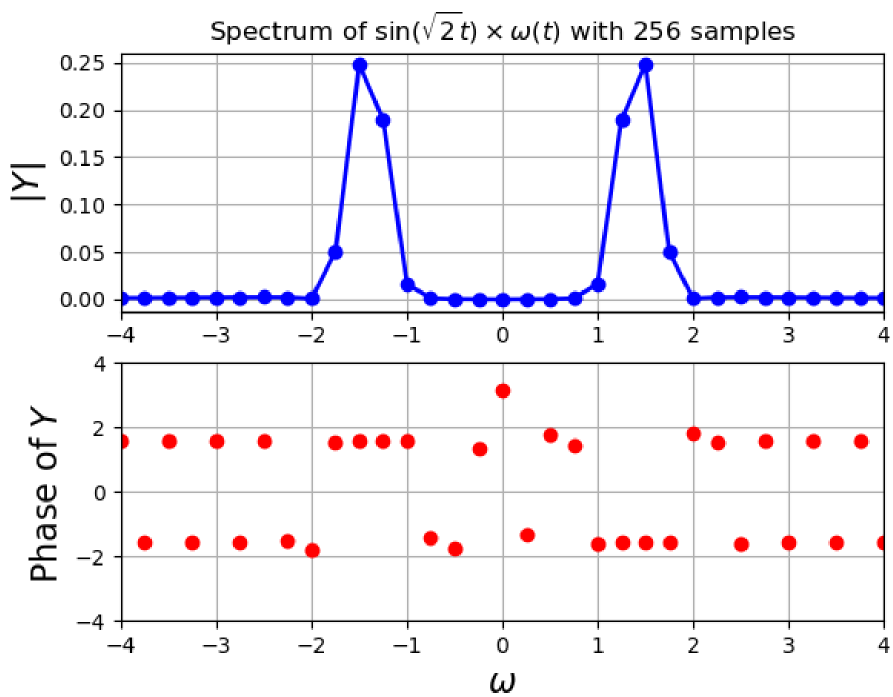
- We observe the plot of $\sin(\sqrt{2}t)w(t)$ with t wrapping every 2π. The plot for the same is shown below.



We now plot the spectrum of $\sin(\sqrt{2}t)$ after windowing. We obtain the spectrum as follows:

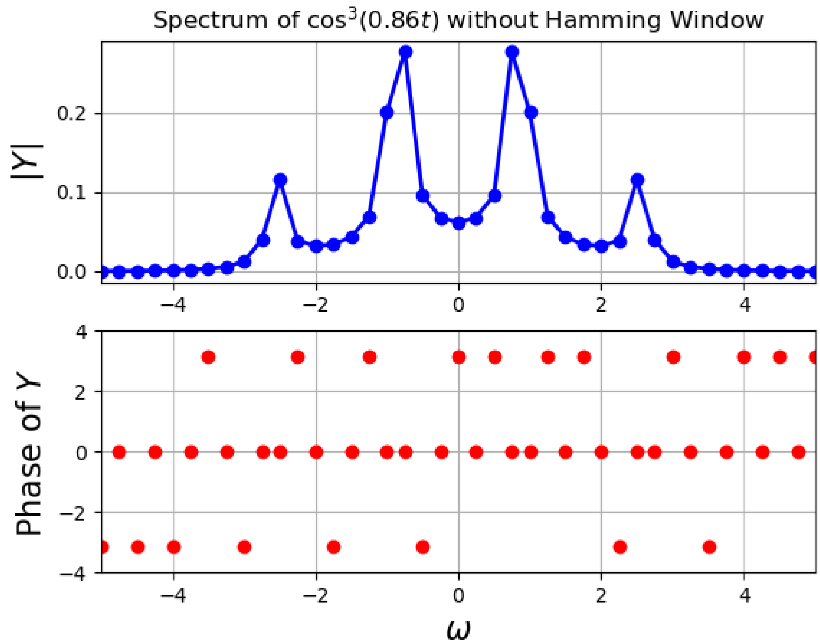


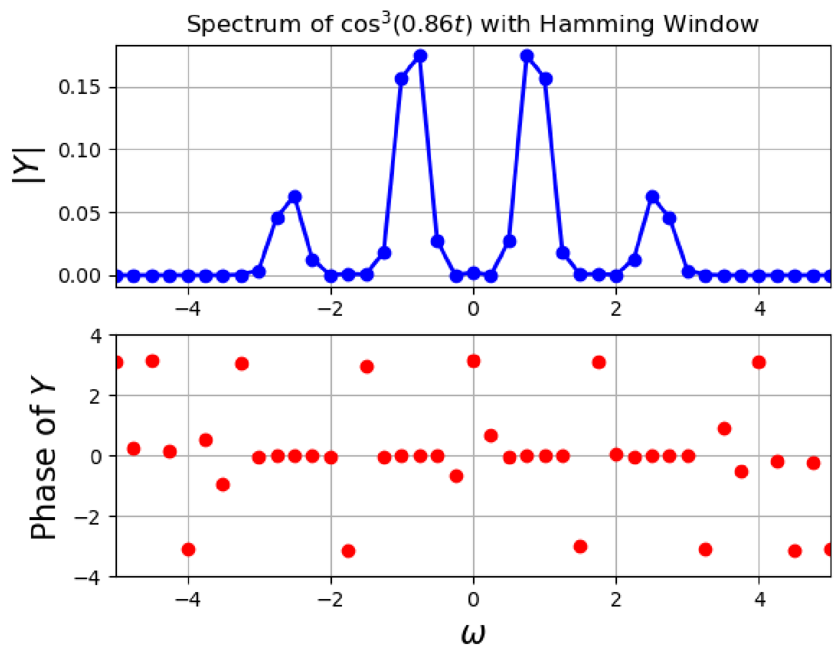
We now increase the number of samples by 4 times. We observe that we obtain an improved and more accurate spectrum as shown below -



5 Spectrum of $\cos^3(0.86t)$

- We plot the spectrum of $\cos^3(0.86t)$ with and without the Hamming window as shown in the below 2 figures.

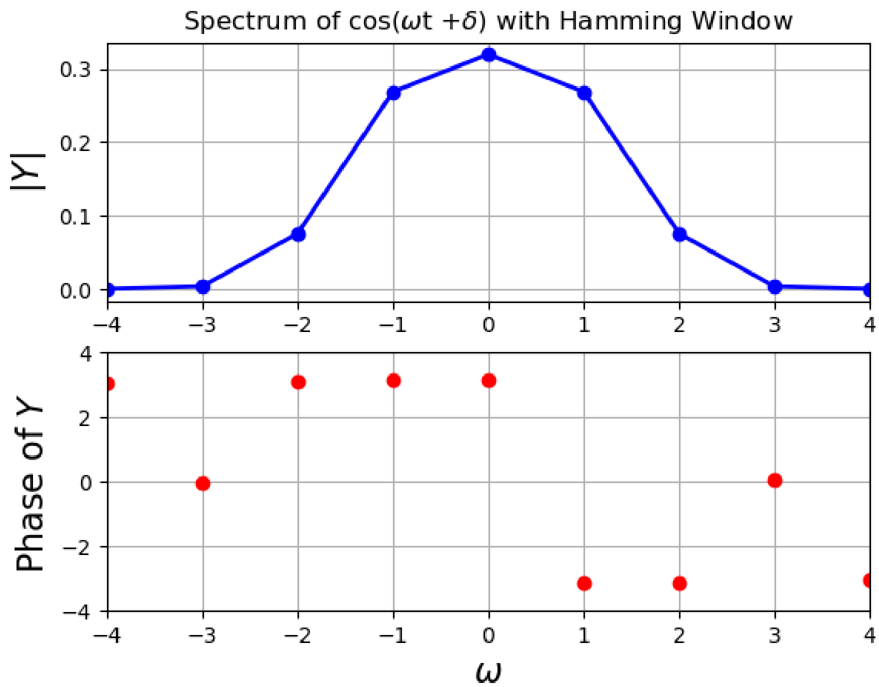




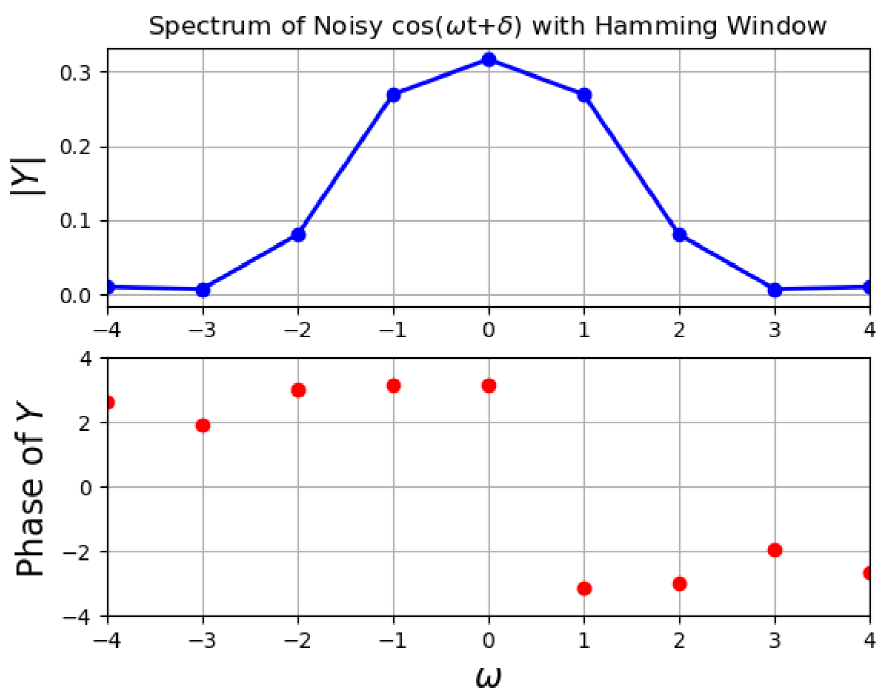
- We observe that the spectrum without the Hamming window has broader peaks, which shows that the peak energy is spread to the nearby frequencies too.
- When we multiply the function with the Hamming window and then obtain the Discrete Fourier Transform spectra, we obtain a more accurate spectrum than the previous one.
- By multiplication with the Hamming window, we attenuate the frequencies around the peak which leads us to obtaining a spectrum with higher accuracy.

6 Plotting and Estimating the spectrum of $\cos(\omega t + \delta)$

- We obtain the DFT of the function $\cos(\omega t + \delta)$ for $\omega = 0.8$ and $\delta = \pi$. We plot the spectrum of this function with the Hamming window as shown below.



We also obtain the spectrum of $\cos(\omega t + \delta)$ with added Gaussian white noise and plot the spectrum as shown below.



6.1 Estimation of ω and δ

- For estimating the values of ω and δ from the extracted Discrete Time Fourier Transform (DFT), we find the weighted average of frequencies.
- The weight of each frequency is taken as the magnitude of DFT at that frequency. We then estimate the value of peak frequency.
- The suitable power of magnitudes was set as $\mathbf{p = 1.6}$. The phase can also be estimated by observing the phase at points where the magnitude peaks, i.e ω_0.
- The code for estimation of the same is shown below:

6.1.1 Code

```
p=1.6
phase = angle(Y[::-1][argmax(abs(Y[::-1]))])
w0 = sum(abs(Y**p*w))/sum(abs(Y)**p)
print(f'Estimated value of frequency: {w0}')
print(f'Estimated value of phase: {phase}')
```

6.1.2 Observations

- **Estimated values of ω and δ for the function $\cos(\omega t + \delta)$ are: ω = 0.75350 and δ = 3.1415**
- **Estimated values of ω and δ for the function $\cos(\omega t + \delta)$ with added Gaussian noise are: ω = 2.55317 and δ = 3.1415**
- As we can see, the mathematically estimated values of ω and δ approximately matches with that of the initially assumed values.

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7 Spectrum of the Chirped Signal

- The chirped signal is given as follows -

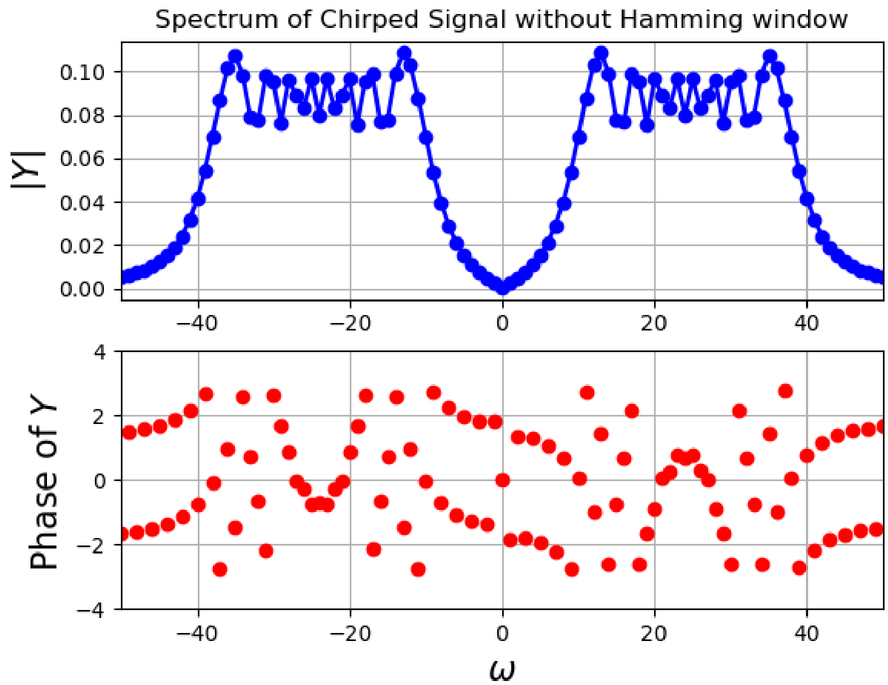
$$x(t) = \cos(16(1.5 + \frac{t}{2\pi})t) \tag{5}$$

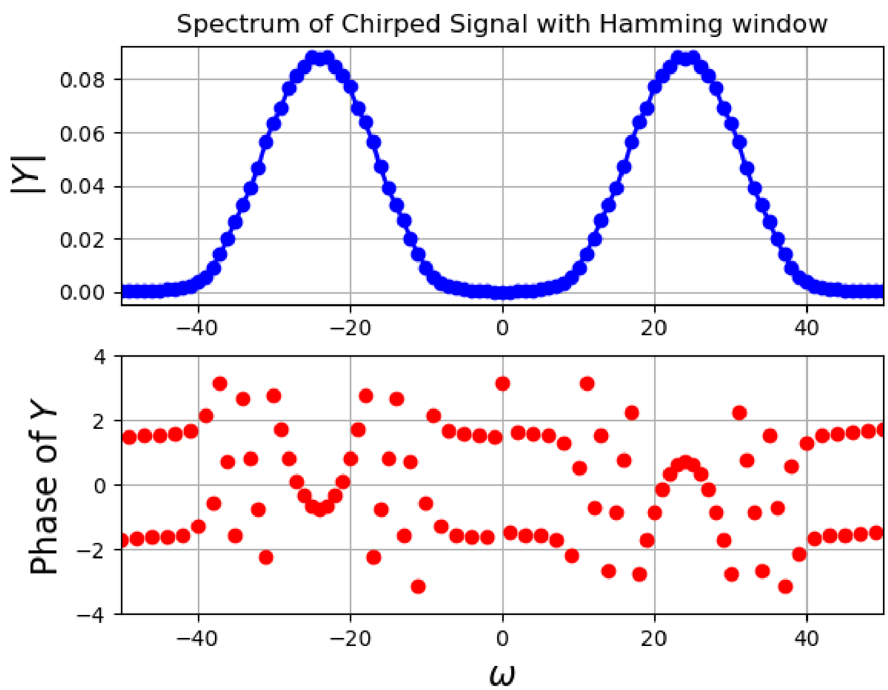
- The frequency of the chirped signal varies from 16 rad/sec at $-\pi$ to 32 rad/sec as we move towards π. We plot and observe the DFT Spectrum of the chirped signal with and without the Hamming Window.
- The plots for the same are shown below.

7.1 Code

```
# QUESTION 5: DFT of Chirped Signal :
#  ## Without Windowing :
HammingWin(4, 1, 1024, 50, heading=r"Spectrum of Chirped Signal without Hamming window", Hamming = False)

## With Windowing :
HammingWin(4, 1, 1024, 50, heading=r"Spectrum of Chirped Signal with Hamming window")
```


Spectrum of Chirped Signal without Hamming window
|Y|
0.10
0.08
0.06
0.04
0.02
0.00
−40
−20
0
20
40
Phase of Y
4
2
0
−2
−4
ω


Spectrum of Chirped Signal with Hamming window
|Y|
0.08
0.06
0.04
0.02
0.00
−40
−20
0
20
40
Phase of Y
4
2
0
−2
−4
ω

8 Surface Plot: Frequency variation with Time-Chirped Signal

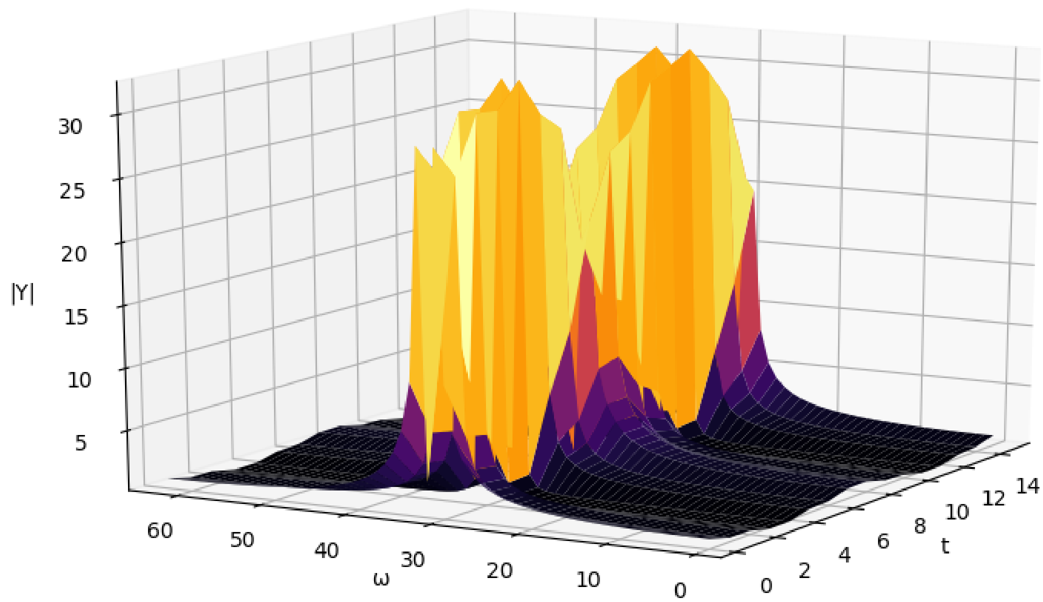
- We plot a Surface Plot to observe the frequency-time variation of the DFT magnitude. As mentioned before, the frequency of this signal varies from 16 rad/sec to 32 rad/sec as time varies.
- The plots are obtained for the chirped signal with and without windowing, as shown in the below figures.
- The code for the same is shown below.

8.1 Code

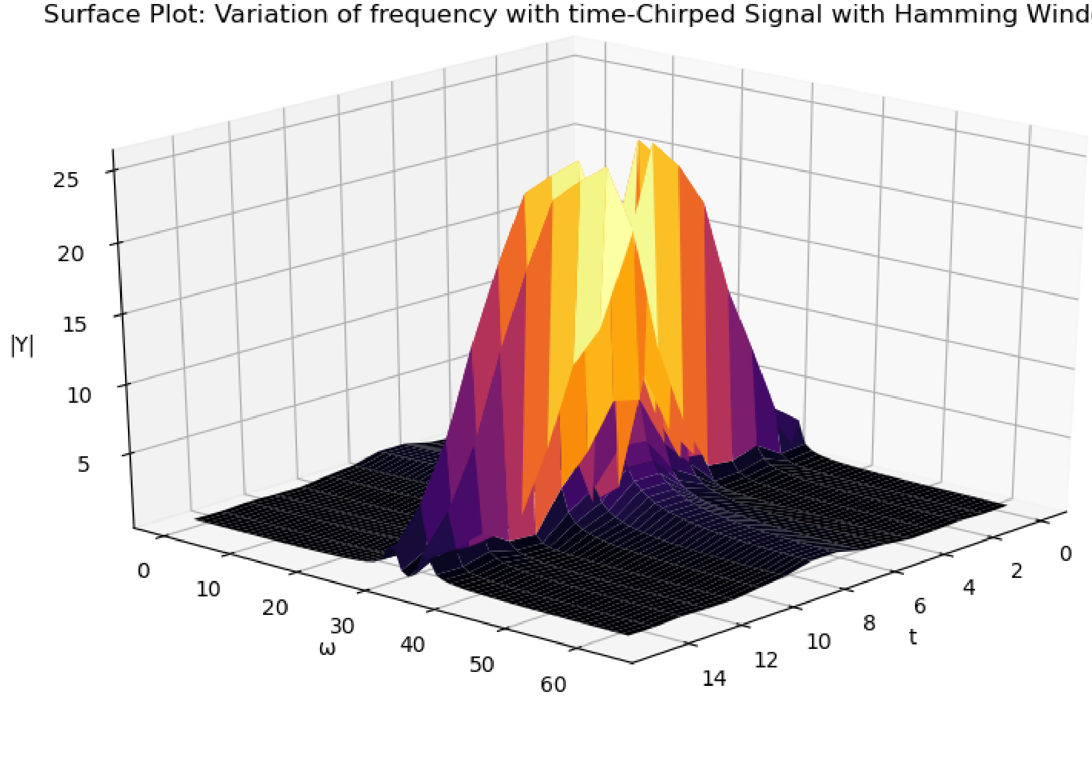
```
# QUESTION 6: Surface Plot - Chirped Signal :
## Without Hamming Window :
t = linspace(-1*pi, 1*pi, 1025); t = t[:-1]
dt = t[1] - t[0]; fmax =1/dt
y = cos(16*(1.5+t/(2*pi))*t)
y_ = np.zeros((16, 64), dtype = complex)
for i in range(16):
    y_[i]=fftshift(fft(fftshift(y[64*i:64*(i+1)])))
w = linspace(-pi*fmax, pi*fmax, 1025); w = w[:-1]
n = arange(64)
t1 = np.array(range(16))
t1, n = meshgrid(t1, n)
ax = Axes3D(figure())
surf = ax.plot_surface(t1, n, abs(y_).T, rstride=1, cstride=1, cmap ='inferno')
ylabel('\u03C9')
xlabel('t')
title("Surface Plot : Variation of frequency with time - Chirped Signal without Hamming Window")
ax.set_zlabel ('|Y|')
show()

 ## With Hamming Window :
t = linspace(-1*pi, 1*pi, 1025); t = t[:-1]
dt = t[1] - t[0]; fmax=1/ dt
n0 = arange(1024)
wnd = fftshift(0.54+0.46*cos(2*pi*n0/(1023)))
y = cos(16*(1.5+t/(2*pi))*t)*wnd
y_= np.zeros((16, 64), dtype = complex)
for i in range(16):
    y_[i]= fftshift(fft(fftshift(y[64*i:64*(i+1)])))
w = linspace(-pi*fmax, pi*fmax ,1025); w = w[:-1]
n = arange(64)
t1 = np.array(range(16))
t1, n = meshgrid(t1, n)
ax = Axes3D(figure())
surf = ax.plot_surface(t1, n, abs(y_).T, rstride=1, cstride=1, cmap='inferno')
ylabel('\u03C9')
xlabel('t')
title("Surface Plot: Variation of frequency with time-Chirped Signal with Hamming Window")
ax.set_zlabel('|Y|')
show()
```

Surface Plot : Variation of frequency with time - Chirped Signal without Hamming Window



Surface Plot: Variation of frequency with time-Chirped Signal with Hamming Window



9 Conclusion

- We can see that non periodic functions, have a distorted FFT due to discontinuities in the periodic extension of the signal (Gibbs Phenomenon). We can correct it by using a `Hamming Window`, which attenuates the discontinuities.
- A Hamming window sharpens the peak in the spectrum and the variation becomes more significant.
- ω was estimated using the square of the magnitude of the frequency spectrum as the probability distribution. δ was estimated by calculating the phase at the peak of spectrum.
- In the case of white noise, the estimated vales are less accurate and vary a lot due to random error.
- The spectrum of the frequency modulated chirped signal is plotted.
- We have also plotted and visualised the frequency-time surface plot of the chirped signal. We see that the peak frequency increases with time.
