Combinatorial game theory

Introduction

Type of Games

Consider a simple game which two players can play. There are N coins in a pile. In each turn, a player can choose to remove one or two coins. The players keep alternating turns and whoever removes the last coin from the table wins.

If N=1 then?

If N=2 then?

If N=3 then?

If N=4 then?

If N=5 then?

If N=6 then?

Once you start playing this game, you will realise that it isn't much fun. You will soon be able to devise a strategy which will let you win for certain values of N. Eventually you will figure out that if both players play smartly, the winner of the game is decided entirely by the initial value of N.

Strategy

Continuing in this fashion, we find that the first player is guaranteed a win unless N is a multiple of 3. The winning strategy is to just remove coins to make N a multiple of 3.

Finders Keepers game

A and B playing a game in which a certain number of coins are placed on a table. Each player picks atleast 'a' and atmost 'b' coins in his turn unless there is less than 'a' coins left in which case the player has to pick all those left.

a. Finders-winners: In this format, the person who picks the last coin wins

b.Keeepers-losers: IN this format, the person who picks last coin loses

Q1. A and B play game of finder-winners with a=2 and b=6. If A starts the game and there are 74 coins on the table initially, how many should A pick?

Sol.

If A picks 2, B will be left with 72 and no matter what number B picks, A can always pick (8-x) and wrap up

Q2.In a game of keepers- losers B started the game when there were N coins on the table. If B is confident of winning the game and a =3, b =5, which of the following cannot be the value of N?

94, 92, 76, 66

Sol.

8k+1,8k+2.......8k+a

IN keeper-losers, the motto is to give opponent ~~8k+1~~ coins to win for sure. with 66 coins, no matter what B does, he cannot give 65 coins to A.

Q3. A and B play Finders-winners with 56 coins, where A plays first anad a=1, b=6. what should B pick immediately after A?

2, 3, 4, cbd

Sol.

A will lose in cany case. But the number of coins B picks depends on what A picks.

Q4. in a game of Keepers-losers played with 126 coins where A plays first ans a=3, b=6, who is the winner?

Sol.

A wins as he can pick convert n to 9k+3

~~IN order to win, A should leave 9k+1 coins on the table, since he cannot do that he will definitely lose.~~

Q5.In a interesting version of game B gets to choose the number of coins on the table and A gets to choose the format of the game it will be as well as pick coins first. If B chooses 144 and a =1, b=5 which format should A choose in order to win?

Sol.

A should choose Keepers-losers and pick 5 coins from the table, leaving 139 coins for B

Properties of the above games

1.The games are sequential. The players take turns one after the other, and there is no passing

2.The games are impartial. Given a state of the game, the set of available moves does not depend on whether you are player 1 or player 2. Chess is a partisan game(moves are not same)

as player 1 can move only black pieces not white, also tic tac toe is partial game

3.Both players have perfect information about the game. There is no secrecy involved

4.The games are guaranteed to end in a finite number of moves.

5.In the end, the player unable to make a move loses. There are no draws. (This is known as a normal game. If on the other hand the last player to move loses, it is called a misère game)

Impartial games can be solved using Sprague-Grundy theorem which reduces every such game to Game of NIM

Game of NIM

Given a number of piles in which each pile contains some numbers of stones/coins. In each turn, a player can choose only one pile and remove any number of stones (at least one) from that pile. The player who cannot move is considered to lose the game (i.e., one who take the last stone is the winner).

/Take Examples/

Solution

Let n1, n2, …nk, be the sizes of the piles. It is a losing position for the player whose turn it is if and only if n1 xor n2 xor … xor nk = 0.

Nim-Sum : The cumulative XOR value of the number of coins/stones in each piles/heaps at any point of the game is called Nim-Sum at that point.

If both A and B play optimally (i.e- they don't make any mistakes), then the player starting first is guaranteed to win if

the Nim-Sum at the beginning of the game is non-zero. Otherwise, if the Nim-Sum evaluates to zero, then player A will lose definitely.

Why does it work?

From the losing positions we can move only to the winning ones:

– if xor of the sizes of the piles is 0 then it will be changed after our move (at least one 1 will be changed to 0, so in that column will be odd number of 1s).

From the winning positions it is possible to move to at least one losing:

– if xor of the sizes of the piles is not 0 we can change it to 0 by finding the left most column where the number of 1s is odd, changing one of them to 0 and then by changing 0s or 1s on the right side of it to gain even number of 1s in every column.

From a balanced state whatever we do, we always end up in unbalanced state. And from an unbalanced state we can always end up in atleast one balanced state.

Now, the pile size is called the **nimber/Grundy number** of the state.

Theorem

A game composed of K solvable subgames with Grundy numbers G1, G2 … GK is winnable iff the Nim game composed of Nim piles with sizes G1, G2 … GK is winnable.

So, to apply the theorem on arbitrary solvable games, we need to find the Grundy number associated with each game state.

But before calculating Grundy Numbers, we need to learn about another term- Mex.

What is Mex?

'Minimum excludant' a.k.a 'Mex' of a set of numbers is the smallest non-negative number not present in the set.

/Take examples/

Calculating Grundy Numbers

Example1
The game starts with a pile of n stones, and the player to move may take any positive number of stones. Calculate the Grundy Numbers for this game. The last player to move wins. Which player wins the game?

Grundy(0)=?

Grundy(1)=?

Grundy(n) = Mex (0, 1, 2, ….n-1) = n

```
int calculateMex(unordered_set<int> Set)
{
    int Mex = 0;

    while (Set.find(Mex) != Set.end())
        Mex++;

    return (Mex);
}

// A function to Compute Grundy Number of 'n'
// Only this function varies according to the game
int calculateGrundy(int n)
{
    if (n == 0)
        return (0);

    unordered_set<int> Set; // A Hash Table

    for (int i=1; i<=n; i++)
            Set.insert(calculateGrundy(n-i));

    return (calculateMex(Set));
}
```

Example2

The game starts with a pile of n stones, and the player to move may take any positive number of stones upto 3 only. The last player to move wins. Which player wins the game? This game is 1 pile version of Nim.

Grundy(0)=?

Grundy(1)=?

Grundy(2)=?

Grundy(3)=?

Grundy(4)=mex(Grundy(1),Grundy(2),Grundy(3))

```
int calculateMex(unordered_set<int> Set)
{
    int Mex = 0;

    while (Set.find(Mex) != Set.end())
        Mex++;

    return (Mex);
}

// A function to Compute Grundy Number of 'n'
// Only this function varies according to the game
int calculateGrundy(int n)
{
    if (n == 0)
```

```
        return (0);
    if (n == 1)
        return (1);
    if (n == 2)
        return (2);
    if (n == 3)
        return (3);

    unordered_set<int> Set; // A Hash Table

    for (int i=1; i<=3; i++)
            Set.insert(calculateGrundy(n - i));

    return (calculateMex(Set));
}
```

The game starts with a number- ‘n’ and the player to move divides the number- ‘n’ with 2, 3 or 6 and then takes the floor. If the integer becomes 0, it is removed. The last player to move wins. Which player wins the game?

```
int calculateGrundy (int n)
{
    if (n == 0)
        return (0);

    unordered_set<int> Set; // A Hash Table

```

```

    Set.insert(calculateGrundy(n/2));
    Set.insert(calculateGrundy(n/3));
    Set.insert(calculateGrundy(n/6));

    return (calculateMex(Set));
}
```

How to apply Sprague Grundy theorem

1.Break the composite game into sub-games.

2.Then for each sub-game, calculate the Grundy Number at that position.

3.Then calculate the XOR of all the calculated Grundy Numbers.

4.If the XOR value is non-zero, then the player who is going to make the turn
(First Player) will win else he is destined to lose, no matter what.

Problems

M&M Game

http://www.spoj.com/problems/MMMGAME/

QCJ3

http://www.spoj.com/problems/QCJ3/

HINT

Each stone at position P, corresponds to heap of size P in NIM

Now,if there are x stones at position n, then n is XORed x times because each stone corresponds to a heap size of n.

Then we use the property of xor operator

```
#include<bits/stdc++.h>

using namespace std;
inline get_int(){
    register int t=0;
    register char c=getchar();
    while(c<'0'||c>'9')c=getchar();
    while(c>='0'&&c<='9'){
        t=(t<<3)+(t<<1)+c='0';
        c=getchar();
    }
    return t;
}

int main()
{
    int n=get_int();
```

```
    while(n--)
    {

     int s=get_int();

    long r=0;
    int x;
    for(int i=1;i<=s;i++)
    {
            x=get_int();
            if(x&1)r=r^i;
    }

    puts(r==0?"Hanks Wins":"Tom Wins");

    }

    return 0;
}
```