RC Bridge Oscillation Memristor Chaotic Circuit

Brief History of Memristor Discovery

The memristor, often considered the fourth fundamental passive circuit element, was first postulated by Professor **Leon Chua** in 1971. Chua proposed the concept of the memristor based on symmetry considerations of the fundamental circuit elements: the resistor, capacitor, and inductor. **The memristor completes the set of relationships among the four fundamental variables in electrical circuits: voltage, current, charge, and magnetic flux**.
The mathematical definition of a memristor is given by:

$$V(t) = M(q(t)) \cdot I(t)$$

where $V(t)$ is the voltage, $I(t)$ is the current, $M(q(t))$ is the memristance, and $q(t)$ is the charge. The memristance M is a function of the charge and can change based on the history of the voltage applied.
For decades, the memristor remained a theoretical concept until 2008, when researchers at Hewlett-Packard (HP), led by Stanley Williams, provided the first experimental evidence of the memristor's existence. The HP team demonstrated that **nanoscale materials exhibiting resistive switching behavior could function as memristors**.

Growing Applications of Memristors

Since their discovery, memristors have opened up numerous applications due to their unique properties, such as non-volatility and the ability to ***remember*** their previous states. The following are some of the key growing applications of memristors today:

- **Non-Volatile Memory (ReRAM)**: Memristor-based resistive random-access memory (ReRAM) provides faster read/write times, higher endurance, and greater storage density compared to traditional flash memory.
- **Neuromorphic Computing**: Memristors are used to emulate synaptic functions in neuromorphic systems. Their ability to store and adapt weights between neurons makes them ideal for implementing energy-efficient artificial neural networks.
- **Analog Computing**: Memristors allow for the direct implementation of analog computations such as integration and differentiation, enabling the solution of differential equations in real-time. This is useful for applications in signal processing and control systems.
- **Chaotic Circuits**: Memristors are employed in chaotic circuits for generating random or pseudo-random signals. These circuits are useful in secure communications, random number generation, and cryptographic applications.
- **Artificial Intelligence (AI) Hardware**: Memristor crossbar arrays enable efficient matrix-vector multiplication, which is fundamental for AI workloads. This leads to speed and power improvements in AI acceleration hardware compared to traditional digital architectures.
- **Edge Computing and IoT**: Memristors are finding applications in edge computing and the Internet of Things (IoT) due to their small size, low power consumption, and ability to perform local data processing, reducing the need for constant cloud connectivity.

The analysis of RC Bridge Oscillation Memristor Chaotic Circuit requires some concepts of chaos theory which is explained in brief below -

Lyapunov Exponents (LE)

Lyapunov exponents λ_i characterize the exponential rates of divergence or convergence of nearby trajectories in a dynamical system. Mathematically, they are defined as:

$$\lambda_i = \lim_{t \to \infty} \lim_{\Delta \mathbf{x}(0) \to 0} \frac{1}{t} \ln \frac{|\Delta \mathbf{x}(t)|}{|\Delta \mathbf{x}(0)|}$$

Where:

- $\Delta\mathbf{x}(0)$ is the initial separation between two nearby trajectories.
- $\Delta\mathbf{x}(t)$ is the separation after time t.

For a system of N-dimensional differential equations, there are N Lyapunov exponents. These exponents measure the growth rate of perturbations along different directions in the state space.

Interpretation:

- $\lambda_i > 0$: Exponential divergence of trajectories, indicating chaos.
- $\lambda_i = 0$: Trajectories remain at a constant distance, indicating neutral stability (often associated with periodic or quasi-periodic behavior).
- $\lambda_i < 0$: Exponential convergence, indicating stable fixed points or attractors.

For chaotic systems, the largest Lyapunov exponent λ_1 is typically positive, signifying sensitive dependence on initial conditions.

Lyapunov Spectrum

The **Lyapunov Spectrum** is the set of all Lyapunov exponents $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ for a system of dimension N. The sum of the Lyapunov exponents is related to the **divergence** or **volume contraction** in the phase space.

- **Dissipative systems**: The sum of the Lyapunov exponents is negative, indicating volume contraction in phase space.
- **Hamiltonian systems**: The sum of the Lyapunov exponents is zero, indicating volume conservation (e.g., energy-conserving systems).

In many physical systems:

- One positive exponent indicates chaos.
- One zero exponent corresponds to time evolution along the trajectory.
- The remaining exponents are typically negative, indicating stability in other directions.

Bifurcation Diagram

A **bifurcation diagram** shows how the steady-state solutions (fixed points, periodic orbits) of a dynamical system change as a parameter μ is varied. Mathematically, bifurcations occur when the stability of a fixed point changes. Consider a one-dimensional system:

$$\dot{x} = f(x, \mu)$$

A bifurcation occurs when the Jacobian $\frac{df}{dx}$ at a fixed point x_0 has zero eigenvalue:

$$\frac{df}{dx}\Big|_{x_0, \mu=\mu_c} = 0$$

Common bifurcations:

- **Saddle-node bifurcation**: Two fixed points (one stable, one unstable) merge and annihilate each other as μ passes through a critical value.
- **Period-doubling bifurcation**: The system transitions from a periodic orbit of period T to one of period $2T$, often leading to chaos via repeated period-doubling.
- **Hopf bifurcation**: A fixed point becomes unstable, and a stable limit cycle arises, leading to periodic oscillations.

In chaotic systems, the bifurcation diagram often shows regions of periodic behavior interspersed with chaotic "bands."

Phase Portrait

A **phase portrait** represents the trajectories of a dynamical system in its state space. For a system of differential equations:

$$\dot{x} = f(x, y, z, \dots)$$
$$\dot{y} = g(x, y, z, \dots)$$
$$\dot{z} = h(x, y, z, \dots)$$

The phase portrait visualizes the evolution of the system as a set of curves, where each curve represents a trajectory for given initial conditions. The direction of the curves indicates the system's dynamics over time.
Fixed points: Points where $\dot{x} = \dot{y} = \dot{z} = 0$ are equilibrium points. The stability of these points can be analyzed by linearizing the system and examining the eigenvalues of the Jacobian matrix J at the fixed point.

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \dots \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

- If the eigenvalues of J have negative real parts, the fixed point is stable.
- If any eigenvalue has a positive real part, the fixed point is unstable.

Limit cycles: Closed loops in the phase portrait represent periodic motion. A limit cycle can be stable or unstable, depending on the behavior of nearby trajectories.

Poincaré Map

The **Poincaré map** reduces the dimensionality of a dynamical system by capturing intersections of trajectories with a lower-dimensional surface, called the **Poincaré section**. For a continuous system:

$$\dot{x} = f(x, y, z)$$

the Poincaré map tracks where the trajectory intersects a plane (e.g., $x = 0$) at regular intervals. Mathematically, the map is a discrete dynamical system:

$$\mathbf{x}_{n+1} = P(\mathbf{x}_n)$$

where P is the Poincaré map, and $\mathbf{x}_n$ are the points at successive intersections.
Interpretation:

- **Fixed points in the Poincaré map**: Correspond to periodic orbits in the original system.
- **Scattered points**: Indicate chaotic behavior, where the system does not return to the same point on the Poincaré section, signifying irregular dynamics.

Equilibrium and Stability Analysis

An **equilibrium point** $\mathbf{x}_0$ in a dynamical system $\dot{\mathbf{x}} = f(\mathbf{x})$ satisfies:

$$f(\mathbf{x}_0) = 0$$

To analyze the stability of this point, we linearize the system near $\mathbf{x}_0$ by expanding $f(\mathbf{x})$ in a Taylor series and keeping the linear terms:

$$\dot{\mathbf{x}} \approx J(\mathbf{x} - \mathbf{x}_0)$$

where J is the Jacobian matrix.
The eigenvalues λ_i of J determine the stability:

- **All** $\lambda_i < 0$: The equilibrium is stable (trajectories converge to $\mathbf{x}_0$).
- **Any** $\lambda_i > 0$: The equilibrium is unstable (trajectories diverge from $\mathbf{x}_0$).

Periodic Limit Cycle

A **periodic limit cycle** is a closed trajectory in phase space where the system returns to the same point after a fixed period T. Mathematically, for a system $\dot{\mathbf{x}} = f(\mathbf{x})$, a periodic solution $\mathbf{x}(t)$ satisfies:

$$\mathbf{x}(t + T) = \mathbf{x}(t) \quad \text{for all } t$$

Limit cycles arise through a **Hopf bifurcation**, where a stable fixed point loses stability and a stable limit cycle emerges. This can occur when a pair of complex conjugate eigenvalues of the Jacobian cross the imaginary axis from negative to positive real parts as a parameter μ changes.
Periodic Limit Cycle vs Chaos:

- **Periodic limit cycle**: The trajectory repeats itself exactly after a fixed period, corresponding to a single point in the Poincaré map.
- **Chaos**: The trajectory never repeats, and the Poincaré map shows a set of scattered points rather than a single point or a few points.

Coexistence Scenarios

Coexistence of Multiple Attractors or Scrolls:

- **Symmetric single-scroll coexistence**: Identical chaotic scrolls that alternate between two regions of phase space.
- **Asymmetric single-scroll coexistence**: Two different chaotic scrolls in the same system.
- **Symmetric double-scroll coexistence**: A pair of scrolls forming a larger structure, often in symmetric systems like Chua's circuit.
- **Asymmetrical limit-cycle coexistence**: A limit cycle coexists with chaotic or quasiperiodic motion, usually in systems with asymmetries.

The RC Bridge Oscillation Memristor Chaotic Circuit that we model is given below -

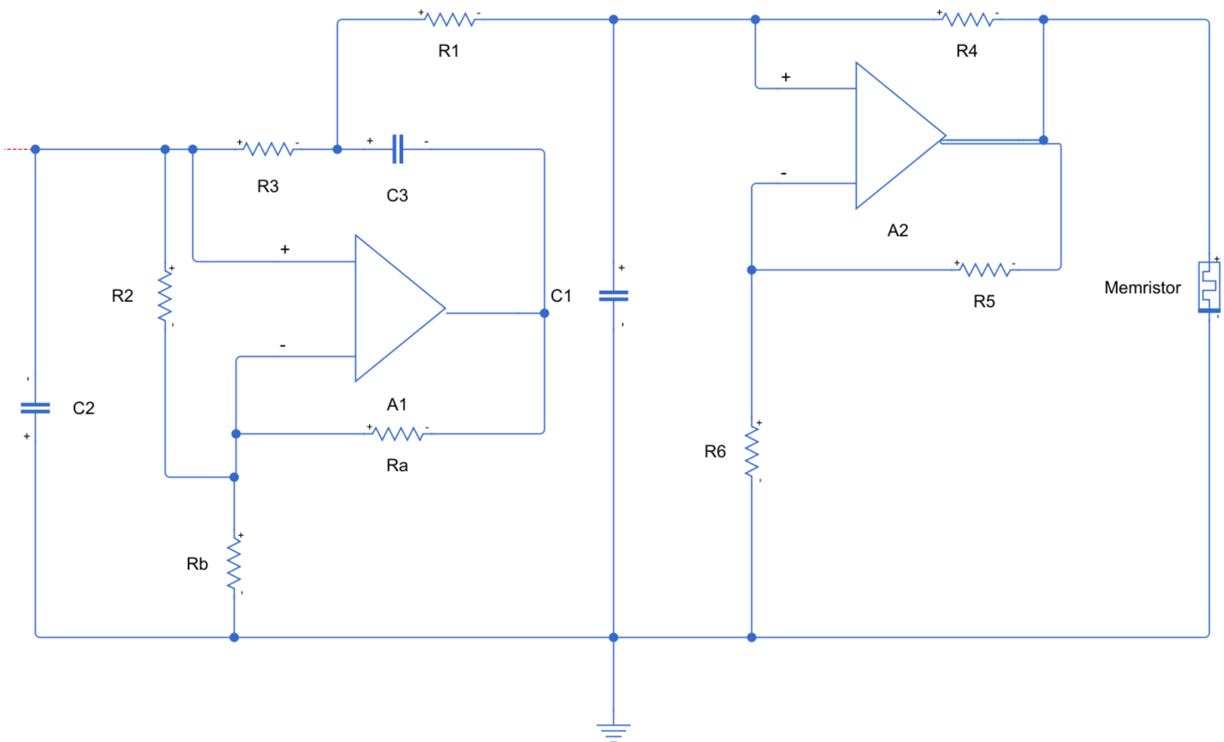


Figure 1: Circuit Diagram

Mathematical Analysis of the Circuit

The state variables are defined as follows:

- $u_1(t)$: Voltage across capacitor C_1
- $u_2(t)$: Voltage across capacitor C_2
- $u_3(t)$: Voltage across capacitor C_3
- $\phi(t)$: Internal flux of the memristor

The equations governing the circuit are:

$$\begin{aligned}
\frac{du_1(t)}{dt} &= \frac{1}{C_1}\left(Au_2 - u_3 - \frac{u_1}{R_{mm}}\right) - u_1 \cdot W(\phi(t)) \\
\frac{du_2(t)}{dt} &= \frac{1}{C_2}\left((A-1)u_2 - u_3 - \frac{u_2}{R_2}\right) \\
\frac{du_3(t)}{dt} &= \frac{1}{C_3}\left(Au_1 - u_3 + \frac{(A-1)u_2}{R_3}\right) \\
\frac{d\phi(t)}{dt} &= u_1
\end{aligned}$$

Simplified Differential Equations

Using the following parameter values:

- $R_1, R_2, R_3, R_4, R_5, R_6$: Resistors in the circuit
- C_1, C_2, C_3: Capacitors in the circuit
- A: Nonlinearity factor in the memristor model
- $W(\phi(t))$: Function representing the memristive behavior, which depends on the internal flux $\phi(t)$

Let:

- $\tau = \frac{t}{C_2 R_2}$
- $E = 1$ V
- $R_2 = R_3$
- $C_2 = C_3$
- $E = 1\,\mathrm{V}$
- $C_2 R_2 = 1\,\mathrm{s}$

Also Let:

- $a = \frac{C_2}{C_1}$
- $b = \frac{R_2}{R_1}$
- $c = \frac{R_2}{R_{mm}}$

Assuming suitable simplifications and relations, we can rewrite the equations as follows:

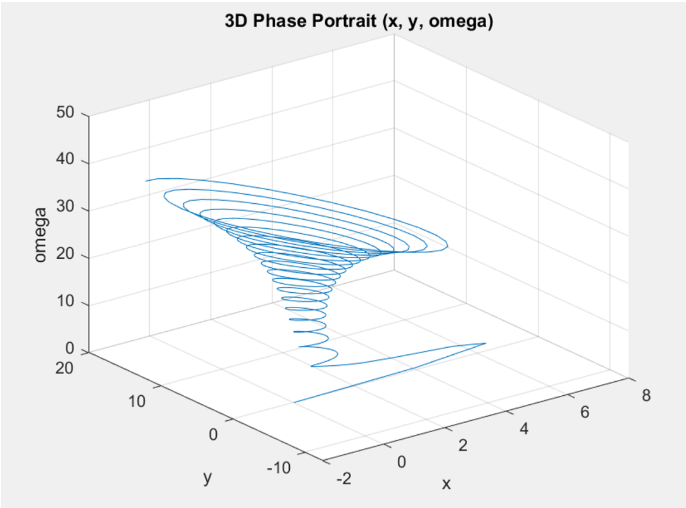
$$\begin{aligned}
\frac{dx}{d\tau} &= a(b(Ay - z - x) - cx - x - W(\omega)) \\
\frac{dy}{d\tau} &= A - 2y - z \\
\frac{dz}{d\tau} &= b(Ay - x - z) + (A-1)y \\
\frac{dx}{d\tau} &= x
\end{aligned}$$

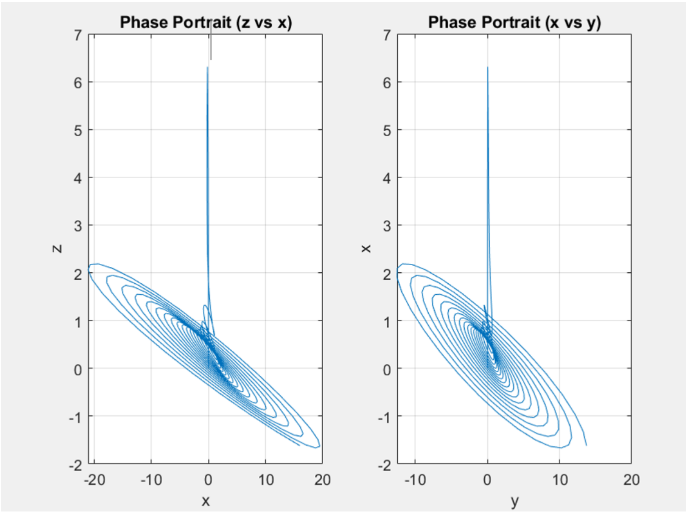
where $W(\omega)$ is taken as:

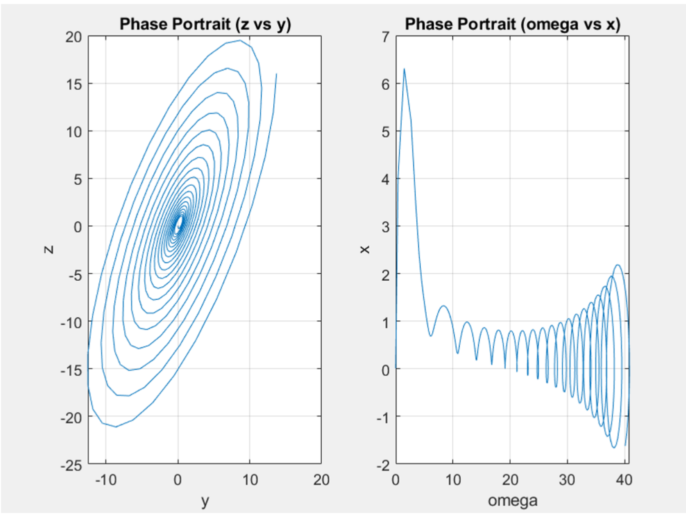
$$W(\omega) = 0.03 + 0.0614|\omega|$$

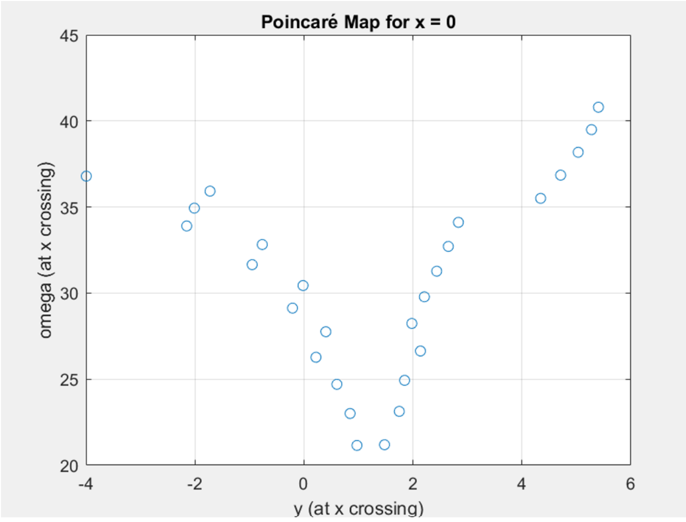
Simulation Results

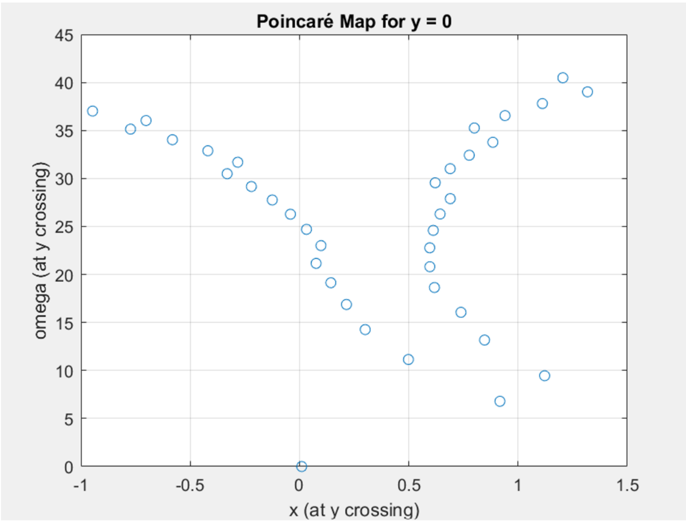
The circuit was simulated according to these equations and parameters in MATLAB. The simulation plots obtained are as follows:

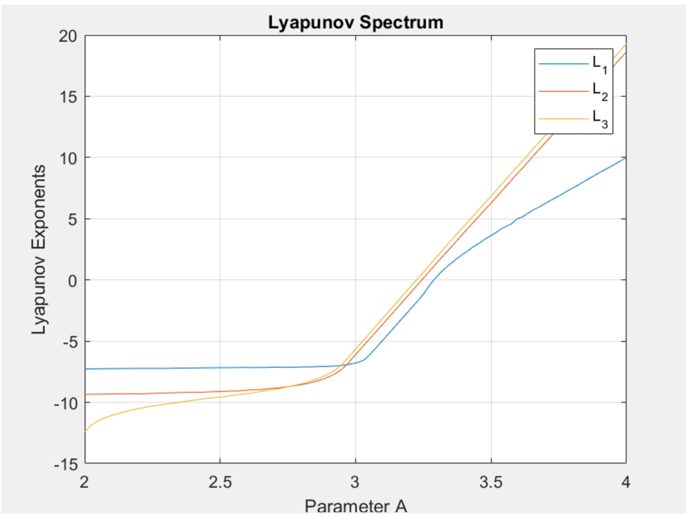


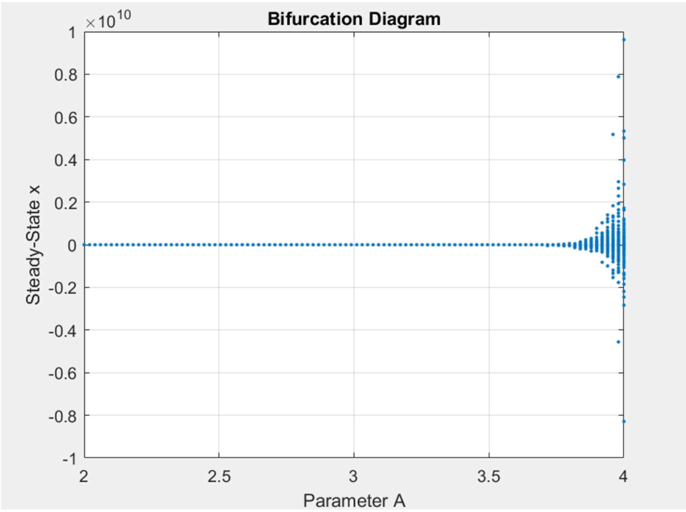





Poincaré Map for x = 0
omega (at x crossing)
y (at x crossing)
45
40
35
30
25
20
-4
-2
0
2
4
6


Poincaré Map for y = 0
omega (at y crossing)
x (at y crossing)
45
40
35
30
25
20
15
10
5
0
-1
-0.5
0
0.5
1
1.5


Lyapunov Spectrum
Lyapunov Exponents
Parameter A
L_1
L_2
L_3
20
15
10
5
0
-5
-10
-15
2
2.5
3
3.5
4



These results are obtained for a very small number of time steps and if we increase the time steps and complexity of the model, we woul expect the results to be the same as below -

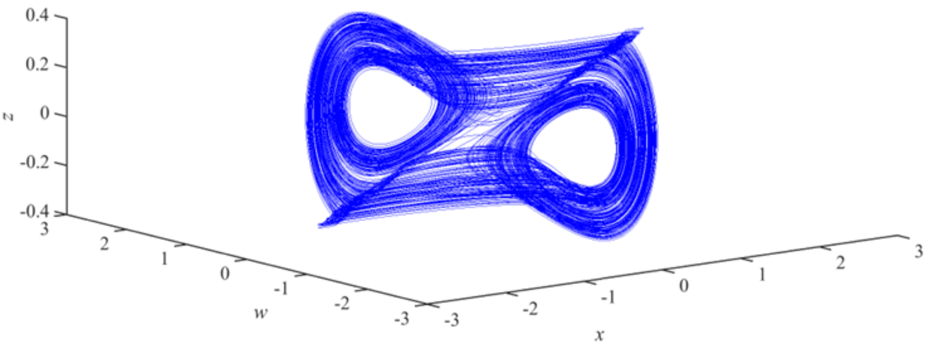


Figure 2: 3D Phase Plot

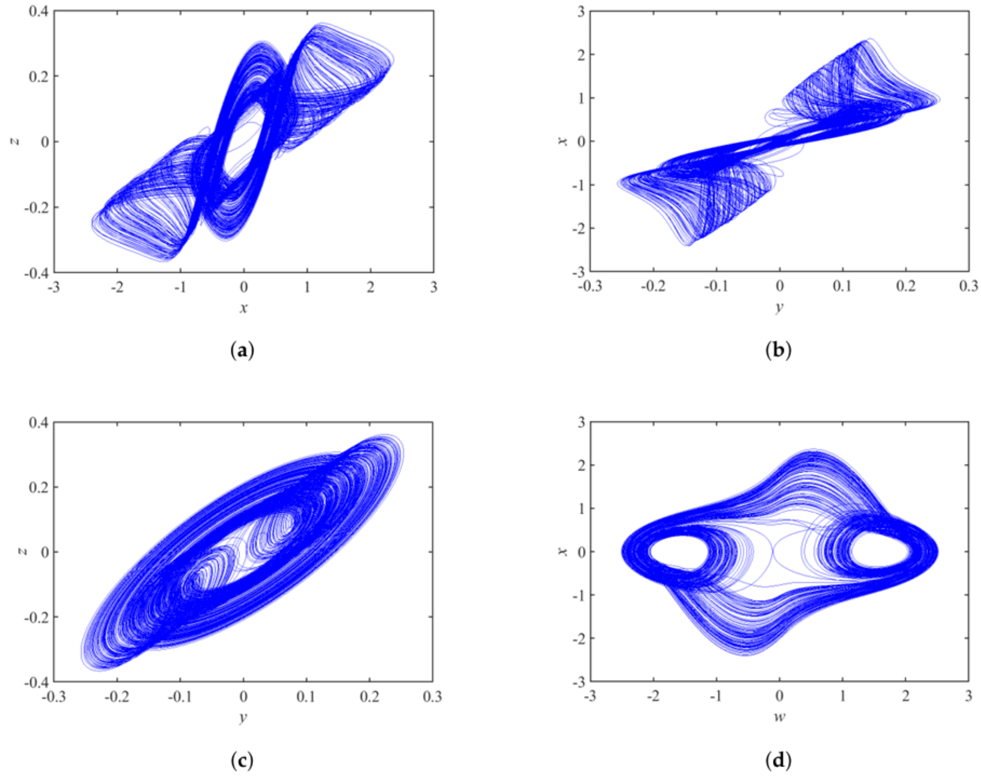


Figure 3: 2D Phase Plots

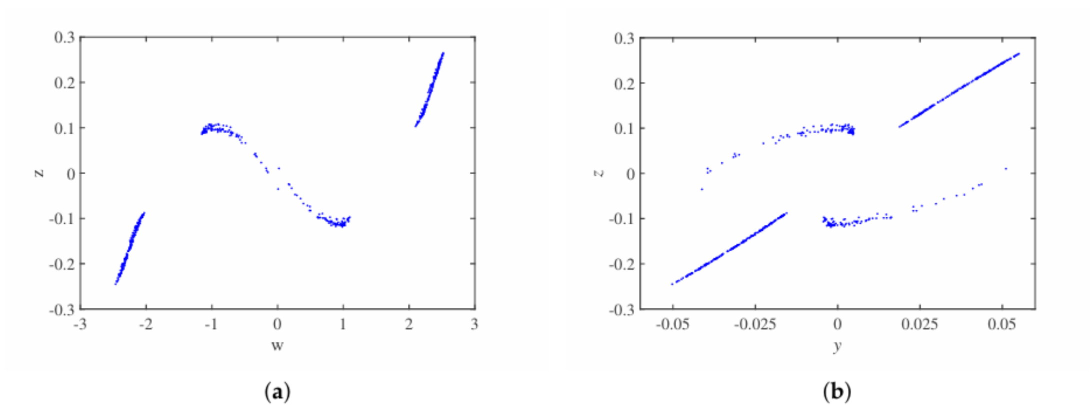
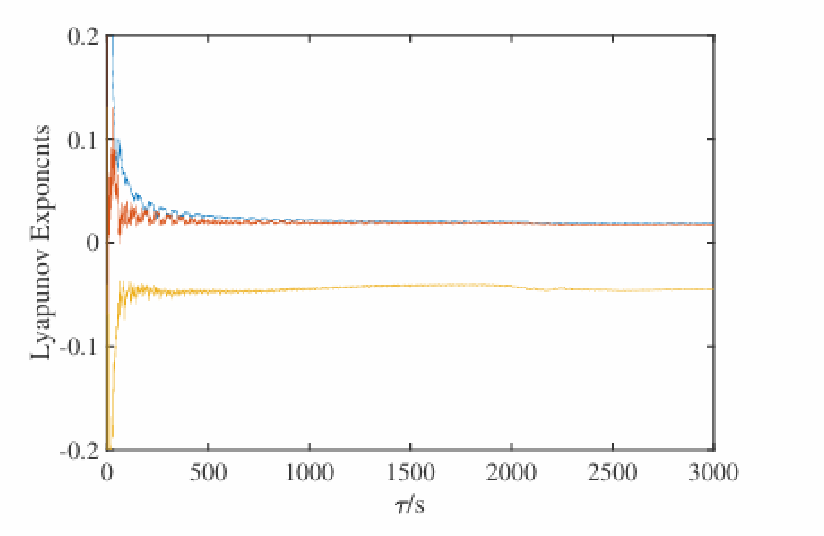


Figure 4: Poincaré Maps



(a)

Figure 5: Lyapunov Exponents Spectrum

Terms for describing the behaviour of a Chaotic System -

- **Attractor**: An attractor is a set of states toward which a dynamical system evolves over time. Attractors can be fixed points, limit cycles, or strange attractors, defining the long-term behavior of the system.
- **Single-Scroll Attractor**: A single-scroll attractor is a type of chaotic attractor characterized by a single, distinct spiral-like structure in the phase space. It exhibits chaotic dynamics, where the trajectory does not repeat and is sensitive to initial conditions.
- **Double-Scroll Attractor**: A double-scroll attractor consists of two intertwined scrolls or spirals in the phase space. It arises from systems with two distinct regions of chaotic behavior, leading to a more complex set of dynamics compared to single-scroll attractors.
- **Basin of Attraction**: A basin of attraction is the region in phase space from which trajectories converge to a particular attractor. The boundaries between basins determine the attractor the system will approach based on initial conditions.
- **Coexistence of Attractors**: The coexistence of attractors refers to scenarios where multiple attractors exist within the same dynamical system, leading to different behaviors depending on initial conditions or parameter values.
- **Strange Attractor**: A strange attractor is a type of attractor found in chaotic systems characterized by a fractal structure and sensitive dependence on initial conditions, where trajectories cover vast areas in a seemingly random manner.
- **Limit Cycle**: A limit cycle is a closed trajectory in phase space that represents periodic motion. It is an attractor that attracts nearby trajectories, leading to stable periodic behavior. Limit cycles can be stable or unstable, with stable limit cycles attracting trajectories and unstable limit cycles repelling them.

Behavior of the System

The behavior of a dynamical system can vary significantly based on its parameters, initial conditions, and the coexistence phenomenon.

0.1 Influence of Parameters

The parameters of a system, such as resistance, capacitance, and nonlinear characteristics of memristors, play a crucial role in determining the system's dynamics. For instance, varying the resistance in a circuit can lead to different attractor behaviors, transitioning from periodic to chaotic dynamics. In systems with memristors, the nonlinear relationship between voltage and current can also cause bifurcations, where a small change in a parameter can lead to a qualitative change in the system's behavior.

0.2 Effect of Initial Conditions

Initial conditions are critical in chaotic systems due to their sensitive dependence on initial states. Slight variations in initial conditions can lead to vastly different trajectories in phase space, resulting in different attractor behaviors. In chaotic circuits, this means that two circuits starting from similar initial states may evolve to entirely different attractors over time.

0.3 Coexistence Phenomenon

The coexistence of multiple attractors occurs when different regions of phase space correspond to distinct attractors. This can lead to complex dynamics where the system's behavior changes based on the initial condition it starts from. For example, a system with both a stable fixed point and a chaotic attractor can exhibit hysteresis, where the path taken to reach a state affects the final attractor.

Applications of Memristors

Some of the applications of memristors that were stated at the start are explained below:

Neuromorphic Computing

Working Principle: Memristors mimic the behavior of biological synapses by dynamically adjusting their resistance based on the history of voltage applied across them. The memristance $M(t)$ depends on the integral of the applied voltage $v(t)$:

$$M(t) = M_0 + \int_0^t v(\tau)\, d\tau$$

where M_0 is the initial memristance.
Applications:

- Synaptic devices for storing connection weights in neural networks.
- Energy-efficient computation due to the non-volatility of memristors.

Mathematical Analysis: In a memristor-based neural network, the update of synaptic weights w_{ij} between neuron i and neuron j is governed by the following equation:

$$\Delta w_{ij} = \eta \cdot x_i \cdot x_j$$

where:

- Δw_{ij} is the change in the synaptic weight connecting neuron i to neuron j.
- η is the learning rate, which determines the magnitude of the weight updates.
- x_i is the activation (or output) of the i-th neuron or input node.
- x_j is the activation (or output) of the j-th neuron or input node.

This equation signifies that the adjustment to the weight w_{ij} is directly proportional to the product of the activations of both connected neurons, reflecting their influence on the strength of their synaptic connection.

Chaotic Signal Generation

Working Principle: Memristor-based circuits, like the RC Bridge Oscillation Memristor Chaotic Circuit, exhibit chaotic behavior described by non-linear differential equations. For an RC memristor chaotic circuit:

$$\frac{dV}{dt} = -\frac{V}{RC} + M(t)I$$

where $M(t)$ is the time-varying memristance.
Applications:

- Secure communication: Chaotic signals can encrypt communication channels.
- Random number generation: Chaotic circuits can generate high-quality random numbers.

Mathematical Description: The Lyapunov exponent λ determines the sensitivity to initial conditions:

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln\left(\frac{d(t)}{d(0)}\right)$$

Non-Volatile Memory (Memristive RAM)

Working Principle: Memristors can be used in non-volatile memory, such as ReRAM, where the resistance state represents stored data. The resistance switching behavior is described by:

$$R(t) = R_0 + \int_0^t f(v(\tau))\, d\tau$$

where R_0 is the initial resistance.
Applications:

- Non-volatile data storage with faster read/write times than flash memory.
- Stacked memory architectures with increased memory density.

Brain-Machine Interfaces (BMIs)

Working Principle: Memristors can act as artificial synapses in BMIs, interfacing with biological neurons. The voltage-current relationship across the memristor is given by:

$$I(t) = G(V(t)) \cdot V(t)$$

where $G(V(t))$ is the conductance of the memristor.
Applications:

- Prosthetic control by decoding neural activity.
- Neural signal processing in medical devices.

Analog Computing

Working Principle: Memristors can be used in analog computing systems for continuous-valued computations, solving differential equations such as:

$$\frac{dV}{dt} = -\frac{V}{R} + I_{input}$$

Applications:

- Signal processing tasks like filtering and integration.
- AI hardware accelerators for efficient matrix-vector multiplication.