Aerodynamics and Fluid Mechanics

Midterm Report

Kushagra Mishra

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Abstract

This document contains a record of the stuff I have learnt in the first four weeks of SoS. The content in this document is based on lectures uploaded on MIT OCW and *Fundamentals of Aerodynamics*, by *J.D. Anderson*.

Contents

1 Introduction

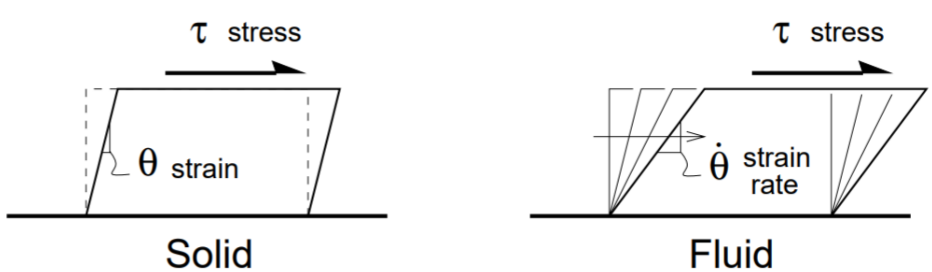
Fluid Mechanics is a branch of physics that is concerned with the behaviour of fluids, and its interactions with other objects, solid or fluid. Fluid mechanics can now be separated into two major sub-branches :

- *Aerostatics*: A branch of statics which deals with equilibrium of gaseous fluids, and solids immersed in them
- *Aerodynamics*: A branch of dynamics which deals with the motion of air and other gaseous fluids and the effects of these motions.

To understand fluid mechanics, we must first understand and important distinction between solids and fluids. When a force is applied *tangentially* to the surface of a solid, the solid experiences a *finite* deformation and the tangential force per unit area- the *shear stress* is usually proportional to the amount of deformation. In contrast, when a tangential force is applied to the surface of a fluid, the fluid experiences a continuously increasing deformation and the shear stress developed is usually proportional to the rate of change of this deformation. In mathematical terms,

$$\tau_{solid} = G\theta$$

$$\tau_{fluid} = \mu\dot{\theta}$$



Aerodynamics is usually aimed at one or more of the following practical objectives :

- The prediction of moments and forces on, along with the heat transfer to, bodies moving through a fluid. An example of this would be the analysis of an airfoil. Quantities like the *Lift*, and *Drag* are essential to actually understanding the flight of an aircraft.

- Determination of flows moving internally through ducts. This comes in handy to understanding the working principles behind wind tunnels, which are again a key part in aircraft and spacecraft design.

2 Properties of Fluids

2.1 Description of fluids

As we know, any substance is made up of discrete molecules. In solids, typically these molecules are always in contact with each other, and they are held together tightly by their strong inter-molecular forces. For a liquid, the molecules are still in contact, the they are not as rigidly held together, and hence, they still act like an uniform fluid material at macroscopic scales.
On the other hand, for a gas, these molecules are **not** in contact with each other, but still, after comparing the *mean free path* of these molecules to the scale of macroscopic system, we still find that we can approximate (Like true and good engineers) gases to be a continuous fluid material. Some known data for air :

- Mean free path at 0 KM (Sea Level) : 0.0001 mm

- Mean free path at 20 KM (U2 Level) : 0.001 mm

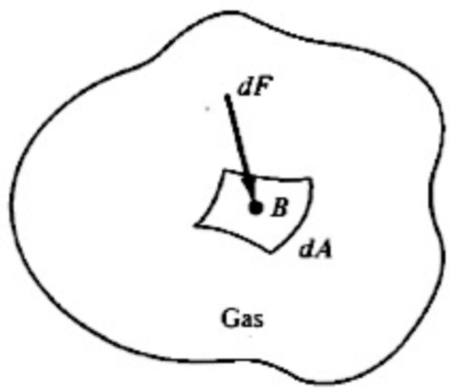
- Mean free path at 50 KM (Weather Balloons) : 0.01 mm

- Mean free path at 150 KM (Low-Earth Orbit) : 1 m

Again, since for low altitude flights, the *mean free path* of molecules is very small when compared to dimensions of any vehicle placed in the fluid, we can assume air to be a continuum. Its only when discussing any satellite or a low-orbit vehicle do we need to consider air to be discrete.

2.2 Pressure

The is pressure at a point is defined as the normal force per unit area exerted on a surface due to the time rate change of momentum of the gas molecules impacting on that surface.



$$pressure, p = \lim_{dA \to 0} \frac{dF}{dA}$$

Pressure can vary in space, and possibly also time. Hence, Pressure $p(x, y, z, t)$ is a *time varying scalar field.*

2.3 Density

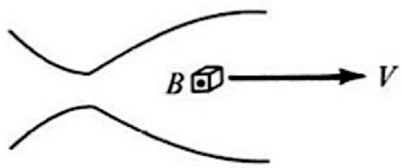
The density of a substance is the mass of that substance per unit volume.

$$density, \rho = \lim_{dV \to 0} \frac{dm}{dV}$$

Just like pressure, density $\rho(x, y, z, t)$ is also a *time varying scalar field.*

2.4 Velocity

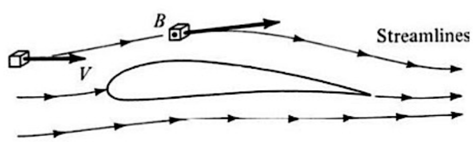
The principal focus of aerodynamics is fluids in motion, hence, flow velocity is an important consideration to be taken into account.



Consider an infinitesimal fluid element flowing through a fixed point B. Now, the instantaneous velocity of this fluid element is defined as the *flow velocity* at point B.
Unlike the previous quantities we've considered, Velocity, $\vec{V}(x, y, z, t)$ is a *time varying vector field*.
An useful quantity to define here is the *speed*, $V(x, y, z, t)$ of the fluid flow, which is defined as the magnitude, $\mid \vec{V} \mid$ of the flow field. Speed is a *time varying scalar field*.

2.5 Steady and Unsteady flows

If a fluid flow is *steady*, then $p, \rho, \vec{V}$ don't change with time. In other words, $p, \rho, \vec{V}$ are strictly functions of coordinates, (x, y, z) only.



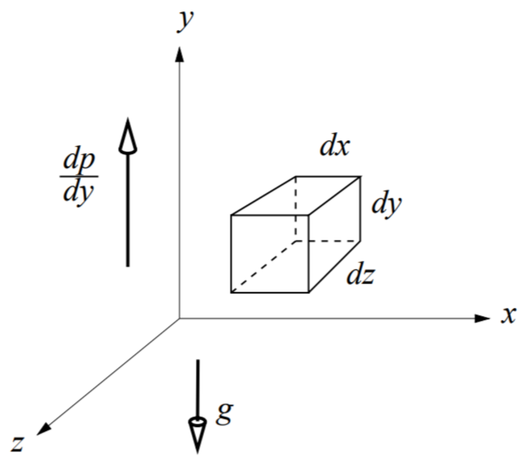
For a steady flow, we can define a *streamline*, as the path traced out by a fluid element over space with time.

3 Hydrostatics

This section will be dealing with the behaviour of a fluid, particularly water, when it is stationary. In such a case, my flow velocity, $\vec{V}$ is 0 for all space.

3.1 Hydrostatic Equation

Exploiting Newton's laws and the conditions of a hydrostatic system, we can arrive at the most useful equation describing the systems behaviour. Consider a fluid element present in a pressure gradient along the $y - axis$ and in the presence of Earth's gravitational field.



Since we're dealing with static conditions here, the net force on this fluid element must be 0. Balancing the forces along the y-direction, we get,

$$PressureForce + GravityForce = 0$$

$$pdA - (p + \frac{dp}{dy}dy)dA - \rho g d\nu = 0$$

$$-\frac{dp}{dy}dydA - \rho g d\nu = 0$$

The pressure acts on the area $dA = dxdz$ and the volume of the element, $d\nu = dxdydz$. Further simplifying, we get

$$dp = -\rho g dy$$

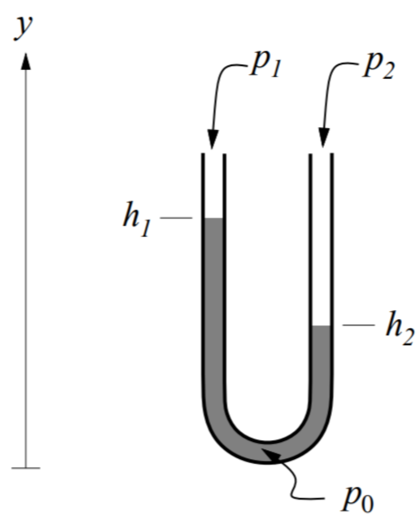
This is known as the differential form of the *Hydrostatic Equation*. If further we assume that density is constant, and that the pressure at $y = 0$ is p_0, we get

$$p(y) = p_0 - \rho g dy$$

This equation has quite a lot of applications, one of which I will be covering in the next subsection.

3.2 Manometer

A manometer is a U-shaped tube partially filled with a liquid of known density (Usually mercury), as show in the figure. Two different pressures, p_1 and p_2 are applied to the open ends of the manometer, causing a height difference to be created between the 2 liquid columns.



We now pick p_0 to be the pressure at some point, say the bottom of the tube. Applying the hydrostatic equation at this point, we get

$$p_1 = p_0 - \rho g h_1$$

$$p_2 = p_0 - \rho g h_2$$

Subtracting these 2 equations, we get

$$p_2 = p_1 + \rho g(h_1 - h_2)$$

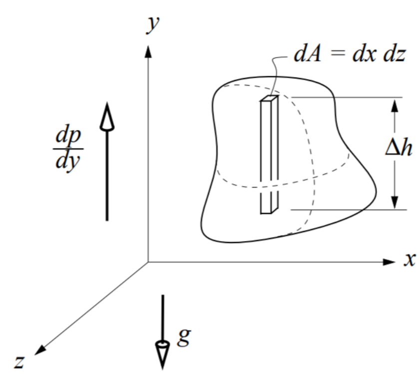
Now, if one end of the tube, say 1, is left open to the air, then $p_1 = p_{atm}$. Now, p_2 can easily be measured by measuring the height difference, $\Delta h = h_1 - h_2$ and applying the equation above.

$$p_2 = p_{atm} + \rho g \Delta h$$

Knowing the density of the fluid, we can calculate p_2 easily.
A manometer is a very widely used instrument in the aviation world and hence is very important.

3.3 Buoyancy

Archemedes Principle states that the buoyant force experiences by an object placed in a static fluid is equal to the weight of the water the object displaces. We can easily prove this using the hydrostatic equation.
Consider a body of arbitrary shape immersed in water. We can divide the object into several infinitesimally small cuboid of equal base area dA and varying heights Δh.



Applying the hydrostatic equation to one such cuboid, we get

$$dF = pdA - (p + \frac{dp}{dy}\Delta h)dA$$

$$dF = -\frac{dp}{dy}\Delta h dA$$

$$dF = \rho g d\nu$$

Integrating over the volume of the object, we get

$$F = \rho g \nu$$

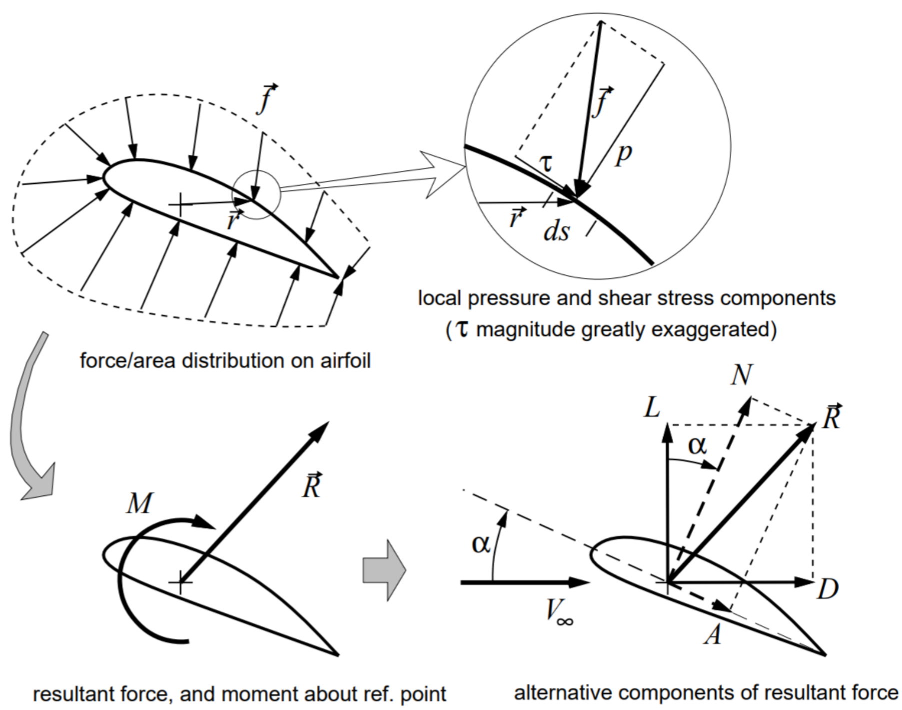
4 Aerodynamic Forces and Moments

Even though the interactions between a body placed in a flowing fluid might seem really complex at first, these aerodynamic forces are generated due to 2 basic sources :

- The shear stress (τ) distribution present on the body surface.
- The Pressure distribution over the body surface.

The *only* way nature has of communicating a force to a body moving through a fluid is entirely through the pressure and shear stress distributions over the body. Both pressure and shear stress (τ) have the same units, force per unit area. The main difference between these 2 forces is that, pressure acts perpendicular to the surface, while the shear stress acts parallel to the surface.

4.1 Surface Force Distribution



The fluid flowing about a body exerts a local *stress* $\vec{f}$ on each point of the body. Its normal and parallel components are p and τ respectively.
In a typical aerodynamic situation, the pressure p is much more larger than the shear stress τ (Usually by 2 orders of magnitude). As a result, the stress, $\vec{f}$ is very nearly perpendicular to the surface of the body.

Unfortunately, the small τ still can't be neglected, and in-fact it provides a significant contribution to the drag on the body.
The stress distribution, $\vec{f}$ when integrated over the whole surface yields the resultant force, $\vec{R}$ and the moment M about some suitably chosen reference point.

4.2 Force components

The resultant force $\vec{R}$ has components along any chosen axes. These axes can be arbitrary, but usually 2 choices are the most convenient to work with.

- *Freestream Axes*: The $\vec{R}$ components parallel to, and perpendicular to the direction of the *freestream*, $\vec{V_\infty}$ yields the drag, D and the lift, L respectively. These are important quantities to be taken into account when analysing a flight.

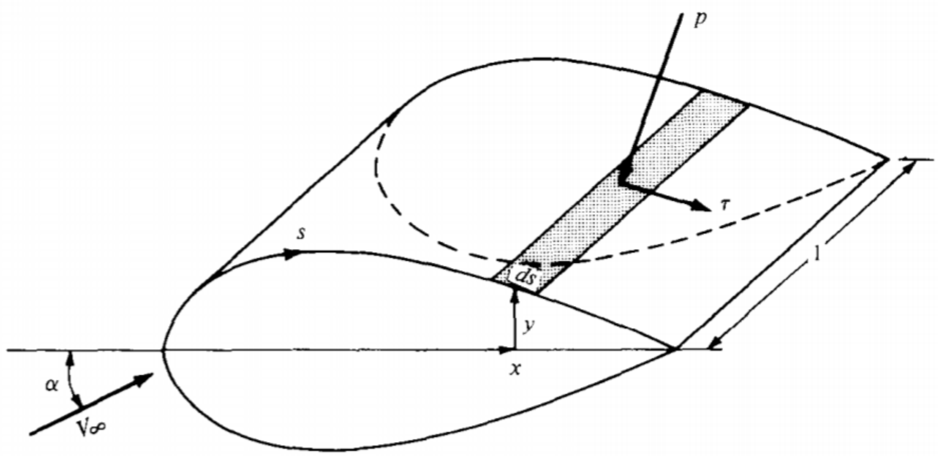
- *Body Axes*: The $\vec{R}$ components perpendicular to, and parallel to the airfoil *chordline* yields the Normal force, N, and Axial force, A, respectively.

The forces components in both frames can be related by

$$L = N\cos\alpha - A\sin\alpha$$

$$D = N\sin\alpha + A\cos\alpha$$

4.3 Force and Moment Calculation



Consider a cylindrical wing section of chord c and span l which has the force components A and N, along with moment M. It is often useful to deal with unit-span quantities.

$$A' \equiv A/l$$

$$N' \equiv N/l$$

$$M' \equiv M/l$$

Consider a small element of width ds_u. The unit-span force components acting on this elemental sections are

$$dN'_u = (-p_u \cos\theta - \tau_u \sin\theta) ds_u$$

$$dA'_u = (-p_u \sin\theta + \tau_u \cos\theta) ds_u$$

Similarly for the Lower Edge,

$$dN'_l = (-p_l \cos\theta - \tau_l \sin\theta) ds_u$$

$$dA'_l = (-p_l \sin\theta + \tau_l \cos\theta) ds_u$$

Integrating this from the *Leading Edge* to the *Trailing Edge*, we get

$$N' = \int_{LE}^{TE} dN'_u + \int_{LE}^{TE} dN'_l$$

$$A' = \int_{LE}^{TE} dA'_u + \int_{LE}^{TE} dA'_l$$

Calculating the moment about the *Leading Edge.* we get

$$M'_{LE} = \int_{LE}^{TE} -xdN'_u + \int_{LE}^{TE} -xdN'_l + \int_{LE}^{TE} ydN'_u + \int_{LE}^{TE} ydN'_l$$

From geometry, we have

$$ds \cos\theta = dx$$

$$ds \sin\theta = -dy$$

Upon simplifying the earlier equations, and taking a few approximations (again, like true and good engineers xD), we arrive at the following formulae

$$L' \approx \int_{0}^{c} (p_l - p_u)dx$$

$$M'_{LE} \approx \int_{0}^{c} -(p_l - p_u)xdx$$

Now, it must be remembered that the above formulae are only valid for low angles of attack.

4.4 Non-dimensional Coefficient

We define the non-dimensional coefficients as follows :

$$C_L \equiv \frac{L}{q_\infty S}$$

$$C_D \equiv \frac{D}{q_\infty S}$$

$$C_{M,LE} \equiv \frac{M_{LE}}{q_\infty Sl}$$

We know that the phenomenon of Lift, Drag and Moments and their variations with various factors is rather complex to understand, and what these coefficients do is basically pack all the complexity into and leave us with these 3 deceptively easy looking formulas. In reality, these coefficients are quite difficult to calculate and they can all be expressed as a function of 3 parameters (α, M, R_e) where M is the mach number of the flow ($M \equiv \frac{V}{V_{sound}}$) and R_e is the Reynolds Number. Mathematically,

$$C_L = f_1(\alpha, M, R_e)$$

$$C_D = f_2(\alpha, M, R_e)$$

$$C_M = f_3(\alpha, M, R_e)$$

Now, even though these formulas may look useless for now, they are able to explain a very important phenomenon. For 2 different flows, with the same value of M and R_e, the coefficients will have the exact same value for 2 similar shapes. This explains why wind tunnel testing of a model aircraft can provide useful data about the actual aircraft. This phenomenon is termed as *Dynamic Similarity.*

5 Effects of Flow conditions

5.1 Effects of Reynolds Number

The Reynolds number can be interpreted as a comparison of *pressure forces* and *viscous shear forces* which act on the body, by taking their ratio.

$$\frac{pressure forces}{shear forces} = \frac{p - p_\infty}{\tau} \approx \frac{\rho u^2}{\mu du/dn} \approx \frac{\rho_\infty V_\infty c}{\mu_\infty} \equiv R_e$$

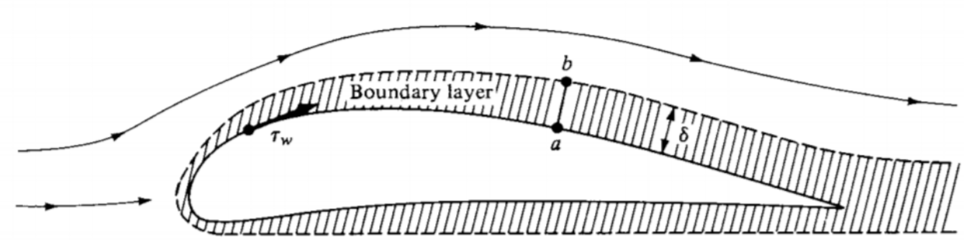
The Reynolds Number is thus a measure of the *pressure forces* relative to the *viscous shear forces*. Thus, if the Reynolds Number *increases*, the viscous effects of the flow become progressively less important.
The ratio $\mu/\rho \equiv \nu$ is called the *kinematic velocity* of the fluid. This quantity is arguably more important than μ since this is what actually occurs in the Reynolds Number formula.

$$R_e = \frac{V_\infty c}{\nu_\infty}$$

Typically, ν has a very small value, which results in a very high Reynolds Number. The typical value of the Reynolds Number for a few object is listed in the table below.

Object	Re
Butterfly	5 K
Pigeon	50 K
RC glider	100 K
Sailplane	1 M
Business jet	10 M
Boeing 777	50 M

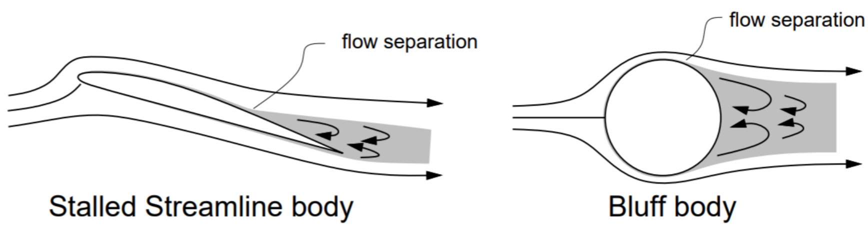
The Reynolds Number is high even for insects, which implies that the flow around them can be assumed to be *inviscid* everywhere. The viscous shear become significant only within the *boundary layer* which forms attached to body surfaces and become a *wake* trailing downstream.



The boundary layer shrinks with an increase in Reynolds Number. This has a few consequences of great physical significance :

- Neglecting the shear stress term in the equations of fluid motions greatly simplifies them and makes them easier to solve.
- If the Reynolds number is large enough, then its effects on some aerodynamic forces and moments can be neglected.

The assumptions that viscosity has negligible effect on the are only valid to *streamline bodies*, which gradually taper to a sharp point or an edge. For *blunt bodies*, which have a blunt front face, or *streamline bodies* with a high angle of attack, we observe an important physical phenomenon known as *flow separaton*. *flow separation* refers to the sudden thickening or break away of the boundary layer, leaving behind a trailing wake. This leads to sudden and dramatic rise in the Drag on the body. Typically, after an airfoil stalls, the drag on it becomes 10 fold.



5.2 Effects of mach number

The mach number of a body is defined as the ratio of the freestream velocity of flow to the speed of the sound. In mathematical terms,

$$M_\infty \equiv \frac{V_\infty}{a_\infty}$$

An interpretation of the mach number is that it represents a measure of the fractional variation in pressure of the flow. In mathematical terms,

$$\frac{\Delta p}{p_\infty} \approx \frac{\gamma}{2} M_\infty^2$$

From this equation, we can observe that for a low-speed flow, the fractional pressure variation will be quite low, and hence variation in other parameter will be quite low as well. This results in an important characteristic of low-speed flows, which is that nearly all fluid properties remain constant throughout,

Another important observation is that low-speed flows are really insensitive to changes in the mach number. The fluid properties for $M_\infty = 0.1$ and $M_\infty = 0.01$ are pretty much the same. This essentially means that for low-speed flows, we can neglect the effects of the mach number on quantities such as the *lift coefficient*.

6 Control Volumes

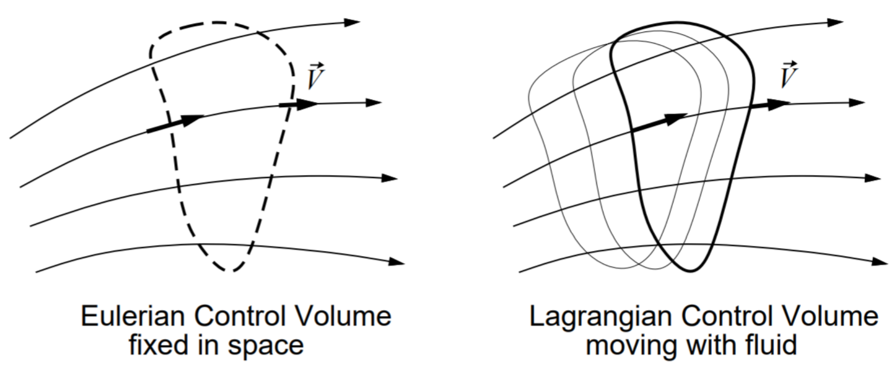
To accurately apply basic laws of physics such as:

- *Law of conservation of Mass*
- *Law of conservation of momentum*
- *Law of conservation of energy*

We invoke the use of a few new concepts. One such concept is *control volumes* which can be both *finite* or *infinitesimal*. There are two types of *control volumes*, namely:

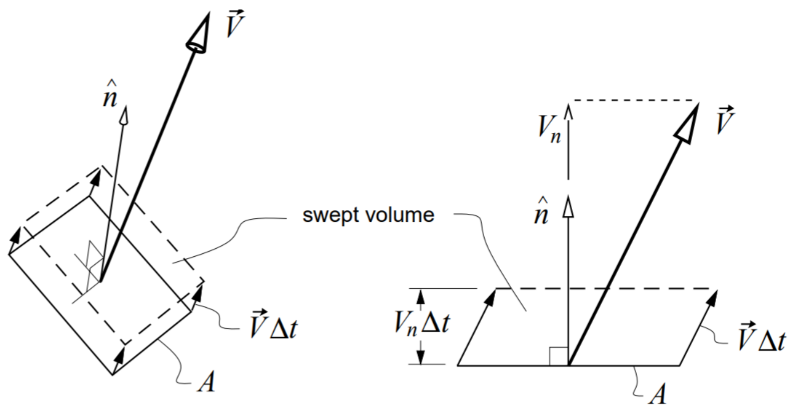
- *Eulerian type*: The control volume which stays fixed in space and allows fluid to flow through it
- *Lagrangian type*: The control volume which moves along with the fluid but doesn't allow fluid to cross its boundaries

For the sake of discussion, we'll limit ourselves to *Eulerian type* control volumes, even though both the types of volumes are physically sound and can be applied anywhere.



6.1 Mass Conservation Application

Consider an infinitesimal patch of area A on the surface of an Eulerian Control Volume present in a fluid flow.



The normal unit vector to the surface is given by $\hat{n}$. The fluid particles present on the patch will move out and sweep a volume

$$\Delta\nu = V_n A \Delta t$$

in the time interval between t and $t + \Delta t$, where V_n is the normal component of the fluid flow velocity given by $V_n = \vec{V}.\hat{n}$.

The mass contained in this small volume, Δm is

$$\Delta m = \rho\Delta\nu = \rho V_n A \Delta t$$

From this, we can define the time rate of mass, also known as *mass flow* as

$$mass\ flow = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} = \rho V_n A$$

$$mass\ flux = \dot{m} = \frac{m}{A} = \rho V_n$$

Conservation of mass can now be applied to a finite Eulerian control volume, and we arrive at

$$\frac{d}{dt}(Mass\ of\ volume) = Mass\ flowing\ into\ volume$$

$$\frac{d}{dt}\iiint \rho d\nu = -\oiint \rho\vec{V}\cdot\hat{n}dA$$

Applying Gauss's Theorem and bringing the time part inside the triple integral, we get

$$\iiint [\frac{d\rho}{dt} + \nabla\cdot(\rho\vec{V})]d\nu = 0$$

Breaking our initial control volume into several infinitesimally small more control volumes and applying the same formula to each volume, we can conclude that the integrand in the above formula must always be zero. Mathematically,

$$\frac{d\rho}{dt} + \nabla \cdot (\rho\vec{V}) = 0$$

This equation is known as the continuity equation and it is one of the fundamental equations used extensively when studying fluid flows. For a steady flow, the equation simplifies to

$$\nabla \cdot (\rho\vec{V}) = 0$$

Which under a further assumption of low-speed flow simplifies to

$$\nabla \cdot \vec{V} = 0$$

A more commonly used form of the continuity equation for a steady-flow is given by,

$$\rho V A = constant$$

7 Momentum Flow

When a fluid flows through a surface, it carries momentum along with mass. We thus define a new quantity, *momentum flow* as

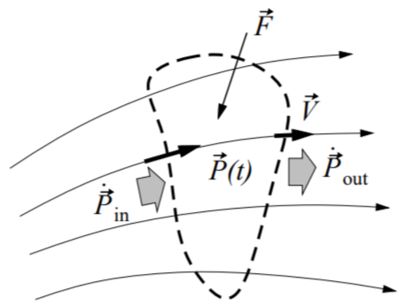
$$momentum\ flow = (mass\ flow) \times (\vec{Momentum}/mass)$$

$$\vec{momentum}\ flow = \dot{m}\vec{V} = \rho A(\vec{V} \cdot \hat{n})\vec{V} = \rho A V_n \vec{V}$$

Note that while mass flow was a scalar quantity, momentum flow is a vector quantity. We thus define *momentum flux* as

$$\vec{momentum}\ flux = \rho V_n \vec{V}$$

7.1 Momentum Conservation



According to Newton's Second Law, The *impulse* provided by a force, $\vec{F}$ in a small time interval Δt produces a change. $d\vec{P}$ in the momentum vector such that

$$\frac{d\vec{P}}{dt} = \vec{F}$$

Applying this equation to a fixed control volume,

$$\frac{d\vec{P}}{dt} + \dot{\vec{P}}_{out} - \dot{\vec{P}}_{in} = \vec{F}$$

where $\vec{P}$ is defined as the instantaneous momentum inside control volume, and is given by

$$\vec{P} = \iiint \rho \vec{V} d\nu$$

The $\dot{\vec{P}}_{out}$ term is added because the out-flowing fluid carries away momentum provided by $\vec{F}$, while the $\dot{\vec{P}}_{in}$ is subtracted because the momentum of the in-flowing fluid is incorrectly accounted for and must be accounted for. Both combined,

$$\dot{\vec{P}}_{out} - \dot{\vec{P}}_{in} = \oiint \rho(\vec{V} \cdot \hat{n})\vec{V} dA$$

Shifting our discussion to the nature of $\vec{F}$, it comes in mainly two types:

1. *Body Forces*: These forces act on the fluid present inside the volume. A good example of this would be gravity, given by

$$\vec{F}_{gravity} = \iiint \rho \vec{g} d\nu$$

2. *Surface Force*: These forces act on the surface of the volume and can be separated into two components, namely *pressure forces* and *viscous forces*.

$$\vec{F}_{pressure} = \oiint -p\hat{n}dA$$

Substituting all these in the Second Law equation, we arrive at the integral form of the *momentum equation*,

$$\frac{d}{dt}\iiint \rho\vec{V}d\nu + \oiint \rho(\vec{V}\cdot\hat{n})\vec{V}dA = \oiint -p\hat{n}dA + \iiint \rho\vec{g}d\nu + \vec{F}_{viscous}$$

Writing $\vec{V}$ as $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$, and applying Gauss's theorem, we arrive at the Differential form of the *Momentum equation*.

$$\frac{\partial(\rho u)}{\partial t} + \nabla\cdot(\rho u\vec{V}) = -\frac{\partial p}{\partial x} + \rho g_x + (F_x)_{viscous}$$

$$\frac{\partial(\rho v)}{\partial t} + \nabla\cdot(\rho v\vec{V}) = -\frac{\partial p}{\partial y} + \rho g_y + (F_y)_{viscous}$$

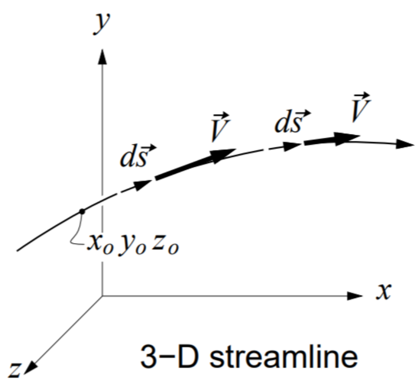
$$\frac{\partial(\rho w)}{\partial t} + \nabla\cdot(\rho w\vec{V}) = -\frac{\partial p}{\partial z} + \rho g_z + (F_z)_{viscous}$$

These equations are the embodiment of Newton's Second Law, applied at every point in the flow field.

8 All the lines

We can define three types of path trajectories for a fluid elements, namely Streamlines, Pathlines and Streaklines. For a steady flow, these trajectories are essentially they change, but they can differ considerably when it comes to unsteady flows.

8.1 streamlines



A *streamline* is defined as a line which is parallel to the fluid velocity vector, $\vec{V}(x, y, z, t) = u\hat{i} + v\hat{j} + w\hat{k}$, everywhere along it.

Since the infinitesimal arc-length vector is parallel to the velocity vector, we have

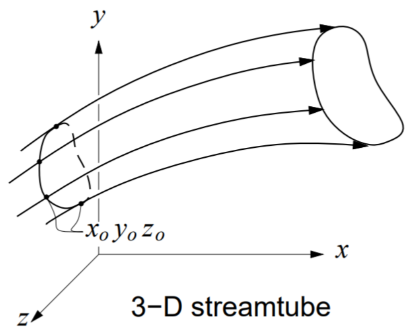
$$d\vec{s} \times \vec{V} = 0$$

$$(wdy - vdz)\hat{i} + (udz - wdx)\hat{j} + (vdx - udy)\hat{k} = 0$$

Setting each component of the above vector to zero, we get the differential equations defining a streamline. Knowing a point (x_0, y_0, z_0) on this streamline, we can find the equation of the entire streamline. If we were to restrict our discussion to only 2-D, we get the differential equation describing a streamline in 2-D, given by

$$\frac{dy}{dx} = \frac{v}{u}$$

A closed set of streamlines comprise the surface of a *streamtube*.



8.2 Pathlines

A *pathline* of a fluid element is defined as the trajectory it traced out over time. An example of a pathline would be the trajectory of a drop as it falls down a tap. The full pathline of a fluid element can be calculated by integrating the 3 velocity field components, $u(x, y, z, t)$, $v(x, y, z, t)$, $w(x, y, z, t)$ over time starting from the coordinates, (x_0, y_o, z_0). Mathematically,

$$x(t) = x_0 + \int_{t_0}^{t} u(x, y, z, \tau) d\tau$$

$$y(t) = y_0 + \int_{t_0}^{t} v(x, y, z, \tau) d\tau$$

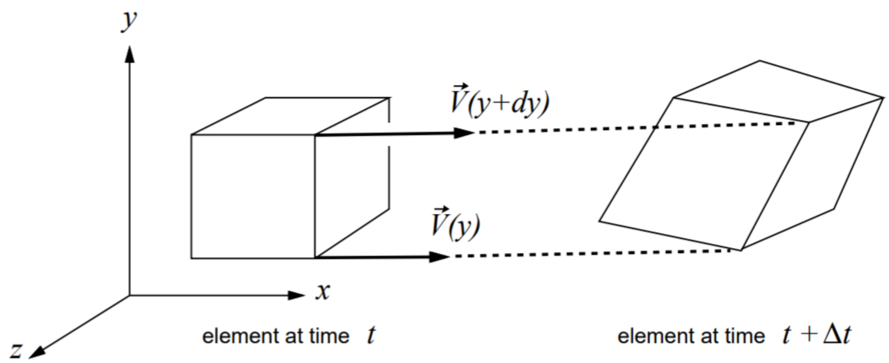
$$z(t) = z_0 + \int_{t_0}^{t} w(x, y, z, \tau) d\tau$$

8.3 Streaklines

A *streakline* is defined as the locus of all points which pass through a fixed point P as the fluid flow progresses.

9 Vorticity and Strain Rate

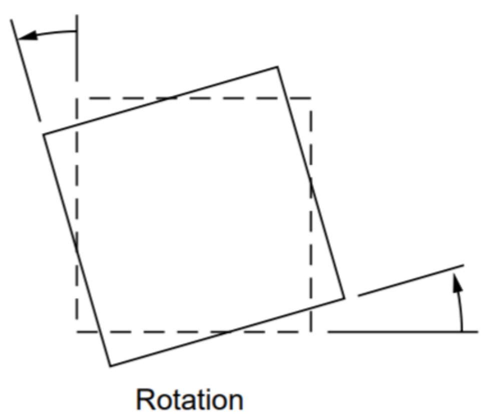
In previous discussions, we had just limited ourselves to discussion about the position and velocity of the fluid element. Its now time to discuss the change in shape and orientation of this fluid element.



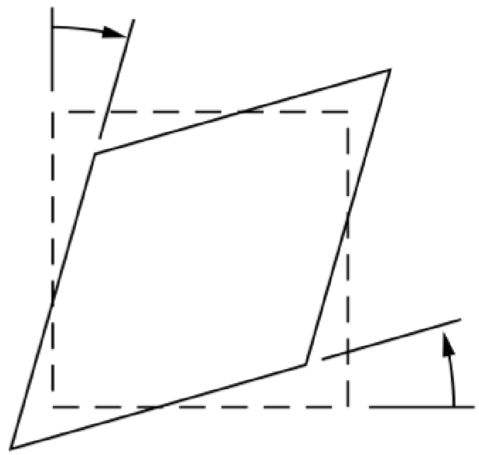
Consider the cuboidal fluid element at time t. If the velocities of the extremities of the element vary significantly, the shape of the element might distort and it might undergo some combination of tilting and stretching. Since tilting motions have a greater impact on the fluid elements, we will limit our discussion to only the tilting motion of a fluid element.

In general, there are two type of *tilting* motions possible for a fluid element.

1. *Pure Rotation*: If the adjacent sides of the element tilt in the same direction by equal amounts, the motion is described as a *pure rotation*.

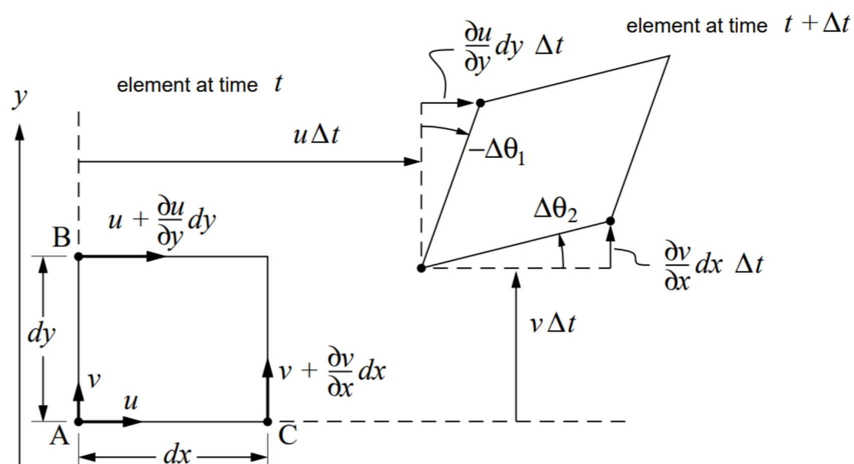


2. *Pure Shear*: If the adjacent sides of the element move in the opposite directions by equal amounts, the motion is described as a *pure shear*.



Both of these motions have great physical implications. Rotation is responsible for complicating the equations of fluid motion quite a bit. Shearing together with fluid viscosity is responsible for phenomenon such as flow separation and drag.

9.1 Side-Tilting analysis



For simplicity's sake, we'll limit our discussion to 2-D for now.

Consider a 2-D fluid element in the $x - y$ plane at two different points of time, t and $t + \Delta t$.

Points A and B differ in the x-component of their velocities by an amount $\frac{\partial u}{\partial y} dy$. Over a time interval of Δt the difference in the x-coordinates of A and B is given by,

$$\Delta x_B - \Delta x_A = \frac{\partial u}{\partial y} dy \, \Delta t$$

This results in the change of angle of side AB made with the vertical by some amount, say θ_1. θ_1 is given by,

$$\begin{aligned} -\Delta\theta_1 &= \frac{\Delta x_B - \Delta x_A}{dy} \\ &= \frac{\partial u}{\partial y} \Delta t \end{aligned}$$

We can now define time rate of change of this angle θ_1 as

$$\frac{d\theta_1}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\theta_1}{\Delta t} = -\frac{\partial u}{\partial y}$$

Similarly, for side AC,

$$\frac{d\theta_2}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\theta_2}{\Delta t} = \frac{\partial v}{\partial x}$$

We will now use these results to define important quantities such as *Vorticity* and *Strain Rate*.

9.2 Vorticity

Vorticity can be thought of as a measure of the local spinning motion of a continuum surrounding some point. Using the results in the previous sections, we can calculate the average angular velocity of sides AB and AC as,

$$\omega_z = \frac{1}{2}(\frac{d\theta_1}{dt} + \frac{d\theta_2}{dt}) = \frac{1}{2}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})$$

Repeating the similar for the other axes as well, we get

$$\begin{aligned} \omega_y &= \frac{1}{2}(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}) \\ \omega_x &= \frac{1}{2}(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}) \end{aligned}$$

We can define the total angular velocity vector, $\vec{\omega}$ as

$$\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

Since we'll be dealing with the quantity, $2\vec{\omega}$ more often, we define the *vorticity vector* of the fluid flow, $\vec{\xi}$ as $\vec{\xi} = 2\vec{\omega}$. It can now be easily verified that $\vec{\xi}$ is simply the curl of $\vec{V}$.

$$\vec{\xi} = \nabla \times \vec{V}$$

An *irrotational flow* is simply one for which $\vec{\xi} = \vec{V}$ is 0 at every point throughout the flow. For such flows, the fluid element may move and deform, but it can never rotate.

9.3 Strain Rate

We define the *strain* on a fluid elements as follows,

$$strain = \Delta\theta_2 - \Delta\theta_1$$

where $\Delta\theta_1$ and $\Delta\theta_2$ are the same angles as the ones defined previously. Now the *strain rate* is simply the derivative of this quantity with respect to time. Mathematically,

$$\varepsilon_{xy} = \frac{strain}{dt} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

Doing the same for the other planes as well,

$$\varepsilon_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\varepsilon_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

9.4 circulation

The instantaneous *circulation* of a flow around any closed curve, C, in a velocity field is defined as

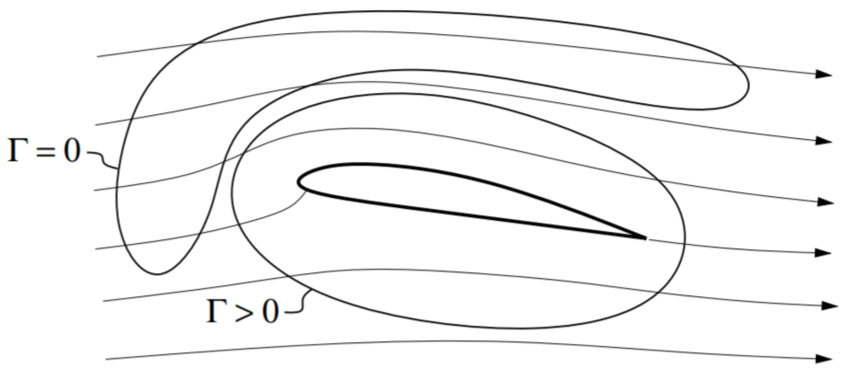
$$\Gamma \equiv -\oint_C \vec{V} \cdot d\vec{s}$$

Applying Stokes' Theorem on the above integral, we get

$$\Gamma \equiv -\oint_C \vec{V} \cdot d\vec{s} = -\iint_A (\nabla \times \vec{V}).\hat{n}dA = \iint_A \vec{\xi}.\hat{n}dA$$

$$\Gamma \equiv \iint_A \xi dA \quad (For\, 2-D)$$

This tells us that circulation is closely related to the vorticity of a flow. For *irrotational flows*, $\vec{\xi} = 0$ by definition. This implies that for *irrotational fields* for any closed contour, provided that it can be reduced to a point while staying inside the flow field, will have $\Gamma = 0$. This means that for a closed loop containing an object, say an airfoil, may have a non-zero *circulation*.



10 Bernoulli's Equation

Just like the previously derived *momentum equation* was just *Newton's Second Law* applied to flow fields, *Bernoulli's Equation* can be thought of as the application of *Conservation of Energy* to *inviscid* and *incompressible* flows. It can be derived for the 1-D flows as follows,

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \rho g_x + (F_x)_{viscous}$$

$$\rho u \frac{du}{dx} + \frac{dp}{dx} = 0$$

$$\frac{1}{2} \rho \frac{d(u^2)}{dx} + \frac{dp}{dx} = 0$$

$$\frac{1}{2} \rho u^2 + p = constant \equiv p_0$$

The above equation makes a few important assumptions about the flow namely,

- A steady flow : $\partial u/\partial t = 0$
- Negligible gravity : $\rho g_x \approx 0$
- Negligible viscous forces : $(F_x)_{viscous} \approx 0$
- An incompressible flow : $\rho = constant$

p_0 in the above equation is known as the *stagnation pressure* of the flow and it is defined as the pressure one would feel if a moving fluid flow were brought to rest *adiabatically* and *reversibly*. This pressure is determined by the upstream conditions.

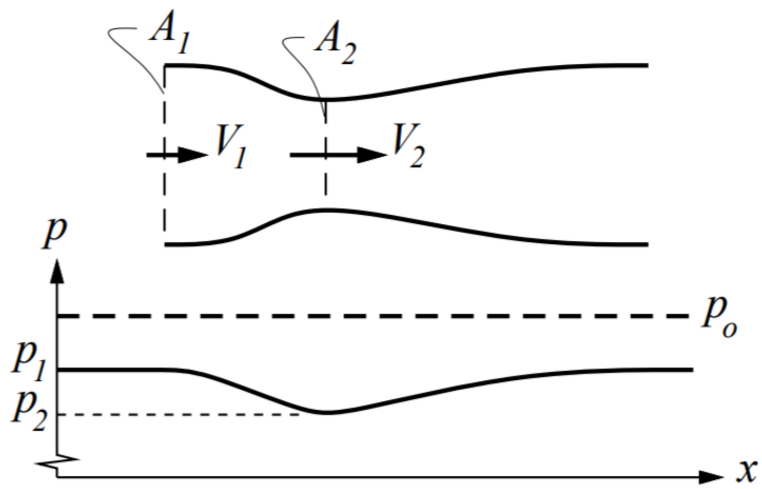
This equation can be extended to higher dimensions as well.

$$\frac{1}{2}\rho V^2 + p = constant \equiv p_0 \quad \text{(along a streamline)}$$

Because of the assumptions made while deriving the *Bernoulli's equation*, it can only relate p and V along any given streamline. Different streamlines can have different constants p_0 and hence cannot be related by this equation.

However, if the field is irrotational, i.e. $\vec{\xi} = 0$ and $\vec{V} = \nabla\phi$, then p_0 takes the same values for all streamlines. This is quite a common occurence in aerodynamics, and it helps relate the velocity and pressure between any two points on the flow field.

11 Venturi Meter



A common use of *Bernoulli's equation* is the *venturi*, which is a device used to find the mass flow rate of a fluid flow. It is essentially a tube with a varying cross-section, with the cross-section area attaining a minimum at some point.

Assuming incompressible flows, the continuity equations gives us

$$A_1 V_1 = A_2 V_2$$

Now, applying *Bernoulli's equations* along points 1 and 2, we get

$$p_1 + \frac{1}{2}\rho V_1^2 = p_0 = p_2 + \frac{1}{2}\rho V_2^2$$

$$V_1 = \sqrt{\frac{2(p_1 - p_2)}{\rho(A_1/A_2)^2 - 1}}$$

12 Updated Plan of Action

- Week 5 (June 18 - June 24) : Endsem break
- Week 6 (June 25 - July 1) : Airfoils and Incompressible Flows
- Week 7 (July 2 - July 8) : Introduction to Compressible Flows
- Week 8 (July 9 - July 15) : Sonic and Supersonic Flows