Unit V
Partial Differential Equations

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5.1 Introduction, Formation of Partial Differential Equations

Partial differential equations are those which involve partial derivatives with respect to two or more independent variables.

For example,

$\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ Two dimensional Laplace equation

$\dfrac{\partial^2 u}{\partial t^2}=c^2\dfrac{\partial^2 u}{\partial x^2}$ One dimensional wave equation

$\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0$ Three dimensional Laplace equation

are partial differential equations.

The order of partial differential equation is the order of the highest derivatives in the equation.

We shall use the following notations:

$$\frac{\partial z}{\partial x}=p,\frac{\partial z}{\partial y}=q,\frac{\partial^2 z}{\partial x^2}=r,\frac{\partial^2 z}{\partial x\partial y}=s,\ \frac{\partial^2 z}{\partial y^2}=t$$

Formation of Partial Differential Equations

Partial differential equations can be formed either by the elimination of

(1) arbitrary-constants present in the functional relation between variables or

(2) arbitrary functions of these variables.

(1) By elimination of arbitrary constants

Consider the function $f(x,y,z,a,b)=0$ (i)

Where a and b are two independent arbitrary constants. To eliminate two constants, we require two more equations. Differentiating equation (i) partially with respect to x and y in turn, we obtain

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial x}=0 \quad or \quad \frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}=0 \qquad \text{(ii)}$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial y}=0 \quad or \quad \frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}=0 \qquad \text{(iii)}$$

Eliminating a and b from the set of equations (i) to (iii) , we get a partial differential equation of the first order of the form

$$F(x,y,z,p,q)=0 \qquad \text{(iv)}$$

(2) By elimination of arbitrary functions

Consider a relation between x, y and z of the type

$$\phi(u,v)=0 \qquad \text{(v)}$$

Where u and v are known functions of x, y and z and ϕ is an arbitrary function of u and v.

Differentiating equation (v) partially with respect to x and y, respectively, we get the equations

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0 \qquad \text{(vi)}$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0 \qquad \text{(vii)}$$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ from equations (vi) and (vii), we get the equation

$$\frac{\partial(u,v)}{\partial(y,z)}p+\frac{\partial(u,v)}{\partial(z,x)}q=\frac{\partial(u,v)}{\partial(x,y)} \qquad \text{(viii)}$$

which is a partial differential equation of the type (iv). Since, the power of p and q are both unity it is also linear equation, whereas equation (iv) need not be linear.

5.2 Classification of second order Partial Differential Equations

General form of second order partial differential equation in variables x, y is

$$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu+G=0 \qquad (1)$$

Where A,B,C,D,E,F,G are constants or functions of $x \& y$.

Principal part of equation (1) is given by

Principal part = $Au_{xx}+Bu_{xy}+Cu_{yy}$

If $B^2-4AC>0$ equation (1) is hyperbolic

$B^2-4AC=0$ equation (1) is parabolic

$B^2-4AC<0$ equation (1) is elliptic

5.3 Integrals of Partial Differential Equations, Solutions of Partial Differential Equations by the Method of Direct Integration

A solution or integral of a partial differential equation is a relation between the dependent and the independent variables that satisfies the differential equation. In general, the totality of solutions of a partial differential equation is very large.

e.g. the functions $u = x^3 - 3xy^2, u = e^x \sin y, u = \log\left(x^2 + y^2\right)$

which are entirely different from each other, are solutions of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Complete Integral

A solution which contains a number of arbitrary constants equal to the independent variables, is called a complete integral.

Particular Integral

A solution obtained by giving particular values to the arbitrary constants in a complete integral is called a particular integral.

Singular Integral

Let $F(x, y, z, p, q) = 0$ (i)

Be the partial differential equation whose complete integral is

$f(x, y, z, a, b) = 0$ (ii)

Eliminating a, b between (ii) and $\frac{\partial f}{\partial a} = 0, \ \frac{\partial f}{\partial b} = 0$

If it exists, is called a singular integral.

General Integral

In (ii),if we assume $b = \phi(a)$, then (ii) becomes

$f\left[x, y, z, a, \phi(a)\right] = 0$ (iii)

Differentiating (ii), partially with respect to a,

$\frac{\partial f}{\partial a} + \frac{\partial f}{\partial b}\phi'(a) = 0$ (iv)

Eliminating a between these two equations (iii) and (iv), if it exists, is called the general integral of (i).

Solutions of Partial Differential Equations by the Method of Direct Integration

This method is applicable to those problems, where direct integration is possible. The solutions of which depend only on the definition of the partial differentiation.

5.4 Separation of variables method to simple problems in Cartesian coordinates, Initial & boundary value problems and solutions by separation of variables

A powerful method of finding solution of second order linear partial differential equation in certain cases is known as the method of separation of variables product method.

In this method, we assume the solution to be the product of two functions, each of which involves only one of the variables.

Class work problems

5.1 Introduction, Formation of Partial Differential Equations

Verify that the following functions are solutions of the Laplace's equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

1. $u=e^x \cos y$
2. $u=\log\left(x^2+y^2\right)$
3. $u=\sin x \sinh y$
4. $u=\tan^{-1}\left(\frac{y}{x}\right)$
5. $u=x^4-6x^2y^2+y^4$

Form the partial differential equations by eliminating the arbitrary constants from the following:

1. $z=(x+a)(y+b)$

   [**Ans.** $z=pq$]
2. $x^2+y^2+(z-c)^2=a^2$

   [**Ans.** $xq=yp$]

Form the partial differential equations by eliminating the arbitrary functions from the following:

1. $z=xy+f\left(x^2+y^2\right)$

   [**Ans.** $qx-py=x^2-y^2$]
2. $f\left(x+y+z, x^2+y^2+z^2\right)=0$

   [**Ans.** $(y-z)p+(z-x)q=x-y$]

5.2 Classification of second order Partial Differential Equations

Classify the following equations in to hyperbolic, elliptic or parabolic:

1. $8u_{xx}+u_{yy}-u_x+\left[\log\left(2+x^2\right)\right]u=0$

   [**Ans.**PDE is elliptic]

2. $\sin^2 x\, u_{xx} + \sin 2x\, u_{xy} + \cos^2 x\, u_{yy} = x$

   [**Ans.**PDE is parabolic]

3. $8u_{xx} + 9u_{xy} + 2u_{yy} + 2u = 0$

   [**Ans.**PDE is hyperbolic]

5.3 Solutions of Partial Differential Equations by the Method of Direct Integration

1. Solve $\dfrac{\partial^2 z}{\partial x \partial y} = x^3 + y^3$

   [**Ans.** $z = \dfrac{x^4 y}{4} + \dfrac{xy^4}{4} + f(y) + g(x)$]

2. Solve $\dfrac{\partial^2 z}{\partial x^2} = \sin x$

   [**Ans.** $z = -\sin x + x f(y) + g(y)$]

3. Solve $\dfrac{\partial^2 z}{\partial x^2} = a^2 z$ given that when $x = 0,\ \dfrac{\partial z}{\partial x} = a \sin y \ \ and \ \ \dfrac{\partial z}{\partial y} = 0$

   [**Ans.** $z = \sin y \sinh ax + A(x) \cosh ax$]

5.4 Separation of variables method to simple problems in Cartesian coordinates, Initial & boundary value problems and solutions by separation of variables

1. Using the method of separation of variables solve $\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial t} + u, \ \ given\ u(x,0) = 6e^{-3x}$

   [**Ans.** $u = 6e^{-(3x+2t)}$]

2. Solve by the method of separation of variables $4\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 3u, \ \ given\ u(x,0) = 4e^{2x}$

   [**Ans.** $u = 4e^{2x-5y}$]

3. Using the method of separation of variables solve

   $\dfrac{\partial^2 u}{\partial x^2} = h^2 \dfrac{\partial u}{\partial t}, \ \ given\ u(0,t) = u(l,t) = 0\ and\ u(x,0) = \sin\left(\dfrac{\pi x}{l}\right)$

   [**Ans.** $u = \sin\left(\dfrac{\pi x}{l}\right) e^{-\frac{\pi^2 t}{h^2 l^2}}$]