Applications of the distribution laws of statistical mechanics

In the following sections we shall discuss some applications of different types of statistical distributions, viz. *MB*, *FD* and *BE* statistics.

Equipartition of energy

The total number of independent quantities required to specify the configuration and position of a system is called the number of *degrees of freedom* of the system. The number is determined by the possibilities of motion of the parts of the system. In the case of a gas, if we regard it as a point, its position is specified by the coordinates x, y and z. thus each gas particle will have three degrees of freedom. If there are N number of gas particles in the system, the number of degrees of freedom of the system will be $3N$. the *law of equipartition of energy* states that the thermal equilibrium energy of the gas particles is uniformly distributed among the various degrees of freedom and for each of them it is $\frac{1}{2}kT$. We shall establish this law here using the MB statistics.

The kinetic energies of a particle associated with its motion along the x, y and z directions are $p_x^2/2m$, $p_y^2/2m$ and $p_z^2/2m$, respectively. Here p_x, p_y and p_z are the x, y and z components of the momentum p. The average energy of the particle for its motion along any of the three directions is

$$\overline{E_j} = \frac{\iiint\left(\frac{p_j^2}{2m}\right)f(E)dp_x dp_y dp_z}{\iiint f(E)dp_x dp_y dp_z} \qquad (j \equiv x, y, z)$$

Where $dp_x\, dp_y\, dp_z$ is a small volume element in the momentum space and the integrals extend over the entire momentum space.

Using the Maxwell-Boltzmann form of the distribution function $f(E)$ and noting that

$E = (p_x^2 + p_y^2 + p_z^2)/(2m)$, we obtain from the above equation

$$\overline{E_x} = \frac{\int_{-\infty}^{\infty}(\frac{p_x^2}{2m})e^{-\frac{p_x^2}{2mkT}}dp_x \int_{-\infty}^{\infty} e^{-\frac{p_y^2}{2mkT}}dp_y \int_{-\infty}^{\infty} e^{-\frac{p_z^2}{2mkT}}dp_z}{\int_{-\infty}^{\infty} e^{-\frac{p_x^2}{2mkT}}dp_x \int_{-\infty}^{\infty} e^{-\frac{p_y^2}{2mkT}}dp_y \int_{-\infty}^{\infty} e^{-\frac{p_z^2}{2mkT}}dp_z}$$

$$= \frac{\int_{-\infty}^{\infty}(\frac{p_x^2}{2m})e^{-\frac{p_x^2}{2mkT}}dp_x}{\int_{-\infty}^{\infty} e^{-\frac{p_x^2}{2mkT}}dp_x} = \frac{\int_0^{\infty}(\frac{p_x^2}{2m})e^{-\frac{p_x^2}{2mkT}}dp_x}{\int_0^{\infty} e^{-\frac{p_x^2}{2mkT}}dp_x}$$

Substituting $x = p_x^2/2mkT$ we obtain

$$\overline{E_x} = kT\frac{\int_0^\infty x^{1/2}e^{-x}dx}{\int_0^\infty x^{-1/2}e^{-x}dx} = kT\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})} = \frac{1}{2}kT$$

since $\Gamma(3/2) = (1/2)\Gamma(1/2)$.

Thus the mean kinetic energy for the motion along the x-direction is $\frac{1}{2}kT$. The same value of $\frac{1}{2}kT$ is also obtained for $\overline{E_y}$ and $\overline{E_z}$. Thus the law of equipartition of energy is seen to be satisfied. The average energy per particle considering the three degrees of freedom is $\frac{3}{2}kT$.

Black body radiation and the Planck radiation law (Statistics of photon gas)

One of the most important applications of Bose statistics is its application to electromagnetic radiation in thermal equilibrium, called the '*black body radiation*'. Black body is assumed to be a cavity whose walls are constantly emitting and absorbing radiation. Planck suggested that the electromagnetic waves emitted from a black body have its energy quantized in units of $h\nu$, where ν is the frequency of the wave and h is the Planck's constant. However, Planck was not very convinced of the physical basis of his derivation. S.N. Bose derived Planck's law of radiation on the basis of quantum statistics. Einstein extended his ideas to the case of material particles obeying Bose statistics. Here, we discuss Bose's derivation of Planck's law.

Let a black body cavity of volume V at temperature T K is filled with large number of indistinguishable photons of various frequencies (photon gas). Photons have zero rest mass and unit spin angular momentum (thus they are Bosons). Photons of same frequency are indistinguishable and behave as a system of non-interacting particles.

To count the photons, we have to estimate the number of cells in the phase space (constructed by positions and momentum of all photons). If each cell has the infinitesimal volume h^3, then the number of cells in the phase space where the momentum lies between p and $p + dp$ is

$$g(p)dp = \frac{\int dxdydzdp_xdp_ydp_z}{h^3}$$

Where,

$\int_v dxdydz = V =$ volume of the cavity, and

$\int_v dp_xdp_ydp_z = 4\pi p^2 dp =$ volume of the spherical cell of radius p and thickness dp in the momentum space.

Therefore,

$$g(p)dp = \frac{4\pi V p^2 dp}{h^3}$$

The photon with a frequency ν has energy $E = h\nu$ and its momentum $p = h\nu/c$. So $dp = (h/c)d\nu$.

Thus, substituting the values of p and $p + dp$, we obtain the no of cells (quantum states) with frequency between ν and $\nu + d\nu$ as

$$g(\nu)d\nu = \frac{4\pi V}{c^3}\nu^2 d\nu$$

Since photon can have two types of polarization and the fact that each cell can be occupied by two photons of two different directions of polarizations, we multiply the above expression by a factor 2 to obtain

$$g(\nu)d\nu = \frac{8\pi V}{c^3}\nu^2 d\nu$$

The number of photons with energy between E and E + dE is

$$N(E)dE = \frac{g(E)}{e^{\alpha}e^{E/kT}-1}dE$$

Here $g(E)dE$ is the number of quantum states of photons of energy between E and $E + dE$.

In the black body chamber at constant temperature photons of different energies are constantly absorbed or re-emitted by the walls of the enclosure, i.e. the number of photons in the system is not constant. Therefore, $\sum_i \delta N_i \neq 0$. This can be taken into consideration by setting the Lagrange multiplier $\alpha = 0$. Thus, for photons of a black body we have (setting $\alpha = 0$)

$$N(E)dE = \frac{g(E)}{e^{\frac{E}{kT}}-1}dE$$

Or, in terms of ν, the number of photons with frequencies between ν and $\nu + d\nu$ in the radiation within a cavity of volume V whose walls are at an absolute temperature T is

$$N(\nu)d\nu = \frac{g(\nu)}{e^{h\nu/kT}-1}d\nu = \frac{8\pi V}{c^3}\frac{\nu^2 d\nu}{e^{\frac{h\nu}{kT}}-1}$$

The corresponding spectral energy density $u(\nu)d\nu$, which is the energy per unit volume in radiation between ν and $\nu + d\nu$ in frequency is

$$\boxed{\boldsymbol{u(\nu)d\nu = \frac{h\nu N(\nu)d\nu}{\nu} = \frac{8\pi h}{c^3}\frac{\nu^3 d\nu}{e^{\frac{h\nu}{kT}}-1}}}$$

Above Eqn. is the Planck's radiation formula in terms of frequency ν, and has been successful in explaining the black body radiation.