Unit 1
Tutorial Problems

Rank of a matrix & System of linear equations

1. Reduce the following matrices to Row Echelon form and find its rank.

a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 3 & 5 & 9 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ d) $\begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ f) $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ g) $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

h) $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 2 \\ 2 & -1 & 2 & 5 \\ 5 & 6 & 3 & 2 \\ 1 & 3 & -1 & -3 \end{bmatrix}$ i) $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$ $[\text{Ans: } 3,3,2,2,2,3,4,3,2]$

2. For what values of p the matrix $\begin{bmatrix} 3 & p & p \\ p & 3 & p \\ p & p & 3 \end{bmatrix}$, has (i) rank 1, (ii) rank 2, or (iii) rank 3.

$\left[\begin{array}{l} \text{Ans: Rank is 1 if } p = 3, \text{ Rank is 2 if } p = \frac{3}{2}, \\ \text{Rank is 3 for } p \neq 3 \text{ or } \frac{3}{2} \end{array}\right]$

3. For what values of k, the matrix $A = \begin{bmatrix} 1 & k+4 & 4k+2 \\ 0 & k-2 & -k+2 \\ 1 & 2k-4 & -k+2 \end{bmatrix}$ has rank 1, rank 2 or rank 3.

$\left[\begin{array}{l} \text{Ans: Rank is 2 if } k = 2 \text{ or } -2, \\ \text{Rank is 3 for all the other values k.} \end{array}\right]$

4. Solve $2x_1 - 2x_2 + 4x_3 + 3x_4 = 9, x_1 - x_2 + 2x_3 + 2x_4 = 6, 2x_1 - 2x_2 + x_3 + 2x_4 = 3, x_1 - x_2 + x_4 = 2$

[Ans. No solution]

5. Solve $5x + 3y + 7z = 4, 3x + 26y + 2z = 9, 7x + 2y + 10z = 5.$

[Ans. $x = \frac{7-16k}{11}, y = \frac{3+k}{11}, z = k$]

6. Solve $x + y + 2z = 8, -x - 2y + 3z = 1, 3x - 7y + 4z = 10.$

[Ans. $x = 3, y = 1, z = 2$]

7. Solve $x + 2y + 3z = 0, 3x + 4y + 4z = 0, 7x + 10y + 12z = 0$.

[Ans. $x = y = z = 0$]

8. Solve $x + y - 3z + 2w = 0, 2x - y + 2z - 3w = 0, 3x - 2y + z - 4w = 0, -4x + y - 3z + w = 0$ [Ans: $x = y = z = w = 0$]

9. Solve $x + 3y + 2z = 0, 2x - y + 3z = 0, 3x - 5y + 4z = 0, x + 17y + 4z = 0$.

[Ans. $x = 11k, y = k, \ z = -7k$]

10. Determine the value of constant b such that the system of homogeneous equations $2x + y + 2z = 0, x + y + 3z = 0, 4x + 3y + bz = 0$ has i) trivial solution, ii) non-trivial solution. Find the non-trivial solution.

[Ans. i) $b \neq 8$, ii) $b = 8$; $x = k, y = -4k, z = k$]

11. Determine the values of b for which the system of equations has non-trivial solutions. Find them.
$(b-1)x + (4b-2)y + (b+3)z = 0$, $(b-1)x + (3b+1)y + 2bz = 0$,
$2x + (3b+1)y + 3(b-1)z = 0$

[Ans. i) $b = 0; x = y = z$, ii) $b = 3; x = -5k_1 - 3k_2, y = k_1, z = k_2$]

12. Investigate for what values of λ and μ the equations $x + 2y + z = 8, 2x + 2y + 2z = 13, 3x + 4y + \lambda z = \mu$ have i) no solution, ii) unique solution and iii) many solutions.

[Ans. i) $\lambda = 3$ & $\mu \neq 21$, ii) $\lambda \neq 3$ & μ has any value, iii)$\lambda = 3$ & $\mu = 21$]

13. Determine the values of λ for which the following equations are consistent. Also solve the system for these values of λ. $x + 2y + z = 3, x + y + z = \lambda, 3x + y + 3z = \lambda^2$.

[Ans. For $\lambda = 3; x = 3 - t, y = 0, z = -2t$. For $\lambda = 2; x = 1 - t, y = 1, z = t$]

Based on Vector Space, Linear independence of vectors, basis, dimension

1) Determine whether $\mathbb{R}^2$ is a vector space with indicated operations of vector addition and scalar multiplication $(x, y) + (x', y') = (x + x', 0)$ and $a(x, y) = (ax, ay)$ **Ans:** Yes
2) Determine whether V is vector space where V = set of all n × n symmetric matrices with real entries over R, with usual addition of matrices and multiplication by a scalar.
3) Let C be set of all complex numbers, C is vector space over $\mathbb{R}$ under the operations defined as
$z_1 + z_2 = (x_1 + i\,y_1) + (x_2 + i\,y_2) = (x_1 + x_2) + i\,(y_1 + y_2)$ and
for any , $\alpha z_1 = \alpha x_1 + i\,\alpha y_1$. Show that C is a vector space under these operations.
4) Show that the set of all skew symmetric matrices of order $m \times n$ is a vector subspace of $M_{m\times n}(\mathbb{R})$
5) If W_1 and W_2 are subspaces of vector space V then prove that $W_1 + W_2 = \{u + v \mid u \in W_1, v \in W_2\}$

is subspace of V .

6) Find the set of all solutions of $x+2y+z=0$ and show that it is a subspace of $\mathbb{R}^3$.

7) Express the vector (5, 9, 5) as linear combinations of (2, 1, 4), (3, 2, 5) & (1, -1, 3) in R^3

Ans: (5, 9, 5) = 3(2, 1, 4) + 1 (3, 2, 5) + (- 4) (1, -1, 3)

8) Determine whether (2, - 1, -8) is in the linear span of S = { (1, 2, 1), (1, 1, -1), (4, 5, -2) } $\subseteq R^3$

Ans: No

9) Determine whether (0, 0, 0) is in the linear span of S = { (1, 2, 1), (1, 1, -1), (4, 5, -2) } $\subseteq R^3$

Ans: Yes

10) Determine whether the set S = { (1, 1, 1), (4, 4, 0), (3, 0, 0) } spans R^3 **Ans:** Yes

11) Examine the following sets of vectors for linear dependence/independence

i) $\left\{x^3 - 4x^2 + 2x + 3,\ x^3 + 2x^2 + 4x - 1,\ 2x^3 - x^2 - 3x + 5\right\}$ in P_3 **Ans:** L.I.

ii) $\left\{\begin{bmatrix}1 & -5\\ -4 & 2\end{bmatrix}, \begin{bmatrix}1 & 1\\ -1 & 5\end{bmatrix}, \begin{bmatrix}2 & -4\\ -5 & 7\end{bmatrix}, \begin{bmatrix}1 & -7\\ -5 & 1\end{bmatrix}\right\}$ in $M_{2\times 2}(R)$ **Ans:** L.D.

12) Show that the set S = { (1, 0, 0), (1, 1, 0), (1, 1, 1) } is a basis of vector space $\mathbb{R}^3$.

13) Determine whether { (1,1,1,1), (1,2,3,2), (2,5,6,4), (2,6,8,5) } form a basis of R^4. If not find the dimension of the subspace they span.

14) Determine a basis and dimension of the solution space of following homogeneous systems

i) $x + y + z = 0$
$3x + 2y - 2z = 0$
$4x + 3y - z = 0$
$6x + 5y + z = 0$

Ans:
B = { (4,- 5,1) }
dim = 1

ii) $x_1 + 2x_2 - 3x_3 + x_4 = 0$
$-x_1 - x_2 + 4x_3 - x_4 = 0$
$-2x_1 - 4x_2 + 7x_3 - 2x_4 = 0$

Ans:
B = { (1,0,0,-1) }
Dim=1

15) Examine whether the following sets of vectors form a basis for the indicated vector space

i) $\left\{x^3 - 2x^2 + 4x + 1,\ x^2 + 6x - 5,\ 2x^3 - 3x^2 + 9x - 1,\ 2x^3 - 5x^2 + 7x + 5\right\}$ in P_3 **Ans :** No

ii) $\left\{\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}, \begin{bmatrix}6 & 0\\ -1 & 4\end{bmatrix}, \begin{bmatrix}3 & 0\\ 1 & 7\end{bmatrix}, \begin{bmatrix}5 & 1\\ 4 & 2\end{bmatrix}\right\}$ in $M_{2\times 2}(R)$

16) Obtain a basis and dimension of the subspace generated by

$$\left\{\begin{bmatrix}1 & -5\\ -4 & 2\end{bmatrix}, \begin{bmatrix}1 & 1\\ -1 & 6\end{bmatrix}, \begin{bmatrix}2 & -4\\ -5 & 7\end{bmatrix}, \begin{bmatrix}1 & -7\\ -5 & 1\end{bmatrix}\right\} \text{ in } M_{2\times 2}(R)$$