Unit 4, Statistical Physics, Lecture 5

Bose-Einstein Statistics

The basic postulates of BE statistics are

1. The associated particles are identical and indistinguishable.
2. Each energy state can contain any number of particles.
3. Total energy and total number of particles of the entire system is constant.
4. The particles have zero or integral spin.
5. The wave function of the system is symmetric under the positional exchange of any two particles. Such particles are known as Bosons. For example, photons, phonons, all mesons (π,κ,η) etc.

[Symmetric and Anti-symmetric wave function: Let the allowed wave function for a *n*-particles system is $\psi(1,2,3,\ldots\ldots,r,s,\ldots n)$, where the integers within the argument of ψ represent the coordinates of the *n*-particles relative to some fixed origin. Now, if we interchange the positions of any two particles, say, *r* and *s*, the resulting wave function becomes $\psi(1,2,3,\ldots.s,r,\ldots\ldots n)$. The wave function ψ is said to be symmetric when $\psi(1,2,3,\ldots\ldots,r,s,\ldots n) = \psi(1,2,3,\ldots.s,r,\ldots\ldots n)$ and anti-symmetric when $\psi(1,2,3,\ldots\ldots,r,s,\ldots n) = -\ \psi(1,2,3,\ldots.s,r,\ldots\ldots n)$.

In B.E. statistics all the particles are indistinguishable. Also, the quantum states are assumed to have equal *a priori* probability. Thus g_i represents the number of quantum states with same energy E_i (g_i is degeneracy). Each quantum state corresponds to a cell in phase space. We shall determine the number of ways in which n_i indistinguishable particles can be distributed in g_i cells.

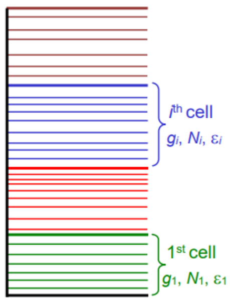

Fig. 9. Energy levels of a system bracketed into cells

Let the g_i number of cells be numbered as $1,2,3,\ \ldots.,\ gi$. Each cell contains 1 or 2 or 3 or … n_i particles at a time. Now one particle can be put in any one cell in g_i ways (i.e. one particle can be put in 1st cell or 2nd cell or g_i-th cell). Two particles can be put in two ways - (i) two particles can be put in a single cell in g_i ways, i.e. we choose one cell out of g_i cells in g_i ways, (ii) each of the two particles can be put in two separate cells, i.e. we choose 2 cells out of g_i cells in ${g_i}_{C_2}$ ways, i.e. $\frac{g_i(g_i-1)}{2}$ ways. Thus two particles can be put in

$$g_i + \frac{g_i(g_i-1)}{2} = \frac{g_i(g_i+1)}{2} = \frac{(g_i+1)!}{(g_i-1)!2!} \quad \text{ways.}$$

Now, three particles can be put in three distinct ways – (i) three particles can be put in a single cell. This can be done in g_i ways, (ii) two particles in one cell and one particle in another cell.

This can be done in $g_i(g_i - 1)$ ways, (iii) three particles can be put in three cells. This can be done in $g_{i_{C_3}}$ ways, i.e. $g_{i_{C_3}} = \frac{g_i!}{3!(g_i-3)!} = \frac{g_i(g_i-1)(g_i-2)}{3!}$ ways. Therefore, the total number of ways will be

$$g_i + g_i(g_i - 1) + \frac{g_i(g_i-1)(g_i-2)}{3!} = \frac{g_i(g_i+1)(g_i+2)}{3!} = \frac{(g_i+2)!}{(g_i-1)!3!}$$

Arguing in this way, the number of ways n_i particles can be put in g_i cells is

$$\frac{(g_i+n_i-1)!}{(g_i-1)!n_i!}$$

If there are

n_1 particles in the energy level E_1 with degeneracy g_1

n_2 particles in the energy level E_2 with degeneracy g_2

…………………………………………………………

n_i particles in the energy level E_i with degeneracy g_i

Hence, all these groups of particles can be distributed in, i.e. the thermodynamic probability

$$W = \prod_i \frac{(g_i+n_i-1)!}{(g_i-1)!n_i!} \tag{43}$$

Now we assume, $(n_i + g_i) >> 1$, so that $(g_i + n_i - 1)! \approx (g_i + n_i)!$

Therefore, Eqn. 43 becomes

$$W = \prod_i \frac{(g_i+n_i)!}{(g_i-1)!n_i!} \tag{44}$$

Taking natural logarithm on both sides we get

$$lnW = \sum_i [\ln (g_i + n_i)! - ln((g_i - 1)! - lnn_i!]$$

$$= \sum_i [(g_i + n_i)ln(g_i + n_i) - (g_i + n_i) - ln(g_i - 1)! - (n_i lnn_i - n_i)]$$

[Here we have used the Sterling's formula $ln\ n! = n\ ln\ n - n$]

$$= \sum_i [(n_i + g_i)ln(n_i + g_i) - ln(g_i - 1)! - n_i lnn_i - g_i] \tag{45}$$

For most probable distribution a small variation δn_i in any n_i does not affect the value of W. For a change δn_i in n_i, $\delta ln\ W_{max} = 0$. Therefore,

$$\delta lnW_{max} = \sum_i [\delta((n_i + g_i)ln(n_i + g_i)) - \delta ln(g_i - 1)! - \delta(n_i lnn_i) - \delta g_i] = 0$$

or $$\sum_i [\delta n_i ln(n_i + g_i) + (n_i + g_i)\frac{1}{(n_i+g_i)}\delta n_i - \delta n_i lnn_i - n_i \frac{1}{n_i}\delta n_i] = 0$$

[Since $\delta g_i = 0$]

or $$\sum_i [ln(n_i + g_i) - lnn_i]\delta n_i = 0 \tag{46}$$

We incorporate the conservation of particles as

$$\delta \sum_i n_i = \delta N = 0 \text{ (N being the total number of particles), i.e.}$$

$$\sum_i \delta n_i = 0 \qquad \text{(i)}$$

and the conservation of energy expressed as

$$\sum_i E_i \delta n_i = 0 \qquad \text{(ii)}$$

[Note: $\sum_i n_i E_i = E\ (total\ energy)$ or $\delta \sum_i n_i E_i = \delta E = 0$, we change n_i not in E_i]

Multiplying *(i)* by $-\alpha$ and (ii) by $-\beta$ and adding to *Eqn. 46*, we get

$$\sum_i [ln(n_i + g_i) - ln n_i - \alpha - \beta E_i]\delta n_i = 0$$

Since the δn_i's are independent, the quantity in bracket of the above equation must vanish for each *i*. Hence

$$ln(n_i + g_i) - ln n_i - \alpha - \beta E_i = 0$$

or $$ln \frac{n_i + g_i}{n_i} = \alpha + \beta E_i$$

or $$\frac{n_i + g_i}{n_i} = e^{\alpha} e^{\beta E_i}$$

or $$1 + \frac{g_i}{n_i} = e^{\alpha} e^{\beta E_i}$$

or $$\boldsymbol{n_i = \frac{g_i}{e^{\alpha} e^{\beta E_i} - 1}} \qquad (47)$$

This is the general form of **Bose-Einstein (*BE*) distribution law**.

From *Eqn. 31*, we have $\beta = 1/kT$. Therefore, *Eq. 47* can be rewritten as

$$\boldsymbol{n_i = \frac{g_i}{e^{\alpha} e^{\frac{E_i}{kT}} - 1}}$$

or $$\boldsymbol{f(E_i) = \frac{n_i}{g_i} = \frac{1}{e^{\alpha} e^{\frac{E_i}{kT}} - 1}} \qquad (48)$$

$f(E_i)$ is known as **Bose-Einstein distribution function**.

Comparison of MB, FD and BE-statistics

Features	**MB**	**BE**	**FD**
Particle	The particle of the system in equilibrium are distunguishable and Pauli's exclusion principle doesn't apply.	The particle of the system in equilibrium are indistunguishable and Pauli's exclusion principle is not obeyed.	The particle of the system in equilibrium are indistunguishable and Pauli's exclusion principle is obeyed.
Particle Spin	Spinless.	0,1,2, …	1/2, 3/2, 5/2, ….
Wave function	-	Symmetric under interchange of the coordinates of any two bosons.	Antisymmetric under interchange of the coordinates of any two fermions.
No. of particles per energy state	No upper limit.	No upper limit as Pauli's exclusion principle is not obeyed.	Maximum one fermion per quantum state is allowed as Pauli's exclusion principle is obeyed.
Distribution function	$f(E_i) = \frac{n_i}{g_i} = e^{-\alpha} e^{-\frac{E_i}{kT}}$	$f(E_i) = \frac{n_i}{g_i} = \frac{1}{e^{\alpha} e^{\frac{E_i}{kT}} - 1}$	$f(E_i) = \frac{N_i}{g_i} = \frac{1}{1 + e^{\frac{E_i - E_F}{kT}}}$
Applies to	Common gases at normal temperature.	Applies to photons, phonons, particles with integral or zero spin, like π-mesons.	Applies to electron gas in metals, particles having half-integer spin like protons, neutrinos etc.

N.B.

Both *FD* and *BE*-statistics reduces to *MB*-statistics when $g_i >> N_i$, i.e. when the particle number is quite small, and when the temperature T is high. The reduction of quantum statistics to the *MB* statistics at sufficiently low concentration or sufficiently high temperature is known as the ***classical limit of quantum statistics***. A gas in the classical limit is called *nondegenerate*, whereas for concentrations and temperatures where *FD* and *BE* distribution function is valid, the gas is called *degenerate*.

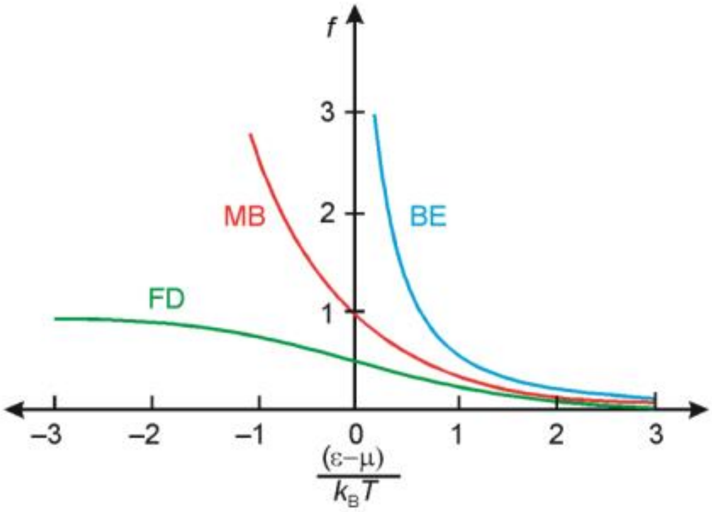

Fig. 10. Plot of MB, FD and BE distribution functions as a function of (ε-μ)/kT. Each system is at the same temperature and has the same number of particles.