Ex 9.1 (Method of Separation of Variables)

(1) $$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0 \quad \text{--- (1)}$$

Let $z = XY$ be the soln.

where $X = X(x)$
$Y = Y(y)$

$$(1) \Rightarrow X''Y - 2X'Y + XY' = 0.$$

$$\Rightarrow \frac{X''}{X} - \frac{2X'}{X} + \frac{Y'}{Y} = 0$$

$$\Rightarrow \frac{X'' - 2X'}{X} = -\frac{Y'}{Y} = K \text{ (say)}.$$

$$\frac{X'' - 2X'}{X} = K \quad \& \quad \frac{Y'}{Y} = -K.$$

$X'' - 2X' - KX = 0$

$(D^2 - 2D - K)X = 0$

$m^2 - 2m - K = 0$

$m = \dfrac{2 \pm \sqrt{4+4K}}{2}$

$m = 1 \pm \sqrt{1+K}$

$X = C_1 e^{(1+\sqrt{1+K})x} + C_2 e^{(1-\sqrt{1+K})x}$

$\frac{Y'}{Y} = -K.$

$\frac{1}{Y} dY = -K\,dy$

Int, $\log Y = -Ky + \log C_3$

$Y = C_3 e^{-Ky}.$

Soln is $z = \left(C_1 e^{(1+\sqrt{1+K})x} + C_2 e^{(1-\sqrt{1+K})x}\right) C_3 e^{-Ky}.$

$$\Rightarrow \boxed{z = \left(A e^{(1+\sqrt{1+K})x} + B e^{(1-\sqrt{1+K})x}\right) e^{-Ky}}.$$

(2) $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$ — (1), where $u(0,y) = 8e^{-3y}$ (2)

Sol^n Let $u = XY$ be the sol^n of (1)

$(1) \Rightarrow \quad X'Y = 4XY'$

$$\frac{X'}{X} = \frac{4Y'}{Y} = k \text{ (say)} \quad (k = \text{const})$$

$$\Rightarrow \quad \frac{X'}{X} = k \quad , \quad \frac{4Y'}{Y} = k.$$

$$\frac{1}{X}dX = k\,dx \quad , \quad \frac{1}{Y}dY = \frac{k}{4}dy$$

$$\log X = kx + \log c_1 \qquad \log Y = \frac{k}{4}y + \log c_2$$

$$X = c_1 e^{kx}. \qquad Y = c_2 e^{ky/4}.$$

$$\Rightarrow \quad u = c_1 e^{kx} c_2 e^{ky/4}.$$

$$\Rightarrow \quad u = c e^{kx} \cdot e^{ky/4} \qquad \text{where } c_1c_2 = c$$

using $u(0,y) = 8e^{-3y}$.

$$\Rightarrow \quad 8e^{-3y} = c e^{ky/4}.$$

$$c = 8, \quad \frac{k}{4} = -3 \quad k = -12$$

$$\Rightarrow \quad \boxed{u = 8e^{-12x} e^{-3y}}.$$

3) $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, where $u(x,0) = 4e^{-x}$

Soln: let $u = XY$ be the solution

$3X'Y + 2XY' = 0.$

$\Rightarrow 3\frac{X'}{X} + \frac{2Y'}{Y} = 0$

$\Rightarrow 3\frac{X'}{X} = -\frac{2Y'}{Y} = K$ (say)

$\frac{X'}{X} = \frac{K}{3}$

$\frac{1}{X}dX = \frac{K}{3}.dx.$

$\log X = \frac{K}{3}x + \log c_1$

$X = c_1 e^{Kx/3}$

$-\frac{2Y'}{Y} = K.$

$\frac{Y'}{Y} = -K/2$

$\frac{1}{Y}dY = -\frac{K}{2}dy$

$\log Y = -\frac{K}{2}y + \log c_2$

$Y = c_2 e^{-Ky/2}$

$\Rightarrow u = c_1c_2 e^{Kx/3} e^{-Ky/2}$

$\Rightarrow u = c e^{Kx/3} e^{-Ky/2}$ — (2) where $c_1c_2 = c$

Using condn, $u(x,0) = 4e^{-x}$

$4e^{-x} = c e^{Kx/3}$

$\Rightarrow c = 4, \quad \frac{K}{3} = -1 \quad \boxed{K = -3}$

(2) $\Rightarrow \boxed{u = 4e^{-x}e^{-\frac{3}{2}y}}$

4 $2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + 4u = 0$ ——— 1

Subject to condition

$u(x,0) = 3e^{-x}(2 + e^{x} - e^{2x})$

Sol^n Let $u = XT$ be the sol^n of 1

$2X'T + XT' + 4XT = 0.$

$\frac{2X'}{X} + \frac{T'}{T} + 4 = 0.$

$\Rightarrow \frac{2X'}{X} + 4 = -\frac{T'}{T} = k$ (say)

$\frac{2X'}{X} + 4 = k$ & $\frac{T'}{T} = -k.$

$\frac{2X'}{X} = k - 4.$

$\frac{1}{X}dX = \left(\frac{k-4}{2}\right)dx$

Integ,

$\log X = \left(\frac{k-4}{2}\right)x + \log c_1$

$\boxed{X = c_1 e^{\left(\frac{k-4}{2}\right)x}}$

$\frac{1}{T}dT = -k\,dt.$

Integ,

$\log T = -kt + \log c_2$

$\boxed{T = c_2 e^{-kt}.}$

Sol^n is $U = c_1 e^{\left(\frac{k-4}{2}\right)x} c_2 e^{-kt}$

$U = C.e^{\left(\frac{k-4}{2}\right)x} e^{-kt}$ ——— 2 $\quad C = c_1 c_2$

Using $u(x,0) = 3e^{-x}(2 + e^{x} - e^{2x})$

eq^n (2) $\Rightarrow$

$$u = c_1 e^{\left(\frac{k_1-4}{2}\right)x} e^{-k_1 t} + c_2 e^{\left(\frac{k_2-4}{2}\right)x} e^{-k_2 t} + c_3 e^{\left(\frac{k_3-4}{2}\right)x} e^{-k_3 t}$$

using condition.

$$3e^{-x}(2+e^{x}-e^{2x}) = c_1 e^{\left(\frac{k_1-4}{2}\right)x} + c_2 e^{\left(\frac{k_2-4}{2}\right)x} + c_3 e^{\left(\frac{k_3-4}{2}\right)x}$$

$$\Rightarrow 6e^{-x} + 3 - 3e^{x} = c_1 e^{\left(\frac{k_1-4}{2}\right)x} + c_2 e^{\left(\frac{k_2-4}{2}\right)x} + c_3 e^{\left(\frac{k_3-4}{2}\right)x}$$

$$c_1 = 6, \quad \frac{k_1-4}{2} = -1 \Rightarrow k_1 = 2$$

$$c_2 = 3, \quad \frac{k_2-4}{2} = 0. \Rightarrow k_2 = 4.$$

$$c_3 = -3, \quad \frac{k_3-4}{2} = 1. \Rightarrow k_3 = 6.$$

Sol^n becomes $\longrightarrow$

$$u = 6e^{-x}e^{-2t} + 3e^{-4t} - 3e^{x}e^{-6t}.$$

(5) $u_x = 4u_t + u$, $u(0,t) = 7e^{3t}$.

Let, $u = XT$ be the solution.

$$X'T = 4XT' + XT.$$

$$\frac{X'}{X} = \frac{4T'}{T} + 1 = k \text{ (say)}$$

(6)

$\frac{X'}{X} = K$

$\frac{1}{X} dx = K dx$

$\log X = Kx + \log C_1$

$X = C_1 e^{Kx}$

$\frac{4T'}{T} = K-1$

$\frac{1}{T} dT = \left(\frac{K-1}{4}\right) dt$

$\log T = \left(\frac{K-1}{4}\right) t + \log C_2$

$T = C_2 e^{\left(\frac{K-1}{4}\right) t}$

$U = C_1 . C_2 e^{Kx} e^{\left(\frac{K-1}{4}\right) t}$

$U = A e^{Kx} e^{\left(\frac{K-1}{4}\right) t}$

using $u(0,t) = 7e^{3t}$

$7e^{3t} = A e^{\left(\frac{K-1}{4}\right) t}$

$\boxed{A = 7}$ $\quad \frac{K-1}{4} = 3$ $\quad \boxed{K = 13}$

$\boxed{U = 7 e^{13x} e^{3t}}$

(b) $\frac{\partial u}{\partial x} + 5\frac{\partial u}{\partial t} + u = 0$ with the condn

$u(x,0) = 3e^{-4x} + 2$

Soln: Let $u = XT$

$X'T + 5XT' + XT = 0$

$\frac{X'}{X} + \frac{5T'}{T} + 1 = 0$

$\frac{X'}{X} + 1 = -\frac{5T'}{T} = K$ (say)

$\frac{X'}{X} = k-1$

$\frac{1}{X} dX = (k-1)dx$

integ, $\log X = (k-1)x + \log c_1$

$X = c_1 e^{(k-1)x}$

$\frac{-5T'}{T} = k$

$\frac{T'}{T} = -k/5$

$\frac{1}{T} dT = \left(\frac{-k}{5}\right) dt$

$\log T = -\frac{kt}{5} + \log c_2$

$T = c_2 e^{-kt/5}$

$u = c_1 c_2 e^{(k-1)x} e^{-kt/5}$

$u = A e^{(k-1)x} e^{-kt/5}$

For using cond^n, $u(x,0) = 3e^{-4x} + 2$

let $u = A_1 e^{(k_1-1)x} e^{-k_1 t/5} + A_2 e^{(k_2-1)x} e^{-k_2 t/5}$

by condition $\longrightarrow$

$3e^{-4x} + 2e^{0x} = A_1 e^{(k_1-1)x} + A_2 e^{(k_2-1)x}$

$A_1 = 3, \quad k_1 - 1 = -4 \Rightarrow \boxed{k_1 = -3}$

$A_2 = 2, \quad (k_2 - 1) = 0 \Rightarrow \boxed{k_2 = 1}$

Solution is

$$\boxed{u = 3e^{-4x} e^{3t/5} + 2e^{-t/5}}$$

(7) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u$ given,

$u(0,y) = 0$

$\frac{\partial u}{\partial x}(0,y) = 1 + e^{-3y}$

Soln $\rightarrow$ let $U = XY$ be the soln

$X''Y = XY' + 2XY.$

$\frac{X''}{X} = \frac{Y'}{Y} + 2 = k.$

$X'' - kX = 0$

$(D^2 - k)X = 0$

$m = \pm\sqrt{k}$

$X = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}$

$\frac{Y'}{Y} = k - 2.$

$\frac{1}{Y} dY = (k-2) \cdot dy.$

$\log Y = (k-2)y + \log C_3$

$Y = C_3 e^{(k-2)y}$

$U = (C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}) C_3 e^{(k-2)y}.$

$U = (A e^{\sqrt{k}x} + B e^{-\sqrt{k}x}) e^{(k-2)y}$ —— (1)

using, $U(0,y) = 0.$

(1) $\Rightarrow 0 = (A + B) e^{(k-2)y}.$

$\Rightarrow \boxed{A = -B}$

$U = A(e^{\sqrt{k}x} - e^{-\sqrt{k}x}) . e^{(k-2)y}.$ —— (2)

using $\frac{\partial u}{\partial x}(0,y) = 1 + e^{-3y}.$

(1)

(2) $\Rightarrow$

$$u = A_1 (2\sinh\sqrt{K_1}x) \cdot e^{(K_1-2)y} + A_2 (2\sinh\sqrt{K_2}x) e^{(K_2-2)y}$$

$$\frac{\partial u}{\partial x} = A_1 (2\cosh\sqrt{K_1}x)(\sqrt{K_1}) e^{(K_1-2)y} + 2A_2\sqrt{K_2}\cosh\sqrt{K_2}x \cdot e^{(K_2-2)y}$$

$$1 + e^{-3y} = 2A_1\sqrt{K_1}\, e^{(K_1-2)y} + 2A_2\sqrt{K_2}\, e^{(K_2-2)y}$$

$2A_1\sqrt{K_1} = 1$, $K_1 - 2 = 0 \Rightarrow$ $\boxed{K_1 = 2}$ $\boxed{A_1 = 1/2\sqrt{2}}$

$2A_2\sqrt{K_2} = 1$, $K_2 - 2 = -3.$ $\boxed{K_2 = -1}$, $\boxed{A_2 = 1/2i}$

$$\Rightarrow u = \frac{1}{2\sqrt{2}}\, 2\sinh\sqrt{2}x + \frac{\cancel{2}}{\cancel{2}i}\sinh ix\, e^{-3y}$$

$\boxed{\sinh i\theta = i\sin\theta}$

$$= \frac{1}{\sqrt{2}}\sinh\sqrt{2}x + \frac{\cancel{i}\sin x}{\cancel{i}} \cdot e^{-3y}$$

$$\boxed{u = \frac{\sinh\sqrt{2}x}{\sqrt{2}} + \sin x\, e^{-3y}}$$