Unit III
First Order Ordinary Differential Equations

Overview:

Differential Equations is an important tool in Mathematics that has many real life applications in Engineering and Technology. If a Differential Equation consists of only ordinary differentiation, then it is termed as Ordinary Differential Equation. In this unit, some standard forms of differential equations and methods of solving various ordinary differential equations are discussed.

Outcome:

After completion of this unit, students would be able to:

Use effective mathematical tools for the solutions of ordinary differential equations that model physical processes.

Pre requisite: Fundamental Knowledge of Differential and Integral Calculus.

3.1 Exact, linear equations:

- **Definition:** Differential equation of the form $M(x,y)dx+N(x,y)dy=0$ is said to be exact if there exists a function of two variables $u(x,y)$ such that $du \equiv Mdx+Ndy$. The general solution of an exact equation is given by $u(x,y)=C$, where C is arbitrary constant.
- **Necessary and Sufficient Condition for an Equation to be an Exact Differential Equation**

  Differential equation $M(x,y)dx+N(x,y)dy=0$ where, M and N are the functions of x and y, will be an exact differential equation, if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

  Solution of exact differential equation is given by

  $$\int_{y\text{ constant}} Mdx+\int(\text{terms of N free from } x)dy=c$$

3.2 Equations reducible to exact equations:

Sometimes the differential equation which is not exact, can be made so on multiplication by a suitable factor is called integrating factor. The rules for finding integrating factors of the differential equation are as follows:

(i) If $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=$ function of x only say $\mathrm{p}(x)$ or constant then $e^{\int Pdx}$ is an Integrating factor.

(ii) If $\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=$ function of y only say $\mathrm{p}(y)$ or constant then $e^{\int Pdy}$ is an Integrating factor.

(iii) If the Differential Equation is in the form of $f_1(xy)\,ydx + f_2(xy)\,xdy = 0$, then then its Integrating factor is $\dfrac{1}{Mx - Ny}$

(iv) If the Differential Equation is Homogeneous Equation then its Integrating factor is $\dfrac{1}{Mx + Ny}$

3.3 Linear Differential Equation:

- **Definition:** A differential equation is said to be linear if the dependent variable and its differential coefficient offer only in the first degree.

  The standard forms of linear differential equation are as follows:

  i) $\dfrac{dy}{dx} + P(x)\,y = Q(x)$, where P, Q are funcions of x or constants. Its solution is given by $y(\text{IF}) = \int Q(x)(\text{IF})\,dx + c$ where IF is $e^{\int P(x)dx}$.

  ii) $\dfrac{dx}{dy} + P(y)\,x = Q(y)$, where P, Q are funcions of y or constants. Its solution is given by $x(\text{IF}) = \int Q(y)(\text{IF})\,dy + c$ where IF is $e^{\int P(y)dy}$.

3.4 Bernoulli's equations:

An equation of the form $\dfrac{dy}{dx} + Py = Qy^n$, --------(1)

where P, Q are funcions of x or constants is called Bernoulli's equation.

To solve Bernoulli's equation we proceed as follows:

Divide the given equation by y^n, so that we get $y^{-n}\dfrac{dy}{dx} + Py^{1-n} = Q$ ---------(2)

Put $y^{1-n} = v$ so that $(1-n)\,y^{-n}\dfrac{dy}{dx} = \dfrac{dv}{dx}$

Equation (2) becomes, $\dfrac{dv}{dx} + P(1-n)\,v = Q(1-n)$,

Which is linear in v and can be solved easily.

3.5 Orthogonal trajectories:

Given one (first) family of curves, if there exists another (second) family of curves such that each curveof the first family cuts each curve of the second family at right angles then the first family is said to be orthogonal trajectories of the second family and viceversa. In other words, each curve in either family is orthogonal (i.e. perpendicular) to every curve in the other family. In such a case the two families are said to be mutually orthogonal, and each family is said to be the orthogonal trajectories of the other family.

Examples of orthogonal trajectories are:

i. Meridians and parallels on world globe.

ii. Curves of steepest descent and contour lines on a map.

iii. Curves of electric force and equipotential lines(constant voltage).

iv. Stream lines and equipotential lines (of constant velocity potential).

v. Lines of heat flow and isothermal curves.

Method:

1. Obtain differential equation for the given family of curves, eliminating constant c, say $\frac{dy}{dx} = f(x, y)$.

2.Consider differential equation of family of orthogonal trajectories as $\frac{dy}{dx} = -\frac{1}{f(x, y)}$.

3.Solving differential equation in Step 2, we obtain family of orthogonal trajectory.

Session 1
Exact Differential equations:

1. Solve $(2x^2 + 3y^2 - 7)\,x dx + (3x^2 + 2y^2 - 8)\,y dy = 0$. **(ans:** $\frac{1}{2}\left(x^4 + 3x^2y^2 - 7x^2 + y^4 - 8y^2\right) = c$)
2. Solve $\frac{dy}{dx} = \frac{y}{2y\log y + y - x}$. (ans: $xy - y^2 \log y = c$)
3. Solve $\left[y\sin(xy) + xy^2\cos(xy)\right]dx + \left[x\sin(xy) + x^2 y\cos(xy)\right]dy = 0$. (ans: $xy\sin(xy) = c$)
4. Solve $\left\{\frac{y^2}{(y-x)^2} - \frac{1}{x}\right\}dx + \left\{\frac{1}{y} - \frac{x^2}{(x-y)^2}\right\}dy = 0$. (ans: $\frac{y^2}{(y-x)} + \log\left(\frac{y}{x}\right) = c$)

Session 2
Non Exact Differential equation reducible to Exact form:

1. Solve $\left(x^4 e^x - 2mxy^2\right)dx + 2mx^2 y\,dy = 0$. (ans: $e^x + m\left(\frac{y}{x}\right)^2 = c$)

2. Solve $\left(x^2 + y^2 + 2x\right)dx + 2ydy = 0$. (ans: $\left(x^2 + y^2\right)e^x = c$)

3. Solve $xe^x\left(dx - dy\right) + e^x\,dx + ye^y dy = 0$. (ans: $\frac{xe^x}{e^y} + \frac{y^2}{2} = c$)

4. Solve $\left(y^4+2y\right)dx+\left(xy^3+2y^4-4x\right)dy=0$. (ans: $xy+2\frac{x}{y^2}+y^2=c$)

5. Solve $y(2xy+1)dx+x(1+2xy-x^3y^3)dy=0$. (ans: $\frac{1}{(xy)^2}+\frac{1}{3(xy)^3}+\log y=c_1$)

6. Solve $\left(y\sin xy+xy^2\cos xy\right)dx+\left(x^2y\cos xy-x\sin xy\right)dy=0$. (ans: $x\left(\sin xy\right)=ye^{c_1}$)

Session 3

Non Exact Differential equation reducible to Exact form:

7. Solve $(3xy^2-y^3)dx-(2x^2y-xy^2)dy=0$. (ans: $Cx^3=y^2e^{-\frac{y}{x}}$)

8. Solve $(x^4+y^4)dx-xy^3dy=0$. (ans: $\log x-\frac{1}{4}\left(\frac{y}{x}\right)^4=c$)

Linear Differential Equations :

1. Solve $\frac{dy}{dx}+2y\tan x=\sin x \quad at \quad y=0;\ x=\frac{\pi}{3}$. (ans: $y\sec^2 x=\sec x-2$)

2. Solve $x\log x\frac{dy}{dx}+y=2\log x$. (ans: $y\log x=\left(\log x\right)^2+c$)

3. Solve $dx+xdy=e^{-y}\sec^2 ydy$. (ans: $xe^y=\tan y+c$)

Session 4

Problems onBernoulli's equations

1. Solve $x\frac{dy}{dx}+y=y^3x^{n+1}$. (ans: $\frac{1}{(xy)^2}+2\left(\frac{x^{n-1}}{n-1}\right)=c$)

2. Solve $y\frac{dx}{dy}=x-yx^2\sin y$. (ans: $-\frac{y}{x}=y\cos y-\sin y+c$)

3. Solve $\frac{dz}{dx}+\frac{z}{x}\log z=\frac{z}{x^2}\left(\log z\right)^3$. (ans: $\frac{1}{\left(x\log z\right)^2}-\frac{2}{3x^3}=c$)

4. Solve $\frac{dy}{dx}=1-x\left(y-x\right)-x^3\left(y-x\right)^2$. (ans: $\frac{1}{\left(y-x\right)}+\left(x^2+2\right)=ce^{\frac{x^2}{2}}$)

Session 5

Problems on orthogonal trajectories:

1. Find the orthogonal trajectories of each of the following family of curves.

i. $xy=c$ ans: $y^2-x^2=2c$

ii. $e^x+e^{-y}=c$ ans: $e^y-e^{-x}=c$

iii. $y^2=cx$ ans: $\frac{y^2}{2}+x^2=c$

2. Show that family of curves $x^2+4y^2=c_1$ and $y=c_2x^4$ are mutually orthogonal.

3. Find particular member of orthogonal trajectories of $x^2+cy^2=1$ passing through the point $\left(2,1\right)$. Ans: $x^2=4e^{-5}e^{x^2+y^2}$