Unit I   **Linear Equations and Vector spaces**
Detailed Syllabus:
1.1 Rank of Matrix
1.2 System of linear equations
1.3 Vector space, Subspace of vector space
1.4 Linear span, Linear independence and dependence
1.5 Basis, Dimension

Prerequisite to the topic

Students must have basic knowledge of matrices and its elementary transformations.

Determinant, Inverse and rank of a matrix

Determinants:

With every square matrix A of order n, we associate a determinant of order n which is denoted by $\det(A)$ or $|A|$. It is defined as

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{j=1}^{n} a_{ij} A_{ij}$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{i=1}^{n} a_{ij} A_{ij}$$, where M_{ij} and A_{ij} are the minors and cofactors of a_{ij} respectively.

Note: The determinant has a value and this value is real if the matrix A is real and may be real or complex, if the matrix is complex.

Properties of determinants

- If all the elements of a row (or column) are zero then the value of the determinant is zero.
- $|A| = |A^T|$.
- If any two rows (or columns) are interchanged, then the value of the determinant is multiplied by (-1).
- If the corresponding elements of two rows (or columns) are proportional to each other, then the value of the determinant is zero.
- If each element of a row (or column) is multiplied by a scalar α then the value of the

determinant is multiplied by the scalar α.

- If β is a factor of each element of a row (or a column) then this factor β can be taken out of the determinant.
- If a non-zero constant multiple of the elements of some row (or column) is added to the corresponding elements of some other row (or column), then the value of the determinant remains unchanged.
- In general, $|A+B| \neq |A|+|B|$

Minors and cofactors

The minor of the elements a_{ij} of a square matrix A is the determinant obtained from $|A|$ by deleting i^{th} row and j^{th} column.

Example: $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$. Minor of $a_1=\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, Minor of $b_1=\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$, Minor of $b_2=\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$, and so on.

The cofactor of the element a_{ij} is denoted by A_{ij} and is defined as the minor of a_{ij} with the sign given by $(-1)^{i+j}\left(\text{minor of } a_{ij}\right)$.

Cofactor of a_1 $=(-1)^{1+1}.\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, Cofactor of b_1 $=(-1)^{1+2}.\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$, Cofactor of b_3 $=(-1)^{3+2}.\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$.

Adjoint of a square matrix

Let $A=\left[a_{ij}\right]$ be a given square matrix and A_{ij} denote the cofactor of a_{ij}. The transpose of the matrix $\left[A_{ij}\right]$ is called the adjoint matrix or adjoint of the matrix A, i.e. the transpose of the matrix of the cofactors is called the adjoint of the matrix.

Inverse of a matrix

If A and B are two non-singular square matrices of the same order such that $AB = BA = I$, where I is the identity matrix, then B is called the inverse of A, i.e. $B = A^{-1}$. We say A^{-1} is the inverse of the matrix A if $AA^{-1} = A^{-1}A = I$.

<u>To find the inverse matrix with the help of adjoint matrix</u>

$$A^{-1}=\frac{adj(A)}{|A|}$$

Note:

- The necessary condition for a square matrix A to possess an inverse is that the matrix

A is non-singular i.e. $|A| \neq 0$.

- A square non-singular matrix A of order n is said to be invertible, if there exists a non-singular square matrix B of order n such that $AB = BA = I$.

Properties of inverse matrices

- If A^{-1} exists, then it is unique.
- $\left(A^{-1}\right)^{-1} = A$.
- $\left(A^{T}\right)^{-1} = \left(A^{-1}\right)^{T}$
- Let $D = diag(d_{11}, d_{22}, d_{33}, d_{44}, ...d_{nn})$, $d_{ii} \neq 0$. Then $D^{-1} = diag(\, ^1/_{d_{11}}, ^1/_{d_{22}}, ^1/_{d_{33}}, ..., ^1/_{d_{nn}})$.
- The inverse of a non-singular symmetric matrix is a symmetric matrix.
- $\left(A^{-1}\right)^{n} = A^{-n}$.

Elementary Transformation

Any one of the following operations on a matrix is called an elementary transformation.

1. Interchange of any two rows (or columns), denoted by $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$.
2. Multiplication of the elements of any row (or column) by any non-zero number k and denoted by $R_i \to kR_i$ or $C_i \to kC_i$.
3. Addition of constant multiple of the elements of any row to the corresponding element of any other row, denoted by $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$.

Echelon form of a matrix

A matrix A is said to be in echelon form if the following hold:

i. Every row of a matrix A which has all its entries zero occurs below every row which has a non-zero entry.
ii. The number of zeros preceding the first non-zero element in a row is less than the number of such zeros in the succeeding rows. Example $A = \begin{bmatrix} 1 & 3 & 2 & 6 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Note:

- A given matrix A is echelon form if one or more elements in each of first r - rows are non-zero and these first r - rows form an upper triangular matrix and the elements in the remaining rows are zero.
- To reduce the matrix to echelon form only row transformations are to be applied.

1.1 Rank of a matrix

Let A be a non-zero matrix. A positive integer r is called the rank of the matrix A if

i. There exists at least one minor of order r of A which is non-zero, and
ii. Every minor of order greater than r is zero.

The rank of matrix A is denoted by $\rho(A)$. The rank in echelon form is equal to the number of non-zero rows of the matrix.

Methods to find rank of a matrix

1. Minor (determinant) method
2. Echelon form

1.2 System of linear equations

Definitions, Formulae and Theory

A linear system of m equations in n unknowns $x_1, x_2, ..., x_n$ is a set of equations of the form

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n = b_m.$$

This is called system of Linear equations. It is called linear because each variable x_j is of first degree only.

The given system can also be written in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & .... & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

i.e. $AX = B$

where A is the matrix of coefficients, X is the matrix of unknown variables and B is the matrix of constants. If $B = 0$, then the system is homogeneous. If $B \neq 0$, then the system is non-homogeneous.

The values of unknowns form a solution set of the given system. It satisfies all the equations of the system.

Augmented Matrix

Note that the matrix $\begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} & b_1 \\ a_{21} & a_{22} & .... & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & .... & a_{mn} & b_m \end{bmatrix}$ is called the augmented matrix. It is represented as $[A \quad B]$.

Consistent

If a system has solution then it is called consistent and vice-e-versa.
If a system has no solution then it is called inconsistent and vice-e-versa.

To test consistency of the system of Linear equations

Method 1:
If in the system there are n unknowns and n equations, then the matrix of coefficients i.e. A is a square matrix.

Consider $AX = B$

$$A^{-1}AX = A^{-1}B$$
$$X = A^{-1}B$$

Applicable only if $|A| \neq 0$ and for square matrix.

Method 2:
This method can be used, if in the system there are n unknowns and m equations.

For Non-Homogeneous Linear Equations:
i) Write down the given system in the matrix form i.e. $AX = B$.
ii) Reduce the matrix A to row echelon form and do same transformation on matrix B also.
iii) As the matrix is in row echelon form, we can determine rank of matrix A and rank of augmented matrix $[A \quad B]$.
iv) If Rank A < Rank $[A \quad B]$, the system is inconsistent i.e. it has no solution.
v) If Rank A = Rank $[A \quad B]$, the system is consistent i.e. it has a solution.
Case a: If rank of A is equal to the number of unknowns, the system has unique solution.
Case b: If rank of A is less than the number of unknowns, the system has infinitely many solutions. Here, some unknowns are assigned arbitrary values called parameters. Number of parameters = (number of unknowns – rank of A).

For Homogeneous Linear Equations:
i) Write down the given system in the matrix form i.e. $AX = 0$.
ii) Reduce the matrix A to row echelon form.
iii) As the matrix is in row echelon form, we can determine rank of matrix A.
iv) If Rank A = number of unknowns, then the only possible solution is zero solution or trivial solution.
v) If Rank A < number of unknowns, then the system has non-trivial solution. Here, some unknowns are assigned arbitrary values called parameters. Number of parameters = (number

of unknowns – rank of A).

1.3.1 Vector Space:

- **Definition :** Let V be a non-empty set of objects whose elements will be called as 'vectors' and let $\mathbb{R}$ be the set of real numbers whose elements will be called as 'scalars'. V is called as ***'vector space'*** over real field $\mathbb{R}$ if and only if the following conditions/axioms are satisfied

A1) Closure under vector addition :

$$u+v\in V \quad \forall\, u,v\in V$$

i.e. to every pair of elements u and v in V there corresponds a unique u + v in V

A2) Associativity Property of vector addition:

$$u+(v+w) = (u+v)+w \quad \forall\, u,v,w\in V$$

A3) Existence of additive identity element w.r.t. vector addition:

There exist a vector called as 'the zero vector' denoted by '0' in V such that $u+0 = u = 0+u \quad \forall\, u\in V$

A4) Existence of inverse element w.r.t. vector addition:

Given any u in V there exist a vector denoted by '-u' in V such that

$$u+(-u) = 0 = (-u)+u \quad \forall\, u\in V$$

A5) Commutativity property of vector addition :

$$u+v = v+u \quad \forall\, u,v\in V$$

A6) Closure under scalar multiplication :

$$\alpha u\in V \quad \forall\, u\in V \text{ and } \forall \alpha\in \mathbb{R}$$

i.e. for any scalar α in R and u in V there corresponds a unique vector 'α u' in V, this operation is known as 'scalar multiplication'

A7) Distributivity property :

$$\alpha(u+v)=\alpha u+\alpha v \quad \forall\, u,v\in V \text{ and } \alpha\in R$$

and $$(\alpha+\beta)u=\alpha u+\beta u \quad \forall\, u\in V \text{ and } \alpha,\beta\in \mathbb{R}$$

A8) $\alpha(\beta u) = (\alpha\beta)u \quad \forall\, u\in V$ and $\alpha,\beta\in \mathbb{R}$

A9) For any u in V , 1u = u , where 1 is the unity of field $\mathbb{R}$

➢ **Notes :**

It is easy to prove that the set of real numbers $\mathbb{R}$ is a vector space over itself.

1.3.2 Subspace of Vector space:

If V is a vector space and W is any non-empty subset of V then W is called as subspace of V if W itself is vector space over field R under the same vector addition and scalar multiplication of vector space V.

Definition : Let V be a vector space over field R. A non-empty subset W of V is subspace of V

iff i) **Closure under vector addition :**

for any $u, v \in W$, $u + v \in W$

and ii) **Closure under scalar multiplication :**

for any $a \in R$ and $u \in W$, $au \in W$

Alternatively ,

Subset W of vector space V is called as subspace iff $au + bv \in W \quad \forall\ u, v \in W\ \&\ a,b \in R$

Note that for a subset to become a subspace, it is necessary that it must contain the zero vector.

e.g. The set C of complex numbers is a vector space over R and as $R \subset C$ and R itself is vector space over R , R is a subspace of C over R

1.4.1 Linear Span of a set :

Definition : Let $S = \{ u_1, u_2, \ldots, u_n\}$ be a non-empty subset of a vector space V. The **linear span** of S is denoted by L(S) and it is defined as

$$L(S) = \{a_1u_1 + a_2u_2 + \ldots + a_nu_n / a_1, a_2, \ldots, a_n \in \mathbb{R}, u_1, u_2, \ldots, u_n \in S, n \in N\}$$

Definition : Let S be a non-empty subset of a vector space V. We say that S spans the vector space V iff L(S) = V i.e. if every vector in V can be expressed as linear combinations of elements of S

Note that

i) In such case L(S) a is subspace of V infact it is the smallest subspace containing S.

ii) S is a subspace of V iff L(S) = S

iii) L[L(S)] = L(S)

1.4.2 Linear dependence and independence of sets :

Definition : Let $S = \{ u_1 , u_2 , . . . , u_n\}$ be a subset of vector space V. S is said to be **linearly dependent** if there exists a non-trivial (non-zero) solution to the homogeneous system $a_1u_1 + a_2u_2 + . . . + a_nu_n = 0$, where $a_1, a_2 , . . . , a_n \in \mathbb{R}$

If the linear system has unique (zero) solution then the set S is known as **linearly independent** set of vectors.

Note that

i) S is linearly dependent set iff one of the vectors in S can be expressed as linear combination of the other vectors of S.
ii) A subset of vector space V containing the zero vector of V is linearly dependent
iii) A superset of linearly dependent set is linearly dependent
iv) A subset of linearly independent set is linearly independent

1.5 Basis and dimension of vector space :

Definition : Let V be a vector space. A finite subset S of V is called a **basis** of V iff

i) S is linearly independent set

and ii) S spans V i.e. L(S) = V

The number of elements in basis set (i.e. cardinality of basis set) of vector space V is known as **dimension** of vector space V.

Note that

i) Every finite dimensional vector space has a basis.

ii) Basis of vector space is not necessarily unique but dimension of vector space is unique.

iii) Any linearly independent subset of finite dimensional vector space is either a basis of V or can be extended to a basis of V.

iv) If dimension of vector space V is 'n' then any subset of V containing '> n' vectors is linearly dependent. Also if dimension of V is 'n' then any linearly independent subset of V containing 'n' vectors is a basis.

Classwork problems

Problems on Rank of matrix:

1. Find the rank of the matrix (by minor form)

a) $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$.

$|A| = 2(-9+8) - 1(0+4) - 1(0-6) = -2-4+6 = 0$. Consider the submatrix

$A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$. Minor of A_1 of order 2, $|A_1| = 6-0 = 6 \neq 0$. Therefore rank of matrix A is 2.

b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ [Ans: Rank of A is 1]

2. Reduce the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \\ 4 & 5 & 2 \end{bmatrix}$ to row-echelon form and find its rank. [Ans: 2]

3. Reduce the matrix $\begin{bmatrix} 1 & -1 & 2 & 6 \\ 3 & -7 & 4 & 8 \\ -2 & 8 & 1 & 9 \end{bmatrix}$ to row-echelon form and find its rank. [Ans: 3]

4. Reduce the matrix $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$ to row-echelon form and find its rank. [Ans: 3]

5. Reduce the matrix $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \\ 5 & 4 & -5 \end{bmatrix}$ to row-echelon form and find its rank. [Ans: 2]

6. Reduce the matrix $\begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$ to row-echelon form and find its rank.

[Ans: 3]

7. Reduce the matrix $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$ to row-echelon form and find its rank. [Ans: 2]

8. Reduce the matrix $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$ to row-echelon form and find its rank.

[Ans: 3]

9. Analyse the rank of the matrix $\begin{bmatrix} 1 & k & 3 \\ 4 & 3 & k \\ 2 & 1 & 2 \end{bmatrix}$ for different values of k.

[Ans: rank is 2 if $k = 0$ or $\frac{9}{2}$, Rank is 3 for all other values of k]

10. For what values of p the matrix $\begin{bmatrix} p & p & 2 \\ 2 & p & p \\ p & 2 & p \end{bmatrix}$, has (i) rank 1, (ii) rank 2, or (iii) rank 3.

Solution: Let the given matrix be A. A is a square matrix. Rank of A is 3 if $|A| \neq 0$.

$|A| = 2(p-2)^2(p+1)$. $|A| = 0$ if and only if $p = 2$ or -1.

If $p = 2$, then $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

Performing $R_2 - R_1$ and $R_3 - R_1$, we get the Echelon form: $\begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Therefore rank=1 when $p = 2$. If $p = -1$, then $A = \begin{bmatrix} -1 & -1 & 2 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{bmatrix}$

Consider the minor $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 2 & -1 \end{vmatrix} = 3 \neq 0$.

There exists a minor of order 2 which is non zero and $|A| = 0$. Therefore rank is 2.

[Ans: Rank is 1 if $p = 2$, Rank is 2 if $p = -1$, Rank is 3 for all the other values $\left(p \neq 2 \,\text{or}\, -1 \right)$]

Problems on System of linear equations

1. Solve $3x + 3y + 2z = 1; x + 2y = 4; 10y + 3z = -2; 2x - 3y - z = 5.$

<u>Solution:</u> Writing the given equations in matrix form, we have

$$\begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$$

i.e. $AX = B$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 3 & 2 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -2 \\ 5 \end{bmatrix}$$

$R_2 - 3R_1; R_4 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 0 & 10 & 3 \\ 0 & -7 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -2 \\ -3 \end{bmatrix}$$

$\dfrac{R_2}{-3}$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{-2}{3} \\ 0 & 10 & 3 \\ 0 & -7 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ \frac{11}{3} \\ -2 \\ -3 \end{bmatrix}$$

$R_3 - 10R_2; R_4 + 7R_2$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{-2}{3} \\ 0 & 0 & \frac{29}{3} \\ 0 & 0 & \frac{-17}{3} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ \frac{11}{3} \\ \frac{-116}{3} \\ \frac{68}{3} \end{bmatrix}$$

$\left(\frac{3}{29}\right)R_3$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{-2}{3} \\ 0 & 0 & 1 \\ 0 & 0 & \frac{-17}{3} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ \frac{11}{3} \\ \frac{-116}{29} \\ \frac{68}{3} \end{bmatrix}$$

$R_4 + \frac{17}{3}R_3$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{-2}{3} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ \frac{11}{3} \\ -4 \\ 0 \end{bmatrix}$$

Rank A = 3 = Rank $[A \quad B]$

The system is consistent.

Rank A = 3 = number of unknowns.

The system has a unique solution.

$\therefore z = -4$

$y - \frac{2}{3}z = \frac{11}{3} \Rightarrow y = 1$

$x + 2y + 0 = 4 \Rightarrow x = 2$

2. Solve $x + 3y - 2z = 0; 2x - y + 4z = 0; x - 11y + 14z = 0$

Solution: Writing the given equations in matrix form, we have

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $AX=0$

$R_2-2R_1; R_3-R_1$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

R_3-2R_2

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank A = 2

Number of unknowns = 3

Rank A < Number of unknowns.

The system has non-trivial solution.

Number of parameters = Number of unknowns - Rank A = 3 – 2 = 1

Let $z=k$

$$\therefore -7y+8z=0 \Rightarrow y=\frac{8k}{7}$$

$$x+3y-2z=0 \Rightarrow x=\frac{-10k}{7}$$

3. Solve $x+2y-z=1; x+y+2z=9; 2x+y-z=2$.

[Ans. $x=2; y=1; z=3$]

4. Solve $x_1+x_2-x_3=0; 2x_1-x_2+x_3=3; 4x_1+2x_2-2x_3=2$.

[Ans. $x_1=1; x_2=k-1; x_3=k$]

5. Solve $x_1+x_2+2x_3+x_4=5; 2x_1+3x_2-x_3-2x_4=2; 4x_1+5x_2+3x_3=7$.

[Ans. No solution]

6. Determine value of a and b for which the system
$x+2y+3z=6; x+3y+5z=9; 2x+5y+az=b$
has (i) no solution (ii) unique solution (iii) infinite number of solutions. Find the solution in case (ii) and (iii).

[Ans. (i) $a=8, b\neq 15$ (ii) $a\neq 8, b$ *any value.* $z=\dfrac{b-15}{a-8}$ (iii) $a=8, b=15.$ $x=k, y=3, z=k$]

7. Solve $x_1+x_2+x_3+x_4=0; x_1+3x_2+2x_3+4x_4=0; 2x_1+x_3-x_4=0$.

[Ans. $x_1 = \frac{-1}{2}k_1 + \frac{1}{2}k_2, x_2 = \frac{-1}{2}k_1 - \frac{3}{2}k_2, x_3 = k_1, x_4 = k_2$]

8. Find the values of k for which the equations $x+y+z=1; x+2y+3z=k; x+5y+9z=k^2$ have a solution. Solve them for these values of k.

[Ans. For $k=1$; $x=1+t, y=-2t, z=t$, For $k=3$; $x=t-1, y=2-2t, z=t$]

9. Solve $3x+2y+z=0; 2x+3z=0; y+5z=0; x+2y+3z=0$.

[Ans. $x=y=z=0$]

10. For what value of λ does the following system of equations possess a nontrivial solution? Obtain the solution for real values of λ.

$3x + y - \lambda z = 0; 4x - 2y - 3z = 0; 2\lambda x + 4y - \lambda z = 0.$

[Ans. For $\lambda = 1; x = -t, y = -t, z = -2t$. For $\lambda = -9; x = -3t, y = -9t, z = 2t$.]

Vector Spaces:

1) Let n be fixed positive integer and $\mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) \ / \ x_i \in \mathbb{R}, 1 \le i \le n \}$, $\mathbb{R}^n$ is set of all n -tuples of real numbers.

Let $u = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $v = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, define vector addition as $u + v = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$ and for any scalar $\alpha \in \mathbb{R}$ define scalar multiplication as $\alpha u = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$. Prove that $\mathbb{R}^n$ is vector space over $\mathbb{R}$ under these operations.

2) Show that the set V of positive real numbers with following operations, is a vector space. addition : $x + y = xy$ and scalar multiplication : $kx = x^k$, where x, y are real numbers and k is any scalar .

3) Let $M_{m\times n}(\mathbb{R})$ denote the set of all $m \times n$ matrices with real entries. Define for $A = (a_{ij}), B = (b_{ij})$, $A + B = (a_{ij} + b_{ij})$ and $\alpha A = (\alpha\, a_{ij})$ where $\alpha \in \mathbb{R}$. Determine whether $M_{m\times n}(\mathbb{R})$ is a vector space under the above operations. Ans: yes

4) Determine whether $V = \left\{ \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \middle| \ x, y \in \mathbb{R} \right\}$ is a vector space over $\mathbb{R}$ with usual addition and scalar multiplication. Ans: yes

5) If $W = \{ (1, x) \in \mathbb{R}^2 / x \in \mathbb{R} \}$, a subset of $\mathbb{R}^2$, Determine whether W is a vector space under operations $(1, x) + (1, y) = (1, x + y)$, $k(1, x) = (1, kx)$, where k is any scalar.

Ans: yes

6) In each case determine whether $\mathbb{R}^2$ is a vector space with indicated operations of vector

addition and scalar multiplication

i) $(x, y) + (x', y') = (x + x', y + y')$ and $a(x, y) = (ax, 0)$ **Ans:** No

ii) $(x, y) + (x', y') = (x + x', y + y')$ and $a(x, y) = (|a|x, |a|y)$ Ans: No

7) Let P_2 denote the set of all polynomials of degree less than or equal to 2, with real coefficients. Define Addition and Scalar multiplication in the usual way: If $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$, then $p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$ and

for any $\alpha \in \mathbb{R}$, $\alpha f(x) = \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2$.

Show that P_2 is a vector space over $\mathbb{R}$.

8) Let P_n denotes set of all polynomials in x with real coefficients

i.e. $P_n = \left\{a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \,/\, a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}\right\}$. Show that P_n is vector space under the operations defined as follows

For $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_n x^n$ and

$$q(x) = b_0 + b_1x + b_2x^2 + \ldots + b_n x^n \in \mathrm{P}$$

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots + (a_n + b_n)x^n$$

and for any $\alpha \in \mathbb{R}$, $\alpha f(x) = \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + \ldots + \alpha a_n x^n$

9) Let F be set of all real valued functions i.e. $F = \{ f : \mathbb{R} \to \mathbb{R} \,/\, f \text{ is a function} \}$.

Define $(f + g)(x) = f(x) + g(x)$ and $(\alpha f)(x) = \alpha f(x)$, then prove that F is a vector space over $\mathbb{R}$.

10) Prove that set of all real valued differentiable functions on (a, b) is a vector space over $\mathbb{R}$.

Vector Subspaces:

1) Let $W = \{ (x, y) \in \mathbb{R}^2 / ax + by = 0 \}$ Show that W is subspace of $\mathbb{R}^2$.

i.e. Show that every line passing through the origin is a subspace of $\mathbb{R}^2$.

2) Let $W = \{ (x, y, z) \in R^3 / ax + by + cz = 0 \}$ Show that W is subspace of $\mathbb{R}^3$.

i.e. Show that every plane passing through the origin is a subspace of $\mathbb{R}^3$.

3) Let $S = \{ (x, y, z) \in \mathbb{R}^3 / ax = by = cz \}$ Show that S is subspace of $\mathbb{R}^3$.

i.e. Show that every line passing through the origin is a subspace of $\mathbb{R}^3$.

4) Let $W = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n / a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0 \right\}$. Show that W is a subspace of $\mathbb{R}^n$.

5) Is the set $W = \left\{ (x, 1, 1) \in \mathbb{R}^3 / x \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^3$ under the usual addition and scalar multiplication.

6) Solve the system of equations: $\begin{matrix} 3x+4y+z=0 \\ x+y+z=0 \end{matrix}$. Let V denote the set of all solutions to the above system. Show that V is a vector subspace of $\mathbb{R}^3$.

7) Show that the set of all symmetric matrices of order $m \times n$ is a vector subspace of $M_{m\times n}(\mathbb{R})$.

8) If W_1 and W_2 are subspaces of vector space V then prove that $W_1 \cap W_2$ is a subspace of V but $W_1 \cup W_2$ is not necessarily a subspace.

Linear Span and Independence:

1) If S = { (1, 0, 0) , (0, 1, 0) } then show that L(S) is subspace of $\mathbb{R}^3$.

2) In $\mathbb{R}^2$, show that the vector (3, 7) is in the linear span of S = { (1, 2), (0, 1) } but (3, 7) is not in the linear span of W = { (1, 2), (2, 4) }

3) Determine whether (1, 1, 0) is in the linear span of S = { (1, 2, 1), (1, 1, -1), (4, 5, -2) } $\subseteq \mathbb{R}^3$

Ans: No

4) Determine whether (-1/3, - 1/3, - 1/3) is in the linear span of S = {(1, 2, 1),(1, 1, -1),(4, 5, -2)} $\subseteq \mathbb{R}^3$ **Ans:** No

5) Examine whether the vectors (2, -1, 4) , (3, 6, 2), (2, 10, -4) in $\mathbb{R}^3$ are linearly dependent.

Ans: L.I.

6) Show that the vectors (0, 3, 1, -1), (6, 0, 5, 1), (4, -7, 1, 3) form a linearly dependent set.

7) In the vector space P_2 , determine whether the vectors $3 - 2x + 4x^2$, $4 - x + 6x^2$ and $7 - 8x + 8x^2$ are linearly dependent. **Ans:** L.D.

8) Examine the following sets of vectors for linear dependence/independence

i) $\{(2,-1,4),\ (3,6,2),\ (2,10,-4)\}$ in $\mathbb{R}^3$ **Ans:** L.I.

ii) $\{(1,3,3),\ (0,1,4),\ (5,6,3)\}$ in $\mathbb{R}^3$ **Ans:** L.I.

iii) $\{(3,0,4,1),\ (6,2,-1,2),\ (-1,3,5,1),\ (-3,7,8,3)\}$ in $\mathbb{R}^4$ **Ans:** L.I.

9) For which values of λ , do the following vectors form linearly dependent set in $\mathbb{R}^3$;
$(\lambda,-1/2,\ -1/2)$, $(-1/2,\ -1/2,\ \lambda)$, $(-1/2,\lambda,-1/2)$

Basis and Dimension:

1) Show that the set $S = \{ (1, 0), (0, 1) \}$ is a basis of vector space $\mathbb{R}^2$ [Infact it is the standard/Euclidean basis of $\mathbb{R}^2$]

2) Show that the set $S = \{ (1, 0, 0), (0, 1, 0), (0, 0, 1) \}$ is a basis of vector space $\mathbb{R}^3$ [Infact it is the standard/Euclidean basis of $\mathbb{R}^3$]

3) Show that the set $S = \left\{ 1,\ x,\ x^2 \right\}$ is basis of vector space P_2 [Infact it is the standard basis of P_2]

4) Show that $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is basis of $M_{2\times2}(\mathbb{R})$

5) Show that the set $S = \{ (1, 2, -3), (1, -3, 2), (2, -1, 5) \}$ is a basis of vector space $\mathbb{R}^3$.

6) Show that the set $S = \left\{-4 + x + 3x^2, 6 + 5x + 2x^2, 8 + 4x + x^2\right\}$ is a basis of P_2.

7) Determine a basis and dimension of the solution space of following homogeneous systems

i) $x_1 - x_2 + x_3 = 0$
$2x_1 + x_2 - x_3 = 0$

Ans:

B = { (0,1,1) }

dim = 1

ii) $x_1 + 3x_2 + x_3 + x_4 = 0$
$2x_1 - 2x_2 + x_3 + 2x_4 = 0$
$x_1 + 11x_2 + 2x_3 + x_4 = 0$

Ans:

B = { (5,1,-8,0) , (-1,0,0,1) }

dim = 2

8) Let V and W be subspaces of $\mathbb{R}^4$. $V = \{(x, y, z, u)/\ y - 2z + u = 0\}$ and $W = \{(x, y, z, u)/\ y = 2z, x = u\}$ Find a basis and dimension of i) V , ii) W , iii) $V \cap W$

$B_V = \{ (1,0,0,0), (0,2,1,0), (0,-1,0,1) \}$, $dim(V) = 3$

Ans: $B_W = \{ (1,0,0,1), (0,2,1,0) \}$, $dim(W) = 2$

$B_{V \cap W} = \{ (0,2,1,0) \}$, $dim(V \cap W) = 1$

9) Examine whether the following sets of vectors form a basis for the indicated vector space

i) $\{(1, 2, 3), (3, 1, 0), (-2, 1, 3)\}$ in $\mathbb{R}^3$ **Ans :** No

ii) $\{(1, 3, -4), (1, 4, -3), (2, 3, -11)\}$ in $\mathbb{R}^3$ **Ans :** No

iii) $\{(1, -1, 0, 1), (0, 0, 0, 1), (2, -1, 0, 1), (3, 2, 1, 0)\}$ in $\mathbb{R}^4$ **Ans :** Yes

10) If W is subspace of $\mathbb{R}^4$ spanned by vectors (1, - 2, 5, - 3), (2, 3, 1, - 4), (3, 8, - 3, - 5). Find basis and dimension of W.

11) Find a basis and dimension of the subspace of indicated vector spaces

i) $\left\{(x, y, z) \in \mathbb{R}^3 \,/\, 2x - 3y + 5z = 0 \right\}$ in $\mathbb{R}^3$.

ii) $\left\{(x, y, z) \in R^3 \,/\, x = 2t, y = -t, z = 4t \text{ and } t \in \mathbb{R}\right\}$ in $\mathbb{R}^3$.