Unit II

Linear Transformation and Eigenvalues

Detailed Syllabus:

2.1Linear transformation

2.2 Matrix associated with linear transformation

2.3 Composition of linear maps, Kernel and Range of a linear map, Rank-Nullity Theorem, Inverse of a linear transformation

2.4 Cayley- Hamilton Theorem

2.5 Eigenvalues, Eigenvectors, Eigenvalues of symmetric, skew-symmetric, Hermitian and Skew-Hermitian matrices

2.6 Diagonalization, Orthogonal Diagonalization of a real symmetric matrix.

2.1 Linear transformations (maps)

Pre-requisites: Vector Space, linear independence of vectors, basis, fundamental knowledge of functions.

Definition of Linear Transformation:

A Linear Transformation from a vector space V to vector space W is a mapping $T : V \to W$ such that, for all v_1 and v_2 in V and for all scalars c ($c \in \mathbb{R}$),

1. $T(v_1 + v_2) = T(v_1) + T(v_2)$
2. $T(cv_1) = cT(v_1)$

This definition is equivalent to the following:

$$T(c_1 v_1 + c_2 v_2 + ... + c_k v_k) = c_1 T(v_1) + c_2 T(v_2) + ... + c_k T(v_k)$$

Remark:

If $T : V \to W$ is a linear transformation, then

i. $T(0) = 0$

ii. $T(-v_1) = -T(v_1)$

iii. $T(v_1 - v_2) = T(v_1) - T(v_2)$

Remark:

Let V and W be vector spaces and let $\{v_1, v_2, ..., v_n\}$ be a basis of V. Recall for every $x \in V$,

$$x = c_1v_1 + c_2v_2 + ...c_nv_n$$

for some scalars $c_1, c_2, ...c_n$. Let $T : V \to W$ be any linear transformation. Since,

$$T(x) = c_1T(v_1) + c_2T(v_2) + ... + c_nT(v_n)$$

we can see that any linear transformation T is completely determined by its action on a basis of V.

2.2 Matrix associated with a linear map.
Matrix associated with Linear Transformation:

Pre-requisites:

Linear transformation, knowledge of matrices

Let V and W be vector spaces of dimension n and m respectively. Let $\{v_1, v_2, ..., v_n\}$ be a basis of V and $\{w_1, w_2, ...w_m\}$ be a basis of W. Let $T : V \to W$ be a linear map. As $T(v_1), T(v_2), ...T(v_n)$ are elements of W, we have

$$\begin{aligned}
T(v_1) &= a_{11}w_1 + a_{21}w_2 + ... + a_{m1}w_m \\
T(v_2) &= a_{12}w_1 + a_{22}w_2 + ... + a_{m2}w_m \\
\vdots \quad & \quad \vdots \qquad \quad \vdots \qquad\qquad\quad \vdots \\
T(v_n) &= a_{1n}w_1 + a_{2n}w_2 + ... + a_{mn}w_m
\end{aligned}$$

where a_{ij} are scalars.

The matrix $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ is called the matrix associated with T with respect to the bases $\{v_1, v_2, ..., v_n\}$ and $\{w_1, w_2, ...w_m\}$.

Remark: For every vector x in V, $A \times \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$ where $x_i, 1 \le i \le n$ are the co-ordinates of x with respect to the basis $\{v_1, v_2, ..., v_n\}$ and $y_j, 1 \le j \le m$ are the co-ordinates of $y = T(x)$ with respect to the basis $\{w_1, w_2, ...w_m\}$.

2.3 Range and kernel of a linear map, Rank-nullity theorem, Composition of linear maps, Inverse of a linear transformation

Range and Kernel of a Linear Transformation:

Pre-requisites: Fundamental knowledge of functions, Vector space and Linear transformations.

Let V and W be vector spaces. Let $T : V \to W$ be any linear transformation. The ***Kernel*** of T**,** denoted as ker(T), is the set of all vectors in V, that are mapped by T to 0 in W. That is

$$ker(T) = \{v \in V : T(v) = 0\}$$

The range of T, denoted as range(T), is the set of all vectors in W that are images of vectors in V under T. That is

$$\begin{aligned} range(T) &= \{T(v) : v \in V\} \\ &= \{w \in W : w = T(v) \text{ for some } v \in V\} \end{aligned}$$

Null Space and Column Space:

Let A be an $m \times n$ matrix.

The null space of A is the subspace of $\mathbb{R}^n$, consisting of solutions of the homogeneous linear system $AX = 0$. It is denoted by $null(A)$.

The column space of A is the subspace $col(A)$ of $\mathbb{R}^m$ spanned by the columns of A.

Note: The kernel of a linear transformation T, (ker(T)) is a subspace of V; and the range of T (range(T)) is a subspace of W.

Rank and Nullity:

Let $T : V \to W$ be any linear transformation. The rank of T is the dimension of the range of T, and is denoted by rank(T). The nullity of T is the dimension of the kernel of T, and is denoted by nullity(T).

Composition of linear maps, Inverse of a linear transformation

Pre-requisites: Fundamental knowledge of functions.

Composition of Linear Transformation:

Let U, V and W be vector spaces of dimension n, m and r respectively. Let $T : U \to V$ and $S : V \to W$ be linear maps. Then, $S \circ T : U \to W$ is also a linear map.

Matrix associated with the composite linear transformation:

Let U, V and W be a vector spaces with bases $B_1 = \{u_1, u_2, ..., u_n\}$, $B_2 = \{v_1, v_2, ..., v_m\}$ and $B_3 = \{w_1, w_2, ...w_r\}$ respectively. Let $T : U \to V$ and $S : V \to W$ be linear maps. Let us denote the matrix associated with the linear map, $T : U \to V$ with respect to the bases B_1 and B_2 as $[A]_{B_2 \leftarrow B_1}$; and the matrix associate with $S : V \to W$ with respect to the bases B_2 and B_3 as $[B]_{B_3 \leftarrow B_2}$

respectively. Then, the matrix associated with the linear map $S \circ T : U \to W$, denoted by $[C]_{B_3 \leftarrow B_1}$ satisfies

(0.1)

$$[C]_{B_3 \leftarrow B_1} = [B]_{B_3 \leftarrow B_2} [A]_{B_2 \leftarrow B_1} .$$

(Right hand side of the above equation is the product of two matrices.)

Inverse of a linear transformation

Definition: Let U and V be vector spaces and $T : U \to V$ be a linear map. A linear map $T^{-1} : V \to U$ is the inverse of T if $T \circ T^{-1} = T^{-1} \circ T = I$.

Note: $T^{-1} : V \to U$ is also linear.

Definition: A linear map $T : U \to V$ which has inverse is called invertible or nonsingular transformation or an isomorphism.

A linear transformation is said to be invertible if the map $T : U \to V$ is one- one and onto.

Matrix associated with inverse of a linear transformation

Let U and V be vector spaces with bases $B_1 = \{u_1, u_2, ..., u_n\}$, $B_2 = \{v_1, v_2, ..., v_m\}$ and let $T : U \to V$ be a linear map. The matrix of $T^{-1} : V \to U$ with respect to given bases is the inverse of matrix of a linear map $T : U \to V$ with respect to the same bases.

2.4 Cayley-Hamilton Theorem and orthogonal transformation

Cayley-Hamilton Theorem

Every square matrix satisfies its own characteristic equation; i.e. if the characteristic equation for the nth order square matrix A is

$|A - \lambda I| = (-1)^n \lambda^n + k_1 \lambda^{n-1} + ... + k_n = 0$ then

$(-1)^n A^n + k_1 A^{n-1} + ... + k_n I = 0$ .....(i)

Note: To find A^{-1} multiply A^{-1} both sides of (i) and simplify we get,

$$A^{-1} = -\frac{1}{k_n}\left[(-1)^n A^{n-1} + k_1 A^{n-2} + ... + k_{n-1} I\right]$$

<u>5.3 Eigenvalues and Eigenvectors:</u>

In this section we will consider only square matrices.

1. Prerequisite to the topic

Basic knowledge of matrices and its elementary transformations, Solving system of Linear system of equations.

2. Definitions, Formulae and Theory

Eigenvalues & Eigenvectors

Let A be a $n \times n$ matrix. Suppose the linear transformation $Y = AX$ transforms X into a scalar multiple of itself i.e. $AX = Y = \lambda X$ where X is an invariant vector.

Then the unknown scalar λ is known as an eigenvalue of the matrix A and the corresponding non-zero vector X is known as eigenvector.

$\therefore AX = \lambda X$ for any $X \neq 0$

$\therefore AX - \lambda IX = 0$

$\therefore (A - \lambda I) X = 0$

This represent a system of n homogeneous equations in the n variables.

It has a non-trivial solution if the coefficient matrix $(A - \lambda I)$ is singular.

i.e. $|A - \lambda I| = 0$

This is known as the characteristic equation of A.

Expansion of the determinant gives a n^{th} degree polynomial known as characteristic polynomial of A.

Thus eigenvalues of matrix A are the roots of the characteristic equation. Hence A can have at least one and at most n eigenvalues.

The eigenvector X corresponding to an eigenvalue λ is obtained by solving the homogeneous system i.e. $(A - \lambda I) X = 0$ with this known eigenvalue λ.

Note 1: If all the n eigenvalues of A are distinct, then there correspond n distinct linearly independent eigenvectors.

Note 2: For an eigenvalue of A, repeated (twice or more), there may correspond one or several linearly independent eigenvectors. Thus the set of eigenvectors may or may not form a set of n linearly independent vectors.

Note 3: Algebraic multiplicity of an eigenvalue λ is the order of the eigenvalue as a root of the characteristic polynomial. (i.e. If λ is a double root then algebraic multiplicity is 2).

Note 4: Geometric multiplicity of λ is the number of linearly independent eigenvectors corresponding to λ.

Note 5: Formula for finding characteristic equation of a 3×3 matrix $\lambda^3 - trace(A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - \det(A) = 0$

Note:

1. The eigenvalues of a real symmetric matrix are real. The eigenvalues of a real skew symmetric matrix are purely imaginary or zero.

2. The eigenvalues of an orthogonal matrix are real or complex conjugates in pairs and have absolute value 1.

Theorems on Eigen Values and Eigen Vectors:

1. The eigen vectors corresponding to distinct eigen values of a real symmetric matrix are orthogonal.
2. If $\lambda_1, \lambda_2, ..., \lambda_n$ are eigen values of A, then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, ..., \frac{1}{\lambda_n}$ are eigen values of A^{-1}.
3. If λ is eigen value of non-singular matrix A, then $\frac{|A|}{\lambda}$ is eigen value of adjoint of A.
4. If A is upper and lower triangular matrix, diagonal elements are eigen values of it.
5. Any two eigen vectors corresponding to two distinct eigen values of a unitary matrix are orthogonal.
6. If $f(x)$ is an algebraic polynomial in x and λ is an eigen value and X is an corresponding eigen vector of a matrix A then $f(\lambda)$ is an eigen value and X is an corresponding eigen vector of a matrix $f(A)$.

2.5 Diagonalization of matrices

Similar Matrices

If A and B are two square matrices of order n then B is said to be a **similar** to A if there exists a non-singular matrix M such that $B = M^{-1}AM$.

A square matrix A is said to be **diagonalisable** if it is similar to a diagonal matrix.

If λ_1 is an eigen value of the characteristic equation $|A - \lambda I| = 0$ repeated t times then t is called **algebraic multiplicity** of λ_1. If s is the number of linearly independent eigen vectors corresponding to the eigen value λ_1 then s is called **geometric multiplicity** of λ_1.

Note

1. The necessary and sufficient condition of a square matrix to be similar to a diagonal matrix is that the **geometric multiplicity** of each of its eigen values coincides with the **algebraic multiplicity.**
2. Every matrix whose eigen values are distinct is similar to a diagonal matrix.
3. A square nonsingular matrix A whose eigen values are all distinct can be diagonalised by a similarity transformation $D = M^{-1}AM$ where M is the matrix whose columns are the eigen vectors of A and D is the diagonal matrix whose diagonal elements are the eigen values of A.

Classwork problems

Problems on Linear transformations (maps)

Show that the following mappings are linear transformations:

1. $T:\mathbb{R}\to\mathbb{R}, T(x)=4x$.
2. $T:\mathbb{R}^2\to\mathbb{R}, T(x,y)=2x+3y$.
3. $T:\mathbb{R}^2\to\mathbb{R}^2, T(x,y)=(3x+y, x-2y)$.
4. $T:\mathbb{R}^3\to\mathbb{R}^2, T(x,y,z)=(2x+z, x-3y)$.
5. $T:\mathbb{R}^3\to\mathbb{R}^3, T(x,y,z)=(x,y,0)$.
6. $T:\mathbb{R}^3\to\mathbb{R}^3, T(x,y,z)=(x+2y-3z, 2x-y+z, -x+2y+z)$.
7. Let A be a 3×3 matrix. Let $T:\mathbb{R}^3\to\mathbb{R}^3, T(X)=AX$ where $X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.
8. Let F denote the set of all differentiable real valued functions defined on the real line $\mathbb{R}$. Let $T:F\to F$, $T(f)=\dfrac{df}{dx}$.
9. Let C denote the set of all continuous real valued functions defined on the real line. Let $T:C\to C, T(f)=\int_0^1 f(x)\,dx$.

Show that the following maps are not linear:

1. $T:\mathbb{R}\to\mathbb{R}, T(x)=x^2$.
2. $T:\mathbb{R}^2\to\mathbb{R}, T(x,y)=xy$.
   (Hint: Give a counter example.)

Continuation- Problems on Linear Transformation:

1. Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be a linear map such that $T(4,1)=(1,1)$, $T(1,1)=(3,-2)$. Compute $T(1,0)$. Find $T(x,y)$, where $(x,y)\in\mathbb{R}^2$.

   $Ans:\left(\dfrac{-2}{3},1\right)$, $T(x,y)=(x+3y, x-2y)$.

2. Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be a linear map such that $T(x,y)=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, where $\theta=\dfrac{\pi}{2}$. Let S be a square whose corners are at $(0,0),(1,0),(1,1)$ and $(0,1)$. Find the image of the corners of S under T. What do you observe as the effect of T on the two sides joining $(0,0),(1,0)$ and $(0,0),(0,1)$. Also observe the effect on the diagonal joining $(0,0),(1,1)$.

   (In general, the above transformation for a particular value of θ is rotation by θ degrees. It is called rotation map.)

   $Ans: T(0,0)=(0,0), T(1,0)=(0,1), T(0,1)=(-1,0), T(1,1)=(-1,1)$

3. Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that $T(x,y)=(x,-y)$. Find image of four points of your choice. What do you observe?

Problems on Matrix associated with a linear map

1. Let $T:\mathbb{R}^2 \to \mathbb{R}^2, T(x,y)=(x+y,x-y)$. Find the matrix associated with T with respect to the standard bases.

   $Ans:\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

2. Let $T:\mathbb{R}^3 \to \mathbb{R}^2, T(x,y,z)=(x+2z,3x-z)$. Find the matrix associated with T with respect to the standard bases.

   $Ans:\begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & -1 \end{pmatrix}$

3. Let $T:\mathbb{R}^2 \to \mathbb{R}^3, T(x,y)=(x,x+y,y)$. Find the matrix associated with T with respect to the bases : $\{(1,1),(0,1)\}$ of $\mathbb{R}^2$ and the standard basis of $\mathbb{R}^3$.

   $Ans:\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}$

4. Let $T:\mathbb{R}^3 \to \mathbb{R}^2, T(x,y,z)=(x-2y,x+y-3z)$. Let $B=\{e_1,e_2,e_3\}$ and $C=\{e_2,e_1\}$ be bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. Find the matrix M with respect to B and C. Verify that

   $M\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}=T(1,3,-2)$.

   $Ans:\begin{pmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{pmatrix}$, $T(1,3,-2)=(-5,10)$

5. Find the matrix associated with T, where T is rotaion by $\frac{\pi}{4}$.

   $Ans:\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

6. Let $D:P_3 \to P_2$ be the differential operator $D(P(x))=p'(x)$. Let $B=\{1,x,x^2,x^3\}$, $C=\{1,x,x^2\}$ be bases for P_3 and P_2 respectively.
   a) Find the matrix M of D with respect to B and C.
   b) Compute $D(2x^3-x+5)$ using part a).

   $a)\ Ans:\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ $\quad b) 6x^2-1$

7. Let $T: P_3 \to P_5$ be the linear transformation given by $T\left(p(x)\right) = \left(1+2x-x^2\right)p(x)$. Find the matrix of T relative to the standard bases of P_3 and P_5.

$$Ans: \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

Problems on range and kernel of a linear map, rank-nullity theorem

1. Find the range and kernel of $T: \mathbb{R}^3 \to \mathbb{R}$ defined by $T(x, y, z) = 3x - 2y + z$.

   $Ans: Ker\,T = \{(x, y, 2y-3x) \mid x, y \in \mathbb{R}\}, Range = \mathbb{R}$

2. Find the range and kernel of $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x, x+y, y)$.

   $Ans: Ker\,T = \{(0,0)\}, Range = \{(r, s, s-r) \mid r, s \in \mathbb{R}\}$

3. Find the range and kernel of $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x+z, x+y+2z, 2x+y+3z)$.

   $Ans: Ker\,T = \{(-z, -z, z) \mid z \in \mathbb{R}\}, Range = \{(r, s, s+r) \mid r, s \in \mathbb{R}\}$

4. Find the range and kernel of $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x-y+z, y-z, 2x-5y+5z)$.

   $Ans: Ker\,T = \{(0, y, y) \mid y \in \mathbb{R}\}, Range = \{(r, s, 2r-3s) \mid r, s \in \mathbb{R}\}$

5. Find the range and kernel of $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(e_1) = e_1 + e_3$, $T(e_2) = e_2 + e_3$ and $T(e_3) = -e_3$.
   $Ans: Ker\,T = \{(0,0,0)\}, Range = \mathbb{R}^3$
6. Find the kernel and range of the differential operator $D: P_3 \to P_2$ defined by $D\left(p(x)\right) = p'(x)$.
   $Ans: Ker\,D = \text{constant polynomials}, Range = P_2$
7. Let $S: P_1 \to \mathbb{R}$ be the linear transformation defined by $S(p(x)) = \int_0^1 p(x)\,dx$. Find the kernel and range of S.
   $Ans: Ker\,S = \{-2bx + b\}, Range = \mathbb{R}$
8. Let $T: M_{22} \to M_{22}$ defined by, $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & 0 \\ 0 & c+d \end{pmatrix}$. Find the kernel and range of T.

   $Ans: Ker\,T = \left\{ \begin{pmatrix} a & -a \\ c & -c \end{pmatrix} \middle| a, c \in \mathbb{R} \right\}, Range = \left\{ \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \middle| r, s \in \mathbb{R} \right\}$

9. Find rank and nullity of the linear transformations given in problem 1 to 8.

	Nullity	Rank
1	2	1
2	0	2
3	1	2
4	1	2
5	0	3
6	1	3
7	1	1
8	2	2

10. Verify Rank- nullity theorem for the linear transformations given in problem 1 to 8.

Problems on Composition of linear maps, Inverse of a linear transformation

1. Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x,y)=(x+y,x-y)$ and $S:\mathbb{R}^2 \to \mathbb{R}$ be defined by $S(a,b)=a+b$. Find $S\circ T(x,y)$.

   $Ans: S\circ T(x,y)=2x$

2. Let $T:\mathbb{R}^3 \to \mathbb{R}^2, T(x,y,z)=(x+2z,3x-z)$; and $S:\mathbb{R}^2 \to \mathbb{R}^3, S(x,y)=(x,x+y,y)$. Find $S\circ T$ and $T\circ S$. Write the matrices of T, S, $S\circ T$ and $T\circ S$ with respect to the standard basis. Verify the result stated by eqution (1.1).

   $Ans:$ $S\circ T(x,y,z)=(x+2z,4x+z,3x-z)$, $T\circ S(x,y)=(x+2y,3x-y)$

   $M_T=\begin{pmatrix}1 & 0 & 2\\ 3 & 0 & -1\end{pmatrix}$ $M_S=\begin{pmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{pmatrix}$ $M_{S\circ T}=\begin{pmatrix}1 & 0 & 2\\ 4 & 0 & 1\\ 3 & 0 & -1\end{pmatrix}$ $M_{T\circ S}=\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix}$

3. Let $T:\mathbb{R}^2 \to P_1$ and $S:P_1 \to P_2$, be the linear transformation defined by $T(y,z)=y+(y+z)x$ and $S(p(x))=x\,p(x)$. Find $S\circ T(a,b)$.

   $Ans: ax+(a+b)x^2$

4. Define linear transformations $S:\mathbb{R}^2 \to M_{22}$ and $T:\mathbb{R}^2 \to \mathbb{R}^2$ by $S(a,b)=\begin{pmatrix}a+b & b\\ 0 & a-b\end{pmatrix}$ and $T(c,d)=(2c+d,-d)$. Compute $(S\circ T)(x,y)$.

   $Ans: (S\circ T)(x,y)=\begin{pmatrix}2x & -y\\ 0 & 2x+2y\end{pmatrix}$

5. Use matrix method and verify the answer in example 1.

$$Ans: M_T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad M_S = \begin{pmatrix} 1 & 1 \end{pmatrix} \qquad M_S M_T = \begin{pmatrix} 2 & 0 \end{pmatrix}$$

6. Let $F:\mathbb{R}^3 \to \mathbb{R}^3$ be defined by $F(x,y,z) = (x+y-2z, x+2y+z, 2x+2y-3z)$ Find if F is nonsingular.
7. Find the linear map $T:\mathbb{R}^2 \to \mathbb{R}^2$ for which $T(1,2)=(2,3)$ and $T(0,1)=(1,4)$. Find a formula for $T(x,y)$. Is T invertible? If so, find T^{-1}.

Ans: $T(x,y) = (y, 4y-5x)$. T is invertible. $T^{-1}(u,v) = \left(\frac{4u-v}{5}, u\right)$

8. Let $\mathrm{T}:\mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\mathrm{T}(x,y,z) = (x+y+z, x+2y-z, 3x+5y-z)$ .Find if T is nonsingular. If not, find $u \neq 0, u$ in R^3 such that $Tu = 0$.

Ans: T is singular. If u= (-3, 2, 1), then Tu=0.

9. Show that $\mathrm{T}:\mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\mathrm{T}(x,y) = (2x+y, 3x-5y)$ is invertible. Find T^{-1}.

Ans: $T^{-1}(u,v) = \left(\frac{5u+v}{13}, \frac{3u-2v}{13}\right)$

10. Show that $\mathrm{T}:\mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x,y,z) = (2y+z, x-4y, 3x)$ is invertible. Find the matrix associated with T^{-1} , with respect to the basis $B = \{(1,1,1), (1,1,0), (1,0,0)\}$.

    Ans: $m(T^{-1}) = \frac{1}{12}\begin{bmatrix} 16 & 18 & 12 \\ -18 & -21 & -12 \\ 6 & 3 & 0 \end{bmatrix}$

Problems on Cayley-Hamilton Theorem

1. Find the characteristic equation of the matrix given below and verify that it satisfies Cayley-Hamilton theorem: i) $A=\begin{bmatrix}1 & 1 & 3\\ 1 & 0 & -3\\ -2 & 1 & 0\end{bmatrix}$, ii) $A=\begin{bmatrix}7 & -1 & 3\\ 6 & 1 & 4\\ 2 & 4 & 8\end{bmatrix}$
2. Verify Cayley-Hamilton theorem for the matrix A and hence find A^{-1}, where $A=\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}$ Ans. $A^3-5A^2+9A-I=0,\ A^{-1}=\begin{bmatrix}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{bmatrix}$.
3. Find the characteristic equation of the matrix A. Show that the matrix A satisfies the characteristic equation and hence find A^{-1} and A^4.

   i) $A=\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 1 & -1\end{bmatrix}$ Ans. $A^{-1}=\frac{1}{40}\begin{bmatrix}-3 & 5 & 11\\ 14 & -10 & 2\\ 5 & 5 & -5\end{bmatrix}, A^4=\begin{bmatrix}248 & 101 & 218\\ 272 & 109 & 50\\ 104 & 98 & 204\end{bmatrix}$

   ii) $A=\begin{bmatrix}1 & 2 & 1\\ -1 & 0 & 3\\ 2 & -1 & 1\end{bmatrix}$ Ans. $A^{-1}=\frac{1}{18}\begin{bmatrix}3 & -3 & 6\\ 7 & -1 & -4\\ 1 & 5 & 2\end{bmatrix}, A^4=\begin{bmatrix}46 & 20 & 10\\ -10 & 36 & 30\\ 20 & -10 & 46\end{bmatrix}$
4. Find the characteristic equation of the matrix A given below and hence, find the matrix represented by $A^5-4A^4-7A^3+11A^2-A-10I$ in terms of A, where $A=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}$. [Ans. $A+5I$]
5. Find the characteristic equation of the matrix A given below and hence, find the matrix represented by $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$, where $A=\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}$. [Ans. $\begin{bmatrix}8 & 5 & 5\\ 0 & 3 & 0\\ 5 & 5 & 8\end{bmatrix}$]
6. Find Characteristic equation of matrix A and hence find the matrix represented by $A^7-4A^6-20A^5-34A^4-4A^3-20A^2-33A+I$ where $A=\begin{bmatrix}1 & 3 & 7\\ 4 & 2 & 3\\ 1 & 2 & 1\end{bmatrix}$.

   [Ans. $\begin{bmatrix}3 & 6 & 14\\ 8 & 5 & 6\\ 2 & 4 & 3\end{bmatrix}$]

Problems on Eigenvalues and eigenvectors:
Symmetric, skew-symmetric and orthogonal matrices

1. Find eigenvalues and eigenvectors of $A=\begin{bmatrix}1 & 0 & -1\\ 1 & 2 & 1\\ 2 & 2 & 3\end{bmatrix}$.

Solution: Characteristic equation of A is given by $\det(A-\lambda I)=0$

$$\begin{vmatrix}1-\lambda & 0 & -1\\ 1 & 2-\lambda & 1\\ 2 & 2 & 3-\lambda\end{vmatrix}=0$$

$\lambda^3-6\lambda^2+11\lambda-6=0$

$(\lambda-1)(\lambda-2)(\lambda-3)=0$

$\therefore \lambda=1,2,3.$

These are the three distinct eigenvalues of A.

Now finding corresponding eigenvectors,

For $\lambda=1$,

$(A-\lambda I)X=0$

$$\begin{bmatrix}0 & 0 & -1\\ 1 & 1 & 1\\ 2 & 2 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=0$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & -1\\ 2 & 2 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=0$$

R_3-2R_1

$$\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & -1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=0$$

$\therefore x_3=0$

And $x_1+x_2+x_3=0$

i.e. $x_1+x_2=0$

Let $x_1=k_1$

$\therefore x_2=-k_1$

Hence, eigenvector is $X_1=k_1\begin{bmatrix}1\\ -1\\ 0\end{bmatrix}$.

Alternate method: (Applicable only for distinct eigenvalue)

For $\lambda = 1$,

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore -x_3 = 0$

$x_1 + x_2 + x_3 = 0$

$2x_1 + 2x_2 + 2x_3 = 0$

Consider two distinct equations,

$x_1 + x_2 + x_3 = 0$

$-x_3 = 0$

Now by using modified Cramer's rule, we get

$$\frac{x_1}{\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}}$$

$$\frac{x_1}{-1} = \frac{-x_2}{-1} = \frac{x_3}{0}$$

Let $\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{0} = k_1$

Hence, eigenvector is $X_1 = -k_1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

For $\lambda = 2$,

$(A - \lambda I)X = 0$

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

On solving, we get eigenvector $X_2 = k_2\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

For $\lambda = 3$,

$(A - \lambda I)X = 0$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

On solving, we get eigenvector $X_3 = k_3 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

2.Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

[Ans. $\lambda = 5,1,1$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$]

3. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

[Ans. $\lambda = 2,2,3$ and for $\lambda = 2, \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix}$; for $\lambda = 3, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$]

4. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

[Ans. $\lambda = 2,2,2$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$]

5. If $A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$ then find eigen values of $6A^{-1} + A^3 + 2I$. [Ans. 31, 13]

6. Find eigen values of $adj(A)$ if $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ [Ans. 48, 24, 12, 8]

7. If $A = \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix}$ find Eigen values of $A^2 + 2A + I$. [Ans. 36, 4]

8. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

[Ans. $\lambda = 0,3,15$ and $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$]

9. Obtain the eigenvalues and corresponding orthogonal eigenvectors of the symmetric matrix $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

[Ans. $\lambda = 1,3,6$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$]

10. Obtain the eigenvalues and corresponding orthogonal eigenvectors of the symmetric matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ [Ans: 2,2,8 and $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$]

11. Obtain the eigenvalues and corresponding orthogonal eigenvectors of the symmetric matrix $A=\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix}$ [Ans: -1,-1,5 and $\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}$]

12. Find Eigen values and Eigen vectors for the following matrices:

a. $\begin{bmatrix}0&1\\-1&0\end{bmatrix}$ [Ans. $\lambda=i,-i$ and $\begin{bmatrix}1\\i\end{bmatrix},\begin{bmatrix}-1\\i\end{bmatrix}$]

b. $\begin{bmatrix}2&-1\\1&2\end{bmatrix}$ [Ans. $\lambda=2+i,2-i$ and $\begin{bmatrix}i\\1\end{bmatrix},\begin{bmatrix}-i\\1\end{bmatrix}$]

13. Find the eigenvalues of the following matrices:

a. $\begin{bmatrix}0&-6&-12\\6&0&-12\\12&12&0\end{bmatrix}$ b. $\begin{bmatrix}0&2&-6\\-2&0&-9\\6&9&0\end{bmatrix}$ [Ans. a. $\lambda=0,18i,-18i$; b. $\lambda=0,11i,-11i$]

14. Find Eigen values and Eigen vectors for the following orthogonal matrix

$\begin{bmatrix}0&1\\1&0\end{bmatrix}$ [Ans. $\lambda=1,-1$ and $\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}$]

15. Find the eigenvalue of the given orthogonal matrix $A=\begin{bmatrix}\frac{4}{9}&\frac{8}{9}&\frac{1}{9}\\\frac{-7}{9}&\frac{4}{9}&\frac{-4}{9}\\\frac{-4}{9}&\frac{1}{9}&\frac{8}{9}\end{bmatrix}$

[Ans. $\lambda=1,\frac{7}{18}+\frac{5\sqrt{11}}{18}i,\frac{7}{18}-\frac{5\sqrt{11}}{18}i$]

Problems on Diagonalization of matrices

Similar Matrices, Diagonalisation

1. Find the algebraic multiplicity and geometric multiplicity of each eigen value of the matrix $\begin{bmatrix}4&6&6\\1&3&2\\-1&-5&-2\end{bmatrix}$.

   [Ans. $\lambda=1,2,2$ *for* $\lambda=1, a.m.=g.m.=1$ & $\lambda=2, a.m.=2, g.m.=1$]
2. Show that the following matrix is diagonalizable. Also find the diagonal matrix and

diagonalising matrix, $A=\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

Answer: Given $A=\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Trace of $A=12$

Sum of minors of diagonal elements of $A=36$

$|A|=32$

$\therefore$ characteristic equation is $\lambda^3-12\lambda^2+36\lambda-32=0$.

$\therefore \lambda=8,2,2$

For $\lambda=2$

Consider $AX=\lambda X \Rightarrow [A-\lambda I]X=0$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow[2R_3-R_1]{2R_2+R_1} \begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of coefficient matrix=1

Number of Variables=3

Number of independent solution= Number of Variables - Rank of coefficient matrix=3-1=2

Considering above matrix form in equation form,

$2x-y+z=0 \Rightarrow z=y-2x$

Put $x=s,\ y=t$ then $z=t-2s$

Solution is $\{(s,t,t-2s)/\ s,t\in R-\{0\}\}$

Linearly independent solution is $(0,1,1)'$ *and* $(1,0,-2)'$.

Arithmetic mean=2=Geometric mean

For $\lambda=8$

Consider $AX=\lambda X \Rightarrow [A-\lambda I]X=0$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Crammers Rule,

$$\frac{x}{\begin{vmatrix} -2 & 2 \\ -5 & -1 \end{vmatrix}}=\frac{-y}{\begin{vmatrix} -2 & 2 \\ -2 & -1 \end{vmatrix}}=\frac{z}{\begin{vmatrix} -2 & -2 \\ -2 & -5 \end{vmatrix}} \Rightarrow \frac{x}{12}=\frac{-y}{6}=\frac{z}{6}$$

Solution is $(2,-1,1)'$

Arithmetic mean=1=Geometric mean

Matrix A is diagonalizable.

$$D = M^{-1}AM \qquad \text{where } D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix},\ M = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}.$$

3. Show that the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is diagonalizable. Find the transforming matrix and the diagonal matrix.

[Ans. $M = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$]

4. Show that the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalizable. Find the transforming matrix M and the diagonal matrix D.

[Ans. $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}, D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$]

5. Show that the matrix $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$ is diagonalizable. Find the transforming matrix M and the diagonal matrix D.

[Ans. $M = \begin{bmatrix} 2 & 1 & 2 \\ -1 & -2 & -2 \\ 2 & 3 & 3 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$]

6. Check whether the following matrix is similar to diagonal matrix: $A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$.

[Ans. No]

7. Show that the matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ is not similar to diagonal matrix.