MTH215A : Number Theory: Assignment - 2

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1 Solutions

1.1 Q27

We have $\{x\} < 1 \; \forall \; x \in \mathbb{R}$ and $y = 1.5$ here. By Euler Summation Formula, we can write $\sum\limits_{1.5<n\leq x} \frac{1}{nlog(n)} = \int\limits_{1.5}^{x} \frac{dt}{tlog(t)} + \int\limits_{1.5}^{x} \frac{\{t\}\cdot(1-log(t))\cdot dt}{(tlog(t))^2} - \frac{\{x\}}{xlog(x)} + \frac{0.5}{1.5log(1.5} = log(log(x)) - \mathcal{O}(1) + \mathcal{O}(\frac{1}{xlogx}) - \mathcal{O}(1) - \mathcal{O}(\frac{1}{xlogx}) + O(1) = log(log(x)) + B + \mathcal{O}(\frac{1}{xlogx})$

$(\because \int\limits_{1.5}^{x} \frac{\{t\}\cdot(1-log(t))\cdot dt}{(tlog(t))^2} < \int\limits_{1.5}^{x} \frac{(1-log(t))\cdot dt}{(tlog(t))^2} = \mathcal{O}(\frac{1}{xlogx}) - \mathcal{O}(1))$.

1.2 Q28

$\sum\limits_{n\leq x} d(n) = xlog(x) + (2C-1)\cdot x + \mathcal{O}(\sqrt(x))$.

By Abel Summation Formula , we have $\sum\limits_{1.5<n\leq x} \frac{d(n)}{n} = \frac{xlog(x)+(2C-1)\cdot x+\mathcal{O}(\sqrt(x))}{x} + \mathcal{O}(1) - (-) \int\limits_{1.5}^{x} \frac{tlog(t)+(2C-1)\cdot t+\mathcal{O}(\sqrt(t))}{t^2} = log(x) + (2C-1) + \mathcal{O}(x^{-\frac{1}{2}}) + \mathcal{O}(1) + \int\limits_{1.5}^{x} \frac{log(t)+(2C-1)}{t} + \mathcal{O}(t^{-1.5}) = \frac{1}{2} \cdot \frac{log^2(x)}{x} + 2C \cdot log(x) + \mathcal{O}(1)$.

1.3 Q30

1.3.1

For the first part, we have $\mu * E = \phi \implies \sum\limits_{d|n} \mu(d) \cdot \frac{n}{d} = \phi(n)$.

$\implies \sum\limits_{n\leq x} \phi(n) = \sum\limits_{n\leq x} \sum\limits_{d|n} \mu(d) \cdot \frac{n}{d}$.

Let $q \cdot d = n$. Then, $\sum\limits_{n\leq x} \sum\limits_{d|n} \mu(d) \cdot \frac{n}{d} = \sum\limits_{d\leq x} \mu(d) \sum\limits_{q\leq \frac{x}{d}} q$

$= \sum\limits_{d\leq x} \mu(d)(\frac{\lfloor \frac{x}{d} \rfloor)^2 + \lfloor \frac{x}{d} \rfloor}{2}) = \frac{1}{2} \cdot \mu(d) \cdot \lfloor \frac{x}{d} \rfloor)^2 + \frac{1}{2}(\because \sum\limits_{n\leq x} \mu(d) \cdot \lfloor \frac{x}{n} \rfloor = 1)$. Hence, proved.

1.3.2

For the second part, $\sum\limits_{n\leq x} \mu(n) \cdot \lfloor \frac{x}{n} \rfloor^2 = \sum\limits_{n\leq x} \mu(n) \cdot (\frac{x}{n} - \{\frac{x}{n}\})^2 = \sum\limits_{n\leq x} \mu(n) \cdot (x^2 - 2x\{\frac{x}{n}\} + \{\frac{x}{n}\}^2) = x^2 \cdot \sum\limits_{n\leq x} \frac{\mu(n)}{n^2} - 2x \cdot \sum\limits_{n\leq x} \frac{\mu(n)}{n} \cdot \{\frac{x}{n}\} + \sum\limits_{n\leq x} \frac{\mu(n)}{n} \cdot \{\frac{x}{n}\}^2$.
$\sum\limits_{n\leq x} \frac{\mu(n)}{n^2} = \sum\limits_{n\leq \infty} \frac{\mu(n)}{n^2} - \sum\limits_{n>x} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} - \mathcal{O}(\frac{1}{x})$.
$\sum\limits_{n\leq x} \frac{\mu(n)}{n} \cdot \{\frac{x}{n}\} \leq \sum\limits_{n\leq x} \frac{1}{n} \cdot 1 (\because \mu(n) \text{ and } \{\frac{x}{n}\} \leq 1) = log(x) + C + \mathcal{O}(1) \implies \sum\limits_{n\leq x} \frac{\mu(n)}{n} \cdot \{\frac{x}{n}\} = \mathcal{O}(log(x))$.
$\sum\limits_{n\leq x} \mu(n) \cdot \{\frac{x}{n}\}^2 \leq x$.
$\implies \sum\limits_{n\leq x} \mu(n) \cdot \lfloor \frac{x}{n} \rfloor^2 = \frac{x^2}{\zeta(2)} - \mathcal{O}(x) + \mathcal{O}(xlogx) + \mathcal{O}(x) = \frac{x^2}{\zeta(2)} + \mathcal{O}(xlogx)$.
So, $\frac{1}{2} \cdot \sum\limits_{n\leq x} \mu(n) \cdot \lfloor \frac{x}{n} \rfloor^2 + \frac{1}{2} = \frac{x^2}{2\zeta(2)} + \mathcal{O}(xlogx) + \frac{1}{2} = \frac{x^2}{2\zeta(2)} + \mathcal{O}(xlogx)$. Hence, the second part is also proved.

1.4 Q32

We have, $\phi(n) = \sum\limits_{d|n} \mu(d) \cdot \frac{n}{d}$.
Observation: $\sum\limits_{1\leq x\leq\infty} \frac{\mu(x)}{x^2} = \sum\limits_{x=p_1p_2...p_k} \frac{\mu(x)}{x^2} = \sum\limits_{x=p_1p_2...p_k} \frac{(-1)^k}{p_1^2p_2^2...p_k^2} = \prod\limits_{p}(1 - \frac{1}{p^2})^{-1} = \frac{1}{\zeta(2)}$
Let $S = \sum\limits_{n\leq x} \frac{1}{n^2} \sum\limits_{d|n} \mu(d) \cdot \frac{n}{d} = \sum\limits_{qd\leq x} \frac{\mu(d)\cdot q}{(qd)^2} = \sum\limits_{qd\leq x} \frac{1}{q} \cdot \frac{\mu(d)}{d^2} = \sum\limits_{d\leq x} \frac{\mu(d)}{d^2} \sum\limits_{q\leq \frac{x}{d}} \frac{1}{q}$
$= \sum\limits_{d\leq x} \frac{\mu(d)}{d^2} \cdot (log(\frac{x}{d}) + C + \mathcal{O}(1))$ where C is Euler's constant.
$S = \sum\limits_{d\leq x} \frac{\mu(d)}{d^2} \cdot (log(x) + C + \mathcal{O}(1) - log(d)) = \sum\limits_{d\leq x} \frac{\mu(d)}{d^2} \cdot (log(x) + C + \mathcal{O}(1)) - \sum\limits_{d\leq x} \frac{\mu(d)log(d)}{d^2}$.
$\sum\limits_{d\leq x} \frac{\mu(d)log(d)}{d^2} = \sum\limits_{d\leq\infty} \frac{\mu(d)log(d)}{d^2} - \sum\limits_{d>x} \frac{\mu(d)log(d)}{d^2}$. Now, $\sum\limits_{d>x} \frac{-\mu(d)log(d)}{d^2} < \sum\limits_{d>x} \frac{1-log(d)-1}{d^2} (\because \mu(d) \leq 1) = \mathcal{O}(\frac{log(x)}{x}) + \mathcal{O}(\frac{1}{x})$. $\implies \sum\limits_{d\leq x} \frac{\mu(d)log(d)}{d^2} = A - \mathcal{O}(\frac{log(x)}{x})$.
Also, $\sum\limits_{d\leq x} \frac{\mu(d)}{d^2} = \sum\limits_{d\leq\infty} \frac{\mu(d)}{d^2} - \sum\limits_{d>x} \frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)} - \mathcal{O}(\frac{1}{x})$.
$S = \frac{1}{\zeta(2)} \cdot (log(\frac{x}{)} + C + \mathcal{O}(1)) - A + \mathcal{O}(\frac{log(x)}{x}) = \frac{log(x)}{\zeta(2)} + \frac{C}{\zeta(2)} - A + \mathcal{O}(\frac{log(x)}{x})$.

1.5 Q33

Since, $\sigma(n)$ and $\phi(n)$ are multiplicative functions, it is sufficient to prove this for $n = p^\alpha$. $\frac{\sigma(n)}{n} = \frac{p^{\alpha+1}-1}{p^\alpha\cdot(p-1)}$. $\frac{n}{\phi(n)} = \frac{1}{1-\frac{1}{p}}$. For the first inequality, we have $\frac{\sigma(n)}{n} < \frac{p^{\alpha+1}}{p^\alpha\cdot(p-1)} = \frac{p}{p-1} = \frac{n}{\phi(n)}$.
For the second inequality, $\frac{\pi^2}{6} = \prod\limits_{p}(1 - \frac{1}{p^2})^{-1}$. So, $\frac{\pi^2}{6} \cdot \frac{\sigma(n)}{n} = \prod\limits_{p}(1 - \frac{1}{p^2})^{-1} \cdot \frac{p^{\alpha+1}-1}{p^\alpha\cdot(p-1)} =$

$\prod\limits_{p'\neq p}(1-\frac{1}{p'^2})^{-1}\cdot\frac{p^{\alpha+1}-1}{p^\alpha\cdot(p-1)}\cdot(1-\frac{1}{p2})^{-1}$.

Now, $p^{\alpha+1}-p^{\alpha-1}\leq p^{\alpha+1}-1 \implies p\cdot p^\alpha\cdot(1-\frac{1}{p^2})\leq p^{\alpha+1}-1 \implies p\leq\frac{p^{\alpha+1}-1}{p^\alpha}\cdot(1-\frac{1}{p2})^{-1}$ $\implies \frac{p}{p-1}\leq\frac{p^{\alpha+1}-1}{p^\alpha\cdot(p-1)}\cdot(1-\frac{1}{p2})^{-1}\leq\prod\limits_{p'\neq p}(1-\frac{1}{p'^2})^{-1}\cdot\frac{p^{\alpha+1}-1}{p^\alpha\cdot(p-1)}\cdot(1-\frac{1}{p2})^{-1}=\frac{\pi^2}{6}\cdot\frac{\sigma(n)}{n}$.

Hence, both inequalities are proved.

By above inequalities, we have $\frac{n}{\phi(n)}$ is asymptotic to $\frac{\sigma(n)}{n}=\mathcal{O}(1)(\because\mathcal{O}(\frac{\sigma(n)}{n})=\mathcal{O}(\frac{p^{\alpha+1}-1}{p^\alpha\cdot(p-1)})=\mathcal{O}(\frac{p^{\alpha+1}-1}{p^{\alpha+1}-p^\alpha})=\mathcal{O}(1))$.

$\sum\limits_{n\leq x}\frac{n}{\phi(n)}=\sum\limits_{n\leq x}\mathcal{O}(1)=\mathcal{O}(x)$.

1.6 Q35

If $x=I\in\mathbb{Z}$, $\{x\}=\{-x\}=0 \implies \{x\}+\{-x\}=0$. Otherwise, $x=I+f, I\in\mathbb{Z}, f\in\mathbb{R}, 0\leq f<1 \implies \{x\}=f$ and $\{-x\}=\{-I-1+(1-f)\}=1-f \implies \{x\}+\{-x\}=1$.

Case I: Let $x=I+f, I\in\mathbb{Z}, f\in\mathbb{R}, 0\leq f<\frac{1}{2}$. $\lfloor 2x\rfloor-2\lfloor x\rfloor=\lfloor 2I+2f\rfloor-2\lfloor I+f\rfloor=0$.

$\lfloor x\rfloor+\lfloor x+\frac{1}{2}\rfloor=I+I=2I=\lfloor 2I+2f\rfloor=\lfloor 2x\rfloor$

Case II: Let $x=I+f, I\in\mathbb{Z}, f\in\mathbb{R}, \frac{1}{2}\leq f<1$. $\lfloor 2x\rfloor-2\lfloor x\rfloor=\lfloor 2I+2f\rfloor-2\lfloor I+f\rfloor=1$.

$\lfloor x\rfloor+\lfloor x+\frac{1}{2}\rfloor=I+I+1=2I+1=\lfloor 2I+2f\rfloor(\because 1\leq 2f<2)=\lfloor 2x\rfloor$.

Hence , proved.

1.7 Q37

Let $a=\sqrt{n}+\sqrt{n+1}$ and $b=\sqrt{4\cdot n+2}$ where $a,b\in\mathbb{R}$.

Observation 1: $b^2-a^2=4\cdot n+2-(2\cdot n+1+2\sqrt{n\cdot(n+1)})=(2\cdot n+1-2\sqrt{n\cdot(n+1)})$. $2\sqrt{n\cdot(n+1)}$ lies between $2n$ and $2n+1 \because n^2<n^2+n<n^2+n+\frac{1}{4}$. $\implies 0<b^2-a^2<1 \implies 4n+2=b^2>a^2>b^2-1=4n+1$. But, there are no perfect squares between $4n+2$ and $4n+1$. So, there are no integers between $\sqrt{4n+2}$ and $\sqrt{4n+1}$ which implies, $\lfloor b\rfloor=\lfloor a\rfloor$ as desired.

1.8 Q39

$\sum\limits_{n\leq x}\lambda(n)\lfloor\frac{x}{n}\rfloor=\sum\limits_{n\leq x}\sum\limits_{d|n}\lambda(d)$. Liouville's function $\lambda(n)=(-1)^{\Omega(n)}$. It is also multiplicative. So, n is square free, it is $\lambda(n)=-1$ otherwise it is 1.

Observation: If $n=p_1^{a_1}\cdot p_2^{a_2}\dots p_k^{a_k}$. Then, $\sum\limits_{d|n}\lambda(n)=\sum\limits_{n\leq x}(\lambda(1)+\lambda(p_1)+\dots+\lambda(p_1^{a_1}))\cdot(\lambda(1)+\lambda(p_2)+\dots+\lambda(p_2^{a_2}))\dots(\lambda(1)+\lambda(p_k)+\dots+\lambda(p_k^{a_k}))=(1-1+1\dots(-1)^{a_1})\cdot(1-1+1\dots(-1)^{a_2})\dots(1-1+1\dots(-1)^{a_k})=0$ if n is square-free otherwise, if n is a perfect square, this sum is 1.

$\sum\limits_{n\leq x}\sum\limits_{d|n}\lambda(d)=\sum\limits_{n\leq x}*$. In $*$, We only count those n which are perfect squares. And, the total number of perfect squares below x is $\lfloor\sqrt{x}\rfloor$. Hence, proved.

1.9 Q41

Take a empty set P.
Take any random number x_1 from the set S, remove it from S and add it to P and add another element x_2 of the set S, remove it and add it to P and so on until we have the all elements of set S in P.
Consider all the states of P that were formed in the intermediate steps to be elements of set T. So, $T = \{\{\}, \{x_1\}, \{x_1, x_2\}, \ldots, \{x_1, x_2 \ldots x_n\}\}$. Consider the n sum of elements of set $T(\text{mod } n) = \{0, x_1(\text{mod } n), (x_1 + x_2)(\text{mod } n), (x_1 + x_2 \cdots + x_n)(\text{mod } n)\}$.
We have $n+1$ remainders and n distinct possibilities of remainder from 0 to $n-1$. By Pigeonhole Principle, two of them must be same. Let the two corresponding set-elements be A and B. Then, the set $C = B - A$ where $|B| > |A|$ (otherwise vice-versa), is the desired subset as the difference of their sum of their elements is the sum of elements of the set $C = 0 \ (\text{mod } n)$.

1.10 Q42

We have $1365 = 3 \cdot 5 \cdot 7 \cdot 13$. By Fermat's Little Theorem, $\forall\ n \in \mathbb{N}$, we have
$n^2 \equiv 1(\text{mod } 3) \implies n^{12} \equiv 1(\text{mod } 3)$, $n^4 \equiv 1(\text{mod } 5) \implies n^{12} \equiv 1(\text{mod } 5)$, $n^6 \equiv 1(\text{mod } 7) \implies n^{12} \equiv 1(\text{mod } 7)$ and $n^{12} \equiv 1(\text{mod } 13)$.
So, as a primes are co-prime with each other, we have $n^{12} \equiv 1(\text{mod } 3 \cdot 5 \cdot 7 \cdot 13) \implies n^{12} \equiv 1(\text{mod } 1365)\ \forall\ n \in \mathbb{N} \implies n^{13} \equiv n(\text{mod } 1365)\ \forall\ n \in \mathbb{N}$

1.11 Q44

The equations are $x \equiv 1(\text{mod}4)$, $x \equiv 2(\text{mod}3)$ and $x \equiv 2(\text{mod}5)$. As $4, 3, 5$ are pairwise co-prime, we can use Chinese Remainder Theorem.
We have $M_1 = 3 \cdot 5, M_2 = 5 \cdot 4, M_3 = 4 \cdot 3$. So, $x = \sum_{1 \le i \le n} b_i \cdot M_i^{\phi(m_i)} = 1 \cdot 15^2 + 2 \cdot 20^2 + 2 \cdot 12^4(\text{mod } 60) = 225 + 800 + 41472(\text{mod } 60) = 17$.
Therefore, the general solution is $x = 60 \cdot t + 17\ \forall\ t \in \mathbb{Z}$.